Geometry

Atli FF

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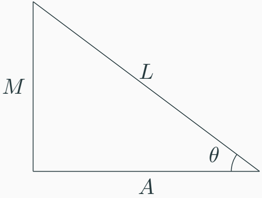
School of Computer Science
Reykjavík University

Today's material

- Trigonometry
- Geometry
- Computational geometry

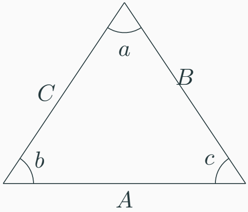
Trigonometry

- Before we even dive into the geometry and how to do it on a computer, let's jog your memories.
- You should all hopefully be familiar with the trigonometric functions.
- We consider a triangle to be right-angled if it has a corner that's 90°.
- For such triangles we have:
  - $\frac{A}{L} = \cos\theta$.
  - $\frac{M}{L} = \sin\theta$.
  - $\frac{M}{A} = \frac{M}{L}\frac{L}{A} = \frac{\sin\theta}{\cos\theta} = \tan\theta$.
- We also have the pythagorean theorem $L^2 = A^2 + M^2$.



More trig

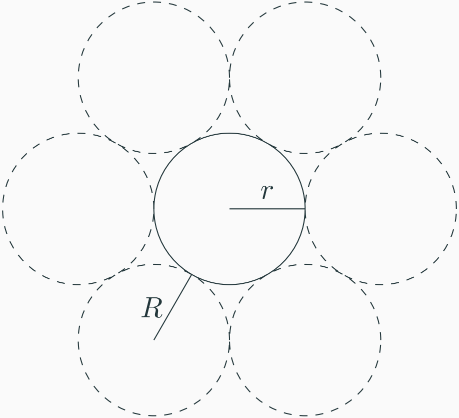
- More generally we have:
  - $\dfrac{\sin a}{A} = \dfrac{\sin b}{B} = \dfrac{\sin c}{C}$ (sine law).
  - $A^2 = B^2 + C^2 - 2BC\cos a$ (cosine law)
- **Exercise:** Prove the pythagorean theorem using the cosine law.



Example: NN and the Optical Illusion

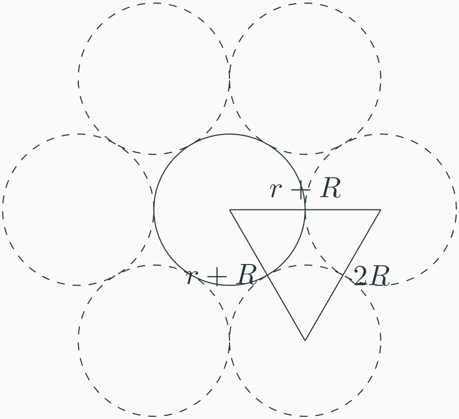
- You are given an integer n and a real number r.
- You then draw a circle of radius r.
- You then want to draw n circles of the same size tangent to the outside of this circle and such that they are tangent to their neighbours.
- What radius will the outer circles have?

$N = 6$ image



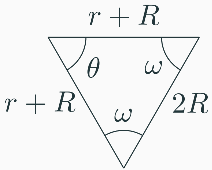
Towards a solution

We see that the distance from the center of the circle in the middle to the center of an outer circle is $r + R$. We thus get an isosceles triangle.



Closer and closer

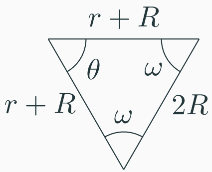
- Now we have $\theta = \dfrac{360^\circ}{n}$ and $\omega = \dfrac{180^\circ - \theta}{2}$.

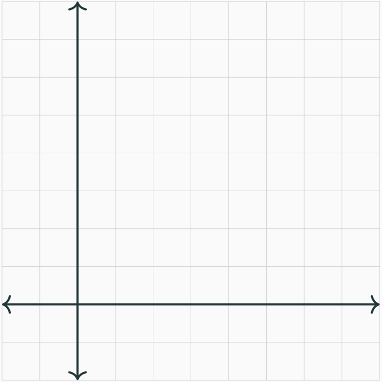


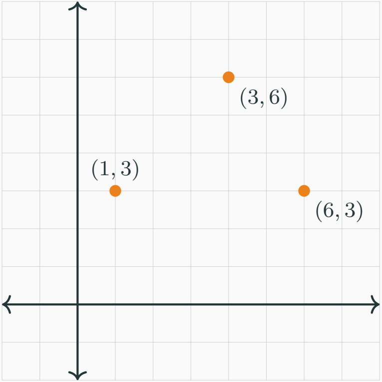
Solution

- Finally the law of sines gives us

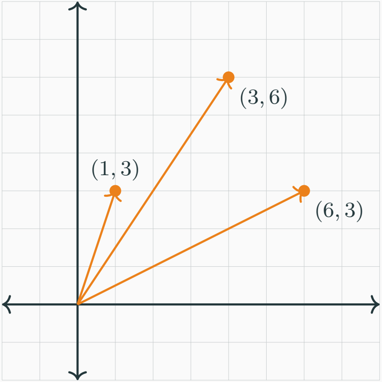
$$\begin{aligned}\frac{2R}{\sin\theta} = \frac{r+R}{\sin\omega} &\Rightarrow 2R\sin\omega = r\sin\theta + R\sin\theta\\ &\Rightarrow 2R\sin\omega - R\sin\theta = r\sin\theta\\ &\Rightarrow R = \frac{r\sin\theta}{2\sin\omega - \sin\theta}.\end{aligned}$$



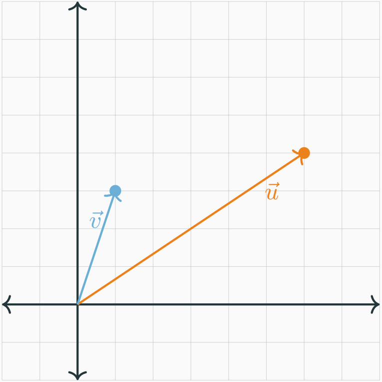




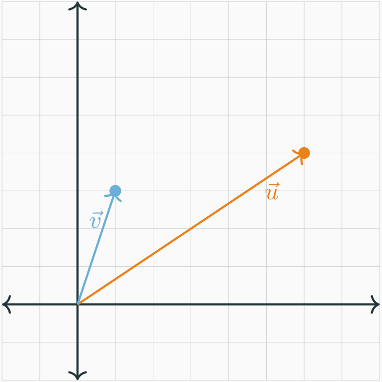
- Points are represented by a pair of numbers, (x, y).



- Points are represented by a pair of numbers, (x, y).
- Vectors are represented in the same way.
- Thinking of points as vectors allows us to do many things.

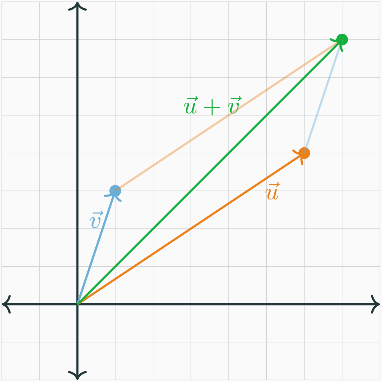


$\vec{v}$
$\vec{u}$



- Simplest operation, addition is defined as

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_0 + x_1 \\ y_0 + y_1 \end{pmatrix}$$



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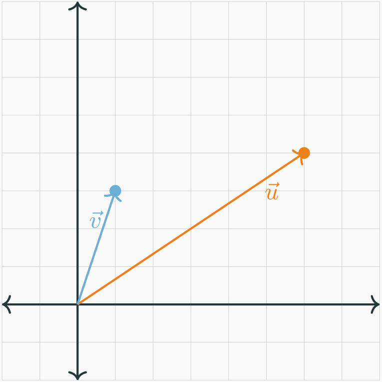

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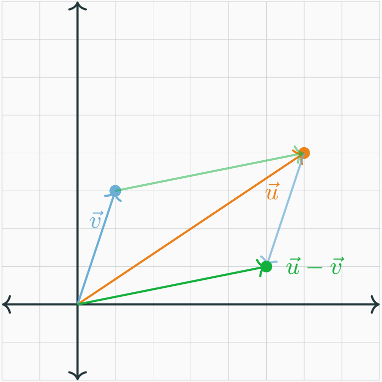
- Subtraction is defined in the same manner

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} - \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_0 - x_1 \\ y_0 - y_1 \end{pmatrix}$$



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```cpp
struct point {
    double x, y;
    point(double _x, double _y) {
        x = _x, y = _y;
    }

    point operator+(const point &oth){
        return point(x + oth.x, y + oth.y);
    }

    point operator-(const point &oth){
        return point(x - oth.x, y - oth.y);
    }
};
```

…or we could use the `complex<double>` class.

```
using points = complex<double>;
```

. . . or we could use the `complex<double>` class.

```
using points = complex<double>;
```

The `complex` class in `C++` and `Java` has methods defined for

- Addition
- Subtraction
- Multiplication by a scalar
- Length
- Trigonometric functions
- And much more!

Complex numbers

- We define $\mathbb{C} := \mathbb{R} \times \mathbb{R}$.
- Then we define addition on $\mathbb{C}$ such that for $(a, b), (c, d) \in \mathbb{C}$ we get
$$(a, b) + (c, d) = (a + c, b + d).$$
- We also define multiplication on $\mathbb{C}$ such that for $(a, b), (c, d) \in \mathbb{C}$ we get
$$(a, b) \cdot (c, d) = (ac - bd, ad + bc).$$
- We usually denote $(0, 1) \in \mathbb{C}$ by i and $(x, y) \in \mathbb{C}$ by $x + yi$.
- Note that $(x, y) = (x, 0) + i \cdot (y, 0)$ here.
- We call these numbers in $\mathbb{C}$ *complex numbers*.

Complex numbers ctd.

- If $z = x + yi \in \mathbb{C}$ then
  - We call x the *real part* of z and y the *imaginary part* of z.
  - We define the *magnitude* of z by $|z| = \sqrt{x^2 + y^2}$.
  - We call $x - yi$ the *conjugate* of z, denoted by $\overline{z}$.
  - We call the angle (x, y) makes with the positive x-axis the *argument* of z and denote it by $\mathrm{Arg}(z)$.

Operations

- Let $w, z \in \mathbb{C}$.
- Then $w + z$ will be z translated by w, as if we were adding vectors.
- If $|w| = 1$ then $z \cdot w$ will be z rotated around 0 by $\operatorname{Arg}(w)$ radians.
- If $|z| = r$ and $\operatorname{Arg}(z) = \theta$ we can write $z = re^{i\theta}$.
- If $z = r_1 e^{i\theta_1}$ and $w = r_2 e^{i\theta_2}$ then $z \cdot w = r_1 r_2 e^{i(\theta_1 + \theta_2)}$.

Using complex in C++

- Usually we do `using point = complex<double>`
- Then we can initialize a point with `point z(x, y)`
  - `real(z)` returns the x-coordinate
  - `imag(z)` returns the y-coordinate
  - `abs(z)` returns the magnitude $|z|$
  - `abs(z - w)` returns the distance from z to w
  - `arg(z)` returns the argument of z
  - `conj(z)` returns the conjugate $\overline{z}$

Example

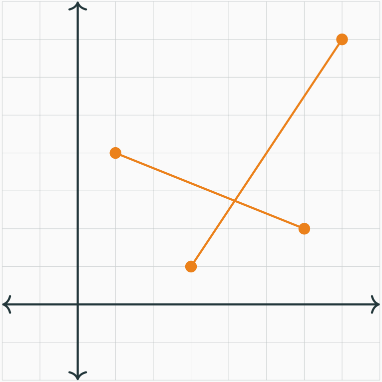
- Let us consider a problem.
- You start at $(0, 0)$ and get a sequence of commands.
- All the commands consist of a single letter and a number. The commands are:
    - ...`f x` you move forward x meters..
    - ...`b x` you move backwards x meters.
    - ...`r x` you rotate x radians to the right.
    - ...`l x` you rotate x radians to the left.
- How far from $(0, 0)$ do you end up after following the commands?

Solution

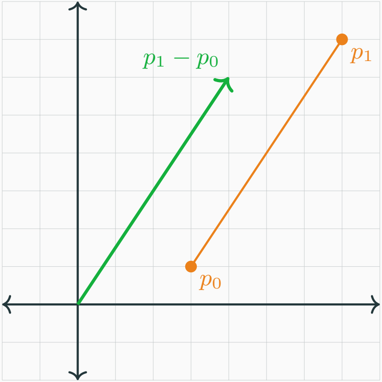
- If we are stood at $p \in \mathbb{C}$ and want to take a step of r meters in the direction θ we simply add $re^{i\theta}$ to p.
- What direction we are facing at the start makes no difference since it gives the same distance at the end.

Code

```cpp
#include <bits/stdc++.h>
using namespace std;
using point = complex<double>;
int main() {
    int n; cin >> n;
    double x, r = 0.0;
    point p(0, 0);
    while (n--) {
        char c; cin >> c >> x;
        if (c == 'f')      p += x*exp(1i*r);
        else if (c == 'b') p -= x*exp(1i*r);
        else if (c == 'l') r += x;
        else if (c == 'r') r -= x;
        else assert(0);
        }
    cout << setprecision(15) << abs(p) << endl;
}
```



- Line segments are represented by a pair of points, $((x_0, y_0), (x_1, y_1))$.



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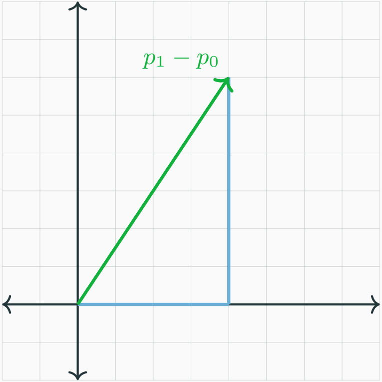

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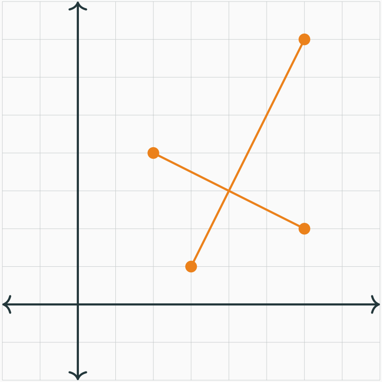
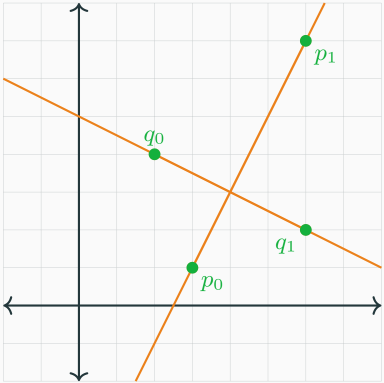
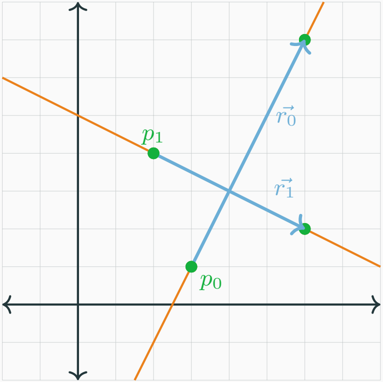


- Line representation same as line segments.

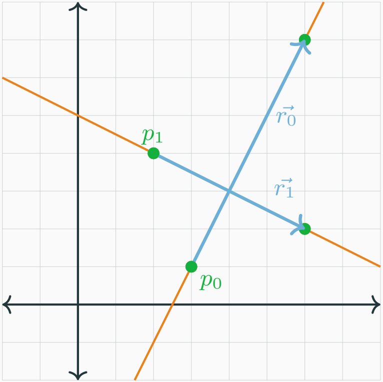


- Line representation same as line segments.
- Treat them as lines passing through the two points.



- Line representation same as line segments.
- Treat them as lines passing through the two points.
- Or as a point and a direction vector.

$$p + t \cdot \vec{r}$$



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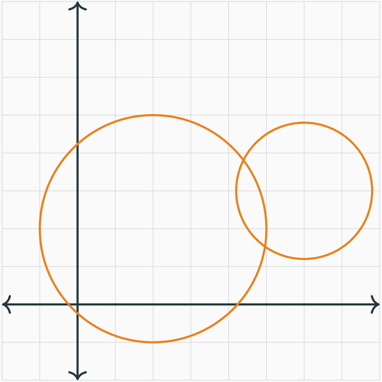
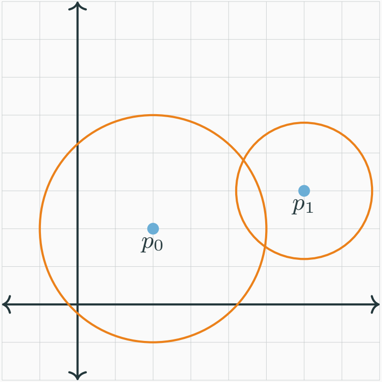
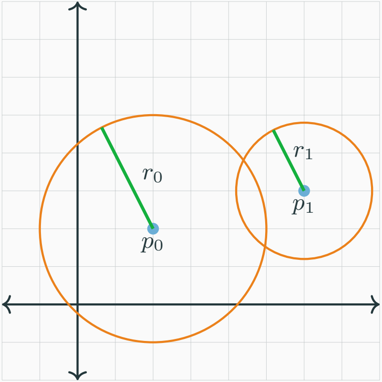
- Either way `pair<point,point>`



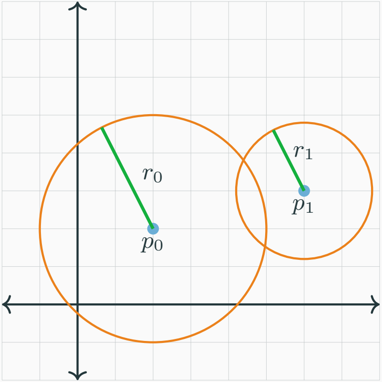
- Circles are very easy to represent.



- Circles are very easy to represent.
- Center point $p = (x, y)$.



- Circles are very easy to represent.
- Center point $p = (x, y)$.
- And the radius r.



- Circles are very easy to represent.
- Center point $p = (x, y)$.
- And the radius r.

`pair<point,double>`

Given two vectors

$$\vec{u} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \qquad \vec{v} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

the dot product of $\vec{u}$ and $\vec{v}$ is defined as

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = x_0 \cdot x_1 + y_0 \cdot y_1$$

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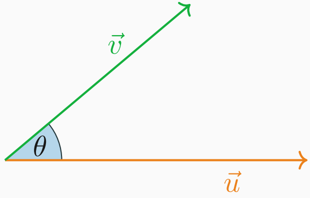
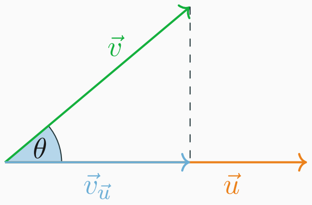
Which in geometric terms is

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos \theta$$



- Allows us to calculate the angle between $\vec{u}$ and $\vec{v}$.

$$\theta = \arccos\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\right)$$

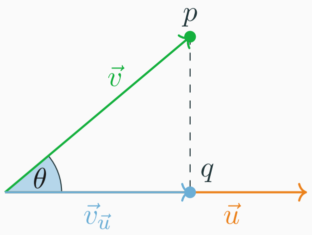


- Allows us to calculate the angle between $\vec{u}$ and $\vec{v}$.

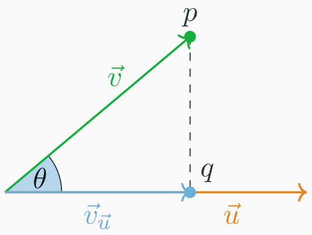
$$\theta = \arccos\left(\frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}\right)$$

- And the projection of $\vec{v}$ onto $\vec{u}$.

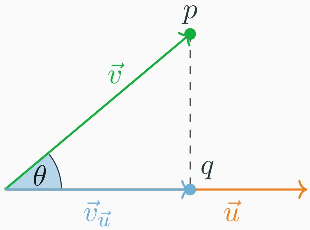
$$\vec{v}_{\vec{u}} = \left(\frac{\vec{u} \cdot \vec{v}}{|u|^2}\right)\vec{u}$$



- The closest point on $\vec{u}$ to p is q.



- The closest point on $\vec{u}$ to p is q.
- The distance from p to $\vec{u}$ is the distance from p to q.



- The closest point on $\vec{u}$ to p is q.
- The distance from p to $\vec{u}$ is the distance from p to q.
- Unless q is outside $\vec{u}$, then the closest point is either of the endpoints.

Rest of the code will use the `complex` class.

```cpp
#define P(p) const point &p
#define L(p0, p1) P(p0), P(p1)
double dot(P(a), P(b)) {
    return real(a) * real(b) + imag(a) * imag(b);
}
double angle(P(a), P(b), P(c)) {
    return acos(dot(b - a, c - b) / abs(b - a) / abs(c - b));
}
point closest_point(L(a, b), P(c), bool segment = false) {
    if (segment) {
        if (dot(b - a, c - b) > 0) return b;
        if (dot(a - b, c - a) > 0) return a;
    }
    double t = dot(c - a, b - a) / norm(b - a);
    return a + t * (b - a);
}
```

Given two vectors

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the cross product of $\vec{u}$ and $\vec{v}$ is defined as

$$\left| \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \times \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \right| = x_0 \cdot y_1 - y_0 \cdot x_1$$

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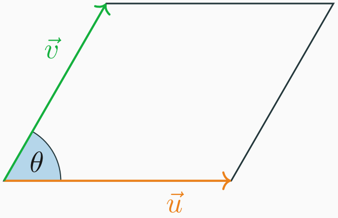
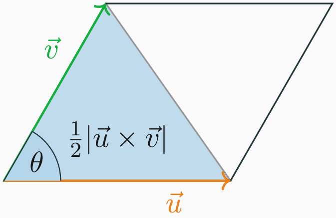
Which in geometric terms is

$$|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin \theta$$



- Allows us to calculate the area of the triangle formed by $\vec{u}$ and $\vec{v}$.

$$\frac{|\vec{u} \times \vec{v}|}{2}$$



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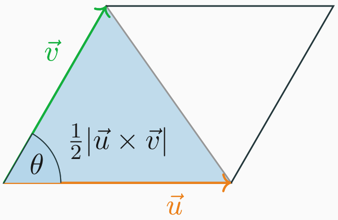
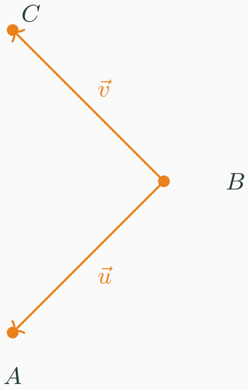


- Allows us to calculate the area of the triangle formed by $\vec{u}$ and $\vec{v}$.

$$\frac{|\vec{u} \times \vec{v}|}{2}$$

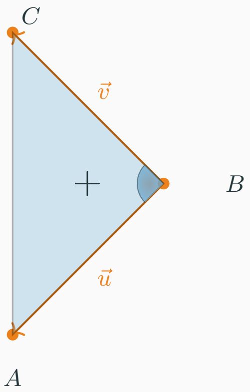
- And can tell us if the angle between $\vec{u}$ and $\vec{v}$ is positive or negative.

$$\begin{aligned} |\vec{u} \times \vec{v}| < 0 \quad &\text{iff} \quad \theta < \pi \\ |\vec{u} \times \vec{v}| = 0 \quad &\text{iff} \quad \theta = \pi \\ |\vec{u} \times \vec{v}| > 0 \quad &\text{iff} \quad \theta > \pi \end{aligned}$$



- Given three points A, B and C, we want to know if they form a counter-clockwise angle in that order.

$$A \to B \to C$$



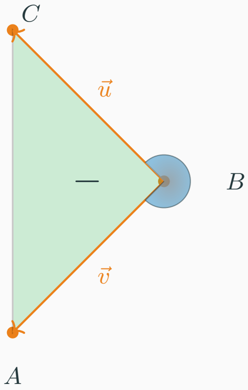
- Given three points A, B and C, we want to know if they form a counter-clockwise angle in that order.

$$A \to B \to C$$

- We can examine the cross product of and the area of the triangle formed by

$$\vec{u} = B - C \quad \vec{v} = B - A$$

$$\vec{u} \times \vec{v} > 0$$



- The points in the reverse order do not form a counter clockwise angle.

$$C \to B \to A$$

- In the reverse order the vectors swap places

$$\vec{u} = B - A \quad \vec{v} = B - C$$
$$\vec{u} \times \vec{v} < 0$$

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- If the points A, B and C are on the same line, then the area will be 0.

```cpp
double cross(P(a), P(b)) {
    return real(a)*imag(b) - imag(a)*real(b);
}
double ccw(P(a), P(b), P(c)) {
    return cross(b - a, c - b);
}
bool collinear(P(a), P(b), P(c)) {
    return abs(ccw(a, b, c)) < EPS;
}
```

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\begin{aligned}
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\end{aligned}
$$

- If we have two lines given by such equations, we simply need to solve for the two unknowns, x and y.

For two lines

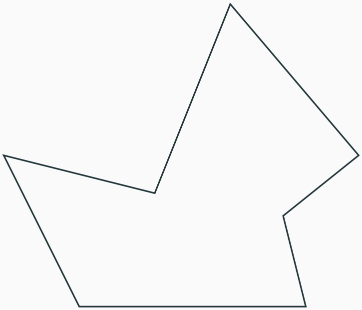
$$A_0x + B_0y = C_0$$
$$A_1x + B_1y = C_1$$

The intersection point is

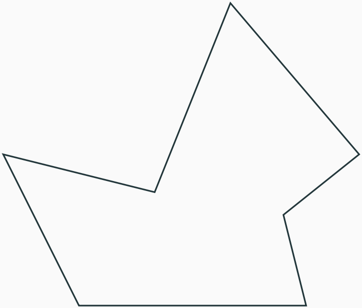
$$x = \frac{(B_1 \cdot C_0 - B_0 \cdot C_1)}{D}$$
$$y = \frac{(A_0 \cdot C_1 - A_1 \cdot C_0)}{D}$$

Where

$$D = A_0 \cdot B_1 - A_1 \cdot B_0$$



- Polygons are represented by a list of points in the order representing the edges.



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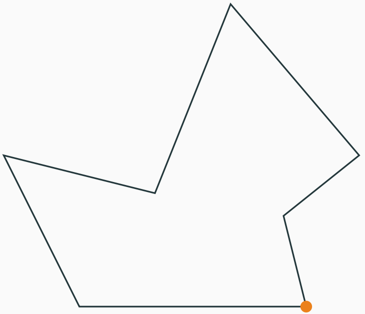

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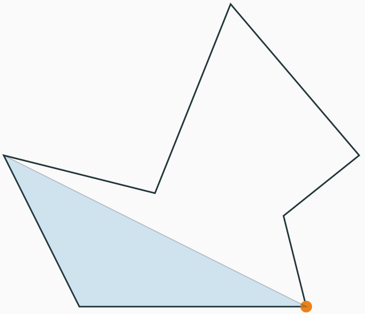

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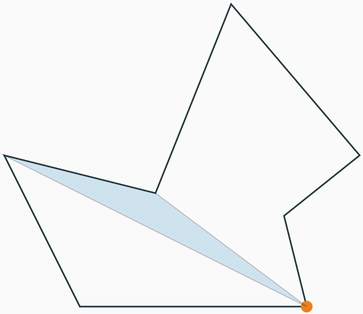

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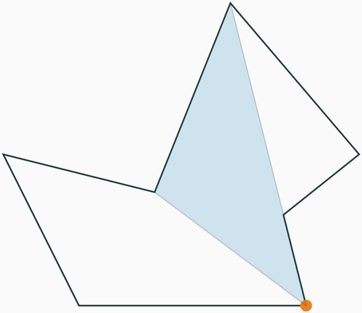

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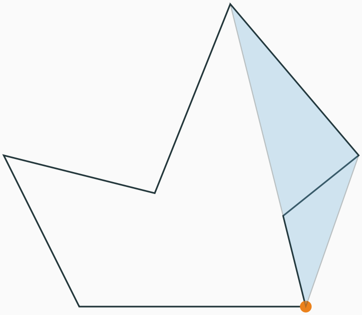

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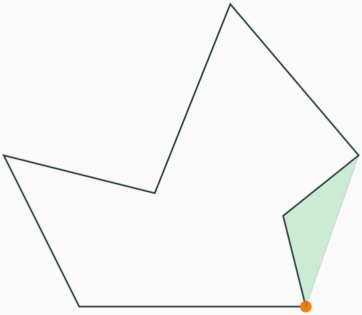

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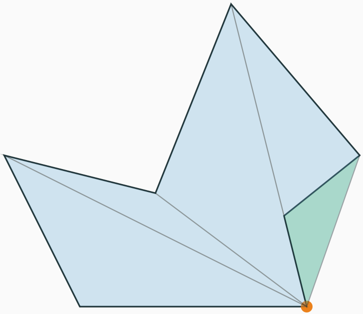

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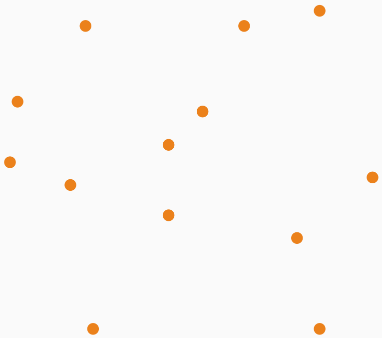

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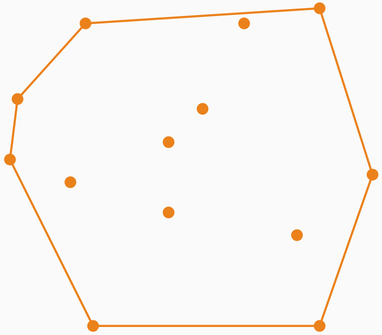

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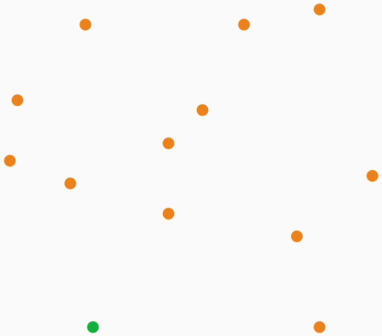
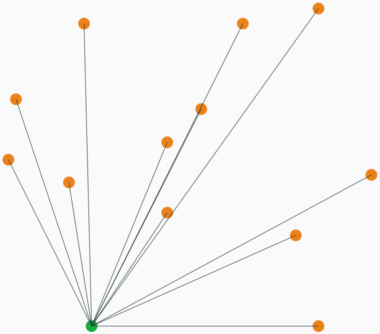
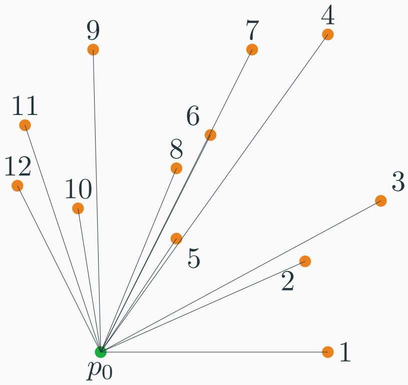
Time complexity $O(N \log N)$.



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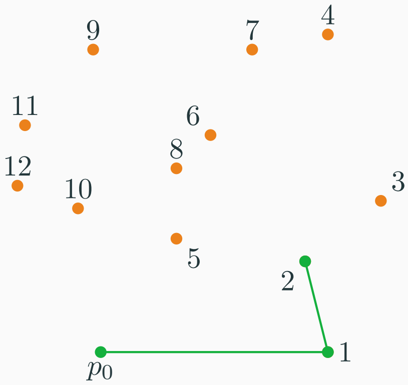

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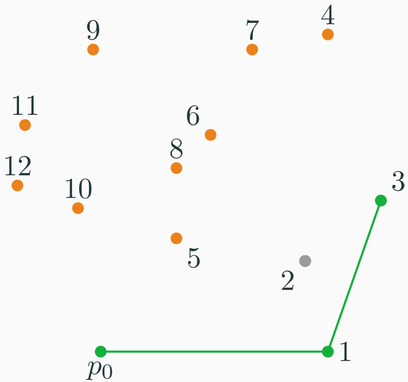
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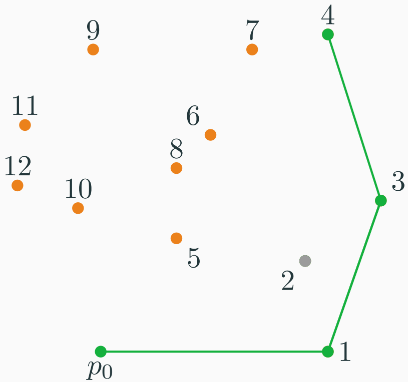

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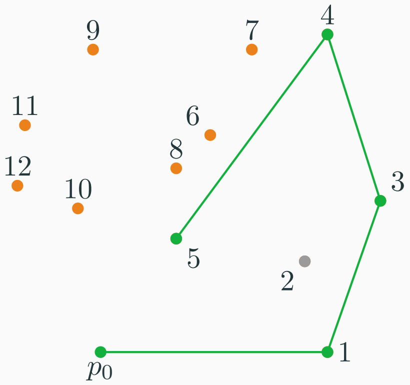

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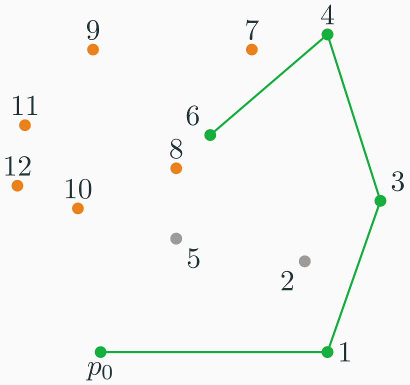

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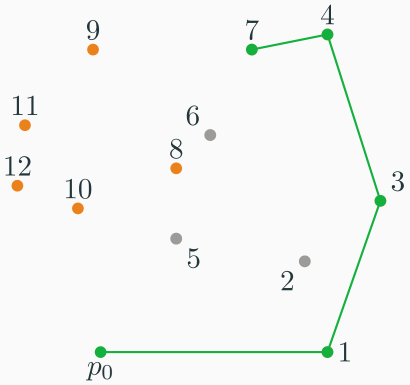

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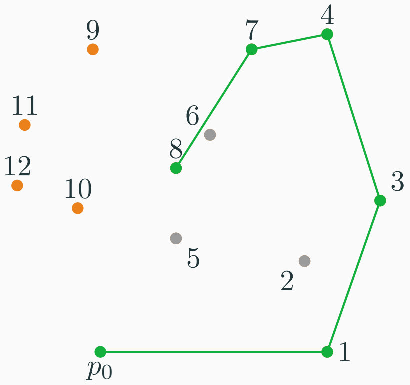
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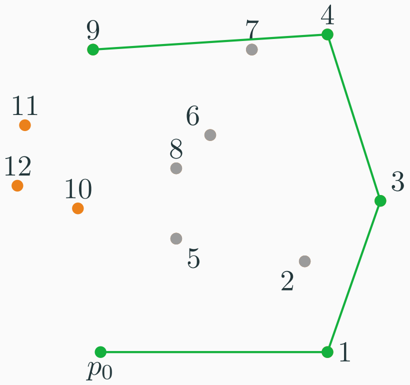

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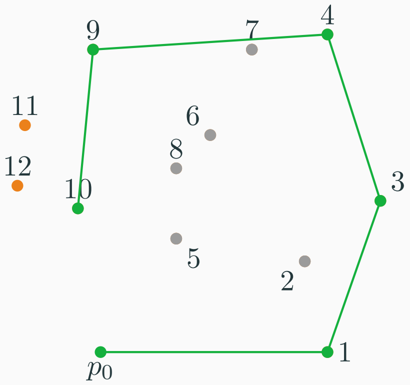

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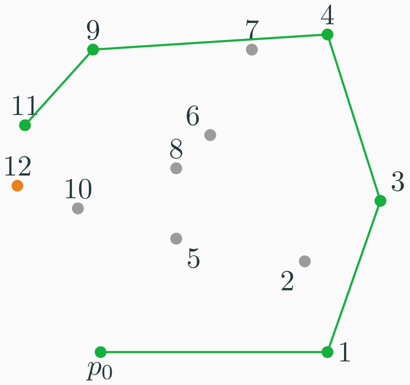

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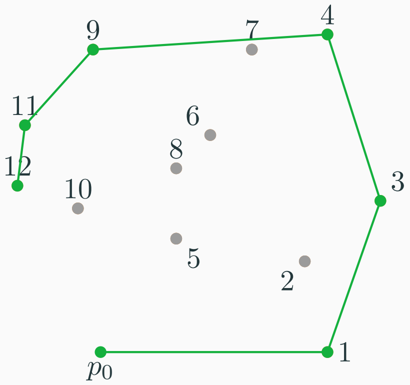

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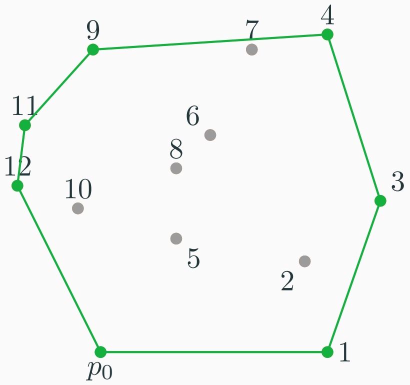

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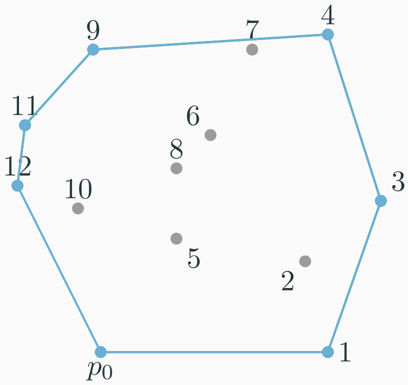
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```
point hull[MAXN];
bool cmp(const point &a, const point &b) {
  return abs(real(a) - real(b)) > EPS ?
    real(a) < real(b) : imag(a) < imag(b); }
int convex_hull(vector<point> p) {
    int n = size(p), l = 0;
    sort(p.begin(), p.end(), cmp);
    for (int i = 0; i < n; i++) {
        if (i > 0 && p[i] == p[i - 1])
            continue;
        while (l >= 2 && ccw(hull[l - 2], hull[l - 1], p[i]) >= 0)
            l--;
        hull[l++] = p[i]; }
    int r = l;
    for (int i = n - 2; i >= 0; i--) {
        if (p[i] == p[i + 1])
            continue;
        while (r - l >= 1 && ccw(hull[r - 2], hull[r - 1], p[i]) >= 0)
            r--;
        hull[r++] = p[i]; }
    return l == 1 ? 1 : r - 1; }
```

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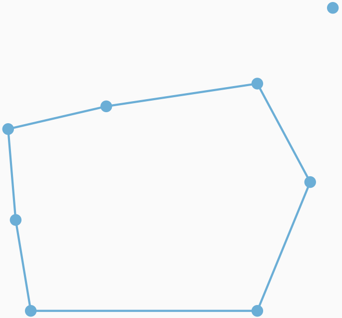
Some can be extended to three dimensions, or higher.

Simple algorithm to check if a point is in a convex polygon.



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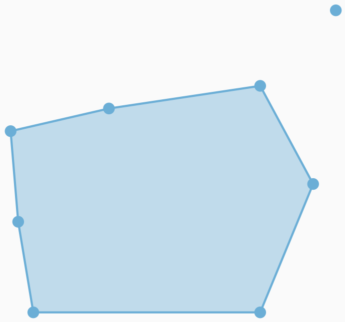

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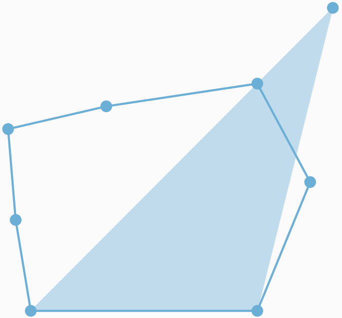

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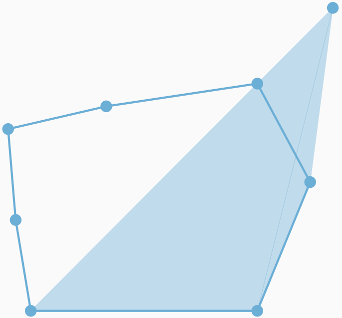

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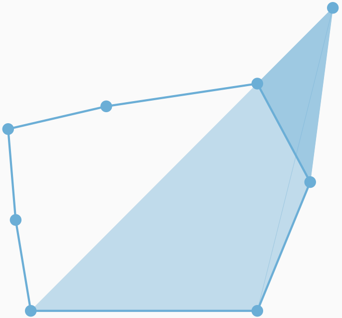

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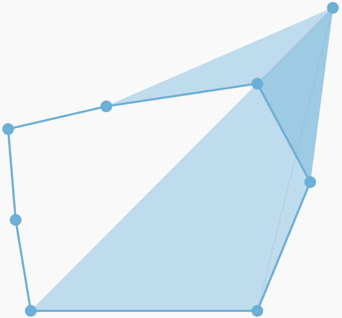

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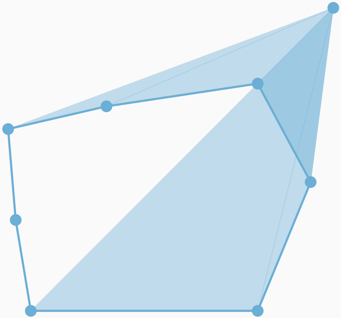

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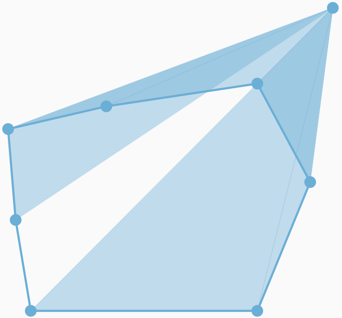

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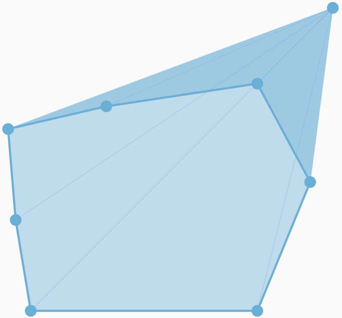

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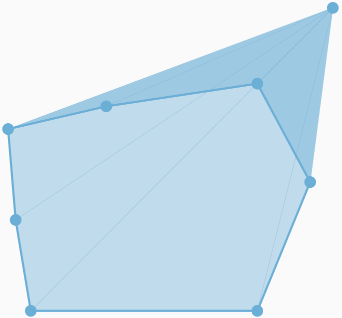

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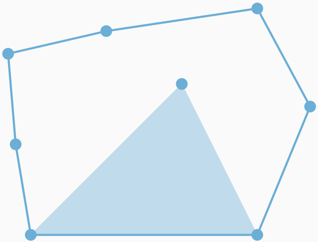

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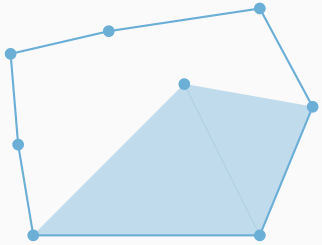

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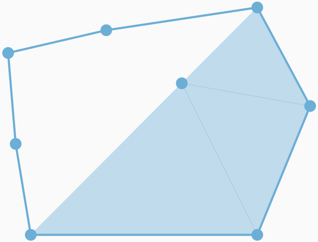

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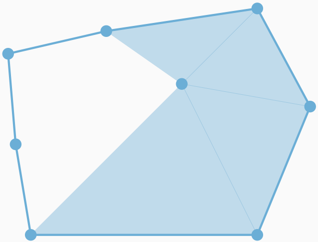

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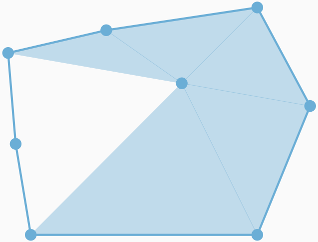

- We start by calculating the area of the polygon.
- To check if our point is contained in the polygon we sum up the area of the triangles formed the point and every two adjacent points.
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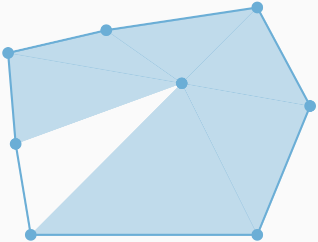

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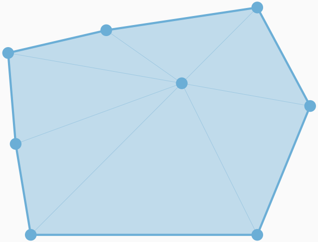

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How about non convex polygon?

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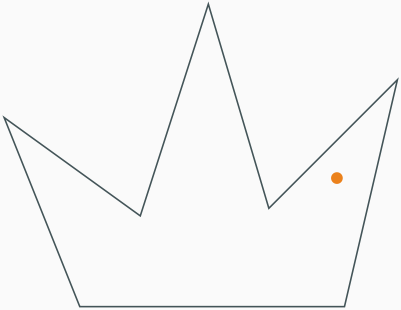
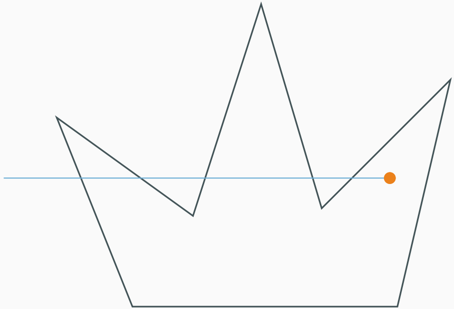
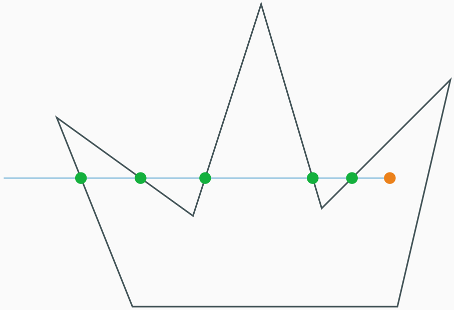
- If the ray crosses the boundary of the polygon, then it alternately goes from outside to inside, and outside to inside.

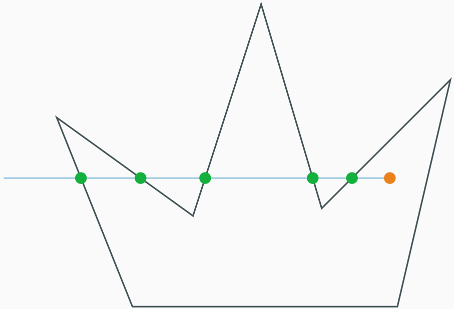




- Ray from the outside of the polygon to the point.



- Ray from the outside of the polygon to the point.
- Count the number of intersection points.



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- If odd, then the point is inside the polygon.
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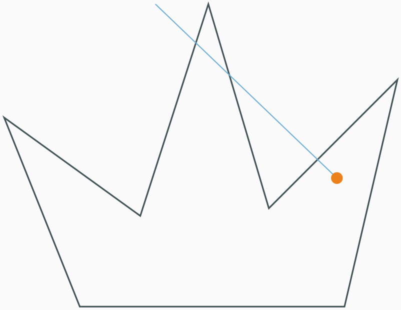

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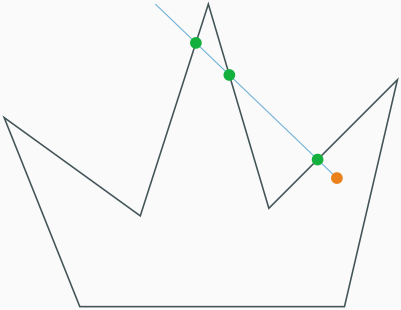

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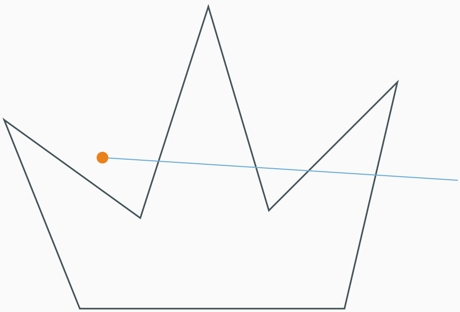

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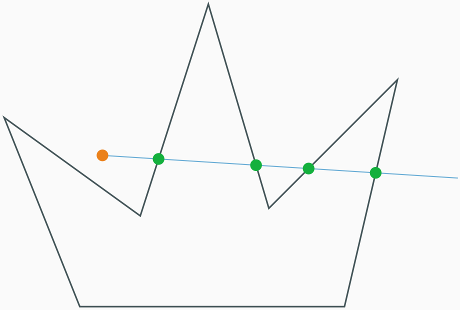

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An algorithm

- Computational geometry has a lot of impressive and technical algorithms.
- The most famous one is probably Delaunay triangulation.
- But that one is a bit too hard for this course, so we will instead look at the classical closest point algorithm.
- We are given n points in the plan, find the pair of points that are closest to one another.
- We can clearly solve this in $\mathcal{O}(n^2)$ time, but can we do better?

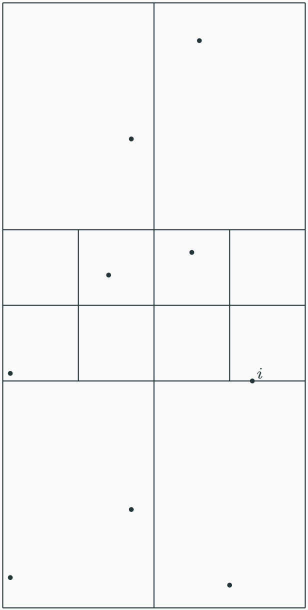
Divide and conquer

- We sort the points by x-coordinate and split the list in half.
- Let x_0 be such that it's between the coordinates of the left and right halves.
- Start by solving each half recursively.
- We now have to find if there's some pair with one point in each half that does better.
- We can't simply try all pairs, that's too slow. Suppose the smallest distance we found recursively was d.
- Then we can ignore all points with x-coordinte outside $[x_0 - d, x_0 + d]$.
- Sort the points inside of this interval by their y-coordinate.
- The big trick is now that we only need to consider a few neighbours for each point.

Neighbours

- Divide the area above x_i into 8 squares, each with side length $d/2$.
- If he distance between all points in each half is at least d, then we can have at most each point per square.
- All points outside these squares are at a distance of at least d from x_i, so we can ignore them.
- Thus we only need to look at the distance from x_i to x_j when $j - i \leq 7$.

Diagram



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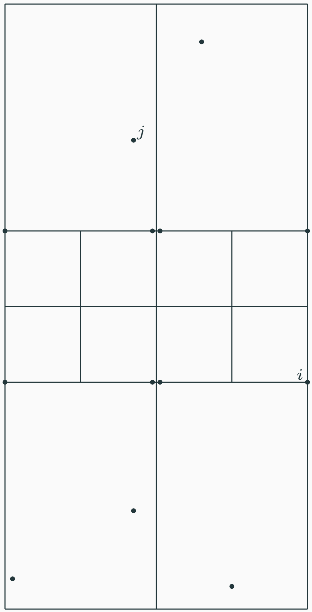

Complexity

- Each recursive call is $\mathcal{O}(n \log(n))$.
- Thus by the master theorem the total complexity is $\mathcal{O}(n \log^2(n))$.
- If we sort the y values as we go using mergesort, we can actually do each call in $\mathcal{O}(n)$.
- This way the complexity is actually $\mathcal{O}(n \log(n))$.