Ch2 Linear Time-Invariant Systems
线性时不变系统

回顾

- 阶跃信号与脉冲信号的两个积分视角
- 系统定义和类型
- 系统表示方法
- 系统分类
- 线性
- 时不变性
- 因果性，记忆性
- 稳定性
- 可逆性

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- **Discrete-Time LTI Systems**
  - The representation of discrete-time signals in terms of impulse
  - The discrete-time unit impulse response and the convolution sum representation of LTI systems
- **离散时间 LTI 系统**
  - 使用脉冲表示离散时间信号
  - 离散时间单位脉冲响应及使用卷积和表示 LTI 系统

Discrete-Time LTI Systems: The Convolution Sum
离散时间 LTI 系统：卷积和表示

- The representation of discrete-time signals in terms of impulse
  - According to the sampling property of impulse signal, discrete-time signals can be represented in terms of
    利用脉冲信号的采样性，离散时间信号可表示为

$$x[n]=\sum_{k=-\infty}^{\infty}x[k]\delta[n-k]$$

Discrete-Time LTI Systems: The Convolution Sum

- The representation of discrete-time signals in terms of impulse
  - According to the sampling property of impulse signal, discrete-time signals can be represented in terms of

$$x[n]=\sum_{k=-\infty}^{\infty}x[k]\delta[n-k]$$

weight
对应权重

Location
脉冲位置

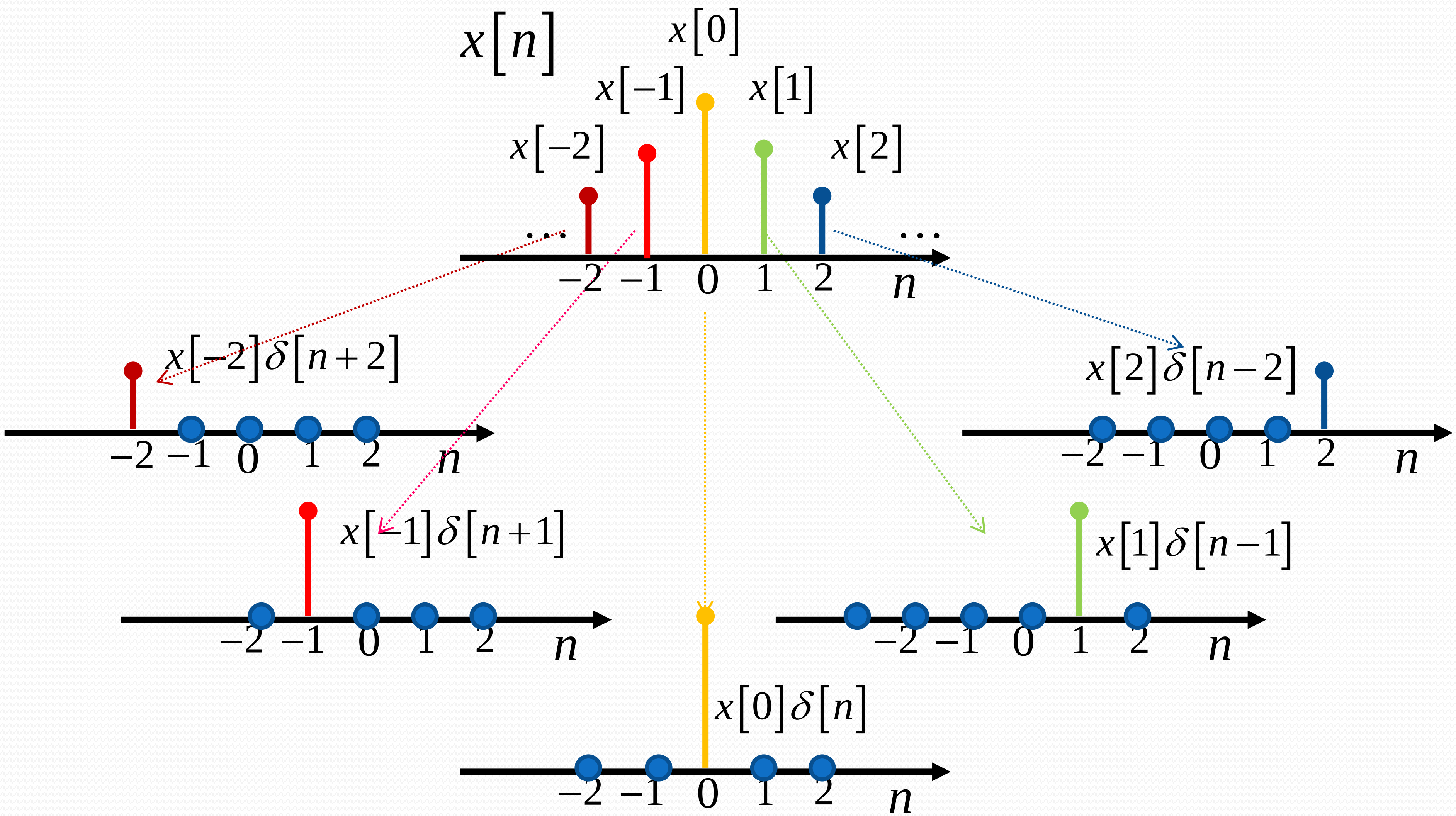
  - Regard a discrete-time signal as a sequence of individual impulses.
    将离散时间信号视为一个个脉冲的序列

Do you remember the property of unit impulse ?

$$x[n]\delta[n-k]=x[k]\delta[n-k]$$

Any discrete-time signal can be represented as the sum of several weighted impulse.

任意离散时间信号都能表示为若干加权脉冲的叠加



$$x[n] = \cdots + x[-2]\delta[n+2] + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + \ldots$$

$$x[n] = \cdots + x[-2]\delta[n+2] + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + \ldots$$

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

$$\Downarrow$$

$$x[n] * \delta[n] \overset{def}{=} \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

The Discrete-Time unit Impulse Response and the Convolution-sum Representation of LTI Systems

- The ***sifting property*** **（*sift v.筛选*）**
  - it represents x[n] as ***a superposition* （*叠加*）*of scaled versions*** of a very simple set of elementary functions, shifted unit impulses.

    将序列 x[n] 表示为单位脉冲平移加权后的叠加
- The response of a linear system to x[n]
  - ***the superposition of the scaled responses*** of the system to each of these shifted impulses.

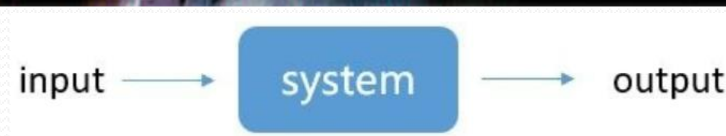
    线性系统对激励 x[n] 的响应可表示为其对各平移后的单位脉冲的响应的加权叠加

- From the property of time invariance
  - the responses of a time-invariant system to the time-shifted unit impulses are simply ***time-shifted versions*** of one another.

    根据时不变性，时不变系统对时移后的单位脉冲的响应是对原单位脉冲的响应的时移
- The convolution-sum（卷积和） representation for a LTI discrete-time system results from putting the above two basic facts together.

  LTI 系统的卷积和表示需要结合以上两点理解

Convolution, what?



Denote $h_k[n]$

the response of the linear system to the shifted unit impulse $\delta[n-k]$

$h_k[n]$ 表示线性系统对平移后的单位脉冲 $\delta[n-k]$ 的响应

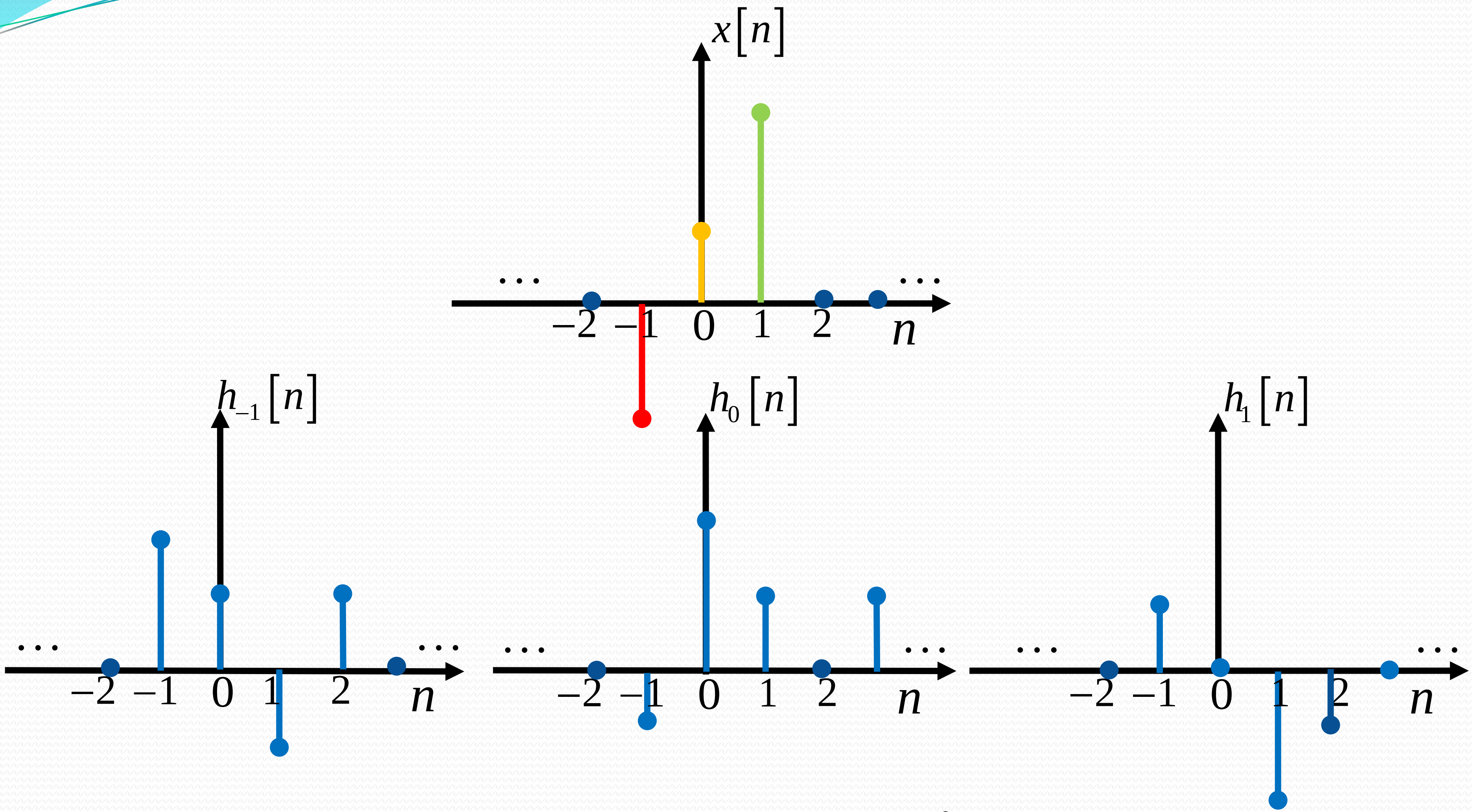
Consider an input:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

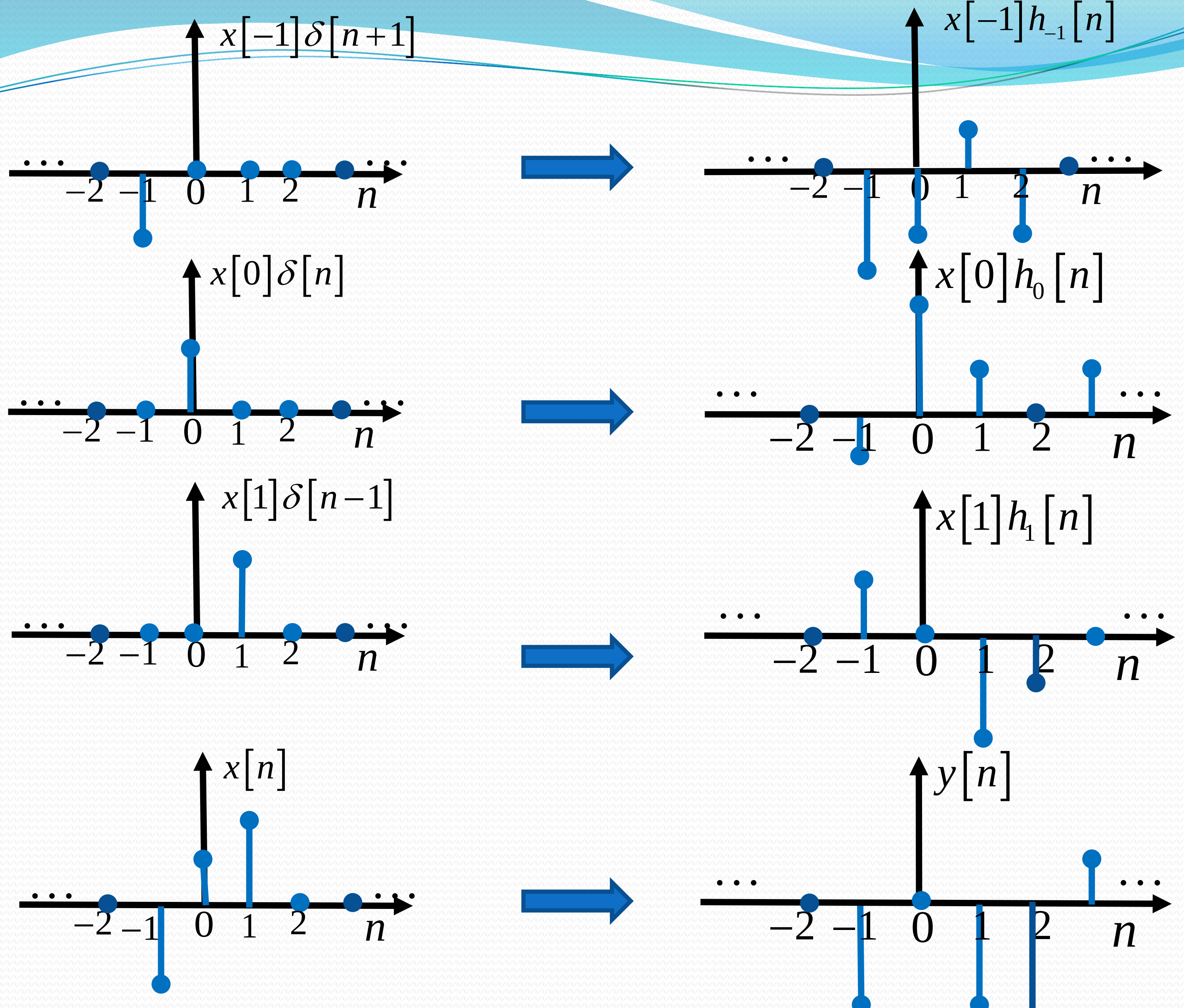
The response of the above system to this input can be expressed as

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h_k[n]$$

It is correct for either time-variant or time-invariant system.
该结论与系统是否具有时不变性无关



Explanation for $y[n] = \sum_{k=-\infty}^{\infty} x[k]h_k[n]$



Continue

If we restrict our consideration onto TIME-INVARIANT systems

我们仅讨论**时不变**系统

$$h_k[n] = h_0[n-k]$$

DO YOU REMEMBER THE TIME INVARIANCE PROPERTY INTRODUCED IN CHAPTER ONE?
时不变性

$$e(t) \longrightarrow r(t)$$

$$e(t-t_0) \longrightarrow r(t-t_0)$$

What would happen if the system is a Linear Time Invariant one?

Denote the input to an LTI system as

输入 $$x[n]=\sum_{k=-\infty}^{\infty}x[k]\delta[n-k]$$

Define the unit impulse response

单位脉冲响应 $$h[n]=h_0[n]$$

The response of the LTI system to the shifted unit impulse $\delta[n-k]$

平移单位脉冲响应 $$h_k[n]=h_0[n-k]$$

The response of this LTI system to the input can be expressed as

$$\boxed{y[n]=\sum_{k=-\infty}^{\infty}x[k]h_k[n]} \Rightarrow \boxed{y[n]=\sum_{k=-\infty}^{\infty}x[k]h[n-k]}$$

THIS IS REFERRED TO AS THE ***CONVOLUTION SUM*** OR SUPERPOSITION SUM. （卷积和）

$$y[n]=\sum_{k=-\infty}^{\infty}x[k]h[n-k]$$

Usually be represented symbolically as
卷积和的符号表示

$$y[n]=x[n]*h[n]$$

Now, let’s call back the definition of LTI system …

Time Invariant 时不变性

$$x_1[n] \longrightarrow \boxed{\text{SYSTEM}} \longrightarrow y_1[n]$$

$$x_1[n-n_0] \longrightarrow \boxed{\text{SYSTEM}} \longrightarrow y_1[n-n_0]$$

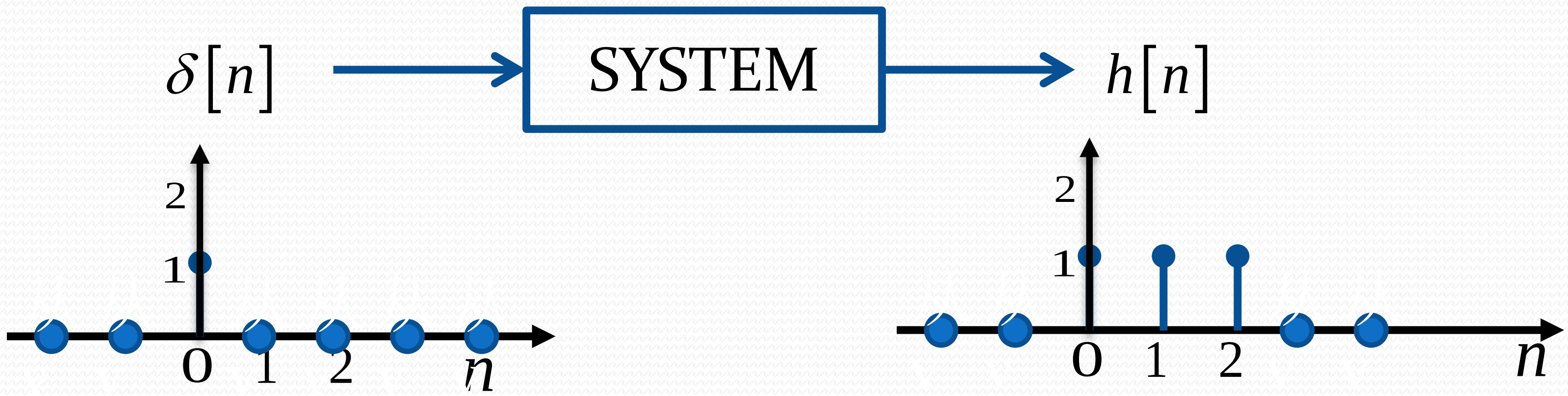
Linear 线性

$$x_1[n] \longrightarrow \boxed{\text{SYSTEM}} \longrightarrow y_1[n]$$

$$x_2[n] \longrightarrow \boxed{\text{SYSTEM}} \longrightarrow y_2[n]$$

$$ax_1[n]+bx_2[n] \longrightarrow \boxed{\text{SYSTEM}} \longrightarrow ay_1[n]+by_2[n]$$

Unit Impulse Response 单位脉冲响应



Unit impulse response
单位脉冲响应

$$\delta[n] \longrightarrow \boxed{\text{SYSTEM}} \longrightarrow h[n]$$

Unit impulse response 单位脉冲响应

$$\delta[n] \longrightarrow \boxed{\text{SYSTEM}} \longrightarrow h[n]$$

Time invariant 时不变

$$\delta[n-k] \longrightarrow \boxed{\text{SYSTEM}} \longrightarrow h[n-k]$$

Unit impulse response 单位脉冲响应

$$\delta[n] \longrightarrow \boxed{\text{SYSTEM}} \longrightarrow h[n]$$

Time invariant 时不变

$$\delta[n-k] \longrightarrow \boxed{\text{SYSTEM}} \longrightarrow h[n-k]$$

Homogeneity 齐次性

$$x[k]\delta[n-k] \longrightarrow \boxed{\text{SYSTEM}} \longrightarrow x[k]h[n-k]$$

Unit impulse response
单位脉冲响应

$$\delta[n] \longrightarrow \boxed{\text{SYSTEM}} \longrightarrow h[n]$$

Time invariant 时不变

$$\delta[n-k] \longrightarrow \boxed{\text{SYSTEM}} \longrightarrow h[n-k]$$

Homogeneity 齐次性

$$x[k]\delta[n-k] \longrightarrow \boxed{\text{SYSTEM}} \longrightarrow x[k]h[n-k]$$

Additivity 可加性

$$\sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \longrightarrow \boxed{\text{SYSTEM}} \longrightarrow \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Unit impulse response 单位脉冲响应

$\delta[n] \rightarrow$ SYSTEM $\rightarrow h[n]$

Time invariant 时不变

$\delta[n-k] \rightarrow$ SYSTEM $\rightarrow h[n-k]$

Homogeneity 齐次性

$x[k]\delta[n-k] \rightarrow$ SYSTEM $\rightarrow x[k]h[n-k]$

Additivity 可加性

$\underbrace{\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]}_{x[n]} \rightarrow$ SYSTEM $\rightarrow \underbrace{\sum_{k=-\infty}^{\infty} x[k]h[n-k]}_{y[n]}$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

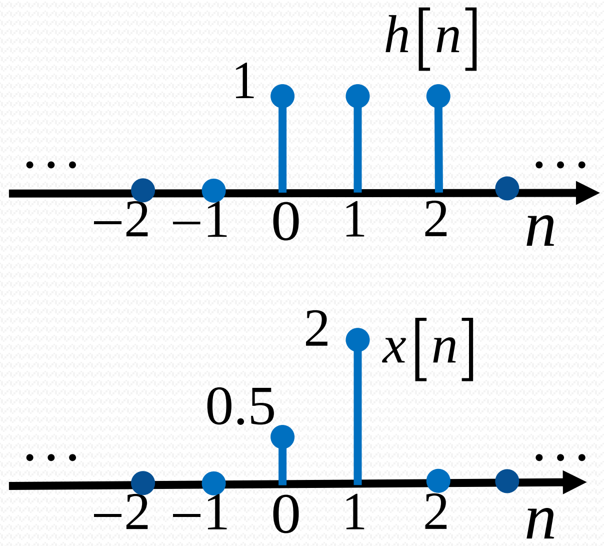
Remember, it is correct for the case when the system is an LTI one.

谨记， 该结论对 LTI 系统成立

WHY?

Example

Consider an LTI system with impulse response h[n] and input x[n], as illustrated in below. What will the response of this system be with input x[n]?



What does h[n] mean?

$$\delta[n] \to h[n]$$

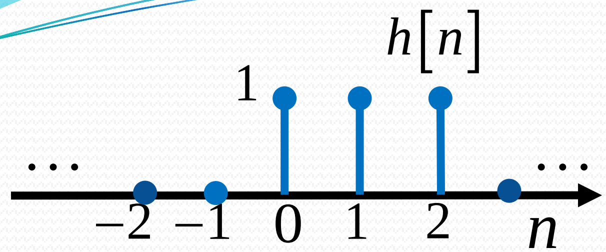
$$\delta[n-1] \to h[n-1]$$

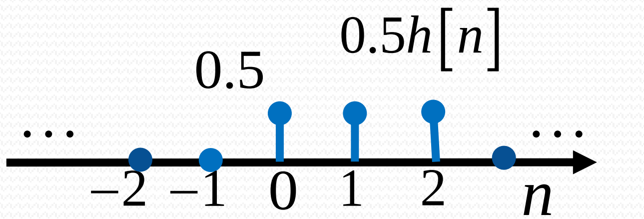
$$\vdots$$

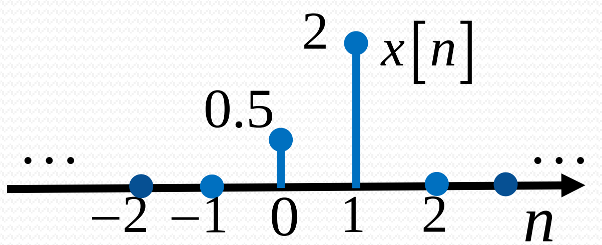
What does x[n] tell us?

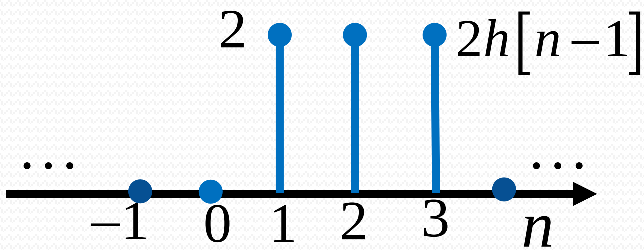
$$x[n] = x[0]\delta[n-0] + x[1]\delta[n-1]$$

$$= 0.5\delta[n-0] + 2\delta[n-1]$$

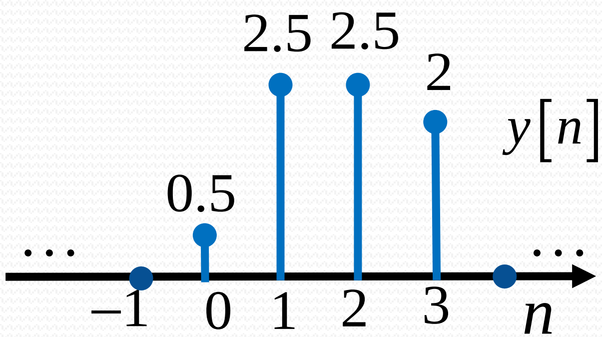








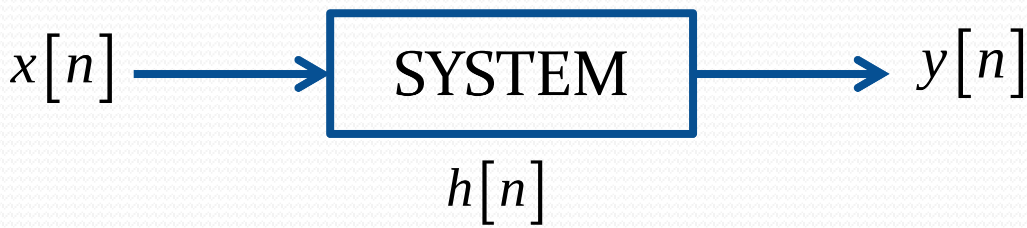
$$x[n] = x[0]\delta[n-0] + x[1]\delta[n-1]$$
$$= 0.5\delta[n-0] + 2\delta[n-1]$$

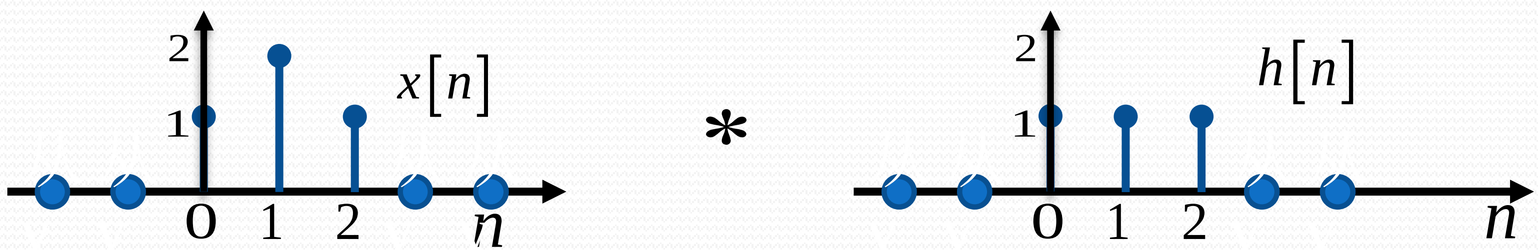


$$y[n] = x[0]h[n-0] + x[1]h[n-1]$$
$$= 0.5h[n] + 2h[n-1]$$

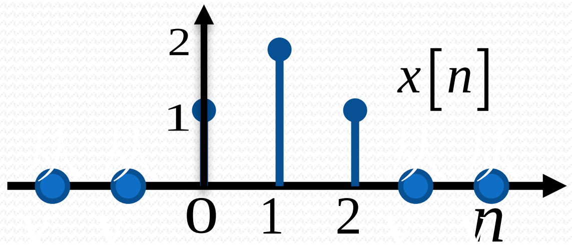
Example: Calculation of convolution

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$





$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$



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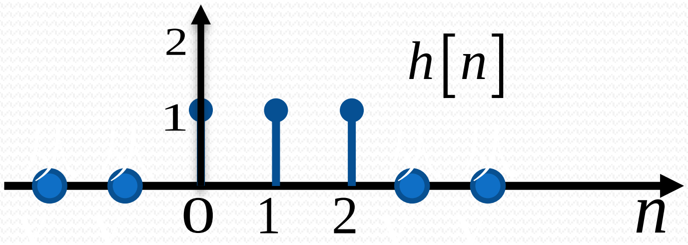
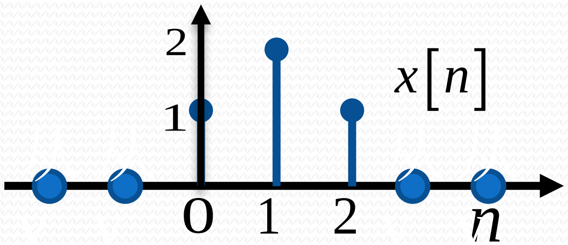


$$y[0] = \sum_{k=-\infty}^{\infty} x[k]h[0-k]$$



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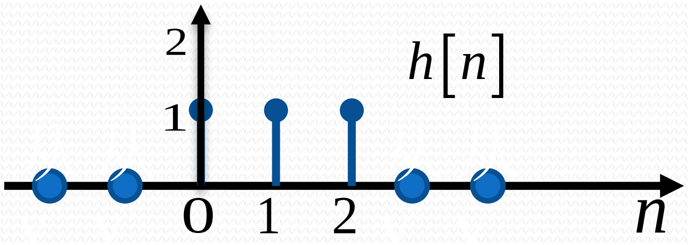
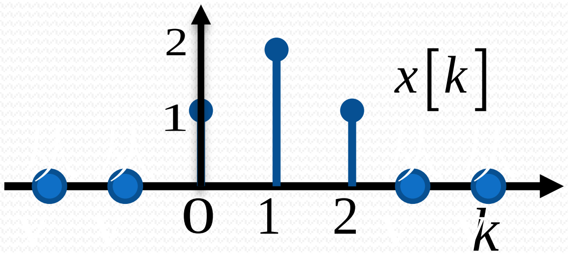


$$y[0]=\sum_{k=-\infty}^{\infty} x[k]h[0-k]$$



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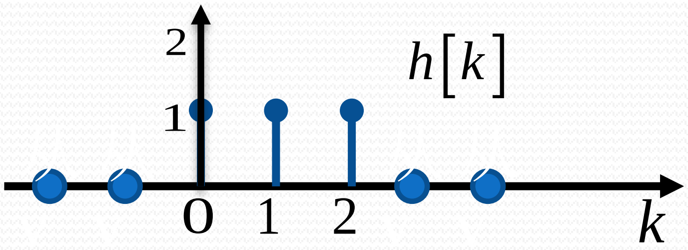
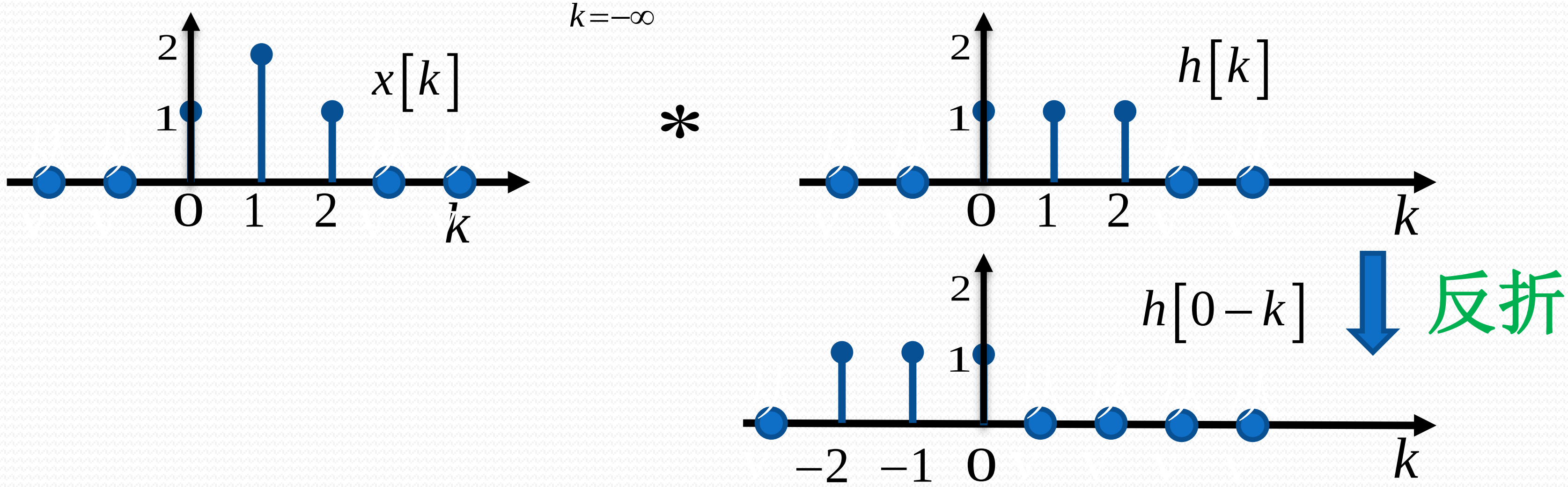
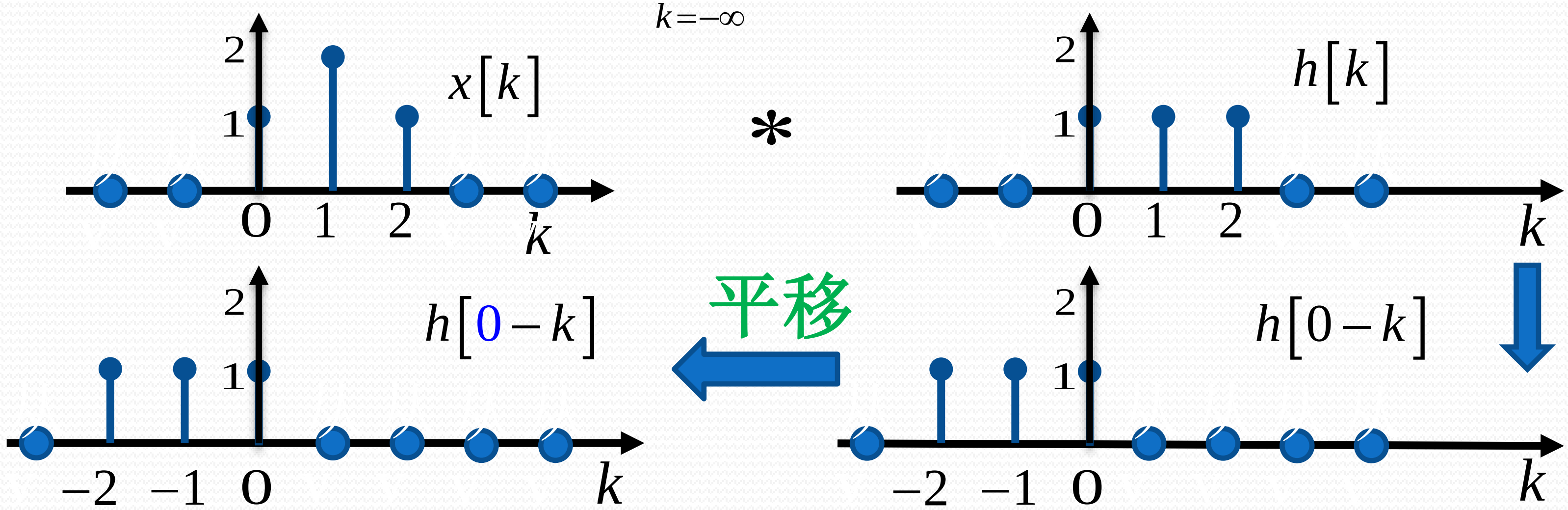


变量代换

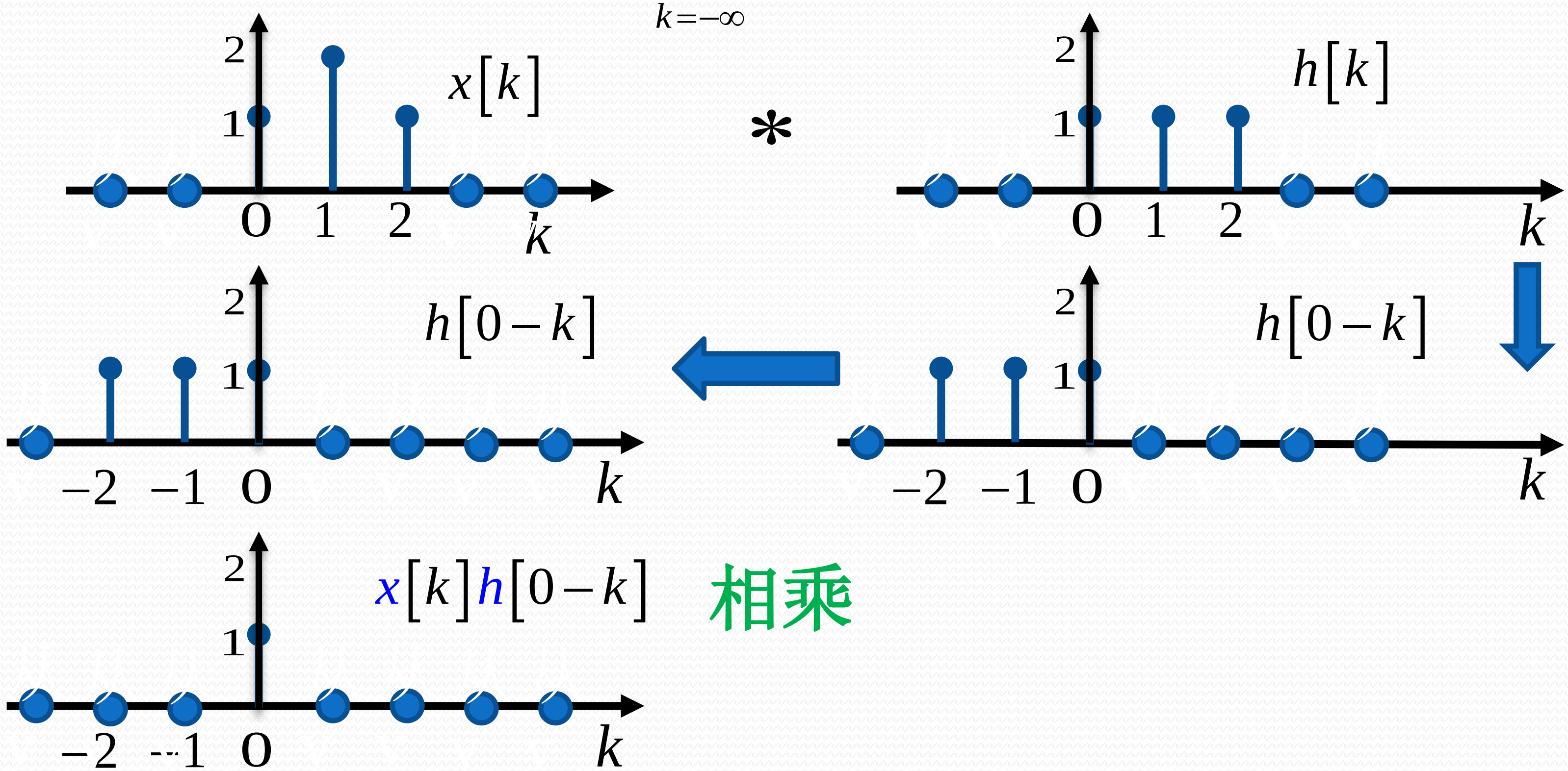
$$y[0]=\sum_{k=-\infty}^{\infty} x[k]h[0-k]$$



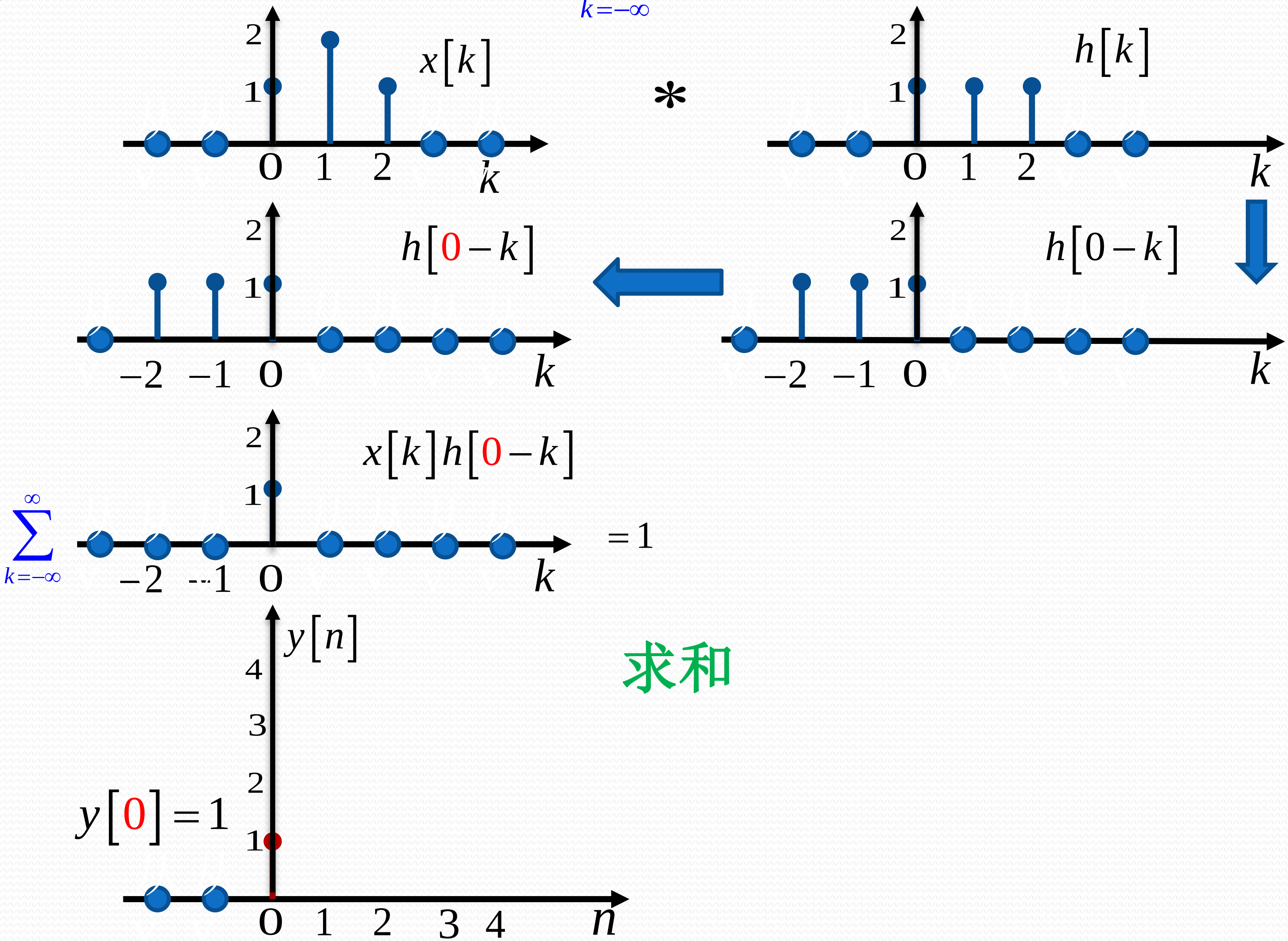
$$y[0]=\sum_{k=-\infty}^{\infty}x[k]h[0-k]$$



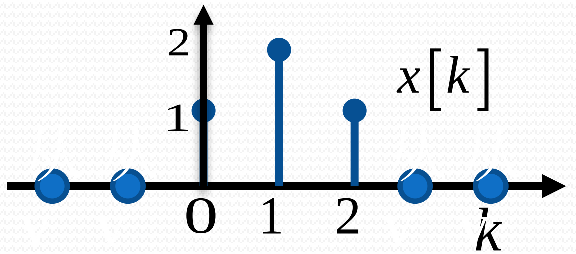
$$y[0]=\sum_{k=-\infty}^{\infty} x[k]h[0-k]$$



$$y[0]=\sum_{k=-\infty}^{\infty} x[k]h[0-k]$$



$$y[1] = \sum_{k=-\infty}^{\infty} x[k]h[1-k]$$



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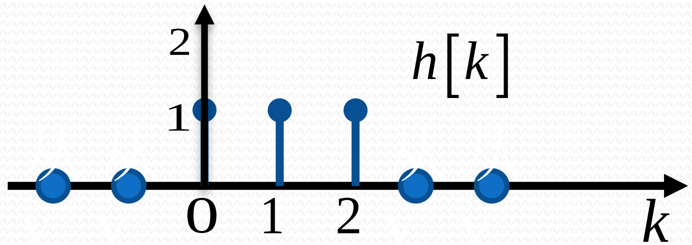
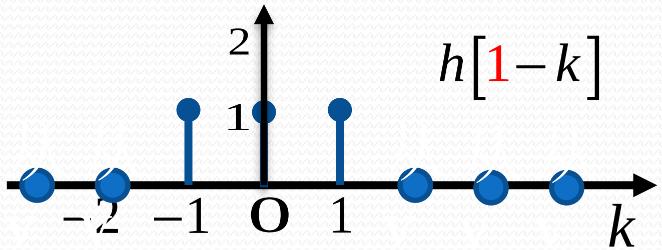
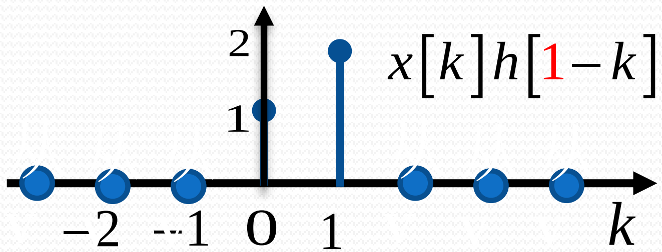
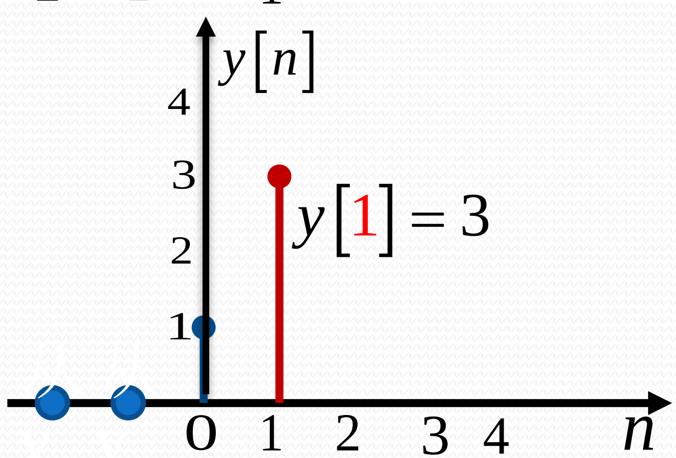




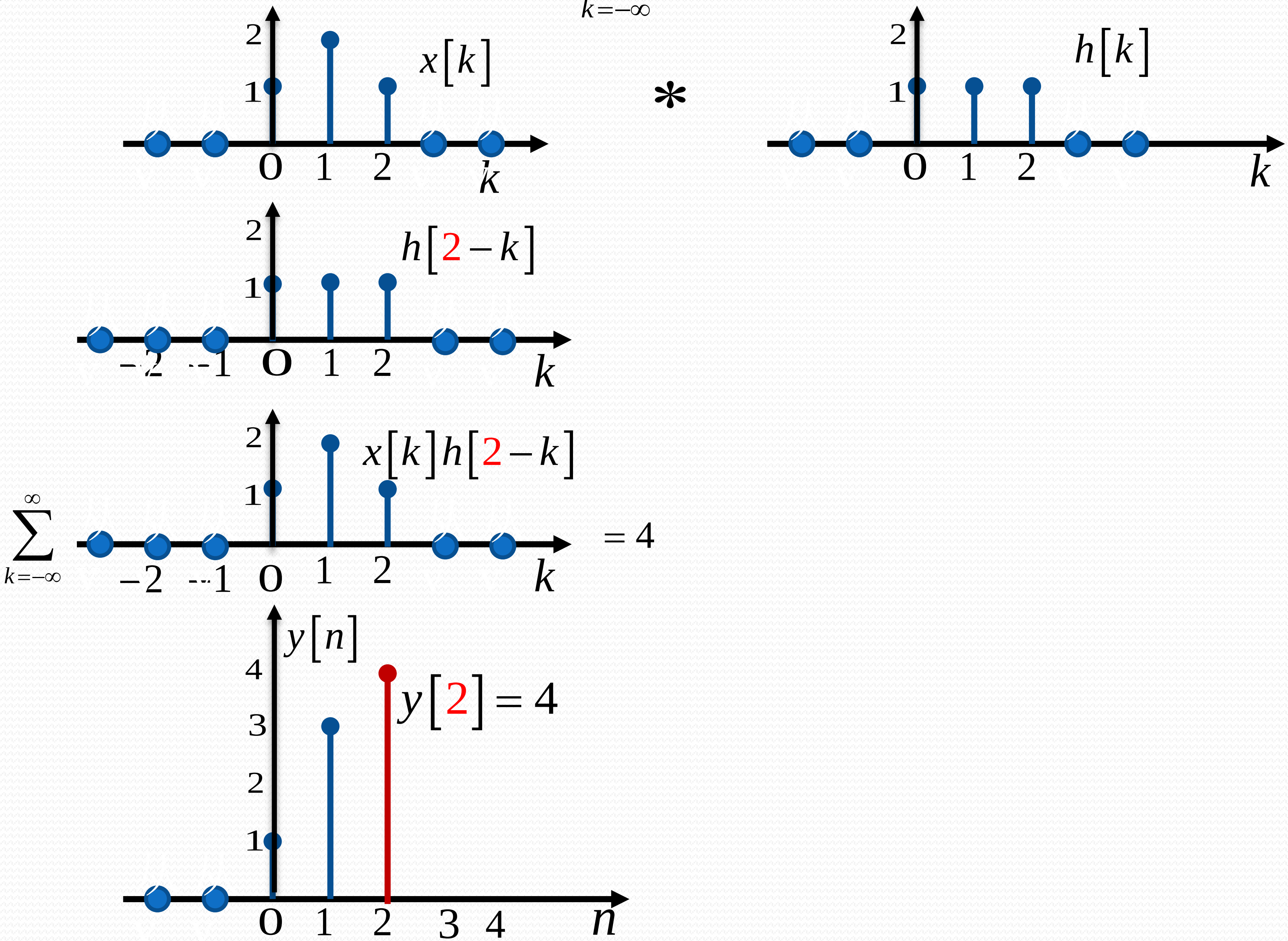
$\sum_{k=-\infty}^{\infty}$



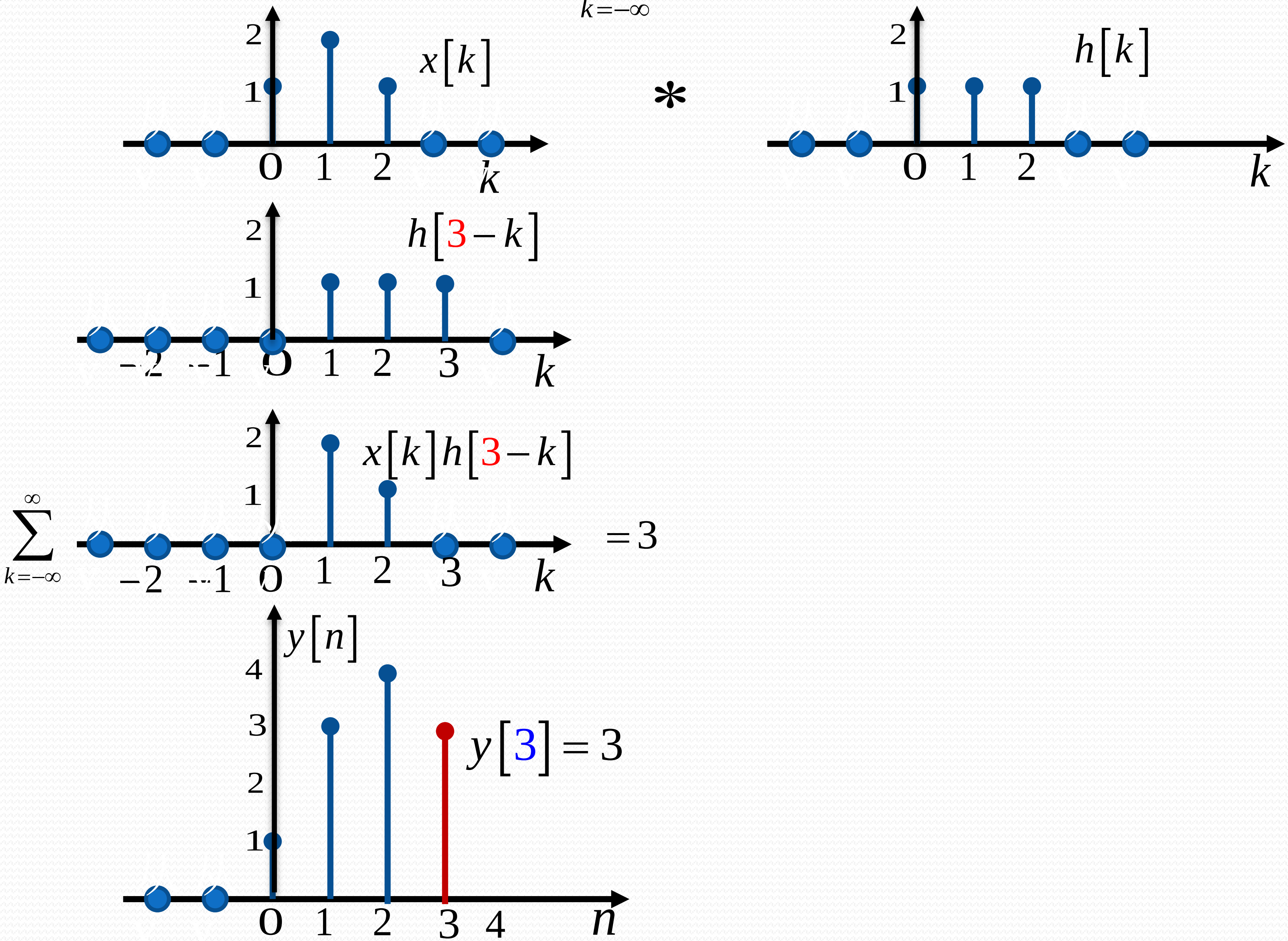
$= 3$



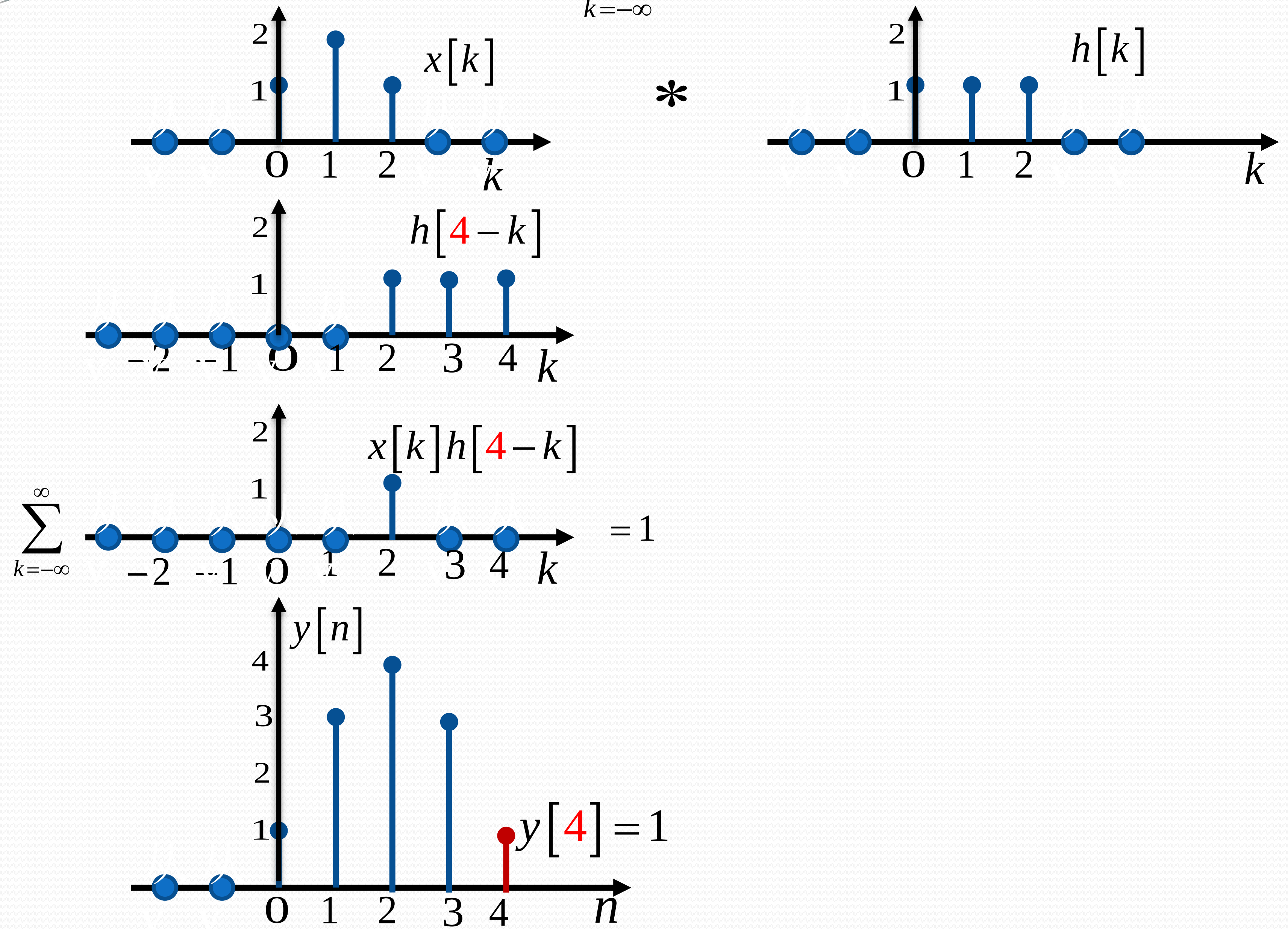
$$y[2]=\sum_{k=-\infty}^{\infty}x[k]h[2-k]$$



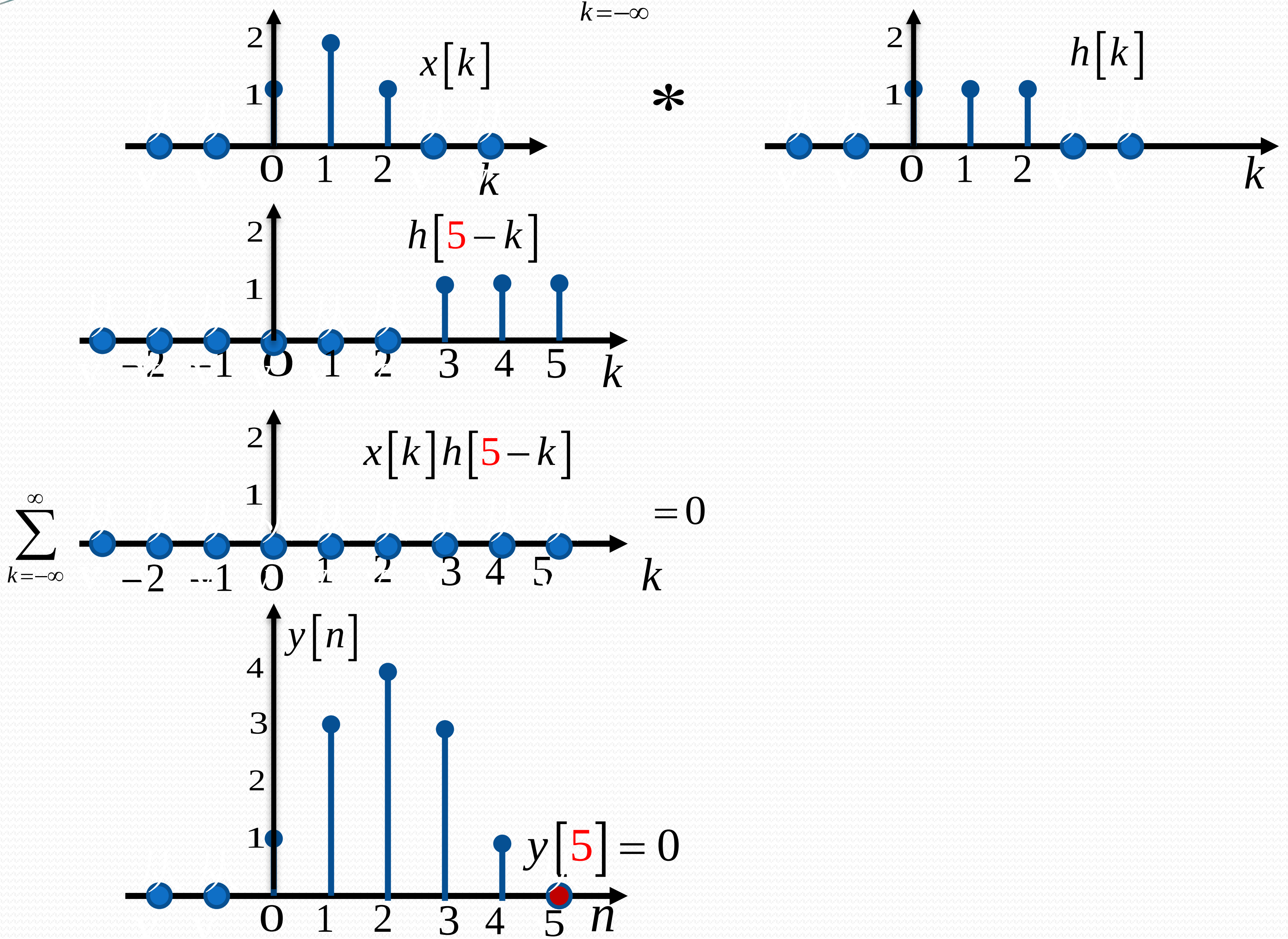
$$y[3]=\sum_{k=-\infty}^{\infty} x[k]h[3-k]$$



$$y[4] = \sum_{k=-\infty}^{\infty} x[k]h[4-k]$$



$$y[5]=\sum_{k=-\infty}^{\infty} x[k]h[5-k]$$

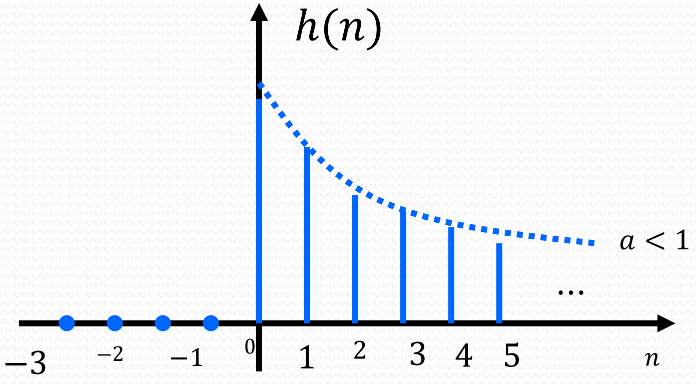


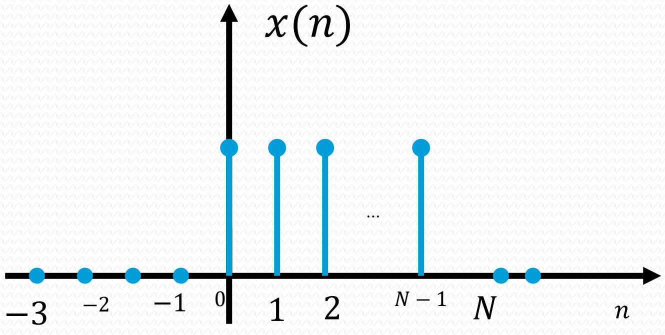
卷积和例题

已知某系统的单位样值响应$\mathrm{h(n)} = \mathrm{a}^{\mathrm{n}}u(n), 0 < a < 1$,
若激励为$\mathrm{x(n)} = u(n) - u(n-N)$，求其响应$\mathrm{y_{ZS}(n)}$

解： $y_{zs}(n) = h(n) * x(n) = \sum_{m=-\infty}^{\infty} x(m)h(n-m)$

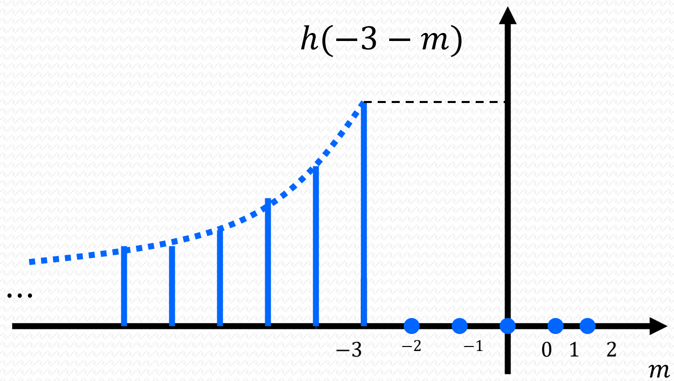
$$= \sum_{m=-\infty}^{\infty} [u(m) - u(m-N)]a^{n-m}u(n-m)$$

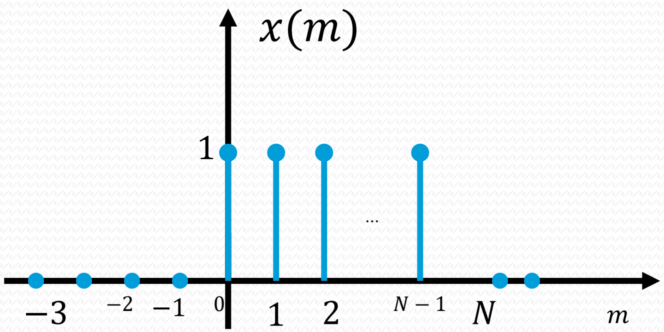




(1)当n<0时， $x(m)$与$h(n-m)$无交叠

即 $y_{zs}(n)=0$

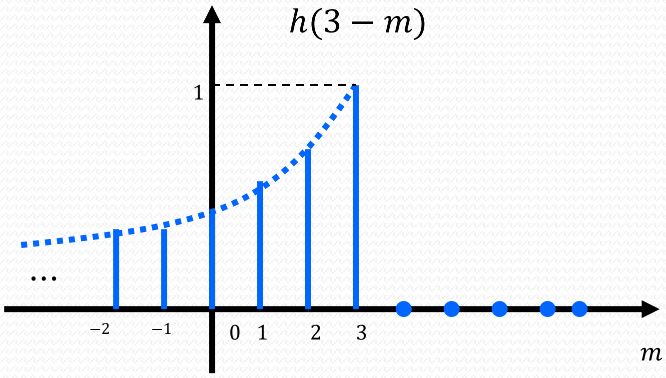


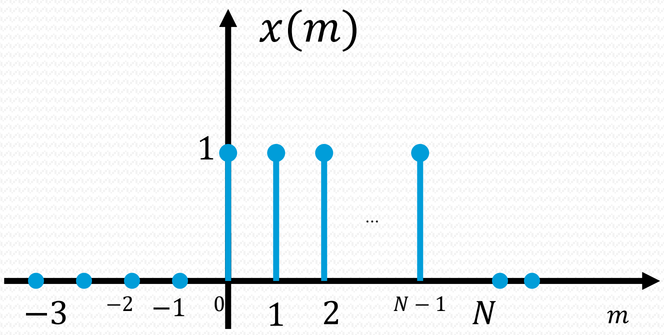


（2）当$0 \le n \le N-1$时，$x(m)$与$h(n-m)$在$[0,n]$区间内重叠非零。

即 $$y_{zs}(n)=\sum_{m=0}^{n}[u(m)-u(m-N)]a^{n-m}u(n-m) \quad 0 \le n \le N-1$$

$$=\sum_{m=0}^{n}a^{n-m} \quad =a^{n}\frac{1-a^{-(n+1)}}{1-a^{-1}} \quad (0 \le \mathrm{n} \le N-1)$$

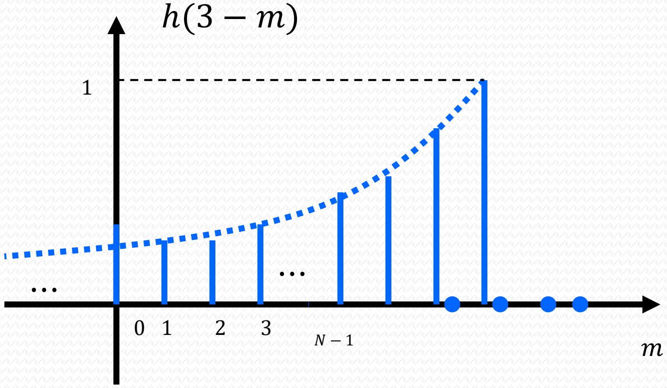


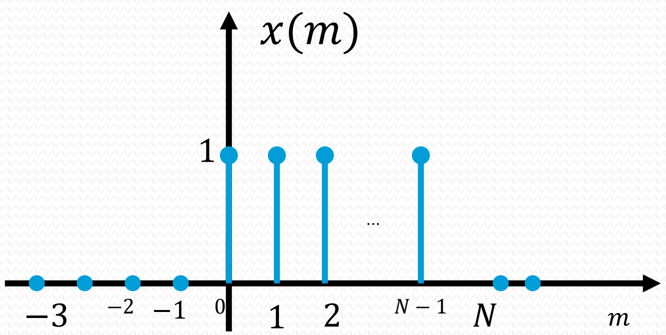


（3）当$n \ge N-1$时，$x(m)$与$h(n-m)$在$[0, N-1]$区间内重叠非零。

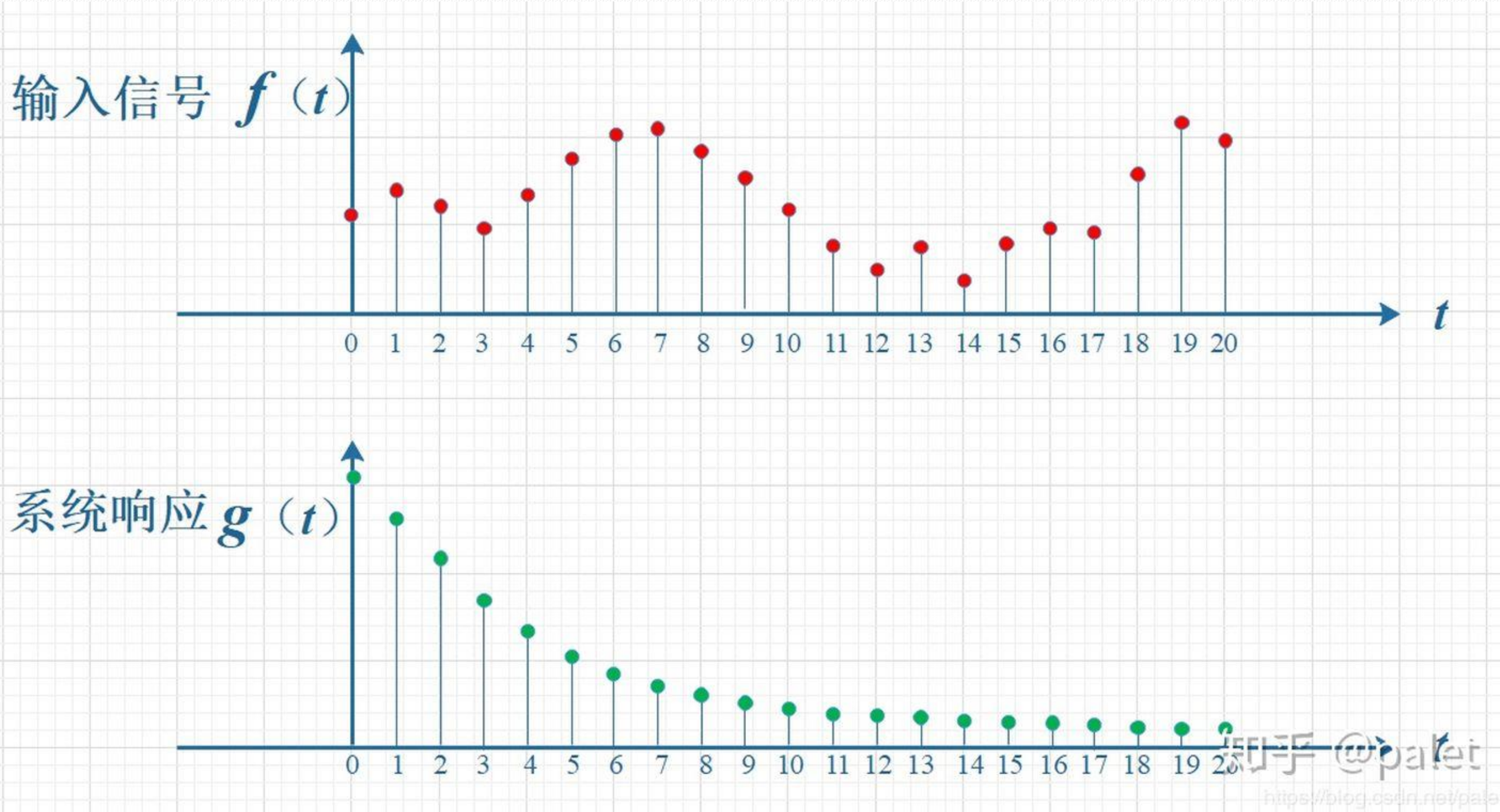
即 $$y_{zs}(n) = \sum_{m=0}^{N-1} [u(m) - u(m-N)] a^{n-m} u(n-m) \quad n \ge (N-1)$$

$$= \sum_{m=0}^{\mathrm{N}-1} a^{n-m} = a^n \frac{1-a^{-N}}{1-a^{-1}} \qquad (\mathrm{n} \ge N-1)$$





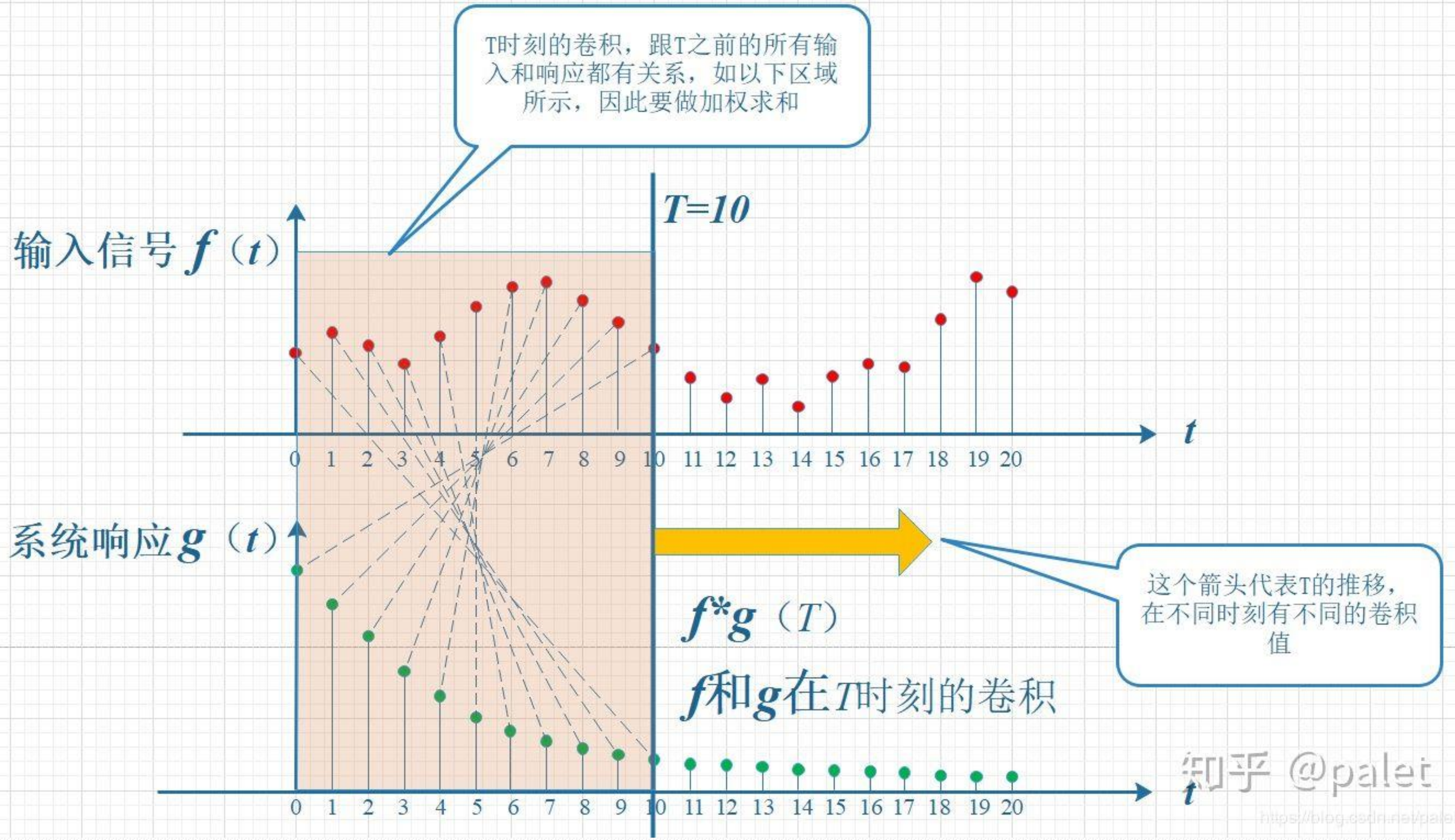
卷积再解读1



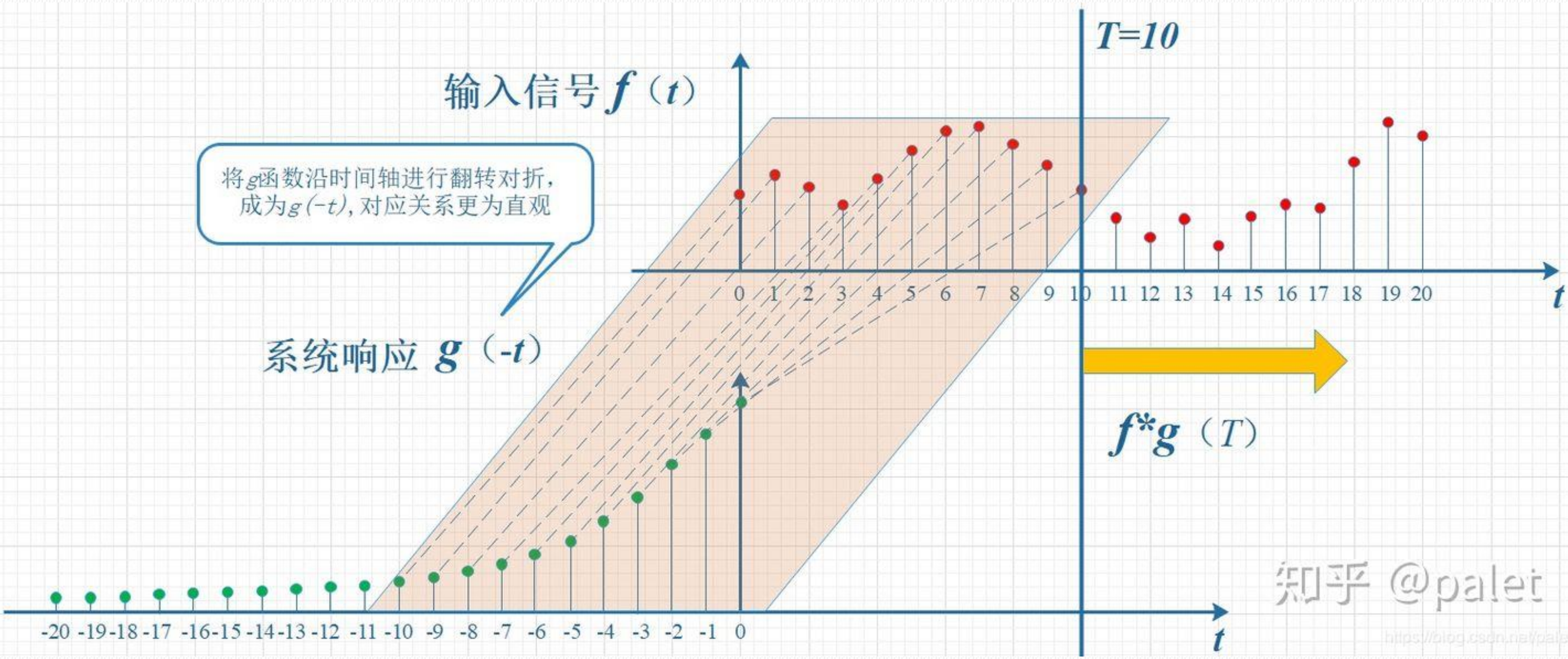
图片源自网络，之后不一一注释

https://www.zhihu.com/question/22298352/answer/637156871

卷积再解读1

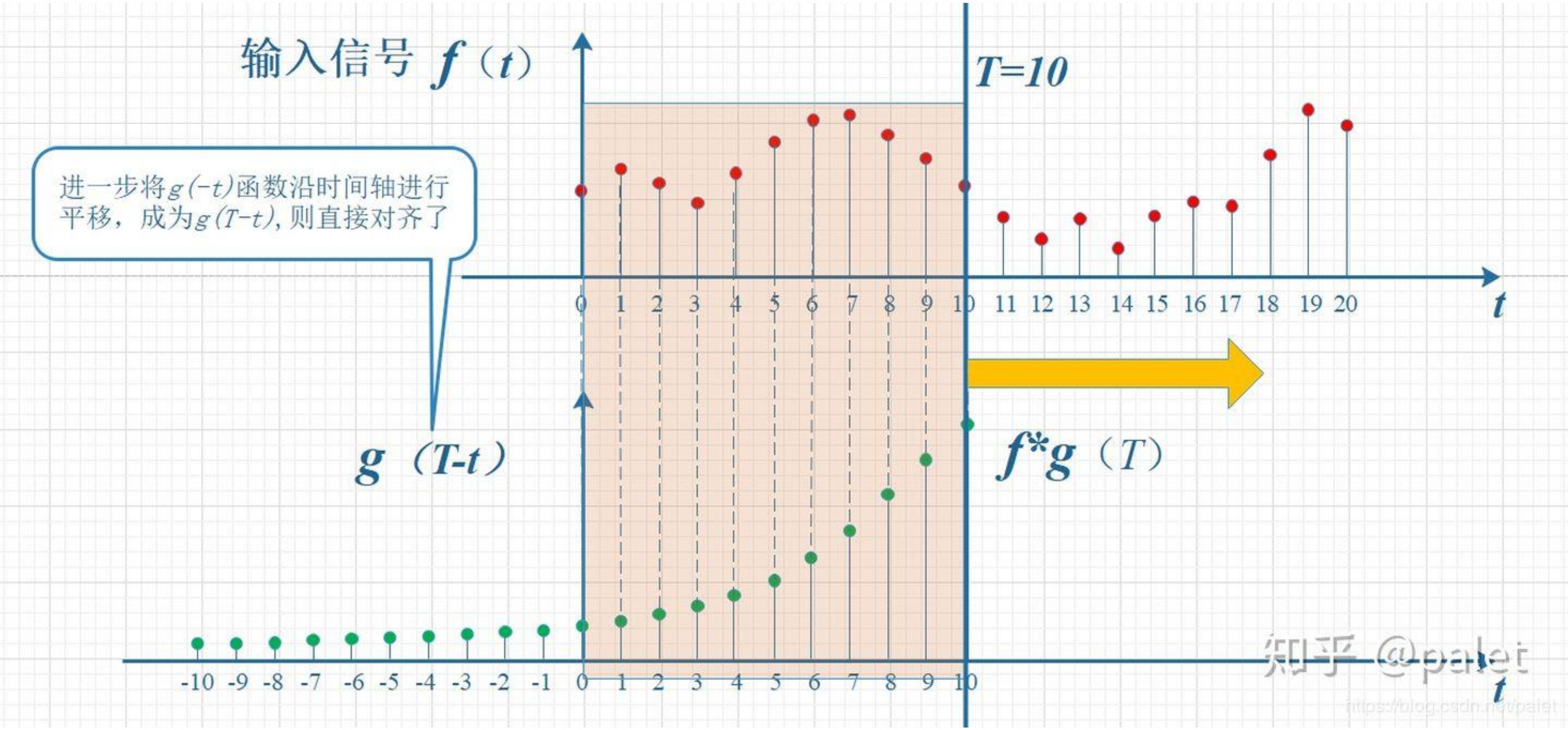

T时刻的卷积，跟T之前的所有输入和响应都有关系，如以下区域所示，因此要做加权求和
T=10
输入信号f（t）
t
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20
系统响应g（t）
f*g（T）
f和g在T时刻的卷积
这个箭头代表T的推移，在不同时刻有不同的卷积值
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20
t
知乎 @palet

卷积再解读1

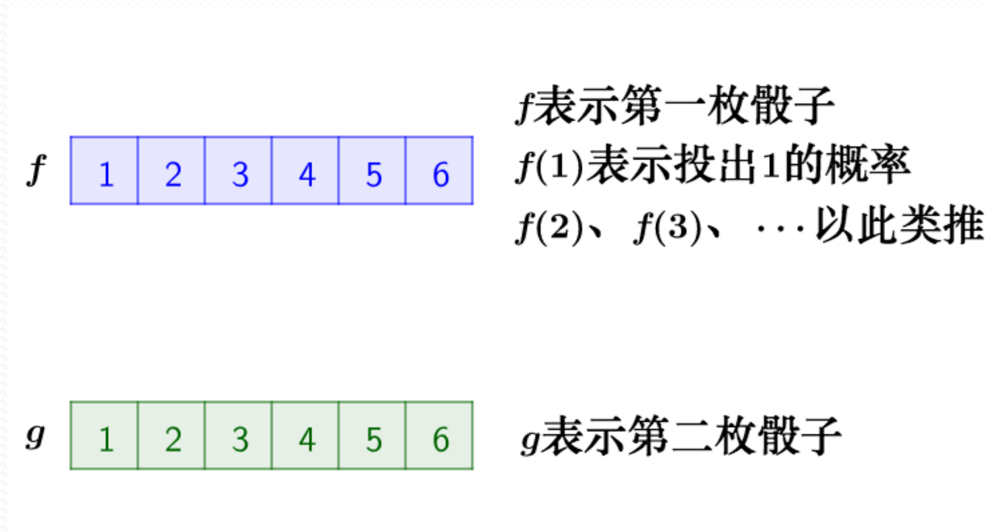


T=10
输入信号 f（t）
将g函数沿时间轴进行翻转对折，
成为g(−t)，对应关系更为直观
系统响应 g（-t）
f*g（T）
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20
t
-20 -19 -18 -17 -16 -15 -14 -13 -12 -11 -10 -9 -8 -7 -6 -5 -4 -3 -2 -1 0
t
知乎 @palet

卷积再解读1


输入信号 f (t)
T=10
进一步将g(-t)函数沿时间轴进行平移，成为g(T-t), 则直接对齐了
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20
t
g (T-t)
f*g (T)
-10 -9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9 10
t

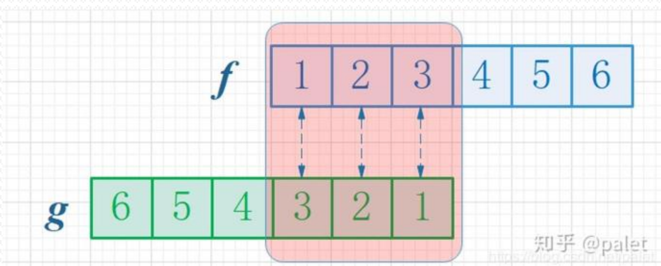
卷积再解读2



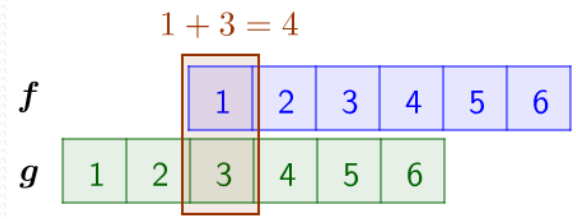
卷积再解读2

两枚骰子点数加起来为4的概率是多少？

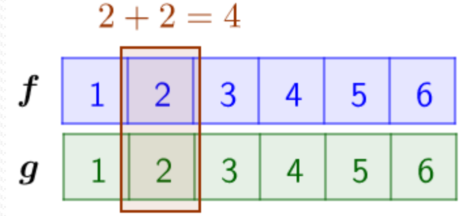
卷积再解读2



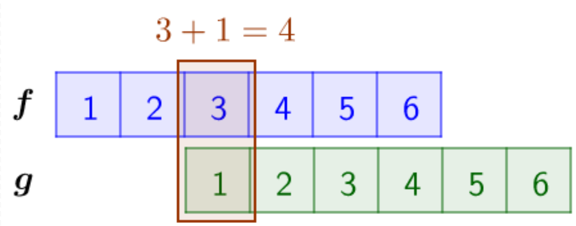
$$f(1)g(3) + f(2)g(2) + f(3)g(1)$$



出现概率为：$f(1)g(3)$

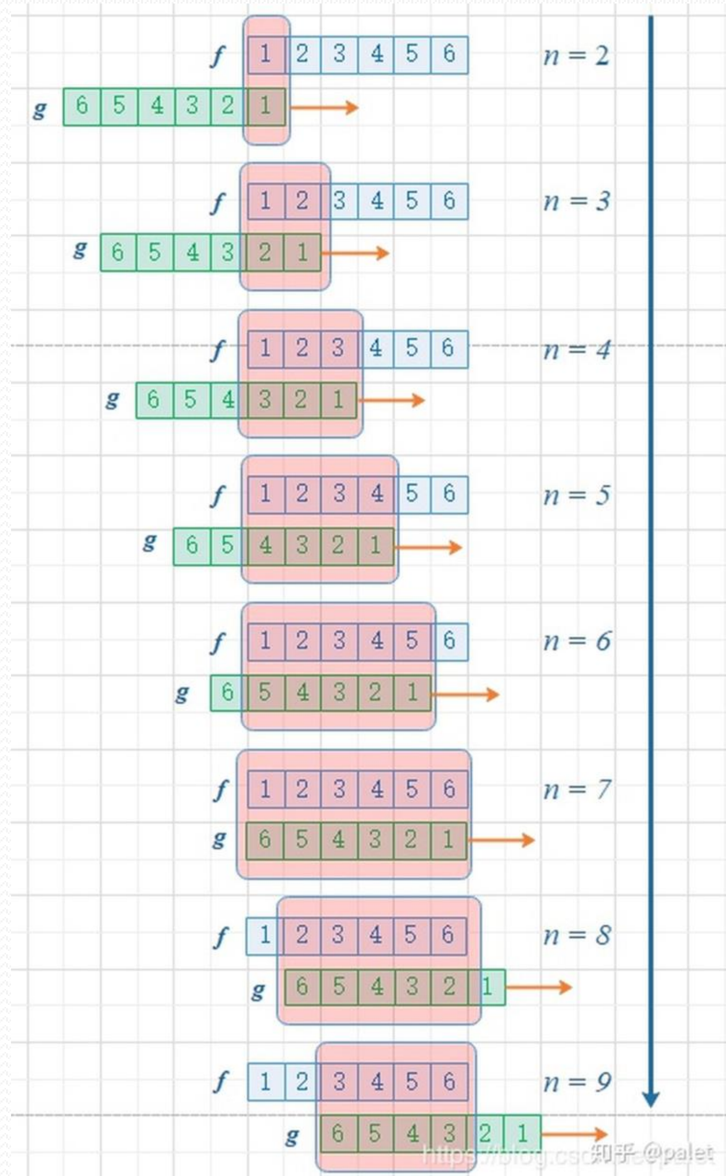


出现概率为：$f(2)g(2)$



出现概率为：$f(3)g(1)$

卷积再解读2



卷积和练习题1

$$x[n] = 0.5\delta[n] + 2\delta[n-1]$$

$$h[n] = \delta[n] + \delta[n-1] + \delta[n-2]$$

标准步骤求解，换元 → 反褶 → 平移 → 相乘 → 取和

卷积和练习题2

$$x[n] = \alpha^n u[n]$$

$$h[n] = u[n]$$

卷积和例题3

$$x[n]=\begin{cases}1, 0\le n\le 4\\0, otherwise\end{cases}$$

$$h[n]=\begin{cases}\alpha^n, 0\le n\le 6\\0, otherwise\end{cases}$$

卷积和例题4

$$x[n] = 2^n u[-n]$$

$$h[n] = u[n]$$