Section 11.10

Penn State University

Math 141 - Section 001 - Summer 2016

11.10: Taylor and Maclaurin Series

In 11.8, we were given a power series, and asked where it converged. In 11.9, we were given a rational function (and its derivative and antiderivative), and asked to find its power series representations. This section is a continuation of 11.9. So far, we have heavily depended on the power series expansion of $\frac{1}{1-x}$, and the limitation of such approach is that it only provides a method to expand rational functions and functions related to rational functions. For example, e^x is not a rational function, and neither is it a derivative nor an antiderivative of a rational function. How do we find a power series expansion of such a function? We study Taylor series expansion in this section, which is a more general method of obtaining power series representations of functions.

Theorem 1. (Tayler series expansion) If f has a power series at a, then it is given by $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ ($|x-a| < R$ for an appropriate R), where $c_n = \frac{f^{(n)}(a)}{n!}$.

The derivation of this formula is discussed in p777. The Theorem gives you a formula to find the cofficient of a power series for $f(x)$. The Theorem comes with a disclaimer, namely that the formula holds if you know that a power series expansion of the function exists. How do we determine whether a power series expansion of f at a given point exists? Another natural question to ask is: how do we know that $\sum_{n=0}^{\infty} c_n(x-a)^n$ equals our function for every point in $(-R+$

$a, R+a)$? Answers to these questions are discussed in p779, but, in my opinion, the discussion is beyond the scope of this course.

In the following examples, don't forget to discuss convergence properties of obtained series.

Exercise 1. Find the Maclaurin series of e^x and $\sin x$.

Exercise 2. Find the Maclaurin series of $\cos x$ using the previous result.

Exercise 3. Find the Taylor series for $f(x) = e^x$ at $a = 2$.

Exercise 4. Find the Taylor series for $f(x) = \sin x$ at $a = \frac{\pi}{3}$.

Exercise 5. Find the Maclaurin series for $f(x) = x \cos x$.

The following two are applications of Taylor series.

Exercise 6. (Ex11) Evaluate $\int e^{-x^2} dx$ as an infinite series. Next, approximate $\int_0^1 e^{-x^2} dx$ within an error of 10^{-3}.

Solution: First, expand the function at 0: $e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$. Then, represent its integral as a power series: $\int e^{-x^2} dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)n!}$. Therefore, $\int_0^1 e^{-x^2} dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)n!}$, which we know how to estimate because it is an alternating series!

Exercise 7. Evaluate $\lim_{x \to 0} \frac{e^x - 1 - x}{x^2}$.

Problems

Expand the following functions at given points. Discuss convergence properties of obtained series.

1. $\frac{1}{\sqrt{4-x}}$ at 0
2. $\ln(1+x)$ at 0
3. $\cos(x^2)$ at 0

4. $x^4 - 3x^2 + 1$ at 1
5. $\ln x$ at 2
6. $\frac{1}{x}$ at -3
7. $\sqrt{x}$ at 4
8. $\cos x$ at π
9. (Sample B) e^{6x} at 0
10. (Sample A) $\cos(2x)$ at 0
11. (Sample D) $1/x^2$ at 1