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Survey papers

Various notions of chaos for discrete dynamical systems. A brief survey

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Dedicated to Professor János Aczél on the occasion of his eightieth birthday

Summary. In this brief survey we present and compare some of the most used notions of chaos for discrete dynamical systems.

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1. Introduction

In the last thirty years a large number of papers and books have been devoted to the study of discrete dynamical systems, i.e., the behaviour of the iterates of mappings from a set into itself. Usually the set involved is a topological space or, more often, a compact metric space and the mappings have some regularity property, usually continuity.

Thus, in this setting we assume that (X, d) is a compact metric space without isolated points and $f : X \to X$ a continuous map. To study the behaviour of the iterates of f means essentially to study the asymptotic properties of the orbits of the points of X, i.e., the sequences $\{f^n(x)\},\ n > 0$, where f^n denotes the n-iterate of f.

In this context naturally arises a notion of *chaotic behaviour*, i.e., roughly speaking, something which has to do with *unpredictability* and *sensitive dependence on initial conditions.*

Many authors made an effort to constrain this quite vague notion in a mathematical definition which, inevitably, depends on the problems it arises from.

The aim of this short survey is to present in a plain way some of these definitions, to compare them and to present some open problems. The target audience are graduate students of mathematics and physics or non-specialists in the field.

2. Various notions of chaos

Throughout this section we assume that f is a continuous map from a compact metric space without isolated points (X, d) into itself.

Topological chaos

An intuitive way to measure the complexity of a dynamical system consists in computing how many points are necessary in order to approximate (in some sense) with their orbits all other orbits.

A formalization of this leads to the notions of *topological entropy* and *sequence topological entropy*.

Let $A = \{t_j\}$ be an increasing sequence of positive integers, let $m > 0$ be an integer and $\varepsilon > 0$. A set $E \subset X$ is an (A, m, f, ε)-span, if for any $x \in X$ there is some $y \in E$ such that

$$d\Big(f^{t_j}(x), f^{t_j}(y)\Big) < \varepsilon, \quad \text{for} \quad 1 \le j \le m.$$

Let $S(A, m, f, \varepsilon)$ be an (A, m, f, ε)-span with minimal possible number of points.

Definition 1 ([14])**.** The *topological sequence entropy of f with respect to A* is

$$h_A(f) = \lim_{\varepsilon \to 0} \limsup_{m \to +\infty} \frac{1}{m} \log \#S(A, m, f, \varepsilon).$$

In the case $A = \mathbb{N}$ we obtain the *topological entropy* $h(f)$ of f.

Definition 2. A continuous map $f : X \to X$ is called **topologically chaotic** if its topological entropy $h(f)$ is positive.

Distributional chaos

Probabilistic metric spaces are generalizations of metric spaces in which the distances between points are probabilistic rather than numerical. Thus, with every pair of points x and y in a probabilistic metric space there is associated a probability distribution function Φ_{xy} whose value $\Phi_{xy}(t)$ is generally interpreted as the probability that the "distance" between x and y is less than t.

Thus, a way to measure the asymptotic behaviour of the distance between the orbits of two different points may use the previous notion. Motivated by this, B. Schweizer and J. Smítal in [31] introduced the concept of distributional chaos.

For any pair of points $x, y \in X$, we define a sequence $\delta_{xy}(m)$ by

$$\delta_{xy}(m) = d\big(f^m(x), f^m(y)\big), \quad m = 0, 1, \cdots .$$

Next we define

$$F_{xy}^{(n)}(t) = \frac{1}{n}\#\{m:\ 0 \le m \le n-1,\ \delta_{xy}(m) < t\ \}.$$

Each $F_{xy}^{(n)}$ is a left-continuous distribution function.

Now consider the functions F_{xy} and F_{xy}^* defined by

$$F_{xy}(t) = \liminf_{n\to+\infty} F_{xy}^{(n)}(t)$$

and

$$F_{xy}^*(t) = \limsup_{n\to+\infty} F_{xy}^{(n)}(t).$$

Clearly, we have $F_{xy}(t) \le F_{xy}^*(t)$.

We shall refer to F_{xy} as the *lower distribution*, and to F_{xy}^* as the *upper distribution* of x and y.

Definition 3. The mapping f is said to be **distributionally chaotic (d-chaotic)** if there exists a pair $x, y \in X$ such that

$$F_{xy}(t) < F_{xy}^*(t)$$

for t in some non-degenerate interval.

Li–Yorke chaos

Definition 4. A set $S \subset X$ (having at least two points) such that for any $x, y \in S$, $x \neq y$,

$$\limsup_{n\to+\infty} d\big(f^n(x), f^n(y)\big) > 0$$

$$\liminf_{n\to+\infty} d\big(f^n(x), f^n(y)\big) = 0$$

is called a **scrambled set**.

This definition addresses, at least in part, the question of *sensitive dependence on initial conditions*.

Thus the following definition seems reasonable.

Definition 5 ([24])**.** The map $f : X \to X$ is **chaotic in the sense of Li and Yorke** if it has a scrambled set S.

ω-chaos

Denote by $\omega_f(x)$ the set of limit points of the sequence $\{f^n(x)\}$.

Definition 6. A set $S \subset X$ (having at least two points) such that for any $x, y \in S$, $x \neq y$,

(i) $\omega_f(x) \setminus \omega_f(y)$ is uncountable,
(ii) $\omega_f(x) \cap \omega_f(y)$ is non-empty,
(iii) $\omega_f(x)$ is not contained in the set of periodic points

is called an ω**-scrambled set**.

In [23] the following definition has been presented (see also [35]).

Definition 7. The map $f : X \to X$ is ω**-chaotic** if there exists an uncountable ω-scrambled set S.

Martelli's chaos

Definition 8. The orbit of a point $x \in X$ is said to be **unstable** if there exists $r > 0$ such that for every $\varepsilon > 0$ there are $y \in X$ and $n \geq 1$ satisfying the two inequalities $d(x, y) < \varepsilon$ and $d(f^n(x), f^n(y)) > r$.

We are led to the following notion of chaos due to Martelli:

Definition 9 ([25])**.** The map f is **chaotic in the sense of Martelli** if there exists $x_0 \in X$ such that

i) the orbit of x_0 is dense in X;
ii) the orbit of x_0 is unstable.

Transitivity and Devaney's chaos

Definition 10. The map f is **(topologically) transitive** if for every pair of non-empty open sets U and V there is a positive integer k such that $f^k(U) \cap V \neq \emptyset$.

This is equivalent to the existence of a point x whose orbit is dense in X. Thus, there is an orbit that passes as close as we like to any point of X.

Now we present the definition of chaos due to R. L. Devaney ([10]), in the formulation of J. Banks, J. Brooks, G. Cairns, G. Davis and P. Stacey ([7]).

Definition 11 ([10], [7])**.** The map f is **chaotic in the sense of Devaney** if

i) f is topologically transitive;
ii) periodic points are dense in X.

These two properties imply that f has *sensitive dependence on the initial condition* in X, i.e., there exists $r > 0$ such that for every $x_0 \in X$ and every $\varepsilon > 0$, we can find a y_0 such that $d(x_0, y_0) < \varepsilon$ and, for some integer m, $d(f^m(x_0), f^m(y_0)) > r$.

It can be proven that, when the orbit of x_0 is dense in X, sensitive dependence on initial conditions is equivalent to the instability of the orbit of x_0. Thus, Devaney's notion of chaos implies Martelli's.

Block–Coppel chaos

Definition 12 ([6])**.** The map f is **chaotic in the sense of Block and Coppel** if there exist disjoint closed subsets X_0, X_1 of X and a positive integer m such that, if $\tilde{X} = X_0 \cup X_1$ and $g = f^m$, then

i) $g(\tilde{X}) \subset \tilde{X}$;

ii) for every sequence $\alpha = (a_0, a_1, a_2, \cdots)$ of 0's and 1's there exists a point $x = x_\alpha \in \tilde{X}$ such that $g^k(x) \in X_{a_k}$ for all $k \geq 0$.

Theorem 1. *A sufficient condition for f to be chaotic in the sense of Block and Coppel is that there exist disjoint non-empty closed subsets Y_0, Y_1 of X and a positive integer m such that*

$$Y_0 \cup Y_1 \subset f^m(Y_0) \cap f^m(Y_1).$$

3. The case of the real interval

Now we restrict ourselves to the case when $X = I$ is a compact interval. This case has been extensively studied and many results have been proven which permit to compare the previous definitions of chaos.

Before doing this it is better to present the celebrated theorem of Sharkovsky on the coexistence of cycles.

On the set $\mathbb{N}_s = \mathbb{N} \cup \{2^\infty\}$ we introduce the *Sharkovsky ordering* as follows:

$3 \succ 5 \succ 7 \succ \cdots \succ 2\cdot 3 \succ 2\cdot 5 \succ \cdots \succ 4\cdot 3 \succ \cdots \succ 2^n\cdot 3 \succ 2^n \cdot 5 \succ \cdots \succ 2^\infty \succ \cdots \succ 2^n \succ \cdots \succ 16 \succ 8 \succ 4 \succ 2 \succ 1$.

For $s \in \mathbb{N}_s$, $S(s)$ denote the set

$$\{\ k \in \mathbb{N}:\ s \succeq k\ \}.$$

The following theorem has been proven by Sharkovsky (1964):

Theorem 2 ([32])**.** *For every continuous map $f : I \to I$ there exists $m \in \mathbb{N}_s$ such that $S(m)$ is the set of periods of f.*

Conversely, for every $m \in \mathbb{N}_s$ there exists a continuous map $f : I \to I$ having exactly $S(m)$ as the set of its periods.

A map is said to be *of type greater than* 2^∞ if it has cycles of order different from a power of 2.

(For interesting historical comments on Sharkovsky's theorem see [28].)

"Distributional" versus "entropy"

The connection between these two notions of chaos is given by the following

Theorem 3 ([31])**.** *A continuous map f from a compact interval I into itself is d-chaotic if and only if it has positive topological entropy.*

"Li–Yorke" versus "entropy" (and "distributional")

First we state a theorem presenting some conditions which are equivalent to Li–Yorke chaos.

Theorem 4 ([15])**.** *For a continuous map f from a compact interval I into itself the following conditions are equivalent:*

i) *f is Li–Yorke chaotic;*

ii) *f has a nonempty perfect scrambled set;*

iii) *f is topologically conjugate to a function g (i.e., $g = h \circ f \circ h^{-1}$, h homeomorphism of I into itself) which has a scrambled set of positive Lebesgue measure.*

Li–Yorke chaos, order of cycles and topological entropy are related by the following results.

The first contains, in some sense, the main result of the original paper of Li and Yorke (1975).

Theorem 5 ([24])**.** *If a map f has a periodic point of period $k \neq 2^i$ $(i \in \mathbb{N})$, then it is Li–Yorke chaotic.*

In other terms, if a map is of type greater than 2^∞, then it is Li–Yorke chaotic.

The next theorem characterizes the maps of type greater than 2^∞ in terms of topological entropy.

Theorem 6 ([27])**.** *Let $f : I \to I$ be continuous. The function f has a cycle of order $k \neq 2^i$, if and only if $h(f) > 0$.*

Thus, all functions with positive topological entropy are Li–Yorke chaotic and, by Theorem 3, all d-chaotic functions are Li–Yorke chaotic.

The question arises for the functions of type less than or equal to 2^∞: Are they Li–Yorke chaotic or not?

The first answer is this:

Theorem 7 ([34]). *There exist functions of type 2^∞ which are Li–Yorke chaotic and there exist functions of type 2^∞ which are not Li–Yorke chaotic. Every function of type less than 2^∞ is non-chaotic.*

At this point the role of the sequence topological entropy appears essential for distinguishing the two types of mappings in the class of maps of type 2^∞.

Theorem 8 ([13]). *A continuous map $f : I \to I$ is Li–Yorke chaotic if and only if there exists an increasing sequence A of non-negative integers such that $h_A(f) > 0$.*

"ω-chaos" versus "entropy"

Theorem 9 ([23]). *A continuous map $f : I \to I$ is ω-chaotic if and only if it has positive topological entropy.*

"Entropy" versus "Devaney"

Arguing exactly as in the proof of Theorem 9 we get the following:

Theorem 10. *A continuous map $f : I \to I$ is chaotic in the sense of Devaney if and only if it has positive topological entropy.*

"Martelli" versus "Devaney"

First note that in the one-dimensional case the topological transitivity of f implies that periodic points are dense in I ([6]). Thus, in this setting Devaney's definition of chaos becomes simply the topological transitivity.

After Definition 10 we have shown that Martelli's chaos implies topological transitivity, thus

Theorem 11. *For a continuous map $f : I \to I$, Martelli's and Devaney's notions of chaos are equivalent.*

"Block–Coppel" versus "entropy"

For a continuous map of a compact interval I into itself, the definition of chaos in the sense of Block and Coppel is equivalent to the following:

The map f is chaotic if there exist compact intervals J and K with at most one common point and a positive integer m such that

$$J \cup K \subset f^m(J) \cap f^m(K).$$

This is (see [6]) equivalent to the condition that f has a periodic point whose period is not a power of 2. Hence, from Theorem 6 we have that a continuous map $f : I \to I$ is chaotic in the sense of Block and Coppel if and only if $h(f) > 0$.

We can summarize the results presented in this section with the following theorem.

Theorem 12. *For a continuous map $f : I \to I$ the following conditions are equivalent:*

(i) *f is topologically chaotic, i.e., has positive topological entropy;*

(ii) *f is distributionally chaotic;*

(iii) *f is ω-chaotic;*

(iv) *f is chaotic in the sense of Martelli;*

(v) *f is chaotic in the sense of Devaney;*

(vi) *f is chaotic in the sense of Block and Coppel.*

All previous properties imply that f is chaotic in the sense of Li and Yorke, but the converse is not true.

4. The general case

In the case of the real interval we have presented theorems relating various notions of chaos. In particular, we stated some equivalences.

What happens in general?

Things change completely and many problems are still open.

We follow the order of presentation of the various notions given in Section 2.

We have seen in Theorem 12 that all notions presented, with the exclusion of Li–Yorke chaos, are equivalent to topological chaos, i.e., the function involved has positive topological entropy.

In [12] it has been proven that, in general distributional chaos need not imply positivity of entropy, while the converse is an open problem.

Continuing, there is a recent paper [8] where the authors solve a long-standing open question by proving that positive topological entropy implies Li–Yorke chaos.

Since Devaney's (and Martelli's) chaos is based on the notion of transitivity, it is natural to ask for the relation between topological entropy and transitivity. Quoting [18] "the question whether the positivity of the entropy implies transitivity does not make sense since transitivity is a global characteristic, while the positivity of the entropy may be caused by the behaviour of the function on an invariant subset of the space." In the other direction, the problem in its generality appears still open (see [3] and [18]).

Open problems are also the relations between the positivity of topological entropy and Block–Coppel and ω-chaos.

Now, we compare distributional chaos with the other notions (except for positive entropy). According to Theorem 12 Li–Yorke chaos does not imply in general the distributional one, while in [5] an example is produced of a map on $\mathbb{R}^2$ which is distributionally chaotic but not chaotic in Li–Yorke sense. Thus, the two notions are independent.

Open problems are the relations between distributional chaos and Martelli's, Devaney's, Block–Coppel and ω-chaos.

It is now the turn of Li–Yorke chaos. By Theorem 12 Li–Yorke chaos does not imply in general ω-chaos, while in [22] it has been proven that an ω-chaotic map is always Li–Yorke chaotic.

We already know that in the case of the real interval Li–Yorke chaos does not imply Devaney's. Very recently Jie–Hua Mai in [16] proved that the reverse implication holds.

A question arises: does Martelli's or Block–Coppel chaos imply Li–Yorke? To my knowledge the problem is still open.

Relations between ω-chaos and Martelli's, Devaney's and Block–Coppel are not known to the author.

We compare now the two very similar notions of Martelli's and Devaney's chaos. As already seen in Section 2, Devaney's chaos implies Martelli's. On the other hand it is proven in [25] that the function

$$F(\rho,\theta) = \begin{cases} (2\rho, \theta+1), & 0 \le \rho < 0.5 \\ (2-2\rho, \theta+1), & 0.5 \le \rho \le 1 \end{cases}$$

which maps the unit disk of the plane into itself, is chaotic in the sense of Martelli, but not in the sense of Devaney.

Open problems appear to be the relations of Martelli's and Devaney's chaos with Block–Coppel chaos.

In order to summarize the previous result, we list the various notions of chaos as follows:

(1) topological chaos, i.e., positive topological entropy;

(2) distributional chaos;

(3) Li–Yorke chaos;

(4) ω-chaos;

(5) Martelli's chaos;

(6) Devaney's chaos;

(7) Block–Coppel chaos.

The known implications are displayed in the diagram in Figure 1, while the implications which are not true are displayed in the diagram in Figure 2.

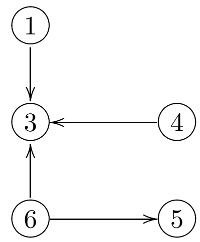

Figure 1

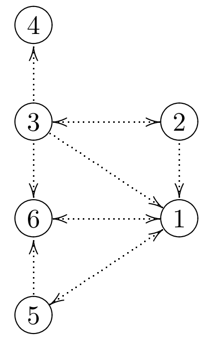

Figure 2

In the previous section Theorems 4–8 give some conditions which, in the case of the real interval, are equivalent to Li–Yorke chaos. If we want to transfer these conditions to the most general case, we immediately realize that some of them are meaningless. Namely those conditions expressed in terms of order of cycles are based on Sharkovsky's theorem which in general is not true.

Hence, we restrict ourselves to the case of *triangular maps* of the square, i.e., continuous maps $F : I^2 \to I^2$ of the form

$$F(x, y) = \big(f(x), g(x, y)\big).$$

These maps are the most similar ones to one-dimensional maps; they transform the fibre $I_x = \{x\} \times I$ into the fibre $I_{f(x)} = \{f(x)\} \times I$. Moreover, P. E. Kloeden proved in [17] that Sharkovsky's theorem is still valid for these maps.

We have the following results which are impossible in the one-dimensional case:

Theorem 13 ([11]). *There exists a triangular map F with the following three properties:*

i) *F is Li–Yorke chaotic with only a two-point scrambled set;*
ii) *the only periodic points of F are fixed points;*
iii) *$h_A(F) = 0$ for* **any** *sequence A.*

Moreover, there exists a triangular map with positive topological sequence entropy for some sequence and **not** *Li–Yorke chaotic.*

In Section 3 we have seen that a continuous map of a real interval into itself has positive topological entropy if and only if it has a cycle whose order is not a power of 2 ([27]). This is no more true for triangular maps. Indeed, the following result holds:

Theorem 14 ([2]). *There exist triangular maps of type 2^∞ either with zero topological entropy or with positive topological entropy.*

5. Final remarks

In this final section we intend to present some developments of the notion of distributional chaos and at least to cite another definition recently introduced.

The notion of distributional chaos presented in Definition 3 requires the existence of a pair $x, y \in X$ such that

$$F_{xy}(t) < F^*_{xy}(t) \tag{DC3}$$

for t in some non-degenerate interval. It is known that in the case of a real interval ([31]) this definition is equivalent to the following:

There exists a pair $x, y \in X$ such that:

$$F^*_{xy} \equiv 1, \text{ and } F_{xy}(t) < F^*_{xy}(t) \tag{DC2}$$

for t in some non-degenerate interval or

$$F^*_{xy} \equiv 1 \text{ and } F_{xy}(t) = 0 \tag{DC1}$$

for some $t > 0$.

This is no more true in higher dimensions. Smítal and Štefánková in [35] proved the existence of a triangular map of type 2^∞ with positive topological entropy satisfying (DC2), but not (DC1) while Paganoni and Smítal proved in [29] that (DC3) does not imply (DC2).

Moreover, it should be noted that (DC2) (and (DC1)) always implies Li–Yorke chaos, while (DC3) and Li–Yorke chaos are independent.

We conclude by presenting the notion of spatio-temporal chaos introduced in [1] and [8]. Given a function $f : X \to X$, a pair $\{x, y\}$ is *proximal* if

$$\liminf_{n\to+\infty} d\big(f^n(x), f^n(y)\big) = 0,$$

while it is *asymptotic* if

$$\lim_{n\to+\infty} d\big(f^n(x), f^n(y)\big) = 0.$$

A function $f : X \to X$ is **spatio-temporally chaotic** if every point is the limit of points which are proximal to but not asymptotic to it.

The relations between spatio-temporal chaos, transitivity and Li–Yorke chaos are studied in [1].

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