Section 7.8

Penn State University

Math 141 - Section 001 - Summer 2016

7.8: Improper Integrals

In a formal and rigorous definition of integration of a single-variable function, an integral is defined only over an interval of a *finite* length.* This is why an integral like $\int_0^\infty e^{-x}dx$, whose domain of integration is infinite, is called "improper."

We refer to an integral in one of the following three forms $\int_a^\infty f(x)dx$, $\int_{-\infty}^a f(x)dx$, and $\int_{-\infty}^\infty f(x)dx$ as an "improper integral of type 1." These are *defined* as follows:

$$\int_a^\infty f(x)dx := \lim_{t\to\infty}\int_a^t f(x)dx$$

if the limit exists;

$$\int_{-\infty}^a f(x)dx := \lim_{t\to-\infty}\int_t^a f(x)dx$$

if the limit exists;

$$\int_{-\infty}^\infty f(x)dx := \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx$$

if the two improper integrals on the right-hand side exist. a is any real number. An improper integral is called ***convergent*** **if the corresponding limit exists**. It is

* Newton and Leipniz invented the concept of integration, but it was Bernhard Riemann who made the theory of integration logically sound. The integration that we use in calculus courses is called "Riemann integration," in order to distinguish it from other definitions of integrations.

called *divergent* otherwise. The third definition requires a further attention, which we will get to later.

From a pragmatic point of view, you already know how to do integrals of this type, so why make a fuss? First of all, you should be aware of the precise definition of what you deal with, because that's the whole point of doing math. Also, there are situations where knowing the exact meaning of "plug in ∞" helps in computing integrals.

Exercise 1. (Example 2) $\int_{-\infty}^{0} xe^x dx$

<u>*Solution*</u>: If we start the problem in the way we are famililar with, $\int_{-\infty}^{0} xe^x dx = xe^x|_{-\infty}^{0} - \int_{-\infty}^{0} e^x dx = 0 \cdot 1 - (-\infty) \cdot 0 - \int_{-\infty}^{0} e^x dx$, we encounter an indeterminate form, which is bad. Try it again using the definition of an improper integral.

We say that an improper integral is *convergent* if the limit in the definition exists (i.e. finite) and *divergent* otherwise (i.e. infinite, the limit does not approach a single value, etc.).

Exercise 2. Is $\int_{1}^{\infty} \frac{1}{x^2} dx$ convergent? What about $\int_{1}^{\infty} \frac{1}{x} dx$?

The fact that $\int_{1}^{\infty} \frac{1}{x^p} dx$ is convergent if and only if $p > 1$ is worth remembering (Example 4). We will use this fact in Chapter 11.

Exercise 3. $\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx$

Exercise 4. Determine whether the integral $\int_{-\infty}^{\infty} x^3 dx$ is convergent.

<u>*Solution*</u>: (Wrong solution)

$$\begin{aligned}\int_{-\infty}^{\infty} x^3 dx &= \int_{0}^{\infty} x^3 dx + \int_{-\infty}^{0} x^3 dx = \lim_{t\to\infty} \int_{0}^{t} x^3 dx + \lim_{t\to-\infty} \int_{t}^{0} x^3 dx \\ &= \lim_{t\to\infty} \frac{t^4}{4} - \lim_{t\to-\infty} \frac{t^4}{4} = \lim_{t\to\infty} \frac{t^4}{4} - \lim_{t\to\infty} \frac{t^4}{4} = \lim_{t\to\infty} \left(\frac{t^4}{4} - \frac{t^4}{4}\right) = 0.\end{aligned}$$

(Correct solution)

$$\int_{0}^{\infty} x^3 dx = \lim_{t\to\infty} \int_{0}^{t} x^3 dx = \lim_{t\to\infty} \frac{t^4}{4}$$

So $\int_0^\infty x^3 dx$ diverges. **If one of the integrals obtained from splitting diverges, then we say that the original integral diverges.** This is how we defined this type of improper integral. Hence, $\int_{-\infty}^\infty x^3 dx$ diverges.

We will now discuss improper integrals of the second type. We have been integrating functions with asymptotes, such as $\frac{1}{x}$, without any caution, but we will do such integrals more carefully from now on.

Definition 1. (Improper integrals of type 2)

1. Suppose f is discontinuous at b. Then,

$$\int_a^b f(x)dx := \lim_{t\to b^-} \int_a^t f(x)dx$$

   if the limit exists.

2. Suppose f is discontinuous at a. Then,

$$\int_a^b f(x)dx := \lim_{t\to a^+} \int_t^b f(x)dx$$

   if the limit exists.

3. Suppose f is discontinuous at $c \in (a, b)$. Then,

$$\int_a^b f(x)dx := \int_a^c f(x)dx + \int_c^b f(x)dx$$

   if the two improper integrals on the right-hand side exist.

Exercise 5. (Example 5) $\int_2^5 \frac{1}{\sqrt{x-2}}dx$

Exercise 6. (Example 6, 7) Does $\int_0^{\pi/2} \sec x dx$ converge? What about $\int_0^3 \frac{dx}{x-1}$?

Exercise 7. Compute $\int_0^1 \ln x dx$.

<u>*Solution*</u>: $\int \ln x dx = x \ln x - x + C$ (integration by parts). So we want to evaluate $x \ln x - x$ at $x = 1$ and $x = 0$. But $x \ln x$ is *not* defined at $x = 0$, and

it is an indeterminate form of type $0 \cdot \infty$. This is where using the definition of an improper integral becomes useful, as in Exercise 1. $\int_0^1 \ln x dx$ is interpreted as $\lim_{t\to 0} \int_t^1 \ln x dx$. So instead of $x \ln x - x|_{x=0}^1$, we need to do $\lim_{t\to 0} x \ln x - x|_{x=t}^1$ to do the evaluations. The result is $\int_0^1 \ln x dx = \lim_{t\to 0} x \ln x - x|_{x=t}^1 = \lim_{t\to 0}(0 - 1) - t \ln t = -1$, because $\lim_{t\to 0} t \ln t = 0$.

In this example, we had a discontinuity at an edge of the region of integration. A more substantial modification is required when the integrand has a discontinuity inside the region of integration.

Exercise 8. Compute $\int_1^3 \frac{dx}{(x-2)^2}$

<u>*Solution*</u>: (Wrong solution) $\int_1^3 \frac{dx}{(x-2)^2} = -\frac{1}{x-2}|_{x=1}^3 = -2$.

This is wrong. In fact, the integral does not converge. The error in the computation above is in ignoring the discontinuity of $\frac{1}{(x-2)^2}$ at $x = 2$ (because the function is not defined there).

To get the correct answer, we need to use the definition of integrals whose integrands have discontinuities. To do so, we split the integral into two at the discontinuity, i.e. $\int_1^3 \frac{dx}{(x-2)^2} = \int_1^2 \frac{dx}{(x-2)^2} + \int_2^3 \frac{dx}{(x-2)^2}$. Note that the preceding equality does not hold unless we can show that the two integrals on the right-hand side are convergent. We have $\int_1^{2^-} \frac{dx}{(x-2)^2} = \lim_{t\to 2^-} \int_1^t \frac{dx}{(x-2)^2} = \lim_{t\to 2^-} -\frac{1}{x-2}|_{x=1}^t = \lim_{t\to 2^-} -\frac{1}{t-2} - 1 = \infty$. So the integral diverges.

The last topic in this section is Comparison Theorem, whose statement is the following:

Theorem 1. If f and g satisfy $f(x) \geq g(x) \geq 0$ for all $x \geq a$, then, we have

(i) If $\int_a^\infty f(x)dx$ is convergent, then $\int_a^\infty g(x)dx$ is convergent.

(ii) If $\int_a^\infty g(x)dx$ is divergent, then $\int_a^\infty f(x)dx$ is divergent.

Exercise 9. (Example 9) Show that $\int_0^\infty e^{-x^2} dx$ is convergent.

Exercise 10. (Example 10) Show that $\int_1^\infty \frac{1+e^{-x}}{x} dx$ is divergent.

Problems

Determine whether the following integrals converge or diverge. If convergent, also discuss what it converges to.

Attention: your solutions must be carefully written. In particular, your writing should indicate that you thoroughly understand the definitions of different kinds of improper integrals.

1. $\int_{-\infty}^{\infty} xe^{-x^2} dx$
2. $\int_{-\infty}^{\infty} \cos \pi t dt$
3. $\int_{1}^{\infty} \frac{\ln x}{x} dx$
4. $\int_{0}^{3} \frac{dx}{x-1}$
5. $\int_{-2}^{3} \frac{dx}{x^4}$
6. $\int_{\pi/2}^{\pi} \csc x dx$
7. $\int_{-\infty}^{0} \frac{dx}{3-4x}$
8. (Sample B #5) For which values of p does the following integral converge? $\int_{2}^{\infty} \frac{dx}{(x-1)^p}$
9. (Sample B #14) $\int_{-\infty}^{\infty} \frac{dx}{x^2+4}$.