Letters to the Editor

Deligne/Zimin Initiative to Support Young Mathematicians

As Russian mathematicians and members of the American Mathematical Society, the undersigned feel it is their pleasant duty to inform fellow members via the *Notices* of a remarkable philanthropic initiative supporting young Russian research mathematicians undertaken by Pierre Deligne and continued by the Russian philantropist D. B. Zimin. In 2004 Deligne wrote in a letter to one of us: "I just won the Balzan Prize. Half the prize amount is for me to spend on a research project agreed to by the Balzan Foundation. I believe that one of the most useful ways to spend this money (500,000 Swiss francs) would be for the benefit of the struggling Russian school of mathematics."

Together with several collaborators of the Independent University of Moscow, Deligne implemented this idea by organizing the "Pierre Deligne Contest for Young Mathematicians", a yearly individual competition of research projects for young Russian, Ukrainian, and Byelorus mathematicians, whose laureates are granted a sizable three-year fellowship. Together with Victor Vassiliev, Deligne heads the jury of the contest, which is run along lines similar to those used by the American NSF. Since 2005 Deligne comes to Moscow each December to supervise the final deliberations. During the past four years, sixteen fellowships have been granted, and the money coming from the Balzan foundation (having in mind the future payments to recent winners) has been entirely exhausted, but Deligne intends to continue the contest by using his personal funds.

In 2006 the Russian philanthropic foundation "Dynasty" has organized the "D. B. Zimin Dynasty Foundation Contest for Young Mathematicians" with the same jury and according to the same rules. According to the corresponding agreement, this contest will run for two more years, after which it may be continued. The continuation of these two contests will undoubtedly play a crucial role in preserving the Russian mathematical school.

We are deeply grateful to Pierre Deligne for his noble initiative, which has already done a great deal to help young Russian mathematicians to survive without giving up research. This initiative is a continuation of the generous international solidarity to Russian scientists which the American Mathematical Society implemented in the 1990s and is continuing today.

We are extremely grateful to Dmitry Zimin for his chivalrous support of Russian fundamental science, in particular mathematics. D. B. Zimin is one of the very few Russian businessmen contributing money for the support of Russian mathematics.

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Response to David Ruelle

Tien-Yien Li and I were delighted to read Freeman Dyson's beautiful article in the February *Notices* that refers to our paper "Period three implies chaos" as "one of the immortal gems in the literature of mathematics". We prove one theorem in the paper, and Dyson reports on what we may assume is his favorite part of it. Assume there is a continuous function F from an interval J to itself, and—to be brief—we assume there is a "period three" point $a \in J$; that is, $F(F(F(a))) = a \neq F(a)$. Dyson simply wrote "An orbit is defined to be chaotic, **in this context** [emphasis added], if it diverges from all periodic orbits."

But then the June *Notices* came with a Letter to the Editor by David Ruelle that made two assertions, that our paper's result was not new, and, worse yet, that he believes that **period three does not imply chaos**!!! Ruelle seems to be aware of only the first of our three conclusions:

(I) (All Periods Exist.) For each positive integer k, there is a point of period k. The Sharkovsky (1964) Theorem stated by Ruelle is indeed a more general theorem than part (I), but our theorem does not stop here.

(II) (Mixing.) There is an uncountable set $S \subset J$ which satisfies: For every $p, q \in S$ with $p \neq q$,

$$\liminf_{n\to\infty} |F^n(p) - F^n(q)| = 0,$$

and

$$\limsup_{n\to\infty} |F^n(p) - F^n(q)| > 0.$$

(III) (Divergence from Periodic Orbits.) The above set S can be chosen so that in addition, (1) is satisfied for each $p \in S$ and each periodic point $q \in J$.

Dyson's elegant statement refers to (III). There is also a large literature that recognizes (II) and refers to it as "Chaos in the sense of Li and Yorke" or "Li-Yorke chaos". We support Dyson's view that the definition of chaos depends on the context; that is, it depends on what you know about a system or what you want to prove. There is a rich tradition of chaos in topology, where exponential divergence of trajectories is not a useful concept.

I fear that Ruelle's version of chaos would leave mathematicians without a method of dealing with the simplest situations when he asserts "chaos occurs if the exponential divergence is present for long-term behavior, i.e., on an attractor." One of the most famous examples of chaos is the map $\alpha - x^2$ which has a chaotic attractor for a set of α having positive measure (M. Jacobson). But there is also an open dense set in which there is an attracting periodic orbit. Since it can be shown that there is only one attractor for this map, such an α has no chaotic attractor. If Ruelle (who is a superb mathematician) can prove there is a chaotic attractor for $\alpha = 3/2$, then I will declare his definition usable! He simply needs to prove an infinite set of lemmas:

for each positive integer N, there is no periodic attractor of period N. I make the following conjecture in the spirit of Gödel.

Conjecture. The map $3/2 - x^2$ has a chaotic attractor and there is no proof of that fact using the usual axioms of set theory.

I believe that this impossibility of proof would apply to the large class of smooth dynamical systems that alternate densely between chaotic attractors and periodic attractors as a parameter is varied. Ruelle says "chaos" should apply only to attractors. (There are proofs for certain very special cases like $2 - x^2$.)

Attractor basins can be fractal—due to the presence of chaos on an uncountable compact invariant set on the basin boundary. He would leave us with no term for such sets!! After Ruelle said that chaos means "neighboring trajectories diverge exponentially," he asserted that period three does not imply chaos! Here is a compromise. We can define chaos in this context as "exponential divergence for *all* trajectories on some uncountable compact invariant set". In the spirit of mathematical fun, I propose the following conjecture using this concept.

Conjecture. Assume F is a continuously differentiable map from an interval J to itself. Then period three implies chaos.

This could be facet (IV) of the improved theorem. And people could still define chaos as they please.

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Reply to Yorke

"Period three implies chaos" by Li and Yorke is a beautiful paper, and much quoted, including by myself. This paper is at the origin of the use of the word chaos in what has become "chaos theory", a multidisciplinary endeavor involving mathematical, numerical, and physical techniques, that has contributed among other things to the explanation of the "Kirkwood gaps" in the rings of asteroids between Mars and Jupiter. Li and Yorke prove that (for suitable maps of the interval) the existence of a periodic point of period three implies a complicated situation reminiscent of the homoclinic tangle discovered by Poincaré in his study of the three-body problem. Clearly, period three implies Li-Yorke chaos. But when chaos in the solar system is discussed (Wisdom, Laskar, ...), or other applications to the real world, another concept of chaos is used, which refers to "chaotic" behavior for initial conditions in a set of positive volume in the (phase) space where the dynamics takes place. This condition (rather than uncountability) must be imposed because sets of measure zero are physically invisible in the present situation. There has been a semantic shift and, if the map $x \mapsto ax(1-x)$ of $[0,1]$ to itself has, for some value of a, an attracting orbit of period 3, most people would not call this map chaotic (this is because the orbit of Lebesgue almost every point tends to the nonchaotic attracting period 3 orbit). This being said, the map under discussion not only exhibits Li-Yorke chaos, but also what Yorke at some point described as transient chaos (this is only transient, but visible on a set of positive Lebesgue measure).

As to the very interesting problem of logical decidability raised by Yorke (to decide if a point belongs to a set of physical interest in parameter space), I think it has to be reinterpreted in case one has physical applications in mind. This is because physical parameters have an interval of uncertainty attached to them. One is thus led to perturbation problems in differentiable dynamics, and Jim Yorke knows that those are usually very hard.

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Calculus and Computers in Mathematical Education

It could well be argued that too much has already been written about this topic. However, it is at least conceivable that some of this hyper senior citizen's memories might yield an alternative perspective of nonnegligible value for discussing the educational opportunities that have arisen from the availability of computers. As will immediately become evident, what follows is primarily oriented toward instruction through so-called "honours" undergraduate calculus classes, assuming (perhaps somewhat unrealistically) that, in such an environment, an uncompromising approach to the subject is permitted.

Most of the relevant memories have their origin in the mathematics department of the University of Capetown and date back to 1934. The offerings in that department included courses in "applied mathematics". There, you learned to formulate simple settings from classical mechanics in terms of differential equations, a pursuit leading to inspiring enlightenment about the true significance of calculus. In stark contrast to this, the introductory courses in formal calculus were as discouraging then as they are now.

In this last context, present educational practice is still operating in a conceptual desert, being focused on unenlightening manipulative tricks of formal differentiation and integration, the life of many of these being supported by the suppression of complex numbers,[1] and the need for most of them having been made obsolete by the computer.

In the context of numerical calculus, educational neglect was justifiable in view of the absurdly large requirements of time and labor. (Memories dating back to 1942 are of spending hundreds of days calculating military firing tables with the help of a Friden desk calculator.) Nowadays, computer-assisted pursuit of numerical calculus could easily be made far more enlightening than the standard introduction to calculus.

Unfortunately, even the modern mathematical computer software is mostly oriented toward applications in the precomputer style. In any case, the current products are much too big for student use.

[1] *The novelist William Styron, whose imagination is of unsurpassable breadth and analytic depth, confesses in one of his short essays that he flunked "trigonometry" four times in succession.*