Section 6.1

Penn State University

Math 141 - Section 001 - Summer 2016

6.1: Inverse functions

An **invertible function** is a function whose effect can be undone (p.385 Figure 5). Note that not every function is invertible. A simple example of a non-invertible function is $f(x) = x^2$. A geometric way to check whether a function is invertible is the **horizontal line test** (p.384). Taking a derivative gives an algebraic method: if $f'(x) > 0$ or $f'(x) < 0$ for all x, then f is one-to-one.

Read Example 1 and 2. Then consider the following problem.

Exercise 1. Is $f(x) = (x-1)(x-3)^2$ invertible?

Solution: Draw the graph (by hand or go to WolframAlpha). You find that the horizontal line $y = 0$ intersects the graph twice. Therefore, f is not invertible.

Read Definition 2. Pay attention to the domains and ranges of f and f^{-1}. We will discuss why it is important to note where the range of domain of the inverse function f^{-1} in the next section.

Read Page 386 carefully. $\boxed{3}$ and $\boxed{4}$ are just reformulations of the definition of invertible functions (i.e. that f^{-1} reverses the effect of f).

$\boxed{5}$ discusses a way to find the inverse function f^{-1}.

Exercise 2. Find the inverse function of $f(x) = \frac{1-\sqrt{x}}{1+\sqrt{x}}$.

Solution: Set $y = \frac{1-\sqrt{x}}{1+\sqrt{x}}$. Our goal is to solve for x. Multiply both sides by $1+\sqrt{x}$ to get

$$(1+\sqrt{x})y = 1-\sqrt{x}.$$

Then, move the terms around

$$\begin{aligned} y + y\sqrt{x} &= 1-\sqrt{x} \\ \iff (1+y)\sqrt{x} &= 1-y \\ \iff \sqrt{x} &= \frac{1-y}{1+y}. \end{aligned}$$

Square both sides to conclude that $x = \left(\frac{1-y}{1+y}\right)^2$. So $f^{-1}(x) = \left(\frac{1-x}{1+x}\right)^2$.

Exercise 3. Check the solution of the previous exercise. In other words, show that $f(f^{-1}(x)) = x = f^{-1}(f(x))$.

Theorem 7 is the most important result from this section. It says that you can evaluate the derivative of f^{-1} at a point indirectly, only using f' and f^{-1}. Take a look at Example 7. The problem asks to find $(f^{-1})'(1)$ for $f(x) = 2x + \cos x$. The first thing that one tries (at least I do) is to find the explicit formula of f^{-1}, differentiate f^{-1}, then evaluate the derivative at 1. So set $y = 2x + \cos x$, then try to solve for x...But how? It turns out there isn't an easy way to find the inverse of this function. In fact, the inverse of the function exists, but we don't have a good way of describing it (input "inverse of $2x + cos(x)$" into WolframAlpha see what it returns). Now, Theorem 7 comes to rescue, which tells you that $(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))}$. We know that $f'(x) = 2 - \sin(x)$, so we can reduce the last expression to $\frac{1}{2-\sin(f^{-1}(1))}$. The last missing piece is the value of $f^{-1}(1)$. We don't know the formula of f^{-1}, so it might seem that we're stuck again. However, there's a way to work around this issue. Set $y = f^{-1}(1)$ for the sake of notation. By definition, y is the value such that $f(y) = 1$. With a little inspection, we see that $y = 0$ works, because $f(0) = 0 + \cos 0 = 1$. So we conclude that $y = 0$. (Since f is one-to-one, we know that there is only one such value.) Therefore, $0 = y = f^{-1}(1)$, and the solution is $\frac{1}{2-\sin(f^{-1}(1))} = \frac{1}{2-\sin 0} = \frac{1}{2}$.

Problems

1. (Exam 1 Sample A) Let $f(x) = 4x + \cos \pi x$. Find $(f^{-1})'(3)$.
2. (Exam 1 Sample C) Suppose that f^{-1} is the inverse function of a differentiable function f such that $f(2) = 5$ and $f'(2) = \frac{1}{3}$. Find $(f^{-1})'(5)$.