Section 11.2

Penn State University

Math 141 - Section 001 - Summer 2016

11.2: Series

Series is the object of study in this chapter. We will study convergence properties of series until the second midterm. You can think of a series as a sum of infinitely many numbers. You can also think of series as a discrete version of integration. In Section 7.8, we discussed what it means for an integral to converge or diverge. The first order of business is to define what a series is, and to define what it means for a series to converge.

One starts with a sequence to define a series. Let $\{a_n\}$ be a sequence. Then the corresponding series is $\sum_{n=1}^{\infty} a_n$ (you don't have to start from $n = 1$; any number works).

Let me start with a few examples before getting into the formal definition.

1. Consider the sequence $a_n = 1$. Then, the corresponding series is $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} 1$. Intuitively, this series diverges to ∞.

2. Consider the sequence $a_n = \left(\frac{1}{2}\right)^n$. The corresponding series is $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$. By a geometric argument, one can show that $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = 1$.

3. This is the last preliminary example and also my favorite one. This example illustrates why we should care about convergence of a series. Let $a_n =$

$(-1)^{n-1}$. Then, the corresponding series is $\sum_{n=1}^{\infty}(-1)^{n-1}$. Let's try to compute the value of this series with an elementary method. (Warning: whatever happens below is **wrong**.) First, write out terms of the series:

$$\sum_{n=1}^{\infty}(-1)^{n-1} = 1 + (-1) + 1 + (-1) + 1 + (-1) \cdots$$

Note that the pattern $1 + (-1)$ repeats. If we group up these pattern and do additions within each patten, we get

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^{n-1} &= (1 + (-1)) + (1 + (-1)) + (1 + (-1)) \cdots \\ &= (0) + (0) + (0) \cdots\end{aligned}$$

So the answer is 0! (Reminder: this is wrong.)

Actually, we can group terms together in a different way. Leave the first term on its own, and for the second term and on group together the pattern $(-1) + 1$:

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^{n-1} &= 1 + ((-1) + 1) + ((-1) + 1) + ((-1) + 1) \cdots \\ &= 1 + (0) + (0) + (0) \cdots \\ &= 1\end{aligned}$$

This time we get 1 as an answer. Therefore,

$$0 = \sum_{n=1}^{\infty}(-1)^{n-1} = 1$$

So we have a proof that $0 = 1$? Something must have went wrong. In fact, we use a definition of convergence of a series such that $\sum_{n=1}^{\infty}(-1)^{n-1}$ is a *divergent* series. It means that we don't assign a value to $\sum_{n=1}^{\infty}(-1)^{n-1}$ at all.

You must keep in mind that $\sum_{n=1}^{\infty}$ means something completely different from $\sum_{n=1}^{3}$, $\sum_{n=1}^{10}$, or $\sum_{n=1}^{k}$ for some finite k. The situation is similar to the distinction we made between improper integrals ($\int_a^\infty$) and integration over a finite domain (e.g. $\int_0^1$). The notation $\sum_{n=1}^{\infty}$ is just a placeholder for what is described below in the definition. You should know that a series is *not exactly* a sum of infinitely many numbers; the $\sum$ notation is there to remind you that it *kind of* behaves like it.

Definition 1. (Convergence of a series) Consider a series $\sum_{n=1}^{\infty} a_n$. We define the n^{th} partial sum s_n as $s_n := \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$. (Note that the n^{th} partial sum is a finite sum, so it always exists.) We say that the series is *convergent* if $\lim_{n\to\infty} s_n$ is convergent, and *divergent* otherwise. If the series is convergent, then $\sum_{n=1}^{\infty} a_n := \lim_{n\to\infty} s_n$.

As in the case of integrals, a series may "diverge to infinity," but we don't say that it "converges to infinity." Again, $\pm\infty$ is not considered a number.

Just like integrals, there aren't many series that we know how to evaluate. The geometric series,

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \quad \text{(if } |r| < 1\text{; the series is divergent otherwise)},$$

is one of the few series that we know the formula for.

Exercise 1. Prove the identity for the geometric series.

Exercise 2. (Example 4) Is the series $\sum_{n=1}^{\infty} 2^{2n} 3^{1-n}$ convergent?

Exercise 3. (Example 5) Write the number $2.3\overline{17} = 2.3171717\cdots$ as a ratio of integers.

Exercise 4. (Example 6) When is $\sum_{n=0}^{\infty} x^n$ convergent?

Exercise 5. (Example 7) Show that $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent, and its sum is 1.

The following theorem says that, if the tail of a sequence does not go to zero, then the series diverges.

Theorem 1. (Divergence Test) If $\lim_{n\to\infty} a_n$ does not converge or $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

The following is an equivalent statement of the Divergence Test, which says: if a series is convergent, then the tail of the corresponding sequence must go to zero.

Theorem 2. If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

The converse of the Divergence Test is not true, and don't ever forget this fact. There are sequences that go to zero, but the corresponding series diverges.

Exercise 6. Show that $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

This series is so important that it's given a name. It's called the harmonic series.

$\boxed{8}$ shows algebraic properties of $\sum_n$. Note that $\sum_n$ behaves like $\int$, i.e. it is linear, and you cannot distribute $\sum_n$ over a product, i.e. $\sum_n a_n b_n \neq \sum_n a_n \cdot \sum_n b_n$.

Problems

1. When is $\sum_{n=0}^{\infty} x^n$ convergent?
2. Compute $\sum_{n=0}^{\infty} x^n$ (assuming it's convergent).
3. Is $\sum_{n=1}^{\infty} \sin n$ convergent?
4. Is $\sum_{n=1}^{\infty} \ln n$ convergent?
5. Is $\sum_{n=1}^{\infty} \sqrt[n]{2}$ convergent?
6. Compute $\sum_{n=2}^{\infty} \frac{2}{n^2-1}$.
7. Compute $\sum_{n=5}^{\infty} \frac{1}{3^n}$.
8. Compute $\sum_{n=1}^{\infty} \frac{7^{n+1}}{10^n}$
9. Compute $\sum_{n=1}^{\infty} \frac{1+2^n}{3^n}$.