2. Finding a Horseshoe on the Beaches of Rio

Steve Smale

City University of Hong Kong

This is an expanded version of a paper to appear in the proceedings of the International Congress of Science and Technology—45 years of the National Research Council of Brazil.

What is Chaos?

A mathematician discussing chaos is featured in the movie *Jurassic Park*. James Gleick's book *Chaos* remains on the best seller list for many months. The characters of the celebrated Broadway play *Arcadia* of Tom Stoppard discourse on the meaning of chaos. What is chaos?

Chaos is a new science which establishes the omnipresence of unpredictability as a fundamental feature of common experience.

A belief in determinism, that the present state of the world determines the future precisely, dominated scientific thinking for two centuries. This credo was based on certain laws of physics, Newton's equations of motion, which describe the trajectories in time of states of nature. These equations have the mathematical property that the initial condition determines the solution for all time. Thus lies the mathematical and physical foundation for deterministic philosophy. One manifestation of determinism was the rejection of free will and hence even of human responsibility.

At the beginning of this century, with the advent of quantum mechanics and the revelations of the German scientists, Heisenberg, Planck, and Schrödinger, the great delusion of determinism was exposed. At least on the level of electrons, protons, and atoms it was discovered that uncertainty prevailed. The equations of motion of quantum mechanics produce solutions which are probabilities evolving in time.

In spite of quantum mechanics, Newton's equations govern the motion of a pendulum, the behavior of the solar system, the evolution of the weather and many situations of everyday life. Therefore the quantum revolution left intact many deterministic dogmas. For example, well after the Second World War, scientists held the belief that long range weather prediction would be successful when computer resources grew large enough.

In the 1970s the scientific community recognized another revolution, called the theory of chaos, which deals a death blow to the Newtonian picture of determinism. As a consequence, the world now knows that one must deal with unpredictability in understanding common experience. The coin-flipping syndrome is pervasive. "Sensitive dependence on initial conditions" has become a catchword of modern science.

Chaos contributes much more than extending the domain of indeterminacy, just as quantum mechanics did so more than half a century earlier. The deeper understanding of dynamics underlying the theory of chaos has shed light on every branch of science. Its accomplishments range from an analysis of electrocardiograms to aiding the construction of computational devices.

Chaos developed not with the discovery of new physical laws, but by a deeper analysis of the equations underlying Newtonian physics. Chaos is a scientific revolution based on mathematics. Deduction rather than induction is the methodology. Chaos takes the equations of Newton, analyses them with mathematics, and uses that analysis to establish the widespread unpredictability in the phenomena described by those equations. Via mathematics, one establishes the failure of Newtonian determinism by using Newton's own equations!

Taxpayers Money

In 1960 in Rio de Janeiro I was receiving support from the National Science Foundation (NSF) of the United States as a postdoctoral fellow, while doing research in an area of mathematics which was to become the theory of chaos. Subsequently, questions were raised about my having used U.S. taxpayers' money for this research done on the beaches of Rio. In fact none other than

President Johnson's science adviser, Donald Hornig, wrote on this issue in 1968 in the widely circulated magazine *Science:*

> This blythe spirit leads mathematicians to seriously propose that the common man who pays the taxes ought to feel that mathematical creation should be supported with public funds on the beaches of Rio . . .

What happened during the passage of time from the work on the beaches to this national condemnation?

This was the era of the turbulent '60s in Berkeley where I was a professor; my students were arrested, tear gas frequently filled the campus air, dynamics conferences opened under curfew; Theodore Kaczynski, the suspected Unabomber, was a colleague of mine in the math department.

The Vietnam War was escalated by President Johnson in 1965, and I was moved to establish with Jerry Rubin a confrontational anti-war force. Our organization, the Vietnam Day Committee (VDC), with its teach-ins, its troop train demonstrations and big marches, put me onto the front pages of the newspapers. These events led to a subpoena by the House Unamerican Activities Committee (HUAC), which was issued while I was enroute to Moscow to receive the Fields Medal (the main prize in mathematics) in 1966. The subsequent press conference I held in Moscow attacking U. S. policies in the Vietnam War (as well as Russian intervention in Hungary) created a long lasting furore in Washington, DC.

Now we are going to see what actually happened in that spring of 1960 on those beaches of Rio de Janeiro.

Flying Down to Rio

Topology is the part of mathematics which is sometimes nicknamed "rubber sheet geometry" since a topologist is allowed to bend rigid objects. In the 1950s there was an explosion of ideas in this subject. Topology, with its developments dominating all of mathematics, caught the imagination of many young research students such as myself. I finished a Ph.D. thesis in that domain at the University of Michigan in Ann Arbor in 1956. During that summer I, with my wife, Clara, attended in Mexico City, a topology conference reflecting this great movement in mathematics with the world wide notables in topology present and giving lectures. There I met a Brazilian graduate student, Elon Lima, writing a thesis in topology at the University of Chicago, and as I was about to take up the position of instructor at that university, I became good friends with Elon.

A couple of years later, Elon introduced me to Mauricio Peixoto, a young visiting professor from Brazil. Mauricio was from Rio, although he came from a northern state of Brazil where his father had been governor. A good humoured pleasant fellow, Mauricio, in spite of his occasional bursts of excitement was conservative in his manner and in his politics. As was typical for the rare mathematician working in Brazil at that time, he was employed as teacher in an engineering college. Mauricio also helped found a new institute of mathematics (IMPA) and his aspirations brought him to America to pursue research in 1957. Subsequently he was to become the President of Brazilian Academy of Sciences.

Mauricio was working in the subject of differential equations or dynamics and showed me some beautiful results. Before long I myself had proved some theorems in dynamics.

In the summer of 1958, Clara and I with our newly born son Nat, moved to the Institute for Advanced Study (IAS) in Princeton, New Jersey. This was the locale made famous as Einstein's workplace in America, with Robert Oppenheimer as its director. I was supposed to spend two years there with an NSF postdoctoral fellowship. However, due to our common mathematical interests, Mauricio and Elon invited me to finish the second year in Rio de Janeiro. So Clara and I and our children, Nat and newly arrived Laura, left Princeton in December, 1959, to fly down to Rio.

The children were so young that most of our luggage consisted of diapers, but nevertheless we were able to realize an old ambition of seeing Latin America. After visiting the Panamanian jungle, the four of us left Quito, Ecuador, Christmas of 1959, on the famous Andean railroad down into the port of Guayaquil. Soon we were flying into Rio de Janeiro, recovering from sicknesses we had acquired in Lima. I still remember vividly, arriving at night, going out several times trying to get milk for our crying children, and returning with a substitute as cream or yogurt. We later learned that, in Rio, milk was sold only in the morning, on the street. At that time Brazil was truly part of the "third world."

However our friends soon helped us settle down into Brazilian life. We arrived in Brazil just after a coup had been attempted by an air force colonel. He fled the country to take refuge in Argentina, and we were able to rent, from his wife, his luxurious 11-room apartment in the district of Rio called Leme. The U.S. dollar went a long way in those days, and we even were able to hire the colonel's two maids, all with our fellowship funds.

Sitting in our upper story garden veranda we could look across to the hill of the favela (called Babylonia) where *Black Orpheus* was filmed. In the hot humid evenings preceeding Carnaval, we would watch hundreds of the favela dwellers descend to samba in the streets. Sometimes I would join their wild dancing which paraded for many miles.

In the front of our apartment, the opposite direction from the hill, lay the famous beach of Copacabana. I would spend my mornings on that wide, beautiful, sandy beach, swimming and body surfing. Also I took a pen and paper and would work on mathematics.

Mathematics on the Beach

Very quickly after our arrival in Rio, I found myself working on mathematical research. My host institution, Instituto da Matematica, Pura e Aplicada (IMPA), funded by the Brazilian government, provided a pleasant office and working environment. Just two years earlier IMPA had established itself in its own quarters, a small colonial building in the old section of Rio called Botafogo. There were no undergraduates and only a handful of graduate mathematics students. There were also a very few research mathematicians, notably Peixoto, Lima, and an analyst named Leopoldo Nachbin. Also there was a good math library. But no one could have guessed that in less than three decades IMPA would become a world center of dynamical systems housed in a palatial building as well as a focus for all of Brazilian science.

In a typical afternoon I would take a bus to IMPA and soon be discussing topology with Elon, dynamics with Mauricio or be browsing in the library. Mathematics research typically doesn't require much, the most important ingredients being a pad of paper and a ballpoint pen. In addition, some kind of library resources, and colleagues to query are helpful. I was satisfied.

Especially enjoyable were the times spent on the beach. My work was mostly scribbling down ideas and trying to see how arguments could be put together. Also I would sketch crude diagrams of geometric objects flowing through space, and try to link the pictures with formal deductions. Deeply involved in this kind of thinking and writing on a pad of paper, the distractions of the beach didn't bother me. Moreover, one could take time off from the research to swim.

The surf was an exiting challenge and even sometimes quite frightening. One time when Lima visited my "beach office," we entered the surf and were both caught in a current which took us out to sea. While Elon felt his life fading, bathers shouted the advice to swim parallel to the shore to a spot where we were able to return. [It was 34 years later just before Carnaval, that once again those same beaches almost did me in. This time a special wave bounced me so hard on the sand it injured my wrist, tore my shoulder tendon, and then that same big wave carried me out to sea. I was lucky to get back using my good arm.]

Letter from America

At that time, as a topologist, I prided myself on a paper that I had just published in dynamics. I was delighted with a conjecture in that paper which had as a consequence (in modern terminology) "chaos doesn't exist"!

This euphoria was soon shattered by a letter I received from an M.I.T. mathematician named Norman Levinson. He had coauthored the main graduate text in ordinary differential equations and was a scientist to be taken seriously.

Levinson wrote me of an earlier result of his which effectively contained a counterexample to my conjecture.

His paper in turn was a clarification of extensive work of the pair of British mathematicians Mary Cartwright and J. L. Littlewood done during World War II. Cartwright and Littlewood had been analyzing some equations that arose in doing war-related studies involving radio waves. They had found unexpected and unusual behavior of solutions of these equations. In fact Cartwright and Littlewood had proved mathematically that signs of chaos could exist, even in equations that arose naturally in engineering. But the world wasn't ready to listen, and even today their important contributions to chaos theory are not well-known. I never met Littlewood, but in the mid-sixties, Dame Mary Cartwright who was head of a women's college (Girton) at Cambridge invited me to high table.

I worked day and night to try to resolve the challenge to my beliefs that the letter posed. It was necessary to translate Levinson's analytic arguments into my own geometric way of thinking. At least in my own case, understanding mathematics doesn't come from reading or even listening. It comes from rethinking what I see or hear. I must redo the mathematics in the context of my particular background. And that background consists of many threads, some strong, some weak, some algebraic, some visual. My background is stronger in geometric analysis, but following a sequence of formulae gives me trouble. I tend to be slower than most mathematians to understand an argument. The mathematical literature is useful in that it provides clues, and one can often use these clues to put together a cogent picture. When I have reorganized the mathematics in my own terms, then I feel an understanding, not before.

Through these kinds of thought processes, I eventually convinced myself that indeed Levinson was correct, and that my conjecture was wrong. Chaos was already implicit in the analyses of Cartwright and Littlewood. The paradox was resolved; I had guessed wrongly. But while learning that, I discovered the horseshoe!

The Horseshoe

The horseshoe is a natural consequence of a geometrical way of looking at the equations of Cartwright-Littlewood and Levinson. It helps understand the mechanism of chaos, and explain the widespread unpredictability in dynamics.

Chaos is a characteristic of dynamics, and dynamics is the time evolution of a set of states of nature. Thus one can think of dynamics as solutions of the equations of motion as given for example by Newton. While time is usually considered as a continuous entity we will suppose that time is measured in discrete units as seconds or minutes. A state of nature will be idealized as a point in the two dimensional plane.

We will start by describing a non-chaotic linear example. The idea is to take a square, Figure 1, and to study what happens to a point on this square in one unit of time, under a transformation to be described.

The vertical dimension is now shrunk uniformly towards the center of the square and the horizontal is expanded uniformly at the same time. Using dots to outline the domain obtained by this process superimposed over the original square yields Figure 2.

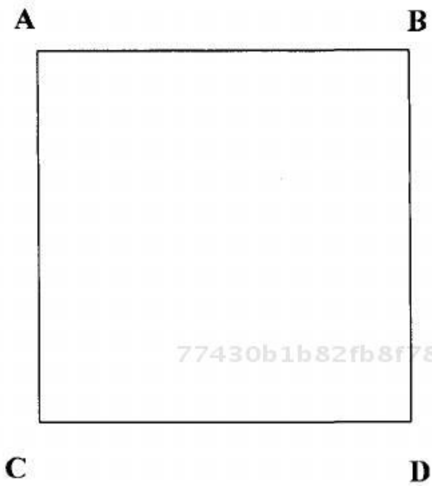

Figure 1

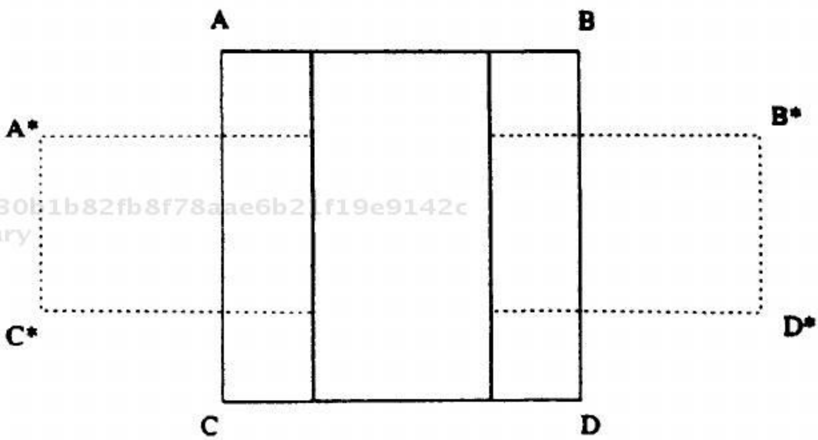

Figure 2

Here superscripts are used to denote the motion so that the corner A moves to A*. We have also shaded in the set of points which don't move out of the square in this process.

The second of our three stages in understanding is the perturbed linear example. Now the square is moved into a bent version of the elongated rectangle of Figure 2. Thus Figure 3 describes the motion of our square obtained by a small modification of Figure 2.

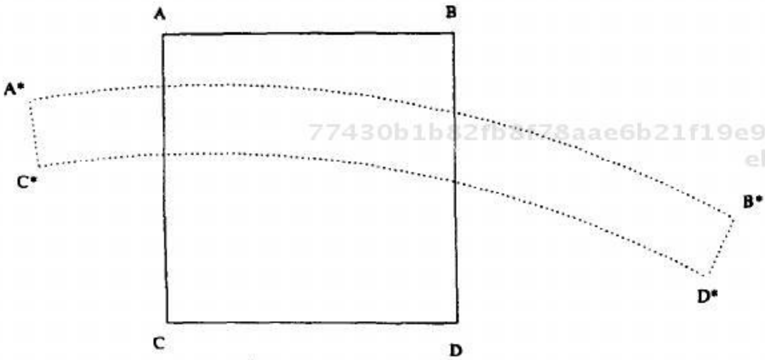

Figure 3

The horseshoe is the fully nonlinear version of what happens to points on the square, by an extension of the process expressed in Figures 2 and 3. This is the situation when motion is expressed by a qualitative departure from the linear model. See Figure 4.

The horseshoe is the domain surrounded by the dotted line.

Instead of a state of nature evolving according to a mathematical formula, the evolution is given geometrically. The full advantage of the geometrical point of view is beginning to appear. The more traditional way of dealing with dynamics was with the use of mathematical, e.g. algebraic, expressions. But a description given by formulae would be cumbersome. It would unlikely lead to insights or to a perceptive analysis, since that form of a description wouldn't communicate as efficiently the information in the figure. My background as a topologist, trained to bend objects as squares helped to make it possible to see the horseshoe.

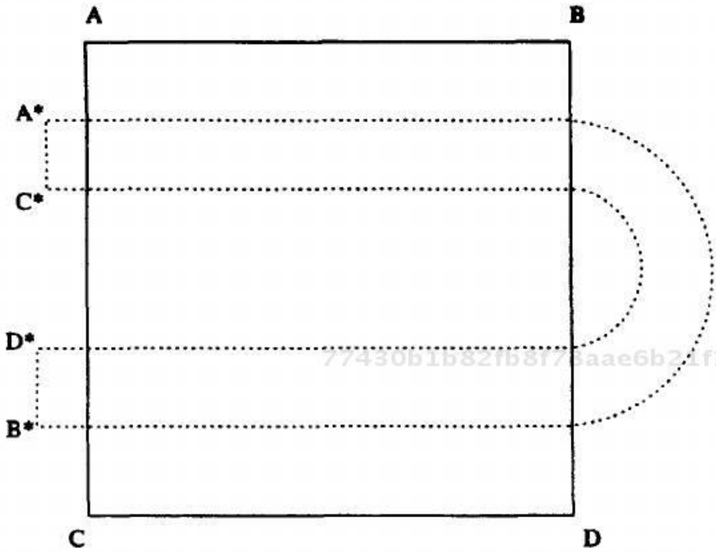

Figure 4

The dynamics of the horseshoe is described by moving a point in the square to a point in the horseshoe according to Figure 4. Thus the corner marked A moves to the point marked A* in one unit of time.

The motion of a general point x in the square is a sequence of points x_0 x_1, x_2 . . . Here $x_0 = x$ is the present state, x_1 is that state a unit of time later, x_2 that state two units of time later, etc.

Now imagine our visual field to be just the square itself. When a point is moved out of the square we will discard that motion. Figure 5 shades in the points which don't leave the square in one unit of time.

We will call a motion which never leaves the square a visual motion. Our results in the next section concern visual motions.

In summary, a fully nonlinear motion finds its realization in the horseshoe. In the next section we will see the consequences to chaos that this picture carries.

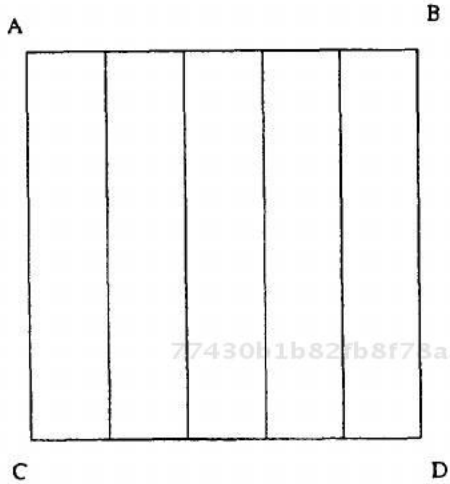

Figure 5

The Horseshoe and Chaos: Coin Flipping

The laws of chance, with good reason, have traditionally been expressed in terms of flipping a coin. Guessing whether heads or tails is the outcome of a coin toss is the paradigm of pure chance. On the other hand it is a deterministic process that governs the whole motion of the coin, and hence the result, heads or tails, depends only on very subtle factors of the initiation of the toss. This is the concept called "sensitive dependence on initial conditions."

A coin-flipping experiment is a sequence of coin tosses each of which has as outcome either heads (H) or tails (T). Thus it can be represented in the form HTTHHTTTTH. . . .[1] A general coin flipping experiment is thus a sequence $s_0 s_1 s_2 \dots$ where each of $s_0, s_1, s_2 \dots$ is either H or T.

Here is the result of the horseshoe analysis that I found on that Copacabana beach. Consider all the motions of the horseshoe construction which stay in the square, i.e., don't drift out of our field of vision. These motions correspond precisely to the set of all coin flipping experiments! This

[1] To give a complete picture in this section, one needs to reverse time and consider sequences of heads and tails which go back in time as well.

discovery demonstrates the occurence of unpredictability in fully nonlinear motion and gives a mechanism of how determinism produces uncertainty.

The demonstration is based on the following construction. To each visual motion there is an associated coin flipping experiment. If $x_0\, x_1\, x_2 \ldots$ is a visual motion of the horseshoe dynamics, at time $i = 0, 1, 2, 3, \ldots$ associate H or heads if x_i lies in the top half of the square and T or tails if it lies in the bottom half.

Moreover, and this is the crux of the matter, every possible sequence of coin flips is represented by a horseshoe motion. Therefore the dynamics is as unpredictable as coin-flipping. In the natural one-one correspondence

$$x_0\, x_1\, x_2 \ldots \rightarrow s_0\, s_1\, s_2 \ldots,$$

$x_0\, x_1\, x_2 \ldots$ is a motion lying in the square and $s_0\, s_1\, s_2 \ldots$ is a sequence of H's and T's. On the left is a deterministically generated motion and on the right a coin-flipping experiment.

The analysis shows that the above association is a complete natural identity between the set of motions lying in the square and the set of all sequences of heads and tails.[2]

We have seen how deterministic fully nonlinear motion, the horseshoe, can be represented as unpredictable coin-tossing experiments. This is chaos.

The Hidden Origins of Chaos

As chaos is a mathematically based revolution, it is not surprising to see that a mathematician first saw evidence of chaos in dynamics.

Henri Poincaré was (with David Hilbert) one of the two foremost mathematicians in the world active at the end of the last century. Poincaré was an originator of topology who had written an article claiming that a manifold with the same algebraic characteristics as the n-dimensional sphere was actually the n-dimensional sphere. When he found a mistake in his proof, restricting himself now to 3 dimensions, he formulated the assertion as a problem, now called Poincaré's Conjecture. This problem is the biggest problem in topology and even one of the three or four great unsolved

[2] Mathematicians say that we have an isomorphism preserving the dynamics, or conjugacy, between the horseshoe motions lying in the square and the coin-flipping experiments.

problems in mathematics today. What concerns us here however is this scientist's contribution to the theory of chaos.

Poincaré made extensive studies in celestial mechanics, that is to say, the motions of the planets. At that time it was a celebrated problem to prove the solvability of those underlying equations, and in fact Poincaré at one time thought that he had proved it. Shortly thereafter however he became traumatized by a discovery which not only showed him wrong but showed the impossibility of ever solving the equations for even three bodies. This discovery was a motion he christened "homoclinic point."

A homoclinic point is a motion tending to an equilibrium as time increases and also to that same equilibrium as time recedes into the past. See Figure 6. Here p is an equilibrium and h marks the homoclinic point. The arrows represent the direction of time.

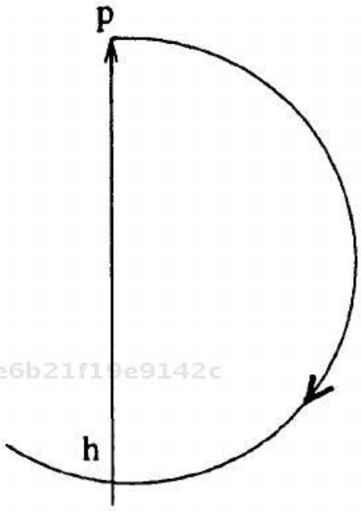

Figure 6

This definition sounds harmless enough but carries amazing consequences. Poincaré wrote concerning his discovery:[3]

[3] my own translation from the French

> One will be struck by the complexity of this figure which I won't even try to draw. Nothing can more clearly give an idea of the complexity of the three body problem and in general of all the problems of dynamics . . .

In addition to showing the impossibility of solving the equations of planetary motion the homoclinic point has turned out to be the trademark of chaos; it is found in essentially every chaotic dynamical system.

It was in the first half of this century that American mathematics came into its own, and traditions stemming from Poincaré in topology and dynamics were central in this development. G. D. Birkhoff was the most well known American mathematician before World War II. He came from Michigan and did his graduate work at the University of Chicago, before settling down at Harvard. Birkhoff was heavily influenced by Poincaré's work in dynamics, and he developed these ideas and especially the properties of homoclinic points in his papers in the '20s and '30s.

Unfortunately, the scientific community soon lost track of the important ideas surrounding the homoclinic points of Poincaré. In the conferences in differential equations and dynamics that I attended in the late '50s, there was no awareness of this work. Even Levinson never showed in his book, papers, or correspondence with me that he was aware of homoclinic points.

It is astounding how important scientific ideas can get lost, even when they are exposed by leading mathematicians of the preceeding decades.

I learned about homoclinic points and Poincaré's work from browsing in Birkhoff's collected works which I found in IMPA's library. It was because of the recently discovered horseshoe that the homoclinic landscape was to sink into my consciousness. In fact there was an important relation between horseshoes and homoclinic points.

If a dynamical system possesses a homoclinic point then I proved that it also contains a horseshoe. This can be seen in Figure 7.

Thus the coin-flipping syndrome underlies the homoclinic phenomenon, and helps to comprehend it.

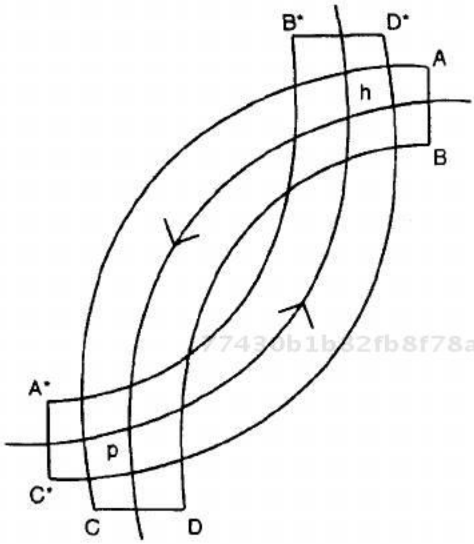

Figure 7

The Third Force

I was lucky to find myself in Rio at the confluence of three different historical traditions in dynamics. These three cultures, while dealing with the same subject, were isolated from each other, and this isolation obstructed their development. We have already discussed two of these forces, Cartwright-Littlewood-Levinson and Poincaré-Birkhoff.

The third had its roots in Russia with the school of differential equations of A. Andronov in Gorki in the 1930s. Andronov had died before the first time I went to the Soviet Union, but in Kiev, in 1961, I did meet his wife, Andronova Leontovitch, who was still working in Gorki in differential equations.

In 1937, Andronov teamed up with a Soviet mathematician L. Pontryagin. Pontryagin had been blinded at the age of 14, yet went on to become a pioneering topologist. The pair described a geometric perspective of differential equations they called "rough," subsequently called structural stability. Chaos, in contrast to the two previously mentioned traditions, was absent in this development because of the restricted class of dynamics.

Fifteen years later the great American topologist Solomon Lefschetz became enthusiastic about Andronov-Pontryagin's work. Lefschetz had also suffered an accident, that of losing his arms, before turning to mathematics, and this perhaps generated some kind of bond between him and the blind Pontryagin. They first met at a topology conference in Moscow in 1938, and again after the war. It was through Lefschetz's influence, in particular, via an article of his student, De Baggis, that Mauricio Peixoto in Brazil learned of structural stability.

Peixoto came to Princeton to work with Lefschetz in 1957 and this is the route which led to our meeting each other through Elon. After this meeting, I studied Lefschetz' book on a geometric approach to differential equations and eventually came to know Lefschetz in Princeton.

Via Pontryagin and Lefschetz there was the specter of topology in the concept of structural stability of ordinary differetial equations. I believe that was why I listened to Mauricio.

Good Luck

Sometimes a horseshoe is considered an omen of good luck. The horseshoe I found on the beach of Rio certainly seemed to have such a property.

In that spring of 1960 I was primarily a topologist, mainly motivated by the problems of that subject, and most of all driven by the great unsolved problem posed by Poincaré. Since I had started doing research in mathematics, I had produced false proofs of the 3-dimensional Poincaré Conjecture, returning again and again to that problem.

Now on those beaches, within two months of finding the horseshoe, I found to my amazement an idea which seemed to succeed provided I returned to Poincaré's original assertion and then restricted the dimension to 5 or more. In fact the idea produced not only a solution of Poincaré's Conjecture in dimensions greater than 4, but it gave rise to a large number of other nice results in topology. It was for this work that I received the Fields Medal in 1966.

Thus ". . . the mathematics created on the beaches of Rio . . ." (Hornig) was the horseshoe and the higher-dimensional Poincaré's conjecture.