Divide and Conquer Optimization

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- Let $S(i, j) = A_i + A_{i+1} + ... + A_{j-1} + A_j$.
- Compute and store all values of $S(0, j)$ for all j such that $0 \leq j < N$.
- Now you can compute $S(i, j) = S(0, j) - S(0, i-1)$ in constant time for any i and j.

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- Each jail cell is assigned to exactly one guard.
- Each guard can only watch over a contiguous range of prisoners.
- If the guard watching prisoner i is watching over k cells, then the prisoner's escaping potential is kS_i.
- Your goal is to assign the cells to guards in a way that minimizes the total escaping potential over all prisoners.

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- Time complexity is $\mathcal{O}(N^2 G)$, which is too slow.

Implementation - Initial Definitions

```cpp
#include <bits/stdc++.h>
using namespace std;
typedef long long ll;
const ll INF = 80'000'000'000'000LL;

ll arr[8000];
ll prefix_sum[8001];
ll mem[3001][8001];
ll range_sum(int left, int right) {
    return prefix_sum[right] - prefix_sum[left-1];
}

ll cost(ll left, ll right) {
    return range_sum(left, right) * (right - left + 1LL);
}
```

Naive Implementation - Computing Each Layer

```
void compute(int level, int n) {
    for (int end = 0; end < n; end++) {
        ll tmp = INF;
        for (int start = 0; start <= end; start++) {
            tmp = min(tmp,
                      (start ? mem[level - 1][start - 1] : 0)
                      + cost(start, end));
        }
        mem[level][end] = tmp;
    }
}
```

Naive Implementation - Main

```
int main()
{
    int n, g;
    cin >> n >> g;
    prefix_sum[0] = 0;
    for (int i = 0; i < n; i++) {
        cin >> arr[i];
        prefix_sum[i+1] = prefix_sum[i] + arr[i];
    }
    for (int i = 0; i < n; i++) {
        mem[0][i] = cost(0, i);
    }
    for (int guards = 2; guards <= g; guards++) {
        compute(guards - 1, n);
    }
    cout << mem[g - 1][n - 1] << endl;
    return 0;
}
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- First compute $\mathrm{dp}(N/2,k)$ and note the value of $\mathrm{opt}(N/2,k)$.

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- First compute $\mathrm{dp}(N/2, k)$ and note the value of $\mathrm{opt}(N/2, k)$.
- With that value in mind, compute $\mathrm{dp}(N/4, k)$ and $\mathrm{dp}(3N/4, k)$.

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- First compute $\mathrm{dp}(N/2, k)$ and note the value of $\mathrm{opt}(N/2, k)$.
- With that value in mind, compute $\mathrm{dp}(N/4, k)$ and $\mathrm{dp}(3N/4, k)$.
- Repeat this process, computing the left and right side, tracking the minimum and maximum possible value of $\mathrm{opt}(j, k)$.

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- At each level we will do $\mathcal{O}(N)$ work, since there is no overlap for values of j at the same level.
- Note it does not matter how balanced $\operatorname{opt}(j, k)$ is, we always do linear work at a level.
- Time complexity is now $\mathcal{O}(NG \log N)$, so fast enough.

Optimized Implementation - Computing Each Layer

```
void compute(int level, int l, int r, int optl, int optr) {
    if (l > r) return;
    int mid = (l+r)/2;
    pair<ll, int> best = {INF, -1};
    for (int k = optl; k <= min(mid, optr); k++) {
        best = min(best,
            {(k ? mem[level - 1][k - 1] : 0LL) + cost(k, mid), k});
    }
    mem[level][mid] = best.first;
    int opt = best.second;
    compute(level, l, mid-1, optl, opt);
    compute(level, mid+1, r, opt, optr);
}
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Optimized Implementation - Main

```
int main()
{
    int n, g;
    cin >> n >> g;
    prefix_sum[0] = 0;
    for (int i = 0; i < n; i++) {
        cin >> arr[i];
        prefix_sum[i+1] = prefix_sum[i] + arr[i];
    }
    for (int i = 0; i < n; i++) {
        mem[0][i] = cost(0, i);
    }
    for (int guards = 2; guards <= g; guards++) {
        compute(guards-1, 0, n-1, 0, n-1);
    }
    cout << mem[g-1][n-1] << endl;
    return 0;
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- It is usually not that difficult to prove the quadrangle inequality holds when it does.
- Once you've proven it for a DP pattern like shown before, you know you can use this method.
- The Convex Hull Trick can often be used in the same tasks to which this method applies.

Try on these problems!

- Guards
- Split the Sequences (Coming to Kattis soon!)
- Partition Game
- The Bakery