Section 10.3

Penn State University

Math 141 - Section 001 - Summer 2016

10.3: Polar Coordinates

Recall that $x^2 + y^2 = 1$ is an equation for the unit circle in the sense that it is the collection of all points satisfying the equation. "The collection of all points satisfying certain conditions (or restrictions)" is a common way of specifying a geometric object. $x^2 + y^2 = 1$ is a way of saying "the collection of all points whose sum of squares of the coordinates is 1." There is a much easier way of describing the unit circle, which is "the collection of all points whose distance from the origin is 1." It feels more natural to describe the unit circle in this way, as the description is the definition of the unit circle.

Now, lets try to formalize this observation. To do so, we need to introduce a new coordinate system. Usually, we use x- and y-coordinates (i.e. the Cartesian coordinate system) as the parameters to describe a point, but this is by no means the only way of parametrizing a point. The second most common coordinate system is the *polar coordinate system*. In this system, a point is descirbed using two parameters: $r =$ distance from the origin; and $\theta =$ angle from the positive x-axis. In the polar coordinate system, the unit circle has a simple representation, namely, $r = 1$, because this is exactly how we say "the collection of all points whose distance from the origin is 1."

Exercise 1. Plot $(r, \theta) = (1, 5\pi/4)$, $(2, 3\pi)$, $(2, -2\pi/3)$, and $(-3, 3\pi/4)$.

(x, y) and (r, θ) are related in the following way: $x = r \cos \theta$, $y = r \sin \theta$ (remember Math26?). These relations also give $r^2 = x^2 + y^2$, $\tan \theta = \frac{y}{x}$. So there is an easy way to go between the Cartesian system and the polar system of a point. (We will see later that changing the coordinate system of a *curve* is not that easy.)

Exercise 2. (Ex2,3) Convert $(r, \theta) = (2, \pi/3)$ to Cartesian. Convert $(x, y) = (1, \sqrt{3})$ to polar.

A *polar curve* is a curve written in the polar coordinates. We already saw an example: $r = 1$ is the unit circle. Similarly, $r = c$ is the circle centered at the origin with radius c. What if we want a circle centered off the origin?

Exercise 3. (Ex6) Sketch the curve with polar equation $r = 2 \cos \theta$. Find a Cartesian equation for this curve.

As this example shows, the polar system isn't great for shifting objects. Shifting (or translation) is better done in Cartesian coordinates. The polar system, on the other hand, is good at describing objects that wrap around the origin (like a planet orbiting around the sun).

Exercise 4. (Ex7,8) Sketch the curves $r = 1 + \sin \theta$ (cardioid) and $r = \cos 2\theta$.

Exercise 5. (Ex9) Sketch the curve $\theta = 1$.

Now, let's do calculus in polar coordinates. First, tangents. Using the derivative formula from the previous section and the chain rule (for example, for the numerator, differentiate $x = r \cos \theta$ with respect to θ to get $\frac{dx}{d\theta} = \frac{dr}{d\theta} \sin \theta - r \sin \theta$), we get

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{dr}{d\theta} \sin \theta + r \cos \theta}{\frac{dr}{d\theta} \cos \theta - r \sin \theta}.$$

Exercise 6. (Ex9) Find the slope of the tangent line of the cardioid ($r = 1 + \sin \theta$) at $\theta = \pi/3$. Find the points where the tangent line is horizontal or vertical.

Problems

Find a polar equation for the curve represented by the given Cartesian equation.

1. $y = -1$

2. $4y^2 = x$

Find a Cartesian equation for the curve.

1. $r = 4 \sec \theta$

2. $r = \cos \theta$

3. $\theta = \dfrac{\pi}{3}$

4. $r = \tan \theta \sec \theta$

Find a polar equation for the curve represented by the given Cartesian equation.

1. $y = x$

2. $4y^2 = x$

3. $xy = 4$

4. $y = 1 + 3x$

Sketch the graph of the polar curves.

1. $r = 1 + \sin \theta$

2. $r = \cos 2\theta$

3. $r = \sin \theta$

4. $r = 1 + \sin \theta$

5. $r = 1 - \cos \theta$

6. $r = 1 + 2\cos\theta$

7. $r = \ln\theta$

8. $r = \cos 5\theta$

9. $r = 2 + \sin\theta$

Find the slope of the tangent line to the given polar curve at the specified point.

1. $r = 2 - \sin\theta, \theta = \frac{\pi}{3}$

2. $r = \cos(\theta/3), \theta = \pi$