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On redefining a snap-back repeller

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Abstract

In this note I correct a minor technical flaw in my original snap-back repeller theorem, and then discuss some recent revisions proposed by other authors that would significantly weaken the theorem.

1. Introduction

In the 1970's, when the field of chaos theory was in its infancy, I developed in [1] criteria for the existence of chaos in multidimensional mappings. In essence, it stated that if a repelling fixed point z of differentiable $f: \mathbf{R}^n \to \mathbf{R}^n$ has an associated homoclinic orbit that is (in some sense) transversal, then f must exhibit chaotic behavior. At the suggestion of James Yorke, one of the founders of chaos theory and originator in [2] of the term *chaos* to describe these dynamics, I called such a fixed point z a *snap-back repeller*. Since then my theorem has been successfully used by a number of researchers, including myself, to investigate the onset of chaos for a variety of multidimensional mappings. See [3–15], for example.

Many years after this work first appeared, it was brought to my attention that there is a minor technical flaw in the reasoning I used in some of my arguments. While my intent was to provide a theorem that could potentially apply to any repelling fixed point, some of the dynamics I employed in its proof are associated only with those that are expanding. Here, a fixed point z is referred to as *repelling* under f if all eigenvalues of $Df(z)$ exceed 1 in magnitude. But z is *expanding* only if

$$\|f(x) - f(y)\| > s\|x - y\| \tag{1}$$

where $s > 1$, for all x, y sufficiently close to z with $x \neq y$.

Although all expanding fixed points are repelling, the converse is not true. As a result, my original definition of a snap-back repeller and proof that the existence of such a point implies chaos are in error. As one might imagine, this came as a shock and was at first a cause of great concern to me. But I quickly realized that the flaw is of a minor technical nature and, as I argue below, easily corrected.

However, during the past decade or so several papers have appeared that first overstate the severity of the error, and then propose correct but profoundly altered and weakened versions of my theorem. The latest of these that I have come

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across, a work by C. Li and G. Chen, appeared in a previous issue of this publication [16]. Although the authors claim to have formulated an *improved* version of my theorem, for reasons cited below, I believe they are mistaken. But before taking up that issue, let me first discuss some corrected versions of my definition and theorem that more closely resemble my original intent.

2. A corrected version of my theorem

I do sadly admit that there is a minor technical error in my original paper, for which I apologize to the mathematical community. I incorrectly assumed: If all eigenvalues of $Df(z)$ exceed 1 in magnitude, then the fixed point z is expanding under f, i.e., Eq. (1) is true with $\|\cdot\|$ denoting the usual Euclidean norm in $\mathbf{R}^n$. Several authors, including Li and Chen, have provided linear counterexamples to demonstrate the flaw in this reasoning.

In my defense, however, it was established long ago that this eigenvalue condition *does* imply the existence of a vector norm in $\mathbf{R}^n$ (which depends on f and z) for which the above inequality is true. See, for example, the discussion by M. Hirsch and S. Smale in [17]. Of course, the principal consequence of this is that z is locally a repeller under f, or, equivalently, an attractor under f^{-1}, which may be uniquely defined here since Eq. (1) also implies that f is locally 1–1. In other words, the local unstable manifold of f, or stable manifold of f^{-1}, includes all points of $\mathbf{R}^n$ sufficiently close to z, and, although f need not be 1–1 over its entire domain, it is 1–1 within this manifold. These are among the most classic and well-known results in the field of dynamical systems.

If one alters slightly my original definition of a snap-back repeller in [1], as well as my proof of the chaos theorem that follows it, by merely substituting this new norm in place of the Euclidean norm, and $B'_r(z)$, the closed ball in $\mathbf{R}^n$ of radius r around z using this new norm, in place of $B_r(z)$, the similarly defined ball using the Euclidean norm, then the error is immediately corrected. In this case, since the Euclidean norm is no longer involved, the terms *expanding* and *contracting* should be replaced with *repelling* and *attracting*, respectively, when they are used to describe z and/or the neighborhood $B'_r(z)$ around it.

In retrospect, I now see that, because of my youth and inexperience at the time, I foolishly confused these issues. However, the trivial nature of this fix is the reason why I never bothered to publish a correction before now. It was only the recent appearance of several works that threaten to significantly weaken my theorem that prompted me to respond at this time.

3. A better version of my theorem

Rather than introduce a somewhat mysterious norm into the definition of a snap-back repeller, which then makes the points of $B'_r(z)$ awkward to visualize in practice, it may be preferable to give an equivalent characterization that avoids any mention of this new norm, except in the seldom checked details of the proof. Since the local unstable manifold of f at z includes all points of R^n sufficiently close to z, identifying a repelling neighborhood of z is not in general a difficult task. In fact, every Euclidean $B_r(z)$ must be one for sufficiently small $r > 0$. This means that for any x in such a $B_r(z)$ the pre-image points $f^{-k}(x)$ remain within the local unstable manifold for all $k \leqslant 0$ (although not necessarily within $B_r(z)$), and $f^{-k}(x) \to z$ as $k \to \infty$.

So, perhaps a better way to define a snap-back repeller of differentiable f may be the more general rephrasing below. Although I originally defined it differently (and incorrectly), from the start many authors, including myself, have adopted this as the most natural and user-friendly definition of the concept:

Definition. Suppose z is a fixed point of f with all eigenvalues of $Df(z)$ exceeding 1 in magnitude, and suppose there exists a point $x_0 \neq z$ in a repelling neighborhood of z, such that $x_M = z$ and $\det(Df(x_k)) \neq 0$ for $1 \leqslant k \leqslant M$, where $x_k = f^k(x_0)$. Then z is called a *snap-back repeller* of f.

It is easily seen that this revised definition still implies that $\{x_k\}_{k=-\infty}^{M}$, where $x_{k+1} = f(x_k)$ for all $k < M$, satisfies $x_M = z$ and $x_k \to z$ as $k \to -\infty$, making this set of points a homoclinic orbit. Also, since all x_k for $k \leqslant 0$ lie within the local unstable manifold of f at z where f is 1–1, and since we assume that $\det(Df(x_k)) \neq 0$ for all $1 \leqslant k \leqslant M$, then this homoclinic orbit is transversal in the sense that f is 1–1 in a neighborhood of each x_k for all $k \leqslant M$.

Using the special norm we have been discussing, which implies that Eq. (1) is true for all x, y in some $B'_r(z)$ with $r > 0$, one may assume without loss of generality that x_0 lies in the interior of this $B'_r(z)$. Otherwise, since $x_k \to z$ as $k \to -\infty$, renumbering the subscripts can make this so.

Observing that all of the criteria associated with my original snap-back repeller are again satisfied, although now using this new norm, the arguments from [1], appropriately modified to accommodate this new norm, may be reused here to prove that f is chaotic. That is, despite the change of norm, one can first derive the existence of a collection of compact sets $\{B_k\}_{k=-\infty}^{N}$ with $N \geqslant M > 0$ satisfying: each B_k is homeomorphic to the unit ball in $\mathbf{R}^n$; each B_k for $k < N$ contains exactly one point of the homoclinic orbit; $B_i \cap B_k = \emptyset$ for all i, $k < N$; $B_k \subset B_N$ for all $k \leqslant 0$; $B_k \cap B_N = \emptyset$ for $1 \leqslant k < N$; and $f(B_k) = B_{k+1}$ in a 1–1 manner for all $k < N$. As before, chaotic dynamics can then be proven by investigating the topological properties exhibited by this collection of sets when iterated under f. (See Remarks 2.2 and 3.1 of [1].)

This therefore establishes the following:

Theorem. *If f has a snap-back repeller then f is chaotic.*

This statement remains essentially unchanged from my original work, as long as one refers to the corrected definition above.

4. An improved version of my theorem?

Since this Theorem is once again valid, modifying my original definition of a snap-back repeller to any greater extent than the above definition does therefore seems rather unnecessary to me. But in their recent efforts to correct the flaw in my original definition, Li and Chen do just that, and then formulate what they claim to be an *improved* version of my theorem using their new definition.

More specifically, the authors begin by proving that if z is a fixed point of f, and all eigenvalues of $Df(z)^{\mathrm{T}} \cdot Df(z)$ exceed 1 in magnitude, then z is expanding. Once this is established, a fixed point z is then defined to be a snap-back repeller of f if there exists a point $x_0 \neq z$ with $x_M = z$ and $\det(Df(x_k)) \neq 0$ for $1 \leqslant k \leqslant M$ where $x_k = f^k(x_0)$, and x_0 lies in some Euclidean $B_r(z)$ with $r > 0$ where all eigenvalues of $Df(x)^{\mathrm{T}} \cdot Df(x)$ exceed 1 in magnitude for all x in $B_r(z)$. Finally, the authors conclude that if such a snap-back repeller exists, then f must be chaotic, relying upon my original proof in [1], which now correctly applies to their situation since z is expanding in $B_r(z)$.

Although they refer to this as an *improved* version of my theorem, I must respectfully disagree for several reasons. First, since fixed points satisfying their eigenvalue criteria constitute only a small subset of all possible repelling fixed points, they end up with a much less general and significantly weaker result than both my original slightly flawed theorem and the corrected one. At best their findings might now be characterized as just a special case of the above definition and theorem. Second, their Marotto-Li-Chen Theorem, as they call it, is less elegant in the sense that my eigenvalue criteria is both simpler to state and consistent with standard analytical techniques that have been used for generations in the stability analysis of fixed points. And third, unlike theirs, my chaos criteria may be viewed as an extension of Smale's transversal homoclinic orbit [18] to the case in which the stable manifold is zero-dimensional, thereby complementing perfectly that classic result and giving a uniform set of criteria for determining chaos in multidimensional mappings.

The excessive measures these authors take in revising my original theorem may possibly stem from a failure to notice that the dynamics actually relied upon when proving the existence of chaos are not unique to expanding fixed points. As argued above, the dynamical properties associated with any repelling fixed point work just as well. Apparently unaware of this, they go looking in what I would consider to be the wrong direction for a means of correcting the flaw in the original definition of a snap-back repeller. Rather than preserve the minimal eigenvalue conditions that ensure z is a repeller and possibly sacrifice its being expanding, they do just the opposite. In their design additional restrictions are placed upon the eigenvalues so that z must definitely be expanding, as if this were an essential aspect of the dynamics that needs to be preserved in order to establish chaotic behavior. They fail to see the benefit of providing a general criteria for chaos that could potentially apply to any repeller, not just those satisfying some overly restrictive eigenvalue conditions that make it expanding.

Instead of narrowly redefining the concept to include only those repelling fixed points that meet their specific criteria, it would have been much better style if Li and Chen had characterized their findings simply as sufficient, but not necessary, conditions for the existence of a more generally defined snap-back repeller. In the event that a repelling fixed point z does satisfy their criteria, it is conceivable that analytically proving the existence of a transversal homoclinic orbit and hence chaos might be simplified a bit using their result. For example, establishing whether the point x_0 of my definition actually lies in a repelling neighborhood of z can be reduced in their case to checking that it is an element of some Euclidean $B_r(z)$ with $r > 0$, where all eigenvalues of $Df(x)^{\mathrm{T}} \cdot Df(x)$ exceed 1 in magnitude for all x in $B_r(z)$. The benefit of this approach remains to be seen, however, since the authors provide no examples.

5. Conclusion

Over the years since it first appeared there have been other revisions of my theorem by a variety of authors with different goals in mind. In some cases corrections are again proposed, while others attempt to generalize the result even further, for example, to non-differentiable mappings. Future authors are certainly welcome, and indeed should be encouraged, to offer their own for consideration. Whether or not my corrected definition and Theorem are superior to all other such re-workings, mathematical history will have to decide.

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