Section 11.5

Penn State University

Math 141 - Section 001 - Summer 2016

11.5: Alternating Series

In sections 11.3 and 11.4, we discussed a few convenient tests for convergence of a series, but note that the tests in those sections are only applicable to series corresponding to *positive* sequences. In the present and next sections, we discuss how to deal with series with negative terms.

We've seen that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. However, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is convergent. When signs of the terms alternate

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \cdots ,$$

$\frac{1}{2}$ cancels a bit of 1, $\frac{1}{4}$ cancels a bit of $\frac{1}{3}$, and so on. It turns out that the cancellations make the terms go to zero fast enough so that the series is convergent. A series of this form, i.e. the signs of terms alternate, is called an *alternating series*.

A generalization of the aforementioned phenomenon is the Alternating Series Test.

Theorem 1. Let $\{b_n\}$ be a positive sequence. If $\{b_n\}$ is decreasing and converges to zero, then the series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ is convergent.

Note that this test can only be used to test for convergence, i.e. you can't use this test to claim that a series is divergent.

Exercise 1. (Ex2,3) Does $\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$ converge? What about $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^3+1}$?

A convergent series can be estimated by its partial sums, and, we saw in Section 11.3 that, for a positive series, we could estimate how good (and bad) a partial sum was. An estimation can be done in a similar manner for alternating series.

Theorem 2. Suppose $\sum_{n=1}^{\infty}(-1)^{n-1} b_n$ satisfies $b_{n+1} \leq b_n$ and $\lim_{n\to\infty} b_n = 0$. Then, $|R_n| \leq b_{n+1}$, where R_n denotes the n^{th} remainder.

Exercise 2. (Ex4) Estimate $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ within 10^{-3}.

Problems

1. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$
2. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+4)}$
3. $\sum_{n=1}^{\infty} (-1)^n \frac{n}{\sqrt{n^3+2}}$
4. $\sum_{n=1}^{\infty} (-1)^n \frac{4n-1}{3n+1}$
5. $\sum_{n=1}^{\infty} (-1)^n \frac{n^n}{n!}$