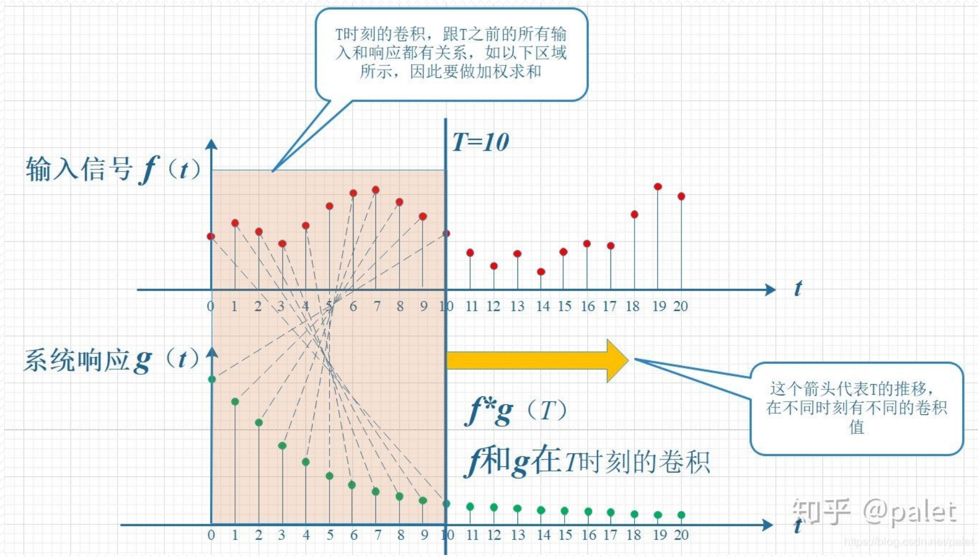
Ch2 Linear Time-Invariant Systems
线性时不变系统

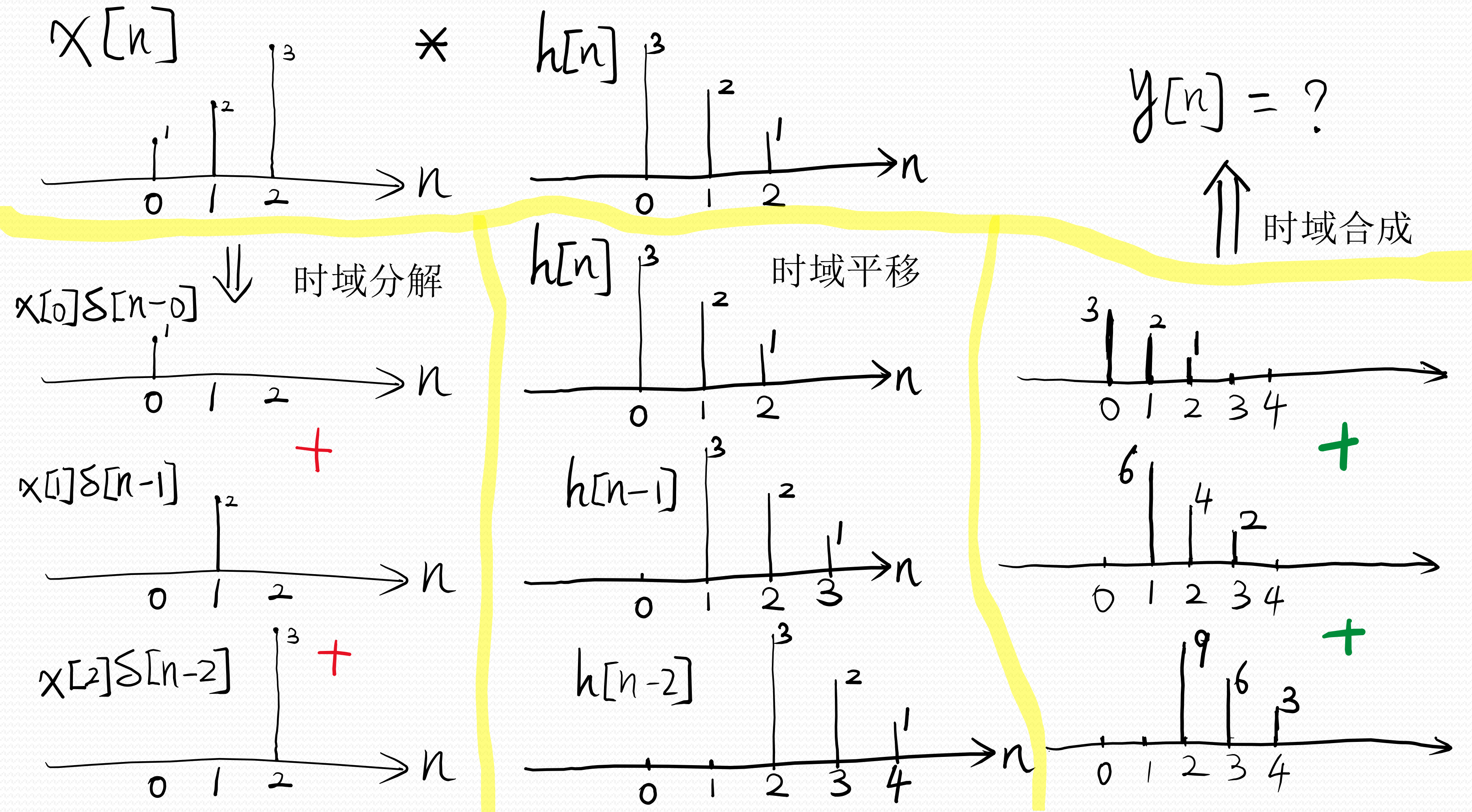
Convolution

卷积

Review

- **Discrete-Time LTI Systems**
  - The representation of discrete-time signals in terms of impulse
  - The discrete-time unit impulse response and the convolution sum representation of LTI systems
- **离散时间 LTI 系统**
  - 使用脉冲表示离散时间信号
  - 离散时间单位脉冲响应及使用卷积和表示 LTI 系统





x[n] * h[n]
y[n] = ?
时域分解
时域平移
时域合成
x[0]δ[n-0]
x[1]δ[n-1]
x[2]δ[n-2]
h[n]
h[n-1]
h[n-2]

Example

$$x[n] = \left(\frac{1}{3}\right)^n u[n] \qquad\qquad h[n] = u[n]$$

```
DiscreteConvolve[(1/3)^n UnitStep[n], UnitStep[n], n, m]
DiscretePlot[%, {m, -8, 8}, PlotRange → All]
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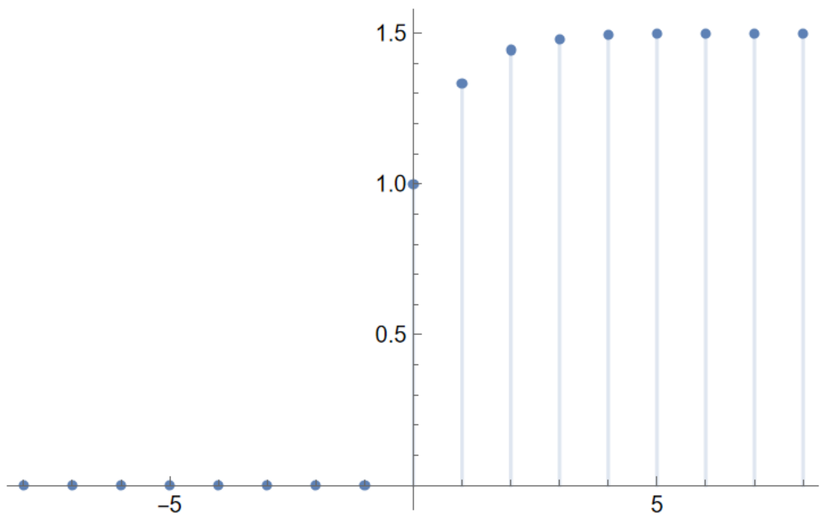
$$\begin{cases} \frac{3}{2} - \frac{3^{-m}}{2} & m \geq 0 \\ 0 & \text{True} \end{cases}$$



Contents

- **Continuous-Time LTI Systems**
  - The representation of continuous-time signals in terms of impulse
  - The continuous-Time unit impulse response and the convolution integral representation
  - Properties of the convolution integral representation
- **连续时间 LTI 系统**
  - 使用脉冲表示连续时间信号
  - 连续时间单位脉冲和卷积积分表示
  - 卷积积分表示的性质

Continuous-Time LTI Systems: The Convolution Integral

连续时间 LTI 系统：卷积积分

- Consider the unit impulse as the idealization of a pulse which is so short that its duration is inconsequential for any real, physical system.

  可将单位脉冲视作一个理想化的冲激，该冲激持续时间极短，对实际的物理系统而言可忽略

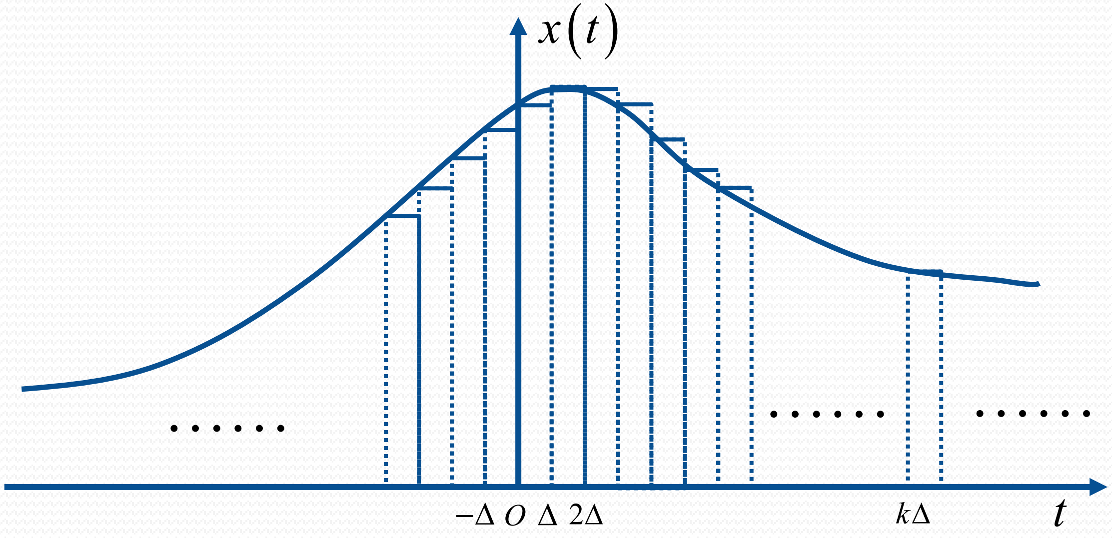
- A representation for arbitrary continuous-time signals in terms of these idealized pulse with vanishingly small duration.

  使用不计持续时间的理想冲激表示任意连续时间信号

- How do you accurately represent an arbitrary signal by a series 'simple' signals?
如何使用一系列简单信号精确地表示任意信号
  - ***Simple***??? 什么是简单的信号
  - 'Small' duration 持续时间短 → accurate 精确
  - Easy to deal with 容易处理 → easy 简单

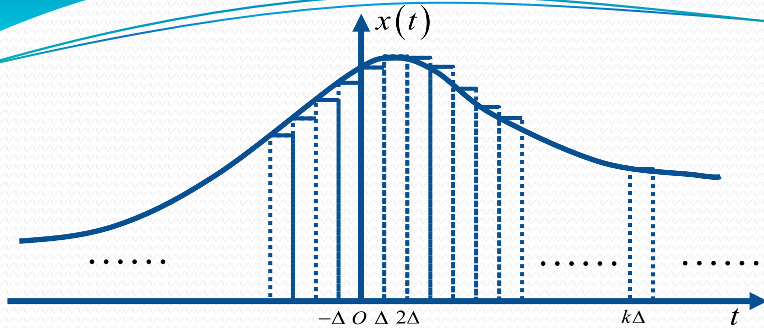
The Representation of Continuous-Time Signals in Terms of Impulse

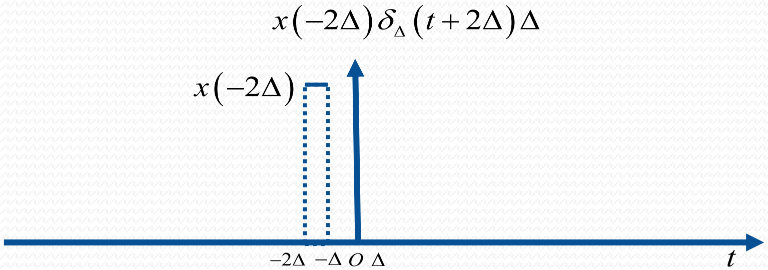
使用脉冲表示连续时间信号

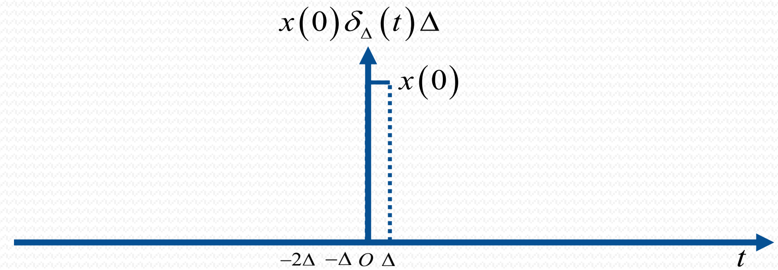


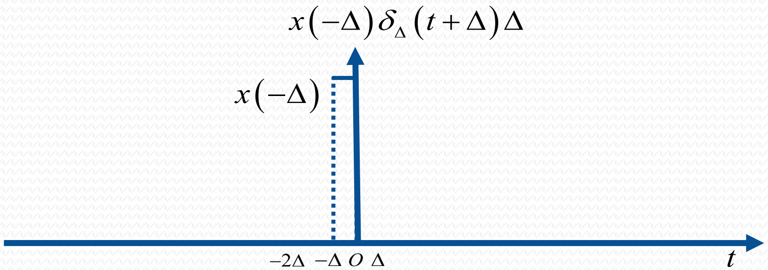
$$\delta_\Delta(t)=\begin{cases}\frac{1}{\Delta} & 0\le t<\Delta\\ 0 & others\end{cases}$$

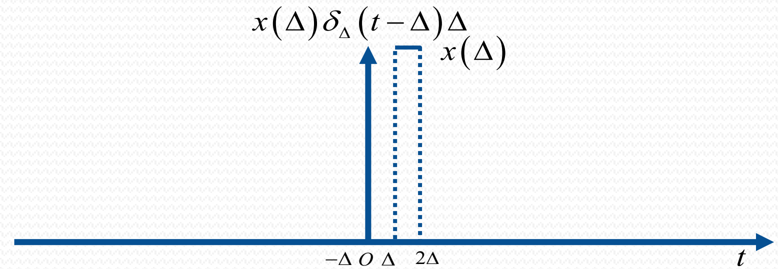
$$\Delta\cdot\delta_\Delta(t)=\begin{cases}1 & 0\le t<\Delta\\ 0 & others\end{cases}$$











$$x(t)=\lim_{\Delta\to 0}\sum_{k=-\infty}^{\infty}x(k\Delta)\delta_\Delta(t-k\Delta)\Delta$$

Represent $x(t)$ as a series of weighted rectangular pulse

将 $x(t)$ 表示为一系列加权矩形冲激

Define $$\delta_\Delta(t) = \begin{cases} \frac{1}{\Delta} & 0 \le t < \Delta \\ 0 & others \end{cases}$$

$$\Delta \cdot \delta_\Delta(t) = \begin{cases} 1 & 0 \le t < \Delta \\ 0 & others \end{cases}$$

Then

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t-k\Delta)\Delta$$

The approximation becomes as Δ approaches 0, and the limit equals x(t).

$$x(t) = \lim_{\Delta\to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t-k\Delta)\Delta$$

From another perspective …

$$x(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)\left[u\left(t - k\Delta\right) - u\left(t - k\Delta - \Delta\right)\right]$$

$$= \lim_{\Delta \to 0} \sum_{k=-\infty}^{\infty} x(k\Delta) \frac{\left[u\left(t - k\Delta\right) - u\left(t - k\Delta - \Delta\right)\right]}{\Delta} \Delta$$

$$= \lim_{\Delta \to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_{\Delta}(t - k\Delta)\Delta$$

As Δ approaches 0

$$k\Delta \to \tau$$

$$x(k\Delta) \to x(\tau)$$

$$\delta_\Delta(t-k\Delta) \to \delta(t-\tau)$$

$$\Delta \to d\tau$$

$$\sum \to \int$$

So, $x(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t-k\Delta)\Delta$

It can be expressed as

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

Sifting Property
筛选性

We can represent an arbitrary signal in terms of integral of impulse function.

我们可将任意信号表示为脉冲函数的积分

Unit step function *单位阶跃函数*

$$u(t)=\int_{-\infty}^{\infty}u(\tau)\delta(t-\tau)d\tau=\int_{0}^{\infty}\delta(t-\tau)d\tau$$

An arbitrary signal x(t) *对任意信号* x(t)

$$\int_{-\infty}^{\infty}x(\tau)\delta(t-\tau)d\tau=\int_{-\infty}^{\infty}x(t)\delta(t-\tau)d\tau$$

$$=x(t)\int_{-\infty}^{\infty}\delta(t-\tau)d\tau=x(t)$$

The Continuous-Time Unit Impulse Response and the Convolution Integral Representation of LTI System

连续时间单位脉冲响应和 LTI 系统的卷积积分表示

- An arbitrary continuous-time signal can be viewed as the superposition of scaled and shifted pulses.

  任意连续时间信号都可视作缩放和平移后的冲激的叠加

- The response of a linear system to this signal will be the superposition of the response to the scaled and shifted versions of pulses.

  线性系统对此信号的响应为该系统对各缩放和平移后的冲激的响应的叠加

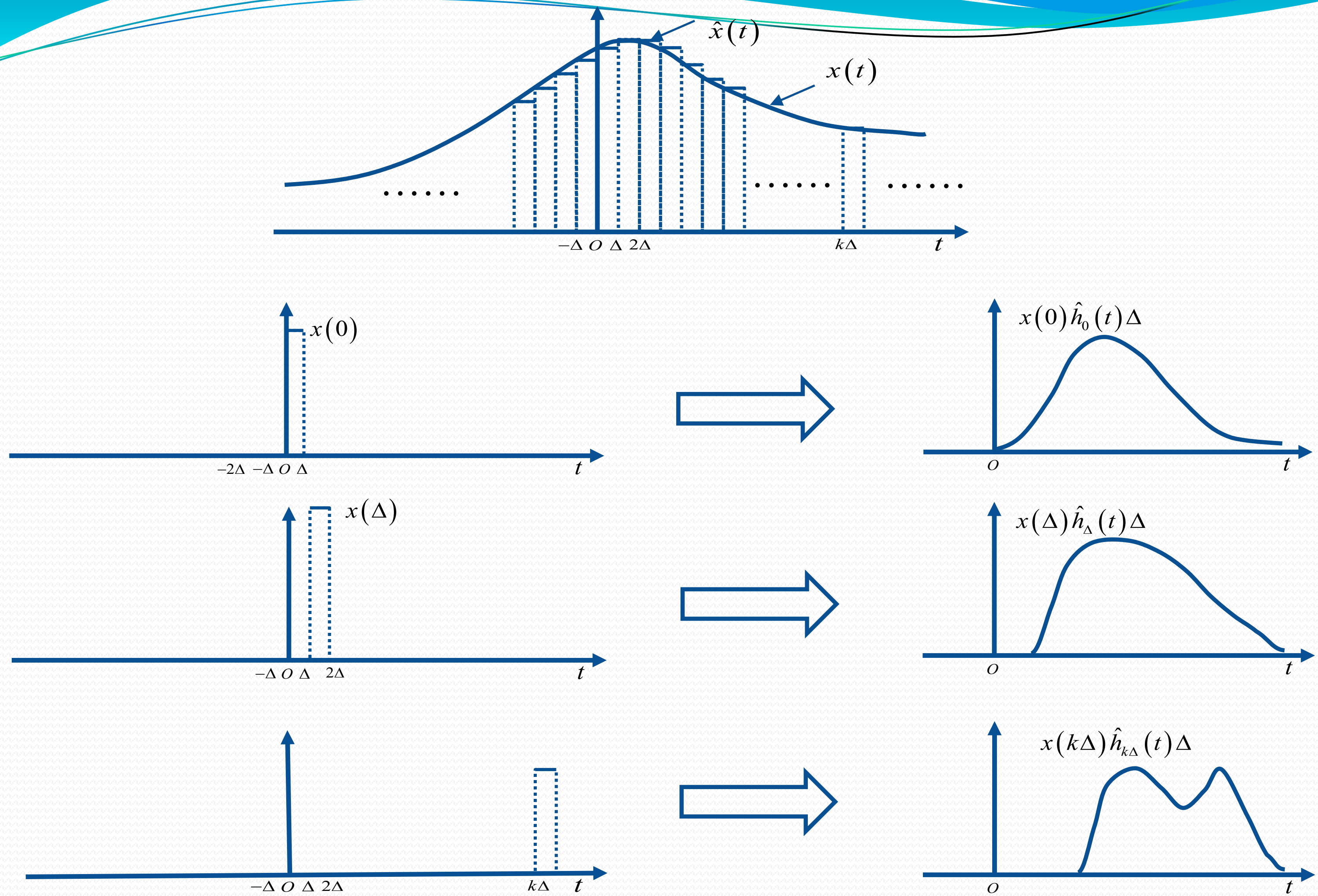
$$x(t)=\lim_{\Delta\to 0}\sum_{k=-\infty}^{\infty}x(k\Delta)\delta_{\Delta}(t-k\Delta)\Delta$$

Define $\hat{h}_{k\Delta}(t)$

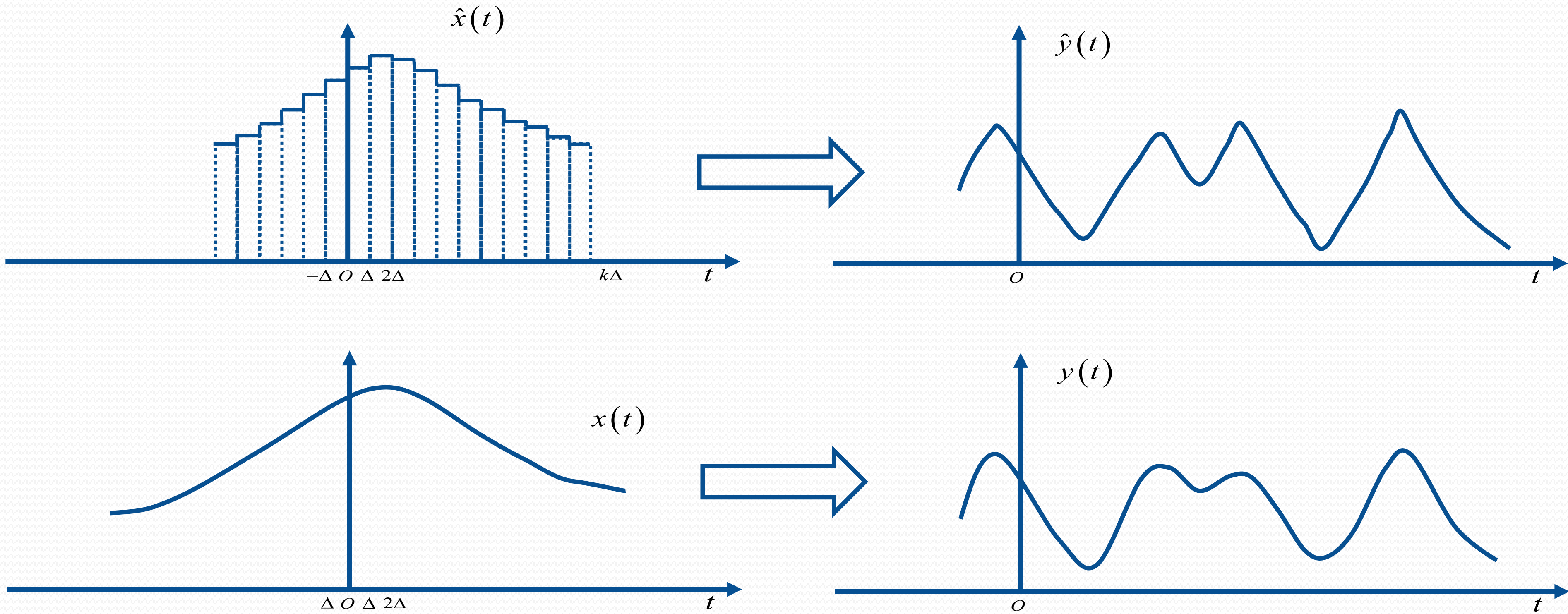
as the response of an linear system to the input $\delta_{\Delta}(t-k\Delta)$

$$\hat{x}(t)=\sum_{k=-\infty}^{\infty}x(k\Delta)\delta_{\Delta}(t-k\Delta)\Delta$$

Then, we can see $\hat{y}(t)=\sum_{k=-\infty}^{\infty}x(k\Delta)\hat{h}_{k\Delta}(t)\Delta$



$\hat{x}(t)$
$x(t)$
$-\Delta$ O Δ 2Δ
$k\Delta$
t
$x(0)$
-2Δ $-\Delta$ O Δ
t
$x(0)\hat{h}_0(t)\Delta$
O
t
$x(\Delta)$
$-\Delta$ O Δ 2Δ
t
$x(\Delta)\hat{h}_\Delta(t)\Delta$
O
t
$-\Delta$ O Δ 2Δ
$k\Delta$
t
$x(k\Delta)\hat{h}_{k\Delta}(t)\Delta$
O
t



$\hat{x}(t)$
$-\Delta$ O Δ 2Δ $k\Delta$ t
$\hat{y}(t)$
O t
$x(t)$
$-\Delta$ O Δ 2Δ t
$y(t)$
O t

As Δ approaches 0,

$$y(t)=\lim_{\Delta\to 0}\hat{y}(t)=\lim_{\Delta\to 0}\sum_{k=-\infty}^{\infty}x(k\Delta)\hat{h}_{k\Delta}(t)\Delta$$

$$y(t)=\int_{-\infty}^{+\infty}x(\tau)h_{\tau}(t)d\tau$$

What is the time invariant property???
回忆时不变性

Consider the time invariant property of LTI system
考虑系统的时不变性

Define $$h(t)=h_0(t),h_{\tau}(t)=h(t-\tau)$$

Then $$y(t)=\int_{-\infty}^{+\infty}x(\tau)h(t-\tau)d\tau$$

THE CONVOLUTION INTEGRAL ***（卷积积分）***
OR THE SUPERPOSITION ITEGRAL

$$y(t)=\int_{-\infty}^{+\infty}x(\tau)h(t-\tau)d\tau$$

Always be represented symbolically as
卷积积分的符号表示

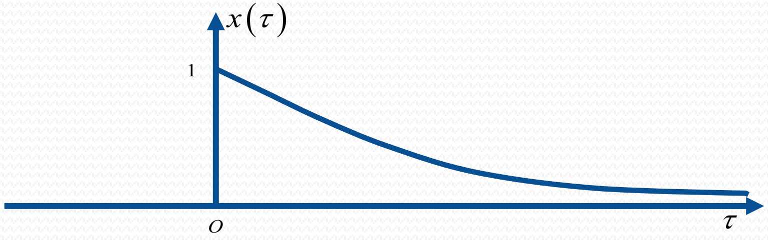
$$y(t)=x(t)*h(t)$$

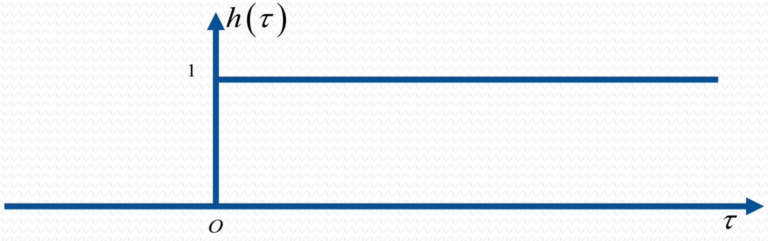
Example

Let $x(t)$ be the input to an LTI system with unit impulse response $h(t)$, where $x(t) = e^{-at}u(t), a > 0$

$$h(t) = u(t)$$

What is the response of this system in such condition?





Since $\quad x(t) = e^{-at}u(t), a > 0$

$$h(t) = u(t)$$

We can see

$$x(\tau)h(t-\tau) = e^{-a\tau}u(\tau)u(t-\tau) = \begin{cases} e^{-a\tau}, & 0 < \tau < t \\ 0, & otherwise \end{cases}$$

Then, for $\boxed{t>0}$

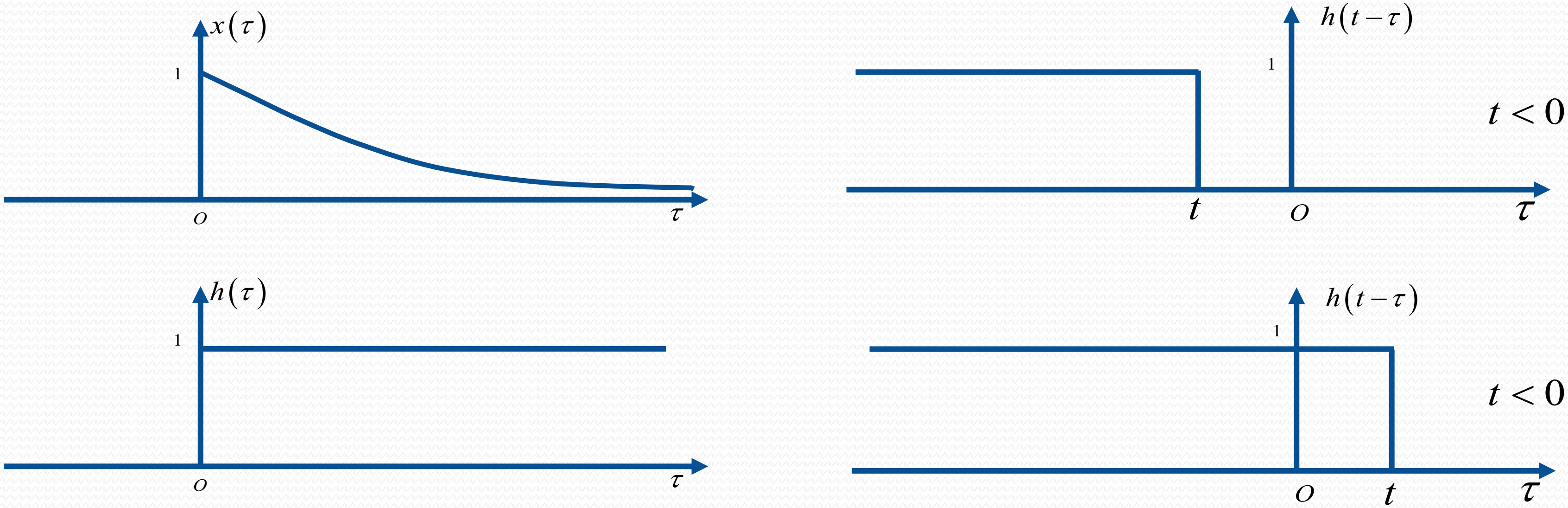
$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} e^{-a\tau}u(\tau)u(t-\tau)d\tau$$

$$= \int_0^t e^{-a\tau}d\tau = -\frac{1}{a}e^{-a\tau}\Big|_0^t = \frac{1}{a}(1-e^{-at})$$

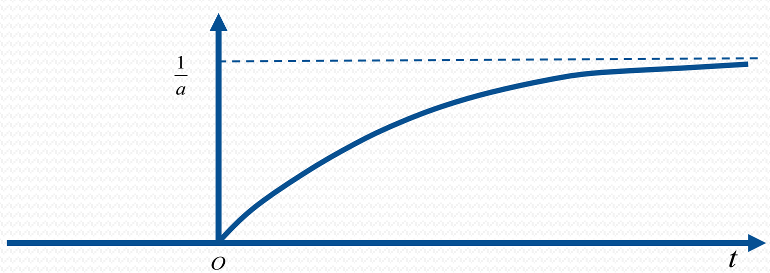
Then, for all *t*,

$$y(t) = \frac{1}{a}(1-e^{-at})u(t)$$

Graphical Interpretation (图示)

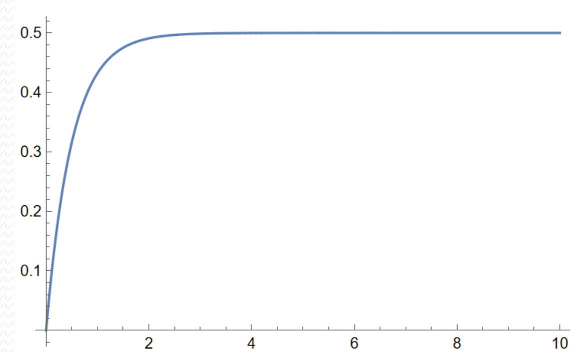


Calculation of the convolution integral




```
Convolve[Exp[-2 x] UnitStep[x], UnitStep[x], x, y]
Plot[%, {y, 0, 10}, PlotRange → All]
```

e^{-y} Sinh[y] UnitStep[y]



Response of the system

How to Calculate Convolution Integral and Convolution Sum?
如何计算卷积积分和卷积和

Step One: Replace the independent variable ‘t’ with ‘τ’
第一步： 把独立变量 ‘t’ 换成 ‘τ’

Step Two: Reverse one signal
第二步： 将信号反折

Step Three: Shifting
第三步： 平移

Step Four: Multiply
第四步： 相乘

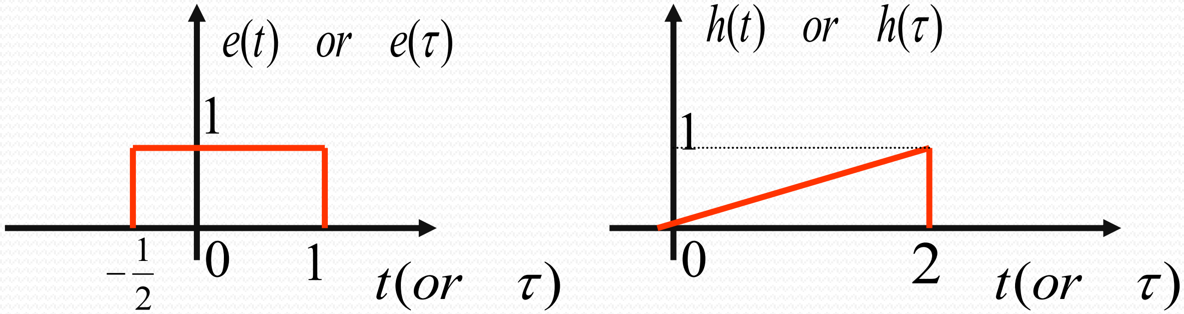
Step Five: Integral or Add
第五步： 积分或求和

Example

Determine and sketch $e(t)*h(t)$

计算并画出 $e(t)*h(t)$

(1)



Example

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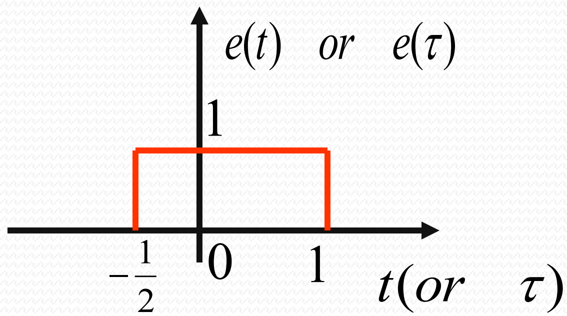
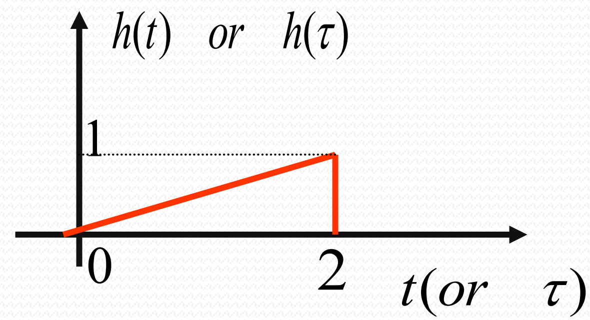
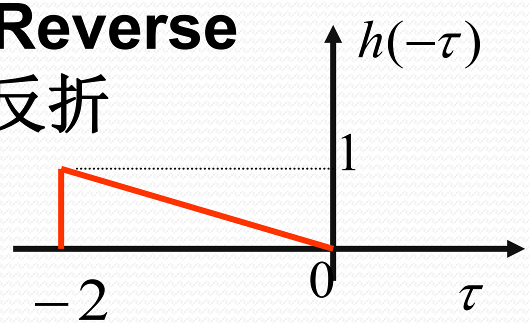
计算并画出 $e(t)*h(t)$

(1)





(2) Reverse
反折

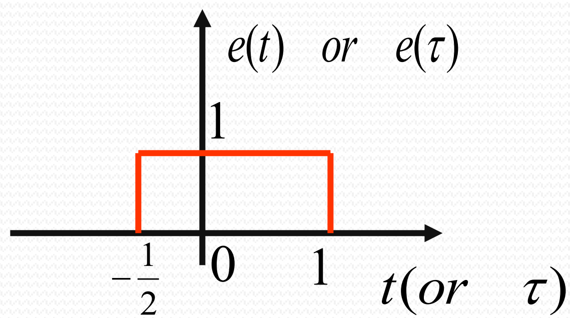


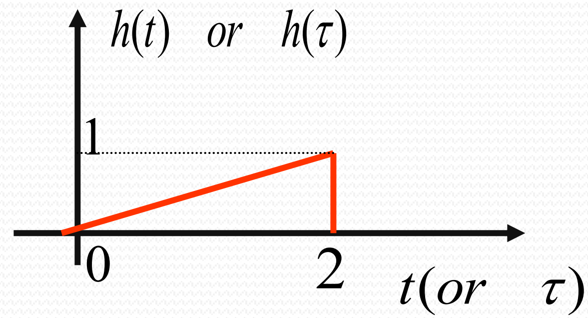
Example

Determine and sketch $e(t)*h(t)$

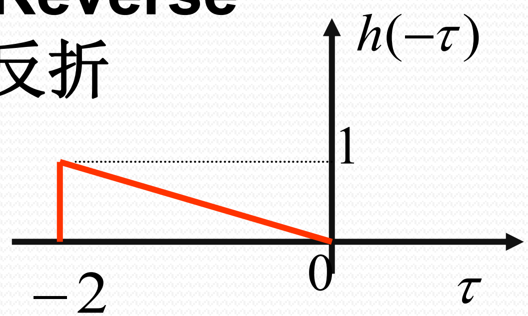
计算并画出 $e(t)*h(t)$

(1)

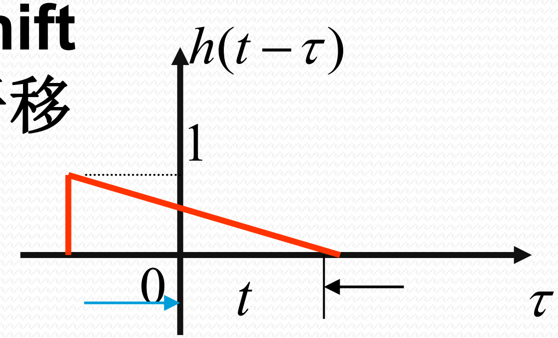




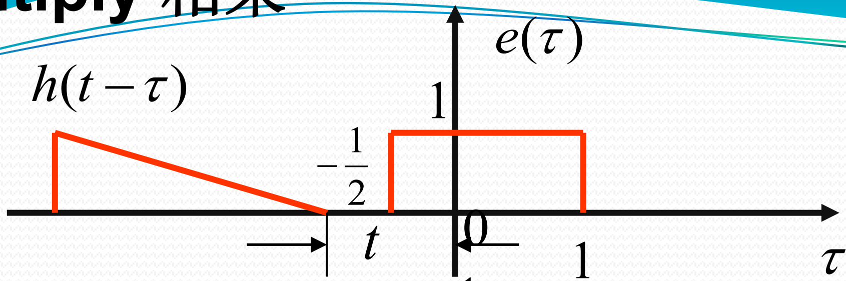
(2) Reverse
反折



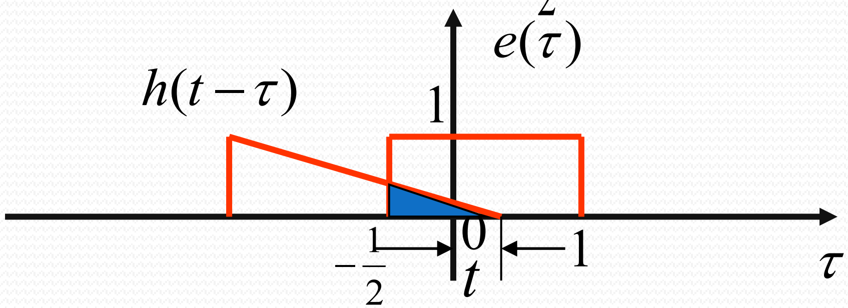
(3) Shift
平移



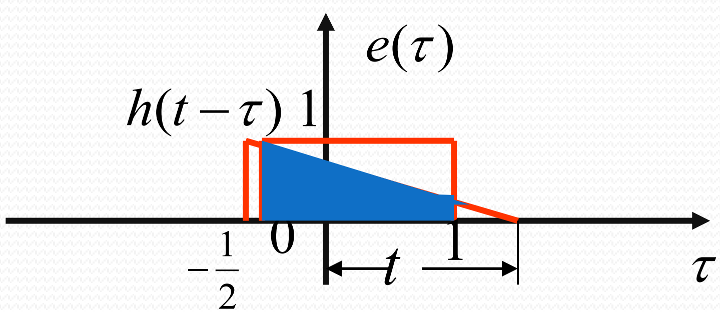
(4)Multiply 相乘



$(a)\quad -\infty < t \leq -\frac{1}{2}$



$(b)\quad -\frac{1}{2} \leq t \leq 1$



$(c)\quad 1 \leq t \leq \frac{3}{2}$

$(a)\quad -\infty < t \leq -\frac{1}{2}$

$$e(t) * h(t) = 0$$

$(b)\quad -\frac{1}{2} \leq t \leq 1$

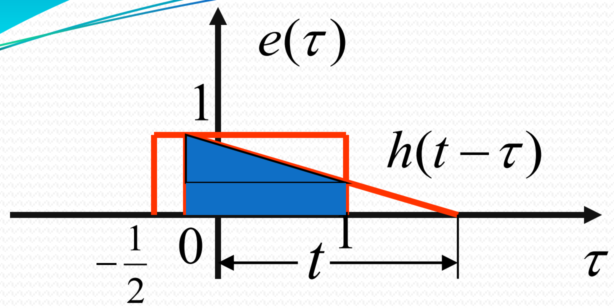
$$e(t) * h(t) = \int_{-\frac{1}{2}}^{t} 1 \times \frac{1}{2}(t-\tau)d\tau$$

$$= \frac{t^2}{4} + \frac{t}{4} + \frac{1}{16}$$

$(c)\quad 1 \leq t \leq \frac{3}{2}$

$$e(t) * h(t) = \int_{-\frac{1}{2}}^{1} 1 \times \frac{1}{2}(t-\tau)d\tau$$

$$= \frac{3}{4}t - \frac{3}{16}$$

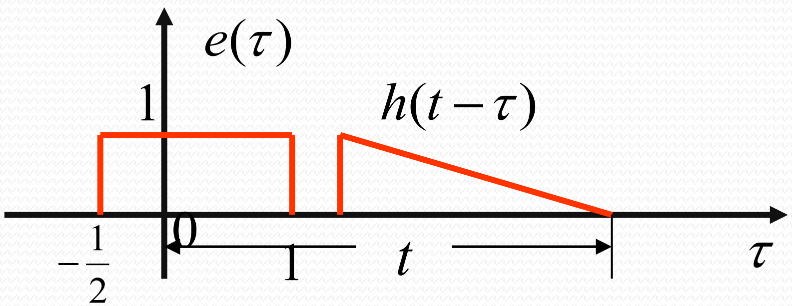


$(d)\quad \frac{3}{2} \le t \le 3$

$(d)\quad \frac{3}{2} \le t \le 3$

$$e(t) * h(t) = \int_{t-2}^{1} 1 \times \frac{1}{2}(t-\tau)\,d\tau$$

$$= -\frac{t^2}{4} + \frac{t}{2} + \frac{3}{4}$$

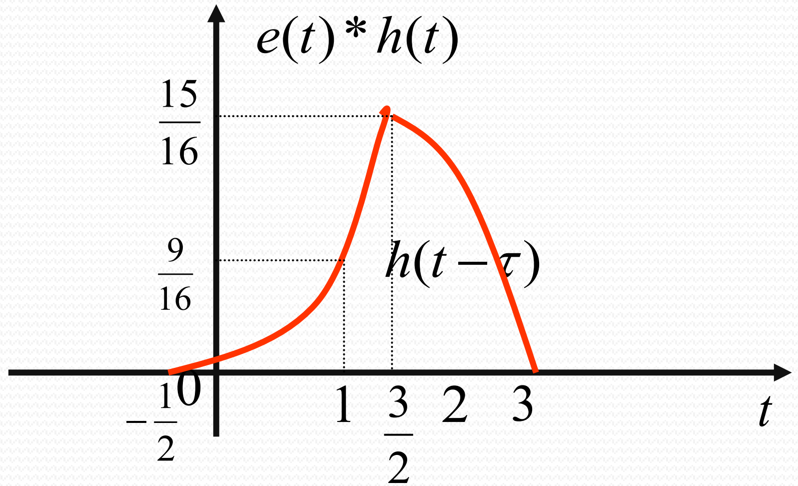


$(e)\quad 3 < t \le \infty$

$(e)\quad 3 < t \le \infty$

$$e(t) * h(t) = 0$$

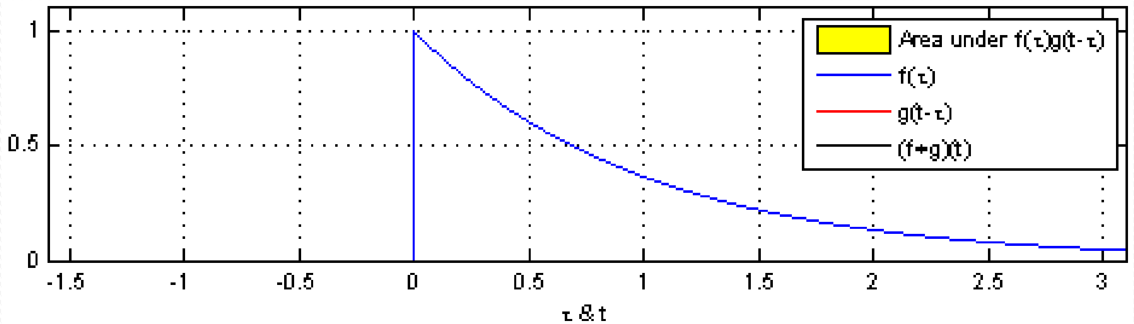
（5）Integral 积分



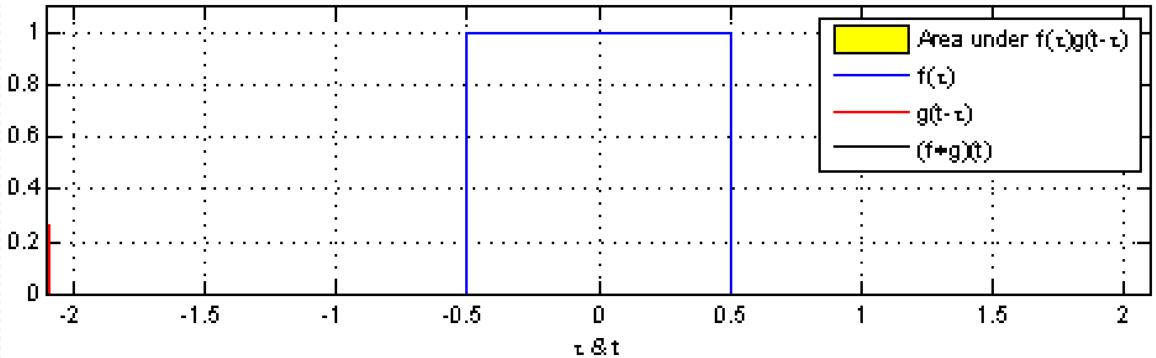
$$e(t)*h(t)$$

Example 1

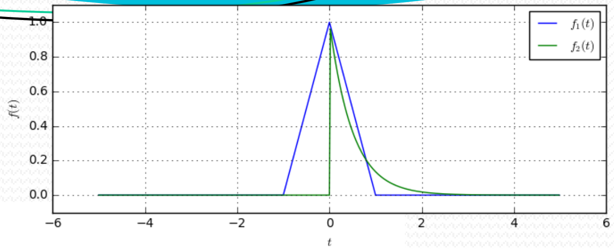
Convolution of a square pulse with the impulse response of an RC circuit to obtain the output signal waveform
通过方形冲激与 RC 电路的脉冲响应卷积来得到输出信号波形



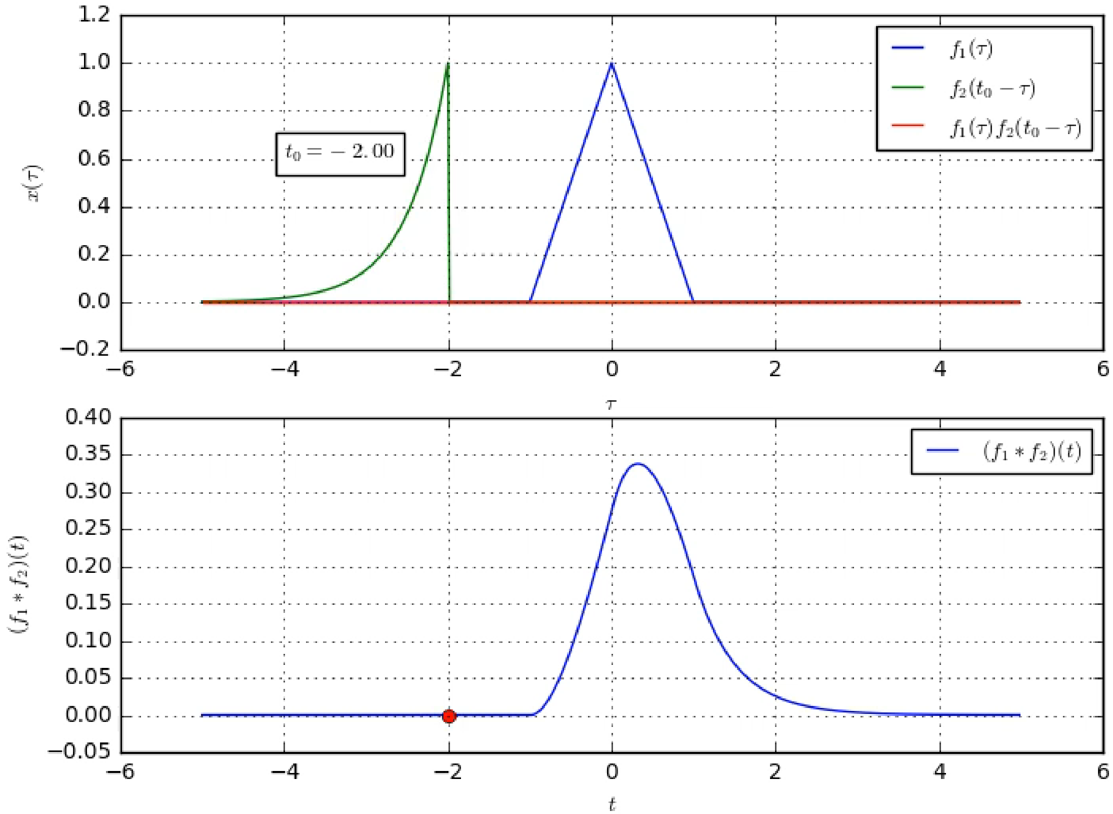
Convolution of two square pulses
两个方形冲激的卷积



Example 2

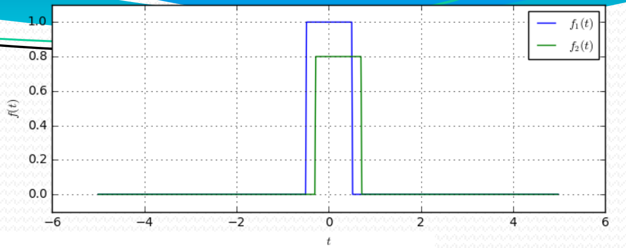


$f_1(t)$
$f_2(t)$
$f(t)$
t

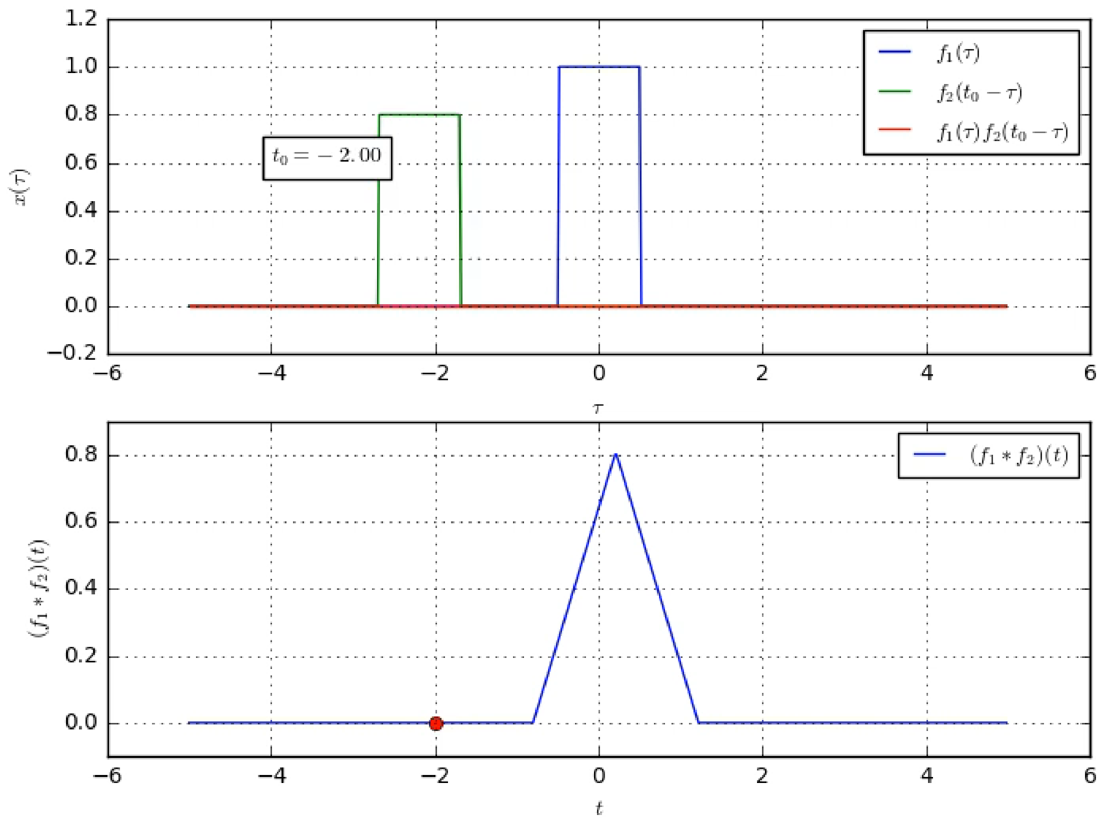


$f_1(\tau)$
$f_2(t_0-\tau)$
$f_1(\tau)f_2(t_0-\tau)$
$t_0=-2.00$
$x(\tau)$
τ
$(f_1*f_2)(t)$
t

Example 3

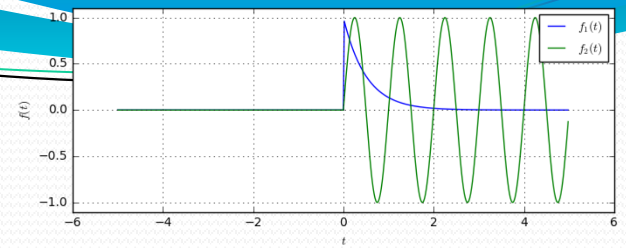


$f_1(t)$
$f_2(t)$
$f(t)$
t

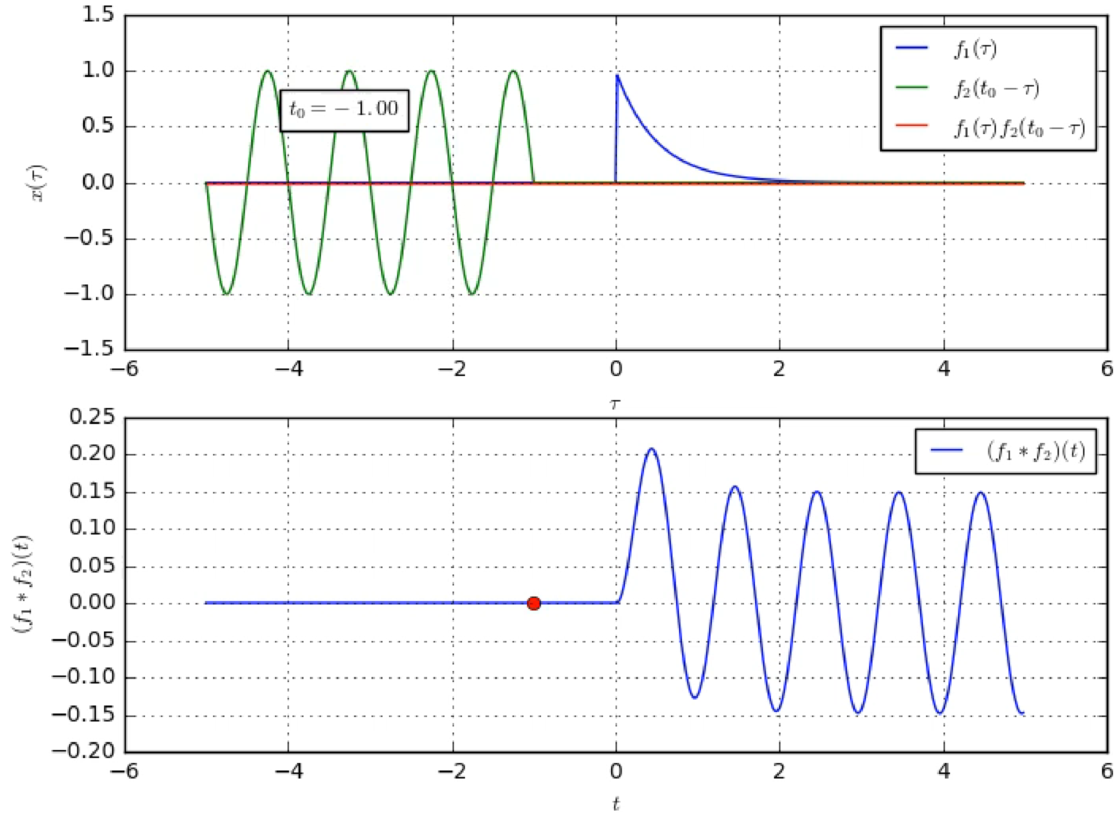


$f_1(\tau)$
$f_2(t_0-\tau)$
$f_1(\tau)f_2(t_0-\tau)$
$t_0 = -2.00$
$x(\tau)$
τ
$(f_1 * f_2)(t)$
t

Example 4



$f_1(t)$
$f_2(t)$
$f(t)$
t



$t_0 = -1.00$
$f_1(\tau)$
$f_2(t_0 - \tau)$
$f_1(\tau) f_2(t_0 - \tau)$
$x(\tau)$
τ
$(f_1 * f_2)(t)$
t

2-D Convolution

$$C(n_1,n_2) = \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty} a(k_1,k_2)b(n_1-k_1+1, n_2-k_2+1)$$

```
% 2-D convolution
A = [1 2 3
    4 5 6
    7 8 9];
B = A;
C = conv2(A,B,'same');
% C=
26  56    54
 84   165   144
 134   236   186
```

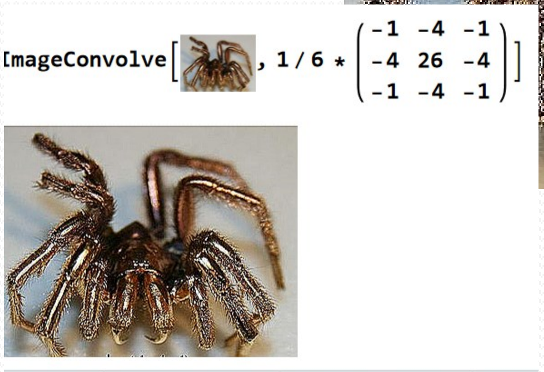
Note that matrix indices in MATLAB always start at 1 rather than 0.

$$C(1,1) = 1\times5+2\times4+4\times2+5\times1 = 26$$
$$C(1,2) = 1\times6+2\times5+3\times4+4\times3+5\times2+6\times1 = 56$$
$$C(1,3) = 2\times6+3\times5+5\times3+6\times2 = 54$$
$$\vdots$$

Applications of convolution

卷积的应用

- Electrical engineering 电子工程
  - Digital signal processing 数字信号处理
  - Image processing 图像处理
- Statistics 统计学
  - Moving average model… 滑动平均模型
- Probability theory 概率论
  - E.g. Pdf of X+Y 例如 X+Y 的概率分布
- …



Some useful properties of Convolution Integral
卷积积分的一些实用性质

(1) $f(t)*\delta(t)=f(t)$

$f(t)*\delta(t-t_0)=f(t-t_0)$

(2) $f(t)*u(t)=f^{(-1)}(t)=\int_{-\infty}^{t}f(\tau)d\tau$

(3) $f_1(t)*f_2(t)=f_2(t)*f_1(t)$

(4) $f_1(t)*[f_2(t)+f_3(t)]=f_1(t)*f_2(t)+f_1(t)*f_3(t)$

(5) $f_1(t)*f_2(t)*f_3(t)=f_1(t)*f_3(t)*f_2(t)$

$$(6)\quad \frac{d[f_1(t)*f_2(t)]}{dt} = f_1(t)*\frac{df_2(t)}{dt} = \frac{df_1(t)}{dt}*f_2(t)$$

$$(7)\quad \int_{-\infty}^{t}[f_1(\tau)*f_2(\tau)]d\tau = f_1(t)*\int_{-\infty}^{t} f_2(\tau)d\tau = \int_{-\infty}^{t} f_1(\tau)d\tau * f_2(t)$$

$$(8)\quad if \quad f_1(-\infty) = f_2(-\infty) = 0$$

$$denote \quad f(t) = f_1(t)*f_2(t)$$

$$then \quad f^{(i)}(t) = f_1^{(j)}(t)*f_2^{(i-j)}(t)$$

Proof.

(3) $f_1(t) * f_2(t) = f_2(t) * f_1(t)$

$$f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau) d\tau$$

$$a = t - \tau$$

$$f_1(t) * f_2(t) = \int_{\infty}^{-\infty} f_1(t-a) f_2(a) d\left(-a\right)$$

$$= \int_{-\infty}^{\infty} f_1(t-a) f_2(a) da$$

$$= f_2(t) * f_1(t)$$

Proof.

$$(7)\ \int_{-\infty}^{t}[f_1(\tau)*f_2(\tau)]d\tau = f_1(t)*\int_{-\infty}^{t}f_2(\tau)d\tau = \int_{-\infty}^{t}f_1(\tau)d\tau * f_2(t)$$

$$\int_{-\infty}^{t}[f_1(\tau)*f_2(\tau)]d\tau = u\left(t\right)*[f_1(t)*f_2(t)]$$

$$f_1(t)*\int_{-\infty}^{t}f_2(\tau)d\tau = f_1\left(t\right)*[u(t)*f_2(t)]$$

$$\int_{-\infty}^{t}f_1(\tau)d\tau * f_2(t) = [u(t)*f_1(t)]*f_2\left(t\right)$$

$$(8) \quad if \quad f_1(-\infty) = f_2(-\infty) = 0$$

$$denote \quad f(t) = f_1(t) * f_2(t)$$

$$then \quad f^{(i)}(t) = f_1^{(j)}(t) * f_2^{(i-j)}(t)$$

Homework: proof it

Example

Suppose that $f_1(t) = (1+t)[u(t) - u(t-1)],$

$f_2(t) = u(t-1)$

Determine $f_1(t) * f_2(t)$

$$f_1(t) * f_2(t) = \int_{-\infty}^{t} f_1(\tau)d\tau * \frac{df_2(t)}{dt}$$

$$= \int_{-\infty}^{t} (1+\tau)\left[u(\tau) - u(\tau-1)\right]d\tau * \delta(t-1)$$

$$= \left[\int_{-\infty}^{t} (1+\tau)u(\tau)\,d\tau - \int_{-\infty}^{t} (1+\tau)u(\tau-1)\,d\tau\right] * \delta(t-1)$$

$$= \left[\int_{0}^{t} (1+\tau)d\tau u(t) - \int_{1}^{t} (1+\tau)d\tau u(t-1)\right] * \delta(t-1)$$

$$f_1(t)*f_2(t)=\left[\int_0^t(1+\tau)d\tau u\left(t\right)-\int_1^t(1+\tau)d\tau u\left(t-1\right)\right]*\delta(t-1)$$

$$=\left[\tau+\frac{\tau^2}{2}\Bigg|_0^t u\left(t\right)-\tau+\frac{\tau^2}{2}\Bigg|_1^t u\left(t-1\right)\right]*\delta(t-1)$$

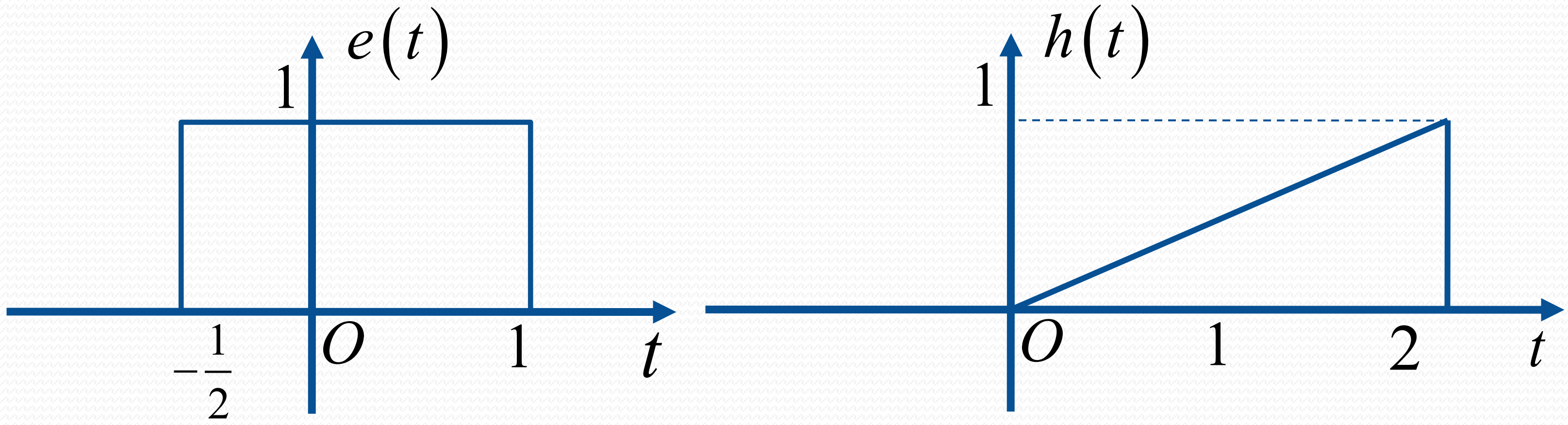
$$=\left[\left(t+\frac{t^2}{2}\right)u(t)-\left(t-1+\frac{t^2-1}{2}\right)u(t-1)\right]*\delta(t-1)$$

$$f_1(t)*f_2(t)=\left[t-1+\frac{(t-1)^2}{2}\right]u(t-1)$$

$$-\left[t-2+\frac{(t-1)^2-1}{2}\right]u(t-2)$$

$$=\frac{t^2-1}{2}u(t-1)-\frac{t^2-4}{2}u(t-2)$$

Exercise. Determine the convolution of e(t) and h(t).



$$e(t) = u\left(t + \frac{1}{2}\right) - u(t-1) \qquad h(t) = \frac{1}{2}t\left[u(t) - u(t-2)\right]$$

$$e(t)*h(t)=\frac{de(t)}{dt}*\int_{-\infty}^{t}h(\tau)d\tau$$

$$\because\ \frac{de(t)}{dt}=\delta\left(t+\frac{1}{2}\right)-\delta(t-1)$$

$$\int_{-\infty}^{t}h(\tau)d\tau=\frac{1}{2}\int_{-\infty}^{t}\tau[u(\tau)-u(\tau-2)]d\tau$$

$$=\frac{1}{2}\left(\int_{0}^{t}\tau d\tau u(t)-\int_{2}^{t}\tau d\tau u(t-2)\right)$$

$$=\frac{1}{4}t^{2}[u(t)-u(t-2)]+u(t-2)$$

$$\therefore e(t)*h(t)=\left[\delta\left(t+\frac{1}{2}\right)-\delta(t-1)\right]*$$

$$\left\{\frac{1}{4}t^2\left[u(t)-u(t-2)\right]+u(t-2)\right\}$$

$$=\delta\left(t+\frac{1}{2}\right)*\left\{\frac{1}{4}t^2\left[u(t)-u(t-2)\right]\right\}$$

$$+\delta\left(t+\frac{1}{2}\right)*u(t-2)$$

$$-\delta(t-1)*\left\{\frac{1}{4}t^2\left[u(t)-u(t-2)\right]\right\}$$

$$-\delta(t-1)*u(t-2)$$

$$e(t)*h(t)=\left\{\frac{1}{4}t^2\left[u(t)-u(t-2)\right]\right\}*\delta\left(t+\frac{1}{2}\right)$$

$$+u(t-2)*\delta\left(t+\frac{1}{2}\right)$$

$$-\left\{\frac{1}{4}t^2\left[u(t)-u(t-2)\right]\right\}*\delta(t-1)$$

$$-u(t-2)*\delta(t-1)$$

$$e(t)*h(t)=\frac{1}{4}\left(t+\frac{1}{2}\right)^2\left[u\left(t+\frac{1}{2}\right)-u\left(t-\frac{3}{2}\right)\right]+u\left(t-\frac{3}{2}\right)$$

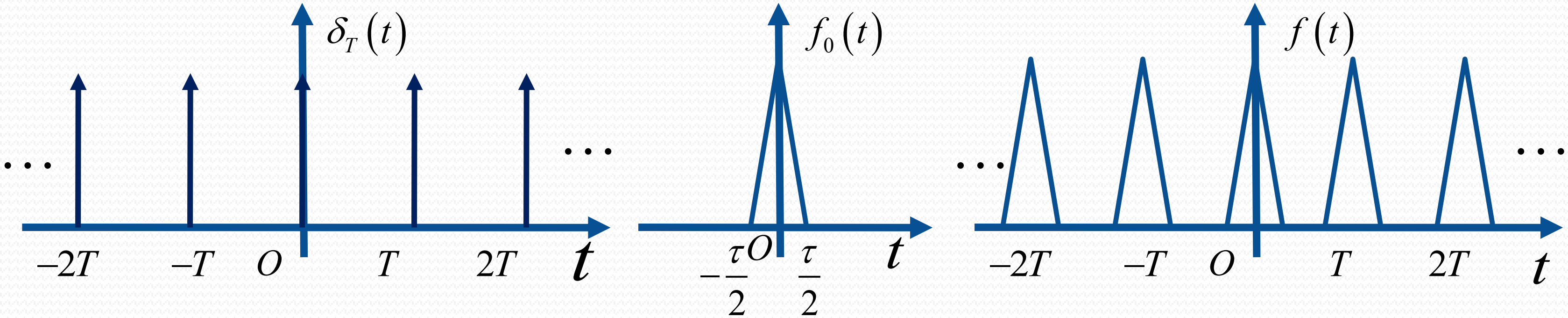
$$-\frac{1}{4}(t-1)^2\left[u(t-1)-u(t-3)\right]-u(t-3)$$

$$\therefore r_{zs}(t)=e(t)*h(t)=\begin{cases}\frac{1}{4}\left(t+\frac{1}{2}\right)^2 & -\frac{1}{2}\le t<1\\ \frac{3}{4}\left(t-\frac{1}{4}\right) & 1\le t<\frac{3}{2}\\ 1-\frac{1}{4}(t-1)^2 & \frac{3}{2}\le t<3\end{cases}$$

Example

Let $\delta_T(t)=\sum_{m=-\infty}^{\infty}\delta(t-mT)$,

$f_0(t)$ is shown in the figure below. Compute $\delta_T(t)*f_0(t)$。



$$\delta_T(t)*f_0(t)=f_0(t)*\delta_T(t)$$

$$=f_0(t)*[\sum_{m=-\infty}^{\infty}\delta(t-mT)]=\sum_{m=-\infty}^{\infty}[f_0(t)*\delta(t-mT)]$$

$$=\sum_{m=-\infty}^{\infty}f_0(t-mT)$$

Summary 小结

- Represent an arbitrary signal by means of unit impulse signal
- Convolution
  - Properties of convolution
  - Properties of unit impulse signal
- 利用单位脉冲信号表示任意信号
- 卷积
  - 卷积的性质
  - 单位脉冲信号的性质