Section 6.8

Penn State University

Math 141 - Section 001 - Summer 2016

6.8: Indeterminate Forms

You might find it useful to read "Relative Rates of Growth," which can be found on the department's course website, alongside this section.

There are seven types of indeterminate forms discussed in this section:

1. $\frac{0}{0}$
2. $\frac{\infty}{\infty}$
3. $0 \cdot \infty$
4. $\infty - \infty$
5. 0^0
6. ∞^0
7. 1^∞

Consider the following: $\lim_{x\to\infty} \frac{\ln x}{x-1}$. Both the numerator and denominator goes to 0, so this is an indeterminate form of type ∞/∞. Remember that this does **not** mean that the limit doesn't exist. To evaluate this expression, we need to manipulate the quotient. Before we get to the details, let's try to guess what the answer is. You should know that $\ln x$ grows very slowly as x tends to ∞, as it is the inverse of e^x, which grows very fast. The growth of $\ln x$ is slower than x, so, as x gets larger, the denominator of $\frac{\ln x}{x-1}$ becomes much larger than the numerator, so the fraction should go to zero. Our guess is that the expression $\lim_{x\to\infty} \frac{\ln x}{x-1}$ should evaluate to 0, and l'Hospital's rule is what we use to do it rigorously.

$$\lim_{x\to\infty} \frac{\ln x}{x-1} = \lim_{x\to\infty} \frac{1/x}{1} = 0.$$

Roughly, l'Hospital's rule is the following: if $f(a) = g(a) = 0$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\text{slope of } f(x) \text{ at } x=a}{\text{slope of } g(x) \text{ at } x=a}$. "Slope of $f(x)$ at $x = a$" is $f'(a)$, so we get the formula $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$. You can use l'Hospital's when the limit has the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, but not for other indeterminate forms.

Exercise 1. (Example 1) $\lim_{x\to 1} \frac{\ln x}{x-1}$

Exercise 2. (Example 2) $\lim_{x\to\infty} \frac{e^x}{x^2}$

Exercise 3. (Example 4) $\lim_{x\to 0} \frac{\tan x - x}{x^3}$

It's important to keep in mind that you can apply l'Hospital's rule only for $0/0$ and ∞/∞. The following is an example where the rule fails.

Exercise 4. Find $\lim_{x\to\infty} \frac{x+\sin x}{x}$.

Solution: (Wrong solution) By l'Hospital, $\lim_{x\to\infty} \frac{x+\sin x}{x} = \lim_{x\to\infty} \frac{1+\cos x}{1}$. This limit does not exist, because $\cos x$ diverges.

This solution is incorrect, because the limit actually does exist! $\lim_{x\to\infty} \frac{x+\sin x}{x} = \lim_{x\to\infty} 1 + \frac{\sin x}{x} = 1$. Why does l'Hospital fail in this example? To apply l'Hospital's rule, the limit in question has to satisfy the following conditions:

1. $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ or $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) \pm \infty$
2. $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists (finite or infinite)
3. $g'(x) \neq 0$ for each $x \neq a$ in some open interval containing a.

The solution is incorrect, because the limit does not satisfy the second condition.

Here's another example where l'Hospital's rule fails.

Exercise 5. (Example 5) $\lim_{x\to\pi^-} \frac{\sin x}{1-\cos x}$

There are ways to use l'Hospital's rule to deal with indeterminate forms other than $0/0$ and ∞/∞. It involves manipulating the indeterminate form so the rule can be applied.

Exercise 6. (Example 6) Show that $\lim_{x\to 0^+} x\ln x = 0$.

Solution: We have $\lim_{x\to 0^+} x = 0$ and $\lim_{x\to 0^+} \ln x = -\infty$, so $\lim_{x\to 0^+} x\ln x$ has the indeterminate form $0\cdot(-\infty)$. So l'Hospital's rule cannot be applied. However, if we write $x\ln x = \frac{\ln x}{1/x}$, then we have an indeterminate form of the type $-\infty/\infty$, to which l'Hospital's rule applies! We get $\lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} -x = 0$.

Exercise 7. (Example 7) Sketch the graph of $f(x) = xe^x$.

Exercise 8. (Example 8) $\lim_{x\to\frac{\pi}{2}^-} \sec x - \tan x$

Indeterminate forms with exponents, i.e. 0^0, ∞^0, and 1^∞, can be solved using a principle that we've used several times in this course: when you see an expression involving powers, use log. Or you could use the definition of general exponential functions.

Exercise 9. (Example 9) $\lim_{x\to 0^+}(1+\sin 4x)^{\cot x}$

Exercise 10. (Example 10) $\lim_{x\to 0^+} x^x$

Problems

1. $\lim_{x\to\infty} \frac{e^{-x}}{x^2}$
2. $\lim_{x\to\infty} \frac{\ln\sqrt{x}}{x^2}$

3. $\lim_{x\to\infty} x\sin(\pi/x)$

4. $\lim_{x\to 1} \frac{x}{x-1} - \frac{1}{\ln x}$

5. $\lim_{x\to 0} \csc x - \cot x$

6. $\lim_{x\to 0^+} x^{\sqrt{x}}$

7. $\lim_{x\to\infty} x^{\frac{1}{\ln x}}$

8. $\lim_{x\to 1}(2-x)^{\tan(\pi x/2)}$