On Devaney's Definition of Chaos

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Chaotic dynamical systems have received a great deal of attention in recent years (see for instance [2], [3]). Although there has been no universally accepted mathematical definition of chaos, the popular text by Devaney [1] isolates three components as being the essential features of chaos. They are formulated for a continuous map $f\colon X \to X$ on some metric space X (to avoid degenerate cases we will assume in this note that X is not a finite set). The first of Devaney's three conditions is that f is *transitive*; that is, for all non-empty open subsets U and V of X there exists a natural number k such that $f^k(U) \cap V$ is nonempty. In a certain sense, transitivity is an irreducibility condition. The second of Devaney's conditions is that the periodic points of f form a dense subset of X. Devaney refers to this condition as an "element of regularity" ([1], p. 50). The final condition is called *sensitive dependence on initial conditions*; f verifies this property if there is a positive real number δ (a *sensitivity constant*) such that for every point x in X and every neighborhood N of x there exists a point y in N and a nonnegative integer n such that the n^{th} iterates $f^n(x)$ and $f^n(y)$ of x and y respectively, are more than distance δ apart. This sensitivity condition captures the idea that in chaotic systems minute errors in experimental readings eventually lead to large scale divergence. Sensitive dependence on initial conditions is thus widely understood as being the central idea in chaos.

Devaney's Definition of Chaos. Let X be a metric space. A continuous map $f\colon X \to X$ is said to be chaotic on X if

1. f is transitive,
2. the periodic points of f are dense in X,
3. f has sensitive dependence on initial conditions.

The aim of this note is to prove the following elementary but somewhat surprising result.

Theorem. *If $f\colon X \to X$ is transitive and has dense periodic points then f has sensitive dependence on initial conditions.*

Before proving this Theorem, let us discuss some of the ideas that motivated it. First of all, any definition of chaos must face the obvious question: Is it preserved under topological conjugation? That is to say, if f is chaotic and if we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ h\downarrow & & \downarrow h \\ Y & \xrightarrow{g} & Y \end{array}$$

where Y is another metric space and h is a homeomorphism, then is g necessarily chaotic? Certainly transitivity and the existence of dense periodic points are preserved as they are purely topological conditions. However, sensitivity is a metric property and in general it is not preserved under topological conjugation, as the following simple example shows. Let X be the subset $(1, \infty)$ of the real line, equipped with the standard metric, let f be multiplication by 2, let Y be the set $\mathbb{R}^+$ of positive reals and let h be *log*. Clearly f has sensitive dependence on initial conditions but g is just a translation and hence is not sensitive for the standard metric on $\mathbb{R}^+$. In fact, as we leave to the reader to verify, it is not difficult to find transitive examples for which sensitivity is not preserved under conjugation. Nevertheless, as the above Theorem shows, transitivity and dense periodic points together (trivially) assure that sensitivity is preserved. Before closing this paragraph on conjugation, let us remark that sensitivity can be regarded as a topological concept if one restricts one's attention to compact spaces X (which is often the case in practice). Indeed, suppose that X is compact and that f is conjugate to g as in the above diagram. Suppose as well that f has sensitive dependence on initial conditions, with sensitivity constant δ. Let D_δ denote the set of pairs (x_1, x_2) of points in X which are separated by distance at least δ. Then D_δ is a compact subset of the Cartesian product $X \times X$ and so its image E_δ in $Y \times Y$ under the map $(x_1, x_2) \mapsto (h(x_1), h(x_2))$ is also compact. Consequently the minimum distance $\delta_Y > 0$ exists between E_δ and the diagonal in $Y \times Y$. It is easy to verify that g has sensitive dependence on initial conditions with sensitivity constant δ_Y.

Proof of Theorem: We suppose that $f: X \to X$ is transitive and has dense periodic points.

First observe that there is a number $\delta_0 > 0$ such that for all $x \in X$ there exists a periodic point $q \in X$ whose orbit $O(q)$ is of distance at least $\delta_0/2$ from x. Indeed, choose two arbitrary periodic points q_1 and q_2 with disjoint orbits $O(q_1)$ and $O(q_2)$. Let δ_0 denote the distance between $O(q_1)$ and $O(q_2)$. Then by the triangle inequality, every point $x \in X$ is at distance at least $\delta_0/2$ from one of the chosen two periodic orbits. We will show that f has sensitive dependence on initial conditions with sensitivity constant $\delta = \delta_0/8$.

Now let x be an arbitrary point in X and let N be some neighborhood of x. Since the periodic points of f are dense, there exists a periodic point p in the intersection $U = N \cap B_\delta(x)$ of N with the ball $B_\delta(x)$ of radius δ centered at x. Let n denote the period of p. As we showed above, there exists a periodic point $q \in X$ whose orbit $O(q)$ is of distance at least 4δ from x. Set

$$V = \bigcap_{i=0}^{n} f^{-i}\big(B_\delta(f^i(q))\big).$$

Clearly V is open and it is non-empty since $q \in V$. Consequently, since f is transitive, there exists y in U and a natural number k such that $f^k(y) \in V$.

Now let j be the integer part of $k/n + 1$. So $1 \leqslant nj - k \leqslant n$. By construction, one has

$$f^{nj}(y) = f^{nj-k}(f^k(y)) \in f^{nj-k}(V) \subseteq B_\delta(f^{nj-k}(q)).$$

Now $f^{nj}(p) = p$, and so by the triangle inequality,

$$d(f^{nj}(p), f^{nj}(y)) = d(p, f^{nj}(y))$$
$$\geqslant d(x, f^{nj-k}(q)) - d(f^{nj-k}(q), f^{nj}(y)) - d(p, x),$$

where d is the distance function on X. Consequently, since $p \in B_\delta(x)$ and $f^{nj}(y) \in B_\delta(f^{nj-k}(q))$, one has

$$d(f^{nj}(p), f^{nj}(y)) > 4\delta - \delta - \delta = 2\delta.$$

Thus, using the triangle inequality again, either $d(f^{nj}(x), f^{nj}(y)) > \delta$ or $d(f^{nj}(x), f^{nj}(p)) > \delta$. In either case, we have found a point in N whose nj^{th} iterate is more than distance δ from $f^{nj}(x)$. This completes the proof.

REFERENCES

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