Section 6.3

Penn State University

Math 141 - Section 001 - Summer 2016

6.3*: The Natural Exponential Function

exp is defined as the inverse of ln ($\boxed{2}$), and its expression is $\exp(x) = e^x$. Important properties of exp are:

1. exp is one-to-one, hence invertible (just like ln).
2. $\exp(\ln x) = x$, $\ln(\exp x) = x$, i.e. exp and ln are inverses of each other ($\boxed{2}$). We can rewrite this as $e^{\ln x} = x$ and $\ln(e^x) = x$ also ($\boxed{4}$ and $\boxed{5}$).
3. The range of exp is $(0, \infty)$, i.e. $\exp(x)$ is always positive.
4. $e^{x+y} = e^x e^y$; $e^{x-y} = \frac{e^x}{e^y}$; $(e^x)^y = e^{xy}$.

Just like ln, it's useful to remember what the graph of exp looks like. Use WolframAlpha to see the graph (try "exp(x)").

1. $\lim_{x\to\infty} e^x = \infty$ i.e. the graph goes to infinity as x gets bigger.
2. $\lim_{x\to-\infty} e^x = 0$, i.e. e^x is close to zero when x is a large negative number.
3. $e^0 = 1$. Compare this with $\ln 1 = 0$.

Two lectures ago, I made a small fuss about the range and domain of the inverse function. This is a good moment to discuss it. Go back to Definition $\boxed{2}$ in p385. Take $f(x) = \exp x$. Then, $f^{-1}(x) = \ln x$, and $f^{-1}(f(x)) = x$ for all x. Note that it does **not** make sense to write $f^{-1}(-1) = \ln(-1)$, because $\ln(x)$ is **not defined** at $x = -1$ or any negative x. So, if $f(x)$ were negative for some x, then we'd in a big trouble, because $f^{-1}(f(x))$ is not defined. But we don't get into such trouble at all. Definition $\boxed{2}$ says the domain of f^{-1} is the same as the range of $f = \exp$, which is $(0, \infty)$, all positive numbers.

Another good example to consider is $\tan x$. $\tan x$ is **not** one-to-one over the entire x-axis (it's a periodic function). However, it is one-to-one when restricted to $(-\pi/2, \pi/2)$, hence invertible. We call the inverse of tan as arctan. The domain and range of tan are $(-\pi/2, \pi/2)$ and $(-\infty, \infty)$, respectively. So the domain and range of arctan are $(-\infty, \infty)$ and $(-\pi/2, \pi/2)$.

Disregarding domains can become the source of all problems when doing integrations. For example, this integral $\int_{-3}^{4} \ln x dx$ makes no sense (why?). Just keep in mind that mindlessly taking inverses of functions can cause troubles. When something is wrong, think once again whether your function is invertible, and, if it is, figure out its domain.

Differential and integral properties of exp are quite simple:

$$\frac{d}{dx} e^x = e^x; \qquad \int e^x dx = e^x + C.$$

Your only task is to learn to use them well in conjunction with the chain rule and product rule.

Exercise 1. Find $\frac{d}{dx} e^{x^2}$.

<u>*Solution*</u>: By the chain rule, $\frac{d}{dx} e^{x^2} = (x^2)' e^{x^2} = 2x e^{x^2}$.

Exercise 2. n is an integer. Find $\frac{d}{dx} e^{x^n}$.

Problems

1. Find $(e^{2x})'$.
2. Find $\int e^{4x} dx$.
3. Find $(e^{x^2})'$.
4. Evaluate $\lim_{x\to\infty} e^{1/x^2}$.
5. Evaluate $\lim_{x\to\infty} \frac{x^3}{e^x}$.
6. Find $(\sin(e^t) + e^{\sin t})'$.
7. $\int e^x(4 + e^x)^5 dx$.
8. $\int e^x\sqrt{1 + e^x} dx$.
9. $\int e^{\tan x} \sec^2 x dx$.
10. (Exam 1 Sample C) Find the inverse function of f, $f^{-1}(x)$, if $f(x) = \frac{e^x}{1+e^x}$.
11. (Exam 1 Sample D) Find the derivative of $y = \sin^{-1}(e^{-x})$.
12. Find $\frac{d}{dx} e^{x \ln a}$ (a is a real number).