UNSTEADY DRAG CORRECTIONS ON A FALLING COFFEE FILTER

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CONTENTS

ABSTRACT

In this experiment, we are exploring different ways for determining the drag coefficient of a falling object, namely a coffee filter. We opted on utilizing two methods of experimentation. Firstly, we performed a drop test from height of approximately 10 meters. And lastly, we captured the displacement of falling coffee filter with a high-speed camera in a drop tunnel. Having collected these physical measurements, we utilized the theoretical model represented by the governing equations of motion to solve for the drag coefficient. The analysis initially resulted in different values for the drag coefficient, but when we accounted for their uncertainties, our values coincided with each other.

INTRODUCTION

In this lab we explored the mechanism of falling objects, namely a coffee filter. We dropped basket style coffee filters (see figure 1) from large heights and measured the time of descent for different masses to determine the coffee filter's coefficient of drag, C_D.

Figure 1: A schematic of a coffee filter.

We opted on conducting our experiments with coffee filters because coffee filters have a geometry that is advantageous for our experiments. Namely, multiple filters can nest inside each other like shown in figure 1, effectively combining their masses without altering their cross-sectional area. Also, we can ignore the effects the extra thickness on the drag force, since thickness of multiple filters is small compared to their diameter.

Theory

Generally, a falling object is subjected to two external forces, the gravitational force expressed as the weight of the object, $W = mg$, and the air resistance expressed as the drag force, F_D. The motion of a falling can be described by newton's second law of motion,

$$\sum F = m\frac{dv}{dt} \tag{1}$$

In aerodynamics the drag force, F_D, is described as the force acting opposite to the relative motion of any moving object respect to its fluid. And this force is proportional to the square of its speed.

$$F_D = \frac{1}{2} C_D \rho v^2 A \tag{2}$$

Where C_D is the drag coefficient, ρ is the fluid density, v is the velocity and A the reference area on with the drag coefficient is based. Furthermore, by assuming steady state then performing a free-body force balance analysis on a coffee filter will yield the following relationship:

$$\sum F_y \equiv W = F_D \tag{3}$$

From equation 3, we see that drag force is equal the weight of our coffee filter, as we reach the maximum velocity, meaning that it we have stopped accelerating, leading the object to achieve terminal velocity. By further simplifying equation 3, we yield the following relationship for the coefficient of drag,

$$C_D = \frac{2mg}{\rho v^2 A} \tag{4}$$

METHODS

Part 1: Drop Time Trial.

In this part of the experiment, we performed a couple drop time trials by going up a staircase and dropping coffee filters of different masses from the same height.

Procedure

1. With one person remaining on the first floor of Baldy to measure the time it takes for the filter to hit the ground; another person went up to the third floor with a tape measure. They then carefully lowered the loose end of the tape measure down the flight of stairs (WITHOUT DROPPING IT) to the person on the ground, and measured the approximate distance between the lowering point and the ground. After measuring the drop height, we patiently winded up the tape measure back up and took notes of the smallest increment on the tape measure for δh.
2. Starting with a single coffee filter, we dropped the filter from the measured height unimpeded for the entire descent and measured the time it took to complete the fall.
3. We collected 3 successful trials and then waited for 20 seconds to allow the forced air current to settle.
4. We repeated step 2 and 3, but with stacked coffee filters of [2, 4, 7, 10] and took note of the smallest increment on the stopwatch for δt.
5. We returned to the lab in Jarvis; then with a ruler, we measured the average diameter of a coffee filter, and with an analytical scale, we measured the average mass for each coffee filter.

Part 2: Measurement with a High-Speed Camera

In this part of the experiment, we captured a falling coffee filter in a drop tunnel with a high-speed camera (see figure 2).

Figure 2: A High-Speed Camera Test Drop Tunnel

Procedure

1. Setting up the high-speed camera:
   - We positioned the camera facing the 0.5″x 0.5″ grid and turned on the continuous LED lights to provide a light source a for our camera.
   - We plugged the camera into the computer and placed a fan box over the camera like shown in figure 3, to keep the camera from overheating, as the camera lacks the means to dissipate the heat by itself.
2. Setting the *U-eye Cockpit* software:
   - We ran the software with optimal then proceeded with performing a few configurations to allow the camera to see and capture images at a large number of frames per second (FPS). Also, we set the exposure time to 1 ms.
   - We then placed the coffee filter in front of the camera and focused on the object by playing with the adjustment dial.

3. Setting up the coffee filter:
   - We lowered the vacuum tube using the white cord, turned on the vacuum pump (see figure 4) and attached the desired number of coffee filters up the pedestal until they held nicely.
   - Using the both the vacuum tube and the white wire, we pulled up the pedestal while keeping level until it attached to the magnetic holder on the ceiling and exited the tunnel, closing the door and waited for 30 seconds to let the air settle.
4. Collecting video sequences
   - We created a file and performed preliminary preparations for capturing the drop, then pressed "Record" and turned off the vacuum pump to allow the filter to drop.
   - After the filter completely landed, we pressed "Stop" then exited. Also, we unplugged the camera to prevent overheating.
5. Measuring the distance between the camera, filter and grid.
   - Sometime the coffee filter appears to be falling faster the closer it is to the camera, which will skew our result. This could be remedied using Trigonometry (see figure 5).
   - We measured the distance between the camera and vertical grid ("L' " in figure 5). Also, we measured the distance between the grid and the middle of the coffee filter ("S") after the it lands for each trial.
6. Reviewing the drop data collection
   - We reviewed the recording, making sure that the filter fell within the field of view. Then saved the AVI file for analysis
7. We repeated step 3 to 6 to collect 3 trials each for stacks of [1, 3, 5] nested coffee filter.

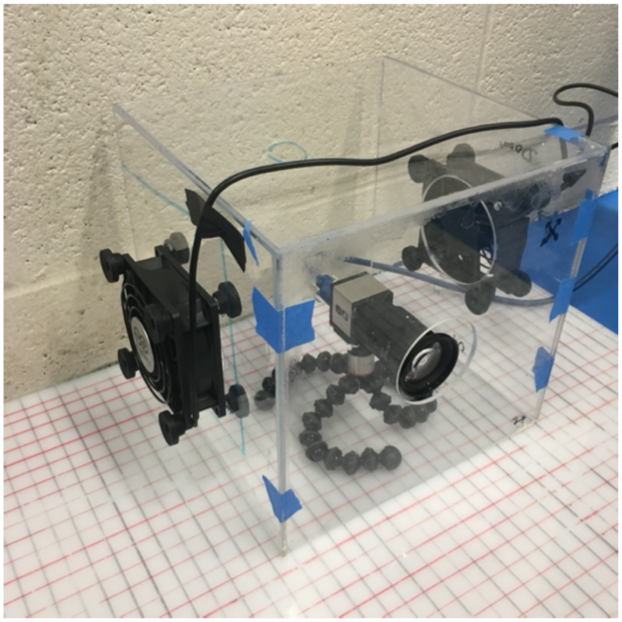


Figure 3: A High-Speed Camera Setup

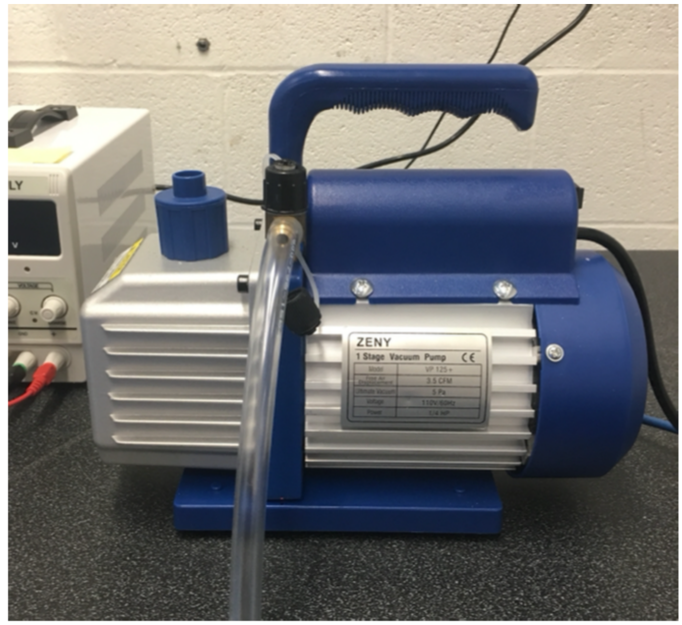


Figure 4: Vacuum Pump

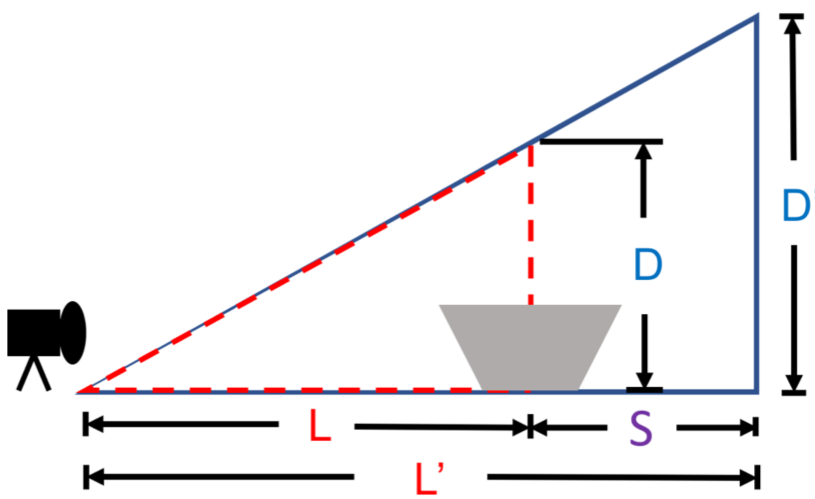


Figure 5: Camera Geometry

RESULTS

Part 1: Drop Time trials

For part 1, we conducted a free body analysis on a falling coffee filter at steady state and isolated the velocity to yield the following relationship:

$$v_{term}^2 = \frac{2mg}{C_D \rho A} \tag{5}$$

Where v_{term} is the terminal speed. Using Equation (5), we plotted the linear relationship between the squared of the terminal velocity (v_{term}^2) with respect to mass (m).

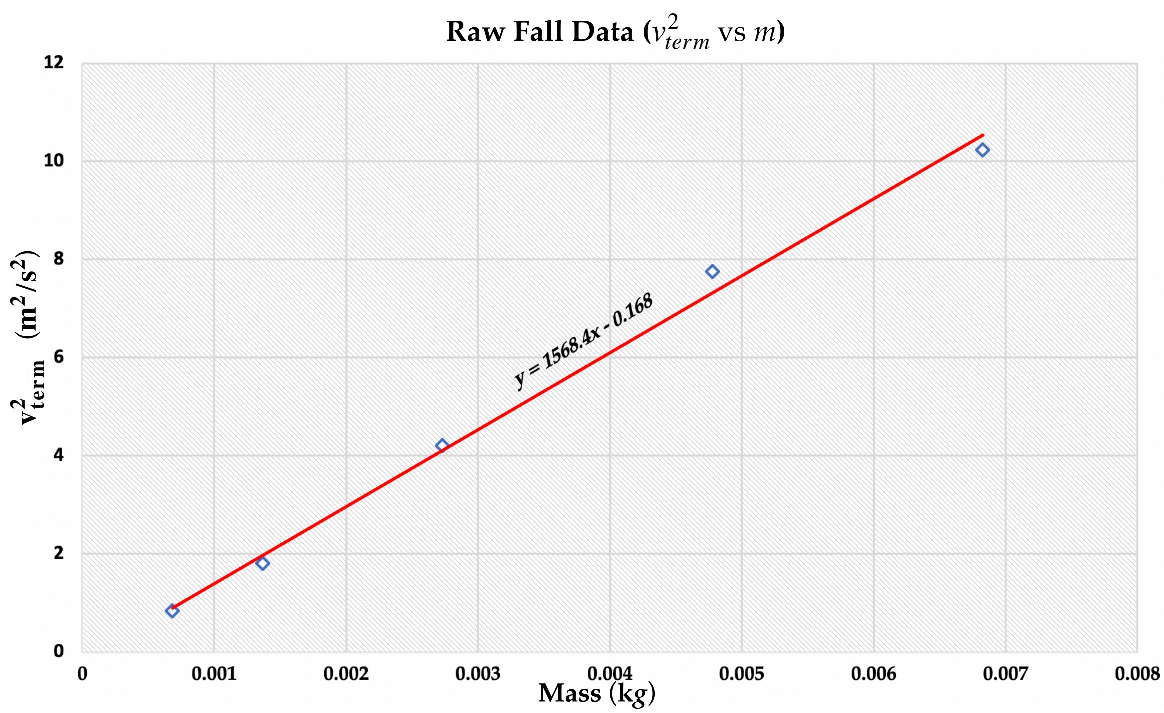


Figure 6: Raw fall data.

From the slope and Equation (5), we extrapolated our parameter drag coefficient, C_D = **0.7967**. However, since coffee filters do not realistically achieve steady state almost immediately, we conducted further free body analysis on the falling filter in order to capture its real behavior. Using equation (1), we assumed unsteady state, where the net force is a non-zero, and completed a free-body analysis on the filter under that assumption and found the following relationship for the unsteady velocity (check the appendix for detailed derivations).

$$\sum F_y = m\frac{dv}{dt} = F_D - W = \frac{1}{2}C_D\rho v^2 A - mg \tag{6}$$

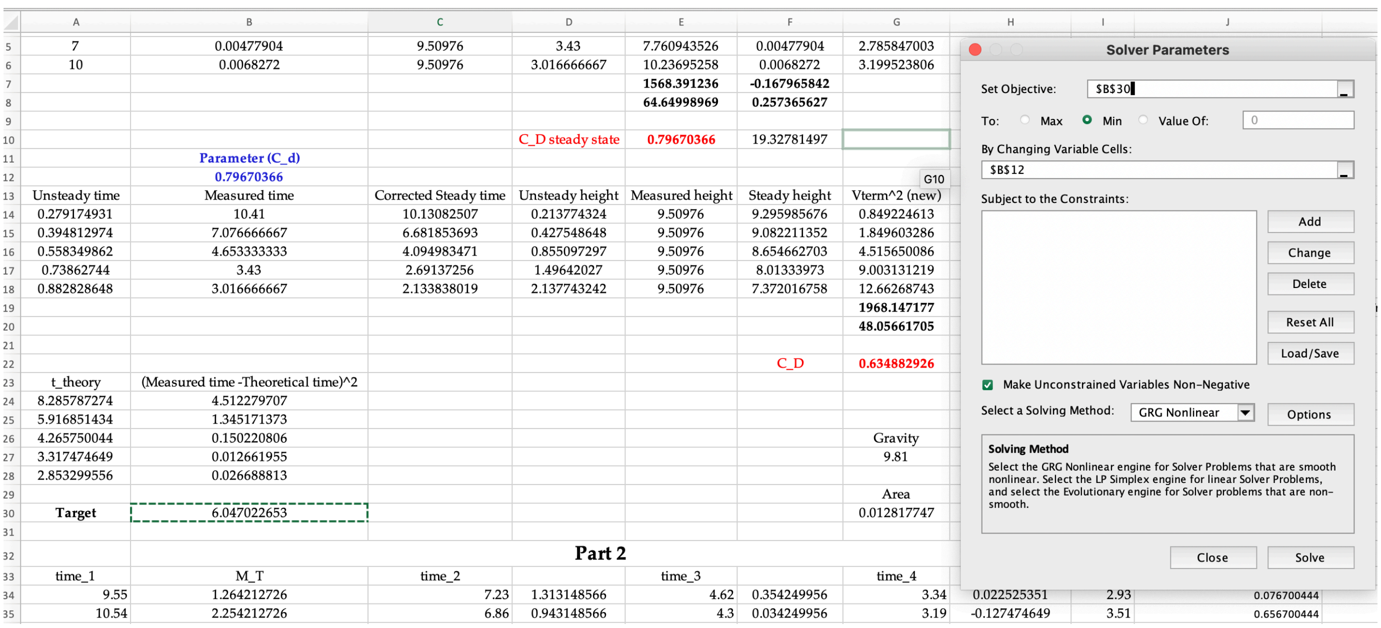
$$v(t) = \frac{-\tanh\left(\sqrt{\frac{\rho A C_D}{2mg}}\, g t\right)}{\sqrt{\frac{\rho A C_D}{2mg}}} \tag{7}$$

And by integrating Equation (7) with respect to time, we obtained the following relationship for the unsteady position:

$$v(t) = \frac{dy}{dt} = -\, v_{term} \tanh\left(\frac{gt}{v_{term}}\right) \tag{8}$$

$$y(t) = y_o - \frac{2m}{\rho C_D A} \ln \left| \cosh\left(\frac{gt}{\sqrt{\frac{2mg}{\rho C_D A}}}\right)\right| \tag{9}$$

Where y_o is the initial position of the filter or the drop height. To extrapolate the drag coefficient from our unsteady relationships, we used Equation (7), to solve for the unsteady time by evaluating how long it takes for the filter to reach 99% of the terminal velocity and subtracted the unsteady time from our measured time and got the steady time. Additionally, we used Equation (9) to determine the unsteady displacement and subtracted the unsteady displacement from the measured height to obtain the steady height. Having obtained both the steady height and steady time, we evaluated for the terminal velocity and calculated for the slope using Excel LINEST function. We applied Equation (5) and found a drag Coefficient, C_D = **0.6349**. However, to get a better approximation for the drag coefficient, we performed additional analysis using Excel's Solver tool, as shown by the figure below.



	A	B	C	D	E	F	G	H	I	J
5	7	0.00477904	9.50976	3.43	7.760943526	0.00477904	2.785847003			
6	10	0.0068272	9.50976	3.016666667	10.23695258	0.0068272	3.199523806			
7					**1568.391236**	**-0.167965842**				
8					**64.64998969**	**0.257365627**				
9										
10				C_D steady state	**0.79670366**	19.32781497				
11		**Parameter (C_d)**								
12		**0.79670366**					G10			
13	Unsteady time	Measured time	Corrected Steady time	Unsteady height	Measured height	Steady height	Vterm^2 (new)			
14	0.279174931	10.41	10.13082507	0.213774324	9.50976	9.295985676	0.849224613			
15	0.394812974	7.076666667	6.681853693	0.427548648	9.50976	9.082211352	1.849603286			
16	0.558349862	4.653333333	4.094983471	0.855097297	9.50976	8.654662703	4.515650086			
17	0.73862744	3.43	2.69137256	1.49642027	9.50976	8.01333973	9.003131219			
18	0.882828648	3.016666667	2.133838019	2.137743242	9.50976	7.372016758	12.66268743			
19							**1968.147177**			
20							**48.05661705**			
21										
22						C_D	**0.634882926**			
23	t_theory	(Measured time -Theoretical time)^2								
24	8.285787274	4.512279707								
25	5.916851434	1.345171373								
26	4.265750044	0.150220806					Gravity			
27	3.317474649	0.012661955					9.81			
28	2.853299556	0.026688813								
29							Area			
30	**Target**	6.047022653					0.012817747			
31										
32				**Part 2**						
33	time_1	M_T	time_2		time_3		time_4			
34	9.55	1.264212726	7.23	1.313148566	4.62	0.354249956	3.34	0.022525351	2.93	0.076700444
35	10.54	2.254212726	6.86	0.943148566	4.3	0.034249956	3.19	-0.127474649	3.51	0.656700444

Figure 7: A Snapshot of Excel's Solver Function

Having obtained the corrected steady height and steady time, we evaluated for the terminal velocity using Equation (5) and plotted the linear relationship between the squared of the terminal velocity (v_{term}^2) with respect to the mass (m). Using the slope on Figure 8 and Equation (5), we calculated our corrected drag coefficient and the uncertainty, C_D = **0.9189 ± 0.0632**.

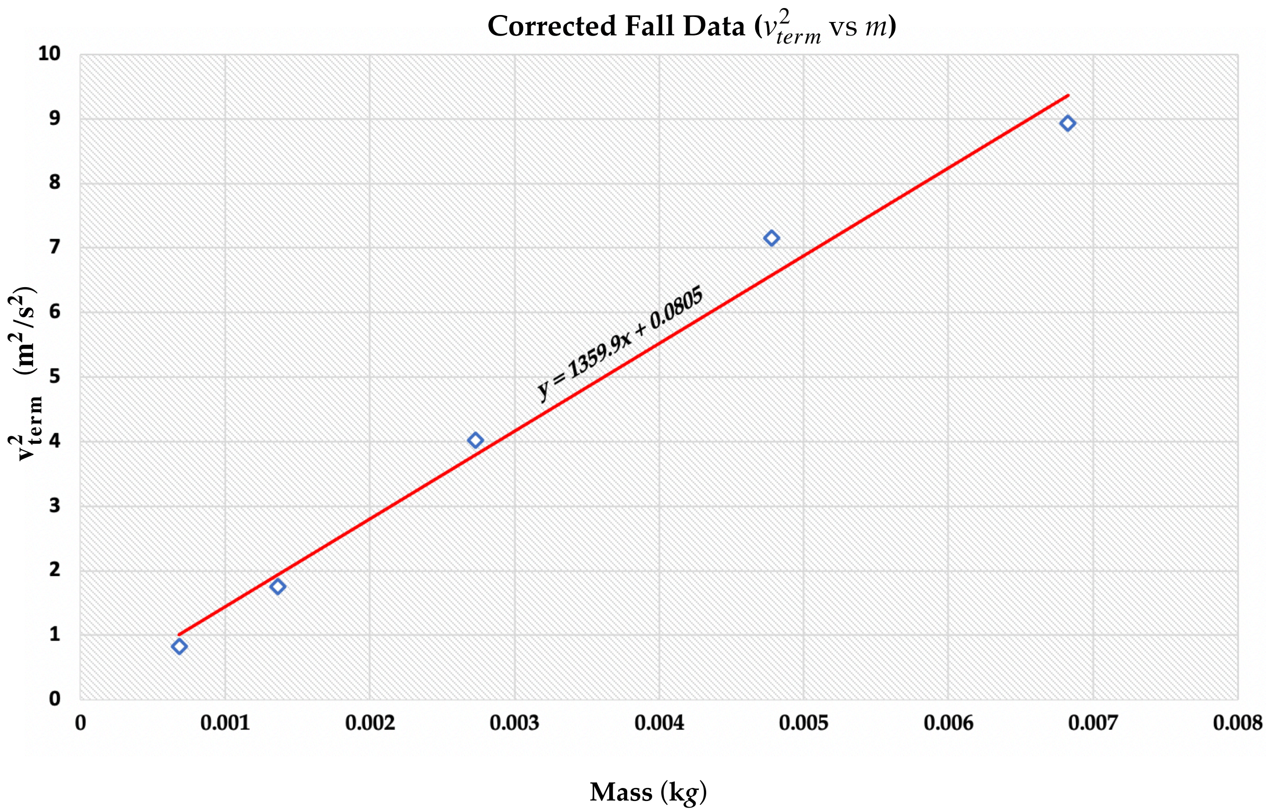


Figure 8: Corrected fall data.

Part 2: Measurements with a High-Speed Camera

For part 2, we captured the motion of a falling coffee filter in a drop tunnel with a high-speed camera and collected data illustrated on the table below:

Table 1: The High-Speed Camera Data

Number of Filters	Trial Number	FPS	Frames	S (m)	L (m)	D (m)
1	1	305	60	0.233	0.334	0.164584832
1	2	275	55	0.192	0.375	0.18478836
1	3	301	67	0.182	0.385	0.189716049
3	1	302	36	0.266	0.301	0.148323457
3	2	299	39	0.209	0.358	0.176411287

3	3	303	37	0.294	0.273	0.134525926
5	1	300	26	0.276	0.291	0.143395767
5	2	302	31	0.211	0.356	0.17542575
5	3	228	27	0.236	0.331	0.163106526

Where FPS is the frame rate per second, S is the distance between the coffee filter and the grid, L is the horizontal from the camera to the grid and D is the vertical distance from the ground to some arbitrary point within the camera's field of view (see figure 5).

To compute for the drag coefficient of the coffee filter, we used Equation (4) to establish following relationship (check the appendix for detailed derivations):

$$C_D = \frac{8L'gmt^2}{D'^2d^2\rho\pi L^2} \tag{10}$$

Where L' is the distance between the camera and the grid, D' is the maximum vertical distance of the Camera's field of view and d is the diameter of the coffee filter. Using Equation (10) and data from Table 1, we calculated the drag coefficient with their respective uncertainties and recorded our results in the following table:

Table 2: The High-Speed Camera Drag Coefficient with their Respective uncertainties

Number of Filters	**Trial Number**	C_D	μ_{C_D}
1	1	1.218756777	± 0.140501672
1	2	0.997143301	± 0.109600617
1	3	1.247177346	± 0.132691911
3	1	1.564562931	± 0.20419345

3	2	1.402852349	± 0.165504627
3	3	1.998924429	± 0.273191597
5	1	1.562263403	± 0.222459168
5	2	1.462386201	± 0.182174139
5	3	2.244488683	± 0.299377029

Taking the average of the drag coefficient and their uncertainties yields drag coefficient, C_D **= 1.5221 ± 0.6634** (check the appendix for detailed work).

DISCUSSION

1. We calculated the uncertainty of steady-state drag coefficient to be $\pm$ **0.0184**. Therefore, the corrected drag coefficient does not lie within the uncertainty of the measured steady-state analysis; with the steady-state drag coefficient yielding an error of approximately 30.9%.

2. We used 99% of the terminal velocity as a cut-off point because it's a point where there is no significant change in velocity, allowing us to assume steady-state, namely terminal velocity or constant velocity.

3. Using MatLab's ODE45, we were able to simulate how well our corrected

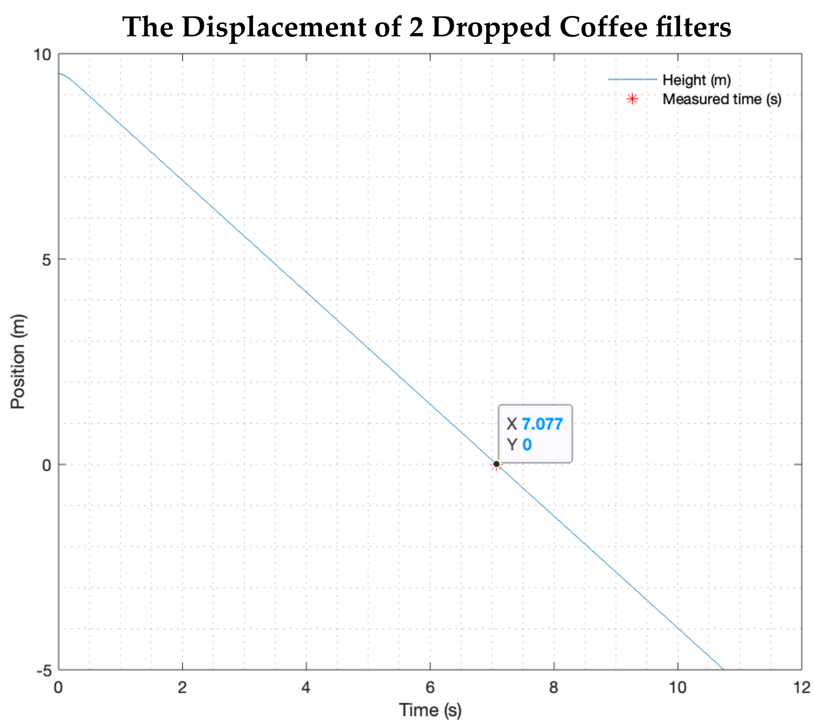


Figure 9: Position vs. Time for 2 Coffee Filters.

From Figure 9, we see can see that our simulation is accurately predicting the measured fall time for two coffee filters.

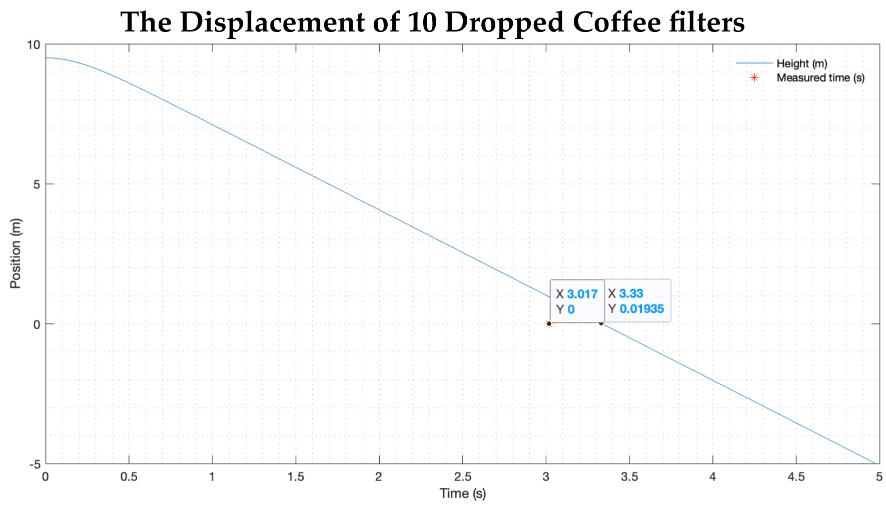


Figure 10: Position vs. Time for 10 Coffee Filters.

From Figure 10, we can see that although the measured time and simulated time are not superimposed, they are nonetheless very close deeming our simulation a success.

4.

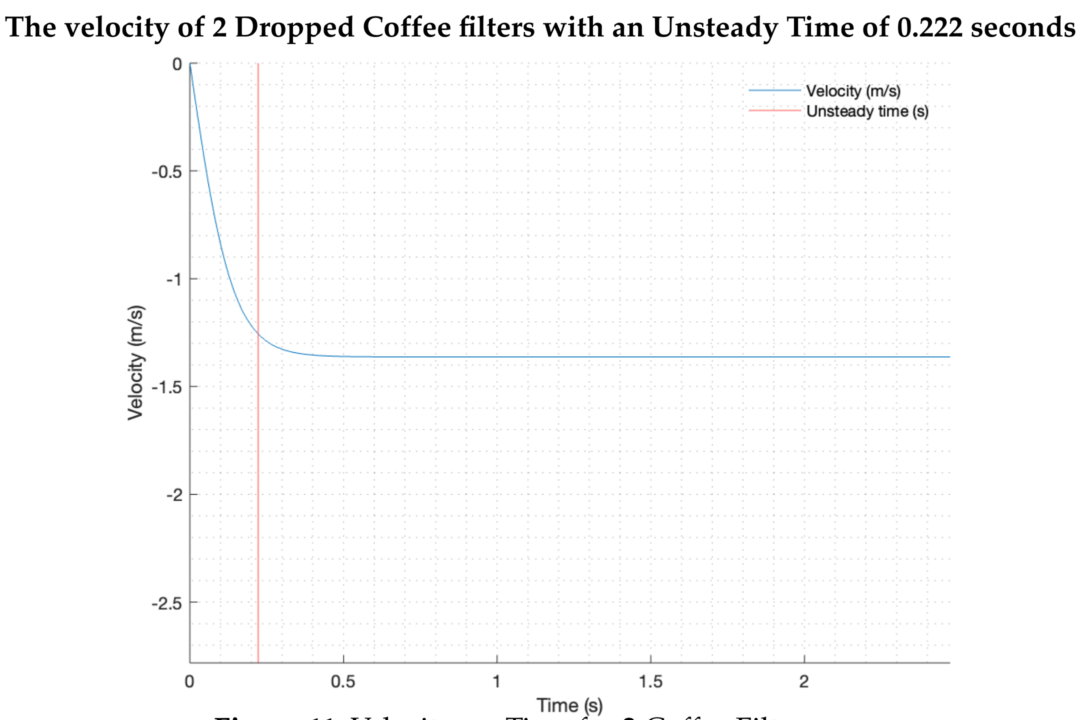


Figure 11: Velocity vs. Time for 2 Coffee Filters.

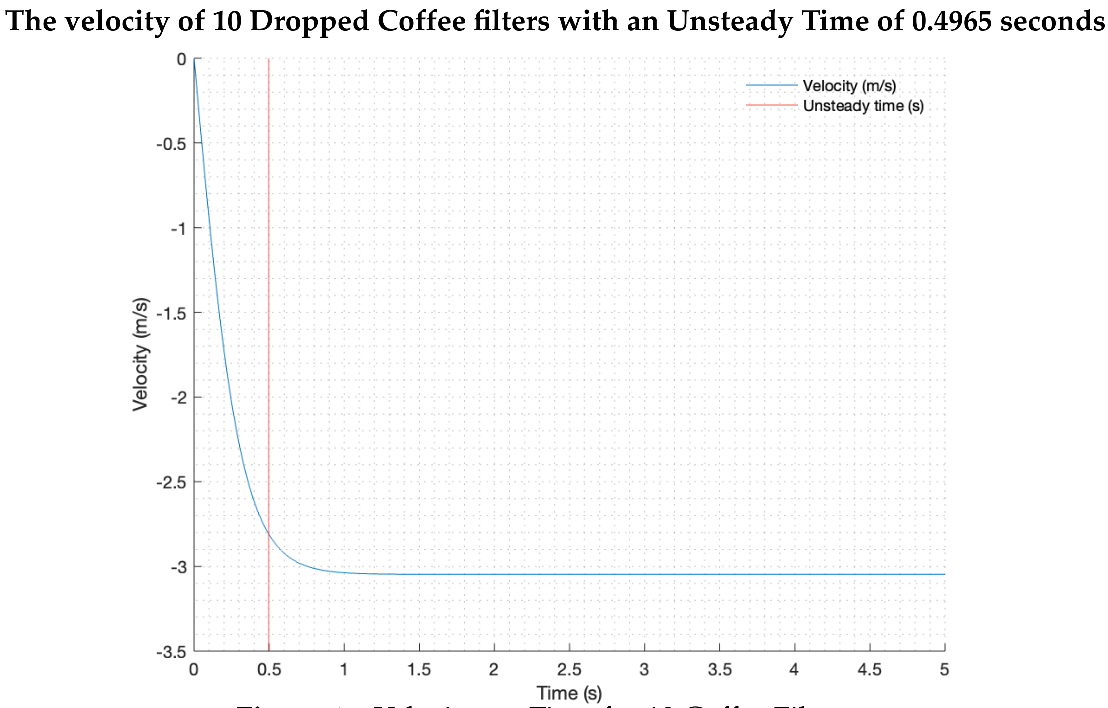


Figure 10: Velocity vs. Time for 10 Coffee Filters.

5. We computed for the drag coefficient by performing a free-fall trial and yielded a drag coefficient of **0.9189 ± 0.0632**. Also, we computed for the drag coefficient by way of capturing the direct velocity measurement with a high-speed camera and yielded a drag coefficient of **1.5221 ± 0.6634**. Comparing these two values, we can clearly see that C_D from the High-speed camera experiment is higher than the C_D from the drop test. However, they're similar in so far as their magnitudes falls within each other's uncertainties. With that said, although we got two consistent estimations for the drag coefficient, we also observed errors in our data acquisition. Namely on the free-fall experiment, the use of a stopwatch for recording time may have not been a great way of doing it; as the device is actively controlled by humans whose reaction time is at best mediocre. Also, for the high-speed camera experiment, errors may stem from parallax error that comes into play when measuring the diameter of the coffee filter, the respective lengths between the camera and the grid.

UNCERTAINTY ANALYSIS

Part 1: Drop Time Trials

We determined the drag coefficient uncertainty using the relationship below:

$$\mu_{C_D} = \sqrt{(\frac{\partial C_D}{\partial s}\mu_s)^2 + (\frac{\partial C_D}{\partial A}\mu_A)^2} \tag{11}$$

Where s is the slope, A is the area and μ is the uncertainty. Using Equation (11), Excel's LINEST Function and a diameter uncertainty of ±1 mm, we obtained the following values (check appendix for detailed work) :

Table 3: Drop Test Uncertainty

Method	Diameter (m)	$\mu_d\ (m)$	S	μ_s	μ_{C_D}	$\mu_A\ (m^2)$	Area (m^2)
Steady State	0.128	± 1E-3	1968.1	± 48.1	**± 0.01841**	0.0002	0.01282
Corrected C_D	0.128	± 1E-3	1359.9	± 91.05	**± 0.06318**	0.0002	0.01282

Part 2: Measurements with a High-Speed Camera

We determined the drag coefficient uncertainty using the relationship below:

$$\mu_{C_D} = \sqrt{(\frac{\partial C_D}{\partial s}\mu_s)^2 + (\frac{\partial C_D}{\partial m}\mu_m)^2 + (\frac{\partial C_D}{\partial d}\mu_d)^2 + (\frac{\partial C_D}{\partial L'}\mu_{L'})^2 + (\frac{\partial C_D}{\partial t}\mu_t)^2 + (\frac{\partial C_D}{\partial D'}\mu_z)^2} \tag{12}$$

Where L' is the distance between the camera and the grid, t is the time, d is the diameter, S is the distance between the coffee filter and the grid, m is the mass and D' is the vertical drop distance. The respective partial derivatives are as follows (Check appendix for detailed work) :

$$\frac{\partial C_D}{\partial s} = \frac{16L'^2gmt^2}{D'^2d^2(L'-s)^3} \tag{13}$$

$$\frac{\partial C_D}{\partial m} = \frac{8L'^2 g t^2}{D'^2 d^2 (L' - s)^2} \tag{14}$$

$$\frac{\partial C_D}{\partial s} = \frac{-16L'^2 g m t^2}{D'^2 d^3 (L' - s)^2} \tag{15}$$

$$\frac{\partial C_D}{\partial L'} = \frac{-16L' g m t^2}{D'^2 d^2 (L' - s)^3} \tag{16}$$

$$\frac{\partial C_D}{\partial t} = \frac{16L'^2 g m t}{D'^2 d^2 (L' - s)^2} \tag{17}$$

$$\frac{\partial C_D}{\partial D'} = \frac{-16L'^2 g m t^2}{D'^3 d^2 (L' - s)^2} \tag{18}$$

Table 4: High speed Camera Parameter and Uncertainties

Mass (kg)	μ_m ***(kg)***	**d (m)**	μ_d ***(m)***	***L′ (m)***	$\mu_{L'}$
0.00068272	± 1E-8	0.128	± 1E-3	0.567	± 0.005
0.00068272	± 1E-8	0.128	± 1E-3	0.567	± 0.005
0.00068272	± 1E-8	0.128	± 1E-3	0.567	± 0.005
0.00204816	± 1E-8	0.128	± 1E-3	0.567	± 0.005
0.00204816	± 1E-8	0.128	± 1E-3	0.567	± 0.005
0.00204816	± 1E-8	0.128	± 1E-3	0.567	± 0.005
0.0034136	± 1E-8	0.128	± 1E-3	0.567	± 0.005
0.0034136	± 1E-8	0.128	± 1E-3	0.567	± 0.005
0.0034136	± 1E-8	0.128	± 1E-3	0.567	± 0.005

Table 5: High speed Camera Parameter and Uncertainties

Time (s)	μ_t **(s)**	**s (m)**	μ_s **(m)**	D' **(m)**	μ_z
0.196721311	± 3.28E-03	0.233	± 1E-3	0.2794	± 0.00127
0.199782056	± 3.63E-03	0.192	± 1E-3	0.2794	± 0.00127
0.229388298	± 3.32E-03	0.182	± 1E-3	0.2794	± 0.00127
0.115970842	± 3.31E-03	0.266	± 1E-3	0.2794	± 0.00127
0.130609511	± 3.35E-03	0.209	± 1E-3	0.2794	± 0.00127
0.118890357	± 3.3E-03	0.294	± 1E-3	0.2794	± 0.00127
0.086782377	± 3.34E-03	0.276	± 1E-3	0.2794	± 0.00127
0.102717031	± 3.31E-03	0.211	± 1E-3	0.2794	± 0.00127
0.118317266	± 4.38E-03	0.236	± 1E-3	0.2794	± 0.00127

CONCLUSION

In this experiment, we adopted two techniques for determining the drag coefficient of a coffee filter, namely a drop time trial and a high-speed camera test. From these two different methods, we were able to determine the drag coefficient their respective uncertainties and our values ended up coinciding with each other's uncertainties. Also, the drop test data provided us with a good estimation for the drag coefficient as it resulted to an accurate Matlab simulation.

REFERENCES

[1] Burge, M. (Fall 2021) Unsteady Drag Corrections On A Falling Coffee FilterBuffalo, NY: University at Buffalo.

[2] Anderson, J. D. (2011). Fundamentals of aerodynamics (6th ed.). New York, NY: McGrawHill.

APPENDIX

Relevant Matlab Code

```
close all; clear

yo=9.50976;
vo=0;
mass=[0.00068272 0.00136544 0.00273088 0.00477904 0.0068272];

time=0:0.01:30;
ini=[yo vo];
options= odeset('AbsTol',1e-8,'RelTol',1e-8);

for i =1:length(mass)
    input=mass(1,i);
    [T,X]=ode45(@diffeq,time,ini,options,input);
    time_m=[10.41 7.076666667 4.653333333 3.43 3.016666667];
    uns_time=[0.156992321 0.222020669 0.313984641 0.415362638
 0.496453308];
%     9.55 6.86 4.3 3.19 2.61
    figure
    plot(T,X(:,1),time_m(1,i),0,'*r');
    ylim([-5 10]);
    grid minor
    ylabel('Position (m)')
    xlabel('Time (s)')
    legend('Height (m)','Measured time (s)')
    legend('boxoff')
%     title(sprintf('Position of %d Dropped Coffee Filters',i))

    figure
    hold on
    plot(T,X(:,2))
    xline(uns_time(1,i),'-r')
    ylim([-3.5 0])
    xlim([0 5])
    hold off
    grid minor
    ylabel('Velocity (m/s)')
    xlabel('Time (s)')
    legend('Velocity (m/s)','Unsteady time (s)')
    legend('boxoff')
%     title(sprintf('Velocity of %d Dropped Coffee Filters with
 Unsteady Time of %.3f seconds',i,uns_time(1,i)))
end


function dx= diffeq(t,x,input)
g=9.81;
rho=1.225;
A=0.012817747;
s=1359.852808;
```

```
C_D=(2*g)/(rho*A*s);
m= input;
y=x(1,1);
v=x(2,1);

dx(1,1)=v;
dx(2,1)=(.5*rho*A*v^2*C_D/m)-g;
end
```

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Excel calculations

Number of filter	Mass (kg)	Height of the fall
1	0.00068272	9.50976
2	0.00136544	9.50976
4	0.00273088	9.50976
7	0.00477904	9.50976
10	0.0068272	9.50976

Parameter (C_d)
2.519375078

Unsteady time	Measured time	Corrected Steady time
0.156992321	10.41	10.25300768
0.222020669	7.076666667	6.854645998
0.313984641	4.653333333	4.339348692
0.415362638	3.43	3.014637362
0.496453308	3.016666667	2.520213358

t_theory	(Measured time -Theoretical time)^2
9.937742945	0.223026726
7.075185718	2.19321E-06
5.070992453	0.17443914
3.910506186	0.230886194
3.336351188	0.102198193
Target	0.730552447

time_1	M_T	time_2
9.55	-0.387742945	7.23
10.54	0.602257055	6.86
11.14	1.202257055	7.14

Number of Filters	Trial Number	FPS
1	1	305
1	2	275
1	3	301
3	1	302
3	2	299
3	3	303
5	1	300
5	2	302
5	3	228

FPS	Frames	Time
305	60	0.196721311
275.3	55	0.199782056
300.8	69	0.229388298
301.8	35	0.115970842
298.6	39	0.130609511
302.8	36	0.118890357
299.6	26	0.086782377
301.8	31	0.102717031
228.2	27	0.118317266

Percent Error
30.90695532

Uncertainties

Part 1

mu_d	0.001
mu_s1	64.64998969
mu_s2	91.04663729
mu_s3	0
mu_C_d_1	0.035129419
mu_C_d_2	0.063181625

Part 2

FPS	***mu_t***	***mu_z***
305	3.28E-03	0.0127
275.3	3.63E-03	0.0127
300.8	3.32E-03	0.0127
301.8	3.31E-03	0.0127
298.6	3.35E-03	0.0127
302.8	3.30E-03	0.0127
299.6	3.34E-03	0.0127
301.8	3.31E-03	0.0127
228.2	4.38E-03	0.0127

ds	0.001
dz	0.00127
dL'	0.005
dm	1.00E-08
dd	0.001
dL	0.01

Time of the fall	Vterm^2	Mass (kg) **Copy**	Vterm
10.41	0.841316709	0.00068272	0.917233181
7.076666667	1.807663267	0.00136544	1.344493684
4.653333333	4.211824256	0.00273088	2.052272949
3.43	7.760943526	0.00477904	2.785847003
3.016666667	10.23695258	0.0068272	3.199523806
	1568.391236	**-0.167965842**	
	64.64998969	**0.257365627**	
C_D steady state	**0.79670366**		

Unsteady height	Measured height	Steady height	Vterm^2 (new)
0.067601997	9.50976	9.442158003	0.828934686
0.135203994	9.50976	9.374556006	1.757429584
0.270407988	9.50976	9.239352012	4.016601755
0.473213979	9.50976	9.036546021	7.153641052
0.67601997	9.50976	8.83374003	8.927360576
			1359.852808
			91.04663729
	0.634882926	C_D	**0.918881096**

Gravity
9.81

Area
0.012817747

Part 2

	time_3		time_4
0.154814282	4.62	-0.450992453	3.34
-0.215185718	4.3	-0.770992453	3.19
0.064814282	5.04	-0.030992453	3.76

Frames	S (m)	L (m)	D (m)
60	0.233	0.334	0.164584832
55	0.192	0.375	0.18478836
67	0.182	0.385	0.189716049
36	0.266	0.301	0.148323457
39	0.209	0.358	0.176411287
37	0.294	0.273	0.134525926
26	0.276	0.291	0.143395767
31	0.211	0.356	0.17542575
27	0.236	0.331	0.163106526

Mass	Velocity	C_D	velocity^2
0.00068272	0.836639565	1.218756788	0.699965762
0.00068272	0.924949735	0.99714331	0.855532013
0.00068272	0.827051995	1.247177357	0.684015002
0.00204816	1.278971979	1.564562945	1.635769323
0.00204816	1.350677191	1.402852362	1.824328874
0.00204816	1.13151251	1.998924447	1.280320561
0.0034136	1.652360456	1.562263417	2.730295076
0.0034136	1.707854555	1.462386214	2.916767182
0.0034136	1.37855219	2.244488703	1.900406141
			647.8931849
			98.59900969
		C_D	**1.928625069**

mu_A	0.000200669		
dC_d|s_1	-0.000507975	dC_d|A_1	-62.15629471
dC_d|s_2	-0.000675721	dC_d|A_2	-71.68819105

mu_L'	***mu_S***	***S (m)***	***L (m)***
0.001	1.00E-03	0.233	0.334
0.001	0.001	0.192	0.375
0.001	0.001	0.182	0.385
0.001	0.001	0.266	0.301
0.001	0.001	0.209	0.358
0.001	0.001	0.294	0.273
0.001	0.001	0.276	0.291
0.001	0.001	0.211	0.356
0.001	0.001	0.236	0.331

C_D_m	C_D_d	C_D_t	C_D_L'
1785.148782	-19.08034093	1.24E+01	-100.7526794
1460.545027	-15.61085402	9.982310935	-73.41965444
1826.777224	-19.52528135	10.87394046	-89.44447354
763.8870652	-24.49413591	26.9820053	-143.5200473
684.9329883	-21.9624634	21.48162623	-108.1969367
975.961072	-31.29431592	33.62635095	-202.1714177
457.6586019	-24.45813547	36.00416274	-148.2338192
428.3999884	-22.8945002	28.47407454	-113.4222113
657.5136757	-35.13876608	37.94017167	-187.2300073
1004.536047	**-23.82875481**	**24.19503741**	**-129.5990274**

C_D_s	C_D_D'	mu_C_D
7.297944771	-68.29041134	0.513193119
5.318097604	-55.87277747	0.376009296
6.478843357	-69.88289675	0.457836953
10.39576699	-87.6669145	0.732152348
7.837163963	-78.60581029	0.555292384
14.64413501	-112.0054256	1.027420274
10.73720552	-87.5380654	0.759504079
8.215652812	-81.94166143	0.584754452
13.56186516	-125.7650898	0.964856684
9.387408354	**-85.28545029**	**0.663446621**

Handwritten derivations

$$\Sigma F_y = m\frac{dv}{dt} = F_D - W$$

$$F_D = W \rightarrow v^2 = \frac{mg}{\frac{1}{2}\rho A C_D}$$

$$\Sigma F_y = m\frac{d\vec{v}}{dt} = \vec{F_D} - \vec{W} = \frac{1}{2}\rho A v^2 C_D - mg \qquad \frac{dv}{\left(\frac{1}{2}\rho A v^2 C_D - 1\right)} = g\,dt$$

$$\frac{dv}{dt} = \frac{\frac{1}{2}\rho A v^2 C_D}{m} - g \qquad \therefore \text{ if } s = \frac{\frac{1}{2}\rho A C_D}{mg}$$

$$= g\frac{\left(\frac{1}{2}\rho A v^2 C_D - 1\right)}{mg}$$

$$\frac{dv}{(sv^2-1)} = g\,dt \quad ; \quad u = \sqrt{s}\,v \quad du = \sqrt{s}\,dv \qquad dv = \frac{du}{\sqrt{s}}$$

$$\frac{du}{u^2-1} = \sqrt{s}\,g\,dt$$

We know that $\int \frac{dx}{1-x_2} = \tanh^{-1}(x) + C$

$$\int \frac{du}{1-u^2} = -g\sqrt{s}\int dt$$

$$\tanh^{-1}(v\sqrt{s}) = -\sqrt{s}\,tg$$

$$v = \frac{\tanh(-\sqrt{s}\,tg)}{\sqrt{s}} \rightarrow \boxed{v(t) = -\frac{\tanh\left(\sqrt{\frac{\rho A C_D}{2mg}}\,tg\right)}{\sqrt{\frac{\rho A C_D}{2mg}}}}$$

$$v_{term} = \sqrt{\frac{2mg}{C_D \rho A}}$$

$$v(t) = \frac{dy}{dt} = -v_{term} \tanh\left(\frac{gt}{v_{term}}\right)$$

$$u = \frac{gt}{v_{term}}$$

$$du = \frac{g}{v_{term}} dt$$

$$\int_{y_0}^{y} dy = -v_{term} \int_0^t \tanh\left(\frac{gt}{v_{term}}\right) dt$$

$$y - y_0 = \frac{-v_{term}^2}{g} \int \tanh(u)\, du$$

We know that $\int \tanh(u)\, du = \ln|\cosh(u)| + c$

$$\Delta y = -\frac{v_{term}^2}{g} \int_0^t \tanh u\, du = \frac{-v_{term}^2}{g} \Big[\ln|\cosh(u)| \Big]_0^t$$

$$y = y_0 - \frac{v_{term}^2}{g} \ln\left(\left|\cosh(gt/v_{term})\right|\right)$$

$$\boxed{y = y_0 - \frac{2m}{\rho A C_D} \ln\left(\left|\cosh\left(\frac{gt}{\sqrt{\frac{2mg}{\rho A C_D}}}\right)\right|\right)}$$

Uncertainty

Part 1

$$\mu_d = \pm 0.001\,m \;;\quad \mu_A = \sqrt{\left(\frac{\partial A}{\partial d}\right)^2} = \sqrt{\left(\frac{\pi d}{2}\mu_d\right)^2}$$

$$C_D = \frac{2g}{s\rho A} \;;\quad \frac{\partial C_D}{\partial s} = -\frac{2g}{s^2\rho A} \;;\quad \frac{\partial C_D}{\partial A} = -\frac{2g}{s\rho A^2}$$

$$\mu_{C_D} = \sqrt{\left(\frac{\partial C_D}{\partial s}\mu_s\right)^2 + \left(\frac{\partial C_D}{\partial A}\mu_A\right)^2}$$

Part 2

$$\mu_t = \pm\frac{1}{FPS} \;;\quad \mu_z = \pm 0.0127\,m$$

$$D = L\left(\frac{D'}{L'}\right) = (L'-s)\left(\frac{D'}{L'}\right) \qquad u_L = u_s = 0.001$$

$$V = \frac{D}{t} = \frac{(L'-s)\left(\frac{D'}{L'}\right)}{t} \qquad C_D = \frac{2mg}{\rho\frac{\pi}{4}d^2\left(\frac{(L'-s)\left(\frac{D'}{L'}\right)}{t}\right)^2}$$

$$C_D = \frac{8L'^2 g m t^2}{D'^2 d^2 \rho\pi (L'-s)^2}$$

$$\mu_{C_D} = \sqrt{\left(\frac{\partial C_D}{\partial m}\mu_m\right)^2 + \left(\frac{\partial C_D}{\partial d}\mu_d\right)^2 + \left(\frac{\partial C_D}{\partial L'}\mu_{L'}\right)^2 + \left(\frac{\partial C_D}{\partial t}\mu_t\right)^2 + \left(\frac{\partial C_D}{\partial s}\mu_s\right)^2 + \left(\frac{\partial C_D}{\partial D'}\mu_z\right)^2}$$

$$\frac{\partial C_D}{\partial m} = \frac{8L'^2 g t^2}{D'^2 d^2 \rho\pi (L'-s)^2} \;;\quad \frac{\partial C_D}{\partial d} = \frac{-16L'^2 g m t^2}{D'^2 d^3 \rho\pi (L'-s)^2}$$

$$\frac{\partial C_D}{\partial t} = \frac{16L'^2 gmt}{D'^2 d^2 \rho \pi (L'-s)^2} \; ; \; \frac{\partial C_D}{\partial L'} = \frac{-16L' gmt^2}{D'^2 d^2 \rho \pi (L'-s)^3}$$

$$\frac{\partial C_D}{\partial s} = \frac{16L'^2 gmt^2}{D'^2 d^2 \rho \pi (L'-s)^3} \; ; \; \frac{\partial C_D}{\partial D'} = \frac{-16L'^2 gmt^2}{D'^3 d^2 \rho \pi (L'-s)^2}$$