Ch3. Fourier Series Representation of Periodic Signals (2b)

Fourier Series Representation of Discrete-Time Periodic Signals

Linear Combinations of Harmonically Related Complex Exponentials (复指数谐波线性组合)

A discrete-time signal x[n] is periodic with period N if

$$x[n]=x[n+N]$$

The set of all discrete-time complex exponential signals that are periodic ***with period N*** is given by

$$\varphi_k[n] = e^{jk\omega_0 n} = e^{jk(2\pi/N)n}, k = 0, \pm 1, \pm 2, \ldots$$

Only *N* distinct signals in this set!

$$\phi_k[n] = \phi_{k+rN}[n]$$

Target is to represent periodic sequences in terms of linear combinations of the harmonically related complex exponential sequences

$$x[n]=\sum_{k=<N>}a_k\phi_k[n]=\sum_{k=<N>}a_k e^{jk\omega_0 n}=\sum_{k=<N>}a_k e^{jk(2\pi/N)n}$$

k varies over a range of N successive integers.

Above equation is referred to as **the *Discrete-time Fourier series***.

Determination of the Fourier Series Representation of a Periodic Signal

$$x[n] = \sum_{k=<N>} a_k e^{jk(2\pi/N)n}$$

Note: The sum of the values of a periodic complex exponential over its period is zero, unless that complex exponential is constant.

$$\sum_{n=<N>} e^{jk(2\pi/N)n} = \begin{cases} N, & k = 0, \pm N, \pm 2N, \ldots \\ 0, & otherwise \end{cases}$$

How to determine the Fourier series coefficients?

$$x[n] = \sum_{k=<N>} a_k e^{jk(2\pi/N)n}$$

$$\sum_{n=<N>} x[n] e^{-jr(2\pi/N)n} = \sum_{n=<N>} \sum_{k=<N>} a_k e^{j(k-r)(2\pi/N)n}$$

$$\sum_{n=<N>} x[n] e^{-jr(2\pi/N)n} = \sum_{k=<N>} a_k \sum_{n=<N>} e^{j(k-r)(2\pi/N)n}$$

$$\sum_{n=<N>} x[n] e^{-jr(2\pi/N)n} = a_r$$

Choose values for r over the same range as that over which k varies in the outer summation, the innermost sum on the right-hand side equals N if $k=r$ and 0 if k does not equal r.

$$a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk(2\pi/N)n}$$

Discrete-time Fourier Series Pair

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n} = \sum_{k=<N>} a_k e^{jk(2\pi/N)n}$$

$$a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\omega_0 n} = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk(2\pi/N)n}$$

It's easy to find

$$a_k = a_{k+N}$$

Properties of Discrete-Time Fourier Series

Multiplication

$$x[n] \overset{FS}{\leftrightarrow} a_k$$

$$y[n] \overset{FS}{\leftrightarrow} b_k$$

$$x[n]y[n] \overset{FS}{\leftrightarrow} d_k = \sum_{l=<N>} a_l b_{k-l}$$

First Difference （一阶差分）

$$x[n] \overset{FS}{\longleftrightarrow} a_k$$

$$x[n] - x[n-1] \overset{FS}{\longleftrightarrow} (1 - e^{-jk(2\pi/N)})a_k$$

Parseval's Relation for Discrete-Time Periodic Signals

$$\frac{1}{N}\sum_{n=\langle N\rangle} |x[n]|^2 = \sum_{k=\langle N\rangle} |a_k|^2$$

$|a_k|^2$ is the average power in the k^{th} harmonic component of $x[n]$.

Fourier Series and LTI Systems

- Fourier series representation can be used to construct any periodic signal in discrete time and essentially all periodic continuous-time signals of practical importance.
- The response of an LTI system to a linear combination of complex exponentials takes a particularly simple form.

$$x(t) = e^{st}$$

$$H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau$$

$$y(t) = x(t) * h(t) = e^{st} H(s)$$

$$x[n] = z^{n}$$

$$H(z) = \sum_{k=-\infty}^{\infty} h(k) z^{-k}$$

$$y[n] = x[n] * h[n] = z^{n} H(z)$$

- When **s** or ***z*** are general complex numbers, $H(s)$ and $H(z)$ are referred to as the ***system functions* （系统函数）** of the corresponding systems.
- In next chapter, we focus on the case in which $Re\{s\} = 0$

$$\mathrm{Re}\{s\} = 0$$

$$s = j\omega, e^{st} = e^{j\omega}$$

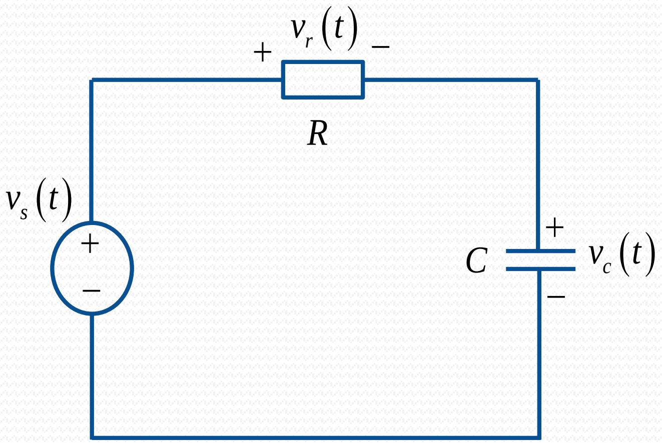
$$H(s) = H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt$$

$H(j\omega)$ viewed as a function of ω is referred as to the ***frequency response* （频率响应）** of the system.

Similar concepts can be derived in the same way for discrete time signals.

Example: A simple RC Low-pass Filter

$$RC\frac{dv_c(t)}{dt}+v_c(t)=v_s(t)$$



$$v_s(t)=e^{j\omega t} \qquad v_c(t)=e^{j\omega t}H\left(j\omega\right)$$

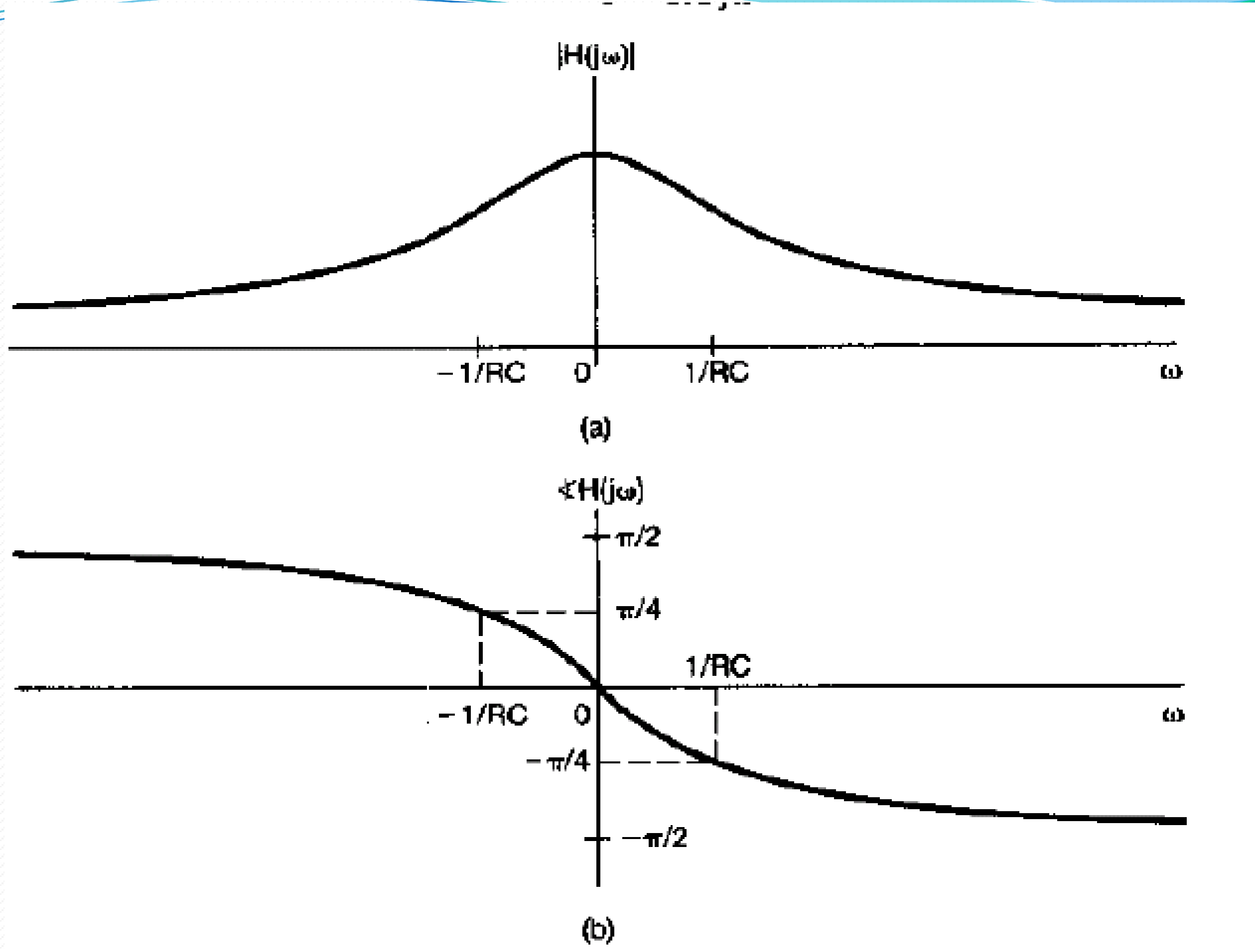
$$\therefore RC\frac{d}{dt}[H(j\omega)e^{j\omega t}]+H(j\omega)e^{j\omega t}=e^{j\omega t}$$

$v_s(t)$: input

$v_c(t)$: output

$$H(j\omega)e^{j\omega t}=\frac{1}{1+RCj\omega}e^{j\omega t}$$

$$H(j\omega)=\frac{1}{1+RCj\omega}$$



(a) Magnitude and (b) phase plots for the frequency response for the RC circuit in the previous page.

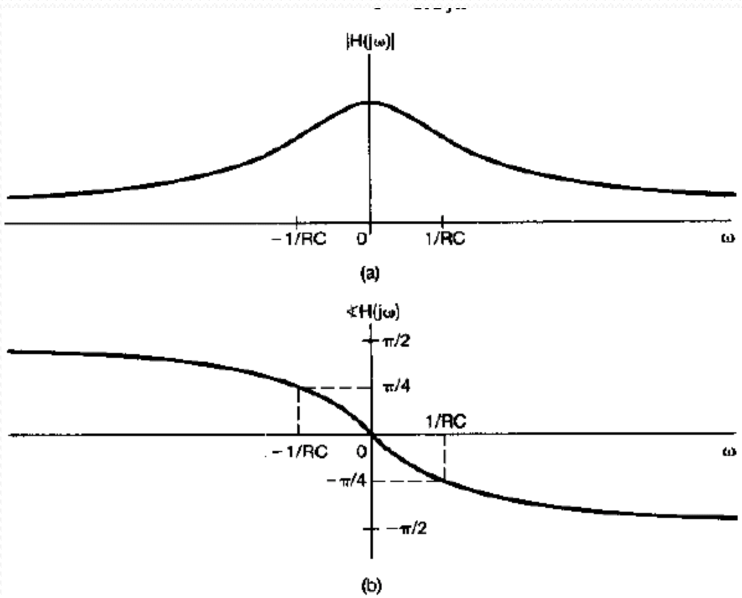
A preliminary discuss on the trade-off in filter design

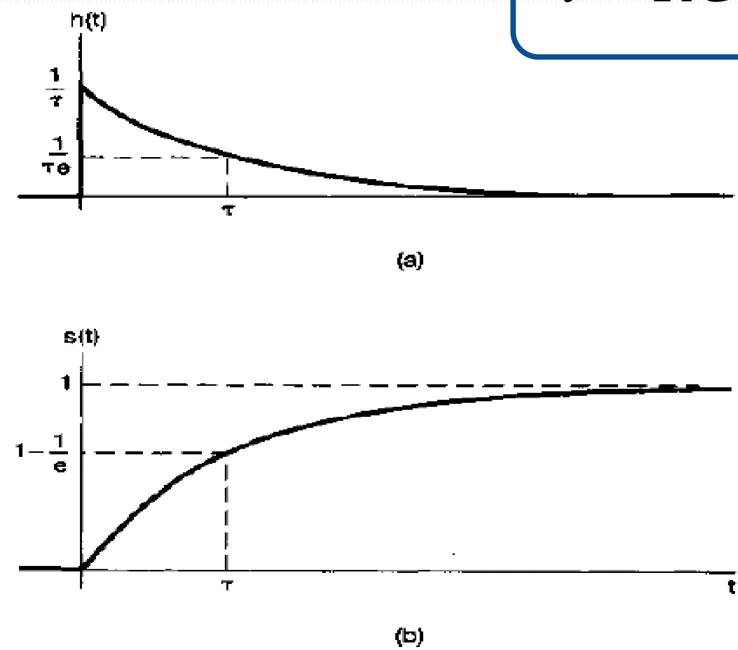
$$H(j\omega) = \frac{1}{1 + RCj\omega}$$

Impulse response: $h(t) = \frac{1}{RC} e^{-\frac{t}{RC}} u(t)$

Step response: $s(t) = [1 - e^{-\frac{t}{RC}}]u(t) = [1 - e^{-\frac{t}{\tau}}]u(t)$

$$\boxed{\tau = RC}$$



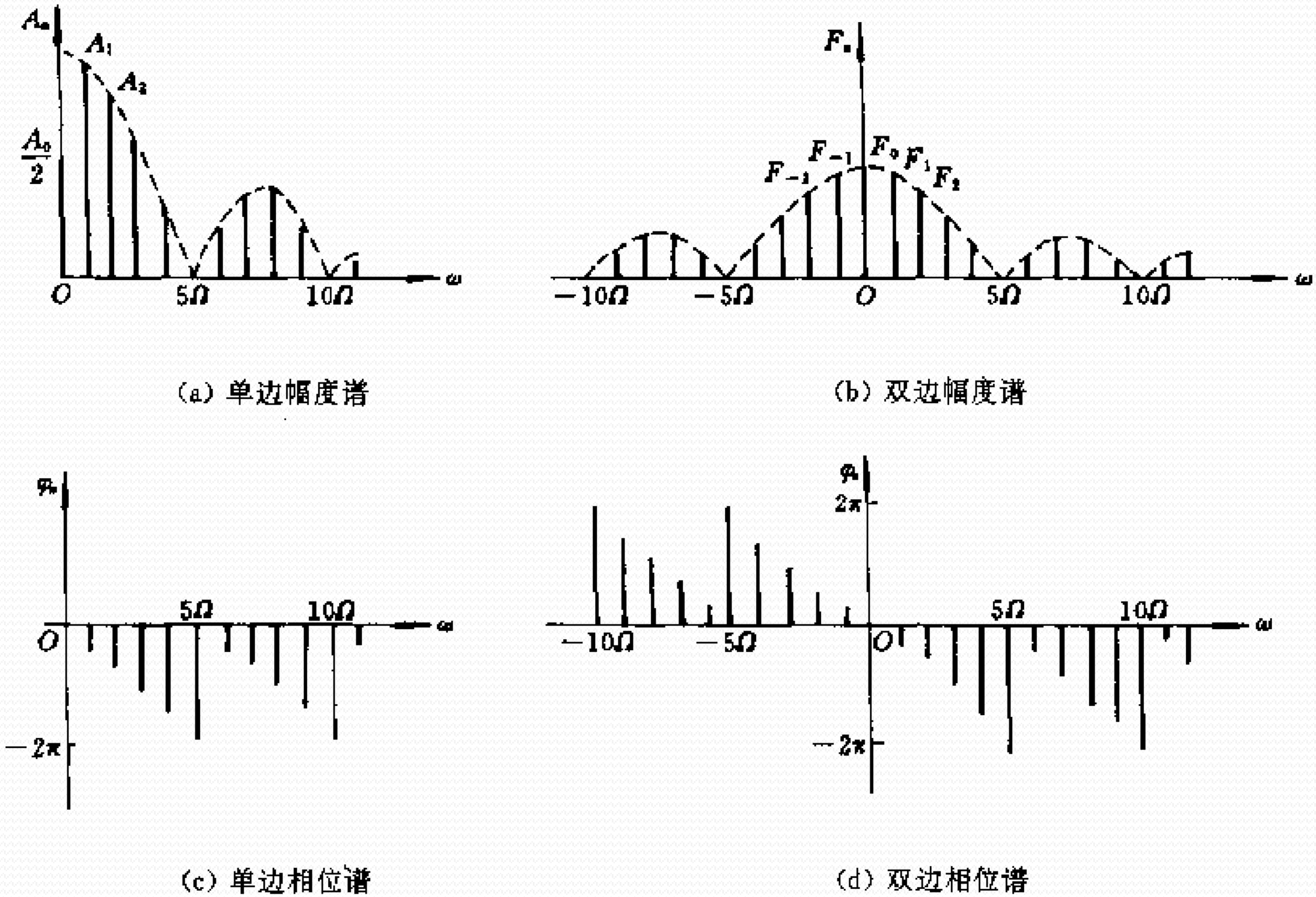


The Spectrum of Periodic Signals

The Spectrum of Periodic Signals (横轴用频率)

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\Omega t}$$

$$f(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(n\Omega t + \phi_n)$$



(a) 单边幅度谱

(b) 双边幅度谱

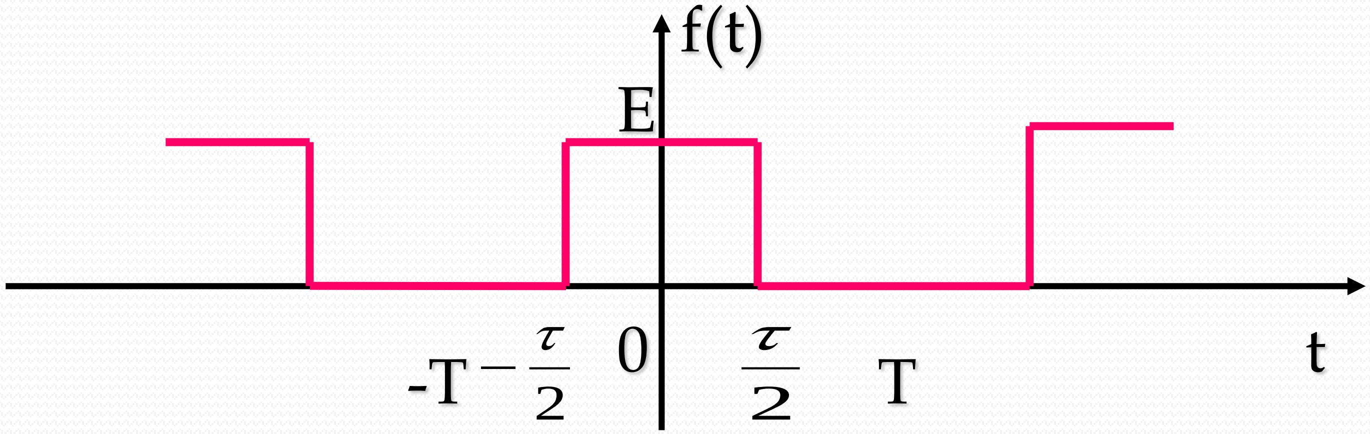
(c) 单边相位谱

(d) 双边相位谱

The Spectrum of Periodic Signals (横轴用频率)

- The spectrum of periodic rectangle signal

$$f(t)=\begin{cases} E & (mT-\frac{\tau}{2}\le t\le mT+\frac{\tau}{2}) \\ 0 & (otherwise) \end{cases}$$



$$f(t)=\sum_{n=-\infty}^{\infty}F_n e^{jn\Omega t}$$

$$F_n=\frac{1}{T}\int_{-\infty}^{\infty}Ee^{-jn\Omega t}dt=\frac{1}{T}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}}Ee^{-jn\Omega t}dt$$

$$=\frac{E}{T(-jn\Omega)}(e^{-jn\Omega\tau/2}-e^{jn\Omega\tau/2})$$

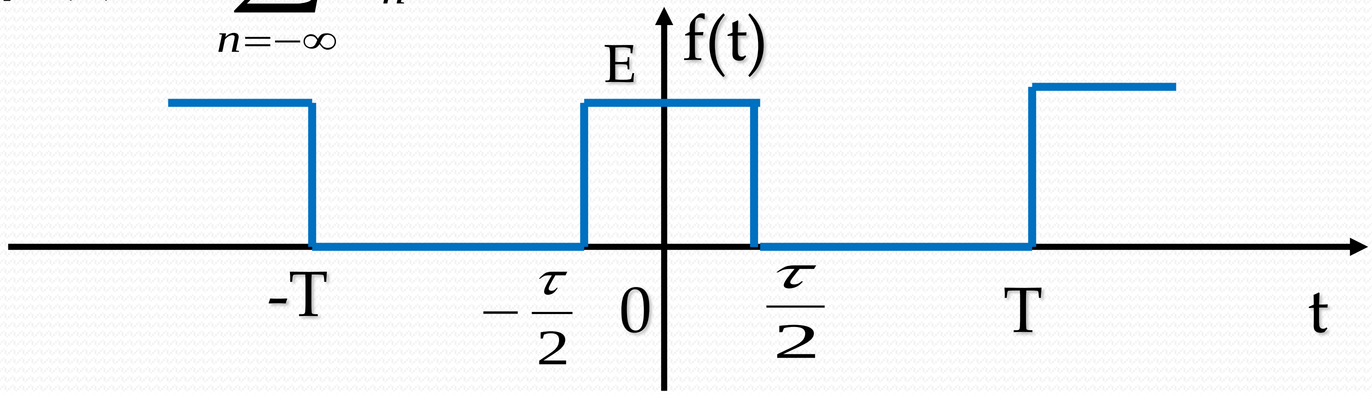
$$=\frac{E\tau}{T}\frac{\sin(\frac{n\Omega\tau}{2})}{\left(\frac{n\Omega\tau}{2}\right)}=\frac{E\tau}{T}Sa(\frac{n\Omega\tau}{2})=\frac{E\tau}{T}Sa(\frac{n\pi\tau}{T})$$

$$n=0,\pm1,\pm2,\cdots$$

$$Sa(x)\overset{def}{=}\frac{\sin x}{x}$$

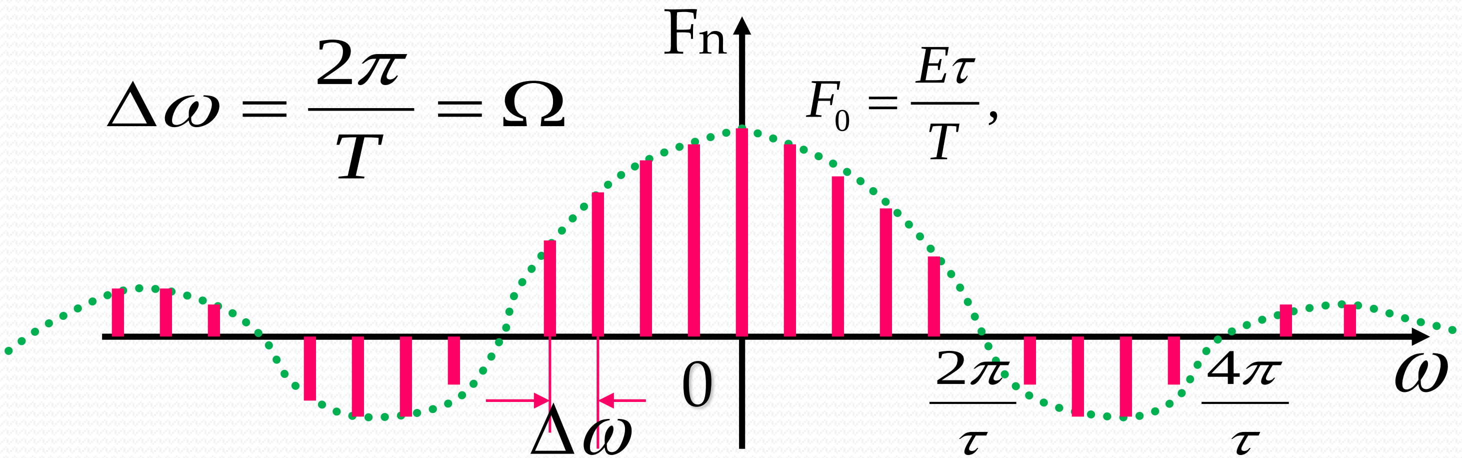
$$Sa(\frac{n\pi\tau}{T})$$

$$f(t)=\sum_{n=-\infty}^{\infty}F_n e^{jn\Omega t}$$



$$\Delta\omega=\frac{2\pi}{T}=\Omega$$

$$F_0=\frac{E\tau}{T},$$

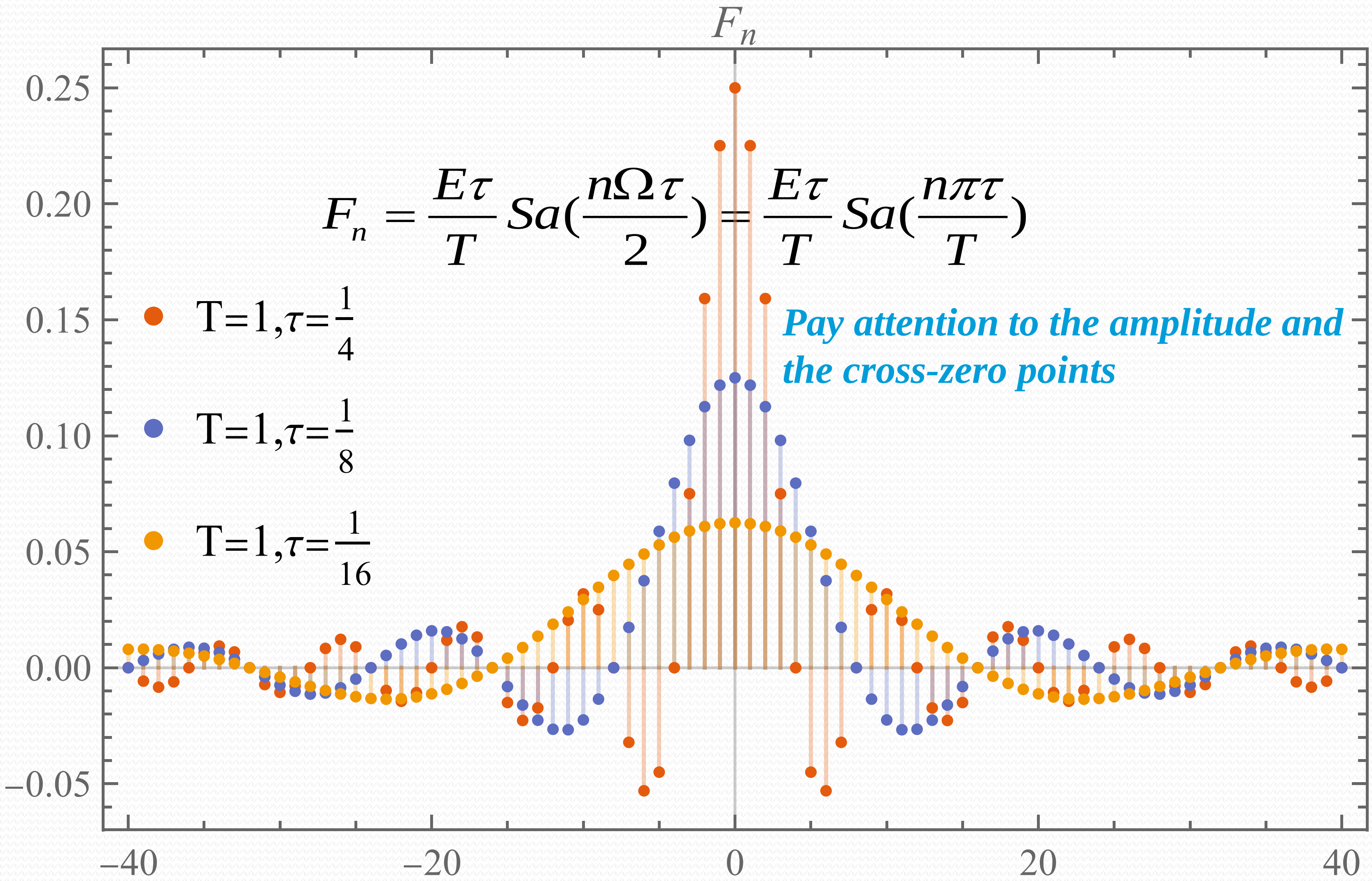


$$F_n=\frac{E\tau}{T}Sa(\frac{n\Omega\tau}{2})=\frac{E\tau}{T}Sa(\frac{n\pi\tau}{T})$$

The Spectrum of Periodic Signals (横轴用频率)

Spectrum features of periodic rectangle wave

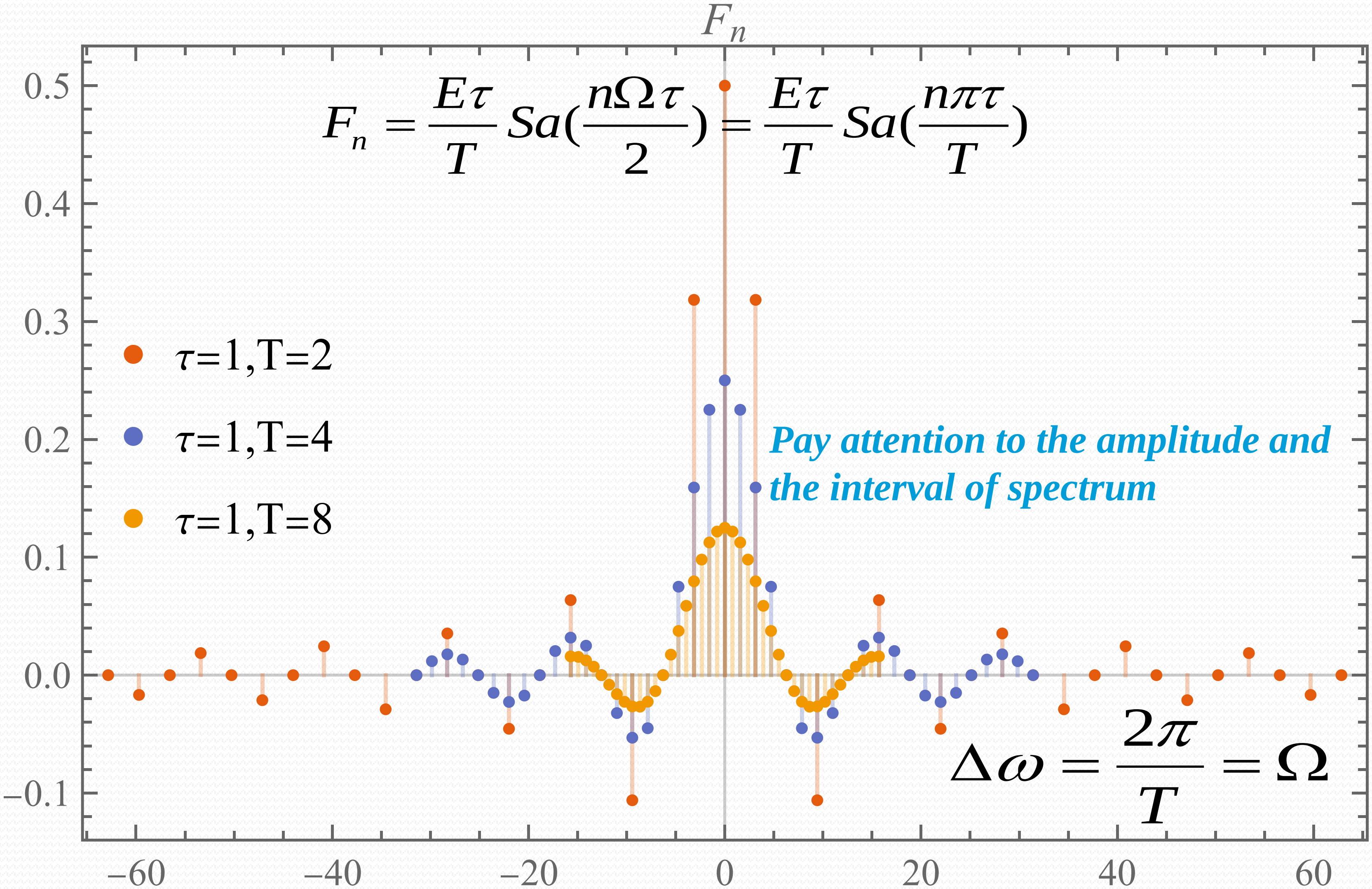
1. period unchanged，width of pulse changes



$$F_n = \frac{E\tau}{T} Sa\left(\frac{n\Omega\tau}{2}\right) = \frac{E\tau}{T} Sa\left(\frac{n\pi\tau}{T}\right)$$

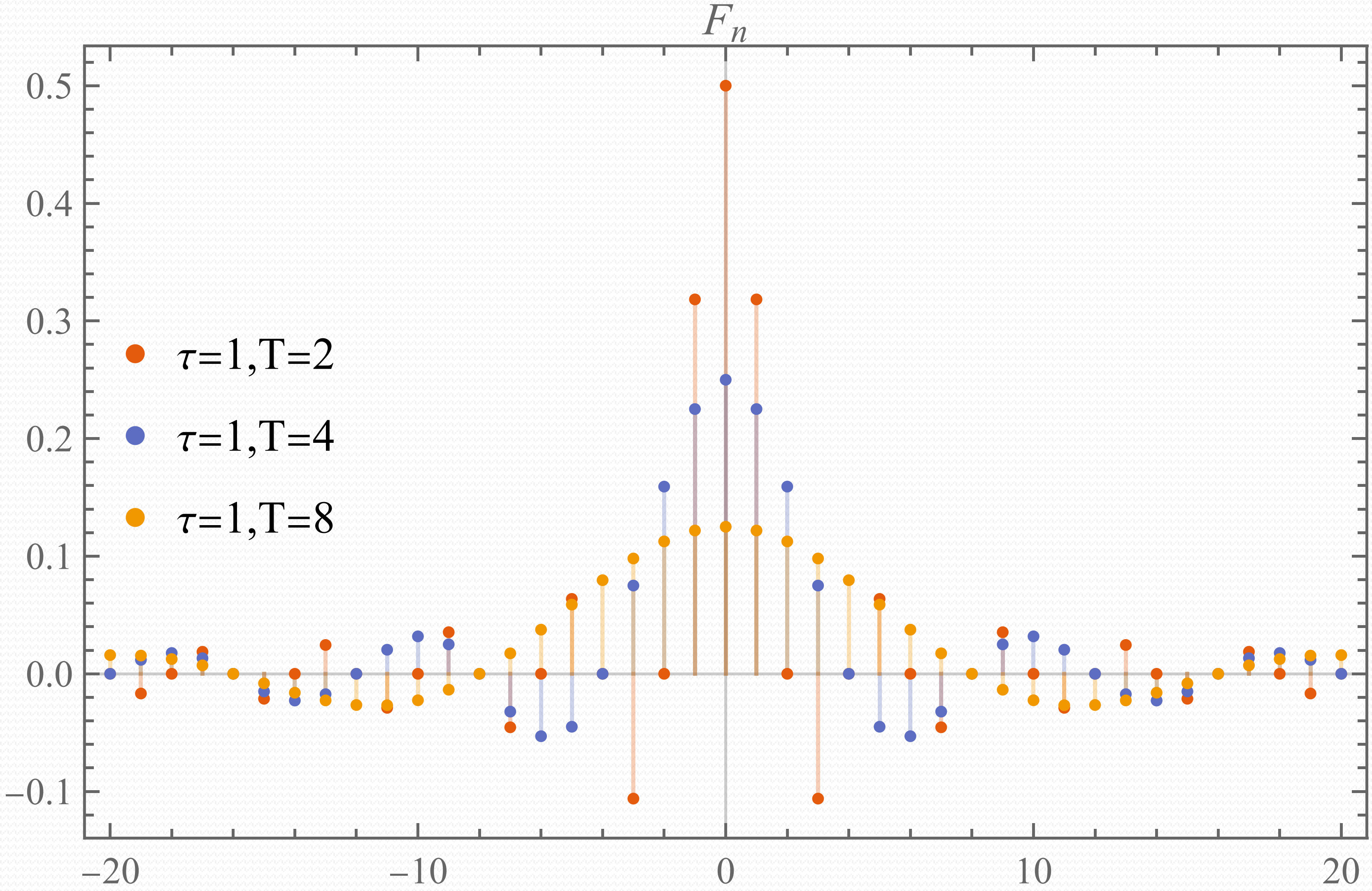
Spectrum features of periodic rectangle wave

2，pulse width unchanged，period changes

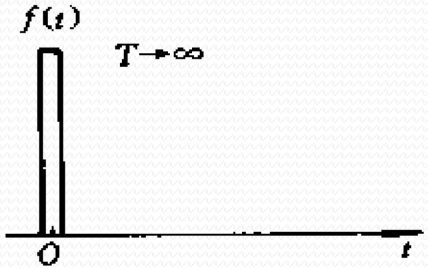


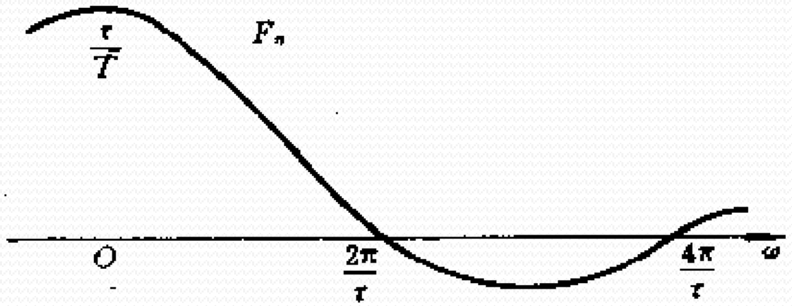
Spectrum features of periodic rectangle wave （若横轴为1）

2，pulse width unchanged，period changes



What will happen if the period approaches to infinite while keeping the width of pulse unchanged?





When the signal's period approaches to infinite, the spectrum of periodic signal turns to be continuous spectrum.

Based on the spectrum analysis for rectangle wave

- The spectrum is in a discrete form.
  - Spectral components only exist on the points which are multiples of the basic frequency.
  - Interval between spectral components is the fundamental frequency.
  - The larger the period, the smaller the spectral interval

$$\Delta\omega = \frac{2\pi}{T} = \Omega$$

- Amplitude of spectral component is proportioned to magnitude and width of impulse, while inversely proportioned to the period.

$$F_n = \frac{E\tau}{T} Sa(\frac{n\pi\tau}{T})$$

- The amplitude of each spectral component varies according to the envelope of $Sa(\frac{n\pi\tau}{T})$

- The cross-zero points are located on $\omega = \frac{2m\pi}{\tau}$

- Most energy distributed with the first cross-zero point. The width of main band is

$$B_\omega = \frac{2\pi}{\tau}$$

Summary

- Discrete time Fourier series
- Fourier series and LTI system
- Spectrum of periodic rectangular wave

Assignments (part2)

- 3.31
- 3.34(b)(c)
- 3.46
  - (a)
  - (b): Only the computation on $x_1(t)$ is required
  - (c)
- 3.61(a)
- 3.64
- 3.65(d)
- 用Python/Wolfram(Mathematica)/Matlab等语言编写计算以及绘制周期连续信号傅里叶级数幅相的代码（可定义截断项，使用内置傅里叶级数函数不算分），重新计算3.22(b)