Section 11.6

Penn State University

Math 141 - Section 001 - Summer 2016

11.6: Absolute Convergence and the Ratio and Root Tests

We learn three convergence tests in this section.

Theorem 1. If $\sum a_n$ is absolutely convergent, then $\sum a_n$ is convergent.

Theorem 2. (The Ratio Test) Given $\sum a_n$, let $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ (note the absolute value).

1. If $L < 1$, then the series is absolutely convergent.
2. If $L = 1$, then the test is inconclusive.
3. If $L > 1$, then the series is divergent.

Theorem 3. (The Root Test) Given $\sum a_n$, let $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$ (again, note the absolute value).

1. If $L < 1$, then the series is absolutely convergent.
2. If $L = 1$, then the test is inconclusive.
3. If $L > 1$, then the series is divergent.

In the previous section, we started our study of series whose terms are not necessarily positive. While the technique of the previous section only applies to a specific type of series, what we learn in this section applies to *any* series. The idea is the following. Say, we have a series $\sum a_n$ where some of a_n are negative. If we take absolute values of the terms, we get a positive series $\sum |a_n|$, which we know very well how to deal with, thanks to Sections 11.3 and 11.4. Thus, we have replaced the problem to something that we are more familiar with. Once we determine whether $\sum |a_n|$ is convergent, our task now is to make conclusions about $\sum a_n$ using the convergence property of $\sum |a_n|$.

If $\sum |a_n|$ is convergent, then we say that $\sum a_n$ is *absolutely convergent*. Note that absolute convergence is different from convergence. In fact, absolute convergence is stronger than convergence, meaning if $\sum a_n$ is absolutely convergent, then $\sum a_n$ is convergent. The converce is not necessarility true, i.e. there are series that are not absolutely convergent, but are convergent.

Exercise 1. Show that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is convergent, but not absolutely convergent.

It is usually the case that $|a_n|$ is easier to deal with than a_n itself.

Exercise 2. Is $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ convergent?

We have emphasized the point that, even if $a_n \to 0$ as $n \to \infty$, may not $\sum a_n$ converge; a_n has to go to zero fast enough for the convergence of series to happen. The idea behind the Ratio Test is to see how fast a_n approaches 0. You should think of $\frac{a_{n+1}}{a_n}$ as the rate of change. Then, $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ measures how fast the sequence a_n decreases towards the tail of the sequence. Intuitively, if the tail of a_n goes to 0 fast, then the series converges, and this is exactly what the Ratio Test states. On the other hand, the ratio is greater than 1, then the series is divergent.

The Ratio Test is gives you a convenient method to test for (absolute) convergence especially when the sequence involves powers and factorials.

Exercise 3. Show that $\sum (-1)^n \frac{n^3}{3^n}$ is absolutely convergent.

Exercise 4. Is $\sum \frac{n^n}{n!}$ convergent? (If each term is positive, convergence and absolute convergence are the same thing. Why?)

In the previous two examples, the terms decay (or grow) at least exponentially. The following example shows that the Ratio Test does not detect sub-exponential decay.

Exercise 5. Show that we cannot conclude the convergence properties of $\sum_{n=1}^{\infty} \frac{1}{n^2}$ or $\sum_{n=1}^{\infty} n$.

<u>*Solution*</u>: For $a_n = \frac{1}{n^2}$,

$$\lim_{n\to\infty} \left| \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} \right| = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^2} = 1.$$

For $a_n = n$,

$$\lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} 1 + \frac{1}{n} = 1.$$

So the tests are inconclusive.

This is a bit disturbing, as $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is obviously (absolutely) convergent, and $\sum_{n=1}^{\infty} n$ is obviously divergent. The Ratio Test is not very sentitive, so you should reserve its use for series involving high-growth expressions (i.e. exponential, factorial, etc.).

Just like the Ratio Test, the Root Test is useful when a_n has the form $(\ \mathit{Blah}\ )^n$, but not so much for factorial.

Exercise 6. Is $\sum \left(\frac{2n+3}{3n+2}\right)^n$ convergent?

Problems

Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

1. $\sum_{n=1}^{\infty} \frac{(-10)^n}{n^5}$

2. $\sum_{n=1}^{\infty} \frac{n!}{100^n}$

3. $\sum_{n=1}^{\infty} \frac{e^n}{2^{n+1}}$

4. $\sum_{n=1}^{\infty} \left(\frac{-2n}{n+1}\right)^{5n}$

5. $\sum_{n=1}^{\infty} \frac{(-1)^n \arctan n}{n^2}$

6. $\sum_{n=1}^{\infty} n \left(\frac{2}{3}\right)^n$

7. (Sample B) $\sum_{n=1}^{\infty} \frac{2^n n!}{e^{n^2}}$

8. (Sample B) $\sum_{n=1}^{\infty}(-1)^n(\ln(2n+1) - \ln(n+3))^n$

9. (Sample D) $\sum_{n=1}^{\infty}(-1)^n \frac{(n+4)2^n}{5^{n+1}}$

10. (Sample D) $\sum_{n=1}^{\infty}(-1)^n \frac{3^{2n+1}(n!)^2}{7^n(2n-2)!}$