Dynamic Programming Optimizations

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Convex Hull Optimization

Kalila and Dimna in the Logging Industry

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- We use the chainsaw charging coefficient of the tree with the highest id which has been cut completely.
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- We want to minimize the total charge cost to cut all trees.

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- Since $b_n = 0$ we must only cut the largest tree, at that point all cuts are free.
- We want to minimize the cost required to cut the largest tree.
- It is also quite clear that once we start cutting a tree, we should finish cutting it before starting to cut others.

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- A line!

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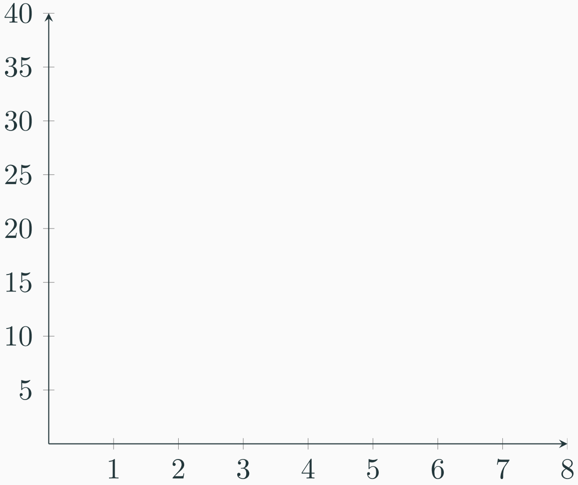
- We need both operations to be sub-linear in time complexity.

Sample 1 - Illustrated



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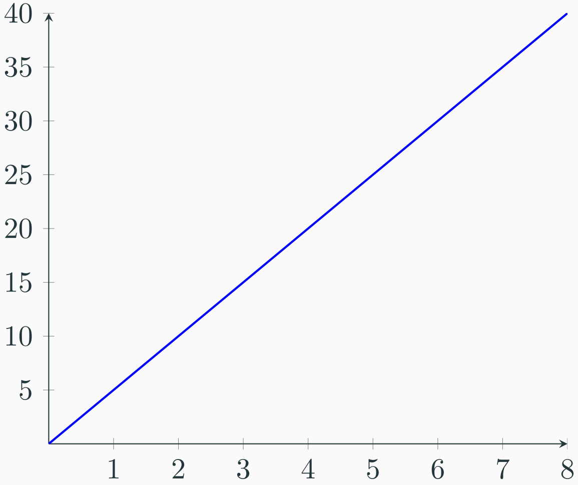

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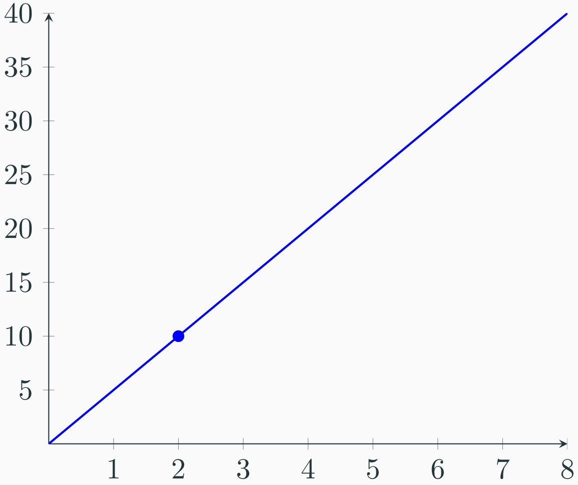

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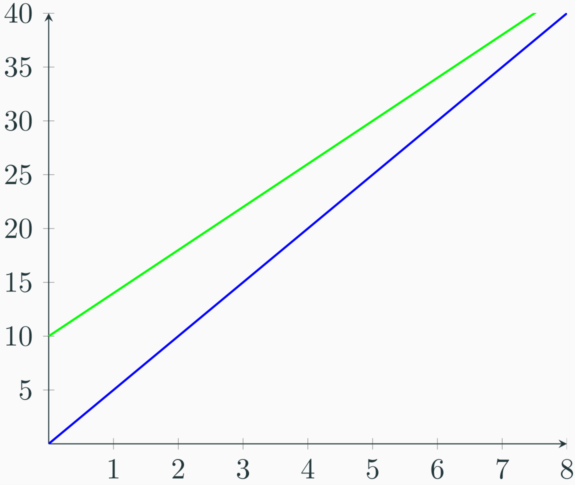

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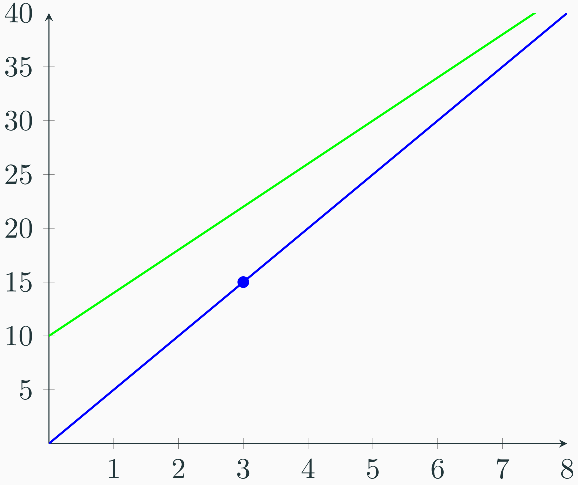

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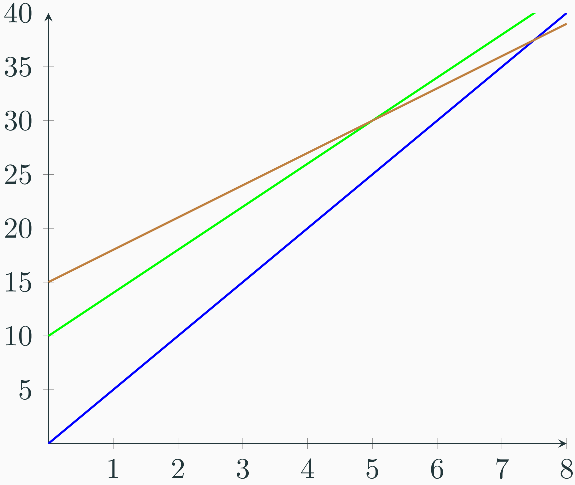

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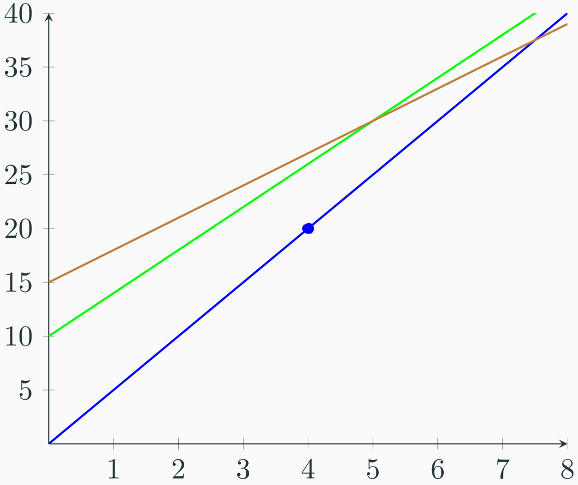

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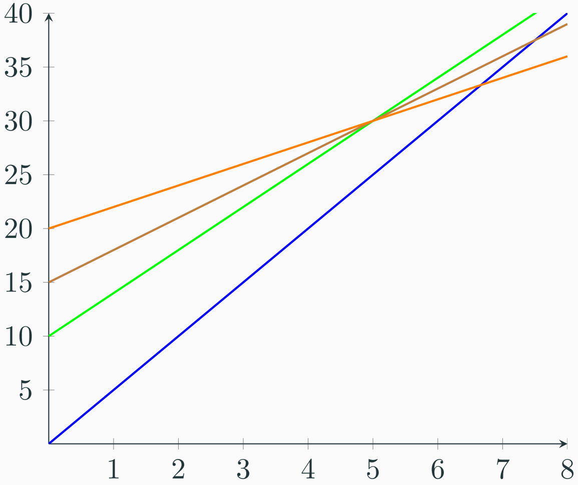

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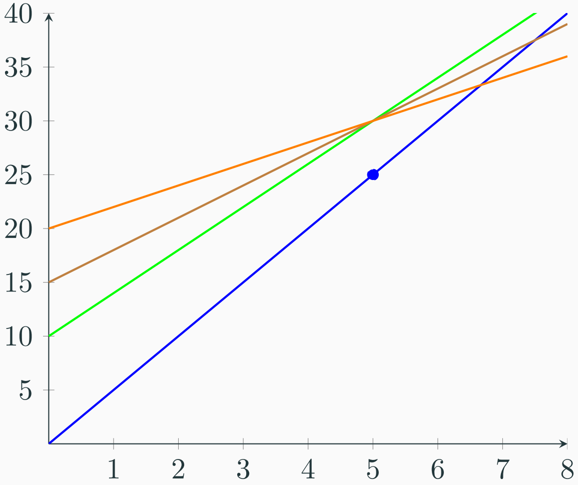

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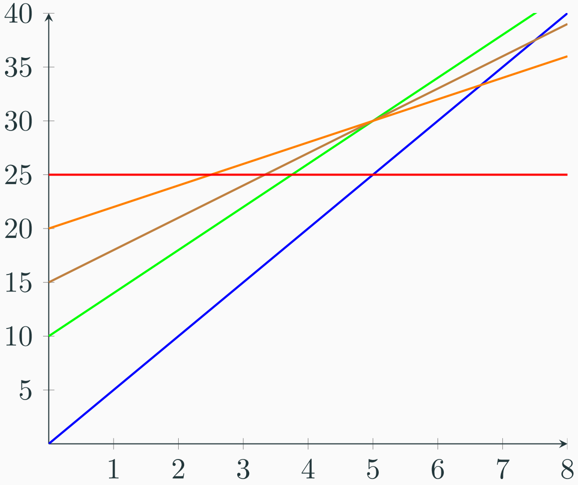
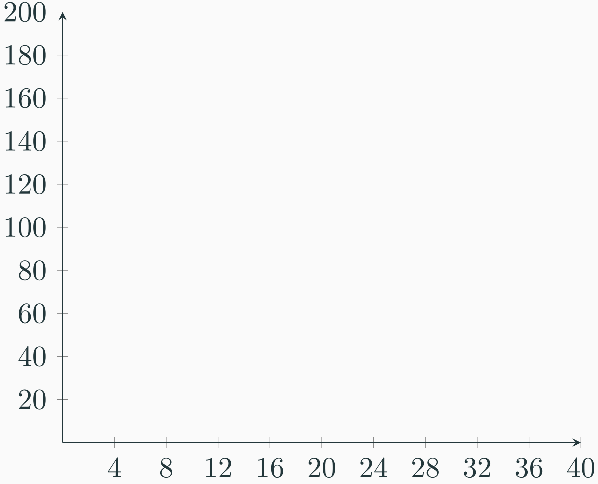


Sample 2 - Illustrated



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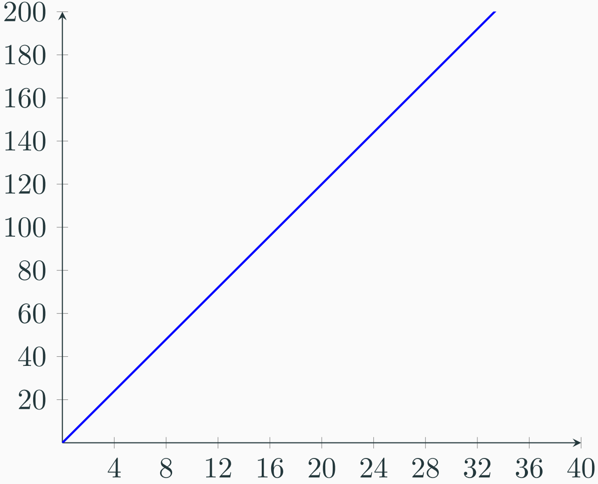

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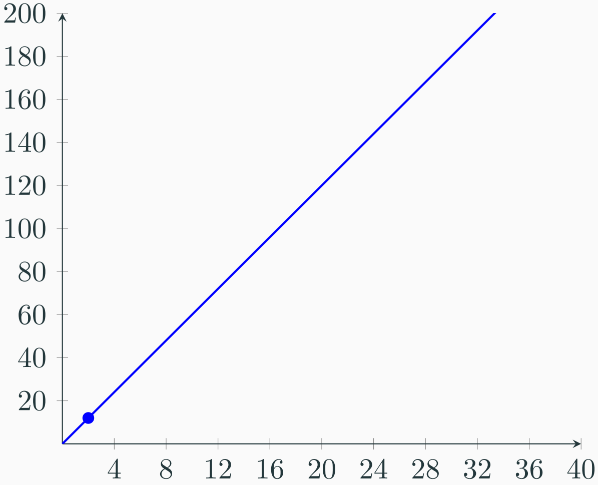

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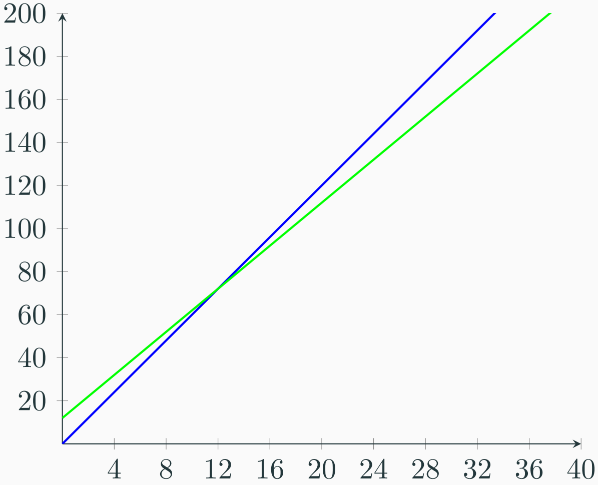

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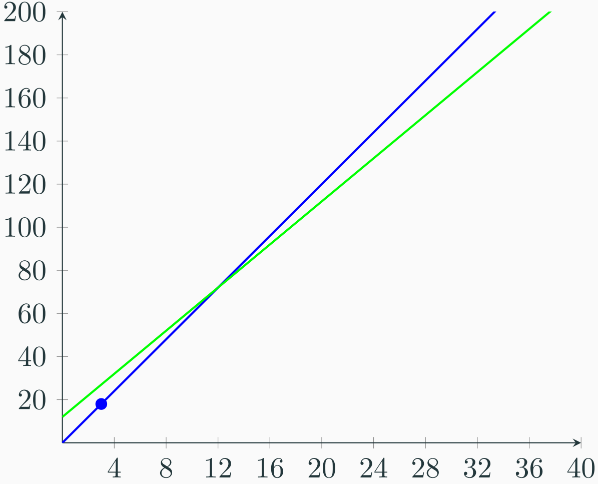

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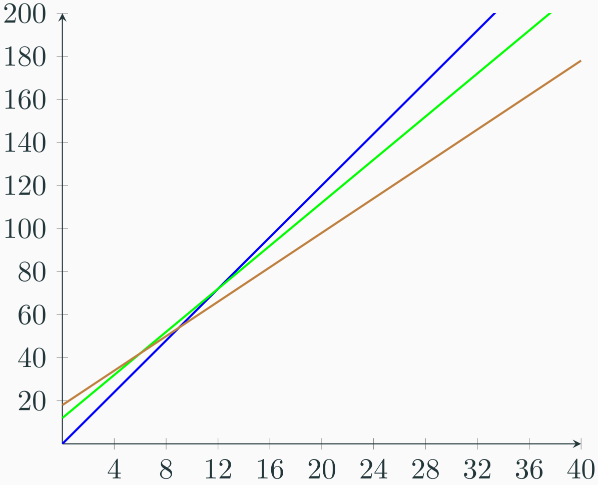

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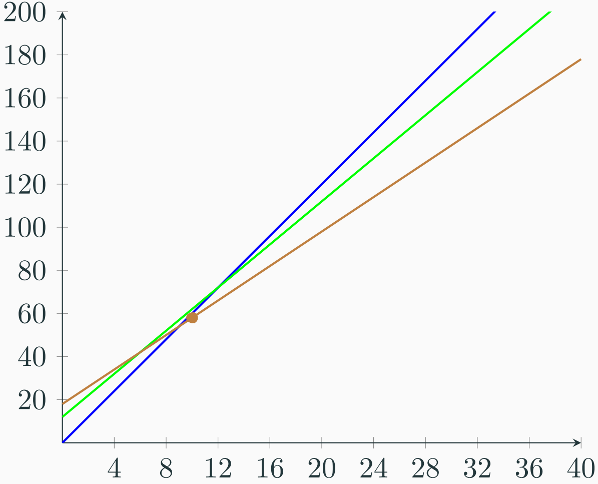

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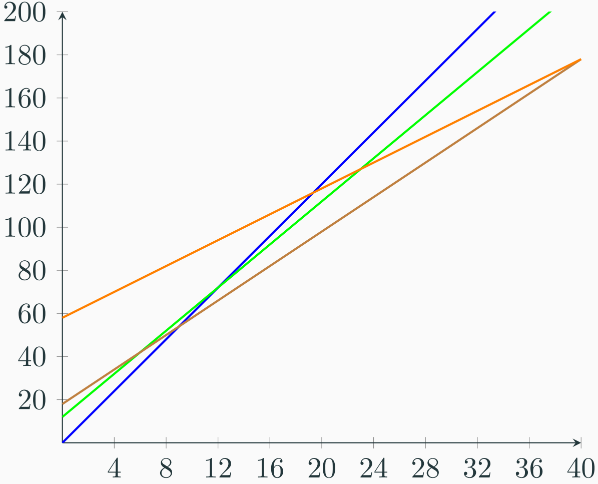

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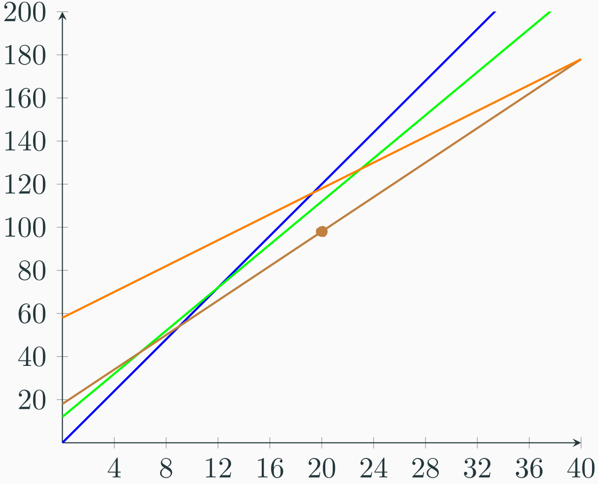

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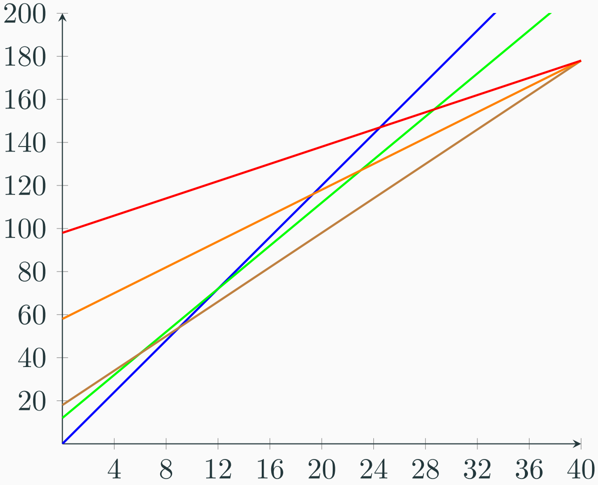

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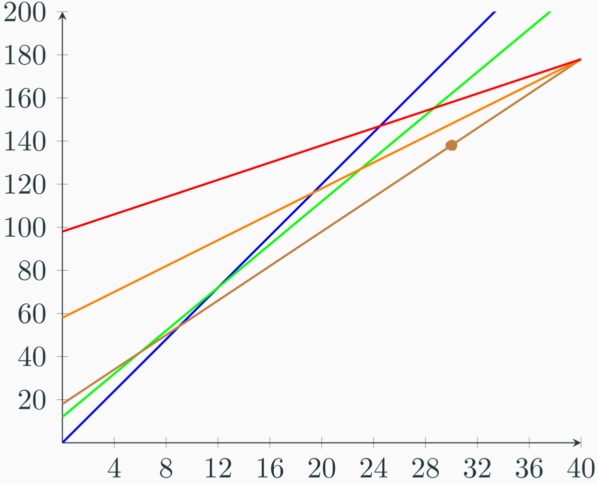

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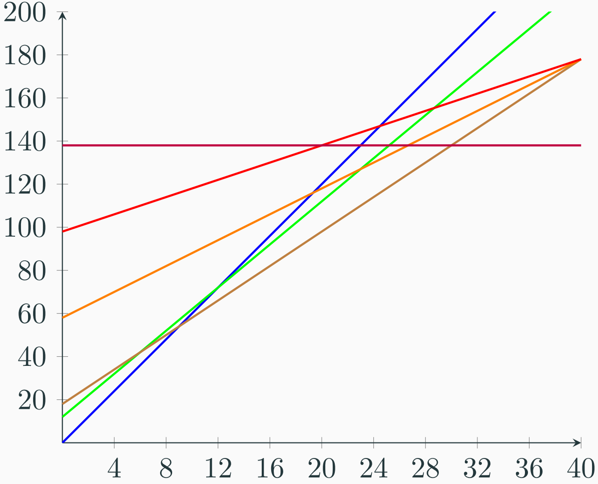

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- If the x-coordinate of a is less than that of b, then the last line is redundant.
- We can therefore iteratively pop redundant lines from the back before adding a line.

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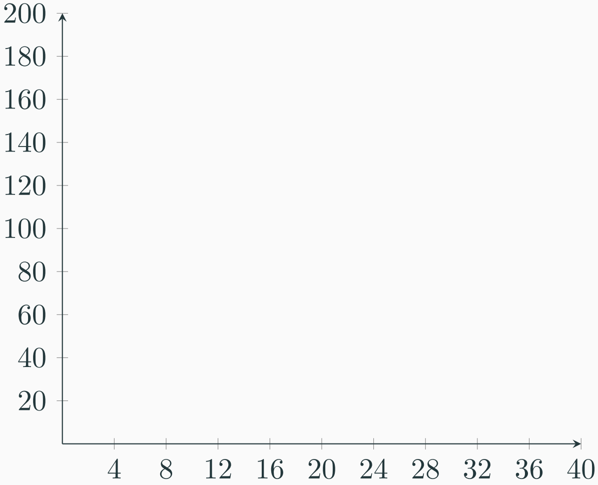

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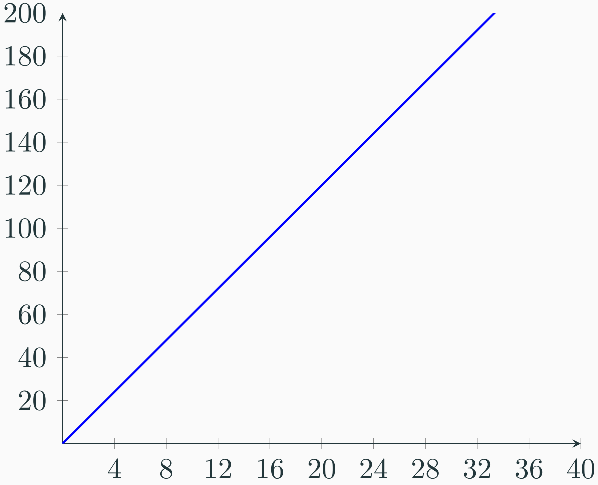

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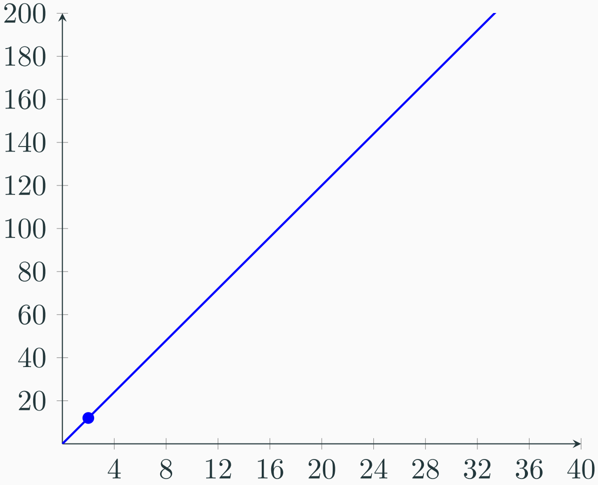

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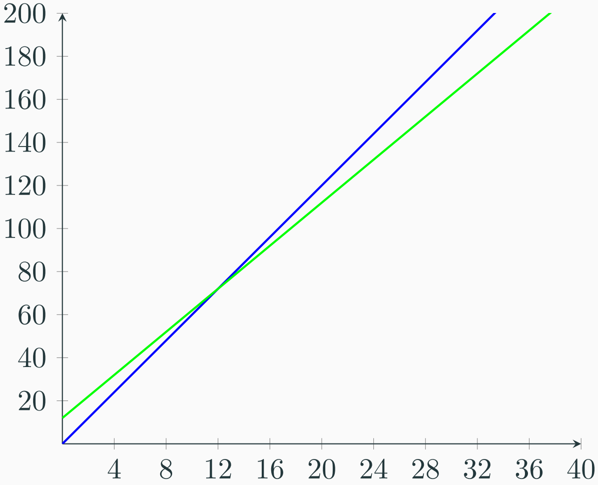

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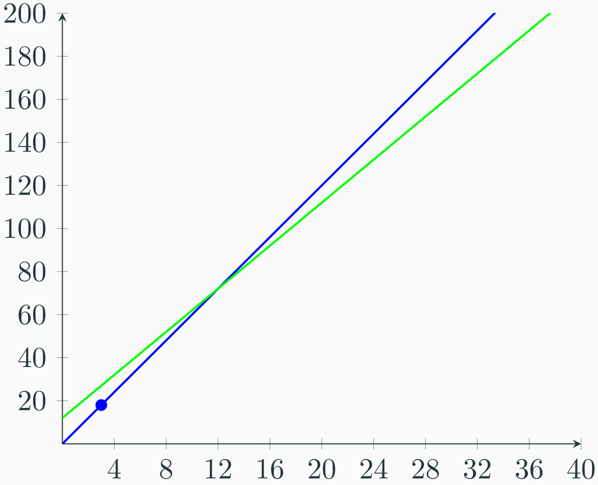

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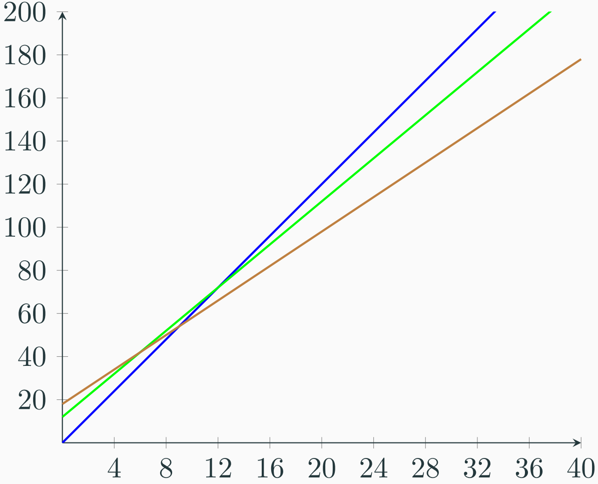

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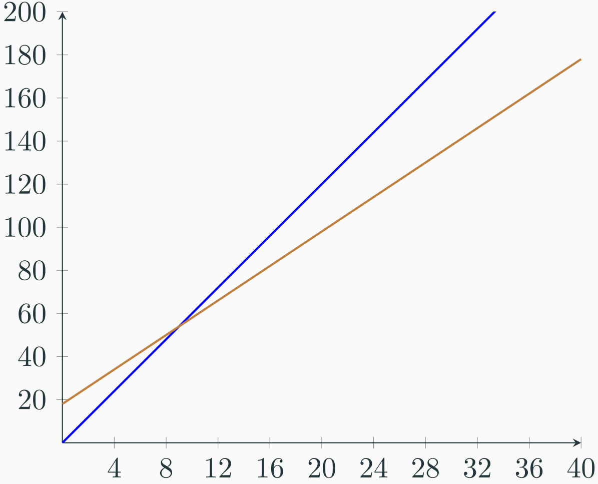

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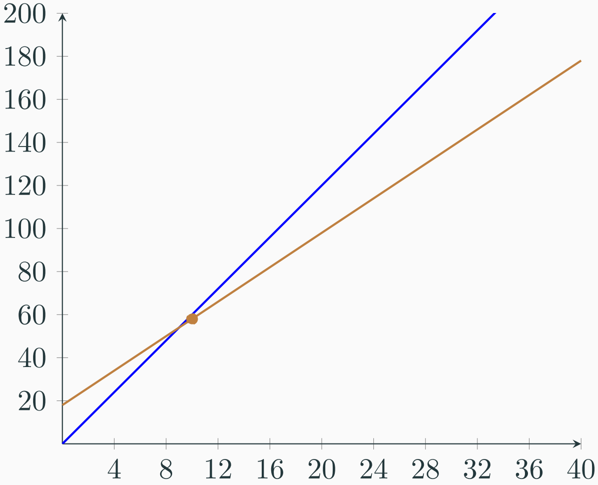

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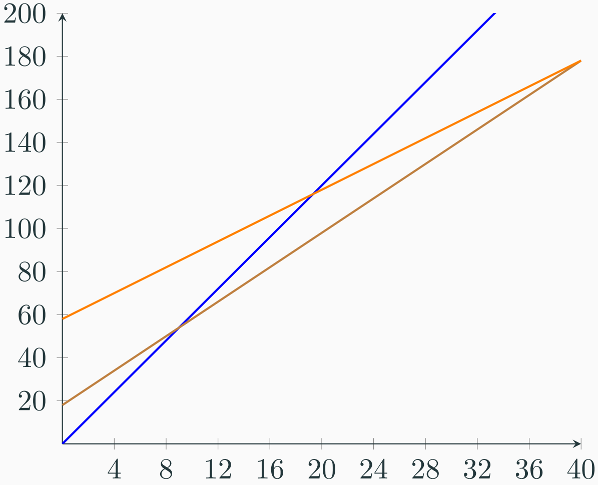

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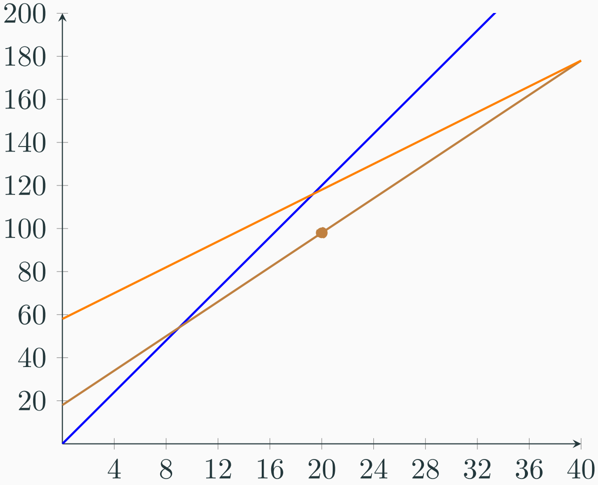

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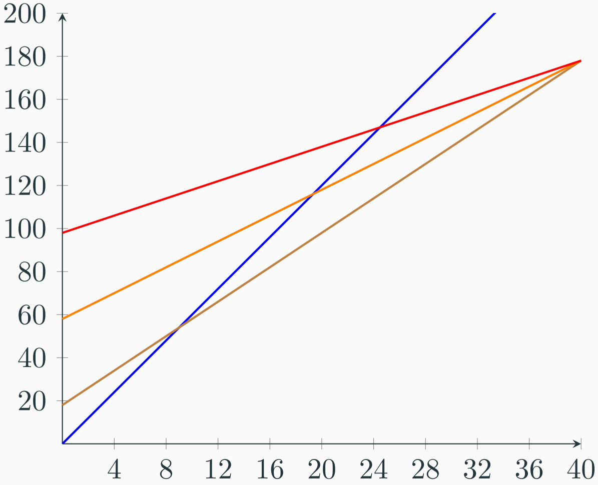

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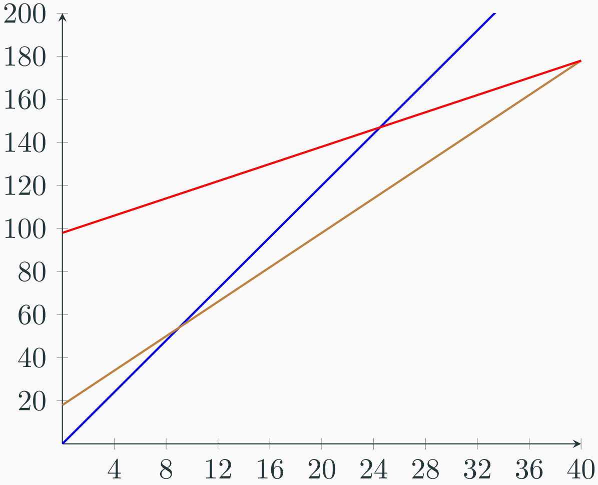

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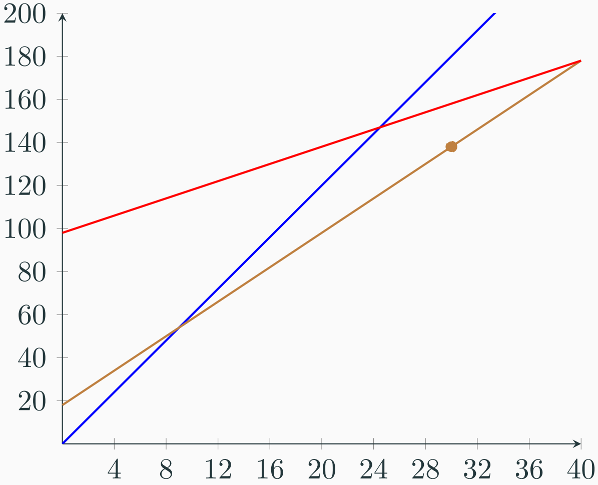

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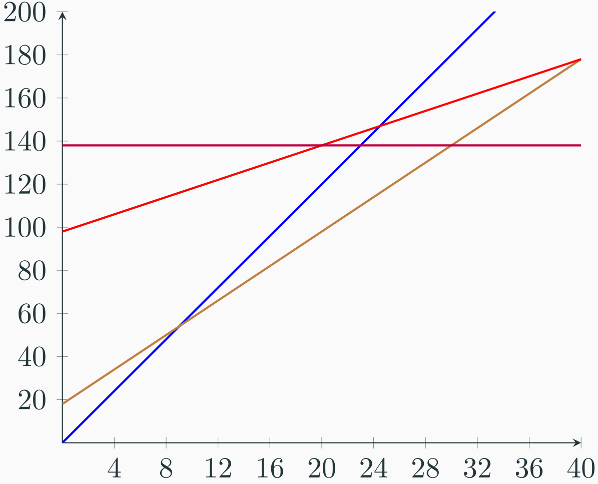

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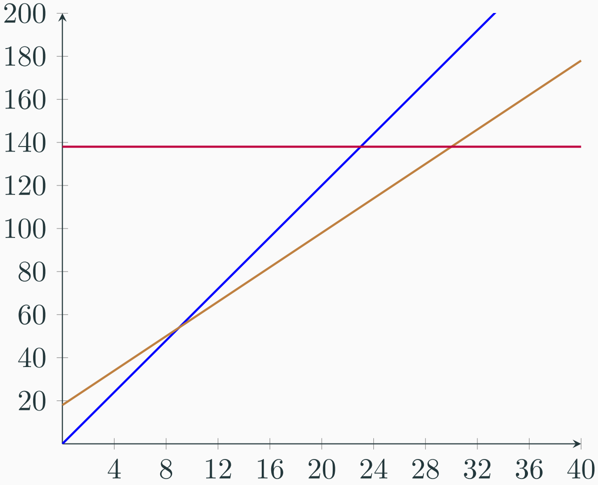

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- Use binary search to find the line in question.
- Construction takes $\mathcal{O}\left(n\right)$ time
- Each query takes $\mathcal{O}\left(\log n\right)$ time.
- We have improved the time complexity from $\mathcal{O}\left(n^2\right)$ to $\mathcal{O}\left(n \log n\right)$.

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- Alternatively, modify such that lines may come in any order while maintaining time complexity.
- Use a multiset instead of a vector/array to keep track of lines and quickly find insertion point. A bit more hassle but powerful.
- Need to consider removing neighbouring lines with higher and lower slopes.

Try on these problems!

- Kalila and Dimna in the Logging Industry
- Covered Walkway (
- Commando

Divide and Conquer Optimization