1. Determination of Fourier Series of Discrete Time Periodic Signal from a Matrix Perspective（矩阵视角下确定离散时间周期函数的傅里叶级数），

$$\begin{pmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{pmatrix} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & \omega^{-1} & \dots & \omega^{-(N-1)} \\ \vdots & \vdots & \dots & \vdots \\ 1 & \omega^{-(N-1)} & \dots & \omega^{-(N-1)^2} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{N-1} \end{pmatrix}, \omega = e^{j\frac{2\pi}{N}}$$

2. Orthogonality of Discrete Complex Exponentials(离散复指数函数的正交性),

$$< \omega^k, \omega^r >= \sum_{n=0}^{N-1} e^{j(k-r)\frac{2\pi}{N}n} = \begin{cases} 0, & k \neq r \\ N, & k = r \end{cases}$$

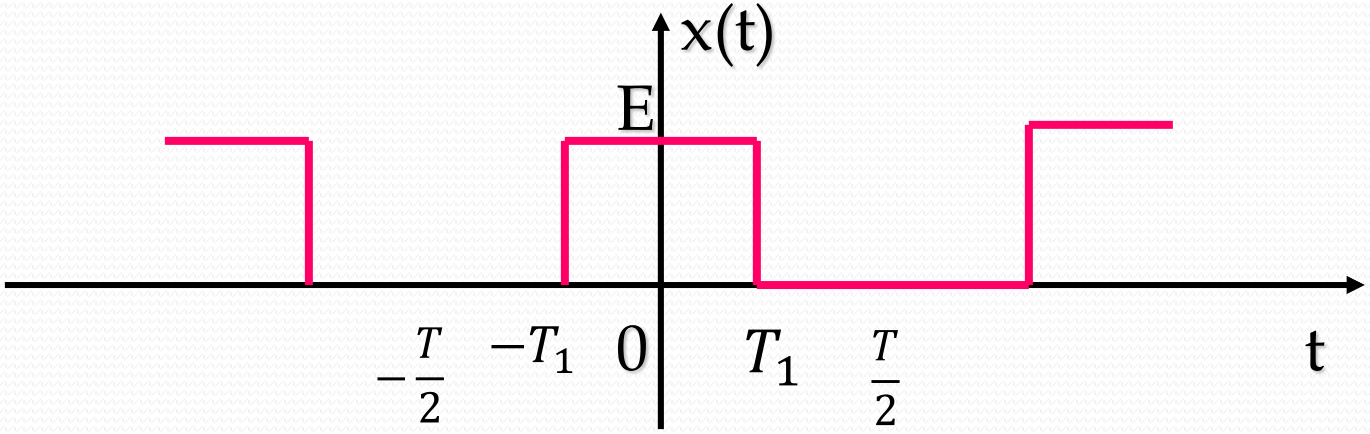
3. Fourier Series from a Rotating Unit Vector Perspective (傅里叶级数的复空间单位向量旋转视角)。
4. Filtering Concept (滤波器概念)。

Ch 4.
Fourier Transform

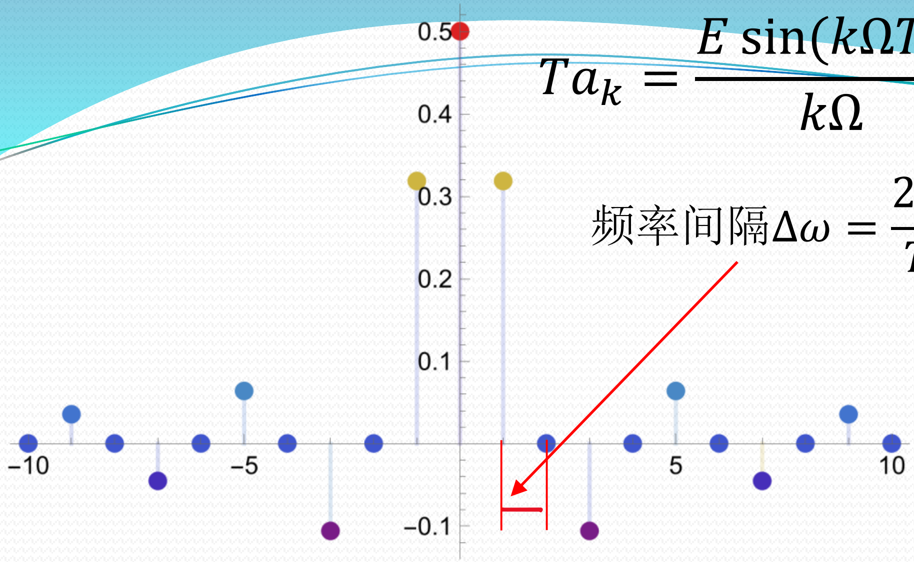
- How do we represent aperiodic（非周期的）signals in a similar way?
  - From Fourier Series to Fourier transform
  - Convergence
  - Examples
  - Properties

4.1 Representation of Aperiodic Signals:
The Continuous-Time Fourier Transform

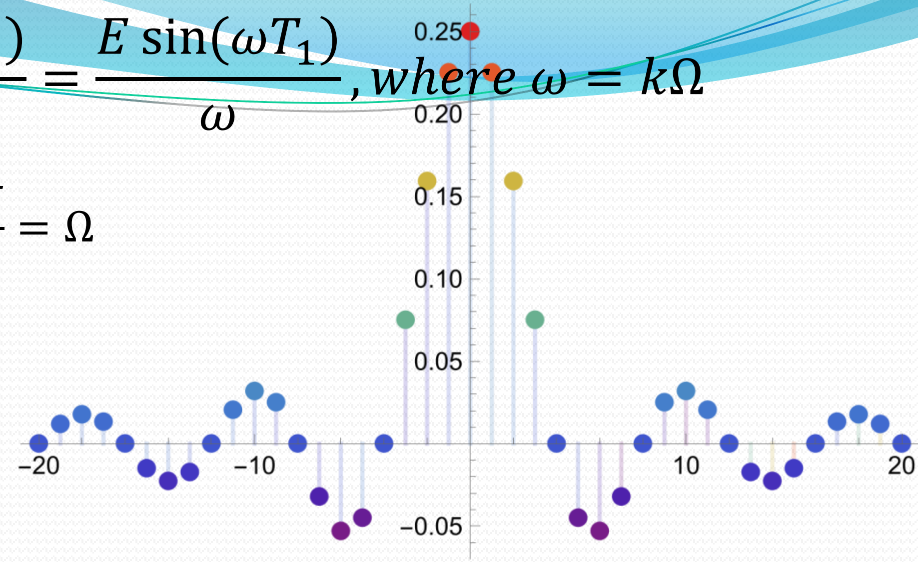
Recall the spectrum of periodic signals

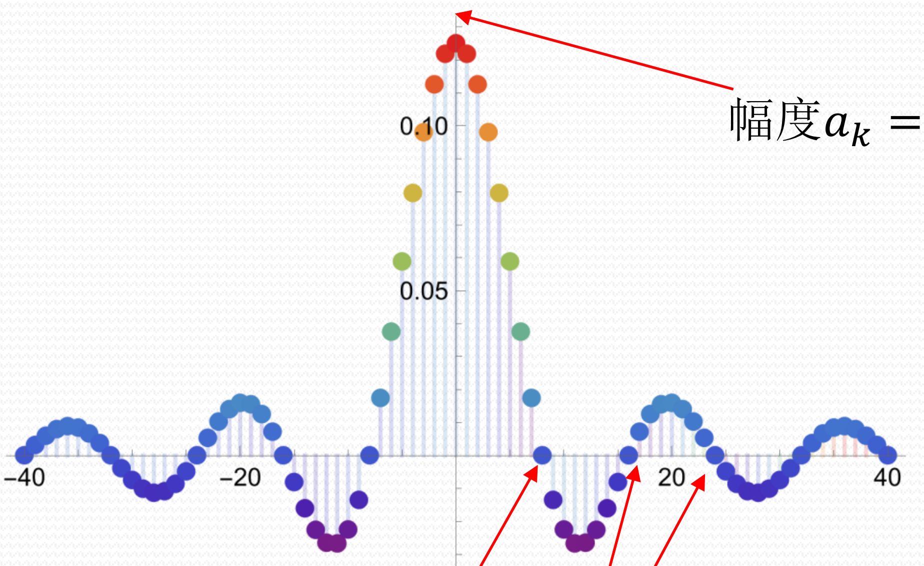


$$Ta_k = \frac{E\sin(k\Omega T_1)}{k\Omega} = \frac{E\sin(\omega T_1)}{\omega}, where\ \omega = k\Omega$$

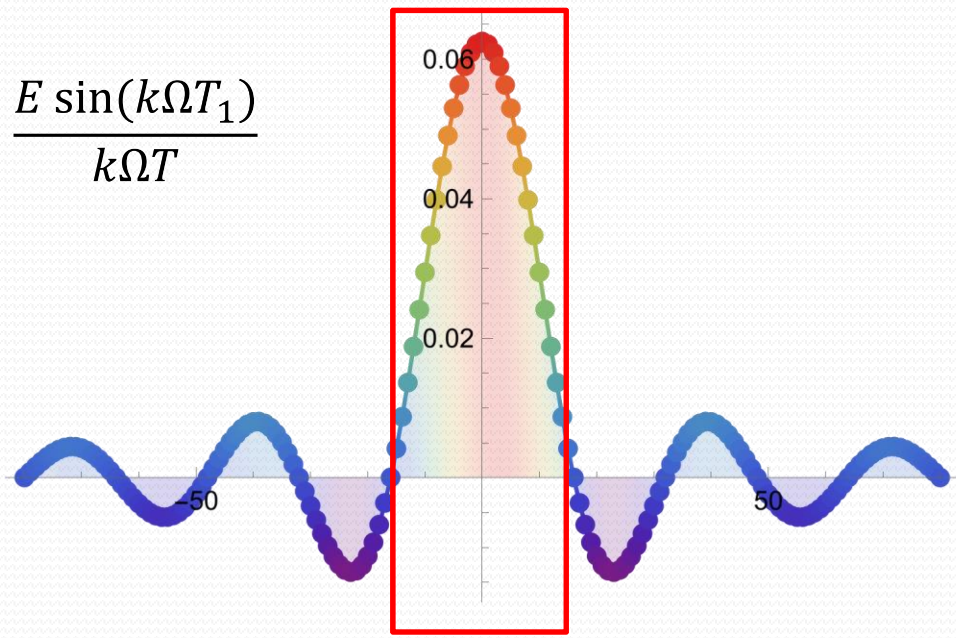


频率间隔$\Delta\omega = \frac{2\pi}{T} = \Omega$



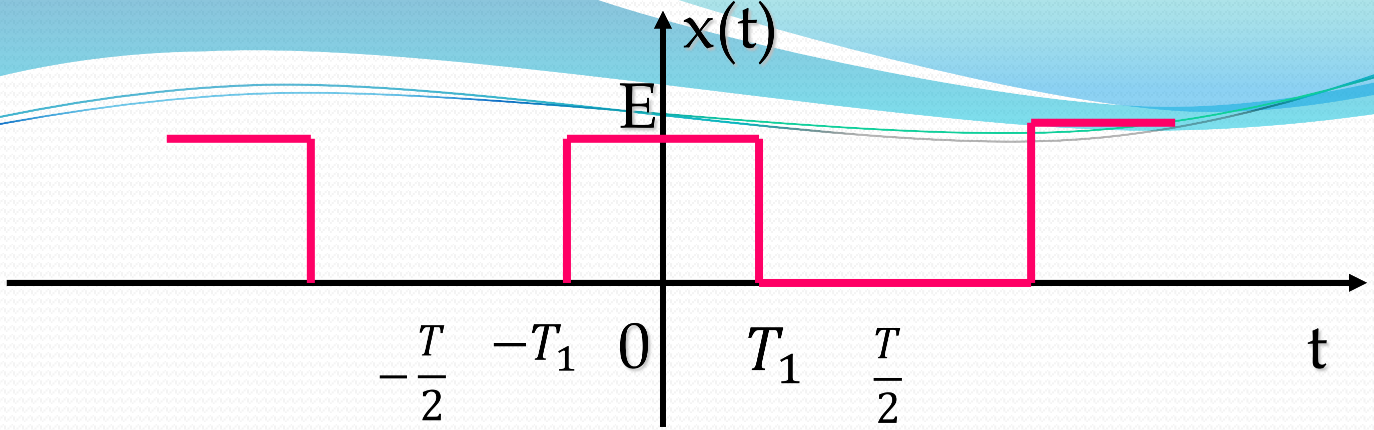


幅度$a_k = \frac{E\sin(k\Omega T_1)}{k\Omega T}$



过零点位置$\omega = \frac{m\pi}{T_1}$

带宽$B_\omega = \frac{2\pi}{T_1}$



From this example, we can see:

1.When the period, T, tends to infinite, periodic signal turns to be an aperiodic impulse signal.

$$T \to \infty$$

2.Meanwhile, the frequency turns to be a continuous independent variable.

$$\Omega = \frac{2\pi}{T} \to 0 \to d\omega \qquad n\Omega \to \omega$$

From Fourier Series to Fourier Transform

$$F_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cdot e^{-jn\Omega t}\cdot dt$$

$$f(t) = \sum_{n=-\infty}^{\infty} F_n \cdot e^{jn\Omega t}$$

As $T \to \infty$

$$F_n T = F_n \cdot \frac{2\pi}{\Omega} = \int_{-\infty}^{\infty} f(t)\cdot e^{-jn\Omega t}\cdot dt$$

$$f(t) = \sum_{n=-\infty}^{\infty} F_n T \cdot e^{jn\Omega t}\cdot \frac{1}{T}$$

Denote: $$F(j\omega) = \lim_{T\to\infty} \frac{F_n}{1/T} = \lim_{T\to\infty} F_n T$$

$$T \to \infty \quad \Omega = \frac{2\pi}{T} \to \rho \quad n\Omega \to \omega \quad \Delta(n\Omega) = \Omega \to d\omega$$

$$\sum_{n=-\infty}^{\infty} \to \int_{-\infty}^{\infty} \quad , \frac{1}{T} = \frac{\Omega}{2\pi} \to \frac{1}{2\pi} d\omega$$

Fourier Transform

$$F_n T = \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cdot e^{-jn\Omega t} \cdot dt$$

$$\Rightarrow F(j\omega) = \int_{-\infty}^{\infty} f(t) \cdot e^{-j\omega t} \cdot dt$$

Inverse Fourier Transform

$$f(t) = \sum_{n=-\infty}^{\infty} F_n T \cdot e^{jn\Omega t} \cdot \frac{1}{T} \Rightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega) e^{j\omega t} d\omega$$

$$F(j\omega)=\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt \qquad f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}d\omega$$

Fourier Transform and Inverse Fourier Transform also can be represented as the following manner

$$F(f)=\int_{-\infty}^{\infty} f(t)e^{-j2\pi ft}dt$$

$$f(t)=\int_{-\infty}^{\infty} F(f)\cdot e^{j2\pi ft}df$$

$$f=\frac{1}{T},\omega=2\pi f$$

- $F(j\omega)$ is different from F_n

- $F(j\omega)$ is defined as a density function;
- $F(j\omega)$ is a continuous function of ω (for aperiodic);
- $F(j\omega)$ contains component with frequency ranging from 0 to infinite;
- No harmonic relations among these components.

$$F(j\omega)=\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt=\left|F(j\omega)\right|e^{j\varphi(\omega)}$$

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}d\omega$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|F(j\omega)\right|e^{j\omega t+j\varphi(\omega)}d\omega$$

If *f(t)* is real, |*F(j ω)*| is an even function of ω, and the phase of *F(j ω)* is an odd function of ω.

WHY?

The sufficient condition for the existence of Fourier Transform

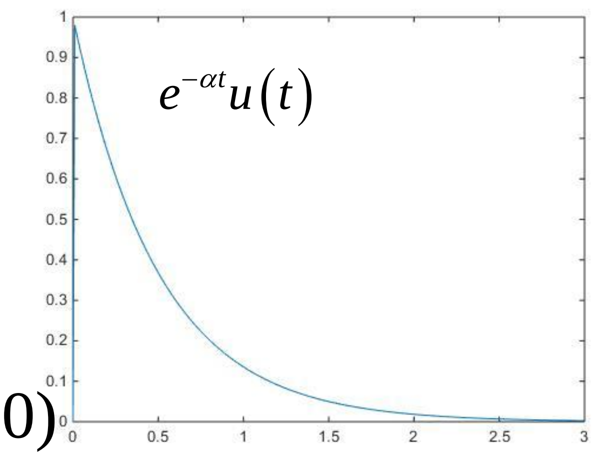
$$\int_{-\infty}^{\infty} |f(t)| dt < \infty$$

- Some singular functions, which do not satisfy the sufficient condition, have corresponding Fourier transforms if unit impulse function is applicable.
- We will interpret it in the chapter of Laplace transform.

4.2 Spectrums of Some Typical Aperiodic Signals

- One-side exponential signal
- Dual-side exponential signal
- Rectangular pulse signal
- Unit impulse function
- Direct current signal

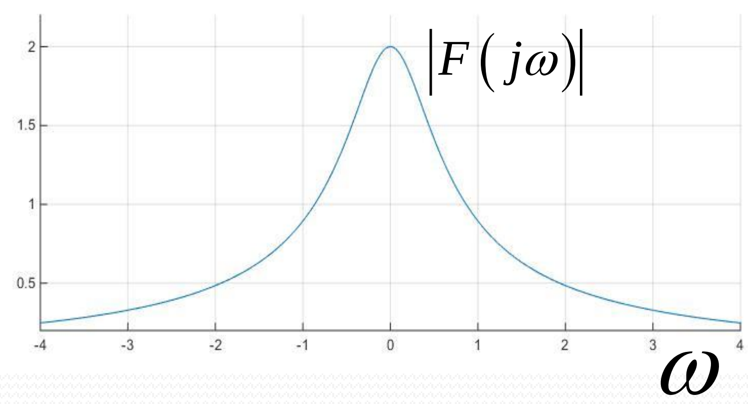
●One-Side Exponential Signal

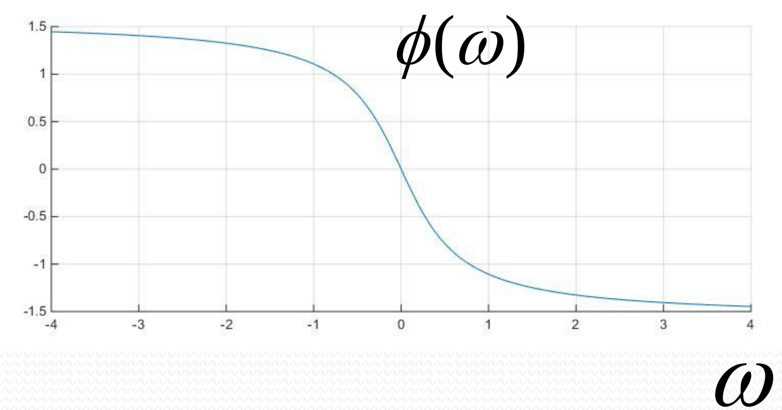


$$f(t) = \begin{cases} e^{-\alpha t} & (t \geq 0) \\ 0 & (t < 0) \end{cases}$$

$$F(j\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt = \frac{1}{\alpha + j\omega} \qquad (\alpha > 0)$$

$$|F(j\omega)| = \frac{1}{\sqrt{\alpha^2 + \omega^2}} \qquad \phi(\omega) = -\arctan(\frac{\omega}{\alpha})$$

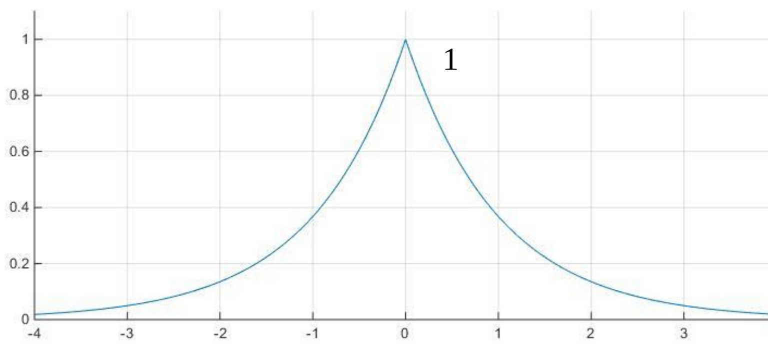




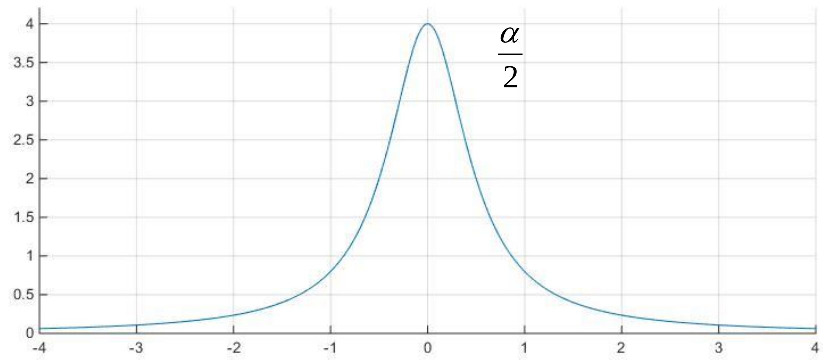
●Dual-Side Exponential Signal

$$f(t) = e^{-\alpha|t|} \qquad (-\infty < t < +\infty)$$

$$F(j\omega) = \frac{2\alpha}{\alpha^2 + \omega^2} \qquad \varphi(\omega) = 0$$



$$f(t) = e^{-\alpha|t|} \qquad (\alpha > 0)$$



$$F(j\omega) = \frac{2\alpha}{\alpha^2 + \omega^2}$$

●Rectangular pulse signal

Very Important!

$$f(t)=\begin{cases} E & (|t|\le \frac{\tau}{2}) \\ 0 & (|t|>\frac{\tau}{2}) \end{cases}$$

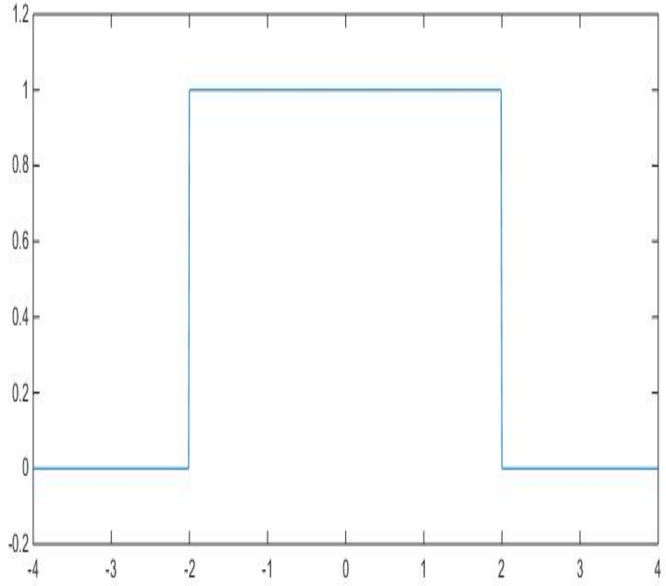
$$F(j\omega)=\int_{-\tau/2}^{\tau/2} Ee^{-j\omega t}dt=\frac{2E}{\omega}\sin(\frac{\omega\tau}{2})$$

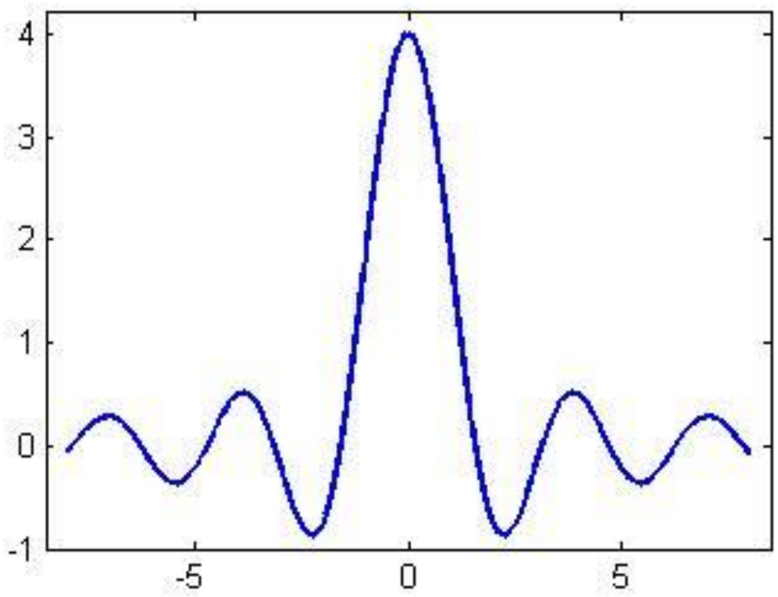
$$=E\tau\left(\frac{\sin(\frac{\omega\tau}{2})}{\frac{\omega\tau}{2}}\right)=E\tau\,Sa(\frac{\omega\tau}{2})$$

$$Sa(x)=\frac{\sin x}{x}$$

$$|F(j\omega)|=E\tau\left|Sa(\tfrac{\omega\tau}{2})\right|$$

$$\varphi(\omega)=\begin{cases} 0 & (\frac{4n\pi}{\tau}<|\omega|<\frac{2(2n+1)\pi}{\tau}) \\ \pi & (\frac{2(2n+1)\pi}{\tau}<|\omega|<\frac{4(n+1)\pi}{\tau}) \end{cases}$$





$$g(t)$$

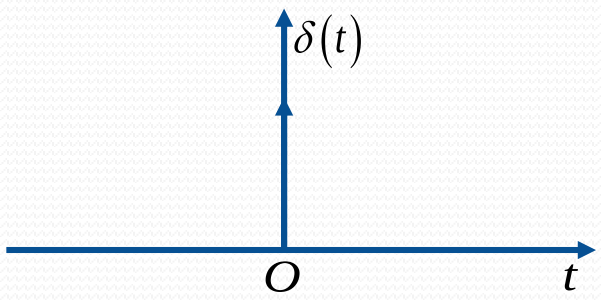
$$\tau = 4$$

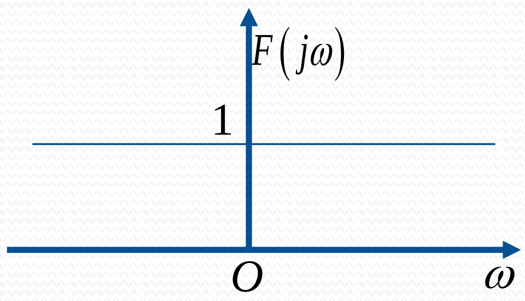
$$E = 1$$

$$G(j\omega) = E\tau Sa\left(\frac{\omega\tau}{2}\right) = 4Sa(2\omega)$$

FT of Unit Impulse Function

$$F(j\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1$$





Simple,

but important.

●Fourier Transform of Direct Current Signal

$$For \quad f(t) = e^{-\alpha|t|} \qquad (-\infty < t < +\infty)$$

$$when \quad \alpha \to 0, \; f(t) \to 1$$

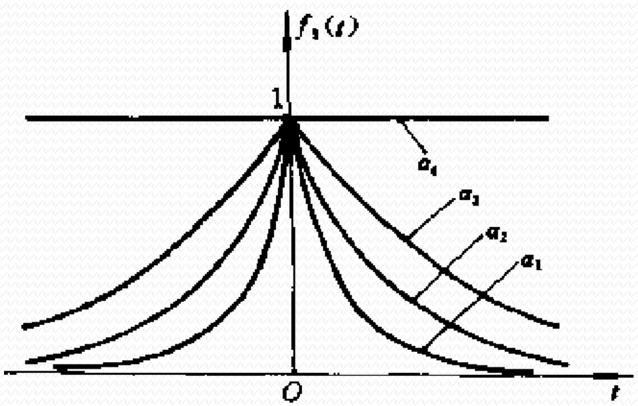
$$then \quad \lim_{\alpha \to 0} F(j\omega) = \lim_{\alpha \to 0} \frac{2\alpha}{\alpha^2 + \omega^2} = \begin{cases} 0, & \omega \neq 0 \\ \infty, & \omega = 0 \end{cases}$$

The strength this impulse has is:

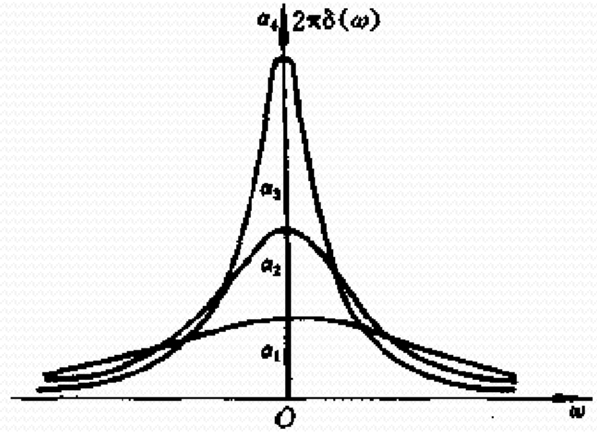
$$\lim_{\alpha \to 0} \int_{-\infty}^{\infty} \frac{2\alpha}{\alpha^2 + \omega^2} d\omega = \lim_{\alpha \to 0} \int_{-\infty}^{\infty} \frac{2}{1 + (\frac{\omega}{\alpha})^2} d(\frac{\omega}{\alpha})$$

$$= \lim_{\alpha \to 0} 2 \arctan(\frac{\omega}{\alpha}) \Big|_{-\infty}^{\infty} = 2\pi$$

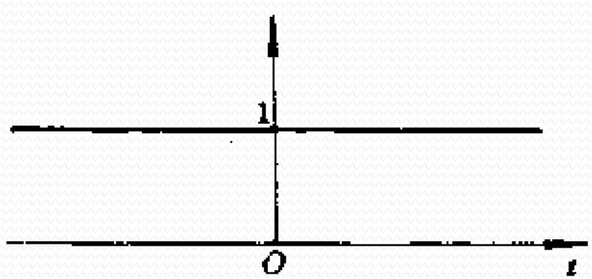
$$\therefore \lim_{\alpha \to 0} \frac{2\alpha}{\alpha^2 + \omega^2} = 2\pi\delta(\omega) \qquad \therefore F[1] = 2\pi\delta(\omega)$$



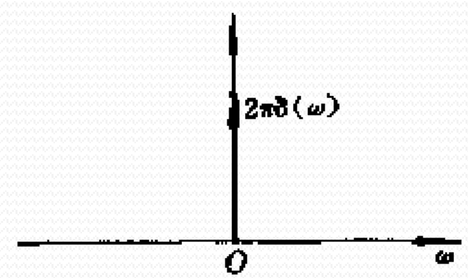
(a) $f_1(t) = e^{-\alpha_i|t|}$



(b) $F_1(j\omega) = \frac{2\alpha_i}{\alpha_i^2 + \omega^2}$



(a)



(b)

Summary

- From Fourier Series to Fourier Transform
- Spectrums of Some Typical Aperiodic Signals

Assignments

- 4.1
- 4.2