Atlas-PS 2

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Problem 1

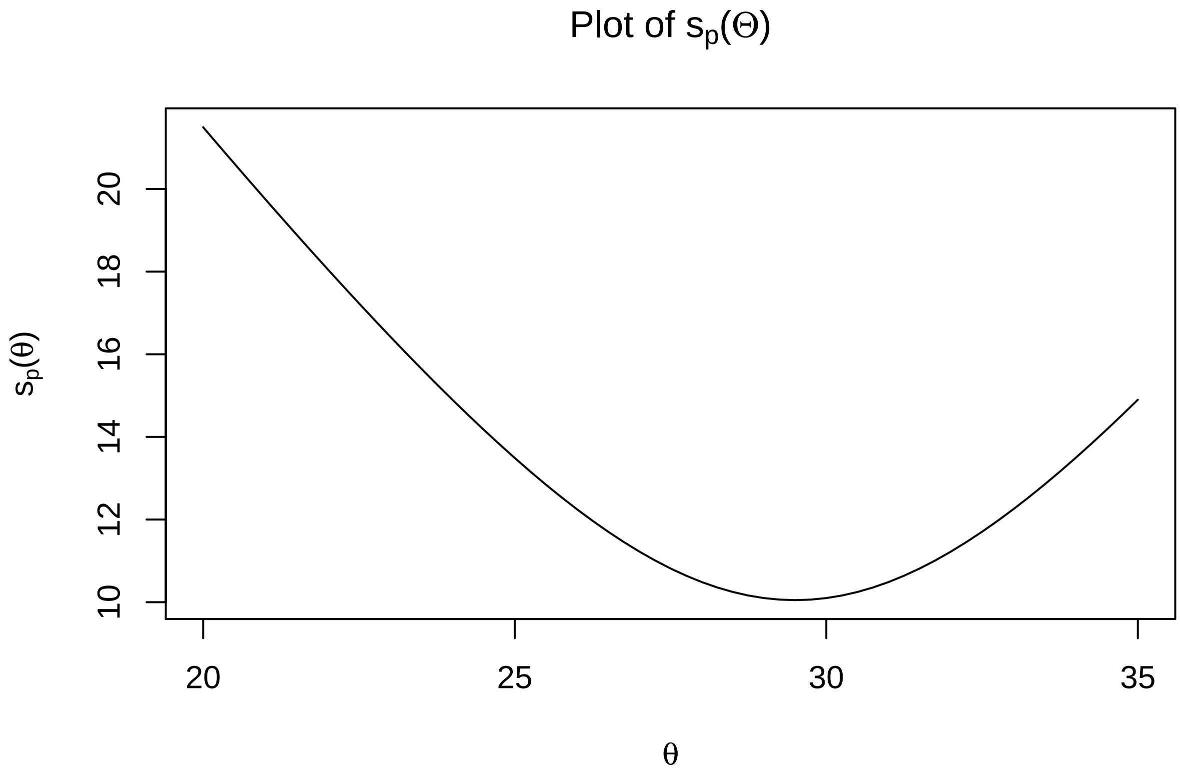
a)

Let $X = (28, 33, 22, 35)$ be our set of i.i.d data points. The function $s_p(\theta) = \sqrt{\Sigma_{x \in X}(\theta - x)^2}$, or the sum of squared residuals.

b)

$s_p(\theta)$ is plotted below, with the R code used to generate the plot.

```
library(latex2exp)
x <- c(28, 33, 22, 35)
s_p <- function(theta, x){
  # We implement our function to minimize
  return(sqrt(sum((theta - x) ^ 2)))
}

# We set up our domain for theta
theta_space <- seq(20, 35, .25)
# We calculate the function value over the space
s_p_theta_space <- sapply(theta_space, function(theta){s_p(theta, x)})

# We plot the function over the space
plot(theta_space, s_p_theta_space, 'l',
  main=TeX('Plot of $s_p(\\Theta)$'),
  xlab=TeX('$\\theta$'), ylab=TeX('$s_p(\\theta)$'))
```



c)

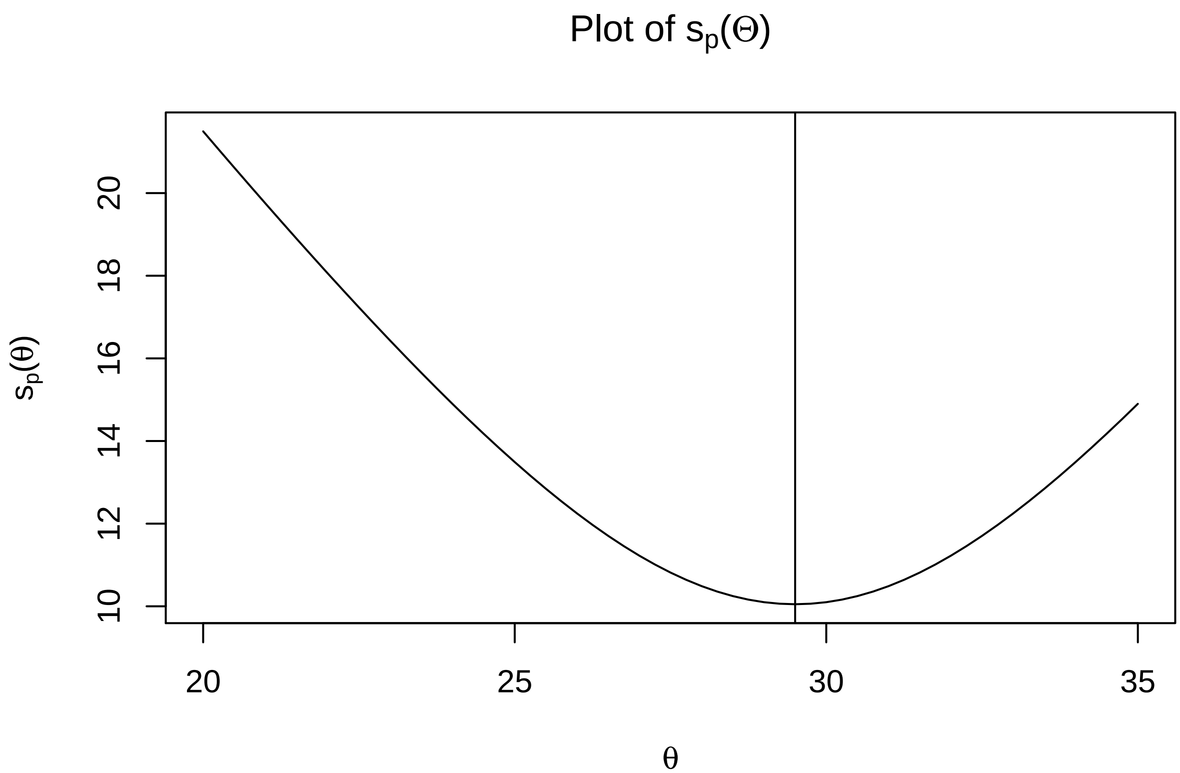
To use the bisection method, we must first compute $s_p\prime(\theta)$.

$$
\begin{aligned}
s_p\prime(\theta) &= \frac{1}{2}(\Sigma_{x\in X}(\theta - x)^2)^{-\frac{1}{2}} \times 2\Sigma_{x\in X}(\theta - x) \\
&= (\Sigma_{x\in X}(\theta - x)^2)^{-\frac{1}{2}}\Sigma_{x\in X}(\theta - x).
\end{aligned}
$$

Next, we implement the bisection method, as well as $s_p(\theta)$ and $s_p\prime(\theta)$. We plot the $s_p(\theta)$ with the Minimum Residual Estimator as a vertical line. The solution to the optimization problem is $\hat{\theta} = 29.50$.

```
x <- c(28, 33, 22, 35)

s_p <- function(theta, x){
  # We implement our function to minimize
  return(sqrt(sum((theta - x) ^ 2)))
}

s_p_prime <- function(theta, x){
  # This is the first derivative of the function
  return(((sum(theta - x) ^ 2) ^ -.5) * sum(theta - x))
}

bisection <- function(a, b, f_prime, tol=.0001, n=0){
  x_t <- .5 * (a + b)
  # Use conditioning to get the next interval
  if(f_prime(a, x) * f_prime(x_t, x) <= 0){
    new_interval <- c(a, x_t)
  }else{
```

```r
    new_interval <- c(x_t, b)
  }

  # if interval is less than the tolerance, stop the recursion.
  if ((b - a) < tol){
    print(paste0("The solution is ", round(x_t, 3) , " and it was found in ", n, " iterations."))
    return(x_t)
  }else{
    # If not, call again on the new interval
    return(bisection(new_interval[1], new_interval[2], f_prime, n=n + 1))
  }
}

plot(theta_space, s_p_theta_space, 'l',
  main=TeX('Plot of $s_p(\\Theta)$'),
  xlab=TeX('$\\theta$'), ylab=TeX('$s_p(\\theta)$'))
abline(v=bisection(20, 35, s_p_prime, tol=.000001))
```




```
## [1] "The solution is 29.5 and it was found in 18 iterations."
```

d)

We already calculated $s_p'(\theta) = (\Sigma_{x \in X}(\theta - x)^2)^{-\frac{1}{2}} \Sigma_{x \in X}(\theta - x)$. We find

$$s_p''(\theta) = -\frac{1}{2}(\Sigma_{x \in X}(\theta - x)^2)^{-\frac{3}{2}} \Sigma_{x \in X}(\theta - x) + (\Sigma_{x \in X}(\theta - x)^2)^{-\frac{1}{2}}.$$

We can then find $h(\theta) = \frac{s_p'(\theta)}{s_p''(\theta)}$.

$$
\begin{aligned}
-\frac{s_p'(\theta)}{s_p''(\theta)} &= -\frac{(\Sigma_{x\in X}(\theta-x)^2)^{-\frac{1}{2}}\Sigma_{x\in X}(\theta-x)}{-\frac{1}{2}(\Sigma_{x\in X}(\theta-x)^2)^{-\frac{3}{2}}\Sigma_{x\in X}(\theta-x)+(\Sigma_{x\in X}(\theta-x)^2)^{-\frac{1}{2}}} \\
&= 2\frac{\Sigma_{x\in X}(\theta-x)}{\Sigma_{x\in X}(\theta-x)^2+2}
\end{aligned}
$$

Problem 2

We maximize the function $f(x) = -\frac{x^4}{4} + \frac{x^2}{2} - x + 2$ using Newton's Method and starting points $x_0 = -1$ and $x_0 = 2$. We also print out the number of iterations needed to converge within 2 decimal places. We define the first 2 derivatives of the function below:

$$
\begin{aligned}
f(x) &= -\frac{x^4}{4} + \frac{x^2}{2} - x + 2 && (1) \\
f'(x) &= -x^3 + x - 1 && (2) \\
f''(x) &= -3x^2 + 1 && (3)
\end{aligned}
$$

We implement Newton's Method:

```
newtons <- function(xt, fprime, f2prime, n=1, tol=0.01){
  # Define the updating equation
  xt_update <- xt - (fprime(xt) / f2prime(xt))

  # If the adjustment value is less than the tolerance, end the iterations
  if(abs(xt_update - xt) < tol){
    print(paste0("The solution is ", round(xt_update, 3) , " and it was found in ", n, " iterations."))
    return(xt_update)
  }else{
    # If not, call the recursive formula again
    return(newtons(xt_update, fprime, f2prime, n=n+1, tol=tol))
  }
}

fprime <- function(x){
  return(-x^3 + x -1)
}

f2prime <- function(x){
  return(-3 * x ^ 2 + 1)
}
```

a)

We solve the optimization using $x_0 = -1$.

```
x0 <- -1
solution <- newtons(x0, fprime, f2prime)
```

```
## [1] "The solution is -1.325 and it was found in 4 iterations."
```

The solution is -1.325, and it took 4 iterations to find it.

b)

We solve the optimization using $x_0 = 2$.

```r
x0 <- 2
solution <- newtons(x0, fprime, f2prime)
```

```
## [1] "The solution is -1.325 and it was found in 64 iterations."
```

The solution is -1.325, and it took 64 iterations to find it.

Problem 3

We solve exercise 2.1 from the textbook:

The following data are an i.i.d. sample from a Cauchy(θ, 1) distribution: 1.77, -.23, 2.76, 3.80, 3.47, 56.75, -1.34, 4.24, -2.44, 3.29, 3.71, -2.40, 4.53, -.07, -1.05, -13.87, -2.53, -1.75, .27, 43.21.

a)

Graph the log likelihood function. Find the MLE for θ using the Newton-Raphson method. Try the following starting point: -11, -1, 0, 1.5, 4, 4.7, 7, 8, 38. Discuss your results. Is the mean of the data a good starting point?

The likelihood function of a Cauchy(θ, 1) distribution:

$$L(\theta) = \prod_{x \in X} \frac{1}{\pi(1 + (x - \theta)^2)}.$$

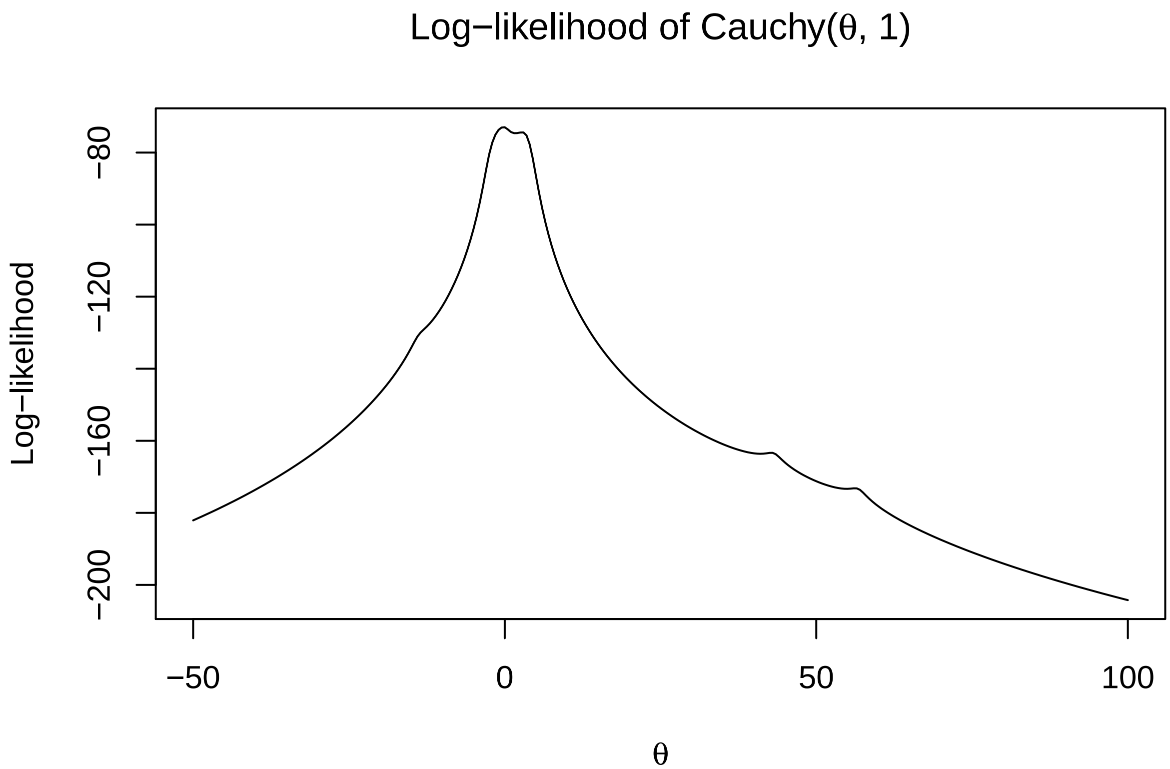
Therefore, the log-likelihood is

$$\begin{aligned} l(\theta) &= \Sigma_{x \in X} \ln\left(\frac{1}{\pi(1 + (x - \theta)^2)}\right) \\ &= \Sigma_{x \in X} - \ln(\pi(1 + (x - \theta)^2)) \\ &= -n\ln(\pi) - \Sigma_{x \in X} \ln(1 + (x - \theta)^2), \end{aligned}$$

where n is the number of observations in X.

We plot the function below.

```r
X <- c(1.77, -.23, 2.76, 3.80, 3.47, 56.75, -1.34, 4.24,
-2.44, 3.29, 3.71, -2.40, 4.53, -.07, -1.05, -13.87,
-2.53, -1.75, .27, 43.21)

log_likelihood <- function(theta, x){
  return (sum(dcauchy(x, location=theta, scale=1, log=TRUE)))
}

theta_space <- seq(-50, 100, .5)
theta_f <- sapply(theta_space, function(theta){log_likelihood(theta, X)})
plot(theta_space, theta_f, 'l',
    main=TeX('Log-likelihood of Cauchy($\\theta$, 1)' ),
    xlab=TeX('$\\theta$'), ylab='Log-likelihood')
```



We note that the log-likelihood values can be negative, as they are not likelihoods, but rather the natural logarithms of those likelihoods.

Next, we find the MLE for θ using Newton's Method for the set of starting values given above. We calculate the first two derivatives of the log-likelihood function.

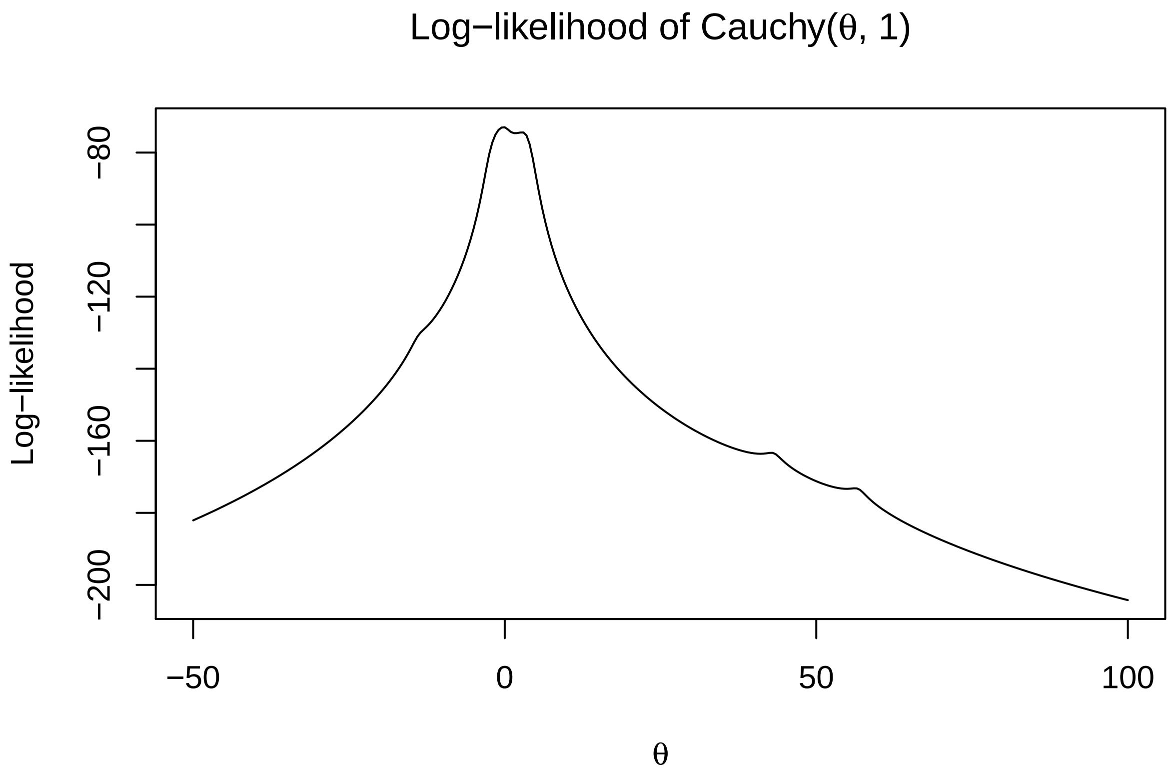
$$
\begin{aligned}
l' &= -\Sigma_{x \in X} 2 \frac{x-\theta}{1+(x-\theta)^2} \\
l'' &= -\Sigma_{x \in X} 2(1+(x-\theta)^2)^{-1} + -4(x-\theta)(1+(x-\theta)^2)^{-2}(x-\theta) \\
&= -\Sigma_{x \in X} \frac{2}{1+(x-\theta)^2} - \frac{4(x-\theta)^2}{(1+(x-\theta)^2)^2}
\end{aligned}
$$

```
newtons <- function(xt, fprime, f2prime, tol=0.01){
  # Define the updating equation
  n <- 0
  xt_update <- xt + 100
  while(abs(fprime(xt)) > tol){
    xt <- ifelse(n == 0, xt, xt_update)

    xt_update <- xt - (fprime(xt) / f2prime(xt))
    n <- n + 1
  }
  # If the adjustment value is less than the tolerance, end the iterations
  print(paste0("The solution is ", round(xt_update, 3) , " and it was found in ", n, " iterations."))
  return(xt_update)
}

fprime <- function(theta){
  return(2 * sum((x-theta) / (1+ (x - theta) ^ 2)))
}
```

```r
f2prime <- function(theta){

  secondderivll <-  2 * sum (((x - theta) ^ 2 - 1) / (1 + (x - theta) ^ 2 ) ^ 2)
  return(secondderivll)
  first_term <- (4 * (x - theta) ^ 2) / (1 + (x - theta) ^ 2) ^ 2
  second_term <- 2 / (1 + (x - theta) ^ 2)
  return(sum(first_term - second_term))
}


starting_points <- c(-11, -1, 0, 1.5, 4, 4.7, 7, 8, 38)

solutions <- sapply(starting_points, function(x0){
  print(paste0("Starting Point: ", x0))
  newtons(x0, fprime=fprime, f2prime=f2prime, tol=.01)
})
```

```
## [1] "Starting Point: -11"
## [1] "The solution is -2507.56 and it was found in 6 iterations."
## [1] "Starting Point: -1"
## [1] "The solution is -1847.745 and it was found in 6 iterations."
## [1] "Starting Point: 0"
## [1] "The solution is -1780.952 and it was found in 6 iterations."
## [1] "Starting Point: 1.5"
## [1] "The solution is -1680.352 and it was found in 6 iterations."
## [1] "Starting Point: 4"
## [1] "The solution is -3052.167 and it was found in 7 iterations."
## [1] "Starting Point: 4.7"
## [1] "The solution is -2956.802 and it was found in 7 iterations."
## [1] "Starting Point: 7"
## [1] "The solution is -2640.641 and it was found in 7 iterations."
## [1] "Starting Point: 8"
## [1] "The solution is -2501.557 and it was found in 7 iterations."
## [1] "Starting Point: 38"
## [1] "The solution is 2999.013 and it was found in 9 iterations."
```

```r
plot(theta_space, theta_f, 'l',
    main=TeX('Log-likelihood of Cauchy($\\theta$, 1)' ),
    xlab=TeX('$\\theta$'), ylab='Log-likelihood')

abline(v=solutions)
```


Log−likelihood of Cauchy(θ, 1)
Log−likelihood
−80
−120
−160
−200
−50
0
50
100
θ

a)