Section 11.11

Penn State University

Math 141 - Section 001 - Summer 2016

11.11: Applications of Taylor Polynomials

(The first half of this lecnote note covers what we skipped in 11.10.) We saw a few applications of series (in limit, differentiation, integration) in the previous sections. The final section of chapter 11 is devoted to what is in my opinion the most important application of series: approximation. Suppose that $f(x)$ can be expanded, say at $x = c$. Then, we get a representation of $f(x)$ as a series, which is almost like a polynomial: $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ (where $a_n = \frac{f^{(n)}(x}{n!}$). This object that is almost like a polynomial, in turn, can be approximated by a polynomial, since a convergent series is approximated by its partial sums. Thus, our goal is to approximate $f(x)$ by a *finite* sum $\sum_{k=0}^{n} a_k (x-c)^k$, that is, to ask how large n must be to ensure $\left| f(x) - \sum_{k=0}^{n} a_k (x-c)^k \right|$ is small.

We define the *n^{th} degree Taylor polynomial of f at a* to be $T_n(x) := \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!}(x-c)^k$. (This is the n^{th} partial sum equivalent of power series.)

Exercise 1. Write down for first few Taylor polynomials of $\frac{1}{1-x}$ expanded at $x = 0$. Do the same for e^x.

<u>*Solution*</u>: $(\frac{1}{1-x})$ $T_0(x) = 1$, $T_1(x) = 1 + x$, $T_2(x) = 1 + x + x^2$, $T_3(x) = 1 + x + x^2 + x^3$.

(e^x) $T_0(x) = 1$, $T_1(x) = 1 + x$, $T_2(x) = 1 + x + \frac{x}{2}$, $T_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!}$.

Plot the polynomials and function to understand how the polynomials approximate the function.

We also define the n^{th} remainder of the Taylor series to be $R_n(x) := f(x) - T_n(x)$. Note that, we have $\lim_{n\to\infty} R_n(x) = 0$ for each x if and only if $T_n(x)$ converges to $f(x)$ for each x. Here is the approximation theorem for R_n:

Theorem 1. (Taylor's inequality) If $\left|f^{(n+1)}(x)\right| \le M$ for $|x-a| \le d$, then we have

$$|R_n(x)| \le \frac{M}{(n+1)!}\,|x-a|^{n+1} \qquad \text{for } |x-a| \le d.$$

We have $R_n(x) = \sum_{k=n+1}^{\infty} \frac{f^{(k)}(x)}{k!}(x-a)^k = \frac{f^{(n+1)}(x)}{(n+1)!}(x-a)^{n+1} + \frac{f^{(n+2)}(x)}{(n+2)!}(x-a)^{n+2} + \cdots$, assuming that the sum converges. Taylor's inequality says that the magnitude of $R_n(x)$ is bounded by the upper bound on the magnitude of the first summand $\left|\frac{f^{(n+1)}(x)}{(n+1)!}(x-a)^{n+1}\right| \le \frac{M}{(n+1)!}\,|x-a|^{n+1}$.

Taylor's inequality has both theoretical and practical significance. This section is devoted to the practical side, but let me discuss briefly how it is useful theoretically before getting deep into estimation.

Exercise 2. (Ex2 of 11.10) Show that $f(x) = e^x$ equals its Maclaurin series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Solution: Fix $d > 0$ We have $f^{(m)}(x) = e^x$ for any m, so, in particular, $\left|f^{(n+1)}(x)\right| \le e^d$ for each n and each x satisfying $|x| \le d$. Therefore, by Taylor's inequality, $|R_n(x)| \le \frac{e^d}{(n+1)!}\,|x|^{n+1}$ for each n and $|x| \le d$. It follows that, if $|x| \le d$, then we have $\lim_{n\to\infty} |R_n(x)| = e^d \lim \frac{|x|^{n+1}}{(n+1)!} = 0$. Hence, if $|x| \le d$, then the Maclaurin series converges to $f(x)$. Since d we chose was arbitrary, it follows that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for any x.

Everything up to here is from 11.10. For a convergent series, s_n converges to the value of the series. For a function that has a convergent power series, $T_n(x)$ converges to $f(x)$ in the interval of convergence. We take $f(x) = \cos x$ as an example. Let's say we are interested in estimating how the function behaves in $[0,1]$ within the error of 10^{-4}. Its Maclaurin series is $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$, where the radius of convergence is ∞. So $T_n(x) = \sum_{k=0}^{n} (-1)^k \frac{x^{2k}}{(2k)!}$. As in the case of a series, the bigger n is, the closer is $T_n(x)$ to $f(x)$. Furthermore, the difference between $T_n(x)$ and $f(x)$ can be estimated by the Taylor's inequality. Since we are interested in the behavior in $[0,1]$ take $a = 0$ and $d = 1$, so we get an estimate in $[-1,1]$, which contains $[0,1]$. We have

$$f^{(n)}(x) = \begin{cases} \cos x & \text{if } n \text{ is a multiple of } 4 \\ -\sin x & \text{if } n = 4m+1 \\ -\cos x & \text{if } n = 4m+2 \\ \sin x & \text{if } n = 4m+3 \end{cases}$$

Therefore, $\left|f^{(n+1)}(x)\right| \leq 1$ for any n and x. In particular, we have $\left|f^{(n+1)}(x)\right| \leq 1$ for any n and $|x-0| \leq d = 1$, which the theorem calls for. Then, by Taylor's inequality,

$$R_n(x) \leq \frac{1}{(n+1)!}\left|x-0\right|^{n+1} \leq \frac{1}{(n+1)!}$$

since $|x| \leq 1$. To ensure $|R_n(x)| \leq 10^{-4}$ in $[-1,1]$, we need n to be at least 8, since $1/7! \approx 1.98 \times 10^{-4}$ and $1/8! \approx 0.248 \times 10^{-4}$.