Section 11.1

Penn State University

Math 141 - Section 001 - Summer 2016

11.1: Sequences

A sequence is a discrete version of a function. The limit of a sequence is defined in essentially the same way (see $\boxed{1}$, $\boxed{2}$, and $\boxed{5}$). (Note that $\lim_{n\to\infty} a_n = \infty$ means something a bit different from the definition of limit when the limit is finite.) Most theorems that hold for limits of functions also hold for limits of sequences. One exception is l'Hospital's rule. However, there is a way to apply l'Hospital's rule indirectly to computing limits of sequences. Theorem $\boxed{3}$ says that a sequence obtained by discretizing a function has the same limit as the function.

Exercise 1. Find $\lim_{n\to\infty} \frac{\ln(1/n)}{n}$.

Solution: Note that this is an indeterminate form of type ∞/∞. Consider $f(x) = \frac{\ln(1/x)}{x}$, which is obtained by replacing n in the sequence with x. Our sequence is the discretization of $f(x)$. Then, the theorem says $\lim_{n\to\infty} \frac{\ln(1/n)}{n} = \lim_{x\to\infty} \frac{\ln(1/x)}{x} = 0$ by l'Hospital's rule.

Not every sequence is convergent. Take $a_n = \cos(n\pi)$ for example. $\lim_{n\to\infty} a_n$ does not exist by the same reason $\lim_{x\to\infty} \cos(\pi x)$ does not converge.

The following exercise shows that the converse of Theorem $\boxed{3}$ is not true.

Exercise 2. Show that $\lim_{n\to\infty}(-1)^{2n}$ (which is a sequence) converges, but $\lim_{x\to\infty}(-1)^{2x}$ (a limit of a function) diverges.

The table in p717 lists algebraic properties of limit. You see that the limit operation for sequences behaves in the same way as that for functions. lim is a nicer than $\int$ in that it distributes over multiplication, i.e. $\lim(a_n \cdot b_n) = \lim a_n \cdot \lim b_n$.

Theorem $\boxed{6}$ is extremely useful, and you will find yourself using it frequently. This is the statement: if the limit of the absolute values of terms is zero, then the original sequence also converges to zero.

Exercise 3. Show that $\lim_{n\to\infty} \frac{\sin n\pi}{n} = 0$.

<u>*Solution*</u>: We have $\left|\frac{\sin n\pi}{n}\right| \le \frac{1}{n}$. So $\lim_{n\to\infty} \left|\frac{\sin n\pi}{n}\right| \le \lim_{n\to\infty} \frac{1}{n} = 0$. Therefore, $\lim_{n\to\infty} \frac{\sin n\pi}{n} = 0$ by Theorem $\boxed{6}$.

Exercise 4. Evaluate $\lim_{n\to\infty} \frac{(-1)^n}{n}$.

Another useful theorem is the Squeeze Theorem (p718, above $\boxed{6}$). (You must've seen this theorem used for functions.)

Exercise 5. Show that $\lim_{n\to\infty} \frac{1}{n} \sin(1/n) = 0$.

<u>*Solution*</u>: Since $-1 \le \sin(1/n) \le 1$, we have $-\frac{1}{n} \le \frac{1}{n}\sin(1/n) \le \frac{1}{n}$. Both $\frac{1}{n}$ and $-\frac{1}{n}$ goes to zero as $n \to \infty$. Then, by the squeeze theorem, $\lim_{n\to\infty} \frac{1}{n}\sin(1/n) = 0$.

The last topic of this section gives a criterion for convergence. We will find this theorem used in determining convergence of series.

Theorem 1. (Monotonic Sequence Theorem) Every bounded, monotonic sequence is convergent.

Exercise 6. Show that $a_n = \tan^{-1}(n)$ is convergent.

Exercise 7. (Example 14) Determine whether the sequence defined by the following recurrence relation is convergent:

$$a_1 = 2, a_{n+1} = \frac{a_n + 6}{2}$$

If it is convergent, find its limit.

Problems

Determine whether the sequence converges or diverges. If it converges, find the limit.

1. $a_n = \frac{n}{n+1}$
2. $a_n = \frac{n}{\sqrt{n+10}}$
3. $a_n = \frac{1+n}{n^2}$
4. $a_n = (-1)^n$
5. $a_n = (-1)^{n^2}$
6. $a_n = (-1)^{2n}$
7. $a_n = \frac{3^n}{1+2^n}$
8. $a_n = 1 + \frac{4^n}{5^n}$
9. $a_n = \frac{5^n}{4^n}$
10. $a_n = n\sin(1/n)$
11. $a_n = ne^{-n}$
12. $a_n = \sqrt{\frac{3+2n^2}{8n^2+n}}$