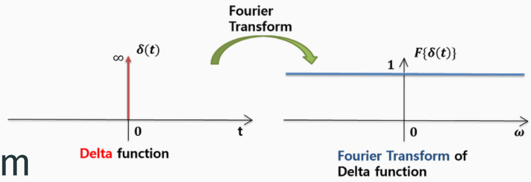

Signal and System

Fourier Series - 2

陆凯

2022 年 3 月 22 日

Feedback via https://www.tapechat.net/uu/A0Z7ZH/6Z00BZMJ

ToC

Recap

Examples: Fourier Series of Continues Time Periodic Signal

Recap

Fourier Series - 1

1. Response of LTI Systems to Complex Exponentials （LTI **系统对复指数函数的响应**），

$$in : e^{st} \Rightarrow out : H(s)e^{st} \tag{1}$$

2. Fourier Series of Continues Time Periodic Signal (**连续时间周期函数的傅里叶级数**),

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t} \tag{2}$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jkw_0 t}dt = \frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}dt \tag{3}$$

3. Convergence of Fourier Series （**傅里叶级数展开收敛性**: Dirichlet **三条件，能量条件**),
4. Properties of Fourier Series（**傅里叶级数的性质**）。

Example 1

Symmetric Periodic Square Wave(对称周期方波信号)

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$

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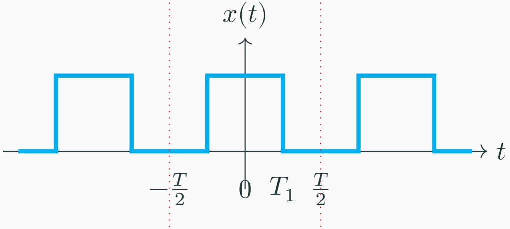

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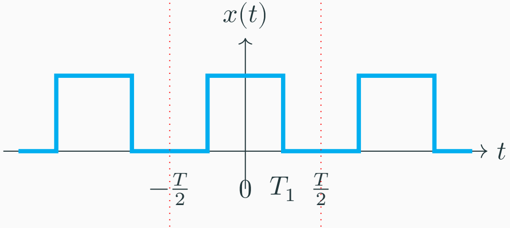

Fourier Series a_k: $a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_{-T_1}^{T_1} x(t)e^{-jk\omega_0 t}dt$

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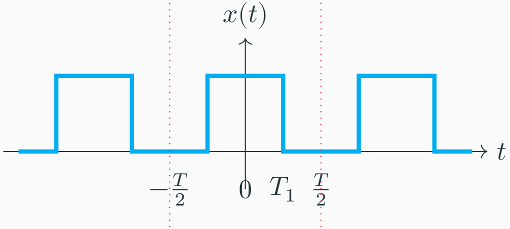

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$$= \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}dt = -\frac{1}{jk\omega_0 T}e^{-jk\omega_0 t}\Big|_{-T_1}^{T_1} = \frac{2}{k\omega_0 T}\left[\frac{e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}}{2j}\right]$$

Example 1

Fourier Series of Symmetric Periodic Square Wave (**对称周期方波信号的傅里叶级数**) a_k

$$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T} = \frac{\sin(k\omega_0 T_1)}{k\pi} = 2\frac{T_1}{T} Sa(2\pi k \frac{T_1}{T}), \quad k \neq 0$$

where, fundamental frequency (**基波角频率**) is

$$\omega_0 = 2\pi/T, Sa(x) = \sin(x)/x$$

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An alternative approach: limit of the general expression

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$$a_0 = \lim_{k\to 0} a_k = \frac{2T_1}{T}$$

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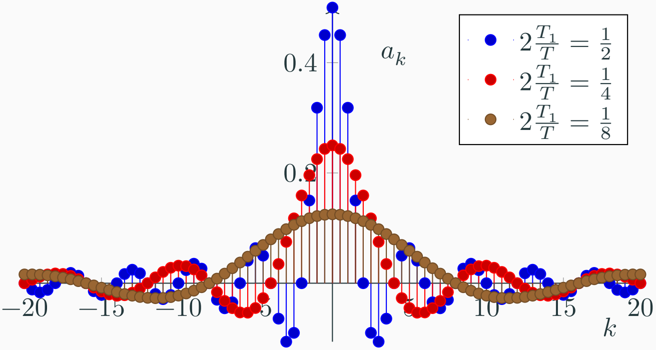
$$a_k = \frac{2T_1}{T} Sa(\pi k \frac{2T_1}{T})$$



Fourier Series with Different Windows Width $\frac{2T_1}{T} \in (0, 1)$

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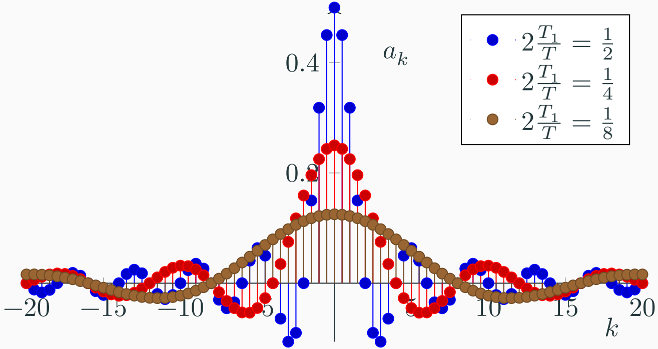

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a_k (or write as $a(k)$) is even, as $x(t)$ is.

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Express $x(t)$ as sum of Fourier Series

$$x(t) = \widetilde{x(t)} \simeq \sum_{k=-N}^{+N} a_k e^{jk(2\pi/T)t}$$

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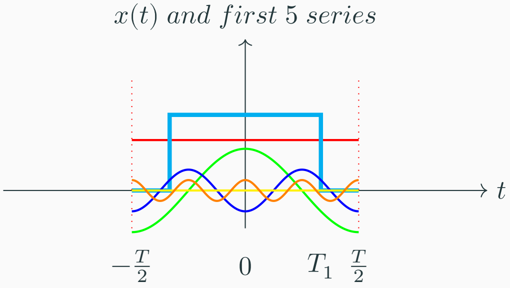

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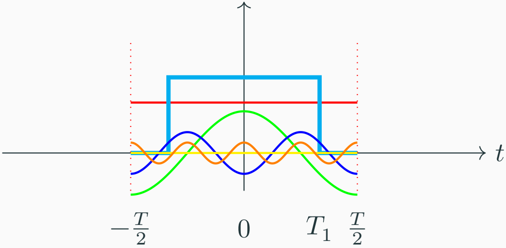
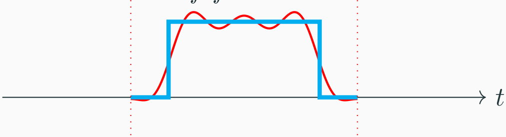
$x(t)$ *and first* 5 *series*



sum of first 5 *series*



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Impulse $\delta(t)$ Train (周期脉冲信号)

$$g(t) = \sum_{k=-\infty}^{+\infty} \delta(t - kT)$$

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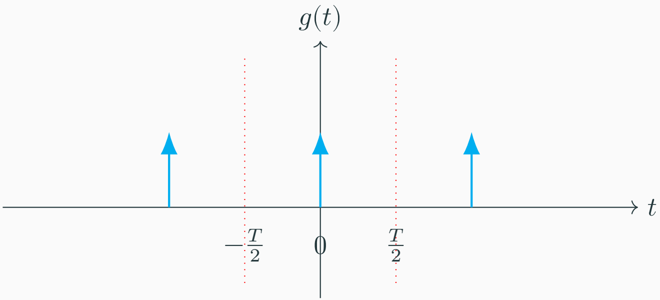

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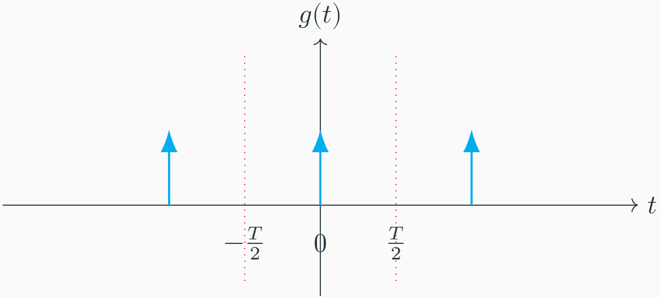

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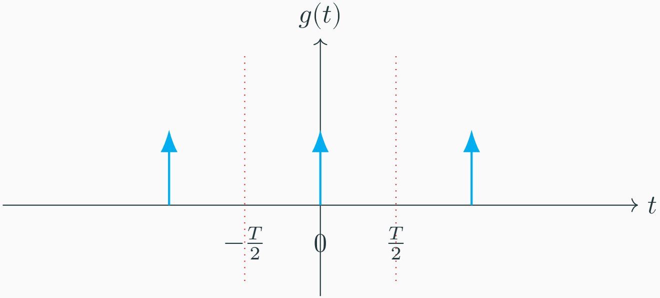

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Fourier Series Coefficients of Impulse $\delta(t)$ Train are Identical for all, $\frac{1}{T}$

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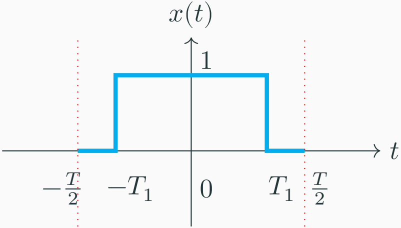
Revisit the periodic square wave, and plot its derivative



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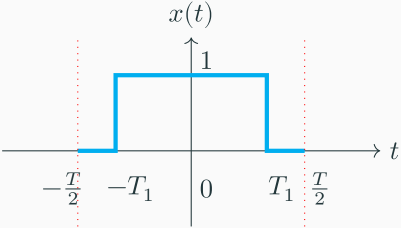

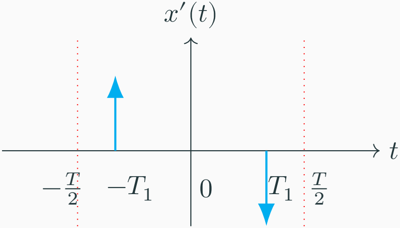

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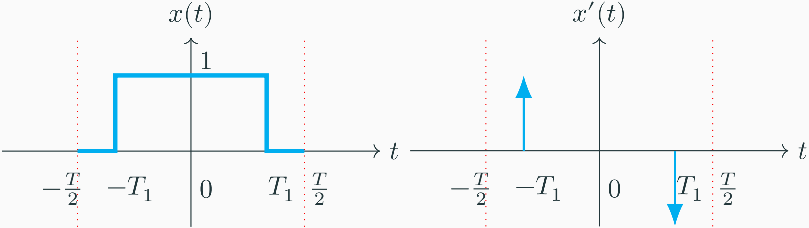

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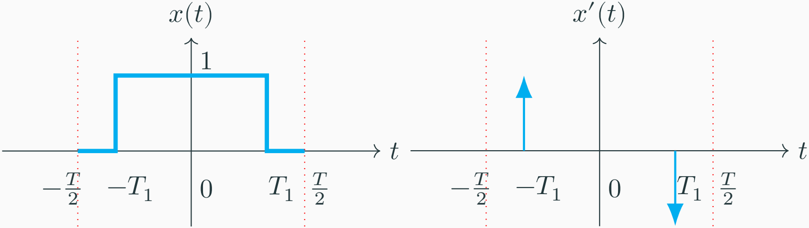

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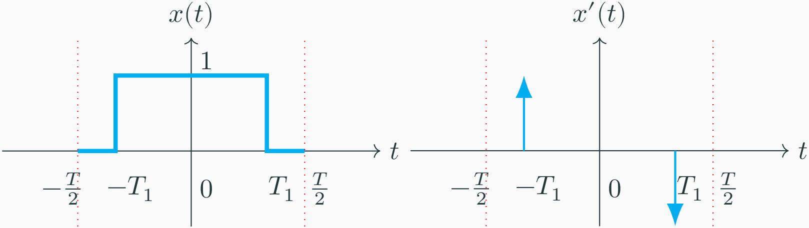

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Using differentiation property, Fourier Series Coefficients of $x(t)$ are

$$c_k = \frac{b_k}{jk(2\pi/T)} = \frac{2T_1}{T}Sa(\pi k\frac{2T_1}{T})$$

Example 3 Extra: Differentiation Property

Assume we have

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k$$

Then proof the differentiation property, i.e.

$$x'(t) \overset{\mathcal{FS}}{\longleftrightarrow} jk\omega_0 a_k$$

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Another approach? Take it as an exercise.

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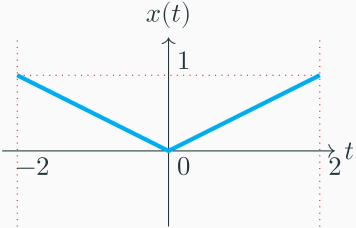
Consider the periodic triangular wave, and its 1st and 2nd derivatives



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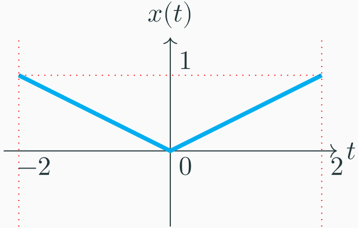

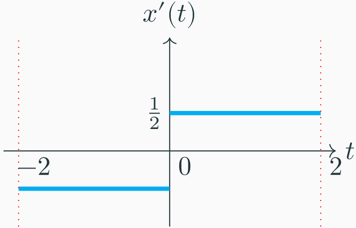

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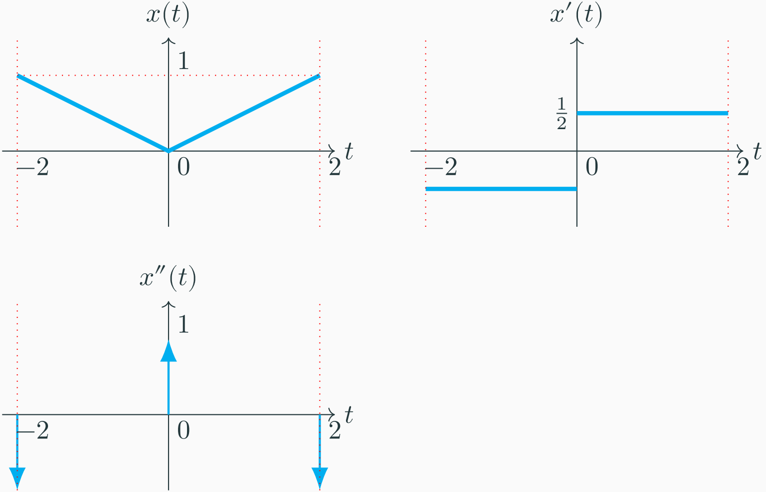

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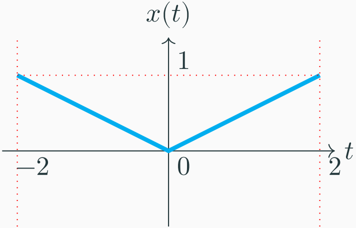

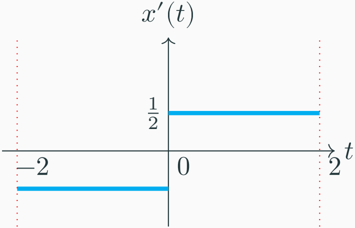

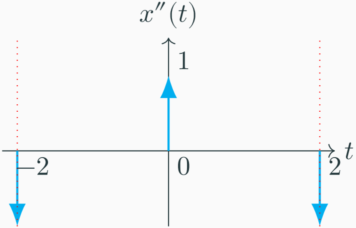

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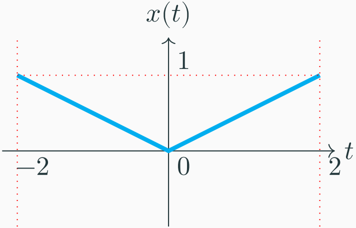

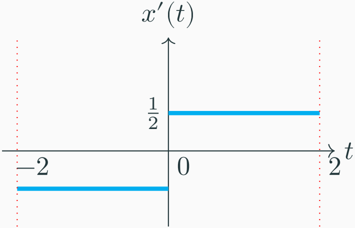

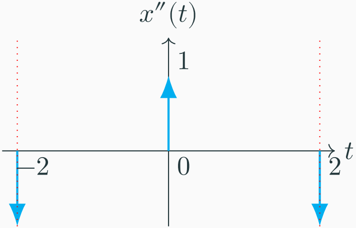

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we have

$$x(t) \xrightarrow{\mathcal{FS}} \frac{a_k(1 - e^{-j2k(2\pi/T)})}{(jk\omega_0)^2}$$

Example 5

Suppose we are given the following facts about a signal $x(t)$:

1. $x(t)$ is a real signal,
2. $x(t)$ is periodic with period of 4, and it has Fourier series coefficients a_k,
3. $a_k = 0$ for $k > 1$,
4. The signal with Fourier coefficients $b_k = e^{-j\pi k/2} a_{-k}$ is odd,
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$$a_k = 0 \text{ for } k > 1 \rightarrow a_{-1}, a_0, a_1 \text{ are the only non-zero items,}$$

Example 5

Suppose we are given the following facts about a signal $x(t)$:

1. $x(t)$ is a real signal,
2. $x(t)$ is periodic with period of 4, and it has Fourier series coefficients a_k,
3. $a_k = 0$ for $k > 1$,
4. The signal with Fourier coefficients $b_k = e^{-j\pi k/2} a_{-k}$ is odd,
5. $\frac{1}{4} \int_T |x(t)|^2 dt = \frac{1}{2}$.

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$$x(t) = cos(\pi t/2)$$