Section 10.4

Penn State University

Math 141 - Section 001 - Summer 2016

10.4: Areas and Lengths in Polar Coordinates

The area of a sector is given by $\frac{1}{2}r^2\theta$, and the following is a generalization of this formula. The area of the region between the origin and a polar curve $r = f(\theta)$ from $\theta = a$ to $\theta = b$ is given by

$$\int_a^b \frac{1}{2}(f(\theta))^2 d\theta = \int_a^b \frac{1}{2}r^2 d\theta.$$

Note that, when $f(\theta)$ is a constant function, we get the area of a sector.

Exercise 1. (Ex1) Find the area enclosed by one loop of $r = \cos 2\theta$.

Exercise 2. (Ex2) Find the area of the region that lies inside the circle $r = 3\sin\theta$ and outside the cardioid $r = 1 + \sin\theta$.

In the example above, the first step is to find points of intersections. However, you cannot find all of them algebraically. In this example, the two curves intersect at the origin, but we cannot find it by solving a system of equations. The origin does not play a role in measuring the area, so this fact was not a nuisance, but it's worth keeping in mind that the best way to see what curves do is to plot them.

Exercise 3. (Ex3) Find all points of intersection of the curves $r = \cos 2\theta$ and $r = \frac{1}{2}$.

We saw the formula for arc length for parametric equations in Cartesian coordinates. All we need to get the polar version of it is to do change of variables by $x = r\cos\theta$ and $y = r\sin\theta$. We get $L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$.

Exercise 4. (Ex4) Find the length of the cardioid $r = 1 + \sin\theta$.

Problems

Sketch the curve and find the area it encloses.

1. $r = 1 - \sin\theta$
2. $r = 4 + 3\sin\theta$

Find the area of the region enclosed by one loop of the curve.

1. $r^2 = \sin 2\theta$

Find the area of the region that lies inside the first curve and outside the second curve.

1. $r = 1 - \sin\theta$, $r = 1$
2. $r = 2 + \sin\theta$, $r = 3\sin\theta$
3. $r = 3\sin\theta$, $r = 2 - \sin\theta$

Find the area of the region that lies inside both curves.

1. $r = 1 + \cos\theta$, $r = 1 - \cos\theta$
2. $r = 3 + 2\cos\theta$, $r = 3 + 2\sin\theta$

Find the length of the polar curve.

1. $r = 5^\theta$, $0 \leq \theta \leq \pi$
2. $r = 2(1 + \cos\theta)$, $0 \leq \theta \leq 2\pi$