Atlas-PS 10

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Problem 1

We define an orthogonal function family to be a set of functions F for which $< f, g >= 0 \forall f \neq g, f \in F, g \in F$. Here, we define the inner product of two functions with respect to a weight function $w(x)$ to be $\int_D f(x)g(x)w(x)dx$. In the given problem, $w(x) = 1$, so we needn't worry about it.

We start by asserting 3 commonly known identities.

$$sin(x)sin(y) = \frac{1}{2}(cos(x-y) - cos(x+y)) \tag{1}$$

$$cos(x)cos(y) = \frac{1}{2}(cos(x+y) + cos(x-y)) \tag{2}$$

$$sin(x)cos(x) = \frac{1}{2}(sin(x+y) + sin(x-y)) \tag{3}$$

As both $\int_0^{2\pi} cos(kx)dx = 0$ and $\int_0^{2\pi} sin(kx)dx = 0$ are trivial for all $k \in \mathbb{N}$, we won't discuss the inner product of 1 and any of the functions in the family.

Next, we invoke the identity to say the following:

Let k, j be natural numbers. Without loss of generalization, we can say that $k < j$.

$$\int_0^{2\pi} cos(jx)cos(kx)dx = \frac{1}{2}\int_0^{2\pi} cos((j+k)x) + cos((j-k)x)$$
$$\frac{1}{2}[\frac{sin((j+k)x)}{j+k} + \frac{sin((j-k)x)}{j-k}|_0^{2\pi}.$$

Note that $\int_0^{2\pi} sin(kx)dx = 0$, implying that $\int_0^{2\pi} cos(jx)cos(kx)dx = 0 \forall j, k \in \mathbb{N}, j \neq k$.

Let k, j be natural numbers. Without loss of generalization, we can say that $k < j$.

$$\int_0^{2\pi} sin(jx)sin(kx)dx = \frac{1}{2}\int_0^{2\pi} cos((j-k)x) - cos((j+k)x)$$
$$\frac{1}{2}[\frac{sin((j-k)x)}{j-k} - \frac{sin((j+k)x)}{j+k}]|_0^{2\pi}.$$

Again, both terms are equal to zero for any natural numbers j and k that aren't equal to each other.

Let k, j be natural numbers. Without loss of generalization, we can say that $k < j$\$.

$$\int_0^{2\pi} cos(jx)sin(kx)dx = \frac{1}{2}\int_0^{2\pi} sin((j+k)x) + sin((j-k)x)$$
$$\frac{1}{2}[-\frac{cos((j+k)x)}{j+k} - \frac{cos((j-k)x)}{j-k}]|_0^{2\pi}$$

We note that $cos(k(2\pi)) = 1 \forall k \in \mathbb{N}$ and $cos(0) = 1$. Therefore, for any natural numbers j and k, $\frac{1}{2}[-\frac{cos((j+k)x)}{j+k} - \frac{cos((j-k)x)}{j-k}]|_0^{2\pi} = -1 - (-1) = 0$.

We should note here that

$$\int_0^{2\pi} sin(kx)cos(kx)dx$$

would result in a divide-by-zero issue above. For this, we note that $\int_0^{2\pi} sin(kx)cos(kx)dx = -\frac{cos^2(kx)}{2k}|_0^{2\pi} = -\frac{1}{2k} - (-\frac{1}{2k}) = 0.$

Therefore, we have show the Fourier Trigonometric Family to be orthogonal on $0 \leq x \leq 2\pi$ with respect to weight $w(x) = 1$.

Problem 2

a)

To show that if $q_i(x)$ is a set of orthogonal functions, then it is a linearly independent set, we use a proof by contradiction. First, note that a set of orthogonal functions can trivialy be normalized, so we can assume without loss of generality that it is an orthonormal set.

Suppose it is not a linearly independent set. Then there must exist a function $q_k(x)$ such that $q_k(x) = \sum_{i=1}^{k-1} c_i q_i(x)$, where $\exists i \in [1, k-1]$ such that $c_i \neq 0$. (Note that we assume here that it is the last function in the set, q_k out of notational convenience, but it doesn't really matter.)

Next, note that we can rewrite the sequence of constants, c_i as

$$c_i =< q_k(x), q_i(x) >, \forall 1 \leq i \leq k-1.$$

Because the set of functions is orthogonal, we know that $< q_k(x), q_i(x) >= 0$ for all $i \in [1, k-1]$. However, this is a contradiction, because $\nexists i \in [1, k-1]$ such that $c_i \neq 0$.

Therefore the set must be linearly independent, and we have show that a set of othogonal functions must be linearly independent too.

b)

We first define $\mathrm{IMSE} = \int_{\mathrm{D}} \mathrm{E}[(\hat{\mathrm{f}}(\mathrm{x}) - \mathrm{f}(\mathrm{x}))^2]$. Also, the expected value of $f(x)$ is a constant with respect to $\hat{f}(x)$.

Noting that the variance of $\hat{f}(x)$ can be written as $\mathrm{V}(\hat{\mathrm{f}}) = \mathrm{E}[\hat{\mathrm{f}}(\mathrm{x})^2] - \mathrm{E}[\hat{\mathrm{f}}(\mathrm{x})]^2$, we rewrite IMSE:

$$\begin{aligned}
\mathrm{IMSE} &= \int_D \mathrm{E}[(\hat{\mathrm{f}}(\mathrm{x}) - \mathrm{f}(\mathrm{x}))^2] \\
&= \int_D \mathrm{E}[(\hat{\mathrm{f}}(\mathrm{x})^2] - 2\mathrm{E}[(\hat{\mathrm{f}}(\mathrm{x})\mathrm{f}(\mathrm{x})] + \mathrm{E}[\mathrm{f}(\mathrm{x})^2] \\
&= \int_D \mathrm{V}(\hat{\mathrm{f}}) - 2\mathrm{E}[(\hat{\mathrm{f}}(\mathrm{x})]\mathrm{f}(\mathrm{x}) + \mathrm{f}(\mathrm{x})^2 + \mathrm{E}[\hat{\mathrm{f}})]^2 \\
&= \int_D \mathrm{V}(\hat{\mathrm{f}}) + (\mathrm{E}[\hat{\mathrm{f}}(\mathrm{x})] - \mathrm{f}(\mathrm{x}))^2 \\
&= \mathrm{IV}(\hat{\mathrm{f}}) + \mathrm{ISB}(\hat{\mathrm{f}}).
\end{aligned}$$

Problem 3

First, $||\sum_{i=1}^{m} q_i||^2$ can be expanded to be equal to $\sum_{i=1}^{m} ||q_i||^2 + \sum_{j \neq k} ||q_j q_k||$. Because the set of functions are orthogonal, $\sum_{j \neq k} ||q_j q_k = 0||$. Therefore, $||\sum_{i=1}^{m} q_i||^2 = \sum_{i=1}^{m} ||q_i^2||$. Under the L2 norm, $\sum_{i=1}^{m} ||q_i^2|| = \sum_{i=1}^{m} ||q_i||^2$.

If each of the q_k are orthonormal, the value of the expression will always evaluate to m, as it will be the sum of m functions equal to 1 over the space.

This expression may not hold under other norms, as the last step above, $\sum_{i=1}^{m} ||q_i^2|| = \sum_{i=1}^{m} ||q_i||^2$ does not necessarily hold under norms that are not L2, as the squared step is not transitive under the norm.

Problem 4

a)

We derive the first 4 Chebyshev Polynomials using the following formula:

$$q_i(x) = x^i - \sum_{j=1}^{i-1} \frac{< q_i, q_j >}{< q_j(x), q_j(x) >}.$$

(Shoutout to Wolfram for assisting with the integrals here.)

Noting that $\int_{-1}^{1} x^k \sqrt{1-x^2} dx = 0$ whenever k is odd, and that $\int_{-1}^{1} x^2 \sqrt{1-x^2} dx = \frac{\pi}{8}$ and $\int_{-1}^{1} x^4 \sqrt{1-x^2} dx = \frac{\pi}{16}$, we can easily define our first four orthogonal polynomials:

$$\begin{aligned} q_0 &= 1 \\ q_1 &= x - 0 = x \\ q_2 &= x^2 - \frac{\frac{\pi}{8}}{\frac{\pi}{2}} - 0 = x^2 - \frac{1}{4} \\ q_3 &= x^3 - 0 - \frac{\frac{\pi}{16}}{\frac{\pi}{2}} - 0 = x^3 - \frac{1}{2}. \end{aligned}$$

Next, we normalize the polynomials via their inner product with themselves.

$$\begin{aligned} q_0 &= 2/pi \\ q_1 &= \frac{8x}{\pi} \\ q_2 &= \frac{32(x^2 - 1/4)}{\pi} \\ q_3 &= \frac{128(x^3 - 1/2)}{21\pi}. \end{aligned}$$

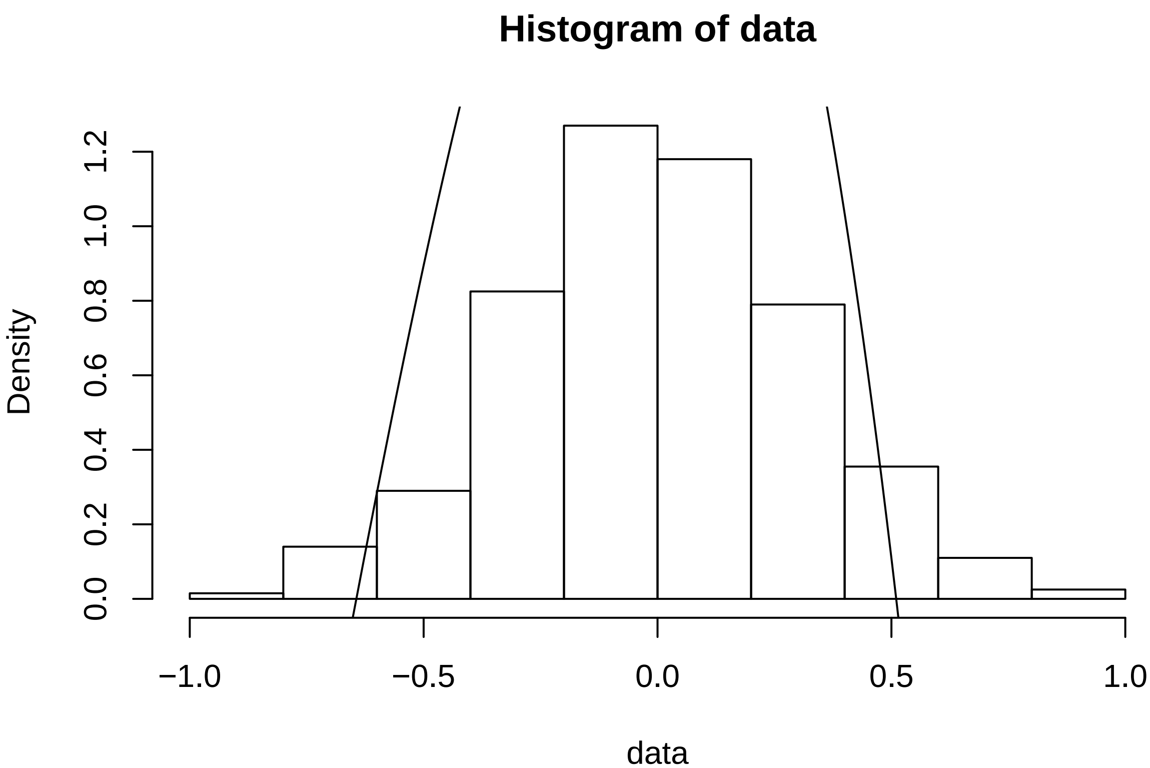
b)

Next, we take the density function described, $f(x) = \frac{1}{\sqrt{.6}} e^{-\frac{x^2}{1.2}}$, and find the inner product of this function with each of the orthonormal functions. We use Wolfram to find the constants below:

$$c_0 = 1.0696$$
$$c_1 = 0.0$$
$$c_2 = -0.799686$$
$$c_3 = -1.62987$$

Next, we read in the data and see if we can approximate the the normal distribution with mean 0 and standard deviation .3.

```
data <- scan("Orthogonal.txt")
approx <- function(x)   -1 * (1.62 * (128 * (x^3 - 1/2)) / (21 * pi)  + .8 * (32 * (x^2 - 1/4)) / (pi) 
X <- seq(-1, 1, .01)
hist(data, freq=F)
lines(X, approx(X))
```



The density is not well approximated by the function. We likely need more than just the 4 terms to get a good approximation.

Problem 5

a)

The cubic spline described is a natural cubic spline, as $f''(-1) = f''(1) = 0$. However, it is not defined beyond the endpoints, and so does not fulfill the requirement that the spline have a zero seond derivative beyond the end knots. The spline is continuous throughout, as is a natural spline. The first derivative is not continuous at the middle knot - not a characteristic of a natrual spline.

b)

We follow the algorithm to calculate the cubic splines, first finding the following quantities.

The differences between points: $w_i = x_{i+1} - x_i \implies (1, 1, 1)$

The first derivatives: $h_i = \frac{y_{i+1} - y_i}{x_{i+1} - x_i} \implies (-\frac{1}{2}, -\frac{1}{6}, -\frac{1}{12})$

Enforcing naturality: $f_1'' = 0$ and $f_4'' = 0$.

The second derivatives: $f'' = 3\frac{h_{i+1} - h_i}{w_{i+1} + w_i} \implies (\frac{1}{12}, \frac{3}{8})$

For each knot, we then find $A_i = \frac{f_{i+1}'' - f_i''}{6w_i}$, $B_i = \frac{f_i''}{2}$, $C_i = h_i - w_i\frac{f_{i+1}'' + 2f_i''}{6}$ and $D_i = y_i$. We use R to help with the computation

```
w_i <- rep(1, 3)
h_i <- c(-.5, -1/3, -1/12)
f2prime <- c(0, 1/4, 3/8, 0)

A <- (f2prime[2:4] - f2prime[1:3]) / 6
B <- f2prime / 2
C <- h_i - w_i * (f2prime[2:4] - 2 * f2prime[1:3]) / 6
D <- c(1, 1/2, 1/3, 1/4)
```

We write the entire spline as

$$s(x) = \begin{cases} 1 & x \leq 1 \\ \frac{1}{24}x^3 - \frac{13}{24}x + 1 & 1 < x \leq 2 \\ \frac{1}{48}x^3 + \frac{1}{8}x^2 - \frac{5}{16}x + \frac{1}{2} & 2 < x \leq 3 \\ -\frac{1}{16}x^3 + \frac{3}{16}x^2 - \frac{1}{24}x + \frac{1}{3} & 3 < x \leq 4 \\ 0 & x > 4 \end{cases}$$

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