Atlas-PS 3

David Atlas

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Problem 1

We define the likelihood function $L(\hat{\alpha}; X)$:

$$L(\alpha; X) = \prod_{i=1}^{n} \frac{(\alpha_1 b_{i,1} + \alpha_2 b_{i,2})^{x_i} \mathrm{e}^{-(\alpha_1 b_{i,1} + \alpha_2 b_{i,2})}}{x_i!},$$

and the log-likelihood function $l(\hat{\alpha}; X)$:

$$l(\alpha; X) = \Sigma_{i=1}^{n} x_i \log \alpha_1 b_{i,1} + \alpha_2 b_{i,2} - \Sigma_{i=1}^{n} \alpha_1 b_{i,1} - \Sigma_{i=1}^{n} \alpha_2 b_{i,2} - \Sigma_{i=1}^{n} \log(x_i!).$$

a)

Derive the Newton Raphson update for finding the MLEs of α_1 and $alpha_2$.

First, we take the first derivative of l' with respect to $\hat{\alpha}$. This leaves us with a 2x1 matrix of first derivatives.

$$\begin{bmatrix} \Sigma_{i=1}^{n} \frac{x_i b_{i,1}}{\alpha_1 b_{i,1} + \alpha_2 b_{i,2}} - \Sigma_{i=1}^{n} b_{i,1} \\ \Sigma_{i=1}^{n} \frac{x_i b_{i,2}}{\alpha_1 b_{i,1} + \alpha_2 b_{i,2}} - \Sigma_{i=1}^{n} b_{i,2} \end{bmatrix}.$$

We find the Hessian:

$$\begin{bmatrix} \Sigma_{i=1}^{n} \frac{-x_i b_{i,1}^2}{(\alpha_1 b_{i,1} + \alpha_2 b_{i,2})^2} & \Sigma_{i=1}^{n} \frac{-x_i b_{i,1} b_{i,2}}{(\alpha_1 b_{i,1} + \alpha_2 b_{i,2})^2} \\ \Sigma_{i=1}^{n} \frac{-x_i b_{i,1} b_{i,2}}{(\alpha_1 b_{i,1} + \alpha_2 b_{i,2})^2} & \Sigma_{i=1}^{n} \frac{-x_i b_{i,2}^2}{(\alpha_1 b_{i,1} + \alpha_2 b_{i,2})^2} \end{bmatrix}$$

The Newton-Raphson update is $h = -\mathbf{l}''(\theta)^{-\mathbf{1}}\mathbf{l}'(\theta)$. We combine the two of them below:

$$h(\alpha) = -\begin{bmatrix} \Sigma_{i=1}^{n} \frac{-x_i b_{i,1}^2}{(\alpha_1 b_{i,1} + \alpha_2 b_{i,2})^2} & \Sigma_{i=1}^{n} \frac{-x_i b_{i,1} b_{i,2}}{(\alpha_1 b_{i,1} + \alpha_2 b_{i,2})^2} \\ \Sigma_{i=1}^{n} \frac{-x_i b_{i,1} b_{i,2}}{(\alpha_1 b_{i,1} + \alpha_2 b_{i,2})^2} & \Sigma_{i=1}^{n} \frac{-x_i b_{i,2}^2}{(\alpha_1 b_{i,1} + \alpha_2 b_{i,2})^2} \end{bmatrix}^{-1} \begin{bmatrix} \Sigma_{i=1}^{n} \frac{x_i b_{i,1}}{\alpha_1 b_{i,1} + \alpha_2 b_{i,2}} - \Sigma_{i=1}^{n} b_{i,1} \\ \Sigma_{i=1}^{n} \frac{x_i b_{i,2}}{\alpha_1 b_{i,1} + \alpha_2 b_{i,2}} - \Sigma_{i=1}^{n} b_{i,2} \end{bmatrix}.$$

b)

Derive the Fisher Scoring update.

We take the Hessian calculated above. We site the textbook for expected value for a $X \sim \text{Poisson}(\lambda)$ distribution: $E(X) = \lambda$. We also point out that the expected value of a sum is equal to the sum of expected values, or $\Sigma_{i=1}^{n} E(X) = E(\Sigma_{i=1}^{n} x)$.

As such, we can write the Fisher Information $I(\alpha) = -\mathrm{E}(\mathrm{l}''(\alpha))$ as:

$$
\begin{aligned}
-\begin{bmatrix} \sum_{i=1}^{n} \frac{-\mathrm{E(x_i)b_{i,1}^2}}{(\alpha_1 b_{i,1}+\alpha_2 b_{i,2})^2} & \sum_{i=1}^{n} \frac{-\mathrm{E(x_i)b_{i,1}b_{i,2}}}{(\alpha_1 b_{i,1}+\alpha_2 b_{i,2})^2} \\ \sum_{i=1}^{n} \frac{-\mathrm{E(x_i)b_{i,1}b_{i,2}}}{(\alpha_1 b_{i,1}+\alpha_2 b_{i,2})^2} & \sum_{i=1}^{n} \frac{-\mathrm{E(x_i)b_{i,2}^2}}{(\alpha_1 b_{i,1}+\alpha_2 b_{i,2})^2} \end{bmatrix} &= -\begin{bmatrix} \sum_{i=1}^{n} \frac{-(\alpha_1 b_{i,1}+\alpha_2 b_{i,2})b_{i,1}^2}{(\alpha_1 b_{i,1}+\alpha_2 b_{i,2})^2} & \sum_{i=1}^{n} \frac{-(\alpha_1 b_{i,1}+\alpha_2 b_{i,2})b_{i,1}b_{i,2}}{(\alpha_1 b_{i,1}+\alpha_2 b_{i,2})^2} \\ \sum_{i=1}^{n} \frac{-(\alpha_1 b_{i,1}+\alpha_2 b_{i,2})b_{i,1}b_{i,2}}{(\alpha_1 b_{i,1}+\alpha_2 b_{i,2})^2} & \sum_{i=1}^{n} \frac{-(\alpha_1 b_{i,1}+\alpha_2 b_{i,2})b_{i,2}^2}{(\alpha_1 b_{i,1}+\alpha_2 b_{i,2})^2} \end{bmatrix} \\
&= \begin{bmatrix} \sum_{i=1}^{n} \frac{b_{i,1}^2}{(\alpha_1 b_{i,1}+\alpha_2 b_{i,2})} & \sum_{i=1}^{n} \frac{b_{i,1}b_{i,2}}{(\alpha_1 b_{i,1}+\alpha_2 b_{i,2})} \\ \sum_{i=1}^{n} \frac{b_{i,1}b_{i,2}}{(\alpha_1 b_{i,1}+\alpha_2 b_{i,2})} & \sum_{i=1}^{n} \frac{b_{i,2}^2}{(\alpha_1 b_{i,1}+\alpha_2 b_{i,2})} \end{bmatrix}
\end{aligned}
$$