Section 11.8

Penn State University

Math 141 - Section 001 - Summer 2016

11.8: Power Series

In the second half of Chapter 11, we discuss an application of series. As I mentioned in passing a couple times, many functions can be represented as series. Informally, you can think of a power series as a polynomial with infinitely many terms (so it is not a polynomial—a polynomial has only finitely many terms). To make a power series, you start with a sequence c_n, which we refer to as the "coefficients." The power series corresponding to c_n is $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$. Note that this is just an expression; *we can't think of this as a function yet*. We will discuss when we can treat a power series as a function later, but you can already guess that convergence has something to do with it.

Take $c_n = n$ for example. The corresponding power series is $\sum_{n=0}^{\infty} n x^n = 0 + x + 2x^2 + 3x^3 + \cdots$. Just like for polynomials, we think of x as a variable. For $x = 0$, the power series evaluates to $\sum_{n=0}^{\infty} n \cdot 0^n = 0 + 0 + 2 \cdot 0^2 + 3 \cdot 0^3 + \cdots = 0$. For $x = 1$, the power series evaluates to $\sum_{n=0}^{\infty} n \cdot 1^n = \sum_{n=0}^{\infty} n$. Lets pause for a moment and think what's going on. When we pick a value for x, we get back a series. So we can consider a power series as a function: set $f(x) = \sum_{n=0}^{\infty} n x^n$. As we showed already, $f(0) = 0$. But what about when $x = 1$? We saw that $f(1) = \sum_{n=0}^{\infty} n$. By the Divergence Test, this series is divergent. Hence, $f(x)$ is *not defined* at $x = 1$. In summary, this is how we define $f(x)$ using the power series $\sum_{n=0}^{\infty} n x^n$:

1. Pick $x = c$.

2. Consider the series $\sum_{n=0}^{\infty} nc^n$, and determine whether it converges.

3. If convergent, the function $f(x)$ is defined at $x = c$, and $f(c) = \sum_{n=0}^{\infty} nc^n$. Otherwise, $f(c)$ is undefined.

Try a few other values, and see if $f(x)$ is defined there.

Exercise 1. Show that $\sum_{n=0}^{\infty} nx^n$ is convergent when $|x| < 1$.

<u>*Solution*</u>: Let $a_n = nx^n$. We have

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)x^{n+1}}{nx^n} \right| = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) |x| = |x| .$$

Therefore, by the Ratio Test, the series is convergent when $|x| < 1$, and divergent when $|x| > 1$. When $|x| = 1$, the Ratio Test is inconclusive, so we need to do extra work for $x = \pm 1$. When $x = \pm 1$, the series is divergent by the Divergence Test. In conclusion, $\sum_{n=0}^{\infty} nx^n$ is convergent when $|x| < 1$, so the series as a function is defined for $|x| < 1$.

Exercise 2. (Ex1-3) Where is $f(x) = \sum_{n=0}^{\infty} n!x^n$ defined? What about $\sum_{n=0}^{\infty} \frac{(x-3)^n}{n}$ and $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$?

I said in the beginning that a power series is like a polynomial of an infinite degree. As much as there is a line between a series (which is like an infinite sum) and finite sum, you must make a clear distinction between a power series and polynomial. I hope the examples have elucidated their differences.

$\sum_{n=0}^{\infty} c_n(x-a)^n$ is called a power series "centered at a." In Examples 1-3, you only had to use the Ratio Test to determine when the series is convergent. When you testing for convergence properties of a power series, the Ratio Test gets the job done in almost every case, though the Root Test may be more convenient in some. This is due to the polynomial-like nature of power series. Theorem $\boxed{3}$ is a generalization of this observation.

Theorem 1. A power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ is either

1. convergent if and only if $x = a$ ($R = 0$);
2. convergent for all x ($R = \infty$); or
3. convergent if and only if $|x - a| < R$.

R above denotes the *radius of convergence*. For $x = a$, the power series is clearly convergent (and it converges to zero). So the question is how *far* x can get from a while maintaining the convergence of the power series, and the radius of convergence captures this distance. Thus, we get an interval centered at a where the power series is convergent. This interval is called the *interval of convergence*. Note that the Ratio Test gives convergence for $|x| < R$ and divergence for $|x| > R$, but it is inconclusive for $|x| = R \iff x = \pm R$. Therefore, there is an extra work to be done after determining the radius, namely to determine whether the power series converges at the end points of the interval of convergence.

Exercise 3. (Ex4,5) Find the radius of convergence and interval of convergence of $\sum \frac{(-3)^n x^n}{\sqrt{n+1}}$. Do the same for $\sum \frac{n(x+2)^n}{3^{n+1}}$.

Problems

For what values of x does the series converge?

1. $\sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$
2. $\sum_{n=0}^{\infty} \frac{(2x-5)^n}{n^4}$
3. $\sum_{n=0}^{\infty} \frac{(-3)^n}{n\sqrt{n}} x^n$
4. $\sum_{n=0}^{\infty} n^n x^n$
5. $\sum_{n=0}^{\infty} \frac{x^n}{3n!}$
6. $\sum_{n=0}^{\infty} \frac{(5x-3)^n}{n^2+1}$
7. $\sum_{n=0}^{\infty} \frac{n}{4^n}(x+1)^n$

8. (Sample A) $\sum_{n=1}^{\infty} \frac{(-1)^n (x-6)^n}{n 2^n}$

9. $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n!}$

10. $\sum_{n=1}^{\infty} \frac{3^n (x-1)^n}{n!}$