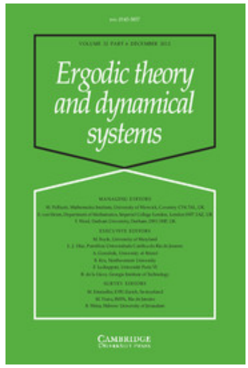

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Measure-theoretic chaos

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Abstract. We define new isomorphism invariants for ergodic measure-preserving systems on standard probability spaces, called measure-theoretic chaos and measure-theoretic$^+$ chaos. These notions are analogs of the topological chaos DC2 and its slightly stronger version (which we denote by DC$1\frac{1}{2}$). We prove that: (1) if a topological system is measure-theoretically (measure-theoretically$^+$) chaotic with respect to at least one of its ergodic measures then it is topologically DC2 (DC$1\frac{1}{2}$) chaotic; (2) every ergodic system with positive Kolmogorov–Sinai entropy is measure-theoretically$^+$ chaotic (even in a slightly stronger uniform sense). We provide an example showing that the latter statement cannot be reversed, that is, of a system of entropy zero with uniform measure-theoretic$^+$ chaos.

1. *Introduction*

The notion of chaos was invented by Li and Yorke in their seminal paper [**LY**] in the context of continuous transformations of the interval. Since then several refinements of chaos have been introduced and extensively studied, for instance three versions of so-called *distributional chaos* (DC1, DC2 and DC3) invented by Smítal *et al* [**BSS, SS, SSt**]. All these notions refer to topological dynamical systems (actions of the iterates of a single continuous transformation T on a compact metric space X) and strongly rely on the observation of distances between orbits, and the existence of so-called *scrambled pairs* (or *scrambled sets*—usually uncountable). There are other notions of chaos, such as Devaney chaos or omega chaos, defined without the notion of scrambling—these are not addressed in our paper.

Unlike in the case of most other notions in dynamics, there have been, to our knowledge, no successful attempts to create a measure-theoretic analog of chaos—a notion applicable to measure-preserving transformations of a standard probability space without any specified topology. Although 'measure-theoretic chaos' appears in the titles of some papers (e.g. [**WW**]), it still applies to topological systems. There are two major reasons why, at a first glance, it seems difficult to create such an analog.

- A standard probability space can be modeled as a compact metric space in many different ways. A pair (or set) scrambled in one metric need not be scrambled in another.
- A scrambled set in a topological dynamical system very often has measure zero for every invariant measure. It is always so, for example, in the case of distributional chaos (in any version)—we will explain this later. Li–Yorke scrambled sets can have positive measure or even be equal to the entire space, but systems with such large scrambled sets are rather exceptional, and any notion of measure-theoretic chaos based on the analogy to these systems would be very restrictive (see [**WW**]). In all other cases, a scrambled set can be easily added to the space (or discarded from it) in a manner that is negligible from the point of view of measure. In other words, chaos based on the existence of a scrambled set is not stable under measure-theoretic isomorphisms.

Inspired by the methods developed in [**D**], in this note we propose a way to overcome these difficulties. We define chaos in measure-theoretic systems using exclusively the measurable structure of the space, and so that it becomes an invariant of measure-theoretic isomorphism. Our new notions maintain their original character—they are defined in terms of uncountable scrambled sets. Moreover, they are related to their topological prototypes and also to positive entropy exactly as one would expect (we will give more details in a moment).

Among the topological notions of chaos we have chosen one—the distributional chaos DC2 (with variants)—as the starting point to define its measure-theoretic analog. This new notion, which we call simply the *measure-theoretic chaos*†, meets all our expectations regarding its relations with the topological prototype, and it inherits the most important implications between chaos and entropy.

Other notions of chaos are not so well adaptable to the measure-theoretic context; the attempted analogs of Li–Yorke and DC1 chaoses fail a key property allowing one to prove that they imply their topological prototypes (see Remark 1 for DC1). It is possible to copy our scheme for DC3 (see Remark 3), but because generally this notion is very weak (it can occur even in distal systems), we have decided to omit it. Nonetheless, for completeness of the survey in the next section, we include the definitions of Li–Yorke, DC1 and DC3 chaoses in topological systems.

Let us recall that Blanchard *et al* have proved that positive topological entropy implies Li–Yorke chaos (see [**BGKM**]). This result was recently strengthened by the first author of this note: positive topological entropy implies distributional chaos DC2 (see [**D**]). Let us also recall that for interval maps all three versions of distributional chaos (DC1, DC2 and DC3) are equivalent to positive topological entropy (see [**SS**]). We can now be more specific about our new notion maintaining these implications. We will prove that:

- a topological dynamical system which is measure-theoretically chaotic with respect to at least one of its invariant measures is DC2 chaotic;

† We have decided to suppress the adjective 'distributional' because 'distribution' is a synonym of 'measure'. For topological chaos, this adjective indicates 'some reference to measures' (maintaining reference to the metric), while here we have reference to a measure and nothing else, so the adjective 'measure-theoretic' should suffice.

- a measure-theoretic system with positive Kolmogorov–Sinai entropy is measure-theoretically chaotic (in particular, a topological system with positive topological entropy is measure-theoretically chaotic for at least one of its invariant measures—and thus DC2 chaotic);
- for a continuous transformation of the interval, positive topological entropy is equivalent to measure-theoretic chaos for some of its invariant measures.

The last statement is a direct consequence of the preceding two, the variational principle and the equivalence between DC2 and positive topological entropy for interval maps, thus we do not need prove it separately. We believe that the above assembly of relations (plus the fact that our notion is an isomorphism invariant) is a good enough reason to consider our notion a successful analog of distributional chaos in measure-theoretic dynamics.

2. *Review of topological chaos*

Let us begin with a review of topological notions of chaos: Li–Yorke, DC1, DC2 and DC3. Later we will also introduce a notion intermediate between DC1 and DC2, which we denote DC$1\frac{1}{2}$. All these notions are defined in the same manner: *there exists an uncountable scrambled set*, where a scrambled set is one whose every pair of distinct elements is scrambled. The only remaining detail is the meaning of a 'scrambled pair' for the above types of chaos. The definitions given below are equivalent to those most commonly appearing in the literature but expressed using a slightly different language (this change is meant for an easy adaptation to the measure-theoretic situation). We will also define uniform versions of DC1 and DC2 (and later of DC$1\frac{1}{2}$).

Traditionally, a pair (x, y) is Li–Yorke scrambled if

$$\liminf_{n\to\infty} d(T^n x, T^n y) = 0 \quad \text{and} \quad \limsup_{n\to\infty} d(T^n x, T^n y) > 0.$$

This can be rephrased as follows.

- A pair (x, y) is *Li–Yorke scrambled* if there exist an increasing sequence n_i such that $d(T^{n_i} x, T^{n_i} y) \xrightarrow{i} 0$, another increasing sequence m_i and a positive number $s > 0$, such that $d(T^{m_i} x, T^{m_i} y) \geq s$ for all i.

Distributional DC1 and DC2 scrambling are similar, except that we put density constraints on the sequences n_i and m_i.

- A pair (x, y) is DC1-*scrambled* if there exist an increasing sequence n_i of upper density 1 such that $d(T^{n_i} x, T^{n_i} y) \xrightarrow{i} 0$, another increasing sequence m_i of upper density 1 and a number $s > 0$ such that $d(T^{m_i} x, T^{m_i} y) \geq s$ for all i.
- A pair (x, y) is DC2-*scrambled* if there exist an increasing sequence n_i of upper density 1 such that $d(T^{n_i} x, T^{n_i} y) \xrightarrow{i} 0$, another increasing sequence m_i of *positive* upper density and a number $s > 0$ such that $d(T^{m_i} x, T^{m_i} y) \geq s$ for all i.

The resulting chaos DC1 is called *uniform* if the constant s can be chosen common to all pairs in the scrambled set. For uniformity of DC2 we will require that all pairs in the scrambled set have in common both the parameter s and a positive lower bound η for the upper density of the sequences m_i.

Scrambling for DC3 has a slightly different structure.

- A pair (x, y) is DC3-*scrambled* if there exists an interval (a, b) such that for every $s \in (a, b)$ the sequence of the times n when $d(T^n x, T^n y) \geq s$ does not have a density (upper and lower densities differ)†.

Requiring the upper density of m_i to be arbitrarily close to 1, we produce a notion intermediate between DC2 and DC1.

- A pair (x, y) is DC$1\frac{1}{2}$-*scrambled* if there exist an increasing sequence n_i of upper density 1 such that $d(T^{n_i}x, T^{n_i}y) \xrightarrow{i} 0$, and, for every $\eta < 1$, an increasing sequence $m_{\eta,i}$ of upper density at least η and a number $s_\eta > 0$ such that $d(T^{m_{\eta,i}}x, T^{m_{\eta,i}}y) \geq s_\eta$ for all i‡.

The meaning of *chaos* DC$1\frac{1}{2}$ is clear: there exists an uncountable DC$1\frac{1}{2}$-scrambled set. *Uniform chaos* DC$1\frac{1}{2}$ occurs when the function $\eta \mapsto s_\eta$ is common to all pairs in the scrambled set§. It is easy to see that

$$\text{DC1} \implies \text{DC}1\tfrac{1}{2} \implies \text{DC2} \implies \text{DC3},$$
$$\text{DC2} \implies \text{Li–Yorke chaos},$$

and that DC1 through DC2 (including the uniform versions) are invariants of topological conjugacy (see [**SSt**]). DC3 does not imply Li–Yorke and, as we mentioned earlier, is not a topological invariant. There are easy examples showing that DC1 is essentially stronger than DC$1\frac{1}{2}$; in fact (as we will show later) every system with positive topological entropy is uniformly DC$1\frac{1}{2}$, while Pikuła provided an example of a system with positive topological entropy which is not DC1 [**P**]. It is not very hard to construct an example showing that DC2 (even uniform) is essentially weaker than DC$1\frac{1}{2}$. We refrain from providing such an example in this note. A more interesting question is whether DC2 (or uniform DC2) *persistent under removing null sets* (see the formulation of Theorem 5) implies DC$1\frac{1}{2}$. At the moment we leave this problem open, with a conjecture that the answer is negative.

Notice that the condition for a DC2-scrambled pair has a beautiful translation to the language of ergodic averages. A pair (x, y) is DC2-scrambled if and only if

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} d(T^i x, T^i y) = 0 \quad \text{and} \quad \limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} d(T^i x, T^i y) > 0.$$

Note how this formulation is analogous to the original condition for Li–Yorke scrambling. It is not hard to see that uniformity of DC2 is equivalent to the upper limit seen above on

† Traditionally, distributional scrambling is defined using the functions $\Phi^*_{x,y}(t)$ and $\Phi_{x,y}(t)$ defined for $t > 0$ as, respectively, the upper and lower densities of the set of times n when $d(T^n x, T^n y) < t$. Clearly, $\Phi^*_{x,y} \geq \Phi_{x,y}$, and both functions increase with t, reaching the value 1 for the diameter of X. One can define both functions at zero as the limit values as $t \to 0^+$. A pair (x, y) is:

– DC1-scrambled if $\Phi^*_{x,y}(0) = 1$ and $\Phi_{x,y}(s) = 0$ for some s;
– DC2-scrambled if $\Phi^*_{x,y}(0) = 1$, $\Phi_{x,y}(0) < 1$;
– DC3-scrambled if $\Phi^*_{x,y}(s) > \Phi_{x,y}(s)$ on an open interval.

‡ In other words, a pair (x, y) is DC$1\frac{1}{2}$-scrambled if $\Phi^*_{x,y}(0) = 1$, $\Phi_{x,y}(0) = 0$.

§ In [**D**] it is proved that positive topological entropy implies uniform chaos DC2. In this paper we will strengthen that result: positive topological entropy implies uniform chaos DC$1\frac{1}{2}$. This is why we think uniform DC$1\frac{1}{2}$ is worth a separate formulation. A similar result is obtained in this paper for the measure-theoretic analog.

the right having a common positive lower bound for all pairs in the scrambled set. The fact that DC2 can be phrased in terms of ergodic averages makes it the best and most natural candidate to become a base for creating a measure-theoretic analog.

3. *Definitions of measure-theoretic chaos*

At this point we leave the topological setup of a compact metric space and we move on to the context of a standard probability space $(X, \mathfrak{B}, \mu)$, where $\mathfrak{B}$ is a complete sigma-algebra and μ is a probability measure on $\mathfrak{B}$, on which we consider the action of a measure-preserving transformation T. As already stated, in order to define measure-theoretic chaos we must overcome two difficulties, the first of which is that the definition of scrambling must not refer to any metric. This is done using refining sequences of finite partitions.

Definition 1. A sequence of finite measurable partitions $(\mathscr{P}_k)_{k\geq 1}$ is called *refining* if $\mathscr{P}_{k+1} \succcurlyeq \mathscr{P}_k$ for every k and jointly they generate $\mathfrak{B}$ (i.e., $\mathfrak{B}$ is the smallest complete sigma-algebra containing all the partitions $\mathscr{P}_k$).

Definition 2. Fix a refining sequence of finite measurable partitions $(\mathscr{P}_k)$. A pair of points (x, y) is $(\mathscr{P}_k)$-*scrambled* if:

- there exists a sequence n_i of upper density 1 such that, for every k and large enough i, $T^{n_i}x$ belongs to the same atom of $\mathscr{P}_k$ as $T^{n_i}y$;
- there exist a sequence m_i, of positive upper density, and k_0 such that, for every i, $T^{m_i}x$ and $T^{m_i}y$ belong to different atoms of $\mathscr{P}_{k_0}$.

The second major difficulty is to ensure that our chaos is an isomorphism invariant. This is achieved by requiring the existence of a scrambled set for *every* refining sequence of finite partitions, as is done in the definitions given below. We will show in the next section that the notions of chaos so constructed are indeed isomorphism invariants.

Definition 3. A measure-preserving transformation T of a standard probability space $(X, \mathfrak{B}, \mu)$ is *measure-theoretically chaotic* if for every refining sequence of finite partitions $(\mathscr{P}_k)$ there exists an uncountable $(\mathscr{P}_k)$-scrambled set.

Definition 4. The above defined chaos is *uniform* if (for any refining sequence $(\mathscr{P}_k)$) all distinct pairs in the scrambled set are $(\mathscr{P}_k)$-scrambled with a common parameter k_0 and with a common positive lower bound η on the upper density of the sequences m_i.

It is also easy to define a stronger version of measure-theoretic chaos, an analog of DC$1\frac{1}{2}$. Precisely this type of chaos is implied by positive entropy, so it is worthy of presentation. It suffices to modify the definition of scrambled pairs.

Definition 5. A pair (x, y) is $(\mathscr{P}_k)^+$-scrambled if:

- there exists a sequence n_i of upper density 1 such that, for every k and large enough i, $T^{n_i}x$ belongs to the same atom of $\mathscr{P}_k$ as $T^{n_i}y$;
- for every $\eta > 0$ there exist a sequence $m_{\eta,i}$, of upper density at least η and k_η such that, for every i, $T^{m_{\eta,i}}x$ and $T^{m_{\eta,i}}y$ belong to different atoms of $\mathscr{P}_{k_\eta}$.

Replacing $(\mathscr{P}_k)$-scrambling in the definition of the measure-theoretic chaos by $(\mathscr{P}_k)^+$-scrambling, we obtain *measure-theoretic*$^+$ *chaos*. For *uniform measure-theoretic*$^+$ *chaos* we require that the function $\eta \mapsto k_\eta$ is common to all pairs in the scrambled set.

4. *Presentation of the measure-theoretic chaos*

In this section we formulate and prove our statements concerning the notion of measure-theoretic chaos, in particular its isomorphism invariance and its relations with the topological counterpart.

THEOREM 1. *Suppose that the systems* $(X, \mathfrak{B}, \mu, T)$ *and* $(Y, \mathfrak{C}, \nu, S)$ *are isomorphic. Then* $(X, \mathfrak{B}, \mu, T)$ *is measure-theoretically (uniformly measure-theoretically, measure-theoretically*$^+$*, uniformly measure-theoretically*$^+$*) chaotic if and only if* $(Y, \mathfrak{C}, \nu, S)$ *is likewise.*

Proof. Let $\pi : X \to Y$ be the isomorphism. Recall that π is a measurable bijection between full sets $X' \subset X$ and $Y' \subset Y$ (i.e., sets of full measure in the respective spaces), intertwining the actions of T and S. By a standard argument we can arrange that X' and Y' are invariant, that is, $T(X') \subset X'$ and $S(Y') \subset Y'$. Let $(\mathscr{Q}_k)$ be an arbitrarily chosen refining sequence of partitions of Y. Denote by $\mathscr{Q}'_k$ the restriction of $\mathscr{Q}_k$ to Y' and let $\mathscr{P}'_k$ be the partition of X' obtained as the preimage by π of $\mathscr{Q}'_k$. Finally, let $\mathscr{P}_k$ denote the partition of X consisting of the elements of $\mathscr{P}'_k$ and the null set $C = X \setminus X'$. It is obvious that $(\mathscr{P}_k)$ is a refining sequence of partitions in X. If the system on X is chaotic (in any of the four considered senses) then there exists an uncountable $(\mathscr{P}_k)$-scrambled set E (for the corresponding meaning of scrambling). By $(\mathscr{P}_k)$-scrambling and invariance of X', for every $x \in E$ there exists n_x such that $T^{n_x} x \in X'$. Since E is uncountable, it has an uncountable subset E' with a common n_x. The set $E'' = T^{n_x}(E')$ is uncountable, $(\mathscr{P}'_k)$-scrambled and contained in X'. Now the set $\pi(E'')$ is obviously $(\mathscr{Q}'_k)$-scrambled in Y', which immediately implies that it is $(\mathscr{Q}_k)$-scrambled in Y. This concludes the proof. □

Although our notions of chaos formally apply to all measure-theoretic systems, we will focus on the most important, ergodic, case. Most of the theorems stated below require ergodicity anyway. Thus, throughout the remainder of the paper we will assume that μ is ergodic.

In this context we will provide conditions equivalent to measure-theoretic chaos (and its variants) referring to only one refining sequence of partitions and 'persistence under removing null sets'.

THEOREM 2. *Let* $(\mathscr{P}_k)$ *be a fixed refining sequence of finite partitions of* X. *The ergodic system* $(X, \mathfrak{B}, \mu, T)$ *is measure-theoretically (measure-theoretically*$^+$*, uniformly measure-theoretically, uniformly measure-theoretically*$^+$*) chaotic if and only if, for any null set* A *(i.e., of measure zero), there exists an uncountable* $(\mathscr{P}_k)$*-scrambled* $((\mathscr{P}_k)^+$*-scrambled, uniformly* $(\mathscr{P}_k)$*-scrambled, uniformly* $(\mathscr{P}_k)^+$*-scrambled) set disjoint from* A.

Proof. One implication is trivial, since removing a null set is in fact an isomorphism. We will focus on the non-trivial implication. Suppose that, no matter what null set is removed from X, there remains a $(\mathscr{P}_k)$-scrambled set. Consider another refining sequence of finite partitions $(\mathscr{P}'_{k'})$. Let A be the null set such that all remaining points satisfy the assertion of the ergodic theorem with regard to all (countably many) elements of the field (note: not sigma-field, just field) $\mathscr{F}$ generated by the partitions $(\mathscr{P}_k)$ and $(\mathscr{P}'_{k'})$ (that is to say, the orbit of every remaining point visits every set $B \in \mathscr{F}$ along a set of times whose density equals $\mu(B)$). By assumption, there exists an uncountable $(\mathscr{P}_k)$-scrambled set disjoint from A. We will show that the same set is $(\mathscr{P}'_{k'})$-scrambled. Take a pair of distinct points (x, y) from this set and fix some k'. For an arbitrarily small $\delta > 0$ there exist k and a set B of measure at most δ such that, relatively on $X \setminus B$, the partition $\mathscr{P}'_{k'}$ is refined by $\mathscr{P}_k$. Clearly, B belongs to $\mathscr{F}$. We know that the sequence of times n when $T^n x$ and $T^n y$ belong to the same element of $\mathscr{P}_k$ has upper density 1. Removing the sequence of times when at least one of the above two points falls into B, we obtain a sequence of times of upper density at least $1 - 2\delta$, when the two points fall in the same element of $\mathscr{P}_k$ within $X \setminus B$ (and hence they fall into the same element of $\mathscr{P}'_{k'}$). Because δ was arbitrarily small, we get that $T^n x$ and $T^n y$ fall into the same element of $\mathscr{P}'_{k'}$ for times n with upper density 1. Further, we know that there exists an index k_0 such that the sequence of times n for which $T^n x$ and $T^n y$ belong to different elements of $\mathscr{P}_{k_0}$ has positive upper density, say η. Fix some $\delta > 0$ much smaller than η and find k'_0 and a set $C \in \mathscr{F}$ of measure at most δ such that $\mathscr{P}'_{k'_0}$ refines $\mathscr{P}_{k_0}$ relatively on $X \setminus C$. Removing the sequence of times when at least one of the above points falls into C, we obtain a sequence of upper density at least $\eta - 2\delta$ (which is positive) when the two points fall into different elements of $\mathscr{P}_{k_0}$ within $X \setminus C$ (hence they are in different atoms of $\mathscr{P}'_{k'_0}$). We have proved that the pair (x, y) is $(\mathscr{P}'_{k'})$-scrambled.

For measure-theoretic$^+$ chaos it suffices to note that if η is close to 1, so is $\eta - 2\delta$.

The same proof applies also to uniform measure-theoretic chaos: if k_0 and η are common to all pairs in the $(\mathscr{P}_k)$-scrambled set, the proof produces common parameters k'_0 and $\eta - 2\delta$ for all pairs in the same set regarded as $(\mathscr{P}'_{k'})$-scrambled.

The argument for uniform measure-theoretic$^+$ chaos is the same as for the uniform measure-theoretic chaos, applied separately for every η (with k_η and k'_η in place of k_0 and k'_0, respectively). □

Remark 1. Even if $\eta = 1$, the proof produces $\eta - 2\delta < 1$. This is the reason why we gave up defining an analog of DC1; it could not be tested using one sequence of partitions. Similarly, it would probably not imply DC1 in topological systems (the proof of Theorem 4 as it is would not go through).

The next theorem replaces the 'persistence under removing null sets' by a much stronger property, 'ubiquitous presence of chaos': scrambled sets exist inside any set of positive measure.

THEOREM 3. *Let $(X, \mathfrak{B}, \mu, T)$ be an ergodic measure-theoretically (uniformly measure-theoretically, measure-theoretically$^+$, uniformly measure-theoretically$^+$) chaotic system. Let $(\mathscr{P}_k)$ be a refining sequence of finite partitions and let $B \in \mathfrak{B}$ be a set of positive measure. Then there exists an uncountable $(\mathscr{P}_k)$-scrambled ($(\mathscr{P}_k)^+$-scrambled, uniformly $(\mathscr{P}_k)$-scrambled, uniformly $(\mathscr{P}_k)^+$-scrambled) set contained in B.*

Proof. Throughout the proof 'scrambled set' stands for either $(\mathscr{P}_k)$-scrambled set, $(\mathscr{P}_k)^+$-scrambled set, uniformly $(\mathscr{P}_k)$-scrambled set, or uniformly $(\mathscr{P}_k)^+$-scrambled set, depending on the version of chaos considered. Let $A \subset X$ be a measurable set containing no uncountable scrambled sets. We need to show that $\mu(A) = 0$. Since the image of a scrambled set is obviously scrambled and has the same cardinality, $T^{-n}(A)$ does not contain uncountable scrambled sets either. This implies that $A' = \bigcup_{n=0}^{\infty} T^{-n}(A)$ does not contain uncountable scrambled sets (otherwise an uncountable subset of the scrambled set would have to fall in one item of the union). But A' is subinvariant (contains its preimage), hence, by ergodicity, its measure is either 1 or 0. The first possibility is excluded by Theorem 2. It follows that $\mu(A') = 0$, in particular $\mu(A) = 0$. □

Let us devote a few lines to better understanding the phenomenon of 'ubiquitous presence of chaos'. This phenomenon is the major difference between how topological and measure-theoretic chaoses are constructed (making the latter much stronger). First of all, let us realize that such presence cannot be achieved by the existence of a scrambled set of full (or even positive) measure. The fact is, every $(\mathscr{P}_k)$-scrambled set must be a null set. The same applies to distributionally scrambled sets in topological dynamical systems.

FACT 1. *Let $(\mathscr{P}_k)$ be a refining sequence of finite measurable partitions. Assume that μ is non-atomic. Then any $(\mathscr{P}_k)$-scrambled set has measure zero. Similarly, any* DC3*-scrambled (and thus also* DC2*-scrambled or* DC1*-scrambled) set in a topological dynamical system is a null set for all non-atomic invariant measures.*

Proof. Let (x, y) be a $(\mathscr{P}_k)$-scrambled pair and let k_0 be the index in the definition of $(\mathscr{P}_k)$-scrambling. Consider the two-element partition of $X \times X$ into two sets: $\bigcup_{A \in \mathscr{P}_{k_0}} A \times A$ and its complement, $\bigcup_{A,B \in \mathscr{P}_{k_0}, A \neq B} A \times B$. The orbit of the pair visits the first set with upper density 1 and the other with positive upper density, so the visits in these sets do not have densities. Such pairs are exceptional (belong to a null set depending on the index k_0) for every ergodic measure on $X \times X$. Since there are countably many choices of k_0, the collection of all $(\mathscr{P}_k)$-scrambled pairs is a null set for any such measure, and hence also for any $T \times T$-invariant measure—in particular, for $\mu \times \mu$. So, if E is a $(\mathscr{P}_k)$-scrambled set, then $(\mu \times \mu)(E \times E \setminus \Delta) = 0$. Since μ is non-atomic, also $(\mu \times \mu)(\Delta) = 0$, which implies that $(\mu \times \mu)(E \times E) = 0$ and hence $\mu(E) = 0$.

The proof for DC3-scrambled sets in topological systems is identical, except that the two-set partition consists of the s-neighborhood of the diagonal and its complement. □

It is clear that the 'ubiquitous presence of chaos' requires the union of all scrambled sets to be a set of full measure. Moreover, by a simple transfinite argument, there must exist a disjoint family of scrambled sets whose union is a full measure set. But even this last condition seems to be insufficient. Although we do not have an example of a dynamical system, it is easy to imagine an abstract family of disjoint uncountable null sets whose union has full measure, yet this measure is supported by a set selecting only countably many points (or just one point) from each member of the family. Then by removing the rest (which is a null set) we destroy all the uncountable sets. So, the 'ubiquitous presence of chaos' requires, most likely, an even more sophisticated configuration of the scrambled sets (than just the existence of a disjoint collection forming a full set). We give up further attempts to find an equivalent condition. What we have just learned for sure is that it is related to *abundance* of scrambled sets rather than their individual largeness.

Next, we take care of the relations between the notions of measure-theoretic chaos and their topological prototypes.

THEOREM 4. *Let (X, T) be a topological dynamical system and let μ be an ergodic T-invariant measure. If the measure-theoretic system $(X, \mathfrak{B}, \mu, T)$ (where $\mathfrak{B}$ denotes the Borel sigma-algebra completed with respect to μ) is measure-theoretically (uniformly measure-theoretically, measure-theoretically$^+$, uniformly measure-theoretically$^+$) chaotic then (X, T) is* DC2 *(uniformly* DC2, DC$1\frac{1}{2}$, *uniformly* DC$1\frac{1}{2}$*) chaotic.*

Proof. Let $(\mathscr{P}_k)$ be a refining sequence of partitions such that the diameter of the largest atom in $\mathscr{P}_k$ decreases to zero with k. For each k we define a sequence of open sets $U_{k,m}$ $(m \geq 1)$ as follows: by regularity of the measure, each atom P of $\mathscr{P}_k$ can be approximated (in measure) by a sequence of its closed subsets, say $(F_{P,m})_{m\geq 1}$. We let

$$U_{k,m} = X \Big\backslash \bigcup_{P \in \mathscr{P}_k} F_{P,m}.$$

We have, for every k, $\mu(U_{k,m}) \underset{m\to\infty}{\longrightarrow} 0$. Also let $s_{k,m}$ denote the (positive) minimal distance between points in different sets $F_{P,m}$, $F_{P',m}$ with $P, P' \in \mathscr{P}_k$.

By Theorem 2, if we remove the null set of points which, for at least one of the sets $U_{k,m}$, do not satisfy the assertion of the ergodic theorem, then in the remaining part there exists an uncountable $(\mathscr{P}_k)$-scrambled set E. We will show that E is DC2-scrambled. Let (x, y) be an off-diagonal pair in E. For every $\epsilon > 0$ the sequence of times n when $d(T^n x, T^n y) < \epsilon$ contains the sequence of times (of upper density 1) when the points $T^n x$, $T^n y$ belong to the same atom of $\mathscr{P}_k$, where k is so large that the diameter of the largest atom of $\mathscr{P}_k$ is smaller than ϵ. This easily implies that (x, y) satisfies the first requirement for being DC2-scrambled.

Further, there exist k_0 and a positive η such that $T^n x$, $T^n y$ belong to different atoms of $\mathscr{P}_{k_0}$ for ns with upper density at least η. Let $\delta > 0$ be much smaller than η. Find m so large that the set $U_{k_0,m}$ has measure smaller than δ. If we now remove from the aforementioned sequence of times n all the times when at least one of the points $T^n x$, $T^n y$ belongs to $U_{k_0,m}$, then we are left with a sequence of upper density at least $\eta - 2\delta$ (still positive) when the two points considered are at least $s_{k_0,m}$ apart. This proves that (x, y) satisfies the second requirement for being DC2-scrambled (with the parameters $s = s_{k_0,m}$ and upper density $\eta - 2\delta$).

If E is uniformly $(\mathscr{P}_k)$-scrambled, $(\mathscr{P}_k)^+$-scrambled, or uniformly $(\mathscr{P}_k)^+$-scrambled, the same proof yields the corresponding topological scrambling, as in the assertion of the theorem. □

There are many examples of DC2 chaotic systems in which the union of all scrambled sets is a null set for all ergodic measures, showing that the implication converse to Theorem 4 need not hold. However, if the topological chaos is 'persistent under removing null sets', it does imply measure-theoretic chaos, as stated below.

THEOREM 5. *Let (X, T) be a topological dynamical system and let μ be an ergodic invariant measure. Then the system $(X, \mathfrak{B}, \mu, T)$ is measure-theoretically (uniformly measure-theoretically, measure-theoretically$^+$, uniformly measure-theoretically$^+$) chaotic if and only if, after removing any set of measure μ zero, there remains an uncountable* DC2-*scrambled (uniformly* DC2-*scrambled,* DC1$\frac{1}{2}$-*scrambled, uniformly* DC1$\frac{1}{2}$-*scrambled) set.*

Proof. Necessity follows from the proof of the preceding theorem; the DC2-scrambled set (or its variants) has been obtained after removing a specific null set, but we could also have additionally removed any other null set. We move on to the proof of sufficiency. Choose a sequence $(\mathscr{P}_k)$ with the diameters of the largest atoms decreasing to zero with k, and define the sets $U_{m,k}$ (and the positive numbers $s_{k,m}$) as in the preceding proof. By Theorem 2, it suffices to fix a null set A and find a scrambled set disjoint from A. Let A_0 be the null set of points which fail the ergodic theorem for at least one of the sets $U_{k,m}$. By assumption, there exists an uncountable DC2-scrambled set E disjoint from $A \cup A_0$. We will show that E is $(\mathscr{P}_k)$-scrambled (and it is obviously disjoint from A). Take a pair (x, y) of distinct points in E and fix some k. Choose an arbitrarily small $\delta > 0$ and let m be such that the measure of $U_{k,m}$ is smaller than δ. As we know, the sequence of times n when $T^n x$ and $T^n y$ are closer together than $s_{k,m}$ has upper density 1. If we disregard the times when at least one of them falls into $U_{m,k}$, we are left with a sequence of upper density at least $1 - 2\delta$. Note that now, at each of these times, the two points belong to the same atom of $\mathscr{P}_k$. Because δ is arbitrarily small, we have shown that $T^n x$ and $T^n y$ belong to the same atom of $\mathscr{P}_k$ for times n of upper density 1.

We also know that $d(T^n x, T^n y)$ is larger than some positive s for times n with positive upper density. It suffices to pick k_0 large enough so that every atom of $\mathscr{P}_{k_0}$ has diameter smaller than s. Then for the same times n, $T^n x$ and $T^n y$ must fall into different atoms of $\mathscr{P}_{k_0}$. This concludes the proof for the usual $(\mathscr{P}_k)$-scrambling.

The same proof works for the other three variants of chaos. □

Remark 2. Using Theorem 3, the following variant of Theorem 5 can be proved: measure-theoretic chaos (and its respective variants) for an ergodic measure μ in a topological dynamical system is equivalent to 'μ-ubiquitous DC2' (and its respective variants): an uncountable DC2-scrambled (uniformly DC2-scrambled, DC1$\frac{1}{2}$-scrambled, uniformly DC1$\frac{1}{2}$-scrambled) set exists within every set of positive measure μ.

Remark 3. Here is an analog of DC3. Call a pair (x, x') $(\mathscr{P}_k)^-$-*scrambled* if there exists k_0 such that the sequence of times n when $T^n x$ belongs to the same atom of $\mathscr{P}_{k_0}$ as $T^n x'$ does not have density (upper and lower densities differ). A system is *measure-theoretically$^-$ chaotic* if, for every refining sequence of finite partitions, there exists an uncountable $(\mathscr{P}_k)^-$-scrambled set. Using slight modifications of the proofs presented in this section, one can prove that:

(1) this notion is an isomorphism invariant;
(2) it suffices to check 'persistence under removing null sets' and just one refining sequence of partitions;

(3) it enjoys the 'ubiquitous presence' property;
(4) in topological systems it is equivalent to DC3 'persistent under removing null sets'—in particular, as an interesting consequence, we get that DC3 'persistent under removing null sets of at least one invariant measure' is a conjugacy invariant.

5. *Measure-theoretic chaos versus entropy*

The most important relation between entropy and chaos is contained in the following theorem, which, combined with our Theorem 4 (and the variational principle), strengthens several former results [**BGKM, D**].

THEOREM 6. *Every ergodic system* $(X, \mathfrak{B}, \mu, T)$ *with positive Kolmogorov–Sinai entropy is uniformly measure-theoretically*$^+$ *chaotic.*

Proof. A large part of the proof is exactly the same as in [**D**]. Let us go directly to a certain point in the proof (skipping all the arguments that lead to that point). We select a sequence of partitions $(\mathscr{P}_k)$ of X in such a way that $(\mathscr{P}_{2k})$ is a refining sequence of partitions (as in the definition of the measure-theoretic chaos), while the odd-numbered partitions $\mathscr{P}_{2k-1}$ are all equal to one finite partition $\mathscr{P}$ with positive dynamical entropy $h_\mu(T, \mathscr{P})$, which we denote by h. We then fix an increasing sequence of integers $S = (a_1, b_1, a_2, b_2, a_3, b_3 \ldots)$. The sequence should grow so fast that b_k/a_k tends to infinity. We introduce the following notation:

$$\mathscr{R}_k = \mathscr{P}_k^{[a_k, b_k-1]} := \bigvee_{i=a_k}^{b_k-1} T^{-i}(\mathscr{P}_k),$$

and

$$\mathscr{R}_{1,2k-1}^{\mathsf{odd}} = \bigvee_{i=1}^{k} \mathscr{R}_{2i-1} \quad \text{and} \quad \mathfrak{R} = \bigvee_{k=1}^{\infty} \mathscr{R}_{2k}$$

(note that $\mathfrak{R}$ is no longer a finite partition, rather a *measurable partition* which can be identified with the collection of atoms of the sigma-algebra generated by the partitions involved in the countable join). We will also denote by n_k the difference $b_k - a_k$. In [**D**] it is shown that, given a sequence of positive numbers δ_k decreasing to zero and a set X' of sufficiently large measure $1 - \epsilon_0$, then, if the sequence S grows fast enough, there exists an atom z of $\mathfrak{R}$, a Borel measure ν supported by $z \cap X'$, and a decreasing sequence of measurable sets V_k, such that:

(A) if B is an atom of $\mathscr{R}_{1,2k-1}^{\mathsf{odd}}$ contained in V_k then B contains at least $2^{n_{2k+1}(h-\delta_k)}$ atoms of $\mathscr{R}_{1,2k+1}^{\mathsf{odd}}$ contained in V_{k+1} and whose conditional measures ν_B range within $2^{-n_{2k+1}(h\pm\delta_k)}$†.

As we shall show in a moment, statement (A) alone suffices to deduce uniform measure-theoretic$^+$ chaos. First of all, we remark that if we subtract a null set from X' the statement will still hold (perhaps on a different atom z and for a different sequence S, but this does

† The measure ν in [**D**] is obtained as a disintegration measure μ_{yz} of μ with respect to $\Pi \vee \mathfrak{R}$ (where Π is the Pinsker sigma-algebra) on an appropriately chosen atom $y \cap z$. The measure is further restricted to the intersection of z with the set X'. In this paper, this set will be chosen differently than in [**D**].

not matter). Thus, in order to complete the proof of Theorem 6, it remains to show that (A) implies the existence of an uncountable uniformly $(\mathscr{P}_k)^+$-scrambled set within the atom z.

It is rather easy to see that statement (A) remains valid if we replace ν by the conditional measure ν_C, where C is any set of positive measure ν. We skip the standard argument here (see [**D**, Fact 1]). We now represent our space X as a subset of a compact metric space (say, of the unit interval) and, using regularity of the measure ν, we can remove a set of small measure from the support of ν in such a way that all atoms of the partitions $\mathscr{R}^{\mathsf{odd}}_{1,2k-1}$ (for all k) intersected with the remaining set C are compact. Replacing ν by ν_C, we obtain the condition (A) with the additional feature that the atoms B (and those to which B splits) are all compact. This will guarantee that the intersection of any nested chain of such atoms (with growing parameter k) is non-empty†.

The rest of our proof deviates from that in [**D**]. The main difference is in obtaining separation along a subsequence of upper density η close to 1 (not just positive). This will be achieved not for the partition $\mathscr{P}$ but for $\mathscr{P}^{[0,m-1]}$ with a suitably selected parameter m.

At this point we specify the set X'. Let ϵ_i $(i \geq 1)$ be a summable sequence of positive numbers with small sum ϵ_0. Using the Shannon–McMillan theorem, we can find integers m_i and a sequence of sets C_i, each being a union of less than $2^{m_i(h+\epsilon_i)}$ cylinders of length m_i, and whose measure exceeds $1-\epsilon_i^2$. Further, using the ergodic theorem, we can find n_i' so large that the set of points whose orbits visit C_i more than $n(1-\epsilon_i^2)$ times within the first n iterates, for every $n \geq n_i'$, has measure at least $1-\epsilon_i$. For points in a set X' of measure larger than $1-\epsilon_0$, this holds for every i.

Fix a number $\eta < 1$. Find the smallest parameter i such that, denoting $\epsilon = \epsilon_\eta = \epsilon_i$ and $m = m_\eta = m_i$ (the notation ϵ_η and m_η will not be used until two pages hence), we have $\epsilon < 1-\sqrt{\eta}$ and

$$\frac{2H(\sqrt{\eta}, 1-\sqrt{\eta})}{m} + \epsilon(3\#\mathscr{P}+1) < (1-\sqrt{\eta})h \tag{1}$$

(here $H(p, 1-p)$ stands for $-p\log p - (1-p)\log(1-p)$). Find k such that $b_{2k+1} \geq n_i'$ and, denoting $n = n_{2k+1}$ we have $2m/n < \epsilon$. Moreover, we require that $2\delta_k$ (δ_k is the parameter occurring in condition (A)) and $(\log m/n)$ can be added on the left-hand side of (1), maintaining the inequality. We remark that the above requirements hold for all sufficiently large k. For future reference we let k_η be the smallest choice of k.

Fix an atom B of $\mathscr{R}^{\mathsf{odd}}_{1,2k-1}$ contained in V_k. By (A), this atom contains at least $2^{n(h-\delta_k)}$ different atoms of $\mathscr{R}^{\mathsf{odd}}_{1,2k+1}$ contained in V_{k+1}. Every such atom has the form $B \cap A$, where A is an atom of $\mathscr{R}_{2k+1}$. We will call the atoms A such that $B \cap A$ is non-empty and contained in V_{k+1} *good continuations* of B. We denote by $\mathcal{A}(B)$ the collection of good continuations of B represented as blocks of length n, over the alphabet $\mathscr{P}$. We will now count how many blocks $A \in \mathcal{A}(B)$ may disagree with one selected block $A_0 \in \mathcal{A}(B)$ on a fraction, smaller than η, of all subblocks of length m.

To do that, we draw the block A_0 m times, and we subdivide the jth copy ($j = 0, 1, \ldots, m-1$) into subblocks of length m by cutting it at positions equal to j mod m

† This part of the proof—ensuring non-empty intersections of nested chains—was handled differently in [**D**], by taking closures of the atoms B. It could also have been handled the same way as we do here.

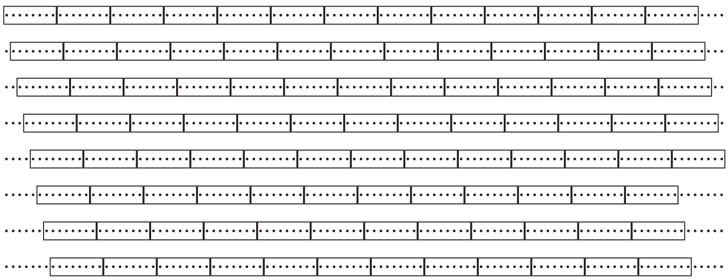

FIGURE 1. All subblocks of length m of A_0 visualized in m copies of A_0.

(there are at most n/m subblocks in each copy). Figure 1 shows all subblocks of length m of A_0 (plus some incomplete 'prefixes' and 'suffixes' at the ends).

Imagine another block A treated the same way and suppose that it disagrees with A_0 on a fraction, smaller than η, of all subblocks. This implies that the fraction of all m copies for which A_0 and A disagree on a larger than $\sqrt{\eta}$ fraction of subblocks visualized in this copy is at most $\sqrt{\eta}$. In other words, for a fraction of at least $1 - \sqrt{\eta}$ of all copies, A_0 and A agree on at least a fraction of $1 - \sqrt{\eta}$ of the subblocks. Further, we know that, in this diagram, at most $b_{2k+1}\epsilon^2 < 2n\epsilon$ subblocks represent cylinders not contained in C_i. Again, in at least a fraction of $1 - \epsilon$ copies the subblocks not contained in C_i constitute a fraction smaller than 2ϵ. Because $\epsilon + \sqrt{\eta} < 1$, there exists at least one copy where we have both a fraction smaller than 2ϵ of subblocks from outside C_i and a fraction larger than $1 - \sqrt{\eta}$ of agreeing subblocks. We can now classify all blocks A that we are counting into at most (not necessarily disjoint)

$$m \cdot 2^{(n/m)H(\sqrt{\eta},1-\sqrt{\eta})} \cdot 2^{(n/m)H(\epsilon_i,1-\epsilon_i)} \leq 2^{(n/m)2H(\sqrt{\eta},1-\sqrt{\eta})+\log m}$$

groups depending on the choice of the copy, the choice of necessarily agreeing subblocks (perhaps there will be more), and the choice of places reserved for visits in C_i (perhaps not all of them will be used).

In every group there are at most

$$2^{3n\epsilon\#\mathscr{P}} 2^{(n/m)\sqrt{\eta}m(h+\epsilon)} \leq 2^{n(\epsilon(3\#\mathscr{P}+1)+h\sqrt{\eta})}$$

blocks. (We allow any symbols from $\mathscr{P}$ on the fraction 2ϵ of subblocks from outside C_i and on the 'prefix' and 'suffix' jointly of length not exceeding $2m$, hence constituting another fraction smaller than ϵ. Otherwise, on a fraction of at most $\sqrt{\eta}$ of all subblocks we have free choice from the collection of at most $2^{m(h+\epsilon)}$ blocks from C_i.) Multiplying this by the number of groups, we get no more than

$$2^{n((2H(\sqrt{\eta},1-\sqrt{\eta})/m)+(\log m/n)+\epsilon(3\#\mathscr{P}+1)+h\sqrt{\eta})} < 2^{n(h-2\delta_k)} = \frac{2^{n(h-\delta_k)}}{2^{n\delta_k}} \tag{2}$$

blocks.

We have shown that blocks A differing from a selected block A_0 on a fraction, smaller than η, of subblocks of length m form a negligibly small fraction (at most $2^{-n\delta_k}$) of the family $\mathcal{A}(B)$.

We are in a position to construct our uncountable uniformly measure-theoretically^{+}-scrambled set E. We begin by constructing a family B_κ, where κ ranges over all finite binary words, such that B_κ is a (non-empty and closed) atom of $\mathscr{R}^{\mathsf{odd}}_{1,2k+1}$ contained in V_{k+1}, where k is the length of κ (for $k=0$, κ is the empty word). We will ensure that if ι extends κ to the right then $B_\iota \subset B_\kappa$. We will also ensure an appropriate separation condition. For that purpose we fix a sequence $(\eta_k)_{k\geq 1}$ with the following properties: the sequence assumes values strictly smaller than but arbitrarily close to 1, each value is assumed infinitely many times, and a value η is allowed to occur only for $k \geq k_\eta$. The inductive separation condition is that if κ and κ' are binary words of the same length, differing at a position k_0, then, for every $k \geq k_0$ (up to the length of κ), the blocks A_k and A'_k differ at a fraction of at least η of all subblocks of length m_{η_k}, where A_k and A'_k are the blocks appearing at the coordinates $[a_{2i+1}, b_{2i+1-1}]$ in the symbolic representation of the atoms B_κ and $B_{\kappa'}$, respectively. We will carry this out by induction on k; at each step we choose two 'children' of every atom of $\mathscr{R}^{\mathsf{odd}}_{1,2k-1}$ so far constructed.

At step $k=0$ we assign $B_\emptyset$ to be an arbitrarily selected atom of $\mathscr{R}_1$ contained in V_1. Suppose that the task has been completed for some $k-1$, that is, that we have selected 2^{k-1} atoms B_κ of $\mathscr{R}^{\mathsf{odd}}_{1,2k-1}$, contained in V_{k-1}, and pairwise separated as required. We order the κs of length $k-1$ lexicographically. Take the first atom B_{κ_1} (assigned for $\kappa_1 = 0\ldots00$). Choose one good continuation A_0 of B_{κ_1}. From every family $\mathcal{A}(B_\kappa)$ (including $\kappa=\kappa_1$) we eliminate (for future choices) all the atoms A which differ from A_0 on a fraction, smaller than η_k, of subblocks of length m_{η_k}. Since $k \geq k_{\eta_k}$, the preceding estimate applies: every family $\mathcal{A}(B_\kappa)$ has 'lost' at most a fraction $2^{-n_{2k+1}\delta_k}$ of its cardinality. Next we choose A_1 from the remaining good continuations of B_{κ_1} and again, from each of the families $\mathcal{A}(B_\kappa)$ we eliminate all the atoms A not sufficiently separated from A_1. Again, the losses are negligibly small. We assign

$$B_{\kappa_1 0} = B_{\kappa_1} \cap A_0 \quad \text{and} \quad B_{\kappa_1 1} = B_{\kappa_1} \cap A_1$$

(here $\kappa_1 0$ and $\kappa_1 1$ denote the two continuations of κ_1).

Next we abandon B_{κ_1}, pass to B_{κ_2} ($\kappa_2 = 0\ldots01$), and repeat the procedure choosing two of its good continuations (say, A'_0, A'_1) not eliminated in the preceding steps, each time eliminating for future choices all blocks insufficiently separated from the chosen ones. We proceed until we have chosen two good continuations for every κ of length $k-1$. Note that near the end of this procedure we will have eliminated from each family $\mathcal{A}(B_\kappa)$ a fraction at most $2^k \cdot 2^{-n_{2k+1}\delta_k}$, which is less than 1 (we decide on the size of n_{2k+1} after fixing δ_k). Hence the procedure will be possible till the end. This completes the inductive step k.

Now let κ denote an infinite binary string, while κ_k is the prefix of length k of κ. The atoms B_{κ_k} form a decreasing sequence of compact sets, hence have a non-empty intersection. We select one point from this intersection and call it x_κ. The set E is defined as the collection $\{x_\kappa : \kappa \in \{0,1\}^{\mathbb{N}}\}$. The following facts are obvious: the set E is uncountable, and all its elements belong to the atom z of $\mathfrak{R}$. The last fact implies that for each k the orbits of all points from E fall in the same element of $\mathscr{P}_{2k}$ (which is a partition in our refining sequence) for all times n belonging to the intervals $[a_{2k'}, b_{2k'-1}]$ for all $k' \geq k$. Because the ratios b_{2k}/a_{2k} tend to infinity, it is clear that such times n have upper density 1 and the first requirement for $(\mathscr{P}_{2k})^{+}$-scrambling is verified for all pairs in E.

Consider a pair of distinct points from E, that is, $x = x_\kappa$ and $x' = x_{\kappa'}$, where $\kappa \neq \kappa'$. Let k_0 denote the first place where κ differs from κ'. Fix some $\eta < 1$ and then, if necessary, replace it by a larger value, so that η occurs as η_k (and then it occurs for infinitely many indices k). Pick such a k larger than k_0 and observe the blocks A_k and A'_k representing the atoms of $\mathscr{R}_{2k+1}$ containing x and x', respectively. Since κ and κ' differ at a position smaller than or equal to k, the blocks A_k and A'_k have been selected as either two different continuations of the same atom of $\mathcal{R}^{\text{odd}}_{1,2k-1}$ or as continuations of two different atoms of this partition. In any case, they have been selected one after another in the inductive step k, which means the latter one (say, A'_k) was chosen after eliminating all blocks that differ from the former (say, A_k) on a fraction, smaller than η, of all subblocks of length m_η. This means that A_k and A'_k differ on a fraction at least η of such subblocks. Moreover, this is true for infinitely many k. Because b_k/a_k tends to infinity, this easily implies that the set of times n for which $T^n x$ and $T^n x'$ belong to different atoms of the partition $\mathscr{P}^{m_\eta}$ has upper density at least η. This is 'almost' the second requirement for $(\mathscr{P}_{2k})^+$-scrambling, except that it refers to a wrong partition. In order to replace the partition $\mathscr{P}^{m_\eta}$ by a partition belonging to our refining sequence, that is, by some $\mathscr{P}_{2k_\eta}$, we apply the same technique as many times before. There exists k_η such that $\mathscr{P}_{2k_\eta}$ refines $\mathscr{P}^{m_\eta}$ except on a set of measure $\delta \ll \eta$. If, at the start of the proof, we eliminate points which do not obey the ergodic theorem for the field generated by all the partitions of the form $\mathscr{P}_{2k}$ and $\mathscr{P}^m$, then our points $T^n x$ and $T^n x'$ will belong to different atoms of $\mathscr{P}_{2k_\eta}$ for powers n with upper density at least $\eta - 2\delta$ which is as close to 1 as we want. We have shown that the pair x, x' is $(\mathscr{P}_{2k})^+$-scrambled. Finally, we note that the assignment $(\eta - 2\delta) \mapsto k_\eta$ arising in the proof does not depend on the pair x, x' and hence the set E is scrambled uniformly, which concludes the proof of Theorem 6. □

Remark 4. Let us say that an increasing sequence n_i *achieves upper density* η *along a subsequence* N *of positive integers* if

$$\limsup_{N\to\infty} \frac{\#\{i : n_i \leq N\}}{N} \geq \eta.$$

In the above construction, all upper densities required in the definition of $(\mathscr{P}_k)^+$-scrambling (the upper density 1 of the sequence n_i and the upper densities η, more precisely $\eta - 2\delta$, of the sequences $m_{\eta,i}$) are achieved along the subsequence b_k (the right ends of the intervals $[a_k, b_k - 1]$). The only constraints on the choice of the sequence S (containing b_k) concern the speed of its growth, thus b_k could have been selected a subsequence of any *a priori* given infinite sequence of positive integers. We will need this observation in the proof of Theorem 7.

Remark 5. We have obtained a specific scrambled set, which, in spite of being 'uniform', has another property one might call 'synchronic'. Let us say that an increasing sequence n_i *achieves lower density* η *along a subsequence* N *of positive integers* if

$$\liminf_{N\to\infty} \frac{\#\{i : n_i \leq N\}}{N} \geq \eta.$$

Clearly, upper density of the sequence a_i equals the supremum of all lower densities that the sequence achieves along various subsequences of the positive integers. For all distinct pairs in our scrambled set E the sequences of times n_i achieve lower densities 1 along the same subsequence, namely along b_{2k}. Further, given $\eta < 1$, for all distinct pairs in our scrambled set, the sequences $m_{\eta,i}$ achieve the lower densities η along a common sequence, (for instance, if η is assumed as η_k then the lower density is achieved along b_{2k+1}, where k denotes only these infinitely many integers for which $\eta_k = \eta$).

Remark 6. In the construction, the sequences $m_{\eta,i}$ obtained for various values of η do not achieve their desired lower densities η along the same sequence (at least this is not assured). This is because with distinct values of η we have associated disjoint sequences of indices k such that $\eta_k = \eta$. It is possible to modify the construction to ensure the existence of a common sequence N along which all the sequences $m_{\eta,i}$ (for varying η and varying pairs) would achieve their desired lower densities η ('synchronic$^+$ scrambling'). This can be done by a different elimination procedure in the construction of the sets B_κ. In that construction we let the sequence (η_k) increase to 1 (without repeating each value infinitely many times) and, in each inductive step (say, k_0), we ensure appropriate separation (on a fraction η_k of all subblocks of length m_{η_k}) *simultaneously* for all $k \le k_0$. This extra property is not worthy of a detailed proof.

Combining Theorem 6 and Remark 2, we obtain the following topological statement, a strengthening of the results from [**BGKM, D**].

COROLLARY 1. *A topological dynamical system (X, T) with positive topological entropy reveals 'ubiquitous uniform chaos* DC1$\frac{1}{2}$*'; an uncountable uniformly* DC1$\frac{1}{2}$*-scrambled set exists within every subset of positive measure μ, for every ergodic measure μ with positive Kolmogorov–Sinai entropy.*

For a more complete picture of relations between entropy and our notions of chaos we give an example showing that Theorem 6 cannot be reversed.

THEOREM 7. *There exists a system $(X, \mathfrak{B}, \mu, T)$ with entropy zero and with uniform measure-theoretic$^+$ chaos.*

Proof. As a matter of fact, such an example exists in a paper of Serafin [**Se**]. Moreover, it is a topological example, with topological entropy zero, in which we will fix an ergodic measure μ. Because the example was created for different purposes, we will need to verify the chaos. This is going to be a tedious task.

Let us first say a few words about certain (invertible) systems (X, T) that have an odometer factor. Consider a two-row symbolic system, where both rows are bi-infinite sequences of symbols. The first row contains symbols from $\{0, 1, 2, \ldots, \infty\}$, the second row is binary (contains symbols from $\{0, 1\}$). The elements $x \in X$ obey the following *odometer rule* with respect to an increasing sequence (N_k) of integers such that, for each k, N_{k+1} is a multiple of N_k (this sequence is called the *base* of the odometer).

- For each $k \ge 1$, the symbols $k' \ge k$ occupy in the first row a periodic set of period N_k having exactly one element in every period. Such symbols will be called *k-markers*. The two-row blocks of length N_k starting with a k-marker will be called k-blocks. Every point in such a system is, for every k, a concatenation of the k-blocks (see Figure 2).

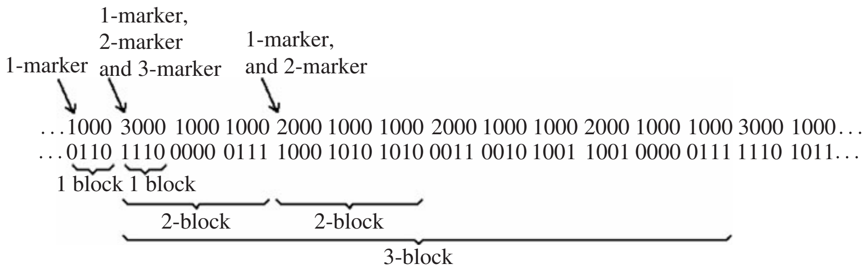

FIGURE 2. An element of a system with an odometer factor to base (N_k), with $N_1 = 4$, $N_2 = 12$, $N_3 = 48$, ...

In such a system we introduce a specific sequence of partitions, which we denote by $(\mathscr{P}_k)$ defined in the following manner: two points belong to the same atom of $\mathscr{P}_k$ if they have the same and identically positioned *central* k-block (by which we mean the k-block covering the coordinate zero). Notice that there are (at most) $N_k \cdot 2^{N_k}$ atoms of $\mathscr{P}_k$ (N_k counts the possible ways the k-block is positioned on the horizontal axis, while 2^{N_k} is the maximal number of possible 'words' in the second row of the k-block).

LEMMA 1. *The sequence of partitions $(\mathscr{P}_k)$ is refining for any invariant measure μ.*

Proof. By general facts concerning standard probability spaces, it suffices to show that after discarding a null set, the partitions $\mathscr{P}_k$ separate points. It is clear that the partitions separate points x belonging to different fibers of the odometer (i.e., differing in the first row). Notice that the symbol ∞ may occur in the first row of an $x \in X$ only once. Thus, by the Poincaré recurrence theorem, the set of elements $x \in X$ in which ∞ occurs is a null set for any invariant measure. After discarding this null set, the partitions $\mathscr{P}_k$ also separate points belonging to the same fiber of the odometer (the central k-blocks grow with k in both directions, eventually covering the entire elements). □

The partitions $\mathscr{P}_k$ have a very specific property (not enjoyed by the 'usual' partitions of symbolic spaces into blocks occurring at fixed positions): if x, x' belong to the same atom of $\mathscr{P}_k$ then $T^i x, T^i x'$ belong to the same atom of $\mathscr{P}_k$, for i ranging in an interval of integers of length N_k containing 0 (namely as long as shifting by i positions maintains the coordinate zero within the same k-block). In particular, if $k' > k$ and x, x' have identically positioned k'-blocks (say, $x[a, b]$ and $x'[a, b]$ are k'-blocks), then the percentage of times $i \in [a, b]$ when $T^i x$ and $T^i x'$ belong to the same (different) elements of $\mathscr{P}_k$ equals the percentage of agreeing (disagreeing) component k-blocks in the k'-blocks $x[a, b]$ and $x'[a, b]$. We will refer to this property at the end of the proof.

Notice that if two points belong to different fibers of the odometer factor then their orbits are separated by some $\mathscr{P}_k$ at all times. Thus every $(\mathscr{P}_k)$-scrambled set (if one exists) is contained in one fiber of the odometer. In particular, the uniform measure-theoretic$^+$ chaos is equivalent to the condition that ν-almost every element y of the odometer, where ν is the unique invariant measure on the odometer, has the property that after discarding any null set for the disintegration measure μ_y, the fiber of y contains an uncountable uniformly $(\mathscr{P}_k)^+$-scrambled set.

$$\begin{array}{l} \ldots 100\,000\ 200\,000\ 100\,000\ 100\,000\ 100\,000\ 100\,000\ 100\,000\ 200\,000\ 100\,000 \ldots \\ \ldots 101\,101\ \underbrace{\underbrace{010\,010}_{\in\mathcal{C}_1}\ \underbrace{110\,110}_{\in\mathcal{C}_1}\ \underbrace{000\,000}_{\in\mathcal{C}_1}\ \underbrace{010\,010\ 110\,110\ 000\,000}_{\text{repetition}}}_{\in\mathcal{C}_2}\ 111\,111\ 011\,011 \ldots \end{array}$$

FIGURE 3. An element of our system (here $q_1 = q_2 = 3$, hence $N_1 = 6$, $N_2 = 36$).

We will now reproduce the construction of the example from [**Se**]. First, we introduce some notation. If $\mathcal{B} \subset \{0, 1\}^k$ (i.e., $\mathcal{B}$ is a family of selected binary blocks of length k) and $q \in \mathbb{N}$ then by $\mathcal{B}^q$ we mean the family of all concatenations of q elements from $\mathcal{B}$, $\mathcal{B}^q = \{B_1 B_2 \cdots B_q : B_1, B_2, \ldots, B_q \in \mathcal{B}\}$ and by $\mathcal{B}_{\mathsf{rep}}$ we will mean $\{BB : B \in \mathcal{B}\}$ (the family of all 2-repetitions of elements of $\mathcal{B}$). Notice that $\mathcal{B}_{\mathsf{rep}} \subset \mathcal{B}^2$, $\#\mathcal{B}^q = (\#\mathcal{B})^q$ and $\#\mathcal{B}_{\mathsf{rep}} = \#\mathcal{B}$.

We fix a sequence of integers $(q_k)_{k\geq 1}$, larger than 1 (for the purposes of this example it suffices to take $q_k = 2$ for all k). The products $q_1 q_2 \cdots q_k$ will be denoted by p_k. Next we define inductively two families of binary blocks: $\mathcal{B}_1 = \{0, 1\}^{q_1}$, $\mathcal{C}_1 = (\mathcal{B}_1)_{\mathsf{rep}}$, and, for $k \geq 2$, $\mathcal{B}_k = (\mathcal{C}_{k-1})^{q_k}$, $\mathcal{C}_k = (\mathcal{B}_k)_{\mathsf{rep}}$. The blocks in $\mathcal{B}_k$ have length $p_k 2^{k-1}$ and those in $\mathcal{C}_k$ have length $N_k = p_k 2^k$. We have $\#\mathcal{B}_k = \#\mathcal{C}_k = 2^{p_k}$.

Now we can define the system (X, T) as a two-row symbolic system with the odometer G to base (N_k) encoded in the first row and such that the 'word' in the second row of every k-block is a block from $\mathcal{C}_k$ (see Figure 3).

The system so defined has topological entropy zero: the first row system has topological entropy zero because it is an odometer and the second row factor has entropy zero by an easy counting argument: the logarithm of the cardinality of the family of k-blocks (equal to p_k) grows much more slowly than their length (equal to $p_k 2^k$). (The entire system has entropy zero, as a topological joining of two systems with entropy zero.)

In order to define an invariant measure μ it suffices to declare all atoms of the partition $\mathscr{P}_k$ (corresponding to the k-blocks positioned around coordinate zero) to have equal measures. (The measure of an atom then equals $1/\#\mathscr{P}_k = 1/N_k 2^{p_k}$.) We skip the standard proof that this indeed determines a shift-invariant measure which is ergodic.

Let H be the odometer to base (p_k) represented similarly to G, in the form of a symbolic system over the alphabet $\{0, 1, \ldots, \infty\}$. Notice that since, for each k, N_k is a multiple of p_k, H is a topological factor of G; the factor map (which we denote by ϕ) consists in inserting more k-markers (we skip the obvious details). The unique invariant measure ν supported by G is sent to the unique invariant measure ξ supported by H.

For ν-almost every $y \in G$ we will now describe a measurable bijection π_y between the fiber of y in our system X and the fiber of $\phi(y)$ in the direct product $(H \times \{0, 1\}^{\mathbb{Z}})$, sending the measure μ_y to the Bernoulli measure $\lambda = \{\frac{1}{2}, \frac{1}{2}\}^{\mathbb{Z}}$. The rigorous definition of the map and the proof of the correspondence of measures, although completely elementary, are lengthy and of little interest. Instead we provide a slightly informal description.

Suppose that we want to create a k-block appearing in the system X. Since the first row is determined (up to the value of the 'leading marker' which can be any number $k' \geq k$) we only need to write a block belonging to $\mathcal{C}_k$. Suppose that we write it from left to right. While doing this, we encounter two kinds of positions: those which can be filled

$$
\begin{array}{l}
3000\ 1000\ 1000\ 1000\ 2000\ 1000\ 1000\ 1000\ 2000\ 1000\ 1000\ 1000\ 2000\ 1000\ 1000\ 1000 \\
\underline{01}01\ \underline{11}11\ 0101\ 1111\ \underline{0\overline{0}}0\overline{0}\ \underline{10}10\ 0\overline{0}0\overline{0}\ 1010\ 0101\ 1111\ 0101\ 1111\ 0\overline{0}0\overline{0}\ 1010\ 0\overline{0}0\overline{0}\ 1010
\end{array}
$$

FIGURE 4. An example of a 3-block (for $q_1 = q_2 = q_3 = 2$). The 'free positions' have underlines and the positions determined by the free position 18 have overlines.

completely arbitrarily, independently of what was filled earlier (we call them the 'free positions'), and others where we have 'forced repetitions' of something that was filled earlier. There are exactly p_k free positions and each of them is repeated $2^k - 1$ times (jointly determining 2^k positions; see Figure 4).

Given a k-block B appearing in X, the free positions read from left to right constitute a new two-row block of length p_k, with markers positioned as in a k-block over the odometer H. We have just described a bijection (denoted π_k) between all k-blocks of X and all k-blocks appearing in $(H \times \{0, 1\}^{\mathbb{Z}})$.

We need to take a closer look at the map π_k. Every $(k+1)$-block C is a concatenation of $2q_k$ k-blocks (half of which is repeated):

$$C = B^{(1)} B^{(2)} \cdots B^{(q_k)} B^{(1)} B^{(2)} \cdots B^{(q_k)}.$$

The reader will easily verify that then

$$\pi_{k+1}(C) = \pi_k(B^{(1)}) \pi_k(B^{(2)}) \cdots \pi_k(B^{(q_k)}).$$

Fix some $y \in G$ and let x belong to the fiber of y (i.e., x has y in the first row). For each k, let $B_k(x)$ denote the central k-block in x. Find the coordinate zero in the block $B_{k+1}(x)$. Since

$$B_{k+1}(x) = B^{(1)} B^{(2)} \cdots B^{(q_k)} B^{(1)} B^{(2)} \cdots B^{(q_k)},$$

there exists an index i such that the coordinate zero falls within (one of two copies) of the k-block $B^{(i)}$ (that is to say, $B_k(x) = B^{(i)}$). In such case we place the image block $\pi_{k+1}(B_{k+1}(x))$ (which equals $\pi_k(B^{(1)}) \pi_k(B^{(2)}) \cdots \pi_k(B^{(q_k)})$) along the horizontal axis in such a way that the zero coordinate falls within the subblock $\pi_k(B^{(i)})$. The precise location of the coordinate zero within the latter block is established analogously, by repeating the same procedure for $k, k-1, k-2$, etc. In this manner we obtain a consistent family of positioned blocks $\pi_k(B_k(x))$ growing either in one or in both directions around the coordinate zero (and this behavior depends only on the first row y of x). By an easy argument, ν-almost surely, the blocks $\pi_k(B_k(x))$ grow around the coordinate zero in both directions, eventually determining a two-sided sequence, which we denote by $\pi_y(x)$. This completes the definition of the map π_y. We skip the verification of the fairly obvious fact that for ys for which the map π_y is defined, π_y is a bijection between the fiber of y and $\{\phi(y)\} \times \{0, 1\}^{\mathbb{Z}}$, sending μ_y to λ.

For ys for which the map π_y is well defined, the partition $\mathscr{P}_k$ (determined by the position and contents of the central k-block) restricted to the fiber of y maps by π_y to an analogous partition of $H \times \{0, 1\}^{\mathbb{Z}}$ (which we denote by $\mathscr{Q}_k$), restricted to $\{\phi(y)\} \times \{0, 1\}^{\mathbb{Z}}$. Since the product system $H \times \{0, 1\}^{\mathbb{Z}}$ with the product measure $\xi \times \lambda$ has obviously positive entropy, by Theorem 6, it is uniformly measure-theoretically$^+$ chaotic. As we have already mentioned, this means that ξ-almost every $z \in H$ (we let H' be the corresponding set of full measure) has the property that after removing any λ-null set, $\{z\} \times \{0, 1\}^{\mathbb{Z}}$ contains an uncountable uniformly $(\mathscr{Q}_k)^+$-scrambled set. We would like to deduce that this property passes to X. We will use the map ϕ between the odometers and the maps π_y on the fibers. For instance, we immediately define $G' = \phi^{-1}(H')$ and note that G' is a set of full measure ν. The difficulty is that the combined map (from X to $H \times \{0, 1\}^{\mathbb{Z}}$) is not shift-equivariant (it cannot be, because a system with positive entropy cannot be a factor of a system with entropy zero), hence the 'percentage of agreement/disagreement' along two orbits is not automatically preserved. We must check it 'manually'.

Notice that, since every free position in a k-block of X has the same number of 'forced repetitions', the percentage of entries where two such k-blocks differ (equal) is the same as the percentage of entries where their images by π_k differ (equal). Moreover, this 'preservation of percentage' passes to higher blocks: pick some $k' > k$. Every k'-block C is a concatenation of k-blocks; some of them are 'free', and some are 'forced repetitions'. Each of the 'free' component k-blocks has the same number of 'forced repetitions'. The reader will easily observe that the image $\pi_{k'}(C)$ equals the concatenation of the images by π_k of the 'free' component k-blocks. Thus, for two k'-blocks, say C, D, the percentage of the component k-blocks which are the same (different) in C and D is the same as the percentage of the component images of the k-blocks which are the same (different) in $\pi_{k'}(C)$ and $\pi_{k'}(D)$.

By an easy argument involving the Borel–Cantelli lemma, ν-almost every y has the following property: there exists a subsequence k_l (depending on y) such that the relative position of the zero coordinate within the central k_l-block divided by its length N_{k_l} converges with l to zero. Similarly, for ν-almost every y, the same holds (and we can assume, along the same sequence k_l) for the images $\pi_y(x)$.

We now fix some $y \in G'$ for which the map π_y is defined and which fulfills the above two 'almost sure' conditions. Let A be a μ_y-null set and let $A' = \pi_y(A)$ (which is a null set for the Bernoulli measure). We already know that $\{\phi(y)\} \times (\{0, 1\}^{\mathbb{Z}} \setminus A')$ contains an uncountable uniformly $(\mathscr{Q}_k)^+$-scrambled set E'_A. Moreover, by Remark 4 we can arrange such a scrambled set E'_A so that all the upper densities required in the definition of scrambling are achieved along the subsequence p_{k_l}.

Let E_A be the preimage by π_y of E'_A. Clearly, E_A is uncountable and disjoint from A. It remains to show that E_A is uniformly $(\mathscr{P}_k)^+$-scrambled. This, however, is an almost immediate consequence of the following facts (we leave the easy deduction to the reader):

- The upper densities required for scrambling of E'_A are all achieved along the subsequence p_{k_l}.
- For large l the central k_l-blocks in x, x' (in the fiber of our selected y) start 'nearly' at the coordinate zero, which implies that the percentage of times $i \in [0, N_{k_l} - 1]$ when $T^i x$ and $T^i x'$ belong to the same (different) atoms of $\mathscr{P}_k$ is nearly the same as for $i \in [a, b]$, where a, b are the ends (common for both points) of the central k_l-block. An analogous statement holds for $\pi_y(x)$ and $\pi_y(x')$ and the times $i \in [0, p_{k_l} - 1]$ (of course, the ends a, b of the central k_l-blocks are now different).
- Due to the specific property of the partitions $\mathscr{P}_k$, the percentage of times $i \in [a, b]$ when $T^i x$ and $T^i x'$ belong to the same (different) atoms of $\mathscr{P}_k$ equals the percentage of agreeing (disagreeing) k-blocks in the central k_l-blocks of these points. An analogous statement holds for the points $\pi_y(x)$, $\pi_y(x')$ and the partitions $\mathscr{Q}_k$.

- For $k' > k$ the percentage of agreeing (disagreeing) k-blocks within the central k_l-blocks of x, x' equals the percentage of agreeing (disagreeing) k-blocks within the central k_l-block of $\pi_y(x), \pi_y(x')$. □

6. *Open problems*

Question 1. As we have already mentioned, we do not know whether DC2 (or uniform DC2) persistent under removing null sets implies DC1$\frac{1}{2}$. Similarly, we have no examples showing that measure-theoretic$^+$ chaos is essentially stronger than the measure-theoretic chaos.

As the results of [**DL1, DL2**] show, some properties necessary for positive topological entropy (such as the existence of asymptotic or forward mean proximal pairs†), if inherited by all topological extensions of the system, also become sufficient. Thus it seems reasonable to ask the following question.

Question 2. Is it true that a topological dynamical system whose every topological extension is DC2 (DC1$\frac{1}{2}$) chaotic has positive topological entropy? Is it true that every ergodic system whose every (measure-theoretic) extension is measure-theoretically (measure-theoretically$^+$) chaotic has positive Kolmogorov–Sinai entropy?

We were unable not only to resolve the above, but even to disprove the following: it might happen that unlike topological chaoses, the measure-theoretic chaos passes to extensions. So we have another question, in a sense opposite to the preceding one.

Question 3. Suppose that $(X, \mathfrak{B}, \mu, T)$ is a (measure-theoretic) factor of $(Y, \mathfrak{C}, \nu, S)$ and that the former system reveals measure-theoretical chaos (in one of the four versions discussed). Does that imply the same chaos for the latter system? Similarly if (X, T) is a topological factor of (Y, S) and (X, T) reveals DC2 persistent under removing null sets. Does that imply DC2 for (Y, S)?

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† A pair x, x' is *forward mean proximal* if there exists a sequence n_i of density 1 along which $d(T^{n_i}x, T^{n_i}x') \to 0$ (in other words, $\Phi_{x,y}(0) = 1$). This terminology goes back to Ornstein and Weiss [**OW**] (or even to an earlier work of Furstenberg). It is regrettable that such pairs are not called 'mean asymptotic'. 'Mean proximal' fits much better for pairs for which the sequence n_i has *upper* density 1 (i.e., with $\Phi^*_{x,y}(0) = 1$), which is the first condition in DC1 and DC2 scrambling. With the present terminology, we have no good name for such pairs.

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