Convex Hull Optimization

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Kalila and Dimna in the Logging Industry

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- We use the chainsaw charging coefficient of the tree with the highest id which has been cut completely.
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- We want to minimize the total charge cost to cut all trees.

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- Since $b_n = 0$ we must only cut the largest tree, at that point all cuts are free.
- We want to minimize the cost required to cut the largest tree.
- It is also quite clear that once we start cutting a tree, we should finish cutting it before starting to cut others.

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- A line!

Enter the Convex Hull Trick

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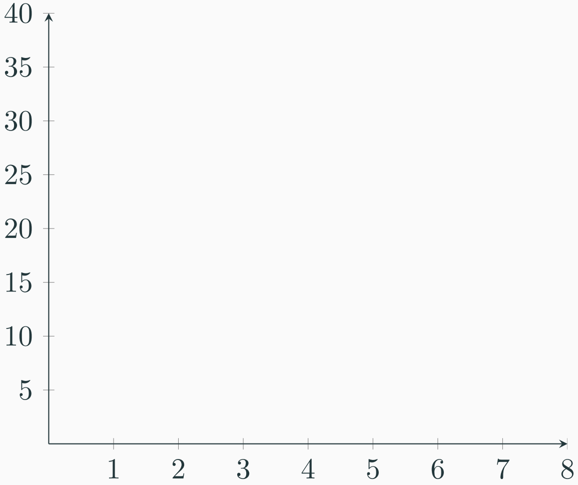
- We need both operations to be sub-linear in time complexity.

Sample 1 - Illustrated



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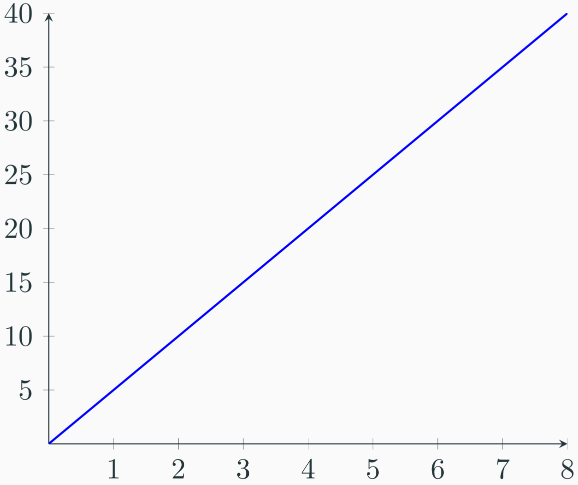

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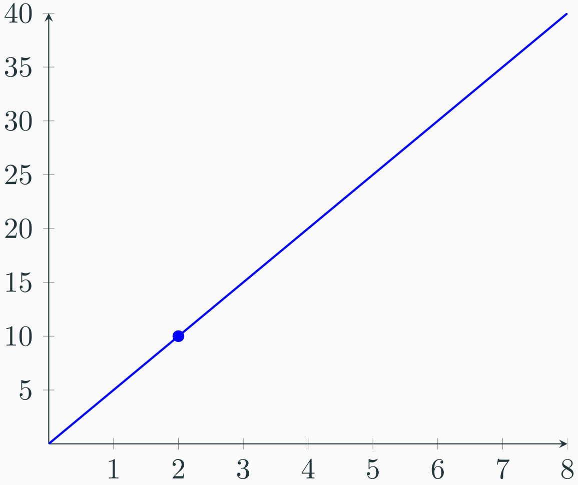

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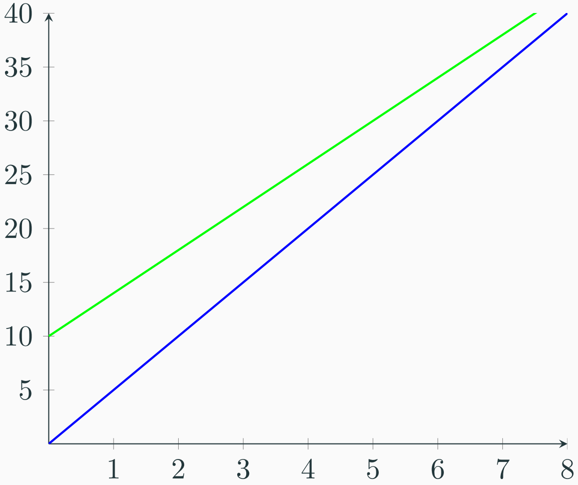

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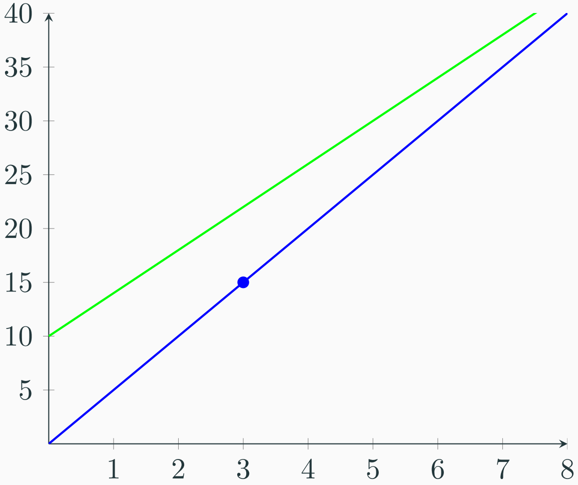

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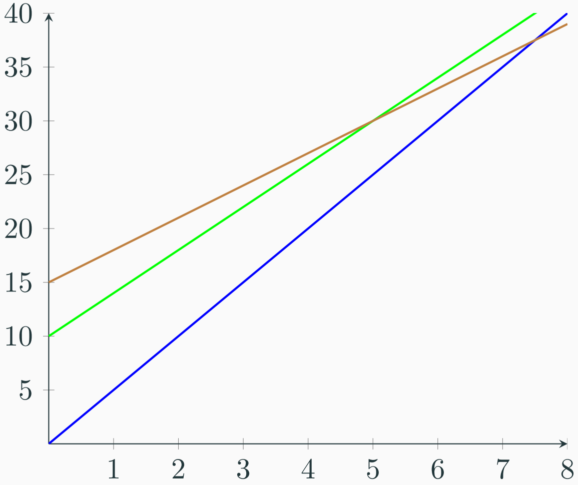

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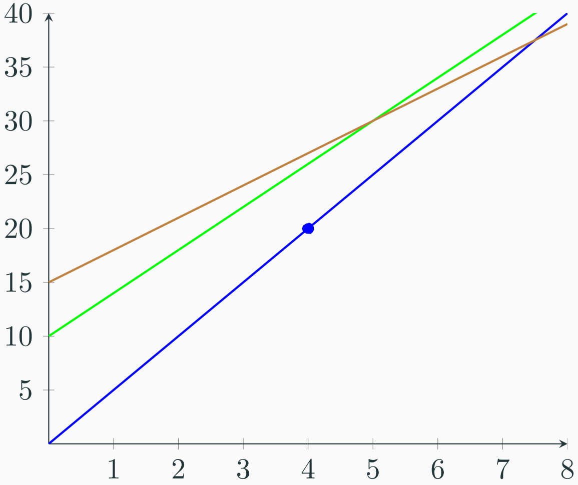

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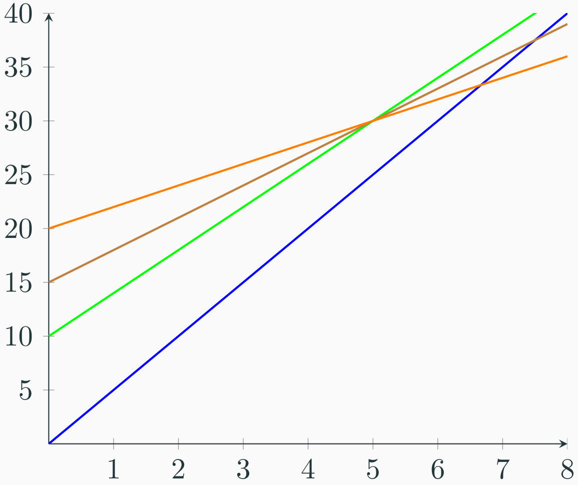

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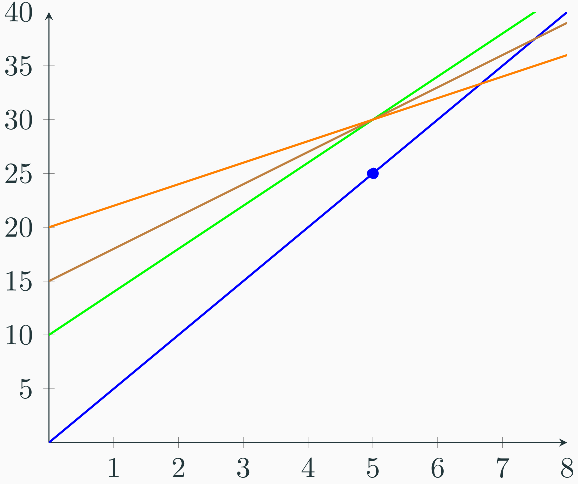

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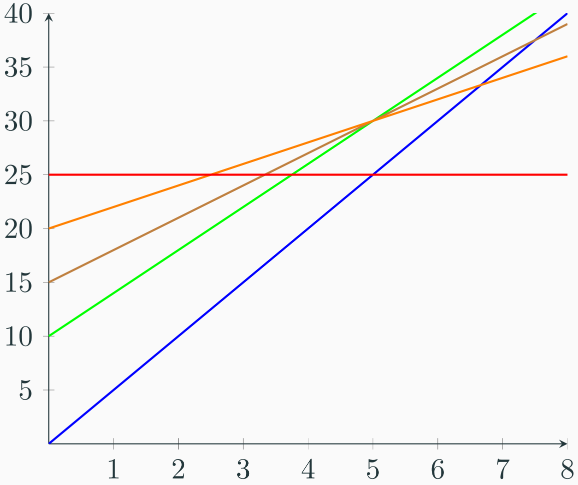

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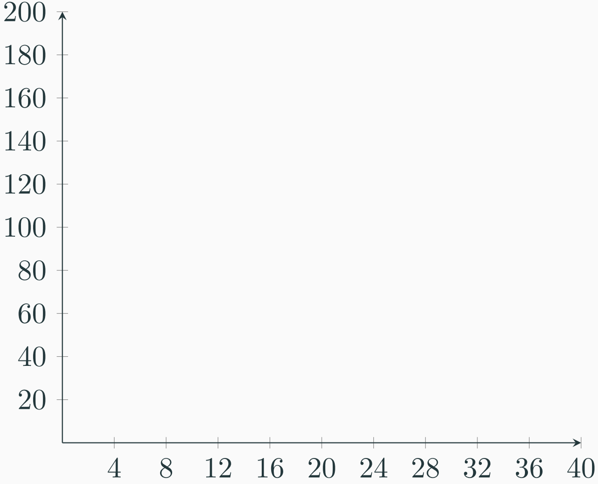

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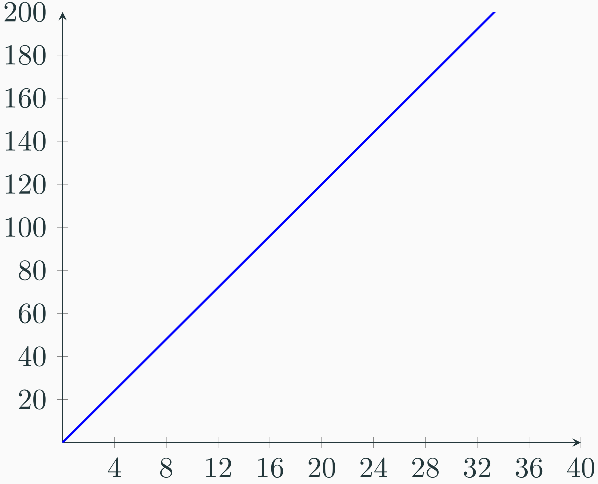

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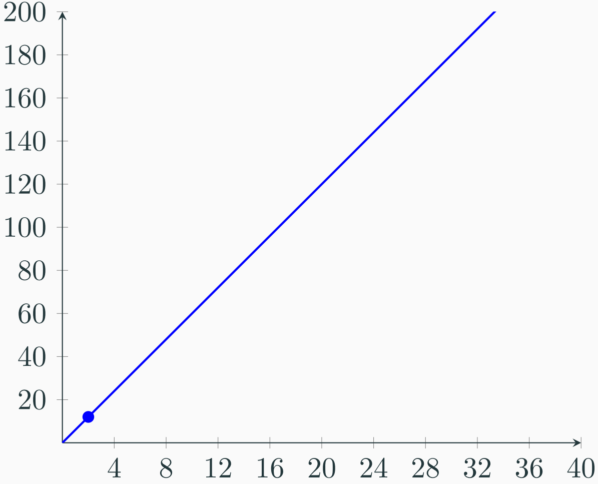

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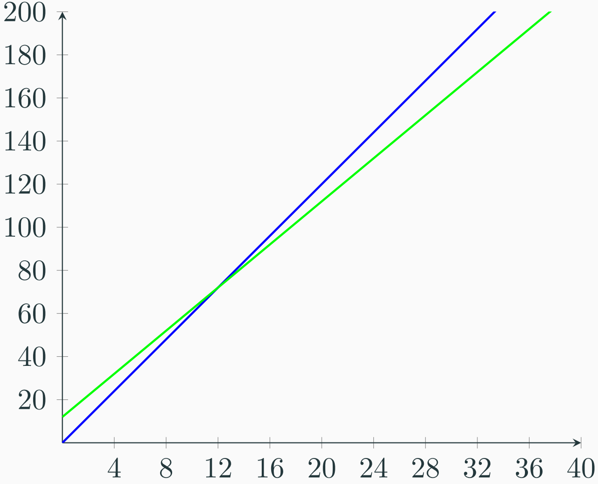

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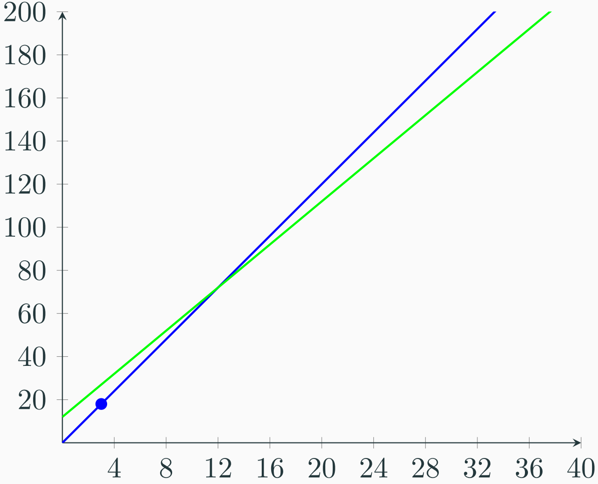

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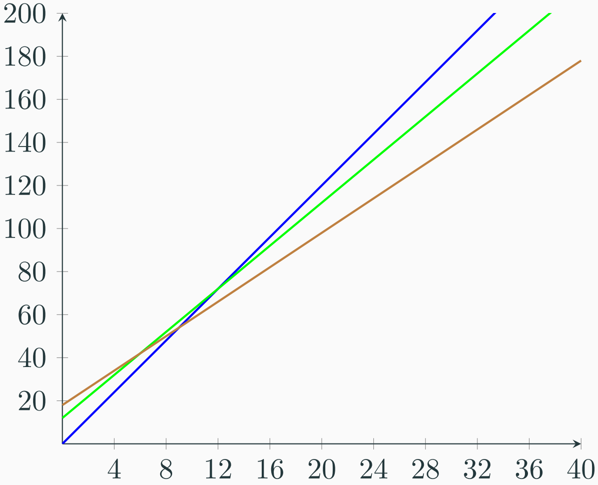

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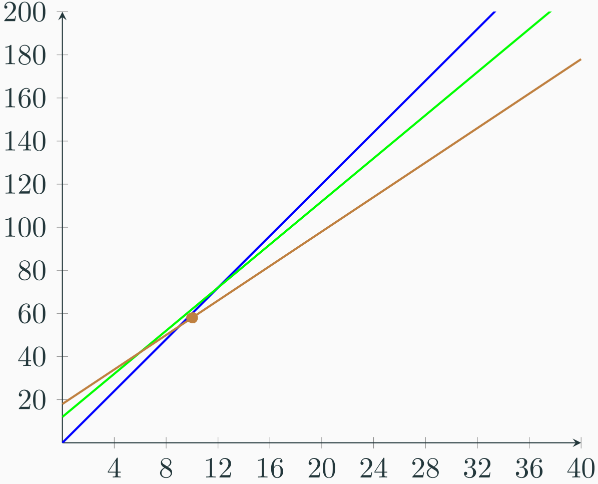

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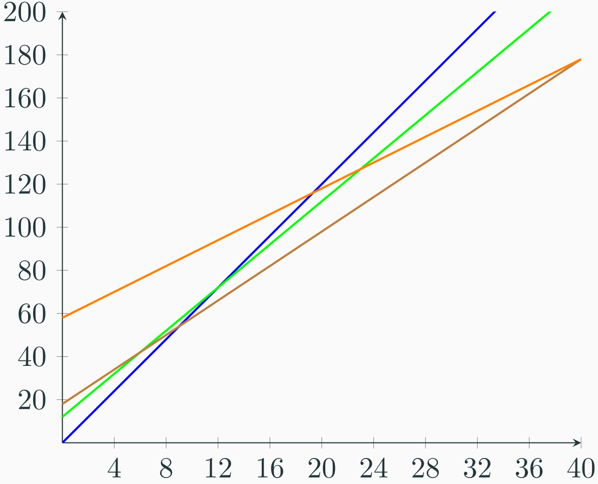

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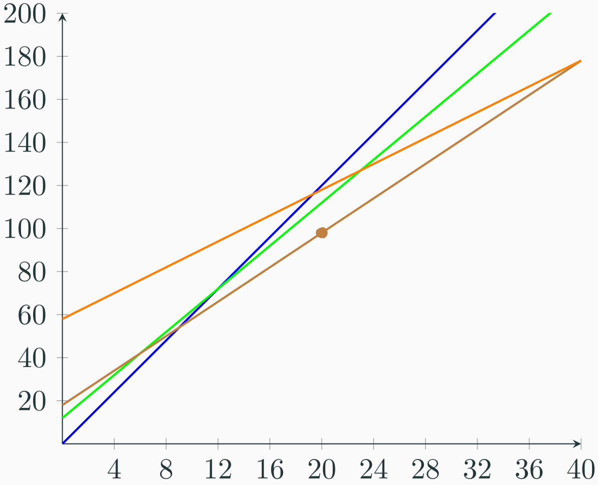

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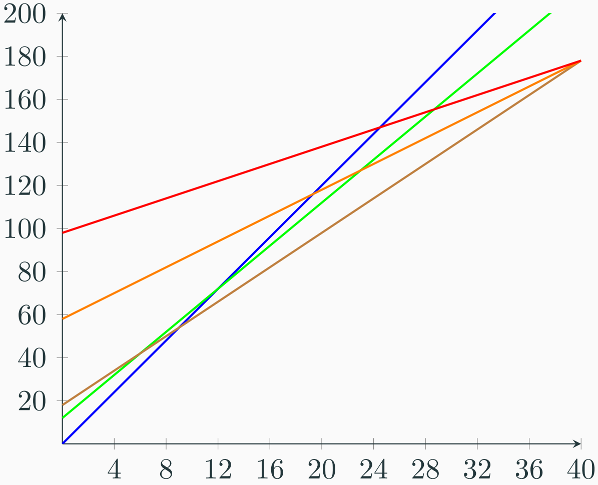

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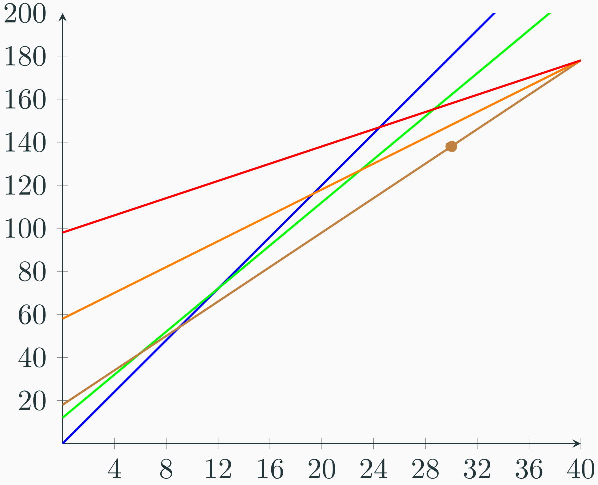

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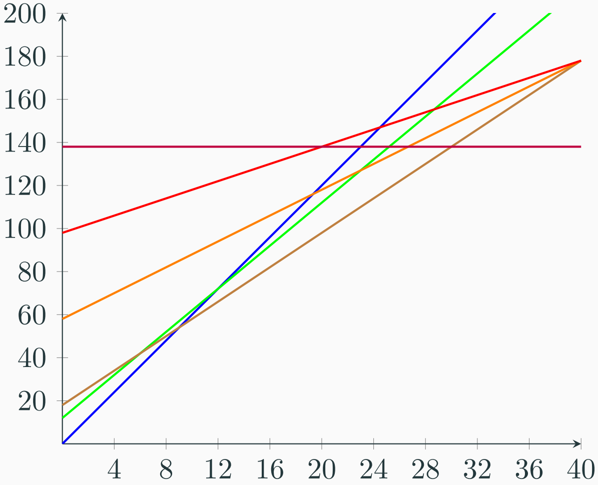

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- If the x-coordinate of a is less than (or equal to) that of b, then the last line is redundant.
- We can therefore iteratively pop redundant lines from the back before adding a line.

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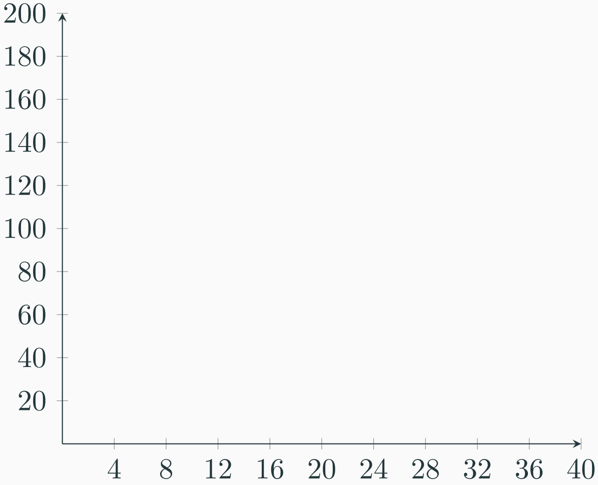

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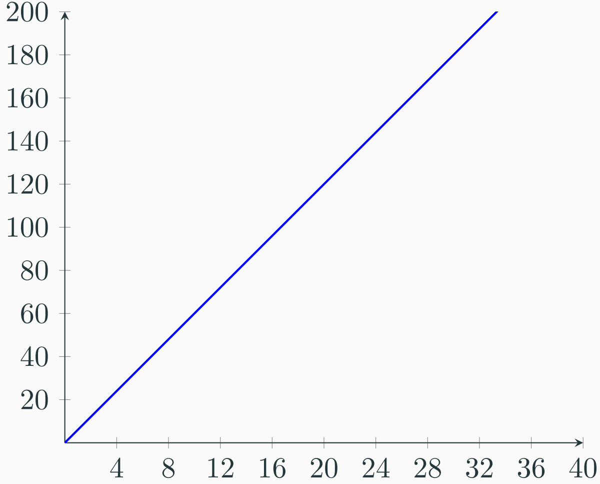

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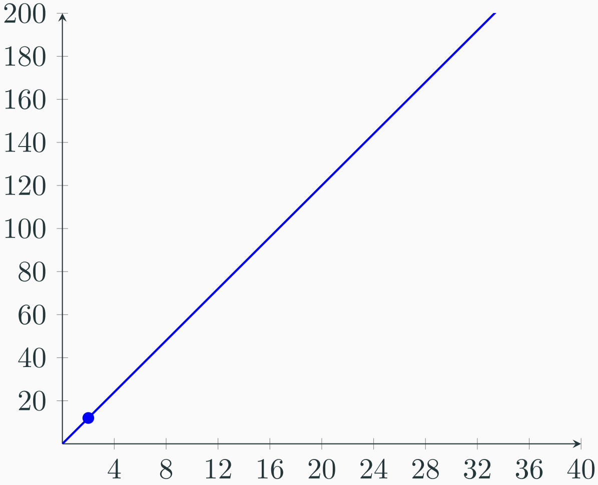

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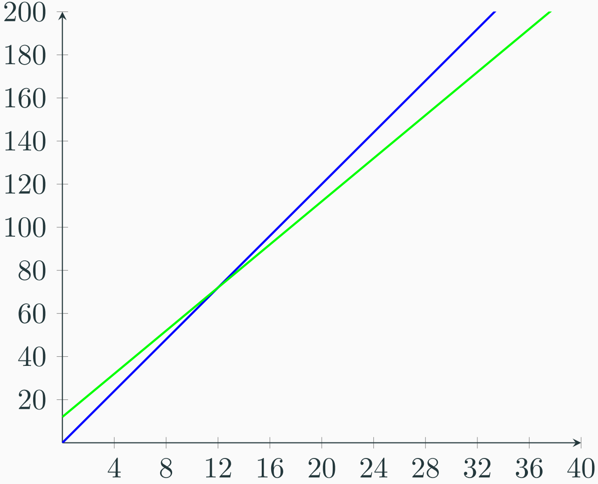

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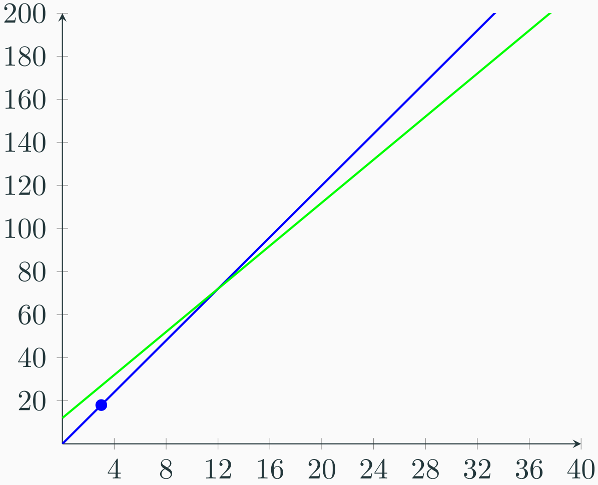

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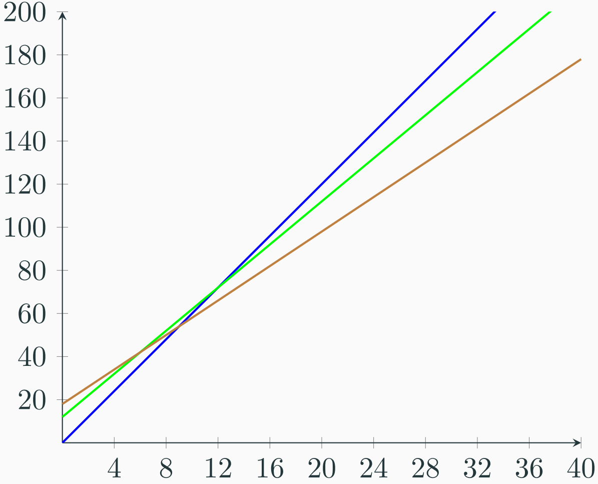

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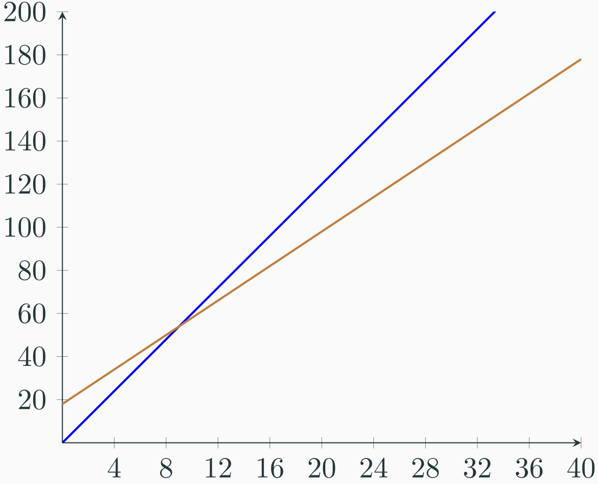

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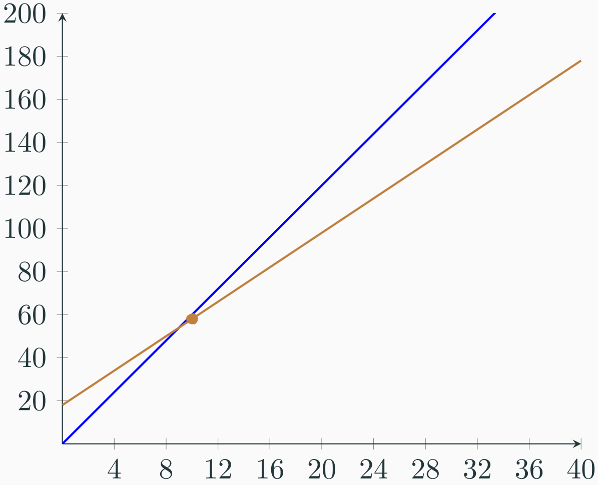

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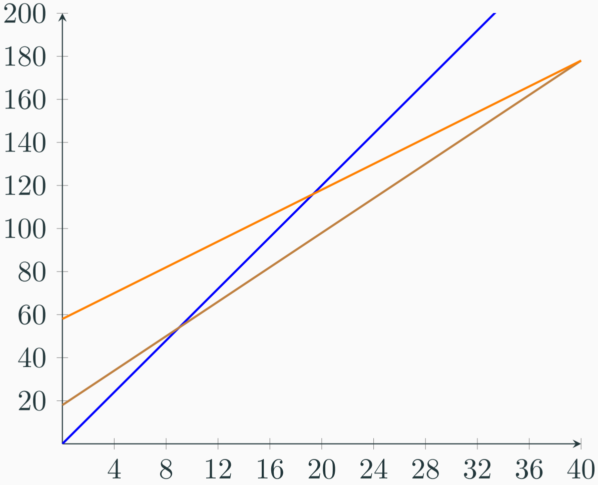

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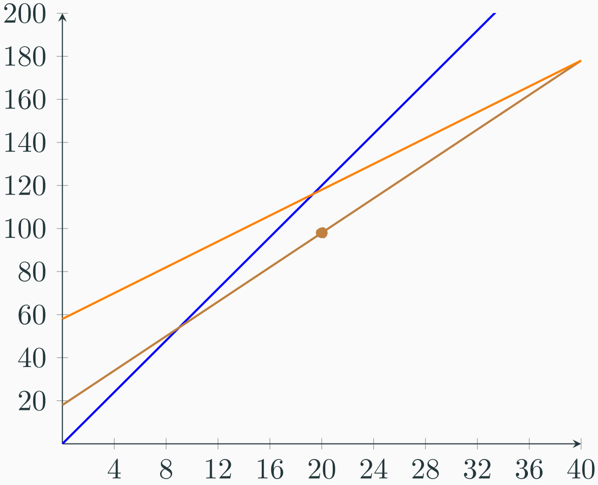

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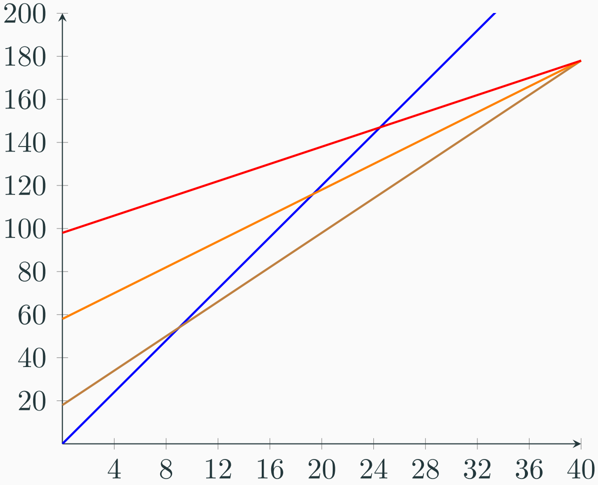

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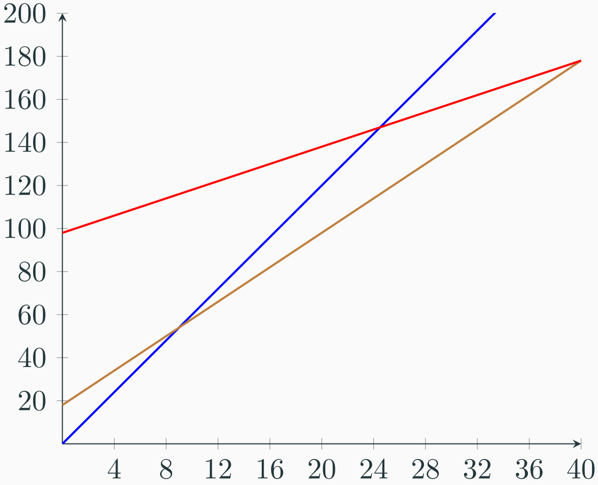

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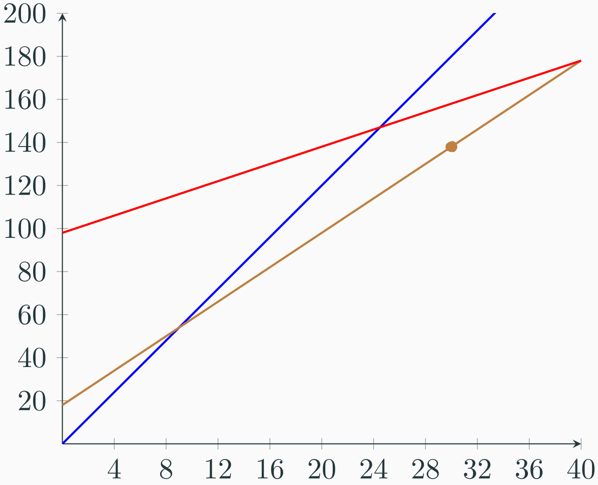

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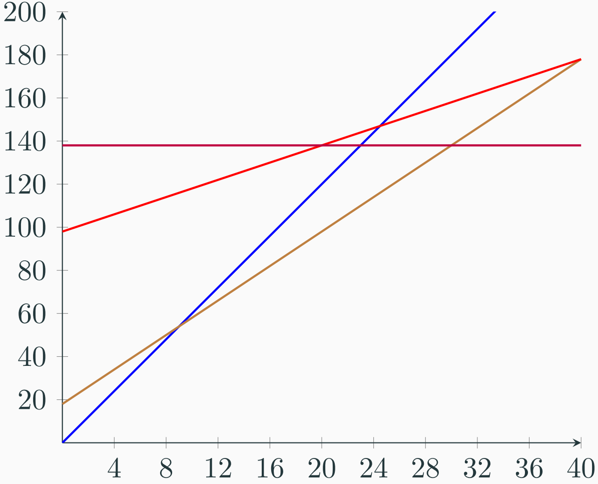

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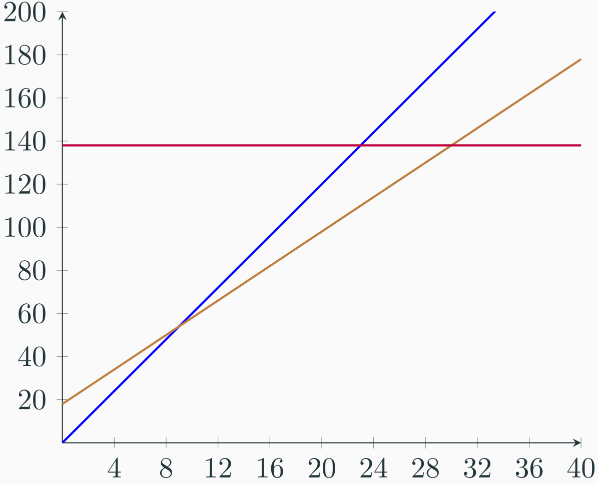

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- Use binary search to find the line in question.
- Construction takes $\mathcal{O}(n)$ time
- Each query takes $\mathcal{O}(\log n)$ time.
- We have improved the time complexity from $\mathcal{O}\left(n^2\right)$ to $\mathcal{O}(n \log n)$.

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- Alternatively, modify such that lines may come in any order while maintaining time complexity.
- Use a multiset instead of a vector/array to keep track of lines and quickly find insertion point. A bit more hassle but powerful.
- Need to consider removing neighbouring lines with higher and lower slopes.

Simple Implementation

```
struct line {
    ll m, b;
    ll get(ll x) { return m*x + b; };
    ll intersect(line other) { return (other.b - b) / (m - other.m); }
};

struct convex_hull_trick {
    vector<line> lines;
    void add(line l) {
        auto sz = lines.size();
        while (sz >= 2 &&
               lines[sz-2].intersect(lines[sz-1]) >= lines[sz-2].intersect(l))
            lines.pop_back();
            sz--;
        }
        lines.push_back(l);
    }
    // to be continued...
```

Simple Implementation - Continued

```
    // ...continued
    ll query(ll x) {
        int lo = 0, hi = static_cast<int>(lines.size()) - 2;
        int ind = hi+1;
        while (lo <= hi) {
            int mid = (lo+hi)/2;
            if (lines[mid].intersect(lines[mid+1]) >= x) {
                ind = mid;
                hi = mid-1;
            }
            else {
                lo = mid+1;
            }
        }
        return lines[ind].get(x);
    }
};
```

Try on these problems!

- Kalila and Dimna in the Logging Industry
- Covered Walkway
- Commando
- Avoiding Airports