Section 10.1

Penn State University

Math 141 - Section 001 - Summer 2016

10.1: Curves defined by parametric equations

There are common geometric objects that are not possible to represent mathematically as the graph of a function. The circle is an archetypical example. The graph of $f(x) = \sqrt{1-x^2}$ in the region $x \in [-1, 1]$ is the upper half of the circle centered at the origin of radius 1. In order to describe the lower half, we need another function $g(x) = -\sqrt{1-x^2}$. The fact that we need two functions to describe the object is not only cumbersome, but also brings about issues; for example, finding the equation of the tangent line at the points where $f(x)$ and $g(x)$ intersect, namely, $(1, 0)$ and $(-1, 0)$ (try it). The main limitation in using graphs to represent a geometric object is that you can assign only one y-value to each x-value. One way to get around this limitation is to describe the circle by a Cartesian equation: $x^2 + y^2 = 1$. Another is to use a *parametric equation*. Let $f(t) = \cos t$ and $g(t) = \sin t$. Then, define a (vector-valued) function $(f(t), g(t))$. This is a function that take a value for t, and returns a point in the plane, where $f(t)$ is the x-coordinate, and $g(t)$ is the y-coordinate. We say that the parametric equation $x = f(t), y = g(t)$ $(0 \leq t \leq 2\pi)$ represents the circle.

You should think of the parametric equation $(f(t), g(t))$ as a function whose domain is *time t*. Suppose that the parametric equation gives a curve $\mathcal{C}$. Imagine that you are a point on $\mathcal{C}$. At time $t = 0$, you sit at the point $(f(0), g(0))$ on the curve, and you move forward as t grows.

Exercise 1. (Ex2) Convert the parametric representation of the circle into Cartesian.

Exercise 2. (Ex3) What curve is represented by $x = \sin 2t$, $y = \cos 2t$?

Exercise 3. Find parametric equations for the circled centered at $(-1, 2)$ and radius 2.

Exercise 4. Sketch $x = \sin t$, $y = \sin^2 t$

A parametric representation has the advantage over Cartesian that it is a collection of functions. This allows us to do calculus on the geometric object it describes, which is the topic of discussion in the next section.

Problems

Eliminate the parameter to find a Cartesian equation of the curve

1. $x = \cos\theta, y = \sin\theta \; (-\pi \leq \theta \leq \pi)$
2. $x = 2\cos\theta, y = 2\sin\theta \; (0 \leq \theta \leq 2\pi)$
3. $x = 3 - 4t, y = 2 - 3t$
4. $x = 1 - 2t, y = \frac{1}{2}t - 1 \; (-2 \leq t \leq 4)$
5. $x = \sqrt{t}, y = 1 - t$
6. $x = \frac{1}{2}\cos\theta, y = 2\sin\theta \; (0 \leq \theta \leq 2\pi)$
7. $x = \sin t, y = \csc t$
8. $x = e^{2t}, y = e^{2t}$