ELSEVIER

Topology and its Applications 117 (2002) 259–272

TOPOLOGY AND ITS APPLICATIONS

www.elsevier.com/locate/topol

Devaney's chaos or 2-scattering implies Li–Yorke's chaos ☆

Wen Huang, Xiangdong Ye *

Department of Mathematics, University of Science and Technology of China, Hefei, Anhui, 230026, People's Republic of China

Received 7 March 2000; received in revised form 14 August 2000

Abstract

Let X be a compact metric space, and let $f:X\to X$ be transitive with X infinite. We show that each asymptotic class (or the stable set $W^s(x)$ for each $x\in X$) is of first category and so is the asymptotic relation. Moreover, we prove that if the proximal relation is dense in a neighbourhood of some point in the diagonal then f is chaotic in the sense of Li–Yorke. As applications we obtain that if f contains a periodic point, or f is 2-scattering, then f is chaotic in the sense of Li–Yorke. Thus, chaos in the sense of Devaney is stronger than that of Li–Yorke. © 2002 Elsevier Science B.V. All rights reserved.

AMS classification: 54H20; 58F03

Keywords: Devaney's chaos; Li–Yorke's chaos; Proximal and asymptotic relation; Scrambled set; Scattering

1. Introduction

By a *dynamical system*, we mean a pair (X, f), where X is a compact metric space and $f:X\to X$ is continuous and surjective. We say (X, f) is trivial, if X is consisting of one point. Chaoticity or complexity is an important property of dynamical systems. In 1975 Li–Yorke gave the first mathematical definition of chaos. Since then, people from different fields try to describe the behaviors by providing definitions of chaos according to their understanding of the subject. Surely, it is an important question to understand the relation among the various definitions.

In this paper we will study the most two popular definitions, given by Devaney, Li–Yorke; and the notion of 2-scattering and scattering given by Blanchard, Host and Maass.

☆ Project supported by NSFC 19625103 and 973 project.

* Corresponding author.
E-mail address: yexd@ustc.edu.cn (W. Huang).

PII: S0166-8641(01)00025-6

Let (X, f) be a dynamical system with metric d. According to Li and Yorke [16], a subset $S \subset X$ is a *scrambled set* (for f), if any different points x and y from S are *proximal* and not *asymptotic*, i.e.,

$$\liminf_{n\to+\infty} d\big(f^n(x), f^n(y)\big) = 0, \quad \text{and} \quad \limsup_{n\to+\infty} d\big(f^n(x), f^n(y)\big) > 0.$$

We say f is *chaotic in the sense of Li–Yorke*, if there exists an uncountable, scrambled set.

A dynamical system is *transitive* if for each pair of non-empty open subsets A, B of X, there is $n \in \mathbb{N}$ such that $f^n(A) \cap B \neq \emptyset$. It is clear that the set of *transitive points* of f, i.e., points with dense orbits, is a dense G_δ subset, which will be denoted by $Tran_f$. A dynamical system is *minimal* if it is transitive and $Tran_f = X$. Denote by $\omega(x, f)$ the ω-limit set of $x \in X$. Then it is easy to see that f is transitive if and only if there is $x \in X$ with $\omega(x, f) = X$. We say (X, f) is *weakly mixing* if $f \times f : X \times X \to X \times X$ is transitive, and (X, f) is *totally transitive* if f^n is transitive for each $n \in \mathbb{N}$. $x \in X$ is a *periodic point* of period n if $f^n(x) = x$, but $f^i(x) \neq x$ for $1 \leqslant i \leqslant n-1$. If $n = 1$, x is a *fixed point*. If (X, f) is transitive and X is finite then it consists of a single periodic orbit, while if X is infinite then it contains no isolated points.

We say that $f : X \to X$ is *sensitive* or *sensitive dependence on initial conditions* if there is $\varepsilon > 0$ such that for each x and each $\delta > 0$ there are y with $d(x, y) < \delta$ and $n \in \mathbb{N}$ such that $d(f^n(x), f^n(y)) > \varepsilon$.

By Devaney [10], f is chaotic, if it satisfies the following three properties:

(1) f is transitive;

(2) the set of periodic points of f is dense in X;

(3) f is sensitive dependence on initial conditions.

We remark that according to [6], or [14], or [4], (1) and (2) implies (3).

In some sense the definitions of Li–Yorke and Devaney describe the complexity a dynamical system using the behaviors of points under iterations. In [8] the authors introduce the notion of scattering and 2-scattering using complexity of covers. Given a cover $\mathcal{C}$ of a dynamical system (X, T), usually open or closed, its complexity function $C(\mathcal{C}, n)$ is the minimal number of a sub-cover of refinement $\mathcal{C}_0^n = \bigvee_{i=0}^n T^{-i}\mathcal{C}$. According to [8], a dynamical system (X, T) is *scattering* if any cover by non-dense open sets has unbounded complexity, and *2-scattering* if the same is true for 2-set covers only.

Scattering evidently implies 2-scattering. Any topologically weakly mixing system is scattering; 2-scattering is total transitivity; (X, T) is scattering if and only if its Cartesian product with any minimal system is transitive (see [8]). Moreover, scattering is strictly weaker than weak mixing [3].

In the context of transitive systems, it seems natural to demand that the scrambled set be large topologically rather than just in cardinality. We call a system *densely Li–Yorke chaotic* when it admits a dense, uncountable scrambled set. Our main result is a characterization of this property in a wide class of systems which includes all transitive systems. Observe that if (X, f) is Li–Yorke chaotic then so is its product with any system (Y, g). On the other hand, if a system has a nontrivial periodic orbit as a factor then it can not be densely Li–Yorke chaotic. In particular if (Y, g) is a nontrivial periodic system, then $(X \times Y, f \times g)$ is not densely Li–Yorke chaotic.

To study scrambled sets for transitive maps, we find that it is very useful to get a deep understanding of the asymptotic class and relation. This will be done in Section 2. It turns out that for transitive systems with infinitely many points they are of first category. In Section 3, we study proximal relation and give some sufficient conditions, each of which implies chaos in the sense of Li–Yorke. Particularly, if the proximal relation is dense in a neighbourhood of some point in the diagonal, and if the system is transitive with infinitely many points, then the system is chaotic in the sense of Li–Yorke. Moreover, we show that in some case the uncountably scrambled set can be chosen both for the map and its inverse. In Section 4, we use the results obtained in Sections 2 and 3 to prove that a transitive system with infinitely many points containing a periodic point must have an uncountable scrambled set. In particular, chaos in the sense of Devaney is stronger than that of Li–Yorke. Moreover, we also show that 2-scattering implies chaos in the sense of Li–Yorke. We mention that in [15,17] the authors show that weak mixing implies chaos in the sense of Li–Yorke, which can be obtained by our results as weak mixing implies 2-scattering. Finally, in the Appendix we prove that some results which hold for homeomorphisms are true for continuous maps.

A point $x \in X$ is called an *equicontinuous point* if for each $\varepsilon > 0$, there is $\delta > 0$ such that *diam* $f^n(B(x,\delta)) < \varepsilon$ for each n, where $B(x,\delta)$ is the open ball centered at x with radius δ. Note that $\overline{B(x,\varepsilon)}$ will be denoted by $\overline{B}(x,\varepsilon)$.

A transitive system with an equicontinuous, transitive point is called *almost equicontinuous*. Finally, f is *uniformly rigid* if the identity is a limit point of the iterates in the uniform topology and f is *2-rigid* if each point in $X \times X$ is recurrent under $f \times f$. Recall that $x \in X$ is a *recurrent point* if there is $n_i \to +\infty$ such that $f^{n_i}(x) \to x$.

The following propositions are proved in [5,14,4], respectively, and we include the proofs here for completeness.

Proposition 1.1. *If $f : X \to X$ is almost equicontinuous, then it is uniformly rigid.*

Proof. Given an $\varepsilon > 0$, there is a transitive point x_0 and an open neighbourhood U of x_0 such that $d(f^n(x_0), f^n(y)) < \varepsilon$ for each n and each $y \in U$. Let now k satisfy $f^k(x_0) \in U$, then $d(f^{n+k}(x_0), f^n(x_0)) < \varepsilon$ for each n, and since x_0 is transitive it follows that $d(f^k(z), z) \leqslant \varepsilon$ for each $z \in X$. Applying this observation to a sequence of ε_i's that tend to zero gives a sequence of k_i's such that f^{k_i} tends to identity uniformly. □

Proposition 1.2. *Assume that (X, f) is a transitive dynamical system. Then f is almost equicontinuous if and only if f is not sensitive.*

Proof. It is clear that if f is almost equicontinuous then f is not sensitive.

Now suppose that f is not sensitive. Then for each n, there is a non-empty open subset U_n of X such that $\mathrm{diam}(f^i(U_n)) < \frac{1}{n}$ for each $i \in \mathbb{N}$. Let $x \in \mathit{Tran}_f$. As f is transitive, for each n, there is a non-empty open subset V_n with $x \in V_n$ and $f^j(V_n) \subset U_n$ for some $j \in \mathbb{N}$ such that $\mathrm{diam}(f^i(V_n)) < \frac{1}{n}$ for each $i \in \mathbb{N}$. Thus, x is an equicontinuous point and hence f is almost equicontinuous. □

2. Asymptotic relation

Let (X, f) be a dynamical system with metric d. Recall that x and y are *asymptotic* if $\lim d(f^n(x), f^n(y)) = 0$, x and y are *proximal* if $\liminf d(f^n(x), f^n(y)) = 0$ and x and y are *distal* if they are not proximal. Denote the asymptotic relation by *AR*, the proximal relation by *PR*, the distal relation by *DR* and the Li–Yorke relation by *LYR*. Sometimes to indicate the space and the map we also use notations $AR(X, f)$, $PR(X, f)$, $DR(X, f)$ and $LYR(X, f)$. Let $\Delta_n = \{(x, y)\colon\ d(x, y) < \frac{1}{n}\}$ and $\Delta = \{(x, x)\colon\ x \in X\}$. Note that for a relation $R \subset X \times X$ and $x \in X$, $R(x) = \{y\colon\ (x, y) \in R\}$. Let $[x] = AR(x)$ be the asymptotic class of $x \in X$. Then we have the following easy facts.

Lemma 2.1. *Let* $A_{k,n} = \bigcap_{i=k}^{\infty}(f \times f)^{-i}\overline{\Delta}_n$. *Then*

(1) $AR = \bigcap_{n=1}^{\infty}(\bigcup_{k=1}^{\infty} A_{k,n})$,

(2) *for each* $x \in X$,

$$[x] = \bigcap_{n=1}^{\infty}\left(\bigcup_{l=1}^{\infty}\bigcap_{j=l}^{\infty} f^{-j}\overline{B}\left(f^j(x), \frac{1}{n}\right)\right) = \bigcap_{n=1}^{\infty}\left(\bigcup_{k=1}^{\infty} A_{k,n}(x)\right),$$

(3) $PR = \bigcap_{k=1}^{\infty}(\bigcup_{n=1}^{\infty}(f \times f)^{-n}\Delta_k)$,

(4) $DR = X \times X \setminus PR$, *and*

(5) $LYR = PR \setminus AR$.

Note that each $A_{k,n}$ is closed, *PR* is a G_δ subset and *AR* is an equivalence relation on X. Concerning the asymptotic relation and class we have

Asymptotic Theorem. *Assume that* $f : X \to X$ *is a dynamical system. Then*

(a) *If* (X, f) *is 2-rigid, then* $AR = \Delta$ *and for each* x, $[x] = \{x\}$,

(b) *If* (X, f) *is sensitive then AR has first category in* $X \times X$ *and for each* $x \in X$, $[x]$ *has first category.*

Proof. (a) Each pair (x, y) is recurrent.

(b) Let $\varepsilon > 0$ be the constant in the definition of sensitivity and $n \in \mathbb{N}$ with $\frac{2}{n} < \varepsilon$. Then for all $k \in \mathbb{N}$ and $x \in X$, $A_{k,n}(x)$ has empty interior. For if U is a non-empty open subset of X contained in $A_{k,n}(x)$ then for y and $z \in U$, $d(f^i(y), f^i(z)) \leqslant \frac{2}{n}$ for all $i \geqslant k$ and so by shrinking U if necessary for all $i \geqslant 0$, contradicting the definition of ε. Thus, for each $x \in X$, $[x]$ has first category and *AR* has first category by Lemma 2.1, as $A_{k,n}$ has empty interior too. $\square$

Corollary 2.2. *Let* (X, f) *be a transitive system with* X *infinite. Then AR has first category and for each* $x \in X$, $[x]$ *has first category.*

Proof. If f is not sensitive, by Propositions 1.1 and 1.2 f is uniformly rigid and hence is 2-rigid. By Asymptotic Theorem the result follows. $\square$

For a dynamical system (X, f) we use $h(f)$ to denote the *topological entropy* of f, which is a non-negative number describing the chaotic behaviors of the system.

Corollary 2.3. *Assume that $\{[x]\colon x \in X\}$ is countable. Then each recurrent point is periodic and the set of periodic points is countable. Consequently, the topological entropy of f is zero.*

Proof. Note that if a recurrent point x is not periodic, then its ω-limit set $\omega(x, f)$ is infinite and $(\omega(x, f), f)$ is transitive. Thus, if $\{[x]\colon x \in X\}$ is countable then each recurrent point is periodic and the set of periodic points is countable by Corollary 2.2. This implies that the topological entropy is zero by the Variational principle. $\square$

Remark 2.4. By Corollary 2.2, for a transitive system (X, f) with X infinite, *AR* has first category. In some cases, *AR* could be dense in $X \times X$: the one-sided full shift with $N \geqslant 2$ symbols is a such example.

Blanchard, Host and Ruette [7] show that the factor induced by the smallest closed invariant equivalence relation containing the asymptotic relation has zero entropy. Thus, it is natural to conjecture that $h(\widetilde{f}) = 0$, where $\widetilde{f}: X/{\sim} \to X/{\sim}$ is the induced map, and $x \sim y$ if and only if $(x, y) \in AR$.

3. Proximal relation and scrambled sets

In this section we study the proximal relation and try to understand how it relates to the scrambled sets. We show that for a transitive system $f: X \to X$ with X infinite if the proximal relation is dense in a neighbourhood of some point in the diagonal, then f is chaotic in the sense of Li–Yorke. To do this we start with the following set-theoretical lemmas. Recall that for a relation $R \subset X \times X$ and $x \in X$, $R(x) = \{y\colon (x, y) \in R\}$.

Lemma 3.1. *Assume that X is a complete separable metric space without isolated points. If R is a symmetric relation with the property that there is a dense G_δ subset A of X such that for each $x \in A$, $R(x)$ contains a dense G_δ subset, then there is a dense, subset B of X with uncountably many points such that $B \times B \setminus \Delta \subset R$.*

Proof. Let $\mathcal{B}$ be the family of all non-empty subsets of A with the property that for each $B \in \mathcal{B}$, $B \times B \setminus \Delta \subset R$. $\mathcal{B}$ is non-empty, as for some $x \in A$ if we take $y \in R(x) \cap A$ with $y \neq x$, then $B_0 = \{x, y\} \in \mathcal{B}$. Suppose that $\{U_i\}$ is a base for the topology of X. We now construct a dense subset $C = \{x_i\colon i \in \mathbb{N}\}$ such that $C \in \mathcal{B}$ and $x_i \in U_i$ for each $i \in \mathbb{N}$. Assume that we have constructed $C_n = \{x_1, \ldots, x_n\}$ for some $n \in \mathbb{N}$ such that $C_n \in \mathcal{B}$ and $x_i \in U_i$, $1 \leqslant i \leqslant n$. As $D_n = A \cap (\bigcap_{i=1}^n R(x_i))$ contains a dense G_δ set, $D_n \cap U_{n+1} \neq \emptyset$. Take any point from the intersection and name it as x_{n+1}. Then $\{x_1, \ldots, x_{n+1}\} \in \mathcal{B}$. In this way we get a dense subset $C \in \mathcal{B}$.

By Zorn's Lemma there is a maximal element $B \supset C$ (under the usual inclusion) of $\mathcal{B}$ with the property. We claim that B is uncountable.

Assume the contrary that $B = \{x_i\}_{i \in K}$ with K countable. As $A \cap (\bigcap_{i \in K} R(x_i))$ contains a dense G_δ subset, we can take $x \in A \cap (\bigcap_{i \in K} R(x_i))$ such that $x \neq x_i$ for each $i \in K$. It is clear that $x \in A$ and $(x, x_i) \in R$ for each i. Let $B' = B \cup \{x\} \subset A$, then $B' \times B' \setminus \Delta \subset R$, a contradiction. Thus, B is uncountable. $\square$

The next lemma is called topological Fubini Theorem in [1, Ch. 7]. Here we provided a direct proof without the assumption of compactness.

Lemma 3.2. *Let R be a relation on a complete separable metric space X, which contains a dense G_δ subset of $X \times X$. Then, there is a dense G_δ subset A of X such that for each $x \in A$, there exists a dense G_δ subset A_x of X with $\{(x, y)\colon x \in A, y \in A_x\} \subset R$.*

Proof. Suppose $R \supset \bigcap_{n=1}^{\infty} U_n$, where U_n is a dense open set of $X \times X$ for each $n \in \mathbb{N}$.

Set $L_n[x] = \{y \in X\colon (x, y) \in U_n\}$. Clearly, $L_n[x]$ is open. Let

$$X_n = \bigl\{x \in X\colon L_n[x] \text{ is a dense set}\bigr\}.$$

We claim that X_n is a dense G_δ set.

Proof of the claim: Let $\{z_i\}_{i=1}^{\infty}$ be a dense subset of X and $F_n = {U_n}^c$. Set

$$F_n(i, r) = \bigl\{x \in X\colon (x, y) \in F_n, \text{ for all } y \in \overline{B}(z_i, r)\bigr\},$$

where $i \in \mathbb{N}$, $r \in Q^+$ and Q^+ is the set of all positive rational numbers. As F_n is closed, it is easy to see that $F_n(i, r)$ is closed. In the meantime, we notice that $F_n(i, r)$ is nowhere dense. Otherwise, if there exists a non-empty open set $U \subset F_n(i, r)$, then we have $U \times B(z_i, r) \subset F_n$, a contradiction as $F_n = U_n^c$ and U_n is dense. Moreover, we have

$$X_n^c = \bigcup_{i \in \mathbb{N},\ r \in Q^+} \bigl\{x \in X\colon\ L_n[x] \cap \overline{B}(z_i, r) = \emptyset\bigr\} = \bigcup_{i \in \mathbb{N},\ r \in Q^+} F_n(i, r).$$

Hence we have proved that X_n is a dense G_δ set. This ends the proof of the claim.

Now, let $A = \bigcap_{n=1}^{+\infty} X_n$. Then A is also a dense G_δ set. For any $x \in A$, $L_n[x]$ is dense and open. Let $A_x = \bigcap_{n=1}^{\infty} L_n[x]$, then A_x is a dense G_δ set. Moreover, $\{(x, y)\colon x \in A,\ y \in A_x\} \subset R$. This ends the proof. $\square$

For a dynamical system (X, f), x and y in X are *regionally proximal* if for $\varepsilon > 0$ and each neighbourhood U of (x, y), one finds $(x', y') \in U$ and $n \in \mathbb{N}$ such that $d(f^n(x'), f^n(y')) \leqslant \varepsilon$. Let RPR be the regionally proximal relation. It is clear that $RPR = \bigcap_{k=1}^{\infty} \overline{(\bigcup_{n=1}^{\infty} (f \times f)^{-n} \Delta_k)}$ is closed. If f is a homeomorphism, we set $RPR_{\mathbb{Z}} = RPR(X, f) \cup RPR(X, f^{-1})$. It is well known that if (X, f) is minimal, then RPR is an equivalence relation, see [2] and Appendix.

It is a natural question: what can we say if (X, f) is invertible and $RPR_{\mathbb{Z}} = X \times X$? It turns out that $RPR_{\mathbb{Z}} = X \times X$ implies that $RPR(X, f) = RPR(X, f^{-1}) = X \times X$ if the system is transitive. To prove this fact we need a simple lemma. Recall that $Rec(X, f)$ is the set of all recurrent points of f. In [11], the author shows that if each point in a dynamical system is non-wandering, i.e., each point is a non-wandering point, then the set of recurrent points is a dense G_δ subset. The similar idea will be used in this section.

Lemma 3.3. *Let (X, f) be a dynamical system. Then*
(1) *If $(x, y) \in DR(X, f)$, then $\omega((x, y), f \times f) \subset DR$. If in addition f is invertible and $(x, y) \in Rec(f \times f)$, then $(x, y) \in DR(X, f^{-1})$.*
(2) *If f is transitive, then $Rec(f \times f)$ is a dense G_δ subset of $X \times X$. Moreover, if f is transitive and does not contain periodic points, then DR is dense in $X \times X$.*

Proof. (1) Set $Z = \omega((x, y), f \times f)$. Then $f \times f(Z) = Z$ and $Z \cap \Delta = \emptyset$. Thus, for each $z \in Z$, $\omega(z, f \times f) \subset Z$. This implies that $\omega(z, f \times f) \cap \Delta = \emptyset$. Hence $z \in DR$. That is, $Z \subset DR$.

If in addition f is invertible and $(x, y) \in Rec(f \times f)$, then $(x, y) \in Z$ and $(f \times f)^{-1}(Z) = Z$. Thus, $\omega((x, y), (f \times f)^{-1}) \subset Z$. This clearly implies that $(x, y) \in DR(X, f^{-1})$.

(2) Note that for each $x \in Tran_f$, $\{(x, f^n(x)): n \in \mathbb{N}\} \subset Rec(f \times f)$ is dense in $\{x\} \times X$. Thus $Rec(f \times f)$ is dense in $X \times X$. By [11], $Rec(f \times f)$ is a dense G_δ subset of $X \times X$.

If f is transitive and does not contain periodic points, then for each $x \in Tran_f$, $\{(x, f^n(x)): n \in \mathbb{N}\} \subset DR$. Thus, DR is dense in $X \times X$. □

Theorem 3.4. *Let (X, f) be a dynamical system.*
(1) *If $RPR(X, f) = X \times X$, then $PR(X, f)$ is a dense G_δ subset of $X \times X$.*
(2) *If (X, f) is an invertible, transitive dynamical system, and if there are non-empty open subsets Y_1 and Y_2 of X with $RPR(X, f) \supset Y_1 \times Y_2$, then $RPR(X, f^{-1}) \supset Y_1 \times Y_2$. Thus if $RPR_{\mathbb{Z}} = X \times X$, then $RPR(X, f) = RPR(X, f^{-1}) = X \times X$.*

Proof. (1) Let $d(,)$ denote the metric on X, and define $F : X \times X \to \mathbb{R}$ such that

$$F(x, y) = \inf_{n \geqslant 0} d\bigl(T^n x, T^n y\bigr).$$

The function F is an upper semicontinuous function. For if $F(x_0, y_0) = u$, then for any $\varepsilon > 0$, some $n \in \mathbb{N}$ with $d(T^n x_0, T^n y_0) < u + \varepsilon$, the same will remain true in a neighborhood of (x_0, y_0). In other words, $F(x, y) < F(x_0, y_0) + \varepsilon$ in a neighborhood of (x_0, y_0). So F is upper semicontinuous.

Now a semicontinuous function on a compact metric space possesses a dense G_δ set of points of continuity. Suppose (x_0, y_0) is a point of continuity of F. If $F(x_0, y_0) = 0$, then $(x_0, y_0) \in PR$. If $F(x_0, y_0) > 0$, then for some $a > 0$ and some neighborhood of (x_0, y_0), $F(x, y) > a$ throughout this neighborhood. Let $U \times V$ be an non-empty open set for which this inequality prevails, then $U \times V \cap RPR = \emptyset$, a contradiction as $RPR = X \times X$. We conclude that if F is continuous at (x_0, y_0) then $F(x_0, y_0) = 0$. Thus PR is a dense G_δ subset.

Another proof of (1) can be obtained by using Lemma 2.1(3) and the fact that $RPR = \bigcap_{k=1}^{\infty} \overline{(\bigcup_{n=1}^{\infty} (f \times f)^{-n} \Delta_k)}$.

(2) Assume that there are non-empty open subsets Y_1 and Y_2 of X such that $RPR(X, f) \supset Y_1 \times Y_2$. Note that to show $RPR(X, f^{-1}) \supset Y_1 \times Y_2$, it is enough to prove $RPR(X, f^{-1}) \cap (Y_1' \times Y_2') \neq \emptyset$ for non-empty open sets $Y_i' \subset Y_i$ with $\overline{Y_i'} \subset Y_i$ for $i = 1, 2$.

As $PR(X, f) \cap \overline{Y_1'} \times \overline{Y_2'}$ is a dense G_δ subset of $\overline{Y_1'} \times \overline{Y_2'}$, we know that

$$\left(Y_1' \times Y_2'\right) \cap Rec(f \times f)^{-1} \cap PR(X, f) \neq \emptyset$$

by (2) of Theorem 3.3, as f is transitive and hence f^{-1} is transitive. Let (x, y) belong to the mentioned set. By Lemma 3.3, $(x, y) \notin DR(X, f^{-1})$. Thus $(x, y) \in PR(X, f^{-1}) \subset RPR(X, f^{-1})$.

If $RPR_{\mathbb{Z}} = X \times X$, then by what we just proved $RPR(X, f) = RPR(X, f^{-1}) = X \times X$. □

Now we are ready to show the main result of the paper.

Theorem 3.5. *Let (X, f) be a dynamical system. Then the following statements are equivalent*:

(1) *$RPR = X \times X$.*

(2) *PR is a dense G_δ set.*

(3) *PR is dense in $X \times X$.*

Moreover, if f is either transitive with X infinite or sensitive, then (1)–(6) *are equivalent*:

(4) *LYR contains a dense G_δ subset of $X \times X$.*

(5) *There is a dense scrambled set for f.*

(6) *f is densely Li–Yorke chaotic.*

If f is transitive invertible with X infinite, then (1)–(7) *are equivalent*:

(7) *There is a dense, uncountable, scrambled set for f and f^{-1}, which is contained in $Tran_f \cap Tran_{f^{-1}}$.*

Proof. By (1) of Theorem 3.4, (1) implies (2).

Clearly, (2) implies (3). As RPR is closed and $PR \subset RPR$, (3) implies (1).

If f is either transitive with X infinite or sensitive, then AR is of first category by Asymptotic Theorem and Corollary 2.2. As $LYR = PR \setminus AR$, (2) implies (4).

Now we assume that (4) holds. By Lemma 3.2, there is a dense G_δ subset A of X such that for each $x \in A$, there is a dense G_δ set A_x with $\{(x, y){:}\ x \in A,\ y \in A_x\} \subset LYR$. Following Lemma 3.1, we can find a dense, uncountable, scrambled set for f. Thus, (4) implies (5) and (6).

It is clear that (5) implies (3) and (6) implies (5).

Now suppose f is transitive invertible with X infinite. It is obvious that (7) implies (3). We show (4) implies (7) now. As f is transitive, f^{-1} is transitive. By Theorem 3.4, $RPR(X, f^{-1}) = X \times X$. Thus $LYR(X, f^{-1})$ contains a dense G_δ subset. Hence

$$LYR(X, f) \cap LYR\left(X, f^{-1}\right) \cap (Tran_f \times Tran_f) \cap (Tran_{f^{-1}} \times Tran_{f^{-1}})$$

contains a dense G_δ subset of $X \times X$. By Lemmas 3.1 and 3.2 we can find a dense, uncountable scrambled set for f and f^{-1}, which is contained in $Tran_f \cap Tran_{f^{-1}}$. □

Let (X, f) be a minimal system. It is known that f is sensitive if and only if f is not equicontinuous [5,4]. Now we have

Corollary 3.6. *If (X, f) is a minimal system with X infinite, then it is densely Li–Yorke chaotic if and only if it is weakly mixing.*

Proof. If f is densely Li–Yorke chaotic, then $RPR = X \times X$ and the maximal equicontinuous factor is trivial. Thus, f is weakly mixing by Proposition A.4(3). Conversely, if f is weakly mixing, then $Tran_{f\times f} \subset LY$ is a dense subset of $X \times X$. Then f is densely Li–Yorke chaotic by Theorem 3.5. □

Corollary 3.7. *Let (X, f) be transitive with X infinite. If there is a non-empty open subset Y of X such that $\overline{PR} \supset Y \times Y$, then there is an uncountable scrambled set contained in Y.*

Proof. Let $Y' \subset Y$ be a non-empty open set with $\overline{Y'} \subset Y$. Then the proofs in Lemmas 3.1, 3.2 and Theorem 3.5 are valid replacing X by $\overline{Y'}$. □

Corollary 3.8. *Let (X, f) be a dynamical system. If there are $n \in \mathbb{N}$ and a non-empty closed f^n-invariant subset Y such that $f^n|_Y$ is transitive, not periodic and $PR(Y, f^n|_Y)$ is dense in some neighbourhood of a point in the diagonal of $Y \times Y$, then f is chaotic in the sense of Li–Yorke. Consequently, if f is transitive, not periodic and not chaotic in the sense of Li–Yorke, then each scrambled set for f is nowhere dense.*

Proof. The first statement of the corollary follows by Corollary 3.7 and the fact that a scrambled set for f^n in Y is a scrambled set for f.

Now we assume that f is not chaotic in the sense of Li–Yorke and S is a scrambled set for f. If S is not no-where dense, then $\overline{PR} \supset Y \times Y$, where Y is a non-empty open subset of X. By Corollary 3.7, f is chaotic in the sense of Li–Yorke, a contradiction. □

Remark 3.9. It is well known that if (X, f) is a dynamical system and R is the smallest closed invariant equivalence relation generated by RPR or $Com(X, f)$ (see Section 4 for the definition), then $(X/R, f_R)$ is the maximal equicontinuous factor of (X, f), see [2,8] and Appendix. A natural question is: if the maximal equicontinuous factor of a transitive, non-periodic system is trivial, is it true that f is chaotic in the sense of Li–Yorke?

Note that if (X, f) is minimal, the assumption implies that f is weakly mixing, and hence f is chaotic in the sense of Li–Yorke, see [15,17] or Section 4 of the paper. In the general case by [9], $LYR \neq \emptyset$.

We have another natural question. Assume that (X, f) is transitive, f is invertible and f is chaotic in the sense of Li–Yorke. Is it true that f^{-1} is chaotic in the same sense? Note that without the transitivity assumption, the question has a negative answer. In [13] an example (X, f) is constructed such that X is a scrambled set for f, but $AR(X, f^{-1}) = X \times X$.

4. Applications

In this section we shall use the results obtained in Sections 2 and 3 to study the scrambled subsets of transitive systems containing periodic points and 2-scattering systems. Note that in the proof of Theorem 4.1, we only use Corollary 2.2 and Lemma 3.1.

Theorem 4.1. *Assume that $f : X \to X$ is transitive with X infinite and contains a periodic point. Then there is an uncountable scrambled set for f. Moreover, if f is totally transitive, then f is densely Li–Yorke chaotic. Particularly, chaos in the sense of Devaney is stronger than that in the sense of Li–Yorke.*

Proof. First assume that f has a fixed point p. For each $x \in X$, it is clear that $PR(x)$ is a G_δ subset. If $x \in Tran_f$, then there is $\{n_i\}$ with $f^{n_i}(x) \to p$. This implies that $f^i(x)$ is proximal to x for each $i \geqslant 1$. Thus $PR(x)$ is a dense G_δ set for each $x \in Tran_f$.

Let $R = LYR(X, f)$ and $A = Tran_f$. For each $x \in A$ it is clear that $R(x) = PR(x) \setminus AR(x)$ contains a dense G_δ subset by Corollary 2.2. By Lemma 3.1, there is subset B of X such that $B \times B \setminus \Delta \subset R$ and B is uncountable. It is clear B is a scrambled set for f.

Now assume that f has a periodic point of period $n > 1$. Let $x \in Tran_f$. Then $\omega(x, f) = X$. Set $D_i = \omega(f^i(x), f^n)$ for each $0 \leqslant i \leqslant n-1$. As $f(D_i) = D_{i+1(\mathrm{mod}\ n)}$, we know that each D_i is uncountable and contains a periodic point of f with period n. As $f^n|_{D_0}$ is transitive, has a fixed point of f^n, we can use the result we just proved. That is, there is an uncountable scrambled set B for f^n. It is clear that B is also a scrambled set for f. Following the above argument it is easy to see that if f is totally transitive, then f is densely Li–Yorke chaotic. This ends the proof. □

In contrast to the results of Lemma 3.3(2) it can happen that the entire space is a scrambled set. For example, if there is a unique minimal set and it is a fixed point then $PR = X \times X$. If, in addition, f is 2-rigid then $AR = \Delta$ and so X is scrambled.

Corollary 4.2. *If $f : X \to X$ is transitive and not minimal. Then there is a factor of (X, f) which has an uncountable scrambled set. Moreover, if f is almost equicontinuous which is not minimal, then it has a factor such that the whole space is a scrambled set.*

Proof. As f is not minimal, there is $x \in X$ such that $A = \omega(x, f)$ is no-where dense. Collapsing A to a point, we get a transitive factor of (X, f), which is not periodic and contains a fixed point. By Theorem 4.1 the factor has an uncountable scrambled set.

If f is almost equicontinuous which is not minimal, then the closure of the union of minimal sets is a proper subset of X [14,4]. When smashed the set to a point we get a factor which is almost equicontinuous and so is uniformly rigid and which has a single fixed point minimal set. Clearly, the whole space is a scrambles set. □

A *standard cover* of X is a pair (U, V) of non-dense open sets of X, covering X. If $x, x' \in X$, and (U, V) is a standard cover of X, we say that x and x' are separated by the cover (U, V) if $x \in Int(U^c)$ and $x' \in Int(V^c)$. Let $Com(X, f)$ be the pairs (x, y) with $x \neq y$

such that each standard cover separating them has unbounded complexity function. Note that if f is invertible, then $Com(X, f) = Com(X, f^{-1})$. It is clear that f is 2-scattering if and only if $Com(X, f) \cup \Delta = X \times X$.

Theorem 4.3. *Let $f : X \to X$ be 2-scattering. Then PR is dense in $X \times X$. Thus, f has a dense, uncountable, scrambled set.*

Proof. As f is 2-scattering, $Com(X, f) \cup \Delta = X \times X$. Then by Proposition A.1 we have $RPR(X, f) = X \times X$. By Theorem 3.5, PR is dense and hence f has a dense, uncountable, scrambled set. □

A very useful remark by Blanchard results the following

Theorem 4.4. *Let (X, f) be a transitive dynamical system. If there is $x \in X$ such that $RPR(x)$ contains a non-empty open subset of X, then there are $n \in \mathbb{N}$ and pair-wise disjoint closed subsets $X_1, \ldots, X_n$ of X such that $\bigcup_{i=1}^n X_i = X$, $f(X_i) = X_{i+1 (\mathrm{mod}\ n)}$, and the maximal equicontinuous factor of $f^n|_{X_i}$ is trivial for $1 \leqslant i \leqslant n$. Thus if (X, f) is minimal, then $f^n|_{X_i}$ is weakly mixing.*

Proof. Let (Y, g) be the maximal equicontinuous factor of (X, f) and $p : X \to Y$ be the factor map. Since f is transitive, (Y, g) is a minimal system. Set $y = p(x)$. Then $Z = p^{-1}(y)$ contains a non-empty open subset of X. As f is transitive, there is m such that $f^m(Z) \cap Z \neq \emptyset$. Thus, (Y, g) is a periodic system. Let n be the period of y, then the first statement of the theorem follows by setting $X_i = p^{-1}(g^i(y))$, $1 \leqslant i \leqslant n$.

If (X, f) is a minimal system, then $f^n|_{X_i}$ is weakly mixing for each i, see Appendix. □

Finally, we have the following remark.

Remark 4.5. It is a long open question: is a positive entropy system chaotic in the sense of Li–Yorke? [1] We mention that the method in the paper does not solve the problem as there exists a positive entropy system, whose proximal relation is "small". For example, point distal systems with positive entropy are such systems, see [12] for details.

Appendix A

In this appendix we show that some properties hold for homeomorphism are true for continuous maps. Though the arguments are simple, the results can be used conveniently.

[1] Recently Blanchard, Glasner, Kolyada and Maass proved that this question has a positive answer [9].

For a dynamical system (X, f) with metric d, we say (X_f, S) is a *natural extension* of (X, f), if $X_f = \{(x_1x_2\cdots):\ f(x_{i+1}) = x_i,\ x_i \in X,\ i \in \mathbb{N}\}$, which is a subspace of the product space $\prod_{i=1}^{\infty} X$ with the compatible metric d_f such that

$$d_f\big((x_1x_2\cdots), (y_1y_2\cdots)\big) = \sum_{i=1}^{\infty} \frac{d(x_i, y_i)}{2^i}.$$

Moreover, $S : X_f \to X_f$ is the shift homeomorphism, i.e., $S(x_1x_2\cdots) = (f(x_1)x_1x_2\cdots)$. We use $\pi : X_f \to X$ to denote the projection to the first coordinate.

In [8] the authors show that for a homeomorphism $Com(X, f) \subset RPR_{\mathbb{Z}}$. We now prove

Proposition A.1. *Let (X, f) be a dynamical system. Then $Com(X, f) \subset RPR(X, f)$.*

Proof. First we note that if f is a homeomorphism, then the proof of Proposition 5.8 of [8] implies $Com(X, f^{-1}) \subset RPR(X, f)$. As $Com(X, f^{-1}) = Com(X, f)$, we have $Com(X, f) \subset RPR(X, f)$.

Assume that (X_f, S) is the natural extension of (X, f). Then we have $Com(X_f, S) \subset RPR(X_f, S)$. Now let $(x, y) \in Com(X, f)$ and $\pi : X_f \to X$ be the projection. As π is continuous, for each $\varepsilon > 0$, there is $\delta > 0$ such that if $d_f(b, c) < \delta$ then $d(\pi(b), \pi(c)) < \varepsilon$.

By [8, Proposition A.1], there are $x', y' \in Com(X_f, S)$ such that $\pi(x') = x$ and $\pi(y') = y$. Suppose U and V are neighbourhood of x and y, respectively. Then $\pi^{-1}(U)$ and $\pi^{-1}(V)$ are neighbourhood of x' and y', respectively. Fix $\varepsilon > 0$. As $(x', y') \in RPR(X_f, S)$, there are $n \in \mathbb{N}$, $x'' \in \pi^{-1}(U)$ and $y'' \in \pi^{-1}(V)$ such that $d_f(S^n(x''), S^n(y'')) < \delta$. Set $x_0 = \pi(x'')$ and $y_0 = \pi(y'')$. It is easy to check that $d(f^n(x_0), f^n(y_0)) < \varepsilon$. That is, $(x, y) \in RPR(X, f)$. □

In [2] it is shown that the maximal equicontinuous factor of an invertible dynamical system is induced by the smallest closed invariant equivalence relation containing $RPR_{\mathbb{Z}}$ and if f is minimal then $RPR_{\mathbb{Z}}$ is an equivalence relation. To show the results hold in the general case we need the following simple lemma whose easy proof is omitted.

Lemma A.2. *Assume that $f : X \to Y$ is a surjective continuous map of compact metric spaces. If there are $y \in Y$ and $y_i \to y$, then there are $x \in f^{-1}(y)$, $n_i \to \infty$ with $x_{n_i} \in f^{-1}(y_{n_i})$ and $x_{n_i} \to x$.*

Another simple lemma we need is the following.

Lemma A.3. *Let (X, f) be a dynamical system. If $(x, y) \in RPR(X, f)$, then $(\pi \times \pi)^{-1}(x, y) \cap \mathrm{RPR}(X_f, S) \neq \emptyset$.*

Proof. As $(x, y) \in RPR(X, f)$, there are $(x_i, y_i) \in X \times X$, $k_i \to \infty$ such that $(x_i, y_i) \to (x, y)$ and $d(f^{k_i}x_i, f^{k_i}y_i) < \frac{1}{k}$.

By Lemma A.2 there are $(\widetilde{x}, \widetilde{y}) \in (\pi \times \pi)^{-1}(x, y)$, $n_i \to \infty$ such that $(\widetilde{x}_{n_i}, \widetilde{y}_{n_i}) \to (\widetilde{x}, \widetilde{y})$ and $\pi \times \pi(\widetilde{x}_{n_i}, \widetilde{y}_{n_i}) = (x_{n_i}, y_{n_i})$. By the definition of d_f it is easy to check that $(\widetilde{x}, \widetilde{y}) \in RPR(X_f, S)$.

Proposition A.4. *Let (X, f) be a dynamical system. Then*

(1) *RPR induces the maximal equicontinuous factor of (X, f). If f is invertible, then $Com(X, f) \subset RPR(X, f^{-1})$ and $RPR(X, f^{-1})$ also induces the maximal equicontinuous factor of (X, f).*

(2) *If (X, f) is minimal, then $Com(X, f) \cup \Delta = RPR(X, f)$ and $RPR(X, f)$ is an equivalence relation. If in addition f is invertible,*

$$Com(X, f) \cup \Delta = RPR(X, f) = RPR\big(X, f^{-1}\big) = RPR_{\mathbb{Z}}.$$

(3) *If (X, f) is minimal and the maximal equicontinuous factor of (X, f) is trivial, then f is weakly mixing.*

Proof. (1) By Proposition A.1, $Com(X, f) \subset RPR(X, f)$ and by [8, Proposition 5.7], $Com(X, f)$ induces the maximal equicontinuous factor of (X, f). Let R be the smallest closed invariant equivalence relation containing $Com(X, f)$. As $(\pi \times \pi)^{-1}(R)$ is a closed invariant equivalence relation containing $Com(X_f, S)$ and $RPR_{\mathbb{Z}}$ induces the maximal equicontinuous factor of (X_f, S), we get that $(\pi \times \pi)^{-1}(R) \supset RPR_{\mathbb{Z}}(X_f, S) \supset RPR(X_f, S)$. Thus, $R \supset RPR(X, f)$. Hence $RPR(X, f)$ induces the maximal equicontinuous factor of (X, f).

If f is invertible,

$$Com(X, f) = Com\big(X, f^{-1}\big) \subset RPR\big(X, f^{-1}\big) \subset RPR_{\mathbb{Z}}\big(X, f^{-1}\big) = RPR_{\mathbb{Z}}(X, f).$$

It is clear that $RPR(X, f^{-1})$ also induces the maximal equicontinuous factor of (X, f).

(2) We first show that for an invertible minimal dynamical system

$$Com(X, f) \cup \Delta = RPR(X, f) = RPR\big(X, f^{-1}\big) = RPR_{\mathbb{Z}}.$$

By Proposition A.1, $Com(X, f) \subset RPR(X, f)$ and by [8, Proposition 5.11], $Com(X, f) = RPR_{\mathbb{Z}}(X, f)$. Note that $Com(X, f) = Com(X, f^{-1})$. The result follows.

Now we assume that (X, f) is minimal. Let $(x, y) \in RPR(X, f)$. By Lemma A.3, there is $(\widetilde{x}, \widetilde{y}) \in RPR(X_f, S)$ such that $\pi \times \pi(\widetilde{x}, \widetilde{y}) = (x, y)$. As $S : X_f \to X_f$ is a homeomorphism and is minimal, we have $Com(X_f, S) \cup \Delta = RPR(X_f, S)$. Thus $(\widetilde{x}, \widetilde{y}) \in Com(X_f, S) \cup \Delta$. By [8, Proposition 5.1], $(x, y) = \pi \times \pi(\widetilde{x}, \widetilde{y}) \in Com(X, f) \cup \Delta$. Thus, $Com(X, f) \cup \Delta = RPR(X, f)$. Finally, we show that $RPR(X, f)$ is an equivalence relation.

Let (x, y) and $(y, z) \in RPR(X, f)$. By Lemma A.3, there are $\widetilde{x} \in \pi^{-1}(x)$, $\widetilde{y}_1, \widetilde{y}_2 \in \pi^{-1}(y)$ and $\widetilde{z} \in \pi^{-1}(z)$ such that both $(\widetilde{x}, \widetilde{y}_1)$ and $(\widetilde{y}_2, \widetilde{z})$ are in $RPR(X_f, S)$. As $RPR(X_f, S)$ is an equivalence relation [2], and $(\widetilde{y}_1, \widetilde{y}_2) \in AR(X_f, S)$, we know that $(\widetilde{x}, \widetilde{z}) \in RPR(X_f, S)$. Thus $(x, z) \in RPR(X, f)$. That is, $RPR(X, f)$ is an equivalence relation.

(3) As $Com(X, f)$ is an equivalence relation and the maximal equicontinuous factor is trivial, we get that $Com(X, f) \cup \Delta = X \times X$. That is, f is 2-scattering and so is S, the natural extension of f. By [8], f is weakly mixing. □

Acknowledgements

The second author would like to thank Francois Blanchard for many useful discussions on the subject during his stay in Marseille in May, 1999 and in the process preparing the manuscript. The authors also thank the referee for many helpful remarks.

References

[1] E. Akin, The General Topology of Dynamical Systems, American Mathematical Society, Providence, RI, 1993.
[2] J. Auslander, Minimal Flows and Their Extensions, North-Holland Math. Stud., Vol. 153, North-Holland, Amsterdam, 1988.
[3] E. Akin, J. Auslander, E. Glasner, Residual properties and almost equicontinuity, Preprint, 1999.
[4] E. Akin, J. Auslander, K. Berg, When is a transitive map chaotic?, in: Convergence in Ergodic Theory and Probability (Columbus, OH, 1993), Ohio State Univ. Math. Res. Inst. Publ., Vol. 5, de Gruyter, Berlin, 1996.
[5] J. Auslander, J.A. Yorke, Interval maps, factors of maps and chaos, Tohoku Math. J. 32 (1980) 177–188.
[6] J. Banks, J. Brooks, G. Cairns, G. Davis, P. Stacey, On Devaney's definition of chaos, Amer. Math. Monthly 99 (1992) 332–334.
[7] F. Blanchard, B. Host, S. Ruette, Asymptotic pairs in positive-entropy systems, Preprint, 2000.
[8] F. Blanchard, B. Host, A. Maass, Topological complexity, Ergodic Theory Dynamical Systems 20 (2000) 641–662.
[9] F. Blanchard, E. Glasner, S. Kolyada, A. Maass, On Li–Yorke pairs, Preprint, 2000.
[10] R. Devaney, Chaotic Dynamical Systems, Addison-Wesley, Reading MA, 1989.
[11] H. Furstenberg, Recurrence in Ergodic Theory and Combinatorial Number Theory, Princeton, 1981.
[12] E. Glasner, A simple characterization of the set of μ-entropy pairs and applications, Israel J. Math. 102 (1997) 13–27.
[13] W. Huang, X. Ye, Homeomorphisms with the whole compacta being scrambled sets, in: Ergodic Theory Dynamical Systems, to appear.
[14] E. Glasner, B. Weiss, Sensitive dependence on initial conditions, Nonlinearity 6 (1993) 1067–1075.
[15] A. Iwanik, Independent sets of transitive points, in: Dynam. Sys. and Ergod. Th., Banach Center Publ., Vol. 23, PWN, Warsaw, 1989, pp. 277–282.
[16] T. Li, J. Yorke, Period 3 implies chaos, Amer. Math. Monthly 82 (1975) 985–992.
[17] J.Xiong, Z. Yang, Chaos caused by a topologically mixing map, in: World Sci. Adv. Ser. Dyn. Syst., Vol. 9, pp. 550–572.