Section 10.2

Penn State University

Math 141 - Section 001 - Summer 2016

10.2: Calculus with Parametric Curves

Tangents

If we are given a curve represented as the graph of a function, we know how to compute the slope of the curve at a point, the area under the curve, the length of the curve, and so on. That is, we know how to do calculus on a curve. However, as we discussed in the previous section, graph is not necessarily the most convenient representation for every geometric object. This section discusses how to do calculus on a curve described by parametric equations.

Suppose we have a curve represented by $x(t)$ and $y(t)$, say, for any t. The slope at time $t = a$ is given by

$$\frac{d}{dx}|_{x(a)}y = \frac{\frac{d}{dt}|_{t=a}y}{\frac{d}{dt}|_{t=a}x} = \frac{y'(a)}{x'(a)},$$

It says that the slope at $(x(a), y(a))$ is obtained by taking the quotient of the speed in the y-direction over the speed in the x-direction at time a, which makes sense if you think geometrically. Note that the formula is ill-defined if $\frac{d}{dt}|_{t=a}x = 0$. Also, the formula is usually summarized as

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)}.$$

The curvature is determined using the following

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}.$$

It's easy to fool yourself into thinking that the curvature is given by $\frac{d^2y/dt^2}{d^2x/dt^2}$, which you might extrapolate from the slope formula, but it is not the case.

Exercise 1. (Ex1) Consider the curve $x(t) = t^2, y(t) = t^3 - 3t$.

1. Find tangents (there are two) at $(3, 0)$.

2. Find points on the curve where the tangent is horizontal or vertical.

3. Where is the curve concave downward?

Areas

Assume for now that the y-coordinate is a function of x, i.e. the way in which you are used to thinking. The area under the curve $y(x)$ from $x = a$ to $x = b$ is given by $\int_a^b ydx$. Now, we want to change our variable from x to t in order to get the area formula for parametric equations. x is a function of t, so $dx = x'(t)dt$. For the domain of integration, we need to find time t_0 and t_1 such that $a = x(t_0)$ and $b = x(t_1)$. Once that is done, we get

$$\int_a^b ydx = \int_{t_0}^{t_1} y(t)x'(t)dt.$$

Exercise 2. Find the area of the upper half unit disk.

<u>*Solution*</u>: The parametric equation for the upper half circle is $x = \cos\theta, y = \sin\theta$ for $0 \le \theta \le \pi$. The area under it is given by

$$\int_\pi^0 \sin\theta(\cos\theta)'d\theta = -\int_\pi^0 \sin^2\theta d\theta = -\int_\pi^0 \frac{1-\cos 2\theta}{2}d\theta = -\left(\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right)_\pi^0 = \frac{\pi}{2}.$$

Exercise 3. (Ex2,3) Consider $x(t) = r(\theta - \sin\theta)$, $y(t) = r(1 - \cos\theta)$. (The curve is called the *cycloid*.)

1. Compute dy/dx and d^2y/dx^2.

2. The curve consists of arches. Compute the area under one arch.

Arc Length

The arc length of $y(x)$ from $x = a$ to $x = b$ is given by

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Using $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ and $dx = \frac{dx}{dt}dt$, we get

$$L = \int_{t_0}^{t_1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

where t_0 and t_1 are chosen so that $a = x(t_0)$, $b = x(t_1)$.

Exercise 4. Find the formula for the circumference of a circle of radius r using parametric equations.

Exercise 5. (Ex5) Find the length of one arch of the cycloid (Exercise 3).

In the same manner, we can get a surface area formula for a surface of rotation, and it is

$$S = \int_{t_0}^{t_1} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Exercise 6. (Ex6) Show that the surface area of a sphere of radius r is $4\pi r$.

Problems

Find an equation of the tangent(s) to the curve at the point corresponding to the given value of the parameter.

1. $x = t\cos t$, $y = t\sin t$, $t = \pi$
2. $x = 6\sin t$, $y = t^2 + t$, $t = 0$

Find dy/dx and d^2y/dx^2.

1. $x = t^3 + 1$, $y = t^2 - t$.
2. $x = \cos 2t$, $y = \cos t$.

1. Find the points on the curve $x = t^3 - 3t$, $y = t^2 - 3$ where the tangent is horizontal or vertical.
2. Find the area enclosed by the curve $x = \sqrt{t}$, $y = t^2 - 2t$ and the x-axis.
3. Find the area enclosed by the curve $x = 1 + e^t$, $y = t - t^2$ and the x-axis.

Set up an integral that represents the length of the curve. Don't evaluate it.

1. $x = t^2 - t$, $y = t^4$, $1 \leq t \leq 4$
2. $x = t + \sqrt{t}$, $y = t - \sqrt{t}$, $0 \leq t \leq 1$

Find the length of the curve.

1. $x = e^t + e^{-t}$, $y = 5 - 2t$, $0 \leq t \leq 3$
2. $x = 3\cos t - \cos 3t$, $y = 3\sin t - \sin 3t$, $0 \leq t \leq \pi$
3. $x = \cos t + \ln(\tan \frac{t}{2})$, $y = \sin t$, $\frac{\pi}{4} \leq t \leq \frac{3\pi}{4}$

Find the area of the surface obtained by rotating the curve about the x-axis.

1. $x = 3t - t^3$, $y = 3t^2$, $0 \leq t \leq 1$