Defining Chaos

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I. Introduction

In the last thirty years scientists have found that unusual and unexpected evolution patterns arise frequently in important deterministic processes of interest to many different fields, including Chemistry, Physics, Biology, Medicine, Engineering, and Economics. Examples of such processes include chemical reactions, pulsation in gas lasers, atmospheric changes, blood cell oscillations, and neural networks. The most peculiar aspect of these patterns is their random-like behavior. The systems are deterministic. Consequently they are, at least in theory, perfectly predictable. Hence, it may seem contradictory to talk about random-like behavior. However, more often than not their evolution appears as a random sequence of events, at least to superficial analysis.

The name "chaotic systems" has been proposed to collect them loosely under a common roof. Biologists, chemists, mathematicians, philosophers, physicists, and others have tried to capture in a formal definition the distinctive and essential features characterizing these systems among all dynamical processes. The success has been limited. On the one hand, everyone recognizes that certain systems cannot be considered chaotic; on the other hand we could say, with a bit of exaggeration, that there are as many definitions of chaos as experts in this new area of knowledge (see, for example **[5]**, **[3]**, **[8]**, **[4]**, **[7]**). Moreover, and this is certainly not a desirable situation, the various definitions are not equivalent to each other.

Many reasons can be given for this state of affairs, and the fact that chaotic behavior is of great interest to many disciplines is certainly one of them. It is difficult to find a common ground that meets the needs and the standards of different fields. For example, an experimental scientist is inclined to adopt a definition that can be tested in a laboratory setting and is less concerned with exceptions. A theoretician, however, is interested in characterizing chaotic behavior uniquely, and does not feel the urgency to provide a definition which can be easily verified by means of numerical or experimental techniques.

The main purpose of this paper is to bring a contribution to the efforts aimed at capturing the distinctive features of chaotic systems in a way that is easily accessible to undergraduates. This purpose is achieved in two ways. The first is by introducing the reader to those definitions of chaotic systems that are more frequently encountered in the literature and do not use advanced mathematical concepts and tools. We illustrate the key components of each definition. We also include a comparison table (Table 3.1) to provide the reader with an "at a glance" overview of the common traits and differences among the various definitions. The second is by analyzing in more detail two simple definitions proposed in recent years, one by S. Wiggins **[8]** and the other by M. Martelli **[6]**. Although formulated in different manner, the two definitions are practically equivalent. Moreover, they seem to embody the essential features which all other definitions are trying to capture. Finally, the characterizing traits of these two definitions are suitable for easy and reliable numerical verification. Therefore, they appear to represent the most effective way to introduce chaotic behavior at an undergraduate level.

The paper is organized as follows. In section 2 we introduce terminology and notation frequently used throughout. In section 3 we present some of the most common definitions of chaos; and we analyze briefly their key components. In section 4 we establish the basic equivalence of the definitions of Wiggins and Martelli. We conclude the paper (section 5) with an analysis of the baker's transformation and with short remarks on numerical tests of chaotic behavior.

Before embarking on the plan we have outlined, we illustrate a simple dynamical system, which is chaotic according to *all* definitions presented later. The purpose of this discussion, conducted mainly by means of graphs, is to make the reader familiar with the characteristic features that each definition of chaos tries to capture.

Example 1.1. Let $f(x)=4x(1-x)$. Notice that f maps the interval $[0,1]$ onto itself. Consider the dynamical system $x_{n+1}=f(x_n)$ governed by the function f in $[0,1]$. Select the point $x_0=0.3$ and study the sequence of iterates of f: $x_1=f(0.3)$, $x_2=f(x_1), \dots, x_{n+1}=f(x_n), \dots$. To see how this sequence behaves, plot the points (x_n, x_{n+1}) for $n = 500, 501, \dots, 1000$ and for $n=1500, 1501, \dots, 2000$ in two side-by-side plots. (See FIGURE 1.1.) The points belong to the graph $G(f)$ of f since $x_{n+1}=f(x_n)$. It appears that they fill up $G(f)$ entirely in both cases. This graphical evidence suggests that no matter how small an interval $[a,b]$ is selected in $[0,1]$, the sequence $x_1=f(0.3)$, $x_2=f(x_1), \dots, x_{n+1}=f(x_n), \dots$ visits $[a,b]$ infinitely often. This is one feature of chaotic systems which all definitions try to capture: the presence of a sequence of iterates (orbit) that passes "as close as we like to any possible state of the system." We shall make this idea more precise in section 2 with the definition of topological transitivity.

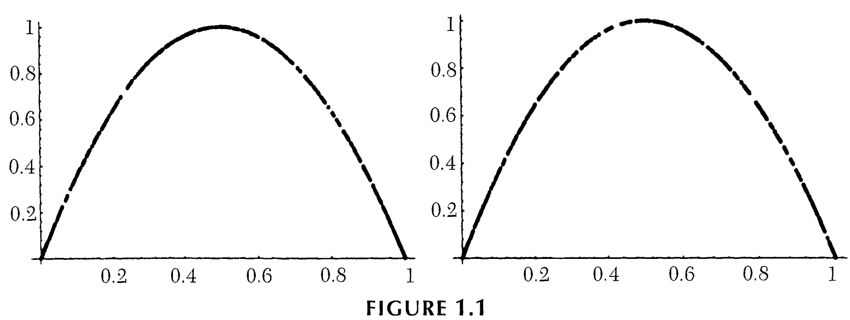

FIGURE 1.1

On the left graph we have plotted (x_n, x_{n+1}) for $n=500, \dots, 1000$ and on the right for $n=1500, \dots, 2000$ from the same sequence of iterates of f starting at $x_0=0.3$. It appears that in both cases the sequence is "reconstructing" the entire graph of f.

To illustrate another important property of the system $x_{n+1}=f(x_n)$, consider two sequences of iterates, one starting (as before) at $x_0=0.3$ and the other starting at a point very close to 0.3. Choose, for example, $y_0=0.300001$. Plot the points $(n, |x_n-y_n|)$, i.e., the iteration number on the horizontal axis and the distance between corresponding iterates of f on the vertical axis. At the beginning (for small values of n) the two sequences are close to each other. Later, they become separated, and the distance $|x_n-y_n|$ oscillates between 0 and 1 in an unpredictable fashion. (See FIGURE 1.2.)

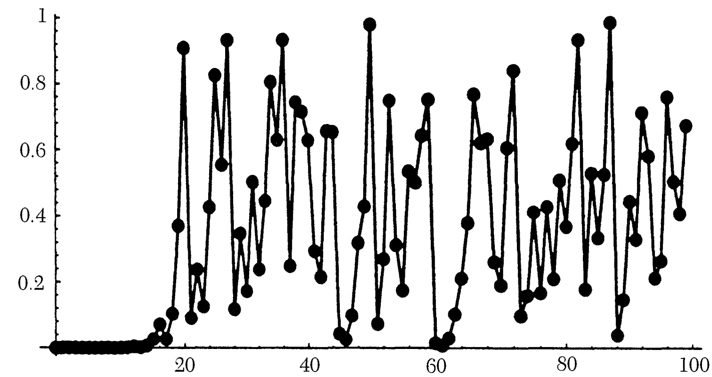

FIGURE 1.2

The distance $|x_n - y_n|$ is plotted versus the iteration number n. Notice that the two sequences of iterates are very close for $n = 0, 1, \ldots, 15$. After that, separation takes over. Sometimes the two sequences are very close (around $n = 45$ and $n = 60$), and sometimes they are as far as they can be.

The graphical evidence suggests that the evolution of the system is very sensitive to small changes. Thus, if this system were a model of a real process, we would be tempted to conclude that its evolution, although governed by a known function, is nevertheless "unpredictable," since it is practically impossible to know the initial state exactly. This is a second feature of chaotic systems which every definition tries to capture, namely the sensitivity to small changes, and the unpredictability that comes with it. We shall make this idea more precise with the definition of unstable orbits and of sensitive dependence on initial conditions.

II. Notations and definitions

Let $F : \operatorname{Dom} F \subseteq \mathbb{R}^q \to \mathbb{R}^q$. A set $X \subseteq \operatorname{Dom} F$ is said to be *invariant* under the action of F if $F(X) \subseteq X$. In the case when $F(X)$ is bounded and F is continuous we can assume that the closure of X is contained in the domain of F. Then the invariance of X implies the invariance of its closure. In this paper we shall always assume, unless otherwise stated, that F is continuous and its invariant sets are closed and bounded.

Let $X \subseteq \operatorname{Dom} F \subseteq \mathbb{R}^q$ and assume that X is invariant. The discrete dynamical system defined by F in X takes the form

$$x_{n+1} = F(x_n). \tag{2.1}$$

Equation (2.1) provides the state x_{n+1} of the system at time $n+1$ once its state x_n at time n is known. Given an initial state $x_0 \in \mathbb{R}^q$, the sequence of iterates of F:

$$x_0,\ x_1 = F(x_0),\ x_2 = F(x_1) = F(F(x_0)) = F^2(x_0), \ldots, x_n = F^n(x_0), \ldots \tag{2.2}$$

is the *orbit* of x_0, denoted by $O(x_0, F)$ or simply $O(x_0)$ when the function F is clearly specified. An orbit $O(x_0)$ is periodic if for some $p \geq 1$

$$x_p = x_0. \tag{2.3}$$

The smallest integer p for which (2.3) holds is called the *period* of the orbit. When $p = 1$ the orbit $O(x_0)$ is *stationary*, and the point x_0, now denoted by x_s, is an *equilibrium point* of the system. An orbit $O(y_0)$ is *asymptotically periodic* if there is a periodic orbit $O(x_0)$ such that

$$\lim_{n \to \infty} \| x_n - y_n \| = 0. \tag{2.4}$$

If, in addition, $y_k = x_k$ for some $k \geq 1$, then $O(y_0)$ is *eventually periodic*.

A point y is a *limit point* of $O(x_0)$ if a subsequence of $O(x_0)$ converges to y. The set of limit points of $O(x_0)$ is denoted by $L(x_0)$. Under our standard assumptions on X and F we have that $L(x_0)$ is closed and bounded and it satisfies the important equality

$$F(L(x_0)) = L(x_0). \tag{2.5}$$

The set $L(x_0)$ is finite if and only if $O(x_0)$ is asymptotically periodic. When $L(x_0)$ is infinite we say that $O(x_0)$ is *aperiodic*.

$O(x_0)$ is said to be *unstable* if there exists $r(x_0) > 0$ such that for every $d > 0$ we can find $y_0 \in \operatorname{Dom} F$ and $n \geq 1$ satisfying the two inequalities $\| y_0 - x_0 \| \leq d$ and $\| y_n - x_n \| > r(x_0)$. An orbit which is not unstable is said to be *stable*. When $O(x_0)$ is contained in an invariant set $X \subset \operatorname{Dom} F$, we say that $O(x_0)$ is unstable with *respect to X* if $y_0 \in X$. Notice that, in this case, the set X has to be infinite.

Let $X \subseteq \operatorname{Dom} F \subseteq \mathbb{R}^q$. F has in X *sensitive dependence on initial conditions* if there exists $r_0 > 0$ such that for every $x_0 \in X$ and $d > 0$ we can find $y_0 \in \operatorname{Dom} F$ and $n \geq 1$ with the property that $\|x_0 - y_0\| \leq d$ and $\|x_n - y_n\| > r_0$. Therefore, every orbit $O(x)$ with $x \in X$ is unstable with the same constant r_0. Consequently, sensitive dependence on initial conditions is stronger than instability. When $X \subset \operatorname{Dom} F$ is an invariant set and we require that $y_0 \in X$, we say that F has in X sensitive dependence on initial conditions *with respect to X*. In this case no point of X is isolated, i.e., for every $x \in X$ and every $c > 0$ we can find $y \in X$, $y \neq x$, such that $\|x - y\| \leq c$.

A set $U \subset X \subseteq \mathbb{R}^q$ is said to be *open in X* if $U = X \cap O$ where O is an open subset of $\mathbb{R}^q$. The function F is *topologically transitive* on an invariant set X if for every pair of sets $U, V \subset X$ which are open in X, there exists an integer $k \geq 1$ such that $F^k(U) \cap V \neq \emptyset$. This property, as we shall see in section 4, guarantees the presence of an orbit "that passes as close as we like to any state of the system."

III. Some common definitions of chaos

In this section we present some definitions of chaos that can be found in the current literature and are accessible to undergraduates.

1. Li–Yorke chaos Let I be an interval and $f: I \to I$ be a continuous function. Assume that f has a periodic orbit of period 3. In a well-known paper Li and Yorke **[5]** proved that

(i) f has periodic orbits of every period;

(ii) there is an uncountable set $S \subset I$ such that $O(x)$ is aperiodic and unstable for every $x \in S$.

Maps of this type have been called *chaotic in the Li–Yorke sense*, without specifying if the chaotic behavior should be considered in the entire interval I or simply in the closure of S.

One of the clear advantages of this definition is that it can be easily verified, by means of graphical techniques, whether a continuous map has a periodic orbit of period 3. Moreover, property (ii) addresses, at least in part, the question of unpredictability of the system, since the orbits starting at points of S are unstable. The following simple example shows that the assumption of continuity is critical in the Li–Yorke approach.

Example 3.1. Let $f:[0,1]\to[0,1]$ be defined by

$$f(x)=\begin{cases} x+.5 & 0\le x\le .5\\ 0 & .5<x\le 1.\end{cases}$$

Notice that f is discontinuous at $x=0.5$. Every orbit of f in $[0,1]$ is eventually periodic of period 3. For example, for $x_0=0.2$ we have $x_1=0.7$, $x_2=0$, $x_3=0.5$, $x_4=1$, $x_5=0,\dots$.

We can also find examples of maps for which the set S is negligible, in the sense that for every $r>0$, S can be covered with a countable family of intervals of total length not exceeding r. Consequently, the probability that an orbit $O(x_0)$ is not asymptotically periodic is zero, and the chaotic behavior is not experimentally observable. The following example illustrates the situation.

Example 3.2. Let

$$f(x)=\begin{cases} 0 & 0\le x<.25\\ 4x-1 & .25\le x<.5\\ -4x+3 & .5\le x<.75\\ 0 & .75\le x\le 1.\end{cases}$$

It can be easily verified that $O(23/65)$ is a periodic orbit of period 3 and $f^2(x)\to 0$ whenever $x<1/3$ or $x>2/3$. Moreover, $f^2(x)=0$ if $x\in \mathrm{I}_1=[5/12,7/12]$, whose length is $1/6$. The inverse image of this interval is made of those points x such that $f^3(x)=0$ and is the union of the two intervals $I_{21}=[17/48,19/48]$ and $I_{22}=[29/48,31/48]$. The total length of the two intervals is $1/12$. The inverse image of $I_{21}\cup I_{22}$ is the union of four intervals $I_{31}=[65/192,67/192]$, $I_{32}=[77/192,79/192]$, $I_{33}=[113/192,115/192]$, $I_{34}=[125/192,127/192]$. Their total length is $1/24$. Every point x of these four intervals has the property $f^4(x)=0$. Proceeding in this way we find a family of disjoint intervals contained in the interval $[1/3,2/3]$ and whose total length is $1/6+1/12+1/24+\cdots=1/6(1+1/2+1/4+1/8+\cdots)=1/3$. Every point x that belongs to one of these intervals satisfies $f^n(x)=0$ for some $n\ge 1$. Hence the set of points S_0 whose orbit does not go to zero is negligible. Since $S\subset S_0$ we see that the orbit of a point x selected at random in $[0,1]$ converges to 0.

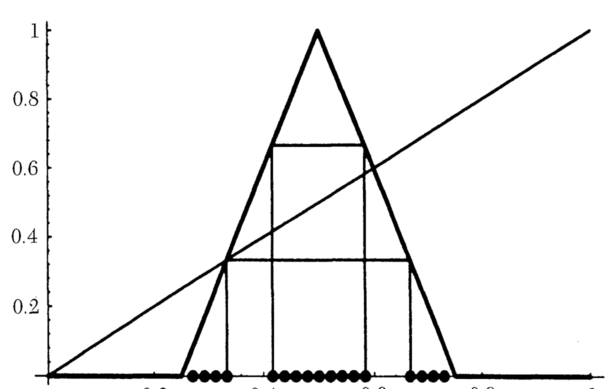

FIGURE 3.1
Shown are the points x such that $f^n(x)=0$ for $n=1,2$.

The result of Li–Yorke does not hold in dimension higher than one. For example, a rotation in $\mathbb{R}^2$ of 120° around the origin has a periodic orbit of period three (all non-stationary orbits are periodic of period 3), but fails to satisfy both (i) and (ii). The orbits of such a system have neither of the two properties we indicated (see Example 1.1) as relevant to chaotic behavior. Table 3.1 compares the definition of chaos according to Li–Yorke to the other definitions listed below.

TABLE 3.1 Comparison among different definitions of chaos

Definition	map	domain	requirements	advantages	weak points
Li–Yorke	continuous	bounded interval	periodic orbit of period 3	easy to check	can be used only in $\mathbb{R}$
Experimentalists'	continuous	$X \subset \mathbb{R}^q$ bounded, closed, invariant	sensitivity on initial conditions	easy to check	defines as chaotic systems which are not
Devaney	continuous	$X \subset \mathbb{R}^q$ bounded, closed, invariant	sensitivity, transitivity, dense periodic orbits	goes to the roots of chaotic behavior	redundancy
Wiggins	continuous	$X \subset \mathbb{R}^q$ bounded, closed, invariant	sensitivity, transitivity	goes to the roots of chaotic behavior	admits degenerate chaos
Martelli	continuous	$X \subset \mathbb{R}^q$ bounded, closed, invariant	dense orbit in X which is unstable	"equivalence" with Wiggins, easy to check numerically	none of the above

2. Experimentalists' definition of chaos (sensitive dependence on initial conditions) According to many non-mathematicians, particularly physical scientists, a dynamical system $x_{n+1} = F(x_n)$ is chaotic in an invariant set X if F has in X sensitive dependence on initial conditions. Therefore, we may obtain very different orbits from two almost identical starting points (see Example 1.1). It follows that the evolution of the system is unpredictable, since it is practically impossible to know the initial conditions exactly (mainly due to unavoidable measurement errors). This is obviously an important feature of the experimentalists' definition of chaos. An additional merit is that sensitive dependence on initial conditions can be checked numerically. However, despite the advantages, this definition of chaos is not satisfactory. The following example illustrates some of the problems which may arise.

Example 3.3. Let $D = \{x \in \mathbb{R}^2 : \|x\| \le 2\}$. Using polar coordinates define F: $D \to D$ by

$$F(x) = F(\rho, \theta) = (\rho, \theta + \rho). \tag{3.1}$$

Notice that for every $\rho \in (0, 2]$ the set $C_\rho = \{x \in \mathbb{R}^2 : \|x\| = \rho\}$ is invariant and the dynamical system defined by F is a rotation in C_ρ. Consequently, it does not seem appropriate to label the system as chaotic in the invariant set C_ρ. However, the system has in C_ρ sensitive dependence on initial conditions with $r_0 = \rho$. In fact, let $x_0 = (\rho_0, \theta_0)$ and $d > 0$. Choose n so large that $\frac{\pi}{n} < d$ and $\rho_0 - \frac{2\pi}{n} > 0$. Let $y_0 =$

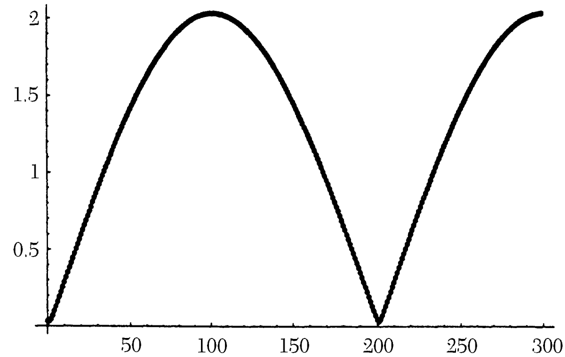

FIGURE 3.2
Plot of $(n, \|x_n - y_n\|)$, with $x_0 = (1, 0)$ and $y_0 = (1 - 0.01\pi, 0)$. We see that the distance can be as large as the diameter of the smaller circle.

$(\rho_0 - \frac{\pi}{n}, \theta_0)$. Then $\|x_0 - y_0\| < d$ and

$$x_n = (\rho_0, \theta_0 + n\rho_0), \quad y_n = \left(\rho_0 - \tfrac{\pi}{n}, \theta + n\rho_0 - \pi\right). \tag{3.2}$$

Consequently $\|x_n - y_n\| > r_0$ and the system is chaotic in C_ρ for every $\rho \in (0, 2]$. However, the system is non-chaotic in the disk D. We do not seem to have a satisfactory situation (see Table 3.1).

3. Wiggins' definition of chaos According to Wiggins **[8]** a map F is chaotic in an invariant set X provided that

(i) F is topologically transitive in X;
(ii) F has in X sensitive dependence on initial conditions.

We shall see in section 4 that topological transitivity implies the existence of an orbit "passing as close as we like to any state" of the system in X. Therefore the definition of Wiggins embodies both properties mentioned in Example 1.1 as fundamental to chaotic behavior. However, Wiggins' approach presents some problems. For example, the map $F(\rho, \theta)$ of Example 3.3 is chaotic in the sense of Wiggins in every circle C_ρ such that ρ/π is irrational. In fact, F has sensitive dependence on initial conditions in C_ρ. Moreover, the orbit $O(x_0)$, $x_0 = (\rho, 0)$ visits every arc of C_ρ, no matter how small. Hence F is topologically transitive in C_ρ. Notice that F is non-chaotic in any annulus $R[a, b] = \{x \in D\colon a \le \|x\| \le b, 0 \le a < b < 2\}$ since F fails to be topologically transitive. An additional problem with Wiggins' definition arises from the so-called "degenerate chaos" (see **[1]**), which is chaotic behavior in a finite set of points. In fact, according to Wiggins a dynamical system can be chaotic in a singleton $X = \{x_0\}$. For example the system governed by the function

$$f(x) = -2|x| + 1 \tag{3.3}$$

is chaotic in the set $X = \{1/3\}$. FIGURE 3.3 illustrates that orbits starting close to $1/3$ move away from the equilibrium point (see Table 3.1 for a summary).

4. Martelli's definition of chaos According to Martelli **[6]**, F is chaotic in an invariant set X provided that there exists $x_0 \in X$ such that

(i) $L(x_0) = X$;
(ii) $O(x_0)$ is unstable with respect to X.

Since $F(L(x_0)) = L(x_0)$ (see Equation 2.5) we obtain that $F(X) = X$, i.e., F is onto.

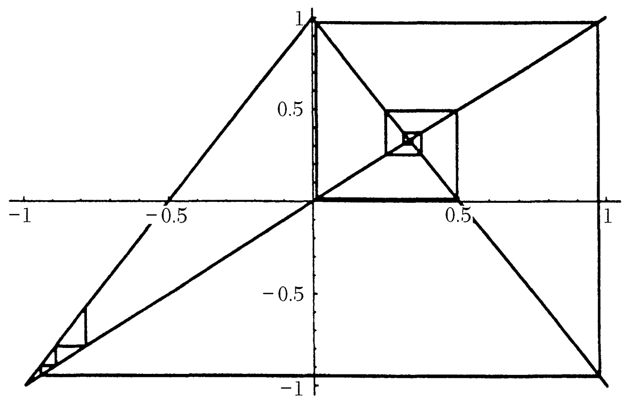

FIGURE 3.3
The orbits of points close to 1/3 more away from the equilibrium point.

The map $F(\rho, \theta)$ of Example 3.3 is non-chaotic in the sense of Martelli in any circle C_ρ or in any annulus $R[a, b]$. In a circle C_ρ the map fails to satisfy (ii), and in an annulus $R[a, b]$ fails to satisfy (i). Moreover, according to Martelli, no map can be chaotic on a finite set, since instability of $O(x_0)$ with respect to X implies that X is infinite.

5. Devaney's definition of chaos A map F is chaotic in the sense of Devaney [**3**] in an invariant set X if

(i) F is topologically transitive in X;
(ii) F has in X sensitive dependence on initial conditions;
(iii) the set P of periodic orbits of F is dense in X.

Devaney adds the density of P in X to the two conditions required by Wiggins, thus bringing back, at least to some extent, a feature of Li–Yorke chaos. Moreover, as Crannell [**2**] points out, the "requirement that periodic orbits be dense appeals to those who look for patterns within a seemingly random system."

It has been shown [**1**] that conditions (i) and (iii) imply (ii). In this sense, Devaney's definition of chaos is redundant. Moreover, as the following example shows, there are systems that seem to deserve the label "chaotic" and do not satisfy the third requirement of Devaney's definition (see Table 3.1 for a summary).

Example 3.4. Let F be given in polar coordinates by $F(\rho, \theta) = (4\rho(1-\rho),\ \theta + 1)$ and let $D(0, 1)$ be the invariant disk centered at the origin, with radius 1. The origin is the only fixed point for F, and F does not have any periodic orbit of period $p > 1$. In fact, F stretches or shrinks the distance of every point of $D(0, 1)$ from the origin, while rotating the point by an angle of 1 radian. Since $1/\pi$ is irrational, no point $x_n \in O(x_0)$ can come back to the same ray which contains x_0. At the end of this paper we will show that the dynamical system governed by F in $D(0, 1)$ is "unpredictable" and has orbits that pass as close as we like to every point of $D(0, 1)$. Thus this system has exactly the two fundamental properties of chaotic behavior mentioned in Example 1.1.

IV. Defining Chaos

Recall that, according to Wiggins [**8**], F is chaotic in an invariant set X if

(i) F is topologically transitive in X;
(ii) F has in X sensitive dependence on initial conditions.

According to Martelli [**6**], F is chaotic in X provided that there exists $x_0 \in X$ such that

(i) $L(x_0) = X$;

(ii) $O(x_0)$ is unstable with respect to X.

These two definitions can be considered equivalent. In fact, (see Theorem 4.1) F is topologically transitive in X if and only if there exists $x_0 \in X$ such that $L(x_0) = X$. In addition, F has in X sensitive dependence on initial conditions *with respect to* X if and only if $O(x_0)$ is unstable *with respect to* X (see Theorem 4.2).

There remains an important difference between the two approaches. Wiggins does not require sensitivity *with respect to* X, while Martelli requires instability *with respect to* X. Theorems 4.1 and 4.2 contain the theoretical results that establish the practical equivalence between these two definitions of chaos. For both theorems we provide a brief sketch of the proof, leaving details to the reader.

THEOREM 4.1. *Let $X \subset \mathbb{R}^q$ be closed and bounded and $F: X \to X$ be continuous. Then F is topologically transitive in X if and only if there exists $x_0 \in X$ such that $L(x_0) = X$.*

Proof. The "if" part is easy. The presence of an orbit $O(x_0)$ such that $L(x_0) = X$ clearly implies topological transitivity.

The "only if" part is a bit more difficult. The basic idea is that given any positive integer m we can cover X with finitely many balls of radius $1/m$ and find a point x_m whose orbit visits each ball of the covering. Moreover, the choice of x_m can be made so that the sequence $\{x_m,\ m = 1, 2, \dots\}$ converges. Let x_0 be its limit. It is easy to verify that $L(x_0) = X$.

THEOREM 4.2. *Let $x_0 \in X$ be such that $L(x_0) = X$. Then F has in X sensitive dependence on initial conditions with respect to X if and only if $O(x_0)$ is unstable with respect to X.*

Proof. This time the "only if" part is immediate. In fact, sensitivity to initial conditions with respect to X clearly implies that $O(x_0)$ is unstable with respect to X.

The "if" part is a bit longer. Given $y_0 \in X$ and $d > 0$, determine an iterate x_n of x_0 such that $\|x_n - y_0\| \le d/2$. This can be done since $L(x_0) = X$. Next, one shows that for every $n > 1$ the orbit $O(x_n)$ has the same instability constant of $O(x_0)$, i.e., $r(x_n) = r(x_0)$. It follows that either some iterate y_p of y_0 is at least as far as $r(x_0)/3$ from x_{n+p}, or this separation happens for some iterate z_p of a point z_0 which is closer than d to both y_0 and x_n. In either case, we obtain that $r(y_0) \ge r(x_0/3)$.

A second look at Example 3.4. With Theorem 4.1 and 4.2 we can establish that the dynamical system of Example 3.4 is chaotic in $D(0,1)$ according to Wiggins and Martelli. We use the fact, well-established in the literature, that the map $f(x) = 4x(1-x)$ of Example 1.1 not only is topologically transitive in $[0,1]$ but has the additional property that given any interval $[a,b] \subset [0,1]$, $a < b$, there is an integer p such that $f^p[a,b] = [0,1]$. Consequently, after finitely many iterations, the F-image of a small open disk in $D(0,1)$ will contain an open set $U \subset D(0,1)$ with a full radius. The rotation of 1 radian spreads U entirely over $D(0,1)$ in finitely many additional iterations. Hence F is topologically transitive in $D(0,1)$. From Theorem 4.1 there is $x_0 \in D(0,1)$ such that $L(x_0) = D(0,1)$. Consequently, using once more the statement from Example 1.1, the orbit passes as close as we like to any point of $D(0,1)$. It is also well known that the map f has sensitive dependence on initial conditions in $[0,1]$. Hence, $O(x_0)$ is unstable in $D(0,1)$ and F is chaotic in $D(0,1)$ according to Wiggins and Martelli. In FIGURE 4.1 we plot $\|x_n - y_n\|$ versus the iteration number n, with $x_0 = (.3, 0)$ and $y_0 = (.300001, 0)$. (The reader should compare the graph with Fig.

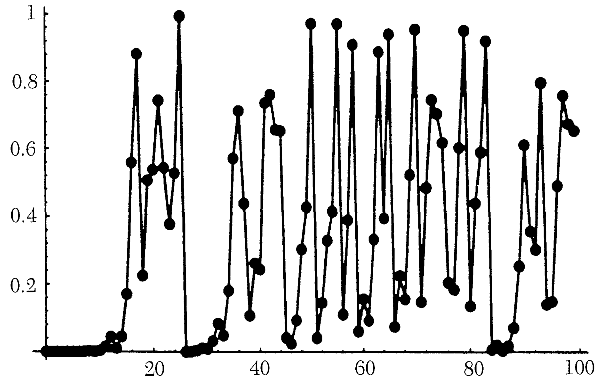

FIGURE 4.1
Plot of $(n, \|x_n - y_n\|)$. The behavior of the distances repeats the situation of Example 1.1.

1.2.) The map F is non-chaotic in $D(0,1)$ according to Devaney. This example seems to suggest that the density of periodic orbits may not be necessary in defining chaos.

V. Conclusion

The definitions of chaos of Wiggins and Martelli, together with the one of Devaney and with the experimentalists' definition, can be applied to a certain class of maps with "admissible" discontinuities. The class is denoted by QC, which stands for "quasi-continuous." It can be shown (the result will appear in a forthcoming paper by A. Crannell and M. Martelli), that the definitions of Wiggins and Martelli remain equivalent in QC. In the following example we present the so-called baker's transformation, which defines a well-known chaotic system in [0, 1], and which belongs to QC.

Example 5.1. Let $B(x) = 2x - [2x]$, where $[2x]$ denotes the greatest integer less than or equal to $2x$. Notice that B maps $[0,1]$ into itself and it is discontinuous at $x = .5$ and $x = 1$. The action of B and its iterates on the elements of $[0,1]$ is better understood if we write them with their binary expansion. Then, for $x \in [0, 0.5)$ we have $x = 0.0a_2a_3\ldots$, while for $x \in [0.5, 1)$ we have $x = 0.1a_2a_3\ldots$ where a_i, $i = 2,3,\ldots$, are either 0 or 1. In both cases we obtain $B(x) = 0.a_2a_3\ldots$. Now we can easily see that the orbit of $x_0 = 0.\ 0\ 1\ 00\ 01\ 10\ 11\ 000\ 001\ 010\ 100\ \ldots$ has the property $L(x_0) = [0,1]$. Moreover, $O(x_0)$ is unstable, since $B'(x) = 2$ for $x \neq 0, 1$.

Hence B is chaotic in $[0,1]$ according to Martelli's definition (applied to QC). Under the action of B the length of every interval $[a,b] \subset [0,1]$, $a < b$ is doubled until, after finitely many iterations, we have $B^k[a,b] = [0,1]$. Thus B is topologically transitive in $[0,1]$. Sensitivity is ensured by $B'(x) = 2$ for $x \neq 0.5$, 1. Hence, B is chaotic in $[0,1]$ according to Wiggins and to the experimentalists' definition (applied to QC). It can be shown that the periodic orbits of B are dense in $[0,1]$. Thus B is chaotic in $[0,1]$ according to the definition of Devaney (applied to QC). B has a periodic orbit of period 3 in $[0,1]$, but the Li–Yorke definition of chaos cannot be applied to B, since we have seen that continuity is critical in the Li–Yorke case.

We close this survey with a remark regarding the possibility of numerically investigating the chaotic behavior of a map. We feel that Martelli's definition is possibly most suitable for this purpose. The property $L(x_0) = X$ can be tested by covering the set X with small boxes (segments in $\mathbb{R}$, squares in $\mathbb{R}^2$, cubes in $\mathbb{R}^3 \ldots$) and by verifying that the orbit "visits" all of them. The instability of the orbit can be tested with the method we used in Example 1.2 and in our second look at Example 3.4. As mentioned in the introduction, chaotic behavior is of great interest to many disciplines. Proving it theoretically, however, is never an easy task, if at all possible. Numerical tests are frequently the only ones available in practical applications and we feel that the simpler they are, the greater their reliability will be.

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A Lambda Slaughter

Mary had a little lamb-
Da curled and curved for show.
And everywhere that lambda went,
The math came out just so.

It followed her to calculus
With multiplier rules,
Which show the way to meet constraints
As in Lagrange's school.

In matrix class it proved itself
To be a trusty pal, whose
Assistance could be counted on
For writing eigenvalues.

So keep an eye on Mary's friend—
Its uses transcend measure.
Beyond a doubt her lambda is
A character to treasure.

—Dan Kalman
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