Section 11.3

Penn State University

Math 141 - Section 001 - Summer 2016

11.3: The Integral Test & Estimates of Sums

You will see in 11.8–11.11 that every reasonably "nice" functions can be represented by a series. Here are some examples:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}; \quad \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \ (|x| < 1)$$

The main topic of our discussion until the second midterm is convergence tests of series, which will be used in 11.8–11.11 to decide whether a representation of a function by a series is valid. So far, we know the convergence condition for the geometric series, and have the Divergence Test .

This section introduces the Integral Test, and more tests are to come in subsequent sections. The Test says that a series is convergent if and only if the corresponding improper integral is convergent. To get an idea of why the Test works, consider the following two examples, which are discussed in the beginning of the section.

Exercise 1. Show that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

Solution: Consider a partial sum as a sum of rectangles below the graph of x^{-2}. Then, compare the area of the rectangles with the area under the graph.

Exercise 2. Show that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergent.

<u>*Solution*</u>: Consider a partial sum as a sum of rectangles *over* the graph of $x^{-1/2}$. Then, compare the area of the rectangles with the area under the graph.

Combining the Integral Test and a fact from 7.8 (the one I marked as extremely important), we get another extremely important fact.

Exercise 3. (Example 2) $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if and only if $p > 1$.

<u>*Solution*</u>: Consider three cases: $p < 0$, $p = 0$, and $p > 0$.

The series is called the *p-series*. Recall the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ from the previous section. The exercise above gives a proof that the harmonic series is divergent, which we left out in 11.2.

Exercise 4. (Example 1) Does $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converge?

<u>*Solution*</u>: This can be solved using the Integral Test or the Comparison Test, which is a topic of the next section.

The Integral Test is useful when the function corresponding to the sequence is easy to integrate. Just like any other nice theorems, it comes with disclaimers. The biggest caveat is that your sequence a_n needs to be *positive and decreasing*, so you can't use the test for series like $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ or $\sum_{n=1}^{\infty} \sin n + 1$.

Exercise 5. Does $\sum_{n=4}^{\infty} \frac{\ln n}{n}$ converge?

<u>*Solution*</u>: First, note that the series starts counting at $n = 4$, instead of the usual $n = 1$. In this case, we want to look at $\int_4^{\infty} \frac{\ln x}{x} dx$ (note the domain of integration). Second, we need to check if the Integral Test applies. To do so, we have to check if the function is positive and decreasing.

The last topic of this section is estimation of a series. Suppose you have a convergent series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ whose value you can't figure out, yet you want to know the value badly. You know that partial sums s_n converge to the value of the series, say L, i.e. s_n gets really close to L when n is large. So why not use a partial sum for some large N as an approximation of L? Now, the question is: how close is s_N to L?

Define the n^{th} *remainder* R_n by $R_n := L - s_n$. Suppose that you want to approximate L within 10^{-2}, i.e. we want to pick N large so $R_N < 10^{-2}$. The issue here is that, we don't know what L is, so we don't have a good way of analyzing $R_n = |s_n - L|$. Here's where math comes to rescue. A theorem (p742) says

$$\int_{n+1}^{\infty} f(x)dx \leq R_n \leq \int_{n}^{\infty} f(x).$$

(The inequalities come from comparing the remainder with approximations by rectangles from below and above.)

We have $\int_n^{\infty} x^{-3}dx = \frac{1}{2n^2}$ and $\int_{n+1}^{\infty} x^{-3}dx = \frac{1}{2(n+1)^2}$. So $\frac{1}{2(n+1)^2} \leq R_n \leq \frac{1}{2n^2}$. Therefore, to guarantee $R_n < 10^{-2}$, we need at least $n = 9$ (by inspection). Thus, by summing only the first nine terms of the sequence, we have a pretty good approximation. Furthermore, the theorem tells you how bad the approximation is: $\frac{1}{2(9+1)^2} = \frac{1}{200} < R_n$. Hence, this approximation is not good enough up to the third decimal point.

Problems

Determine whether the series converges.

1. $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n}$
2. $\sum_{n=1}^{\infty} \frac{1}{n^3}$
3. $\sum_{n=1}^{\infty} \frac{1}{3n+2}$
4. $\sum_{n=1}^{\infty} \frac{1}{n^{\pi}}$
5. $\sum_{n=5}^{\infty} n^{1-\sqrt{2}}$
6. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+4}}$
7. $\sum_{n=1}^{\infty} \frac{1}{n \ln n}$
8. $\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$