Lecture 16
Properties of the Continuous-Time Fourier Transform

Basic Properties of Fourier Transform

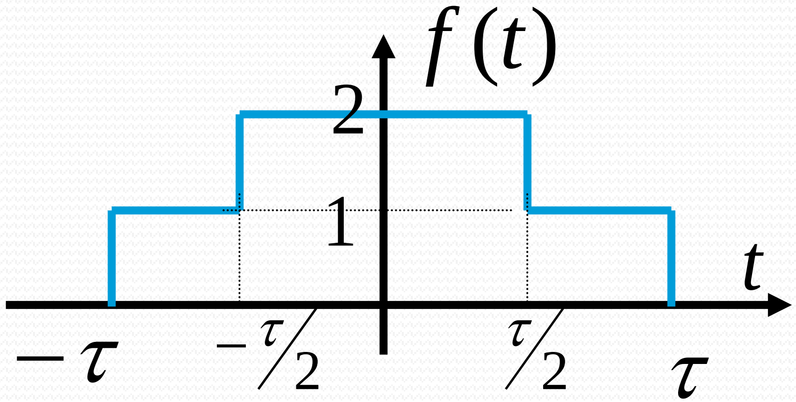
- Linearity
- Odd-Even and Real-Imaginary
- Time and Frequency Scaling
- Shifting in Time and in Frequency
- Differentiation and Integration
- Parseval's Relation
- Duality
- Convolution
- Differentiation and Integration

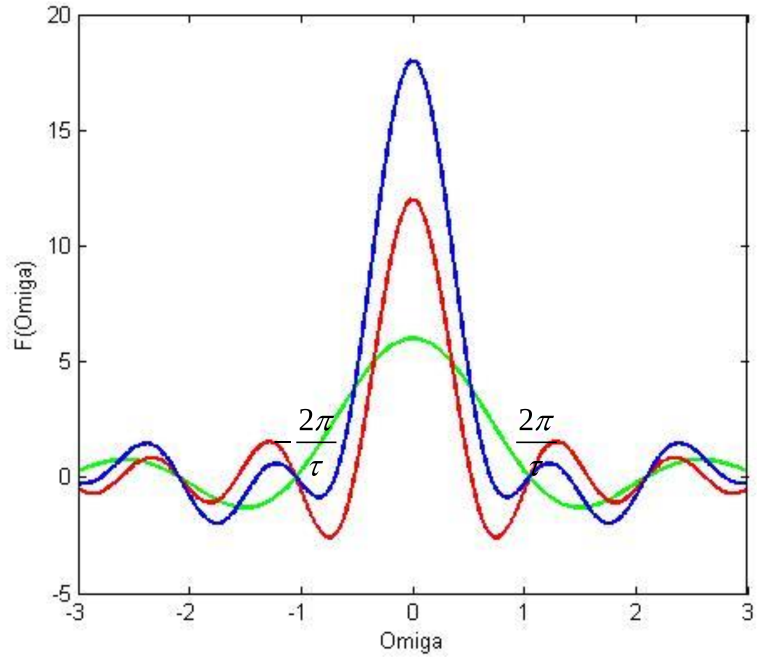
•Linearity

$$FT\left[f_i(t)\right]=F_i(j\omega)$$

$$FT\left[\sum_{i=1}^{n}a_i f_i(t)\right]=\sum_{i=1}^{n}a_iF_i(j\omega)$$

Example Determine the Fourier Transform of f(t)





$$f(t) = [u(t+\tfrac{\tau}{2}) - u(t-\tfrac{\tau}{2})] + [u(t+\tau) - u(t-\tau)]$$

$$F(j\omega) = \tau[Sa(\omega\tau/2) + 2Sa(\omega\tau)]$$

•Odd-Even and Real-Imaginary

$$FT[f(t)] = F(j\omega)$$

$$\boxed{FT[f(-t)] = F(-j\omega)}$$

Reverse in time domain, then reverse in frequency domain

$$\boxed{FT[f^*(t)] = F^*(-j\omega)}$$

Conjugate in time domain, then conjugate and reverse in frequency domain.

$$FT[f(-t)] = \int_{-\infty}^{\infty} f(-t)e^{-j\omega t}dt = \int_{\infty}^{-\infty} f(\tau)e^{j\omega\tau}d(-\tau)$$

$$= \int_{-\infty}^{\infty} f(\tau)e^{j\omega\tau}d\tau = \int_{-\infty}^{\infty} f(\tau)e^{-j(-\omega)\tau}d\tau = F(-j\omega)$$

$$FT[f^*(t)] = \int_{-\infty}^{\infty} f^*(t)e^{-j\omega t}dt = \left[\int_{-\infty}^{\infty} f(t)e^{j\omega t}dt\right]^* = F^*(-j\omega)$$

for real $f(t)$, we have $F(jw) = F^*(-jw)$

or $F(-jw) = F^*(jw)$

$F(-j\omega) = \overset{*}{F}(j\omega)$

- If *f(t)* is a real function of *t*

$$F(j\omega) = \underbrace{\int_{-\infty}^{\infty} f(t)\cos\omega t dt}_{R(\omega)} - j\underbrace{\int_{-\infty}^{\infty} f(t)\sin\omega t dt}_{X(\omega)}$$

$R(\omega) = R(-\omega)$ Even function

$X(\omega) = -X(-\omega)$ Odd function

$|F(j\omega)| = \sqrt{R^2(\omega) + X^2(\omega)}$ Even function

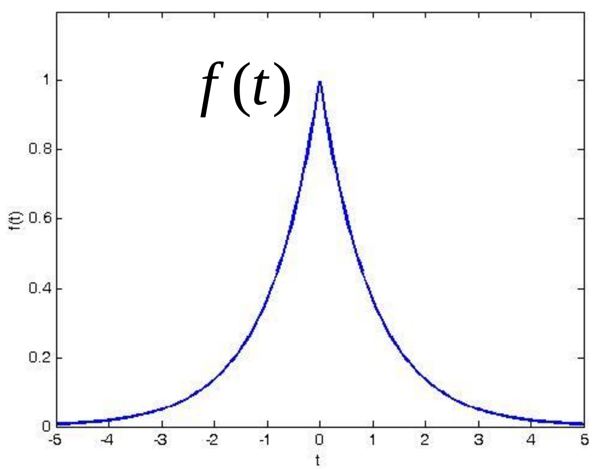
$\phi(\omega) = \arctan(\frac{X(\omega)}{R(\omega)})$ Odd function

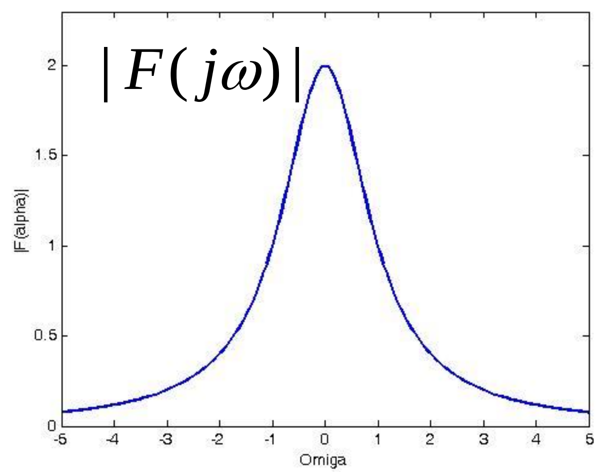
$F(-jw) = F^*(jw)$ real $\qquad F(-jw) = F(jw)$ even $\Rightarrow F^*(jw) = F(jw)$

The FT of a real-even function still is a real-even function

$$f(t) = e^{-\alpha|t|} \qquad (-\infty < t < +\infty)$$

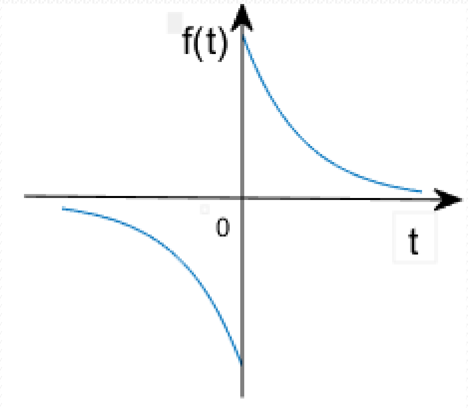
$$F(j\omega) = \frac{2\alpha}{\alpha^2 + \omega^2} \qquad \varphi(\omega) = 0$$

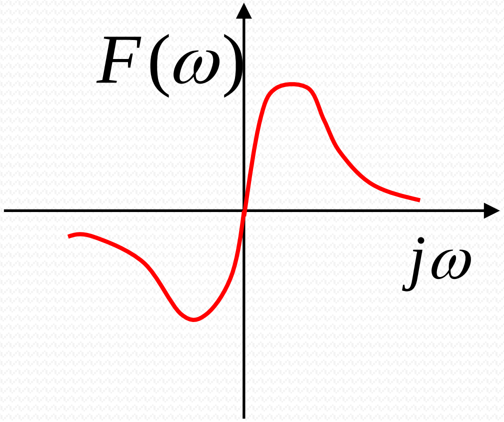




The FT of a real-odd function is an imaginary-odd function

$$f(t)=\begin{cases} e^{-at} & (t>0) \\ -e^{at} & (t<0) \end{cases}$$

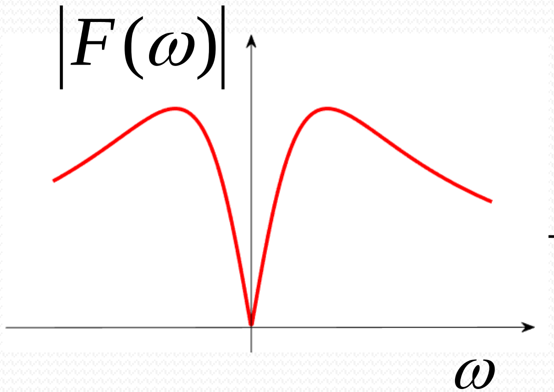


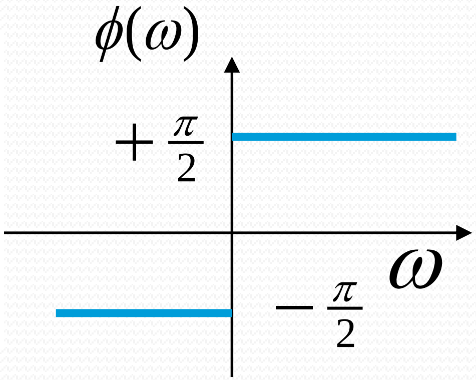


$$F(j\omega)=\frac{-2j\omega}{\alpha^2+\omega^2}$$

$$|F(j\omega)|=\frac{2|\omega|}{\alpha^2+\omega^2}$$

$$\varphi(\omega)=\begin{cases} -\dfrac{\pi}{2} & (\omega>0) \\ +\dfrac{\pi}{2} & (\omega<0) \end{cases}$$





- If f(t) = jg(t) is an imaginary function

$$F(j\omega) = \underbrace{\int_{-\infty}^{\infty} g(t)\sin\omega t dt}_{R(\omega)} + j\underbrace{\int_{-\infty}^{\infty} g(t)\cos\omega t dt}_{X(\omega)}$$

$R(\omega) = -R(-\omega)$ odd function $X(\omega) = X(-\omega)$ even function

The magnitude of FT for imaginary function is an even function; its spectrum of phase is an odd function.

•Time and Frequency Scaling

$$If \quad FT[f(t)] = F(j\omega)$$
$$Then \quad FT[f(at)] = \frac{1}{|a|}F(\frac{j\omega}{a})$$

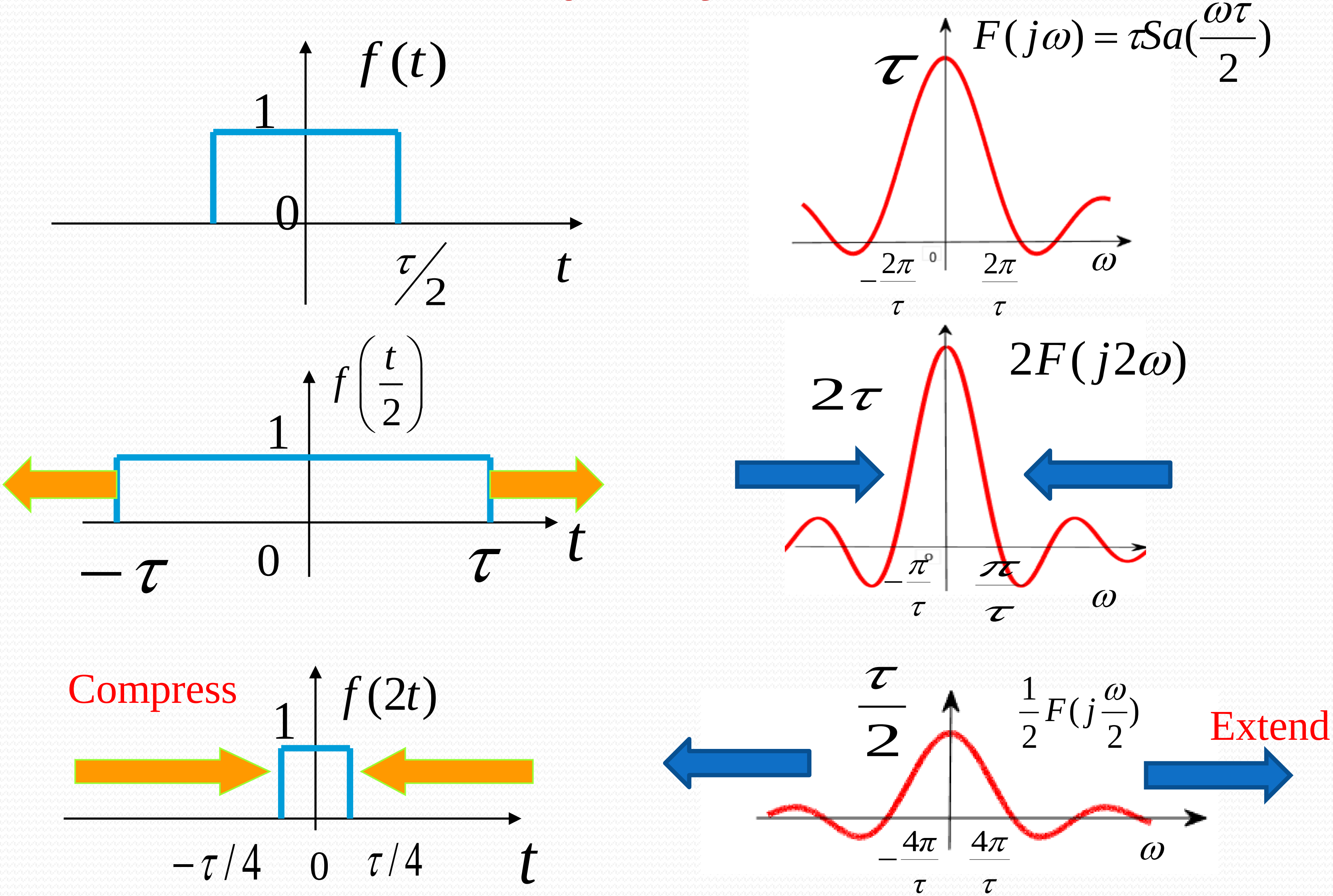
$$a > 0 \quad FT[f(at)] = \int_{-\infty}^{\infty} f(at)e^{-j\omega t}dt$$

$$= \frac{1}{a}\int_{-\infty}^{\infty} f(x)e^{-j\omega\frac{x}{a}}dx = \frac{1}{a}\int_{-\infty}^{\infty} f(x)e^{-j\frac{\omega}{a}x}dx = \frac{1}{a}F(j\frac{\omega}{a})$$

$$a < 0 \quad FT[f(at)] = \int_{-\infty}^{\infty} f(at)e^{-j\omega t}dt$$

$$= \frac{-1}{a}\int_{-\infty}^{\infty} f(x)e^{-j\omega\frac{x}{a}}dx = \frac{-1}{a}\int_{-\infty}^{\infty} f(x)e^{-j\frac{\omega}{a}x}dx = \frac{-1}{a}F(j\frac{\omega}{a})$$

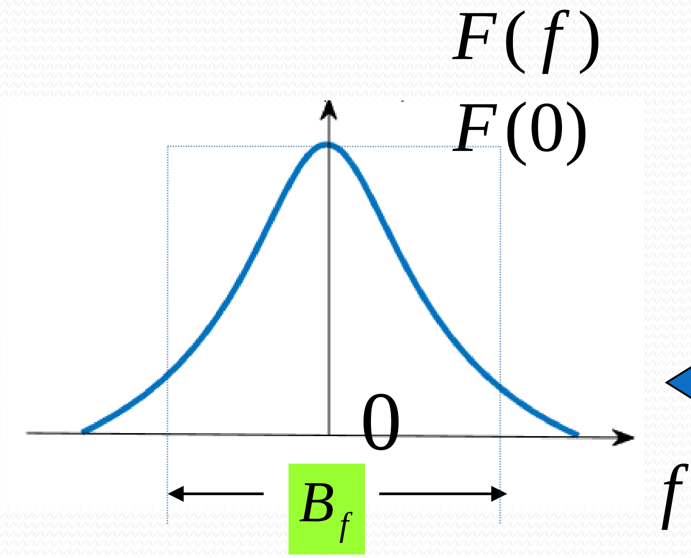
Compressing in time-domain implies extending in frequency-domain



Pulse Width and Band Width

$$F(f) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi f t} dt \qquad f(t) = \int_{-\infty}^{\infty} F(f) e^{j2\pi f t} df$$

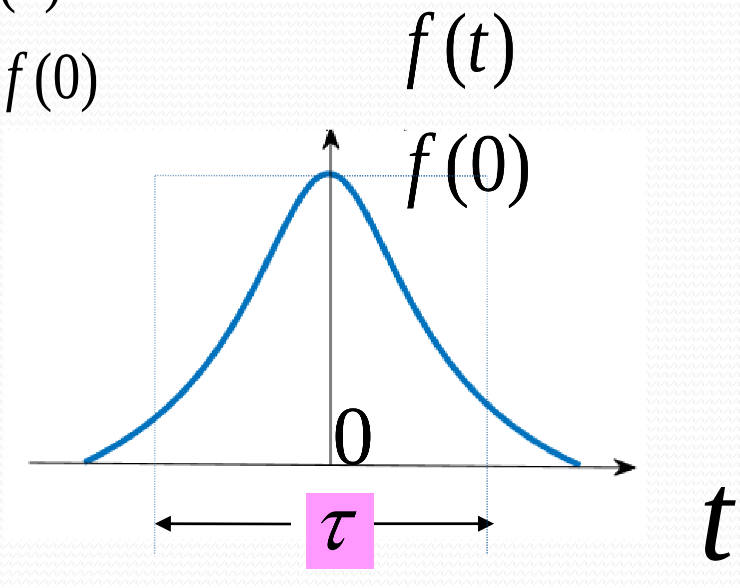
$$F(0) = \int_{-\infty}^{\infty} f(t) dt \qquad f(0) = \int_{-\infty}^{\infty} F(f) df$$



$$f(0) \cdot \tau = F(0)$$

$$F(0) \cdot B_f = f(0)$$

$$B_f = \frac{1}{\tau}$$



•Shifting in Time

$$if \quad FT\left[f(t)\right]=F(j\omega)$$

$$then \quad FT\left[f(t-t_0)\right]=F(j\omega)e^{-j\omega t_0}$$

$$FT\left[f(t-t_0)\right]=\int_{-\infty}^{\infty} f(t-t_0)e^{-j\omega t}dt$$

$$Let \quad x=t-t_0$$

$$FT\left[f(x)\right]=\int_{-\infty}^{\infty} f(x)e^{-j\omega(x+t_0)}dx$$

$$=e^{-j\omega t_0}\int_{-\infty}^{\infty} f(x)e^{-j\omega x}dx=e^{-j\omega t_0}F(j\omega)$$

$$\therefore \quad FT\left[f(t-t_0)\right]=e^{-j\omega t_0}F(j\omega)$$

Shifting in Time with Scaling

$$if \quad FT\left[f(t)\right]=F(j\omega)$$

$$FT\left[f(at-t_0)\right]=\frac{1}{|a|}F(j\frac{\omega}{a})e^{-j\frac{\omega t_0}{a}}$$

$$Proof \quad FT[f(at-t_0)]=\int_{-\infty}^{\infty} f(at-t_0)e^{-j\omega t}dt \qquad a>0$$

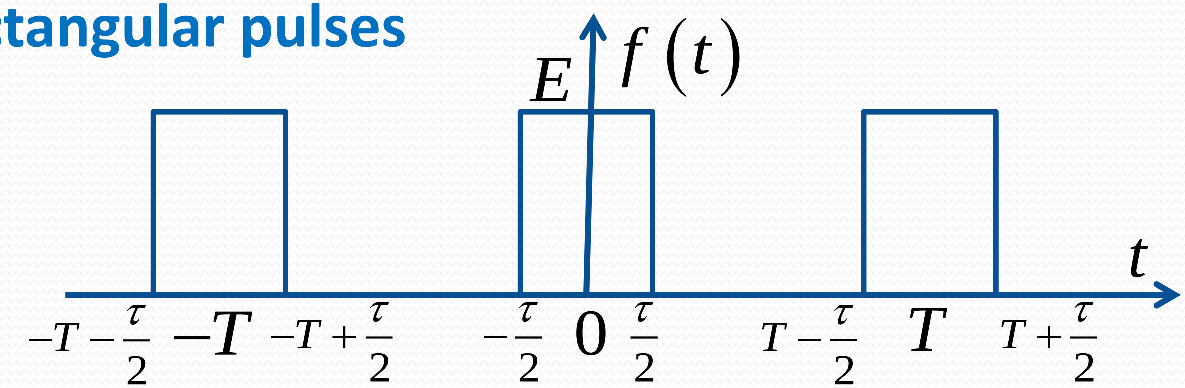
$$x=at-t_0 \qquad =\frac{1}{a}\int_{-\infty}^{\infty} f(x)e^{-j\omega(x+t_0)/a}dx$$

$$t=(x+t_0)/a \qquad =\frac{1}{a}e^{-j\frac{\omega t_0}{a}}\int_{-\infty}^{\infty} f(x)e^{-j(\frac{\omega}{a})x}dx$$

$$=\frac{1}{a}e^{-j\frac{\omega t_0}{a}}F(j\frac{\omega}{a})$$

$$FT[f(at-t_0)]=-\frac{1}{a}e^{-j\frac{\omega t_0}{a}}F(j\frac{\omega}{a}) \qquad a<0$$

Example Determine the spectrum of a signal composed by three rectangular pulses



It should be ***EASY*** to recall the FT of single rectangular pulse, $f_0(t)$, is

$$F_0(j\omega) = E\tau\, Sa(\frac{\omega\tau}{2})$$

f(t) can be expressed as:

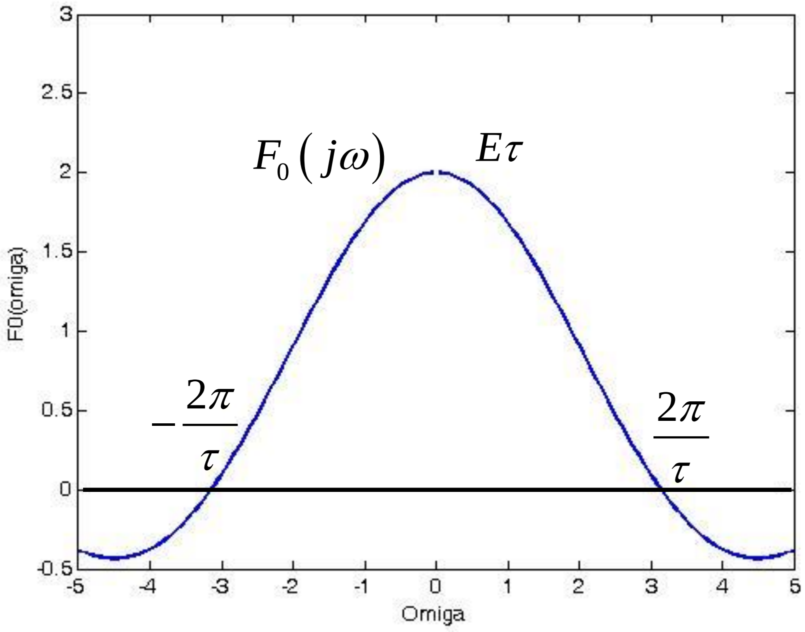
$$f(t) = f_0(t) + f_0(t+T) + f_0(t-T)$$

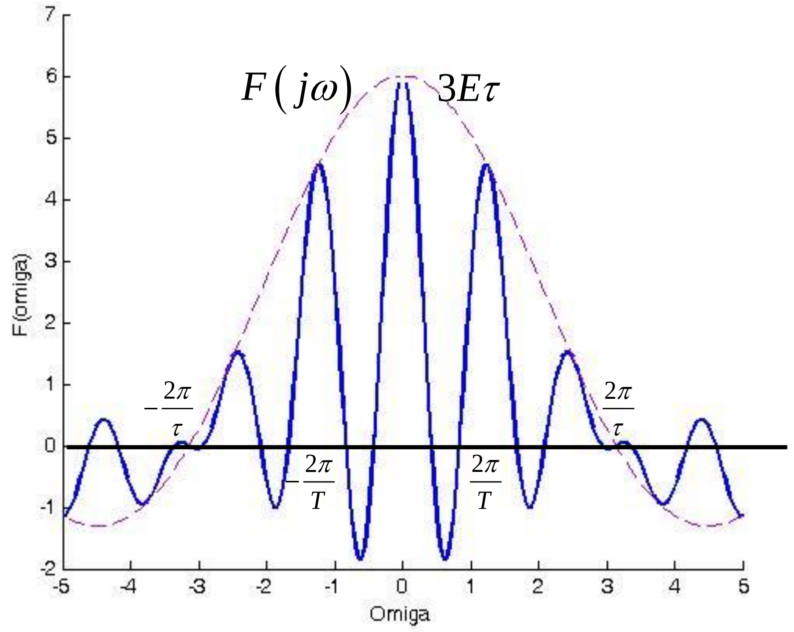
It's spectrum is:

$$F(j\omega) = F_0(j\omega)(1 + e^{j\omega T} + e^{-j\omega T})$$

$$= F_0(j\omega)(1 + 2\cos\omega T)$$

$$= E\tau\, Sa(\frac{\omega\tau}{2})(1 + 2\cos\omega T)$$


$F_0(j\omega)$
$E\tau$
$-\frac{2\pi}{\tau}$
$\frac{2\pi}{\tau}$
F0(omiga)
Omiga


$F(j\omega)$
$3E\tau$
$-\frac{2\pi}{\tau}$
$\frac{2\pi}{\tau}$
$-\frac{2\pi}{T}$
$\frac{2\pi}{T}$
F(omiga)
Omiga

•Shifting in Frequency

$$if \quad FT\left[f(t)\right] = F(j\omega)$$

$$then \quad FT[f(t)e^{j\omega_0 t}] = F\left(j\left(\omega - \omega_0\right)\right)$$

$$Proof$$

$$FT[f(t)e^{j\omega_0 t}] = \int_{-\infty}^{\infty} f(t)e^{j\omega_0 t}e^{-j\omega t}dt$$

$$= \int_{-\infty}^{\infty} f(t)e^{-j(\omega-\omega_0)t}dt = F\left(j\left(\omega - \omega_0\right)\right)$$

$$Similarly \quad FT[f(t)e^{-j\omega_0 t}] = F\left(j\left(\omega + \omega_0\right)\right)$$

Example FT of Sinusoidal and Cosine Signals

-- By the Shifting Property in Frequency

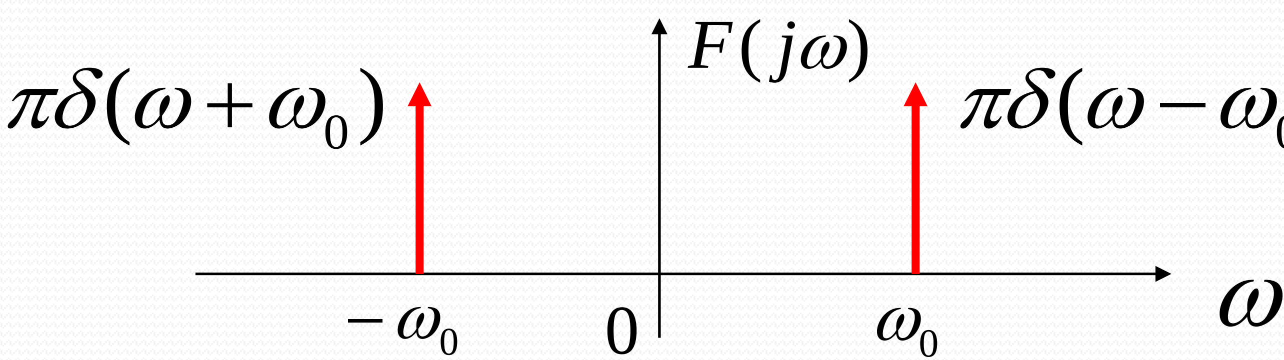
$$f_0(t) = 1$$

$$F_0(j\omega) = FT[1] = 2\pi\delta(\omega)$$

$$FT[f_0(t)\cdot e^{j\omega_0 t}] = F_0(j(\omega - \omega_0))$$

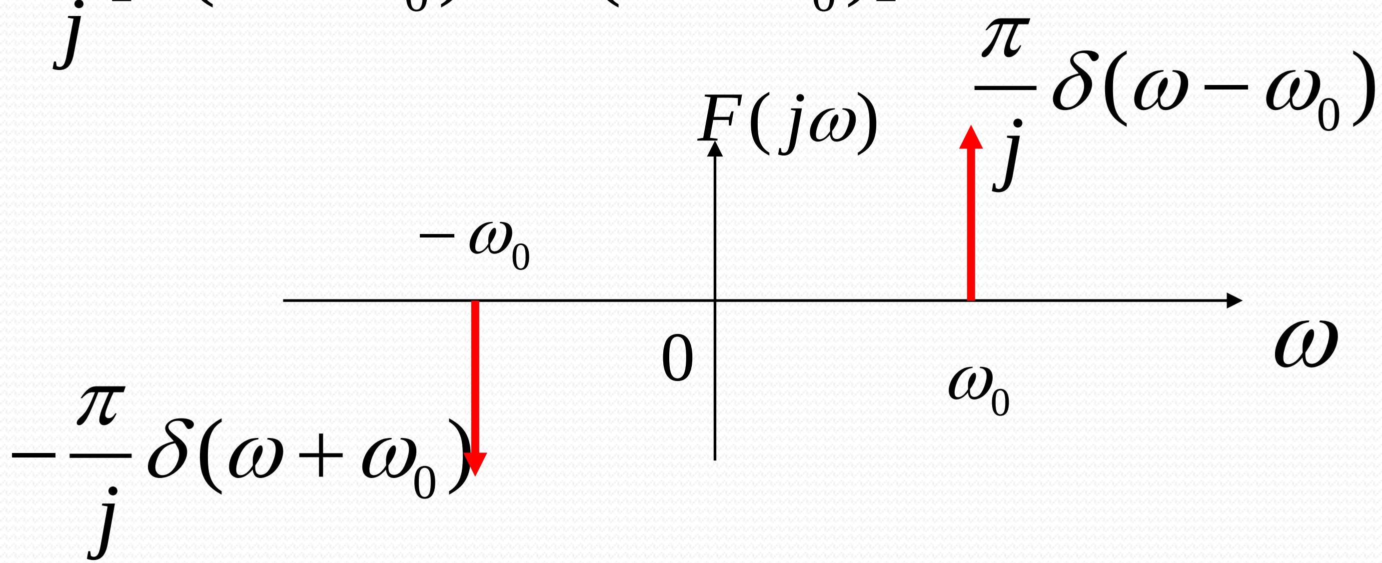
$$FT[\cos\omega_0 t] = FT[\tfrac{1}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t})]$$

$$= \pi[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)]$$



$$FT[\sin\omega_0 t] = FT[\tfrac{1}{2j}(e^{j\omega_0 t} - e^{-j\omega_0 t})]$$

$$= \frac{\pi}{j}[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)]$$



Example Determine the spectrum of $f(t)\cos\omega_0 t$

$$\cos\omega_0 t = \frac{1}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t})$$

$\cos\omega_0 t$ is modulated by $f(t)$.

$$FT[f(t)\cos\omega_0 t] = FT[\frac{1}{2}f(t)e^{j\omega_0 t}] + FT[\frac{1}{2}f(t)e^{-j\omega_0 t}]$$

$$= \frac{1}{2}[F(j(\omega - \omega_0)) + F(j(\omega + \omega_0))]$$

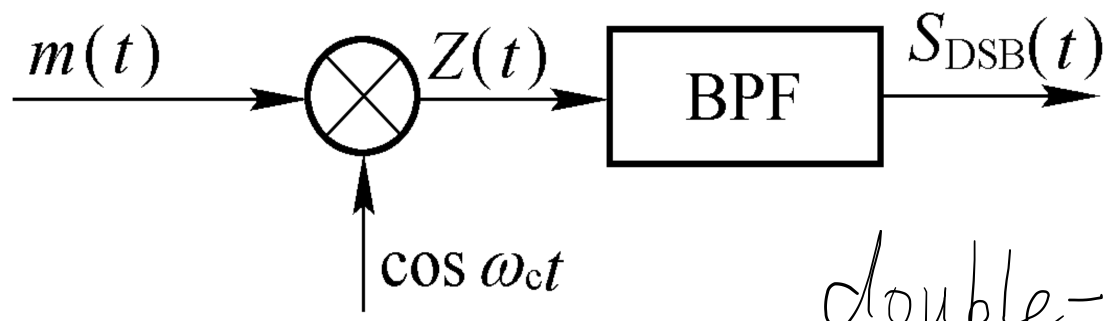
Question: How do you interpret the meaning of this result?

Similarly:

$$\sin\omega_0 t = \frac{1}{2j}(e^{j\omega_0 t} - e^{-j\omega_0 t})$$

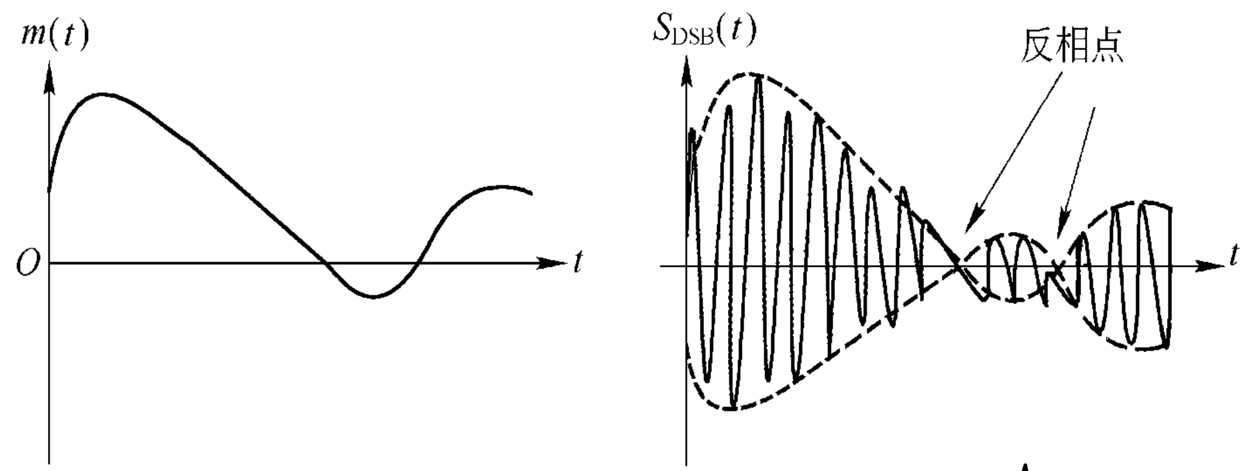
$$FT[f(t)\sin\omega_0 t] = \frac{1}{2j}[F(j(\omega-\omega_0)) - F(j(\omega+\omega_0))]$$

Application in communication:
Spectra of Amplitude-Modulation Signals

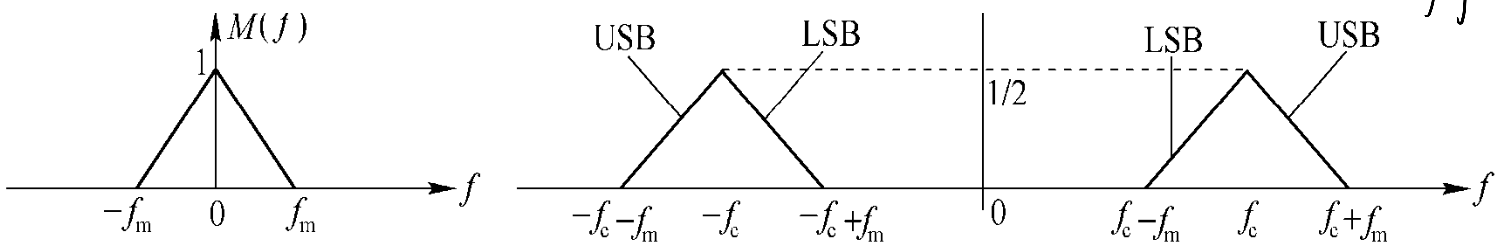


double-side band

$$Z(t) = m(t)\cos(\omega_c t)$$



(a) (b) lower upper



(a) (b)

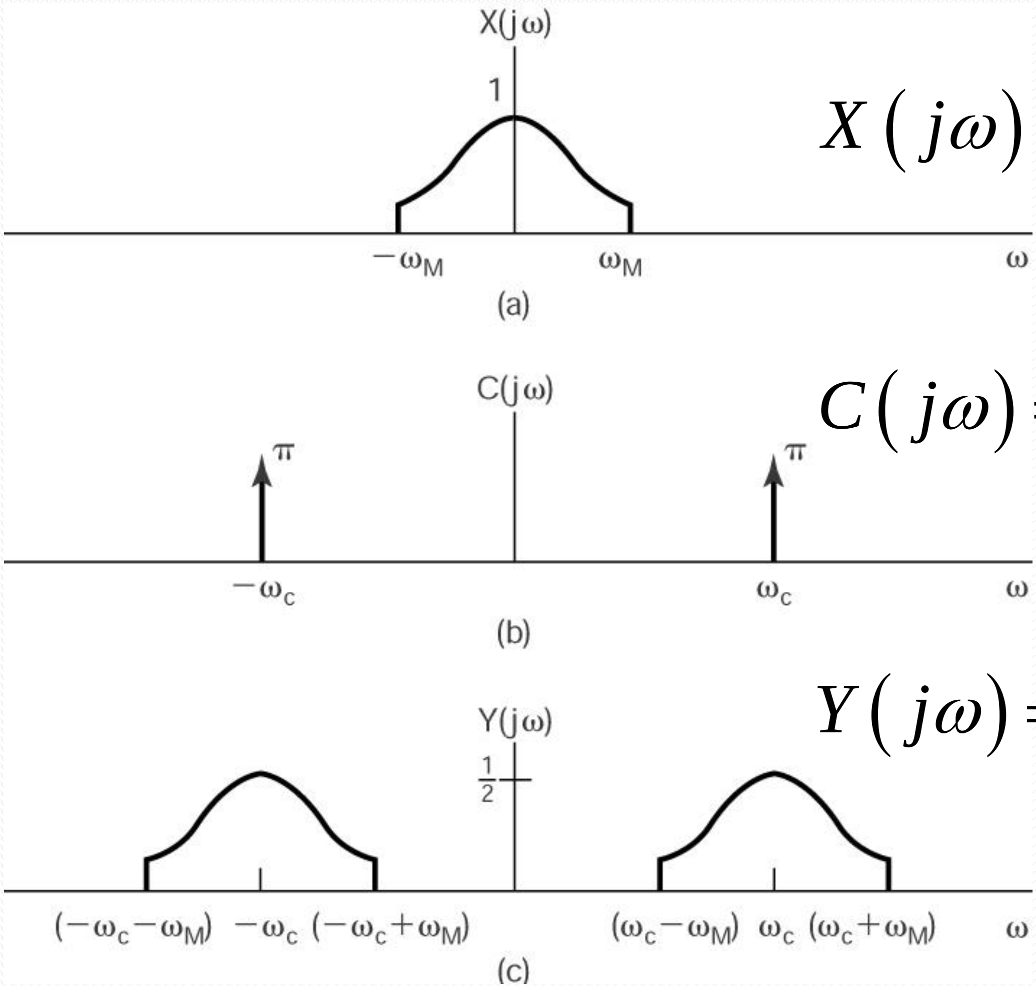
$$FT[m(t)\cos\omega_0 t] = FT[\frac{1}{2}m(t)e^{j\omega_0 t}] + FT[\frac{1}{2}m(t)e^{-j\omega_0 t}]$$

?

$$= \frac{1}{2}[M(j(\omega-\omega_0)) + M(j(\omega+\omega_0))]$$

mirror vs. shift

An Example of sinusoidal amplitude modulation



$$X(j\omega) = FT\left[x(t)\right]$$

$$C(j\omega) = FT\left[\cos\omega_c t\right]$$

$$Y(j\omega) = FT\left[x(t)\cos\omega_c t\right]$$

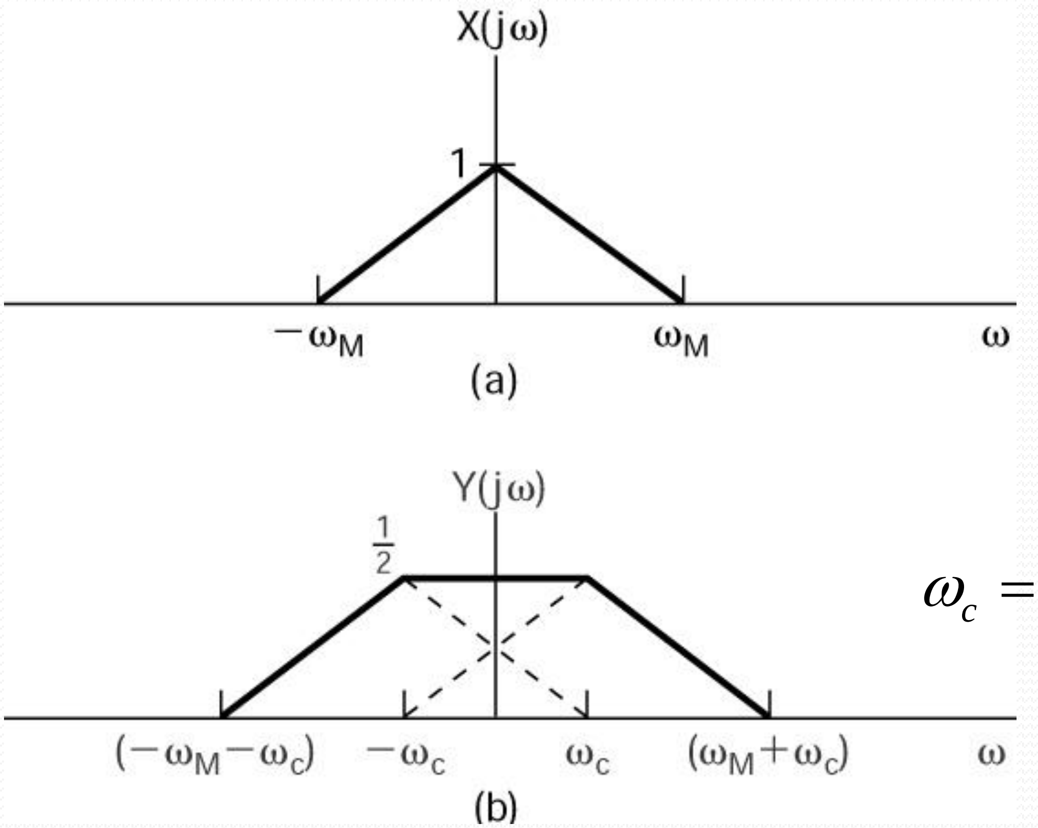
Question: How do you demodulate the message m(t) or x(t) from the modulated signal?

Question: How do you demodulate the message m(t) or x(t) from the modulated signal?

- Envelope Detection
- Coherent Demodulation

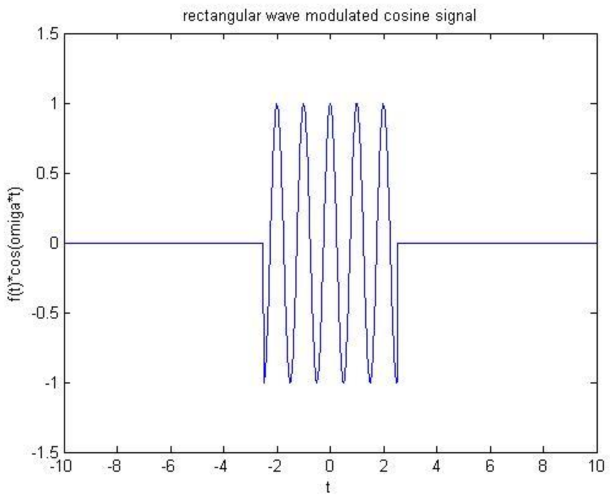
$$m(t)\cos(\omega_c t)\cdot\cos(\omega_c t)=\frac{1}{2}m(t)+\underline{\frac{1}{2}m(t)\cos(2\omega_c t)}$$

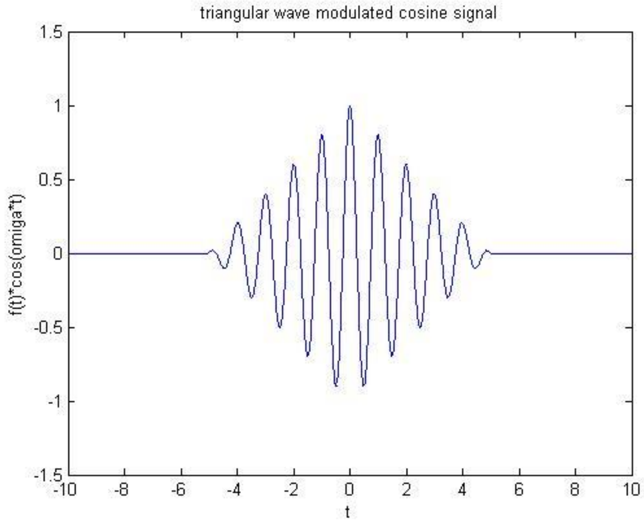
Another example of sinusoidal amplitude modulation



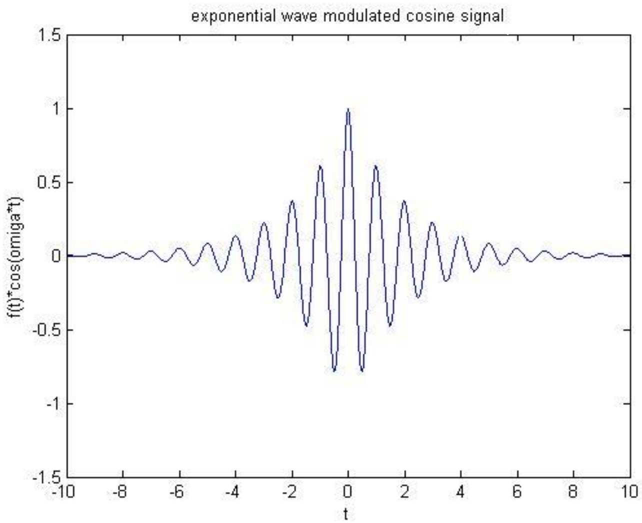
$$\omega_c = \frac{\omega_M}{2}$$

Every Amplitude-modulated signal can be viewed as a product $f(t)\cos\omega_0 t$





$$G(t)\cos\omega_0 t$$



$$\left(1-\frac{2|t|}{\tau}\right)\cos\omega_0 t$$

$$e^{-at}\cos\omega_0 t$$

Can you determine their spectra?

•Differentiation

$$If \quad FT[f(t)] = F(j\omega)$$

$$Then \quad FT\left[\frac{df(t)}{dt}\right] = j\omega F(j\omega)$$

$$Proof$$

$$\because \; f'(t) = f'(t) * \delta(t) = f(t) * \delta'(t)$$

$$\therefore F[f'(t)] = F[f(t)]F[\delta'(t)] = j\omega F(j\omega)$$

$$FT\left[\frac{d^n f(t)}{dt^n}\right] = (j\omega)^n F(j\omega)$$

•Integration

$$If \quad FT[f(t)] = F(j\omega)$$

$$Then \quad FT\left[\int_{-\infty}^{t} f(\tau)d\tau\right] = \frac{F(j\omega)}{j\omega} + \boxed{\pi F(0)\delta(\omega)}$$

DC value

$$If \quad F(0) = 0$$

$$Then \quad FT\left[\int_{-\infty}^{t} f(\tau)d\tau\right] = \frac{F(j\omega)}{j\omega}$$

Proof of the integration property of FT

$$Denote \quad f^{-1}(t) = \int_{-\infty}^{t} f(x)dx$$

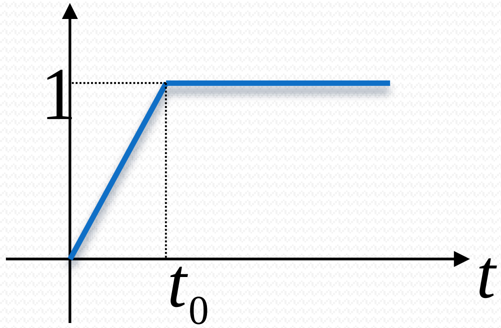
$$F[f^{-1}(t)] = F[f^{-1}(t) * \delta(t)] = F[f(t) * u(t)]$$

$$= F(j\omega)FT[u(t)] = F(j\omega)[\pi\delta(\omega) + \frac{1}{j\omega}]$$

$$= \pi F(0)\delta(\omega) + \frac{F(j\omega)}{j\omega}$$

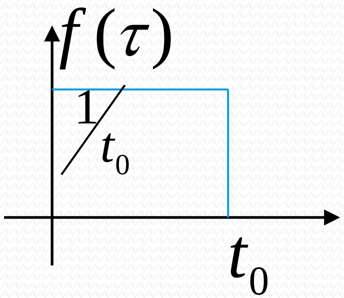
Example Determine the spectrum of *y(t)*

$$y(t)=\begin{cases}0 & (t<0)\\ t/t_0 & (0<t<t_0)\\ 1 & (t>t_0)\end{cases}$$



y(t) can be regarded as an integral of a rectangular pulse, *f(τ)*, with the height $1/t_0$ and width t_0.

$$f(\tau)=\begin{cases}0 & (\tau<0)\\ 1/t_0 & (0<\tau<t_0)\\ 0 & (\tau>t_0)\end{cases}$$

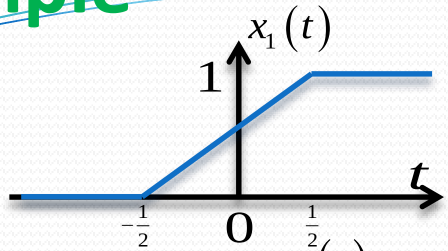


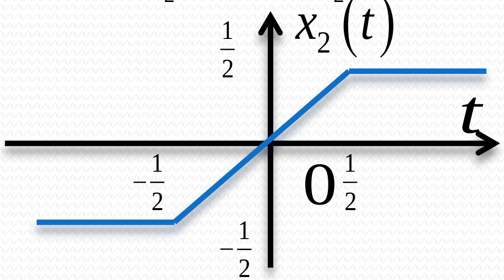
$$y(t)=\int_{-\infty}^{t} f(\tau)d\tau$$

$$Y(j\omega)=FT[y(t)]=FT[f(t)*u(t)]=\frac{1}{j\omega}F(j\omega)+\pi F(0)\delta(\omega)$$

$$=\frac{1}{j\omega}Sa(\frac{\omega t_0}{2})e^{-j\frac{\omega t_0}{2}}+\pi\delta(\omega)$$

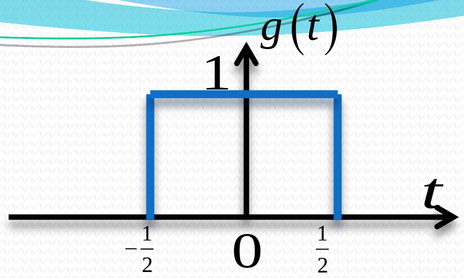
Example





$$x_2'(t) = g(t)$$

$$x_1'(t) = g(t)$$



$$\int_{-\infty}^{t} g(t)dt = x_1(t) = x_2(t) + \frac{1}{2}$$

Given $G(j\omega) = Sa\left(\frac{\omega}{2}\right)$

We have

$$X_1(j\omega) = \frac{1}{j\omega}G(j\omega) + \pi G(0)\delta(\omega)$$

$$= \frac{1}{j\omega}Sa\left(\frac{\omega}{2}\right) + \pi\delta(\omega)$$

$$X_2(j\omega) = X_1(j\omega) - \pi\delta(\omega) = \frac{1}{j\omega}Sa\left(\frac{\omega}{2}\right)$$

Given $X_1(j\omega) = \frac{1}{j\omega}Sa\left(\frac{\omega}{2}\right) + \pi\delta(\omega)$

We have

$$G(j\omega) = j\omega X_1(j\omega) = Sa\left(\frac{\omega}{2}\right)$$

Given $X_2(j\omega) = \frac{1}{j\omega}Sa\left(\frac{\omega}{2}\right)$

We have

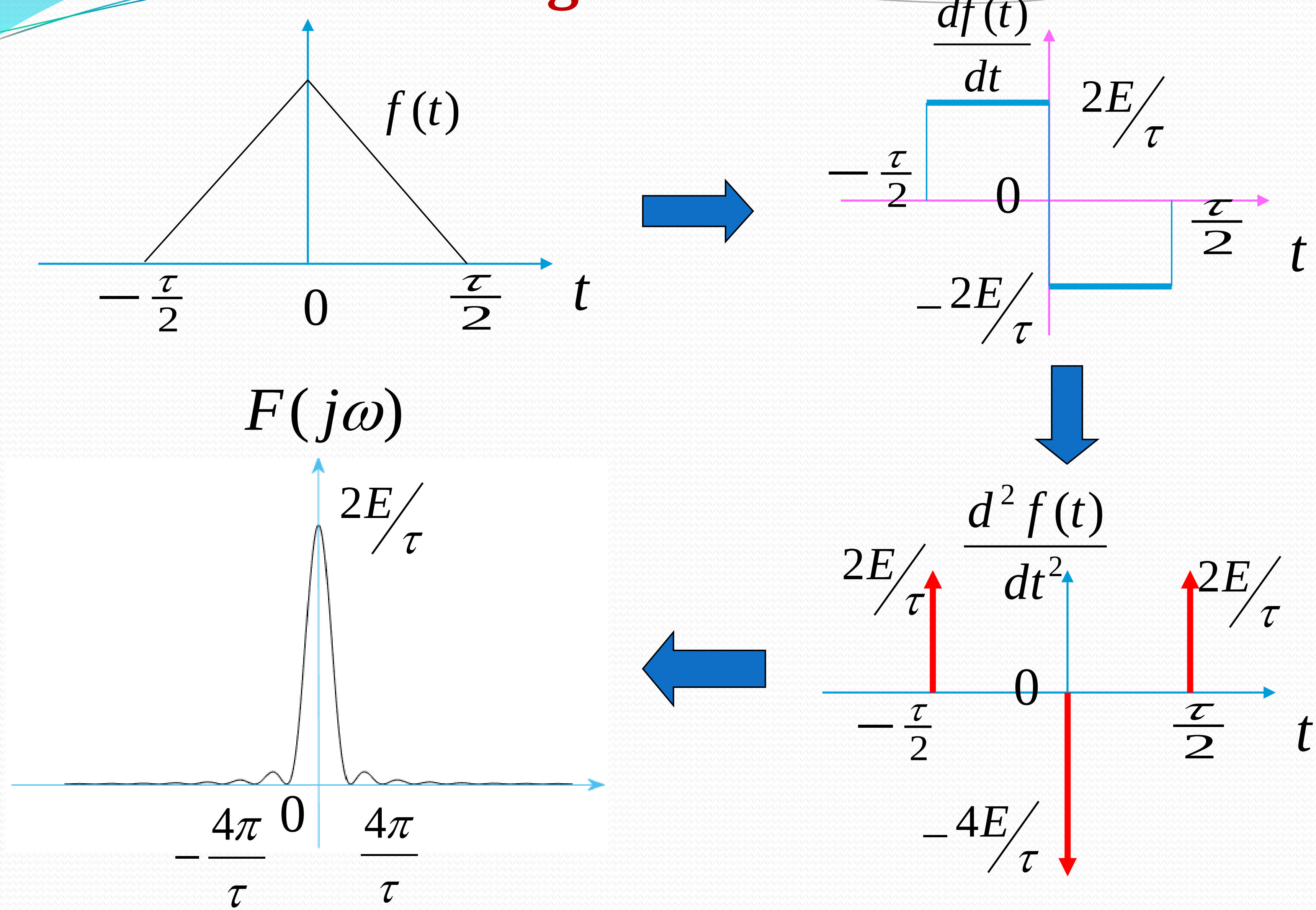
$$G(j\omega) = j\omega X_2(j\omega) = Sa\left(\frac{\omega}{2}\right)$$

Spectrum of a triangular pulse signal

$$f(t)=\begin{cases} E(1-\frac{2}{\tau}|t|) & (|t|<\frac{\tau}{2}) \\ 0 & (|t|>\frac{\tau}{2}) \end{cases}$$

- Method I: By definition
- Method II: Use the property of integration of FT

Triangular Pulse



$$f(t)=\begin{cases}E(1-\frac{2}{\tau}|t|) & (|t|<\frac{\tau}{2})\\ 0 & (|t|>\frac{\tau}{2})\end{cases}$$

$$y(t)=f''(t)=\frac{2E}{\tau}\left[\delta(t+\tfrac{\tau}{2})+\delta(t-\tfrac{\tau}{2})-2\delta(t)\right]$$

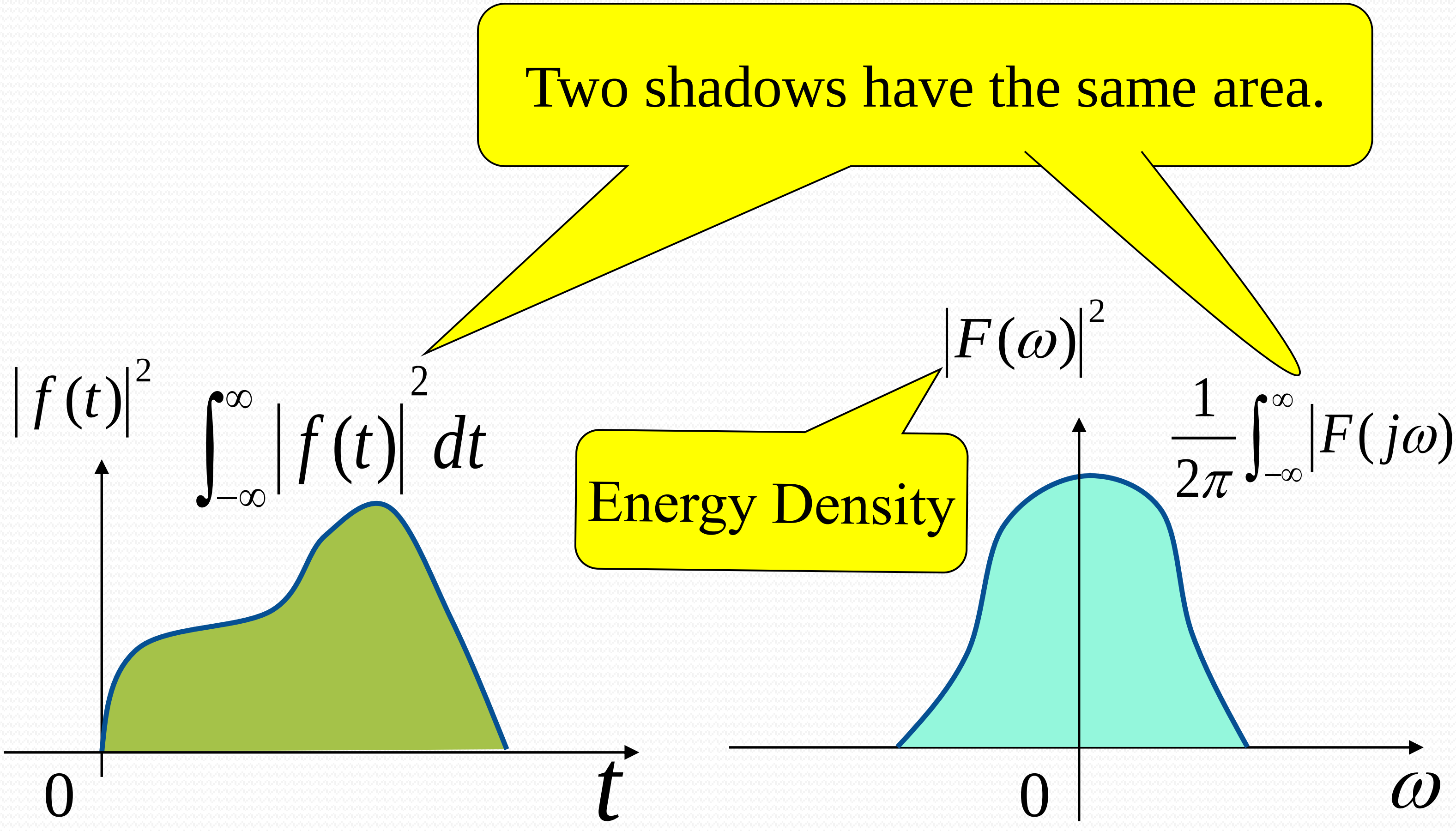
$$Y(j\omega)=\frac{2E}{\tau}(e^{j\omega\frac{\tau}{2}}+e^{-j\omega\frac{\tau}{2}}-2)=-\frac{8E}{\tau}\sin^2\left(\frac{\omega\tau}{4}\right)=-\frac{\omega^2E\tau}{2}Sa^2(\frac{\omega\tau}{4})$$

$$Y(j\omega)\Big|_{\omega=0}=\int_{-\infty}^{\infty}f''(t)e^{-j\omega t}dt\Big|_{\omega=0}=\int_{-\infty}^{\infty}f''(t)dt=0$$

$$\int_{-\infty}^{\infty}f'(t)dt=0$$

$$F(j\omega)=\frac{Y(j\omega)}{(j\omega)^2}=\frac{E\tau}{2}Sa^2(\frac{\omega\tau}{4})$$

•Energy Spectrum
– Parseval's Relation



$$E = \int_{-\infty}^{\infty}|f(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}|F(j\omega)|^2 d\omega$$

Proof

$$E = \int_{-\infty}^{\infty} |f(t)|^2 dt$$

$$= \int_{-\infty}^{\infty} f(t) f^*(t) dt$$

$$= \int_{-\infty}^{\infty} f(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega) e^{j\omega t} d\omega \right]^* dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) \left[\int_{-\infty}^{\infty} F^*(j\omega) e^{-j\omega t} d\omega \right] dt$$

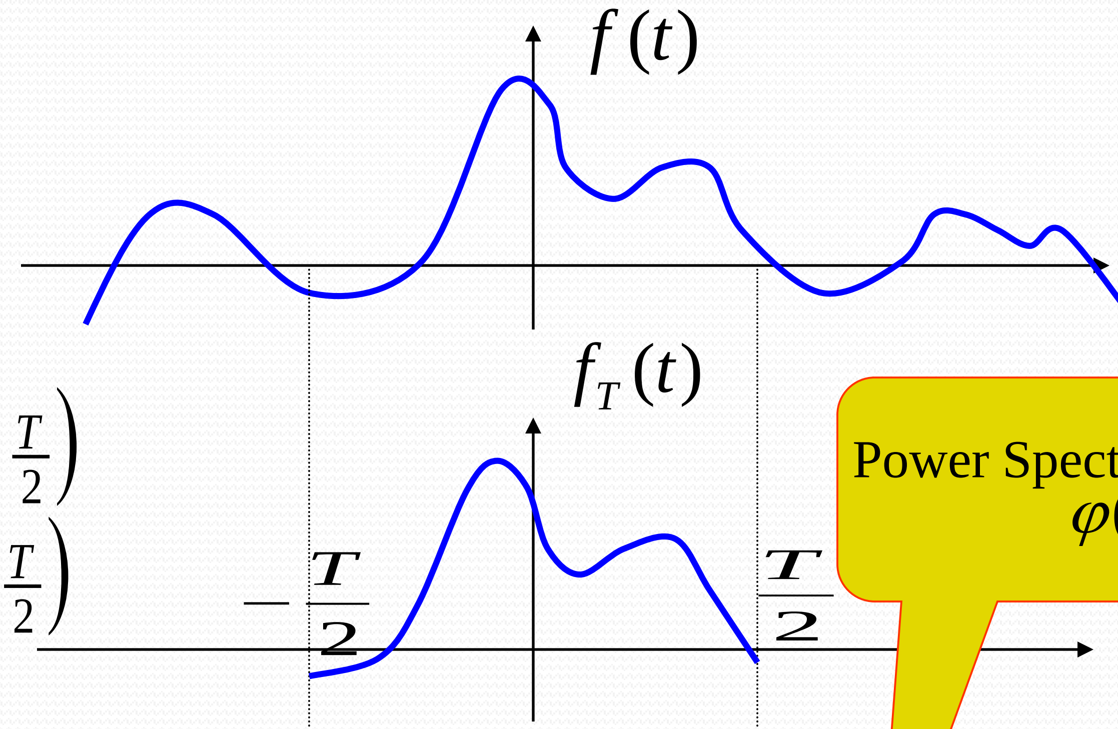
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F^*(j\omega) \left[\int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \right] d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F^*(j\omega) F(j\omega) d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(j\omega)|^2 d\omega$$

•Average Power and Power Spectrum

Signal with limited power



$$f_T(t)=\begin{cases} f(t) & \left(|t|\le \frac{T}{2}\right) \\ 0 & \left(|t|> \frac{T}{2}\right) \end{cases}$$

Power Spectrum $\varphi(\omega)$

Average power $$P=\lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f^2(t)dt$$

Parseval's Relation

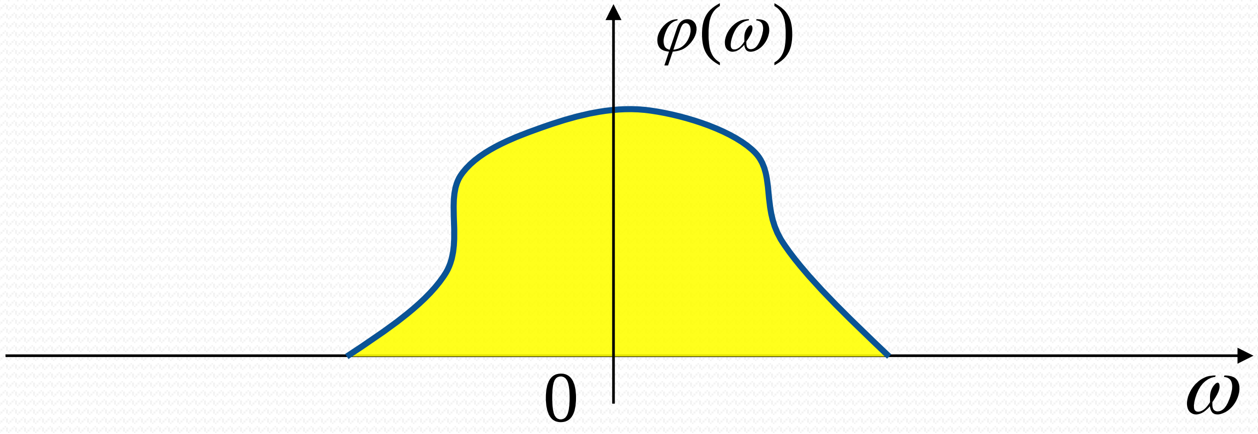
$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\boxed{\lim_{T\to\infty}\frac{|F_T(j\omega)|^2}{T}}d\omega$$

Power Spectrum Function

$$\phi(\omega) = \lim_{T\to\infty} \frac{\left|F_T(j\omega)\right|^2}{T}$$

Average Power

$$P = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi(\omega)\, d\omega$$



•Duality （对偶性）

$$\text{Denote} \quad F(j\omega) = FT\left[f(t)\right]$$

$$\text{Then} \quad FT\left[F(jt)\right] = 2\pi\, f(-\omega)$$

Proof

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}\,d\omega$$

$$f(-t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{-j\omega t}\,d\omega,$$

$$f(-\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(jt)e^{-j\omega t}\,dt$$

$$2\pi f(-\omega) = \int_{-\infty}^{\infty} F(jt)e^{-j\omega t}\,dt$$

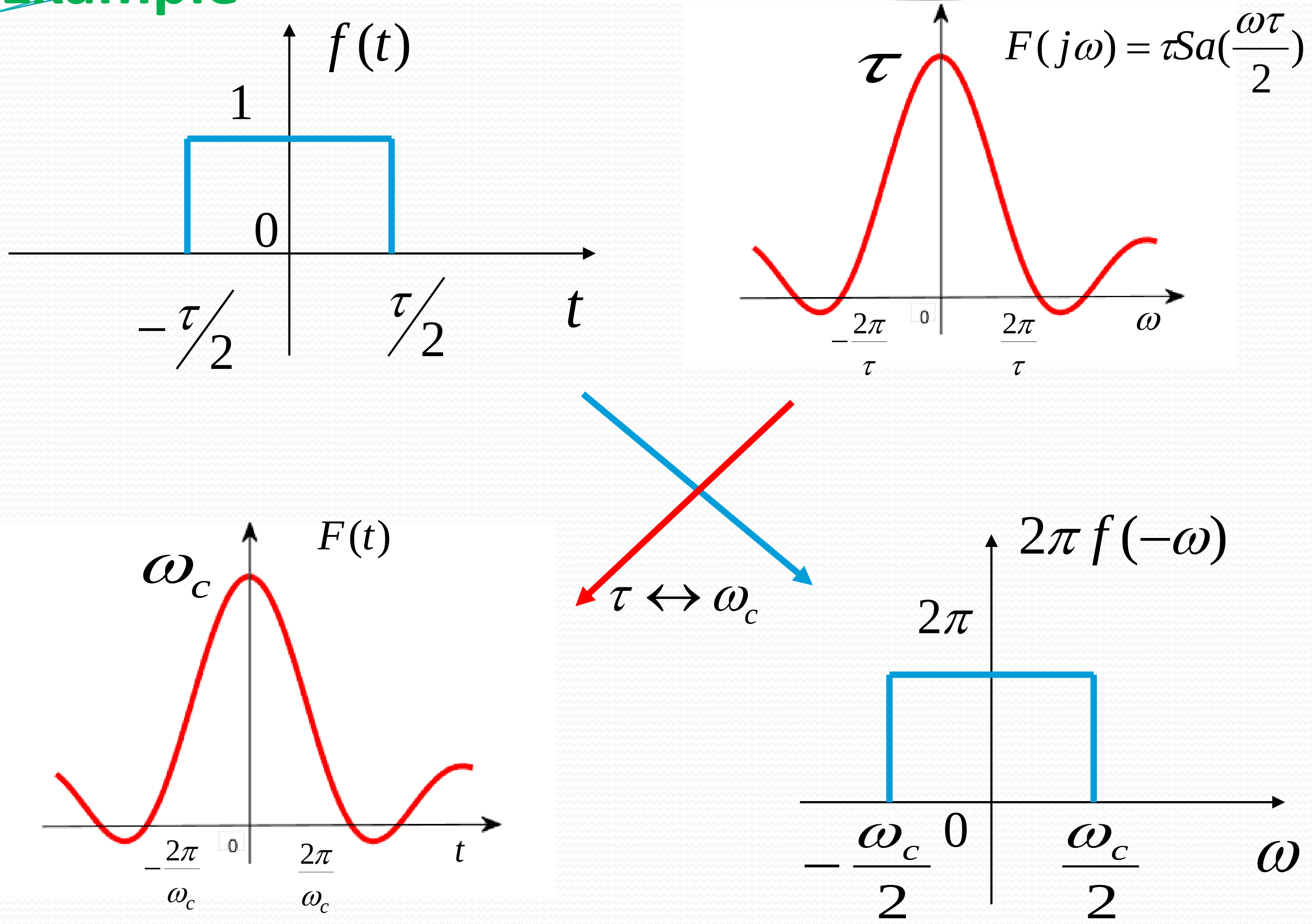
$$FT\left[F(jt)\right] = 2\pi f(-\omega)$$

$$f(t) \xleftrightarrow{\mathscr{F}} F(jw)$$

$$\Downarrow$$

$$F(jt) \xleftrightarrow{\mathscr{F}} 2\pi f(-w)$$

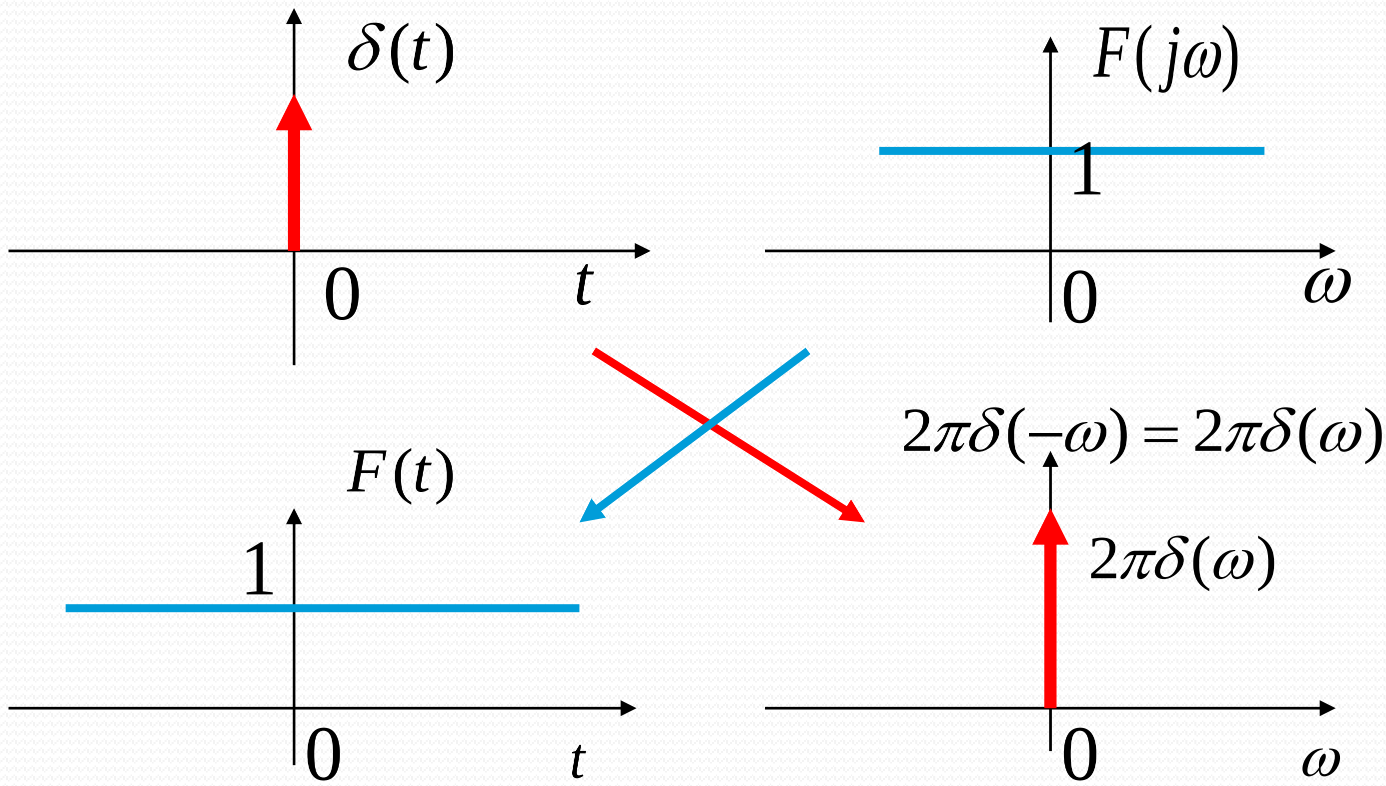
Example



f(t)
1
0
-τ/2
τ/2
t
F(jω) = τSa(ωτ/2)
τ
-2π/τ
0
2π/τ
ω
τ ↔ ω_c
F(t)
ω_c
-2π/ω_c
0
2π/ω_c
t
2πf(-ω)
2π
-ω_c/2
0
ω_c/2
ω

Example

The duality of DC signal and impulse signal



Example

$$F_1(j\omega) = FT\left[\frac{1}{a+jt}\right] = ? \qquad a > 1$$

$$f(t) = e^{-at} \xrightarrow{\text{FT}} F(j\omega) = \frac{1}{a+j\omega}$$

$$\text{Duality} \quad F(t) \to 2\pi f(-\omega)$$

$$\text{then} \quad \frac{1}{a+jt} \to 2\pi f(-\omega) = 2\pi e^{a\omega}$$

$$\therefore F_1(j\omega) = FT\left[\frac{1}{a+jt}\right] = 2\pi e^{a\omega}$$

replace t with ω

Example

$$\because \delta'(t) \leftrightarrow j\omega$$

$$\therefore\ jt \leftrightarrow 2\pi\delta'(-\omega) = -2\pi\delta'(\omega)$$

$$\therefore t \leftrightarrow j2\pi\delta'(\omega)$$

Example

$$\because \operatorname{sgn}(t) \leftrightarrow \frac{2}{j\omega}$$

$$\therefore \frac{2}{jt} \leftrightarrow 2\pi \operatorname{sgn}(-\omega) = -2\pi \operatorname{sgn}(\omega)$$

$$\therefore \frac{1}{t} \leftrightarrow -j\pi \operatorname{sgn}(\omega)$$

•Convolution Theorem in Time-Domain

- **if** $$FT[f_1(t)] = F_1(j\omega)$$

$$FT[f_2(t)] = F_2(j\omega)$$

- **then** $$FT[f_1(t) * f_2(t)] = F_1(j\omega) \cdot F_2(j\omega)$$

Proof

$$FT[f_1(t) * f_2(t)] = \int_{-\infty}^{\infty} [\int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau) d\tau] e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} f_1(\tau) [\int_{-\infty}^{\infty} f_2(t-\tau) e^{-j\omega t} dt] d\tau$$

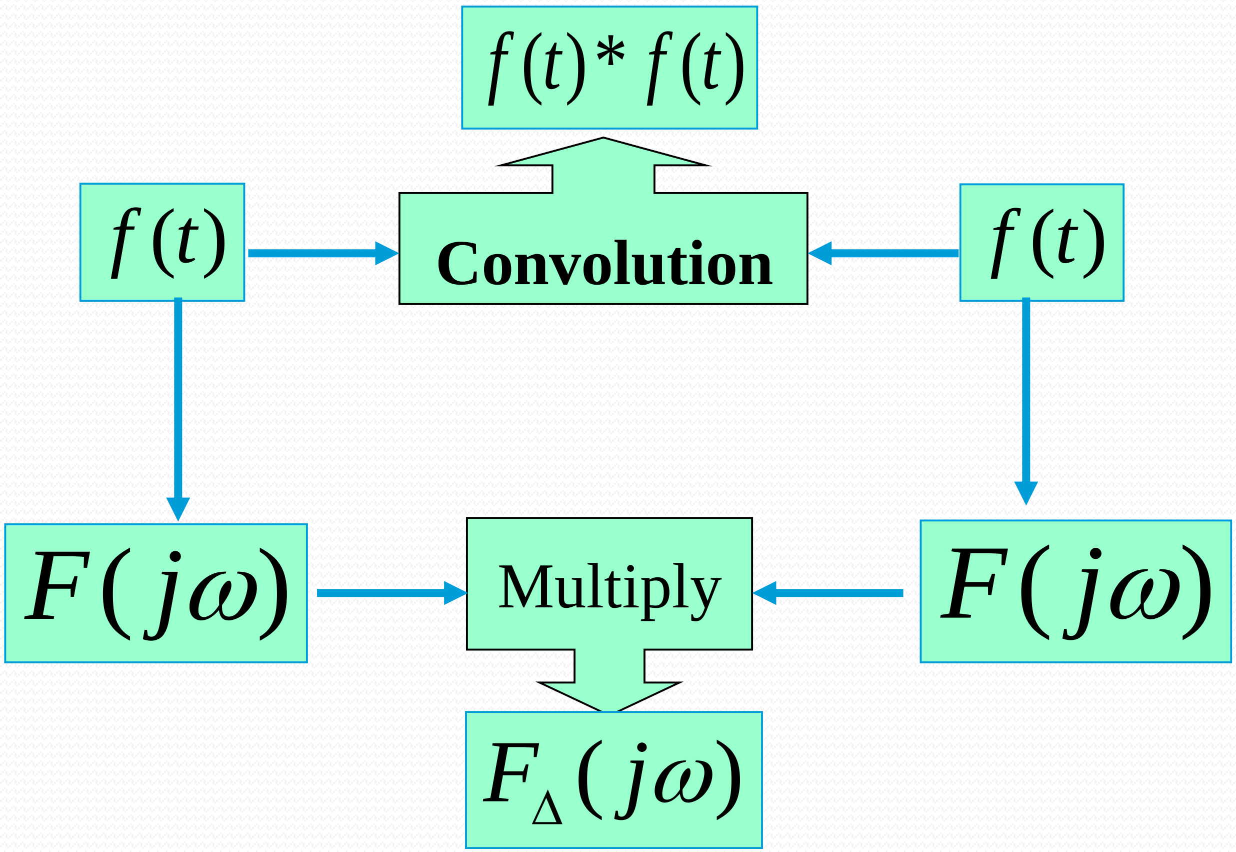
$$= \int_{-\infty}^{\infty} f_1(\tau) [\int_{-\infty}^{\infty} f_2(t-\tau) e^{-j\omega(t-\tau)} dt] e^{-j\omega\tau} d\tau$$

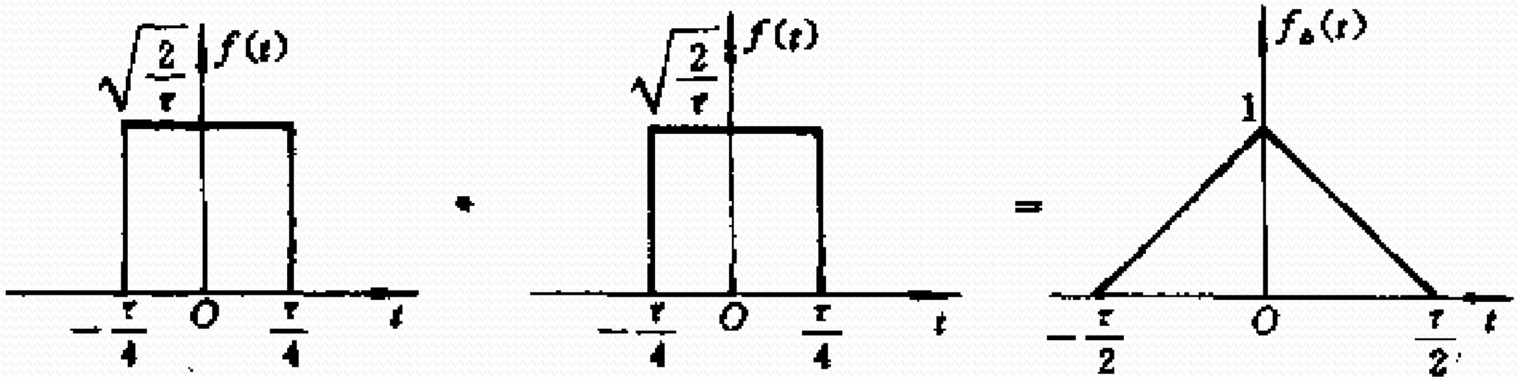
$$= \int_{-\infty}^{\infty} f_1(\tau) F_2(j\omega) e^{-j\omega\tau} d\tau = F_2(j\omega) \int_{-\infty}^{\infty} f_1(\tau) e^{-j\omega\tau} d\tau$$

$$= F_2(j\omega) F_1(j\omega)$$

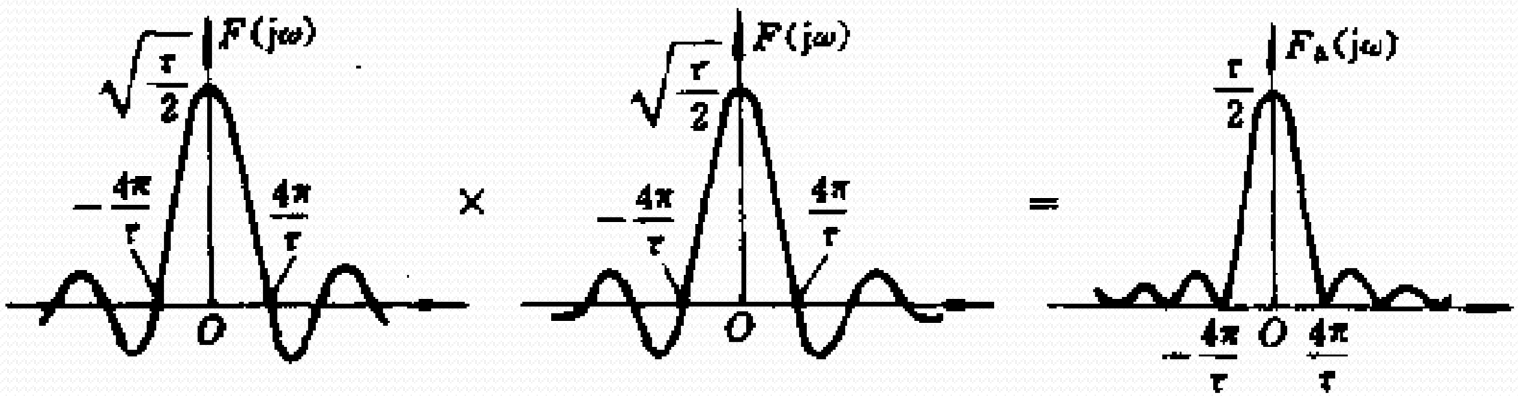
Example: **Determine the spectrum of a triangular signal**

Note: A triangular signal can be viewed as a convolution of two rectangular pulses.





(a) 时域 $f(t) * f(t) = f_\Delta(t)$



(b) 频域 $F(j\omega)F(j\omega) = F_\Delta(j\omega)$

$$F(j\omega) = \sqrt{\frac{2}{\tau}}\frac{\tau}{2}Sa\left(\frac{\omega\tau}{4}\right)$$

$$F_\Delta(j\omega) = \frac{\tau}{2}Sa^2\left(\frac{\omega\tau}{4}\right)$$

•Convolution Theorem in Frequency-Domain

- **If**

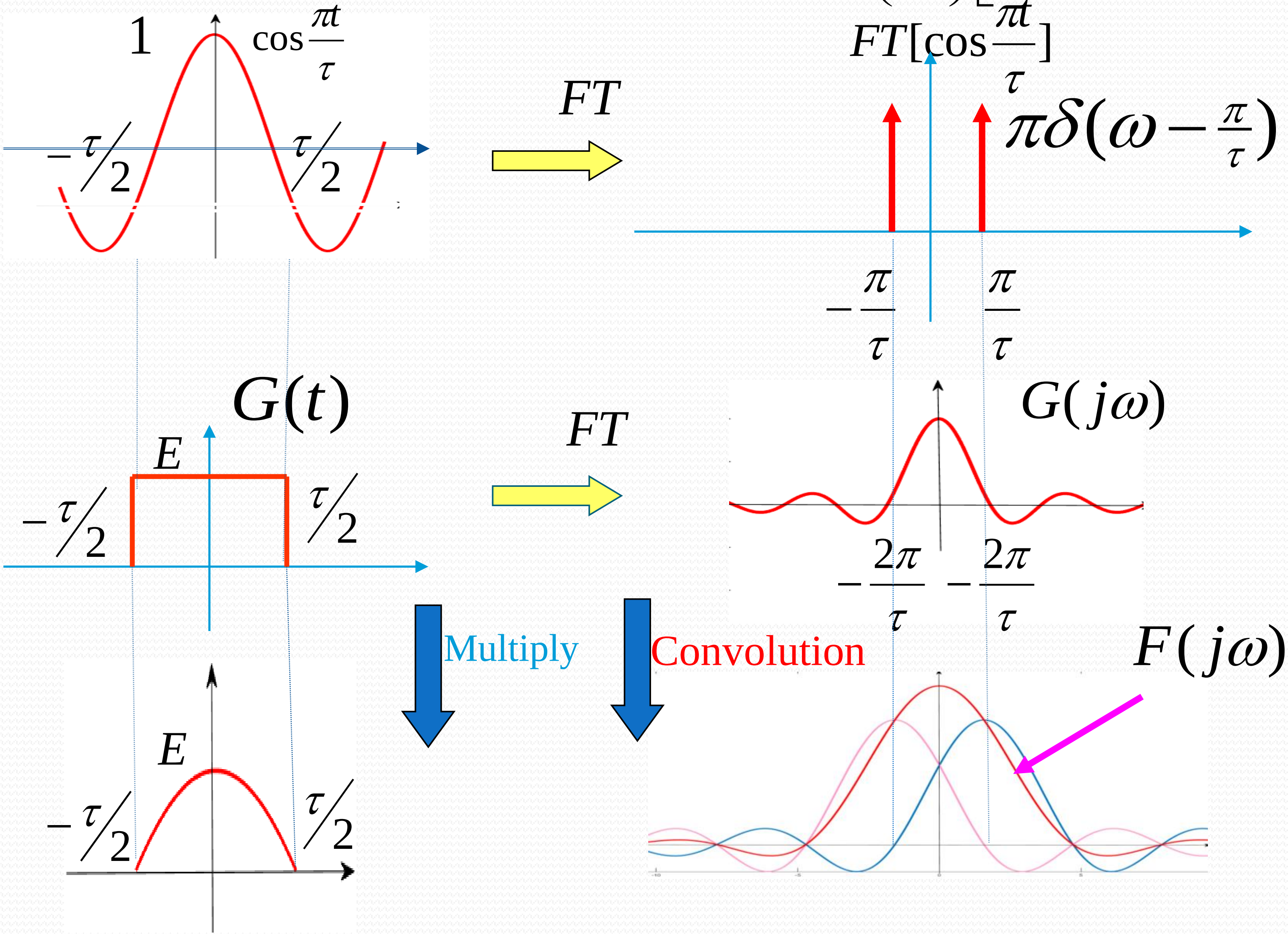
$$FT[f_1(t)] = F_1(j\omega)$$

$$FT[f_2(t)] = F_2(j\omega)$$

- **Then**

$$FT[f_1(t)\cdot f_2(t)] = \frac{1}{2\pi}F_1(j\omega) * F_2(j\omega)$$

Example Determine the spectrum of $f(t) = E\cos\left(\frac{\pi t}{\tau}\right)\left[u(t+\frac{\tau}{2}) - u(t-\frac{\tau}{2})\right]$

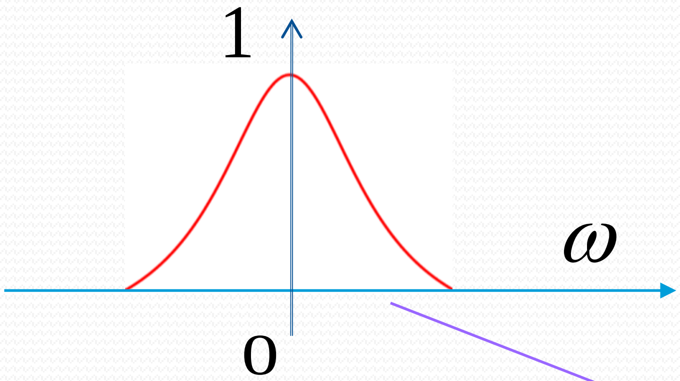


$$\begin{tikzcd}
 & f(t)=g(t)\cdot\cos\frac{\pi t}{\tau} & \\
g(t) \arrow[r] \arrow[d, leftrightarrow, "\text{FT/IFT}"'] & \times \arrow[u] & \cos\frac{\pi t}{\tau} \arrow[l] \arrow[d, leftrightarrow, "\text{FT/IFT}"] \\
G(j\omega)=E\tau Sa(\frac{\omega\tau}{2}) \arrow[r] & * \arrow[d] & \pi\delta(\omega+\frac{\pi}{\tau})+\pi\delta(\omega-\frac{\pi}{\tau}) \arrow[l] \\
 & F(j\omega)=\frac{2E\tau\cos(\frac{\omega\tau}{2})}{\pi\left[1-(\frac{\omega\tau}{\pi})^2\right]} &
\end{tikzcd}$$

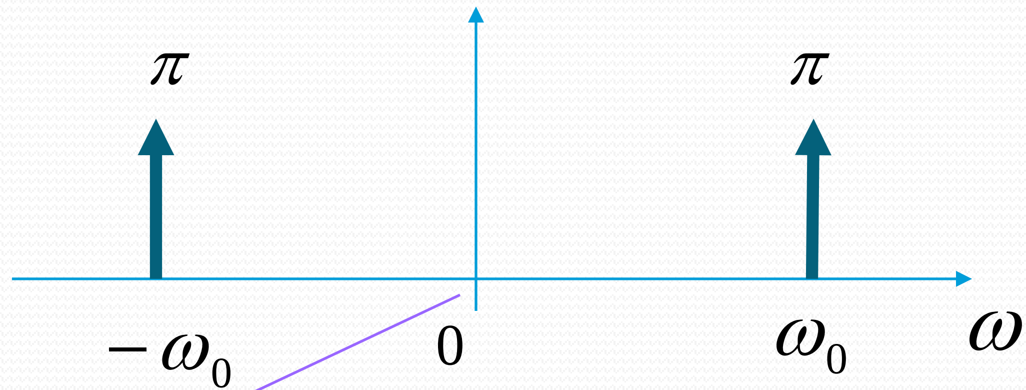
Explain a previous example with the convolution theorem.

$$FT[f(t)\cos\omega_0 t]$$

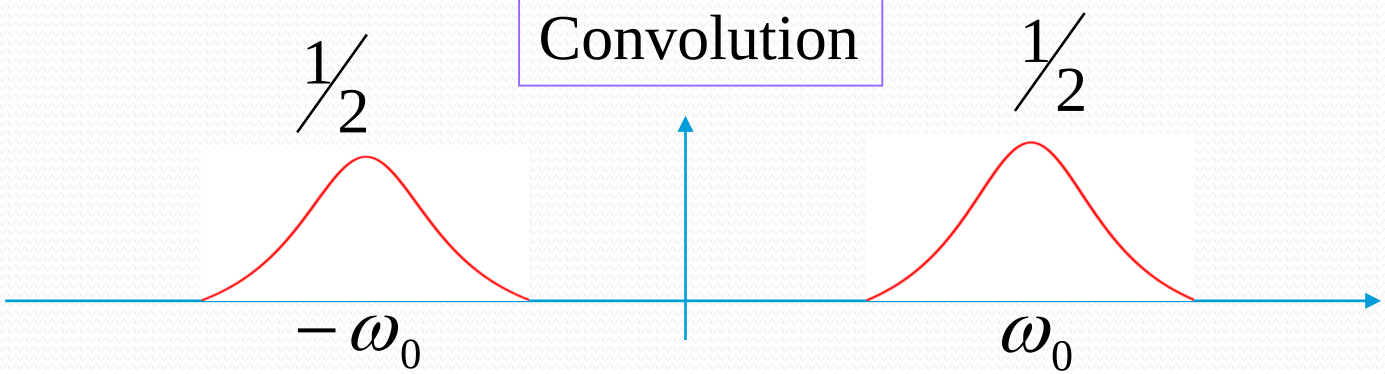
$FT[f(t)]$



$FT[\cos\omega_0 t]$

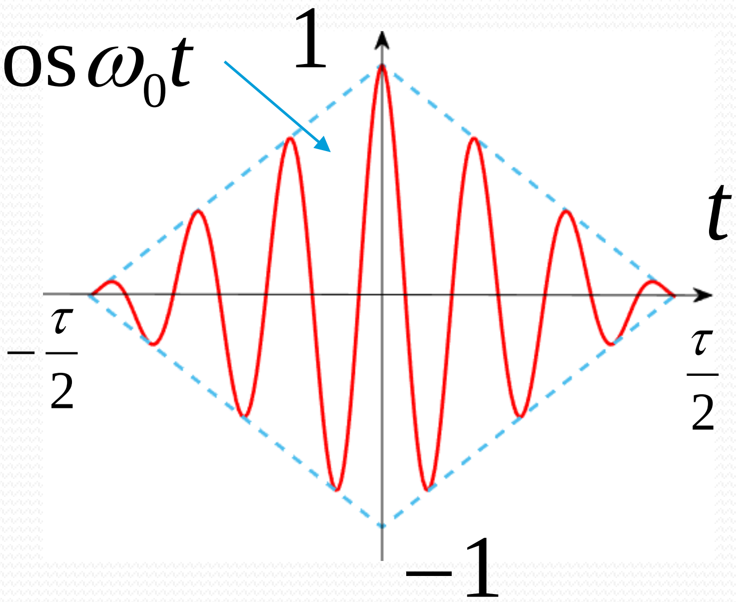


Convolution



$$\frac{1}{2}[F(j(\omega-\omega_0))+F(j(\omega+\omega_0))]$$

Example: **Determine the spectrum of an AM signal modulated by a triangular signal as shown in following figure.**



$$\cos\omega_0 t = \frac{1}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t})$$

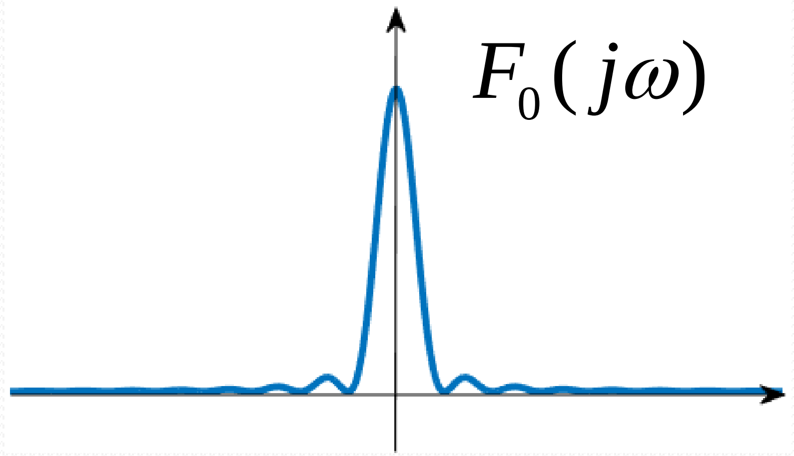
$$f_0(t) = 1 - \frac{2|t|}{\tau}$$

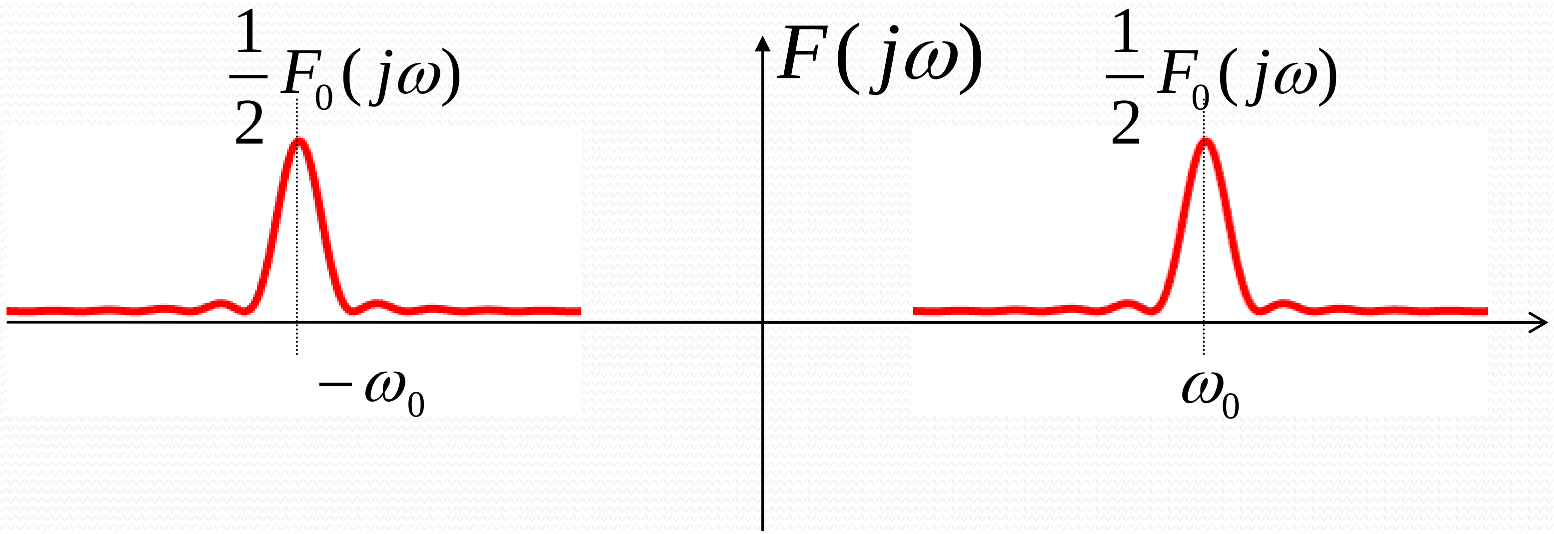
Triangular Signal

$$F_0(j\omega) = \frac{E\tau}{2} Sa^2\left(\frac{\omega\tau}{4}\right)$$

$$E = 1$$

$$F(j\omega) = \frac{E\tau}{4}\left\{Sa^2\left(\frac{(\omega-\omega_0)\tau}{4}\right) + Sa^2\left(\frac{(\omega+\omega_0)\tau}{4}\right)\right\}$$


$F_0(j\omega)$


$\frac{1}{2}F_0(j\omega)$
$F(j\omega)$
$\frac{1}{2}F_0(j\omega)$
$-\omega_0$
ω_0