Selection 2

ON THE BEHAVIOR OF SELF-OSCILLATORY SYSTEMS WITH EXTERNAL FORCE

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Abstract

Nonlinear systems may exhibit oscillatory phenomena which are essentially different from those in linear systems. Among them there are nonlinear resonance with hysteresis, higher harmonic and subharmonic oscillations, and almost-periodic oscillations arising through self-excitation. This paper deals with nonlinear oscillations which occur in self-oscillatory systems under the action of a periodic external force by applying the mapping procedure based on the qualitative theory of differential equations.

1. INTRODUCTION

Let us consider the second-order nonlinear differential equation

$$\frac{d^2x}{dt^2} + f\left(x, \frac{dx}{dt}\right)\frac{dx}{dt} + g(x) = e(t) \tag{1}$$

where $e(t)$ is periodic function of period L. Oscillatory phenomena governed by this type of differential equation, i.e., nonlinear oscillations, received much attention through the oscillatory phenomena in mechanical and electrical systems first studied by Duffing and van der Pol. Since then, the phenomena have been studied by many engineers and mathematicians.

From both the engineering and mathematical points of view, the types and numbers of steady solutions become a first subject of discussion in relation to the form of non-linearity and the parameter values in the equation. Then the global behavior of solutions both in time and in phase space attracts our interest. In many cases, the relatively simple form of the equation is deceptive, and closer study reveals complicated behavior. In this paper the steady solutions for a few equations of the form (1) are discussed by applying the method of harmonic balance and by using electronic computers. Further, global behavior of solutions is examined topologically by applying the transformation theory of differential equations.

2. METHOD OF ANALYSIS

In this section, the transformation theory for the topological analysis of global behavior of solutions is briefly explained.

2.1 Transformation T

Let us rewrite Eq. (1) as a system of first-order equations

$$\begin{aligned} \frac{dx}{dt} &= y && \equiv X(x, y, t) \\ \frac{dy}{dt} &= -f(x, y)y - g(x) + e(t) && \equiv Y(x, y, t) \end{aligned} \tag{2}$$

where the functions f and g are differentiable functions of sufficient order with respect to x and y, and $e(t)$ is a periodic function of t with period L.

Let $\{x(x_0, y_0, t), y(x_0, y_0, t)\}$ be a solution of Eqs. (2) which when $t = 0$ is at the point $P_0(x_0, y_0)$ of the xy plane. Let us focus our attention on the location of the point $P_n(x_n, y_n)$ whose coordinates are given by

$$x_n = x(x_0, y_0, nL), \qquad y_n = y(x_0, y_0, nL) \tag{3}$$

for $n = 0, \pm 1, \pm 2, \cdots$. Then the mapping

$$P_n = T^n P_0 \tag{4}$$

is defined which takes $P_0(x_0, y_0)$ into $P_n(x_n, y_n)$ of the xy plane. The mapping T thus defined is known to be a one-to-one, continuous and orientation-preserving transformation of the xy plane into itself [1, 2].

A point in the xy plane which is invariant under the mapping T is called a fixed point. Thus, if P_0 is a fixed point, then $P_0 = P_1 (= TP_0)$. Let m be the smallest positive integer for which $P_0 = P_m (= T^m P_0)$, then P_0 will iterate periodically through a set of m distinct points and is therefore called an m-periodic point. The set of these m-periodic points is called an m-periodic group. A solution of Eqs. (2) which passes through a fixed point of T at $t = 0$ is periodic with period L. Similarly, a solution associated with an m-periodic point is a subharmonic solution of period mL.

2.2 Maximum Finite Invariant Set

Under conditions generally met in practice [3], through any point in the xy plane sufficiently remote from the origin, there exist a simple closed curve having the property that every solution $(x(t),\, y(t))$ of Eqs. (2) which passes through the curve at time t can intersect the curve only by crossing it from the domain exterior to the curve into the domain interior to the curve with increasing t. Let Γ_0 denote a simple closed curve of this type, $T^n\Gamma_0$ be denoted by Γ_n and Δ_n be the closed domain bounded by Γ_n in the xy plane; then $\Delta_{n+1} \subset \Delta_n$ follows from the property of the curve Γ_0. The closed set defined by the intersection of all $\Delta_n (n \geq 0)$ is called the maximum finite invariant set for Eqs. (2). Let this set be denoted by Δ. It is known that Δ is a bounded and connected closed set and possesses the property that images of a point which is not contained in Δ tend to the maximum finite invariant set under iteration of the mapping T. Moreover, fixed points and periodic points, if they exist, are all contained in Δ. There exists at least one fixed point in Δ [1, 2].

2.3 Fixed Points, Periodic Points and Related Properties

In connection with the type and the number of periodic solutions of Eqs. (2), let us describe the classification of fixed points and periodic points and the number of these points contained in the maximum finite invariant set Δ.

Inasmuch as any neighboring point of the fixed point P_0 is taken into a neighborhood of the point P_0 under T, the type of fixed points can be classified by investigating the movement of neighboring images under the mapping T. Let a neighboring point $Q_0(x_0 + u_0,\, y_0 + v_0)$ of the fixed point $P_0(x_0,\, y_0)$ be transformed into $Q_1(x_0 + u_1,\, y_0 + v_1)$ under the mapping T, then the following relation results,

$$\begin{aligned} x_0 + u_1 &= x(x_0 + u_0,\, y_0 + v_0,\, L) \\ y_0 + v_1 &= y(x_0 + u_0,\, y_0 + v_0,\, L). \end{aligned} \tag{5}$$

Since P_0 is a fixed point under T, $u_0 = v_0 = 0$ means $u_1 = v_1 = 0$. For small values of u_0 and v_0, u_1 and v_1 can be expanded into power series in u_0 and v_0 as follows,

$$\begin{aligned} u_1 &= au_0 + bv_0 + \cdots \\ v_1 &= cu_0 + dv_0 + \cdots \end{aligned} \tag{6}$$

where

$$
\begin{aligned}
a &= \frac{\partial x(x_0,\, y_0,\, L)}{\partial x_0}, \qquad & b &= \frac{\partial x(x_0,\, y_0,\, L)}{\partial y_0} \\
c &= \frac{\partial y(x_0,\, y_0,\, L)}{\partial x_0}, \qquad & d &= \frac{\partial y(x_0,\, y_0,\, L)}{\partial y_0}.
\end{aligned} \tag{7}
$$

The terms not explicitly given in the right sides of Eqs. (6) are of degree higher than the first in u_0 and v_0. Equations (6) express the mapping $(u_0,\, v_0) \rightarrow (u_1,\, v_1)$ in the neighborhood of the fixed point P_0, and this transformation is characterized by the roots m_1 and m_2 of the characteristic equation*

$$
\begin{vmatrix} a-m & b \\ c & d-m \end{vmatrix} = 0. \tag{8}
$$

The fixed point P_0 is called simple if both $|m_1|$ and $|m_2|$ are different from unity. Simple fixed points are classified as follows [1]:

Completely stable if	$\lvert m_1 \rvert < 1, \quad \lvert m_2 \rvert < 1$
Completely unstable if	$\lvert m_1 \rvert > 1, \quad \lvert m_2 \rvert > 1$
Directly unstable if	$0 < m_1 < 1 < m_2$
Inversely unstable if	$m_1 < -1 < m_2 < 0.$

The same classification also applies to periodic points.

Images of any point in the neighborhood of a completely stable fixed point tend to the fixed point under iteration of the mapping T. In the completely unstable case images move away from the fixed point. In the directly and inversely unstable cases there exist two invariant curves which traverse the fixed point, and successive images of the mapping T approach the fixed point along one of the invariant curves, while they move away from the fixed point along the other invariant curve. The former invariant curve is called the ω-branch and the latter the α-branch.

N. Levinson [1] and J. L. Massera [4] have discussed the number of fixed points and periodic points of Eqs. (2). Let $N(n)$ be the total number of n-periodic points (fixed points for $n = 1$) and $C(n)$ the total number of completely stable and completely unstable n-periodic points. Similarly, let $D(n)$ and $I(n)$ be the number of directly and inversely unstable n-periodic points, respectively.

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*The product of the roots of Eq. (8) is given by

$$
m_1 m_2 = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = exp \int_0^L \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} \right) dt > 0.
$$

If Eqs. (2) have a maximum finite invariant set and all periodic points are simple, the following relations hold.

For $n = 1$,

$$C(1) + I(1) = D(1) + 1, \qquad N(1) = 2D(1) + 1$$

For $n = 2, 4, 6, \cdots$,

$$C(n) + I(n) = D(n) + 2I(n/2), \qquad N(n) = 2[D(n) + I(n/2)] \tag{9}$$

For $n = 3, 5, 7, \cdots$,

$$C(n) + I(n) = D(n), \qquad N(n) = 2D(n).$$

2.4 Invariant Closed Curves and Almost Periodic Solutions

A closed curve invariant under the mapping T is called an invariant closed curve. Clearly, if such curves exits they are contained in the maximum finite invariant set Δ. Now consider a solution of Eqs. (2) correlated with the invariant closed curve C. The solutions of Eqs. (2) which start from C at $t = 0$ form a surface S in the xyt space. Since C is invariant under T, the part of this surface lying between $t = 0$ and $t = L$ can be mapped onto a closed torus. Therefore the solutions of Eqs. (2) emanating from C can be investigated by the following differential equation on a torus [1]

$$\frac{d\theta}{dt} = p(\theta, t) \tag{10}$$

where p is periodic in θ with period 1 and in t with period L. This type of differential equation has been studied by H. Poincaré [5], A. Denjoy [6] and P. Bohl [7]. Here let us briefly describe their results concerning the transformation of C into itself [8].

Associated with the solution curves of the equation (10) on the torus is a rotation number ρ. This number is the average advance of θ for an advance of t by L. If ρ is rational and of the form p/q, where p and q have no common factors, then there exist some periodic groups (which consist of q-periodic points) on C. The images of any point on C approach one of these periodic groups under iteration of the mapping T. In this case the solutions of Eqs. (2) include subharmonics of order q. If ρ is irrational there are two possibilities. One of them is termed the ergodic or transitive case. This is the case in which the derived set, or closure, of the sequence $\{P_i\}$ ($P_i = T^i P_0$, for $i = 0, \pm 1, \pm 2, \cdots$, where P_0 is an arbitrary point on C) coincides with C itself. The corresponding solution of Eqs. (2) is known as an almost periodic function of time t defined by H. Bohr

[9]. The other possibility is termed the singular or intransitive case. Here the derived set of $\{P_i\}$ is nowhere dense on C.

2.5 Doubly Asymptotic Points

As previously stated there exist invariant α- and ω-branches for a directly or inversely unstable fixed or periodic point. Here let us explain the behavior of these branches studied by Poincaré [10]. Let us consider the totality of α- and ω-branches of periodic points of all orders in the xy plane. It is readily seen from the uniqueness of solutions of differential equations that no α- (or ω-) branch can intersect another α- (or ω-) branch. However, an α-branch may intersect an ω-branch, and images of a point of intersection are themselves points of intersection, which converge along the ω-branch toward the unstable point on indefinite iteration of T, and along the α-branch on iteration of T^{-1}. Based on this property, the points of intersection are called doubly asymptotic points. A doubly asymptotic point is said to be of general type if the α- and ω-branches are not coincident or merely tangent at the doubly asymptotic point; in the contrary case the point is of special type. A doubly asymptotic point is called homoclinic if the α- and ω-branches on which it lies issue from the same point or from two points belonging to the same periodic group. A homoclinic point of the former type is called simple. A doubly asymptotic point is called heteroclinic if the α- and ω-branches on which it lies issue from two fixed and/or periodic points, each of them belonging to different periodic groups.

3. ENTRAINMENT OF FREQUENCY AND BEAT OSCILLATIONS

By applying transformation theory described above, here let us consider oscillatory phenomena which occur in periodically-forced self-oscillatory systems composed of a negative-resistance element or vacuum-tube with a feedback circuit. When a periodic force is applied to a self-oscillatory system, the frequency of the self-exited oscillation, that is, the natural frequency of the system, either falls in synchronism with the driving frequency, or else it fails to synchronize, resulting in the occurrence of a beat oscillation. The former phenomenon is known as synchronization or entrainment of frequency. Regions of the external force giving rise to frequency entrainment and global phase-plane structures representing oscillatory phenomena are investigated below for some forced self-oscillatory systems taking the form of Eq. (1).

3.1 Entrainment of Frequency

As a specific example of Eq. (1) let us consider the differential equation (see Appendix I)

$$\frac{d^2x}{dt^2} - \mu(1-\gamma x^2)\frac{dx}{dt} + x^3 = B\cos\nu t, \qquad \mu > 0. \tag{11}$$

The regions of frequency entrainment for this equation will be obtained by applying the method of harmonic balance and by using an analog computer. Further, phase-plane analysis is carried out by applying the transformation theory.

(a) Regions of Frequency Entrainment

To begin with let us consider the self-excited oscillation for Eq. (11) with $B = 0$, that is,

$$\frac{d^2x}{dt^2} - \mu(1-\gamma x^2)\frac{dx}{dt} + x^3 = 0. \tag{12}$$

Let us express approximately the self-excited oscillation by the form

$$x(t) = a_0 \cos\omega_0 t. \tag{13}$$

Substituting Eq. (13) into Eq. (12) and equating the coefficients of the terms containing $\sin\omega_0 t$ and $\cos\omega_0 t$ separately to zero, the following relations are obtained for the amplitude a_0 and the frequency ω_0,

$$a_0 = \sqrt{4/\gamma}, \qquad \omega_0 = \sqrt{3/\gamma}. \tag{14}$$

When the external force is applied and its frequency ν is nearly equal to ω_0, an approximate solution of the following form

$$x(t) = b_1(t)\sin\nu t + b_2(t)\cos\nu t \tag{15}$$

is introduced for the Eq. (11), where the functions $b_1(t)$ and $b_2(t)$ are constants for the entrained states; they are slowly varying functions of t when the system is not entrained. Substituting Eq. (15) into Eq. (11) and equating the coefficients of the terms containing $\cos\nu t$ and $\sin\nu t$ separately to zero leads to

$$\begin{aligned} \frac{dx_1}{dt} &= \frac{\mu}{2}\left[(1-r_1^2)x_1 - \sigma y_1 + \frac{B}{\mu\nu a_0}\right] \\ \frac{dy_1}{dt} &= \frac{\mu}{2}\left[\sigma x_1 + (1-r_1^2)y_1\right] \end{aligned} \tag{16}$$

where

$$x_1 = \frac{b_1}{a_0}, \quad y_1 = \frac{b_2}{a_0}, \quad r_1^2 = x_1^2 + y_1^2, \quad \sigma = \frac{\omega_0^2 r_1^2 - \nu^2}{\mu\nu}. \tag{17}$$

In deriving the autonomous equations (16), the following assumptions are used:

1. The amplitudes $b_1(t)$ and $b_2(t)$ are slowly varying functions of t; therefore, d^2b_1/dt^2 and d^2b_2/dt^2 are neglected.
2. Since μ is a small quantity, $\mu db_1/dt$ and $\mu db_2/dt$ are also discarded.

When the system is entrained, the amplitudes x_1 and y_1 become constant. The constants are obtained by putting $dx_1/dt = 0$ and $dy_1/dt = 0$, and a singular point representing an equilibrium state is given by

$$x_1 = -\frac{\mu\nu a_0}{B}(1 - r_1^2)r_1^2, \qquad y_1 = \frac{\mu\nu a_0}{B}\sigma r_1^2 \tag{18}$$

where the amplitude r_1^2 is determined by the relation

$$[(1 - r_1^2)^2 + \sigma^2]r_1^2 = \left(\frac{B}{\mu\nu a_0}\right)^2. \tag{19}$$

The periodic solution thus obtained is actually sustained only when it is stable. In order to discuss the stability of the periodic solution, let us consider a small variation (ξ, η) from the singular point given by Eqs. (18). The variational equation is written as

$$\frac{d\xi}{dt} = a_1\xi + a_2\eta, \qquad \frac{d\eta}{dt} = b_1\xi + b_2\eta \tag{20}$$

where

$$\begin{aligned} a_1 &= \frac{\mu}{2}\left(1 - r_1^2 - 2x_1^2 - 2\frac{\omega_0^2}{\mu\nu}x_1y_1\right), & a_2 &= \frac{\mu}{2}\left(-2x_1y_1 - \sigma - 2\frac{\omega_0^2}{\mu\nu}y_1^2\right) \\ b_1 &= \frac{\mu}{2}\left(-2x_1y_1 + \sigma + 2\frac{\omega_0^2}{\mu\nu}x_1^2\right), & b_2 &= \frac{\mu}{2}\left(1 - r_1^2 - 2y_1^2 + 2\frac{\omega_0^2}{\mu\nu}x_1y_1\right). \end{aligned} \tag{21}$$

If the real parts of the two roots of the characteristic equation

$$\begin{vmatrix} a_1 - \lambda & a_2 \\ b_1 & b_2 - \lambda \end{vmatrix} = 0 \tag{22}$$

are negative, the corresponding equilibrium state is stable. This stability con-

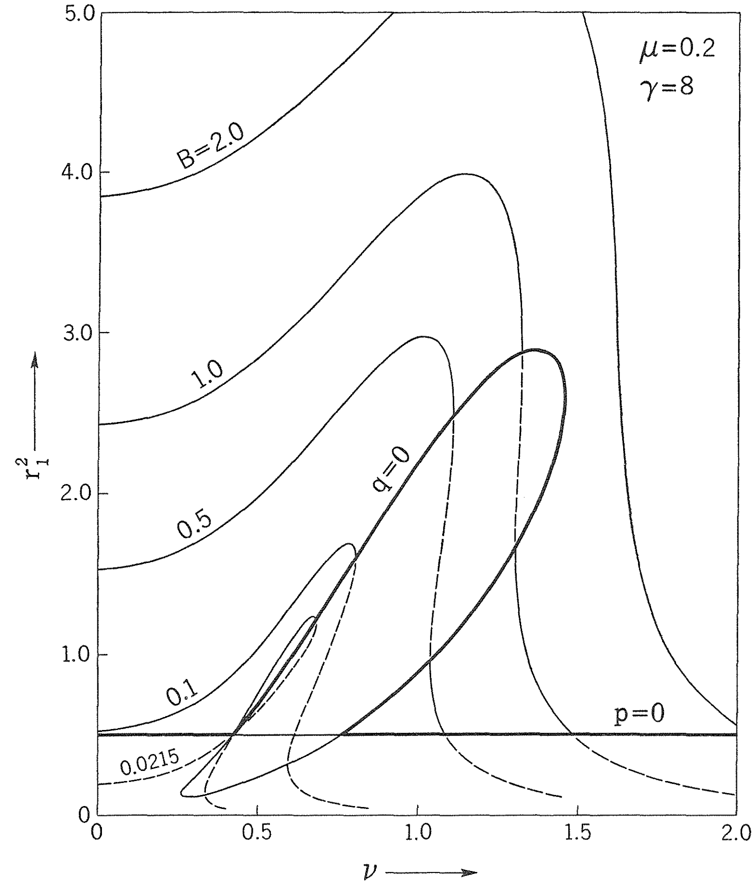

Fig. 1 Resonance curves for the approximate harmonic solution of Eq. (11).

dition is given by the Routh-Hurwitz criterion, namely,

$$
\begin{aligned}
p &= -(a_1 + b_2) = \mu(2r_1^2 - 1) > 0 \\
q &= a_1 b_2 - a_2 b_1 = \frac{\mu^2}{4}\left[(1 - r_1^2)(1 - 3r_1^2) + \sigma^2 + 2\frac{\omega_0^2}{\mu\nu}\sigma r_1^2\right] > 0.
\end{aligned}
\tag{23}
$$

Figure 1 shows the resonance curves obtained by plotting Eq. (19) in the $(\nu,\ r_1^2)$ plane for several values of the amplitude B. The system parameters in Eq. (11) are given by $\mu = 0.2$ and $\gamma = 8$. The stability limits are given by $p = 0$ and $q = 0$ of Eqs. (23) and they are also shown in the figure. Hence the solid lines of the resonance curves represent stable states, while the dashed portions indicate

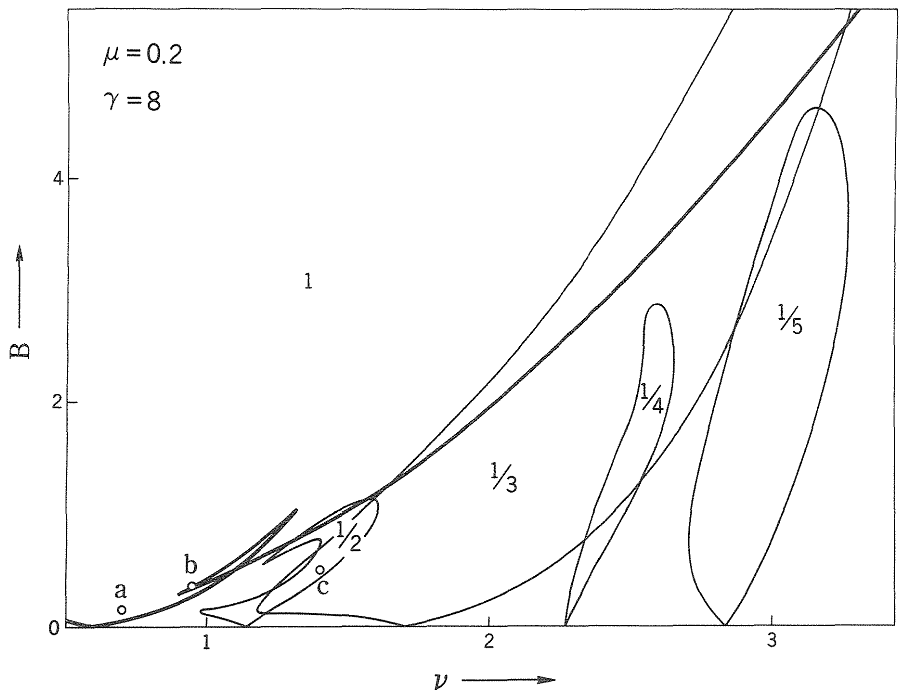

Fig. 2 Regions of frequency entrainment for Eq. (11) obtained by analog computer analysis.

unstable states. When the driving frequency ν is nearly equal to the natural frequency $\omega_0\,(= 0.61\cdots)$, the amplitude of the oscillation becomes large even for small B owing to the resonance between these frequencies. As the driving frequency ν leaves the natural frequency ω_0, the amplitude becomes small. Thus once the response curves of the entrained oscillations are obtained, the approximate region of harmonic entrainment can be obtained in the $B\nu$ plane, as the region in which Eqs. (16) has at least one stable equilibrium point.† In order to check these results, solutions of Eq. (11) are sought by using an analog computer. The region of harmonic entrainment, and some of the main regions of subharmonic entrainment, are shown in Fig. 2. In the area common to entrainment regions of different frequencies, final oscillations are determined depending on the initial conditions. If the amplitude B and the frequency ν are given outside these regions, non-periodic beat oscillations occur.

†The foregoing analysis of the entrained oscillation is based on the autonomous equations (16); phase-portraits for Eqs. (16) showing generation and extinction of the singular points caused by changes of the external force are reported in Refs. [11] and [12].

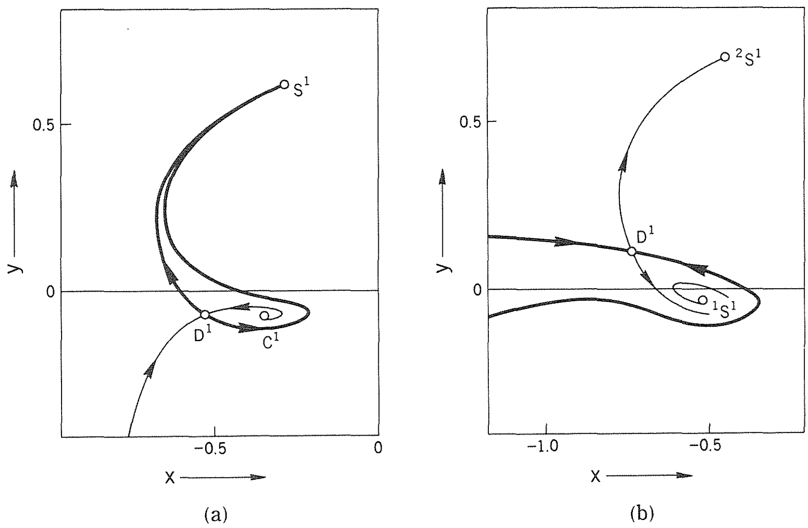

Fig. 3 Fixed points and invariant curves representing solutions of Eqs. (24), the parameters being
(a) $B = 0.1,\ \ \nu = 0.7$
(b) $B = 0.36,\ \ \nu = 0.95.$

(b) Phase-Plane Analysis by Applying Mapping Method

Equation (11) is transformed into a set of simultaneous equations

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = 0.2(1 - 8x^2)y - x^3 + B\cos\nu t. \tag{24}$$

By applying the mapping method described in Sec. 2, the topological properties of the solutions of Eqs. (24) are examined by using computers.

(i) Harmonic Entrainment   Let us consider the solutions of Eqs. (24) in which the parameters are $B = 0.1$ and $\nu = 0.7$. The location of the parameters is chosen in the region of harmonic entrainment of Fig. 2 (indicated by point a). Figure 3(a) shows the phase-plane portrait for this case. In the figure point D^1 is a directly unstable fixed point; S^1 and C^1 indicate completely stable and completely unstable fixed points, respectively. Besides these fixed points the invariant curves which abut D^1 are also shown. The arrows on the invariant curves indicate the direction of the movement of successive images under the mapping T. The two α-branches (heavy line) which start from D^1 both terminate at the point S^1; the closed region bounded by these α-branches is the maximum finite invariant set Δ. Next let us consider a case in which the external force is chosen in the region where two types of harmonic entrainments

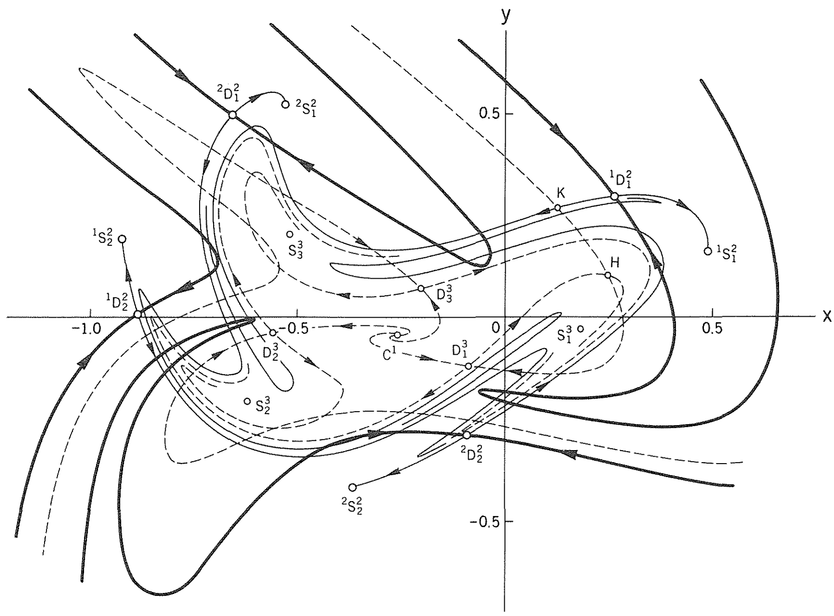

Fig. 4 Fixed point, periodic points, and invariant curves of the mapping for Eqs. (24), the parameters being $B = 0.5$ and $\nu = 1.4$.

coexist. The parameters are $B = 0.36$ and $\nu = 0.95$ (point b in Fig. 2). The phase-plane portrait is shown in Fig. 3(b). Point D^1 is a directly unstable fixed point and points ${}^1S^1$ and ${}^2S^1$ are both completely stable fixed points corresponding to the non-resonant and resonant states, respectively. The ω-branches (heavy line) are the boundaries of the two domains of attraction which contain fixed points ${}^1S^1$ and ${}^2S^1$.

(ii) 1/2-Harmonic and 1/3-Harmonic Entrainments Figure 4 shows the phase-plane portrait for the case in which the external force lies in the area common to the regions of 1/2- and 1/3-harmonic entrainments. The parameters of Eqs. (24) are $B = 0.5$ and $\nu = 1.4$ (indicated by point c in Fig. 2). In this figure the symbol ${}^iD^n_j$ denotes a directly unstable n-periodic point which belongs to the i-th periodic group and the subscript j shows the order of movement of images under the mapping T, that is,

$$T({}^iD^n_j) = {}^iD^n_{j+1}, \qquad T^n({}^iD^n_j) = {}^iD^n_j. \tag{25}$$

Similar notations are used for completely stable periodic points (S), completely unstable periodic points (C), and inversely unstable periodic points (I). Invariant curves of the mapping T^2 are shown by solid lines and those of the mapping T^3 by dashed lines. It is seen in the figure that the α- and ω-branches of the directly unstable periodic points intersect one another and that there exist some doubly asymptotic points. For example, since the point H is an intersection of the α-branch and ω-branch both emanating from D_1^3, this point is a simple homoclinic point. The images of the point H converge toward D_1^3 on indefinite iteration of both T^3 and T^{-3}. Also since K is an intersection of the α-branch emanating from ${}^1D_1^2$ and the ω-branch converging to D_1^3, this point is a heteroclinic point. The images of the point K converge toward D_1^3 on indefinite iteration of T^3 and toward ${}^1D_1^2$ under T^{-2}. The points H and K are examples of doubly asymptotic points. It can be seen from the figure that there must exist an infinite number of doubly asymptotic points.[‡] In this case the maximum finite invariant set and the domains of attraction for different completely stable periodic groups exhibit extremely complicated configurations and they cannot be drawn in the straightforward manner of Fig. 3.

3.2 Analysis of Beat Oscillations

Let us consider beat oscillations which occur in a system described by the differential equation

$$\frac{d^2x}{dt^2} - \mu\left[1 - \gamma\left(\frac{dx}{dt}\right)^2\right]\frac{dx}{dt} + x^3 = B\cos\nu t, \quad \mu > 0. \tag{26}$$

Following the same procedure as before, the resonance curves are drawn for approximate solutions which consist only of the fundamental component. The result is shown in Fig. 5. The system parameters are given by $\mu = 0.2$ and $\gamma = 4$ in Eq. (26). In the figure the amplitude r_1^2 shows a normalized quantity in the same way as in Fig. 1 (see Appendix II). From this result we can obtain the approximate region of harmonic entrainment. Then the solutions of Eq. (26) are examined by using an analog computer. In the course of this study, when the external force is given just outside the region of harmonic entrainment, different types of beat oscillations are observed depending on the magnitude of the amplitude of the external force. Let us examine the aspect of these solutions in the phase plane by applying the transformation theory.

‡The following theorems have been proved by G. D. Birkhoff [13-15]:

(a) An arbitrary neighborhood of a homoclinic point contains an infinite number of homoclinic points.

(b) An arbitrary neighborhood of a homoclinic point contains an infinite number of periodic points.

Some examples of periodic points are shown in Refs. [16-18].

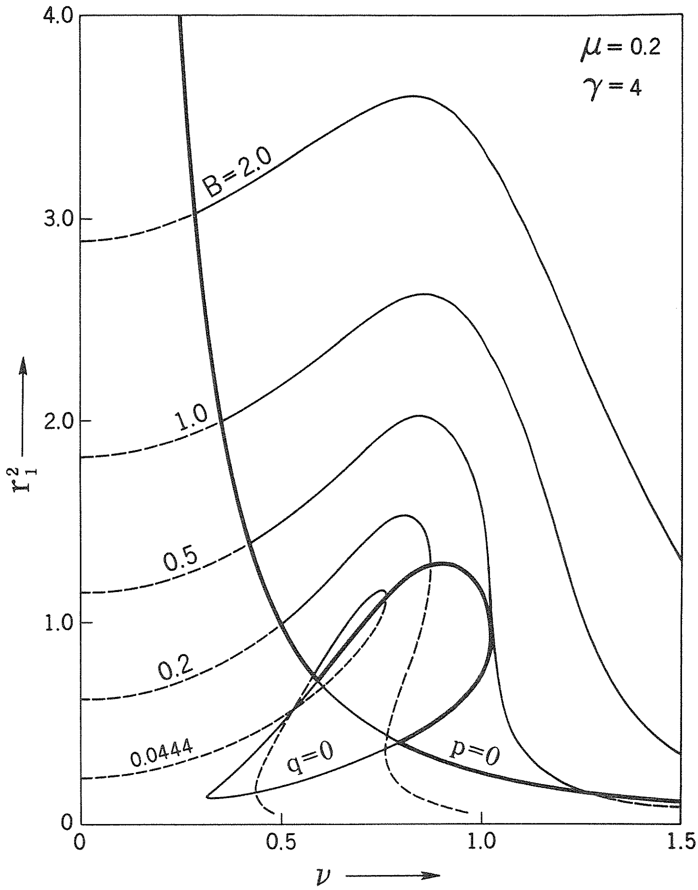

Fig. 5 Resonance curves for the approximate harmonic solution of Eq. (26), the system parameters being $\mu = 0.2$ and $\gamma = 4$.

First let us consider the case in which the amplitude B of the external force is comparatively small. Figure 6(a) shows the successive movement of images obtained by using an analog computer. The system under consideration is described by

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = 0.2(1 - 4y^2)y - x^3 + B\cos\nu t \tag{27}$$

with $B = 0.1$ and $\nu = 1.1$. In this figure the numbers attached to the points indicate the order of successive images under the mapping T, and these are counted after the transient has decayed. It is seen in the figure that these images form a smooth invariant closed curve, which is the boundary of the

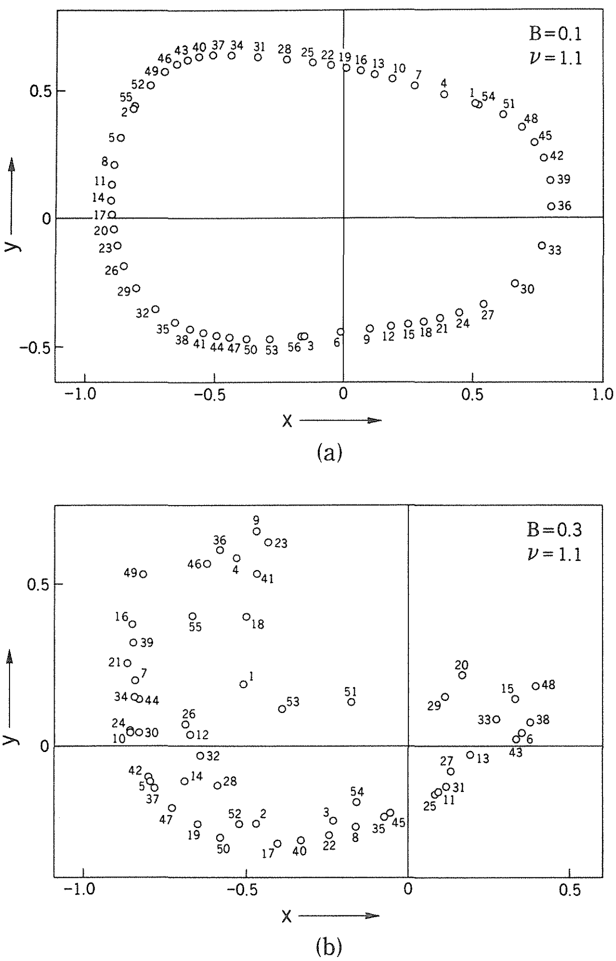

Fig. 6 Point sequences representing beat oscillations for Eqs. (27), the parameters being (a) $B = 0.1,\ \ \nu = 1.1$ and (b) $B = 0.3,\ \ \nu = 1.1$.

maximum finite invariant set in this case. The rotation number ρ associated with this curve is approximately equal to 0.66. It cannot be concluded by computer experiment whether it is a rational or an irrational number. If we assume that ρ is an irrational number, then the mapping is either ergodic or singular. However, it is known that the singular case cannot occur when an invariant closed curve is sufficiently smooth, as is the case in the figure [8, 19]. The mapping is thus ergodic and the solution of Eqs. (27) is almost periodic.

Next let us consider the case in which the amplitude B is increased. Figure 6(b) shows the behavior of successive images in the system (27) with $B =$

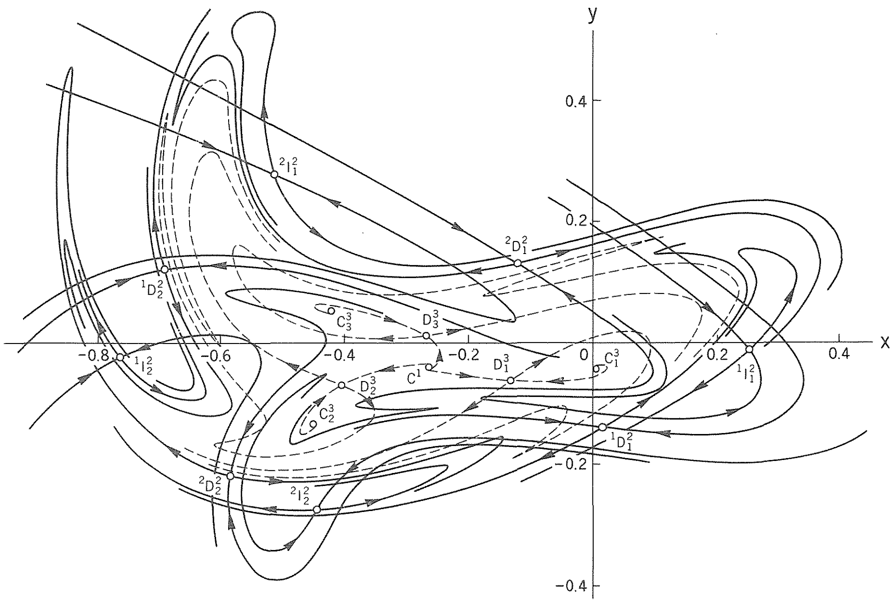

Fig. 7 Fixed point, periodic points and invariant curves of the mapping for Eqs. (27), the parameters being $B = 0.3$ and $\nu = 1.1$.

0.3 and $\nu = 1.1$. The numbers attached to the images are counted after the transient has decayed in the same way as in Fig. 6(a). The movement of the images is irregular and complicated as is seen in the figure. In this case we cannot infer the existence of a simple invariant closed curve as we did in Fig. 6(a). Figure 7 shows the phase-plane portrait of this case. The same notation is applied to the fixed and periodic points, and solid and dashed lines are used as in Fig. 4. Again the existence of homoclinic and heteroclinic points is observed.

Let us consider the number of fixed and periodic points, using the same notation as in Sec. 2.3. Since $C(1) = 1, D(2) = I(2) = 4$ and $D(3) = C(3) = 3$, the relations (9) for $n = 1, 2$ and 3 are satisfied. For $n = 4$, $C(4) + I(4) = D(4) + 2I(2) \neq 0$. Computer analysis revealed that there are no completely stable or unstable 4-periodic points. That is, $C(4) = 0$, and therefore $I(4) \neq 0$ must hold. Similarly, $I(8) \neq 0, \cdots$ hold and the existence of 2^n-periodic points (n: an integer) is inferred.

Let us draw a simple closed curve Γ_0 in the xy plane as shown in Fig. 8 which possesses the property mentioned in Sec. 2.2. Then successive mapping is applied to Γ_0. For simplicity, only the curve Γ_3 $(= T^3\Gamma_0)$ is shown in the figure. From this result it can be seen that images Γ_i become more and more complicated as i increases, and the configuration of Γ_i for $i \to \infty$ cannot even

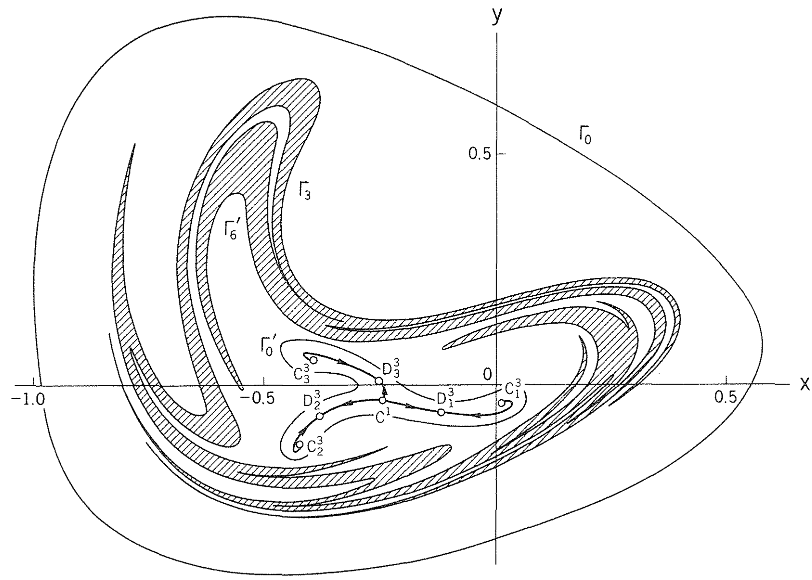

Fig. 8 Ring domain within which the images under the mapping representing steady state irregular beat oscillations are confined.

be imagined. Next let us draw the closed curve Γ_0' which encloses the points C^1, C_i^3, D_i^3 (i = 1, 2, 3). Let us consider the closed domain S_0 bounded by Γ_0' and apply successive mapping on S_0 to obtain the closed set $\bigcup_{i=0}^{6} T^i S_0$. However, the configuration of this boundary curve is much too intricate, and we are compelled to draw the closed curve Γ_6' inside this closed domain in somewhat simplified form. It is seen from this construction that successive images P_i, which represent steady oscillation, must remain in the shaded ring domain bordered by the curves Γ_3 and Γ_6'. The full implications of this point sequence $\{P_i\}$ have not yet been clarified.

4. CONCLUSION

Nonlinear oscillatory phenomena have been studied which occur in periodically-forced self-oscillatory systems containing a negative-resistance element or vacuum-tube with feedback circuit.

First, the resonance curve of the harmonic entrainment was discussed by the method of harmonic balance; then the regions of entrainment were obtained at fundamental and some principal subharmonic frequencies by using an analog computer. From these results, shapes and mutual relations of the regions at various frequencies are clarified.

Second, in order to understand the oscillatory phenomena more clearly, some representative phase-plane portraits were obtained by applying the transformation theory based on the qualitative theory of differential equations. For certain values of the external force, the existence of doubly asymptotic points was observed in the phase plane. In this case, the maximum finite invariant set and the domains of attractions for completely stable fixed or periodic points exhibit complicated configurations. Further, for the external force prescribed outside the regions of entrainment, different types of beat oscillations were observed. One of them is a almost-periodic oscillation and the corresponding successive images form a smooth invariant closed curve on the phase plane as shown in Fig. 6(a). The other type of beat oscillations occurs depending upon the external force. The genesis of the oscillations is not fully grasped. In the neighborhood of the point sequence in the phase plane representing steady states of this type of oscillations there exist unstable 2^n-periodic points as well as doubly asymptotic points as shown in Fig. 7. Further, since all points of this sequence are found inside the ring domain of Fig. 8 and there are no stable periodic points, the resulting oscillation behaves non-periodically. The nature of this type of oscillation deserves attention as material for further study from the point of view of topological dynamics [20].

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APPENDICES

I. Derivation of Eq. (11)

The schematic diagram in Fig. A(a) shows a forced self-oscillatory circuit which contains a negative-resistance element N. With the notation of the figure,

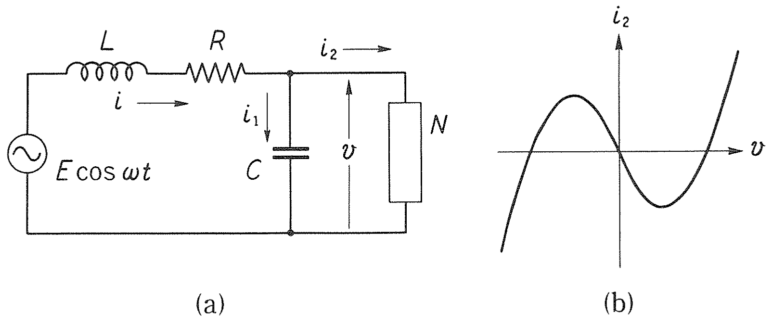

Fig. A (a) Negative-resistance oscillator with external force.
(b) Nonlinear characteristic of the negative-resistance element N.

the equations for the circuit are written as

$$L\frac{di}{dt} + Ri + v = E\cos\omega t, \quad i = i_1 + i_2, \quad i_1 = C\frac{dv}{dt}. \tag{A.1}$$

The nonlinear characteristic of the negative resistance N is shown in Fig. A(b) and is expressed by

$$i_2 = f(v). \tag{A.2}$$

Retaining v and eliminating the other variables, we obtain

$$LC\frac{d^2v}{dt^2} + \left(RC + L\frac{df}{dv}\right)\frac{dv}{dt} + v + Rf(v) = E\cos\omega t. \tag{A.3}$$

For simplicity, let the voltage-current characteristic be expressed by

$$i_2 = f(v) = Sv\left(-1 + \frac{v^2}{V_s^2}\right) \tag{A.4}$$

with the relation $S = 1/R$; then Eq. (A.3) is transformed into

$$\frac{d^2x}{d\tau^2} - \mu(1 - \gamma x^2)\frac{dx}{d\tau} + x^3 = B\cos\nu\tau \tag{A.5}$$

where

$$\begin{gathered} x = \frac{v}{V_s}, \quad \tau = \frac{t}{\sqrt{LC}}, \quad B = \frac{E}{V_s}, \quad \nu = \omega\sqrt{LC} \\ \mu = \frac{LS - RC}{\sqrt{LC}}, \qquad \gamma = \frac{3LS}{LS - RC}. \end{gathered} \tag{A.6}$$

II. Approximate Harmonic Solution of Eq. (26) and Its Stability

Here let us give the approximate harmonic solution and its stability condition for the equation (26), i.e.,

$$\frac{d^2x}{dt^2} - \mu\left[1 - \gamma\left(\frac{dx}{dt}\right)^2\right]\frac{dx}{dt} + x^3 = B\cos\nu t, \quad \mu > 0. \tag{A.7}$$

Let the approximate solution of Eq. (A.7) with $B = 0$ be

$$x(t) = a_0 \cos\omega_0 t, \tag{A.8}$$

then the amplitude a_0 and the frequency ω_0 are given by

$$a_0 = \sqrt{\frac{4}{3\sqrt{\gamma}}}, \qquad \omega_0 = \sqrt{\frac{1}{\sqrt{\gamma}}}. \tag{A.9}$$

When $B \neq 0$, let us assume an approximate harmonic solution of the form

$$x(t) = b_1(t)\sin\nu t + b_2(t)\cos\nu t. \tag{A.10}$$

By a procedure similar to that in Sec. 3.1, a set of autonomous equations is derived, that is,

$$\begin{aligned}\frac{dx_1}{dt} &= \frac{\mu}{2}\left[(1 - \sqrt{\gamma}\nu^2 r_1^2)x_1 - \sigma y_1 + \frac{B}{\mu\nu a_0}\right]\\ \frac{dy_1}{dt} &= \frac{\mu}{2}\left[\sigma x_1 + (1 - \sqrt{\gamma}\nu^2 r_1^2)y_1\right]\end{aligned} \tag{A.11}$$

where

$$x_1 = \frac{b_1}{a_0}, \quad y_1 = \frac{b_2}{a_0}, \quad r_1^2 = x_1^2 + y_1^2, \quad \sigma = \frac{\omega_0^2 r_1^2 - \nu^2}{\mu\nu}. \tag{A.12}$$

Any singular point (for which x_1 and y_1 are constants) of Eqs. (A.11) is given by

$$x_1 = -\frac{\mu\nu a_0}{B}(1 - \sqrt{\gamma}\nu^2 r_1^2)r_1^2, \qquad y_1 = \frac{\mu\nu a_0}{B}\sigma r_1^2 \tag{A.13}$$

where r_1^2 is determined by solving the equation

$$[(1 - \sqrt{\gamma}\nu^2 r_1^2)^2 + \sigma^2]r_1^2 = \left(\frac{B}{\mu\nu a_0}\right)^2. \tag{A.14}$$

The stability condition of the singular point is given by

$$
\begin{aligned}
p &= \mu(2\sqrt{\gamma}\nu^2 r_1^2 - 1) > 0 \\
q &= \frac{\mu^2}{4}\left[(1-\sqrt{\gamma}\nu^2 r_1^2)(1-3\sqrt{\gamma}\nu^2 r_1^2) + \sigma^2 + 2\frac{\omega_0^2}{\mu\nu}\sigma r_1^2\right] > 0.
\end{aligned}
\tag{A.15}
$$

Figure 5 in Sec. 3.2 was plotted by making use of Eqs. (A.14) and (A.15).