Section 11.9

Penn State University

Math 141 - Section 001 - Summer 2016

11.9: Representations of Functions as Power Series

(When we say "expand $f(x)$ around $x = a$," it means "get a power series representation of $f(x)$ centered at $x = a$.")

The power series representation $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ was proved in an earlier section. This formula can be used to find power series representations of some rational functions (see Examples 1-3). Note that $\frac{1}{1-x}$ is defined everywhere except $x = 1$, but its power series representation is defined *only for* $|x| < 1$. Pick $x = -1$, for example. The function $f(x) = \frac{1}{1-x}$ evaluates to $f(-1) = \frac{-1}{2}$, but $\sum_{n=0}^{\infty}(-1)^n$ doesn't converge.

Let's play around with this function a little bit more. When we expand $f(x) = \frac{1}{1-x}$ around $x = 0$, the power series didn't carry any information of f at $x = -1$. Suppose that, for whatever reason, we are interested in knowing how f behaves near $x = -1$. To do so, we expand f at $x = -1$.

Exercise 1. Find the power series expansion of $\frac{1}{1-x}$ at $x = -1$, i.e. find coefficients a_n of $\sum a_n(x+1)^n$.

<u>*Solution*</u>:

$$\frac{1}{1-x} = \frac{1}{2-(x+1)} = \frac{1}{2} \cdot \frac{1}{1-\frac{x+1}{2}} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x+1}{2}\right)^n.$$

This series converges when $\left|\frac{x+1}{2}\right| < 1 \iff |x+1| < 2 \iff -3 < x < 1$. Note that the interval of convergence for this expansion is larger than the expansion at $x = 0$. The expansion at $x = -1$ agrees with the expansion at $x = 0$, i.e. $\sum_{n=0}^{\infty} \left(\frac{x+1}{2}\right)^n = \sum_{n=0}^{\infty} x^n$, when $|x| < 1$.

Exercise 2. (Ex1-3) Find a power series representation of $\frac{1}{1+x^2}$, and discuss its interval of convergence. Do the same for $\frac{1}{x+2}$ and $\frac{x^3}{x+2}$

Theorem $\boxed{2}$ says that the derivative of a function defined by a power series is obtained by term-by-term differentiation, and that integration works in the same manner. In other words, you can treat power series like polynomials when differentiating and integrating. Moreover, these operations do not change the interval of convergence.

Combining Theorem $\boxed{2}$ and $\boxed{1}$, we can find power series representations for a wider variety of functions.

Exercise 3. (Ex5) Find a power series representation of $\frac{1}{(1-x)^2}$ by differentiating $\frac{1}{1-x}$.

Exercise 4. (Ex6) Find a power series representation of $\ln(1+x)$ by integrating $\frac{1}{1+x}$.

Exercise 5. (Ex7) Find a power series representation of $\tan^{-1} x$ by integrating $\frac{1}{1+x^2}$.

Problems

Find a power series representation of the function and determine the interval of convergence.

1. $\frac{1}{x+1}$
2. $\frac{1}{x+2}$
3. $\frac{x^2}{x+2}$
4. $\frac{x}{2x^2+1}$
5. $\frac{5}{1-4x^2}$

6. $\frac{3}{x^2-x-2}$ (use partial fraction)

7. $\ln(3-x)$

8. $\ln(x^2+4)$

9. $x^2\tan^{-1}(x^3)$

10. $\frac{x}{(1+4x)^2}$

11. $\frac{(x^2+x)}{(1-x)^3}$