

Data Structures

Arnar Bjarni Arnarson

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School of Computer Science

Reykjavík University

Today's material

- Built-in data structures and their applications
- Augmenting a data structure
- Union-Find
- Precomputations like prefix sums
- Square root decomposition
- Segment trees
- Sparse tables

Applications of Arrays and Linked Lists

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- Most problems require storing data, usually in an array
- On modern machines, arrays are almost always a better choice than a linked list
- There are however a few cases where linked lists are better

Example problem: Broken Keyboard

- <http://uva.onlinejudge.org/external/119/11988.html>

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- And a lot more

Applications of Queues

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Applications of Priority Queues

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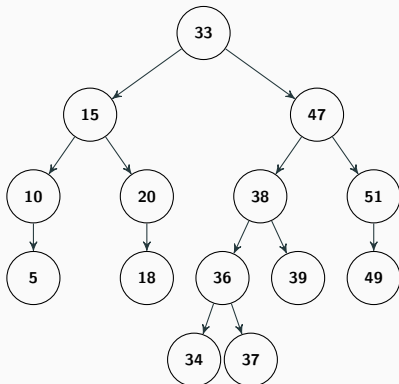
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- Example: Augmenting binary search trees

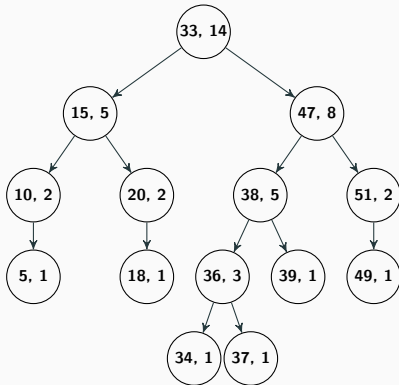
Augmenting Binary Search Trees

- We have a binary search tree and want to efficiently:
 - Count number of elements $< x$
 - Find the k th smallest element
- Naive method is to go through all vertices, but that is slow: $O(n)$



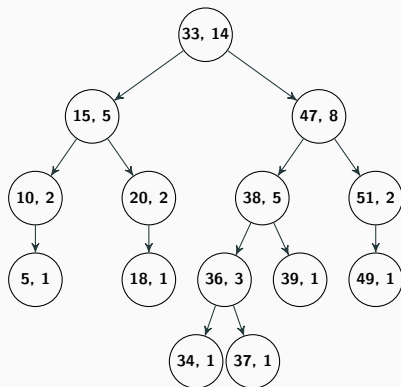
Augmenting Binary Search Trees

- Idea: In each vertex store the size of the subtree
- This information can be maintained when we insert/delete elements without increasing time complexity



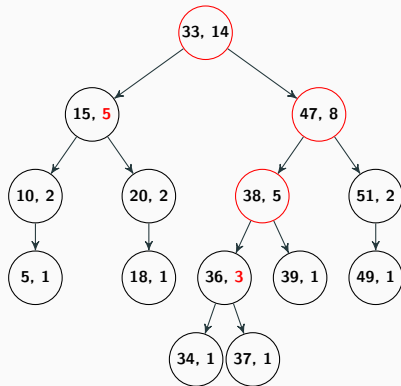
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- Count number of elements < 38
 - Search for 38 in the tree
 - Count the vertices that we pass by that are less than x
 - When we are at a vertex where we should go right, get the size of the left subtree and add it to our count



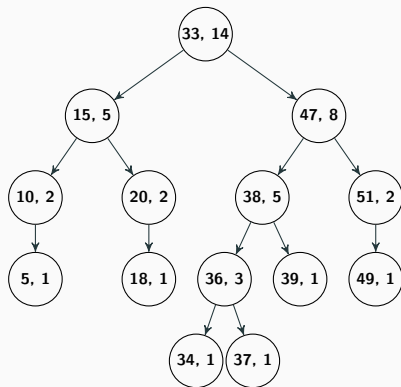
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- Time complexity $O(\log n)$



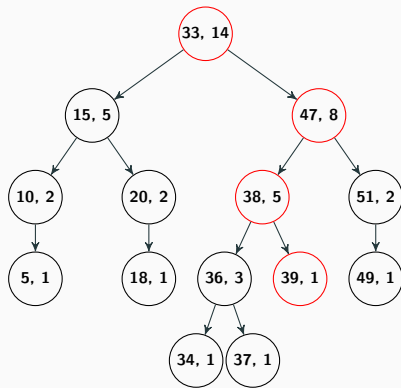
Augmenting Binary Search Trees

- Find k th smallest element
 - We're on a vertex whose left subtree is of size m
 - If $k = m + 1$, we found it
 - If $k \leq m$, look for the k th smallest element in the left subtree
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- Example: $k = 11$



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- Operation `union(x, y)` unions the sets of which x and y are members.

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- $\text{join}(1, 4)$ finally results in $\{\{1, 2, 3, 4, 5\}\}$.
- At any given point $\text{find}(x)$ returns some value in the same set as x .
- The important bit is that $\text{find}(x)$ returns the same value for all elements of the same set, the representative.

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- To get the representative of x we go to the parent of our current item (starting at x) until the item has no parent.
- Then to unite x, y we simply make the representative of x the parent of the representative of y .

Naïve Union-Find implementation

```
struct union_find {  
    vector<int> parent;  
    union_find(int n) {  
        parent = vector<int>(n);  
        for(int i = 0; i < n; i++) {  
            parent[i] = i;  
        }  
    }  
    int find(int x) {  
        return parent[x] == x ? x : find(parent[x]);  
    }  
    void unite(int x, int y) {  
        parent[find(x)] = find(y);  
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- The key to making this more efficient is making those chains shorter.
- We do this by flattening the chain each time we query `find`, so the amortized complexity becomes good.
- The worst case is still $\mathcal{O}(n)$ but the amortized complexity is $\mathcal{O}(\alpha(n))$ which may as well be a constant, as it is < 5 for n equal to the number of atoms in the observable universe.

Path compressed Union-Find implementation

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        }  
    }  
    int find(int x) {  
        if(parent[x] == x) return x;  
        return parent[x] = find(parent[x]);  
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    void unite(int x, int y) {  
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 - An iterable list of all elements of the set containing x
- When tracking size you can use it to always perform small-to-large merges for $\mathcal{O}(\log n)$ time complexity.

Example problem: Skolavslutningen

- <https://open.kattis.com/problems/skolavslutningen>

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- We want to answer these queries efficiently, or in other words, without looking through all elements.
- Sometimes we also want to update elements.

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- Simplification: only support queries of the form $\text{sum}(0, j)$
- Notice that $\text{sum}(i, j) = \text{sum}(0, j) - \text{sum}(0, i - 1)$

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- Can we support updating efficiently? No, at least not without modification

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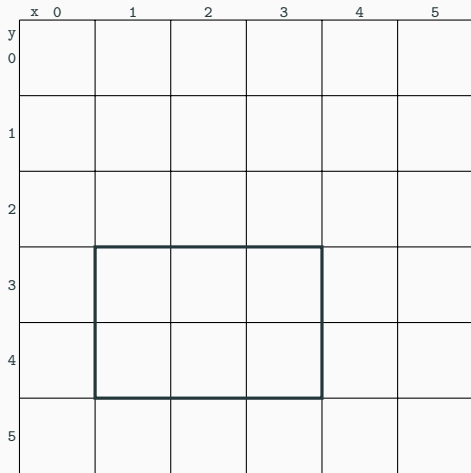
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- This also works for multidimensional arrays, but the math is more involved.
- We let $\text{sum}(x_i, x_j, y_i, y_j)$ denote the query for the sum from x_i to x_j along the x -dimension, and the same for y .

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- Then the formula becomes

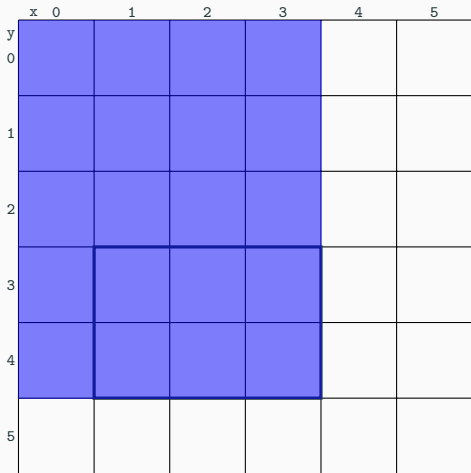
$$\begin{aligned}\text{sum}(x_i, x_j, y_i, y_j) &= \text{sum}(0, x_j, 0, y_j) \\ &\quad - \text{sum}(0, x_{i-1}, 0, y_j) \\ &\quad - \text{sum}(0, x_j, 0, y_{i-1}) \\ &\quad + \text{sum}(0, x_{i-1}, 0, y_{i-1})\end{aligned}$$

2D sum



`query(1, 3, 3, 4)`

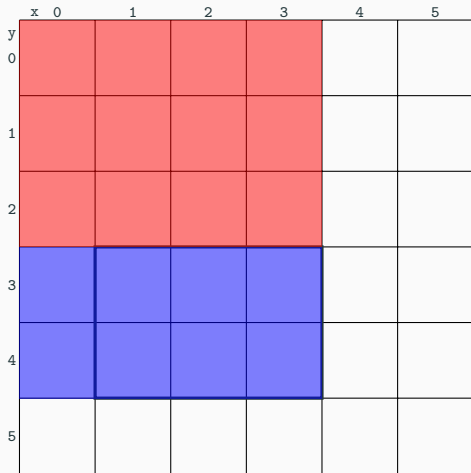
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`query(1, 3, 3, 4)`

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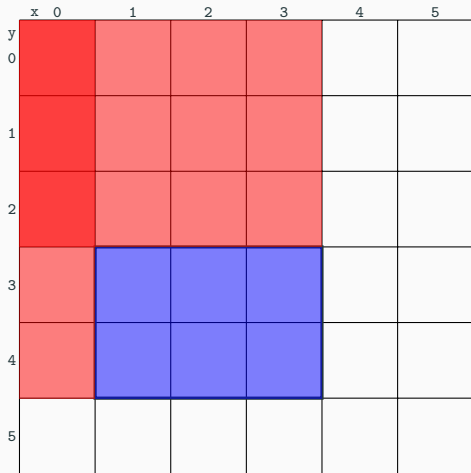


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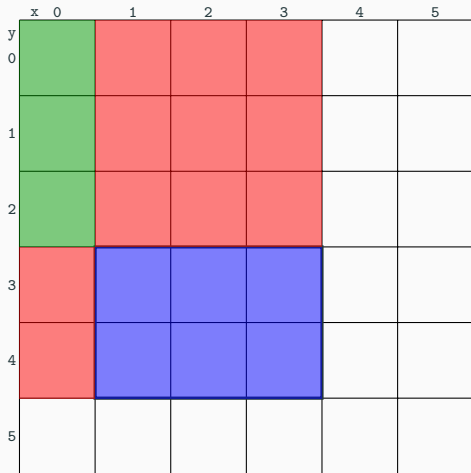
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Range sum on a mutable array

- What if we want to support:
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- How do we support these queries efficiently?

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- Time complexity is minimized for $k = \sqrt{n}$:
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 - Querying in $O(n/\sqrt{n} + \sqrt{n}) = O(\sqrt{n})$
- Also known as square root decomposition, and is a very powerful technique

Example problem: Supercomputer

- <https://open.kattis.com/problems/supercomputer>

Range queries

- Now we know how to do these queries in $O(\sqrt{n})$

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- Now we know how to do these queries in $O(\sqrt{n})$
- May be too slow if n is large and many queries
- Can we do better?

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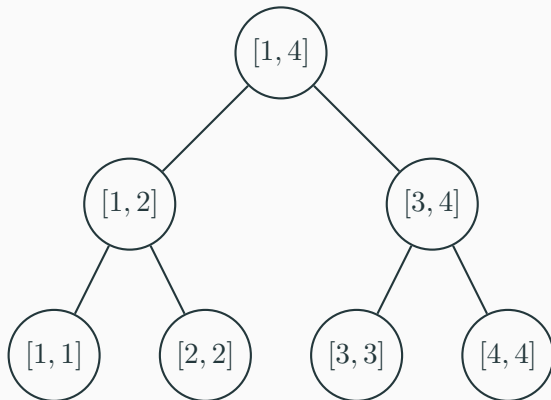
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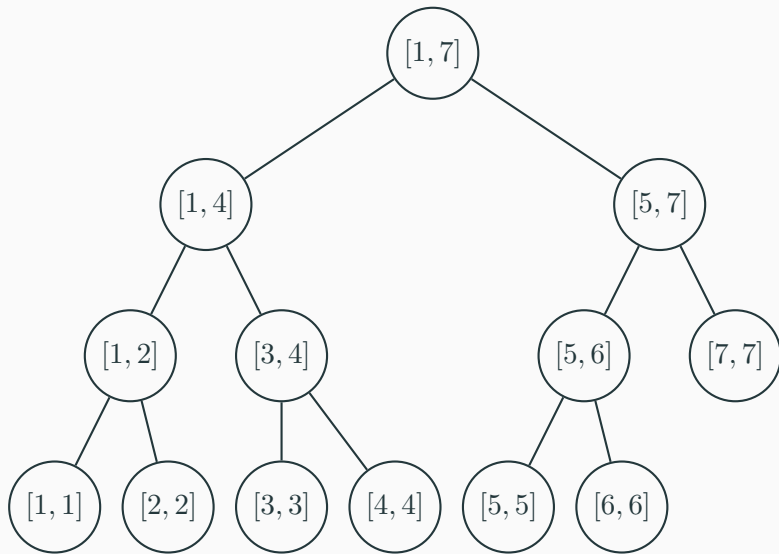
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- We travel down the tree looking for the left and right endpoints, adding intervals that are completely inside our query range.
- When we update a value we only need to update the parents of that node up to the root, at most $\mathcal{O}(\log(n))$ nodes.

Drawn Segment Tree, $n = 4$



Drawn Segment Tree, $n = 7$

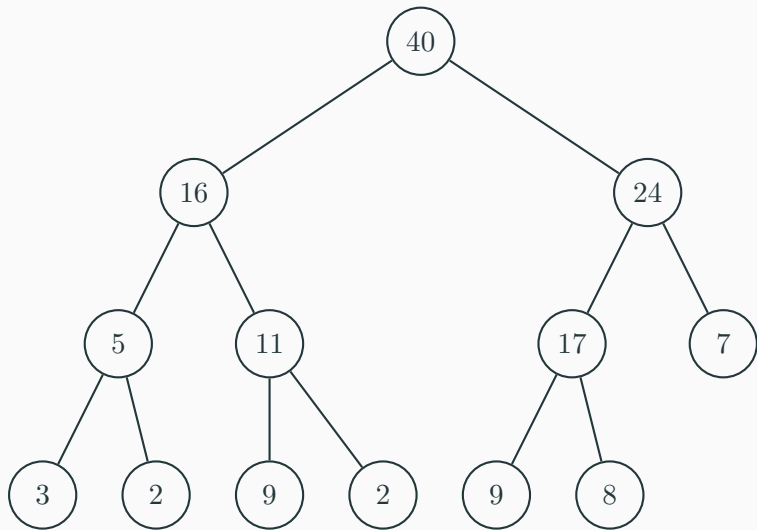


Segment Tree - Code

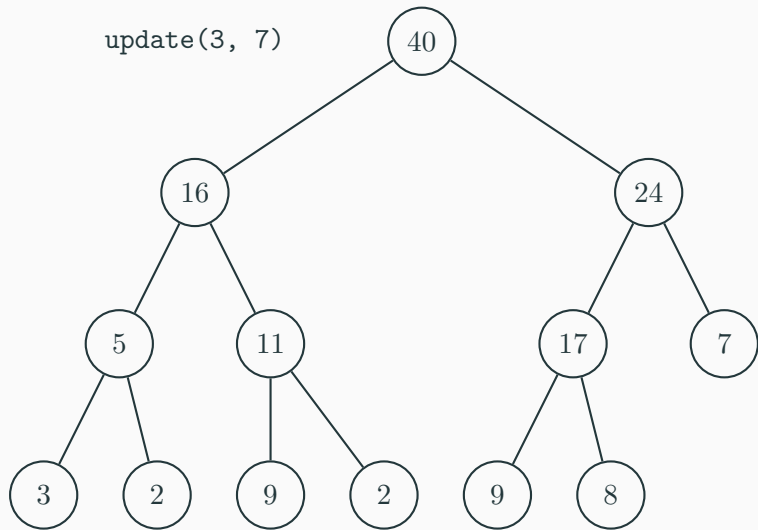
```
struct segment_tree {
    segment_tree *left, *right;
    int from, to, value;
    segment_tree(int from, int to)
        : from(from), to(to), left(NULL), right(NULL), value(0) { }
};
```

```
segment_tree* build(const vector<int> &arr, int l, int r) {
    if (l > r) return NULL;
    segment_tree *res = new segment_tree(l, r);
    if (l == r) {
        res->value = arr[l];
    } else {
        int m = (l + r) / 2;
        res->left = build(arr, l, m);
        res->right = build(arr, m + 1, r);
        if (res->left != NULL) res->value += res->left->value;
        if (res->right != NULL) res->value += res->right->value;
    }
    return res;
}
```

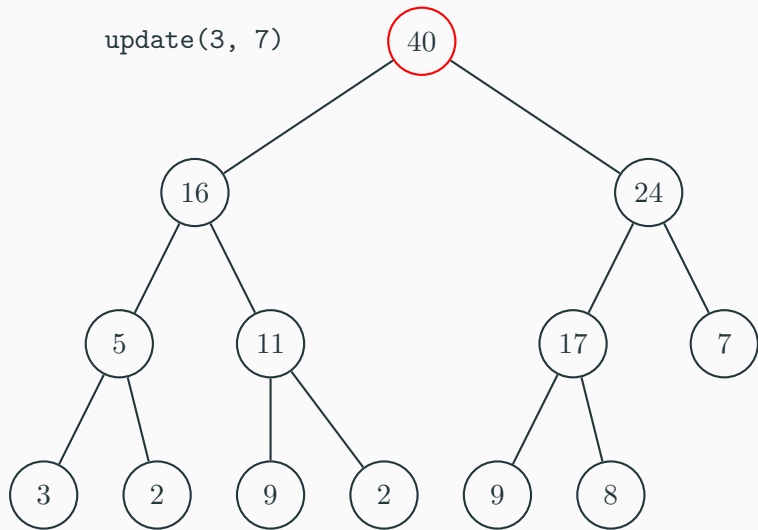
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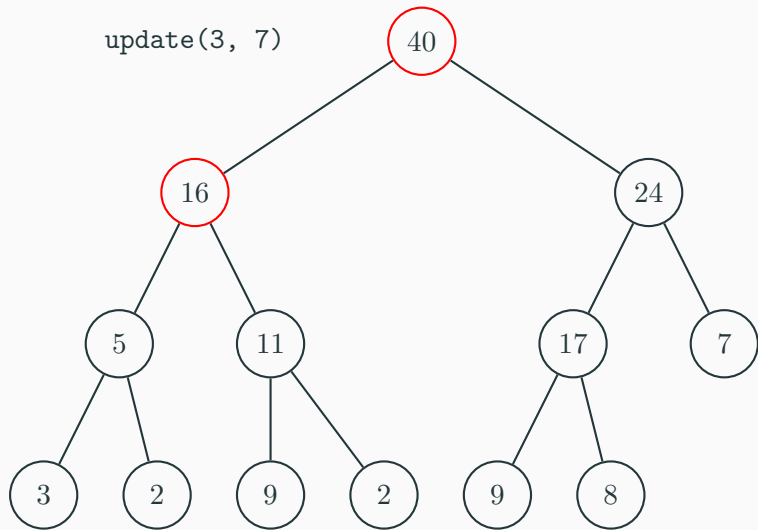
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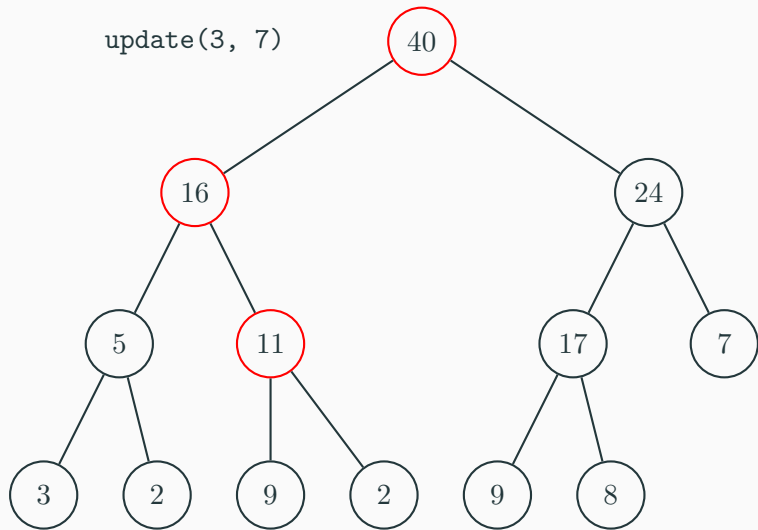
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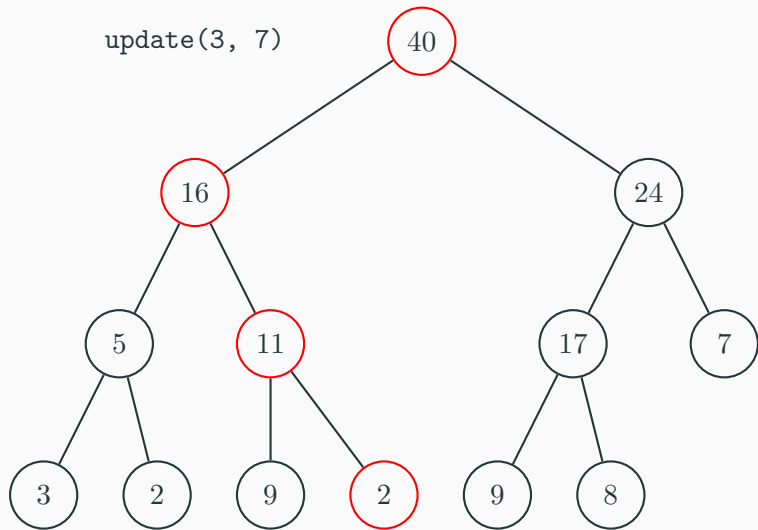
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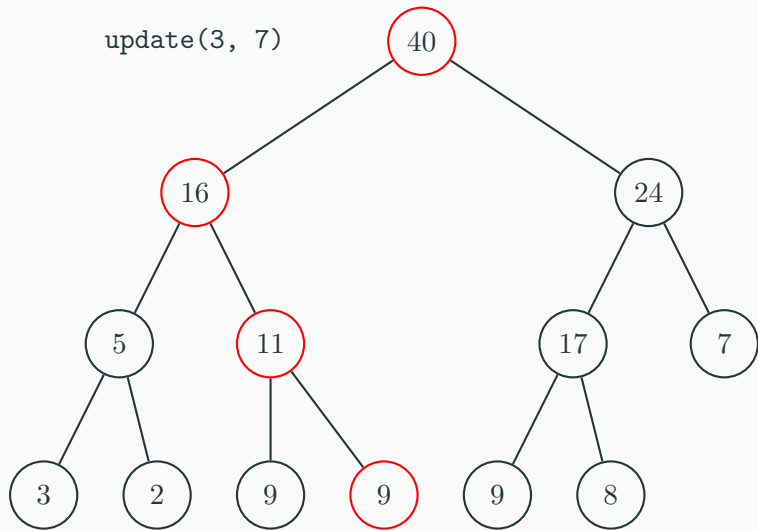
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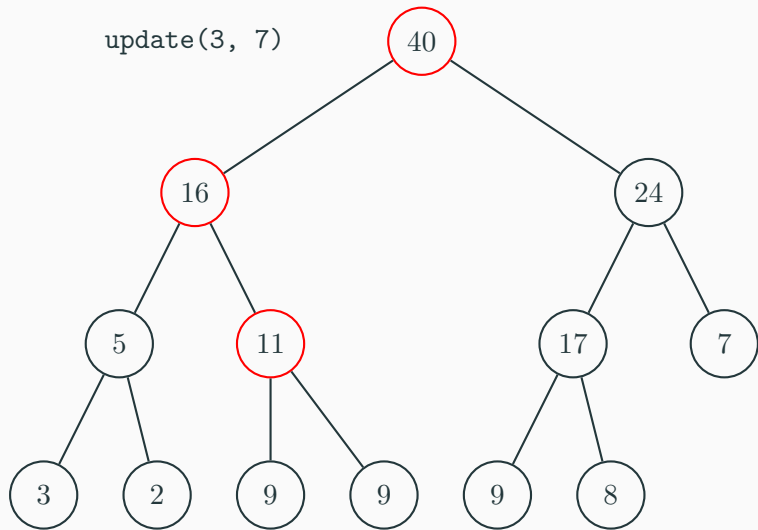
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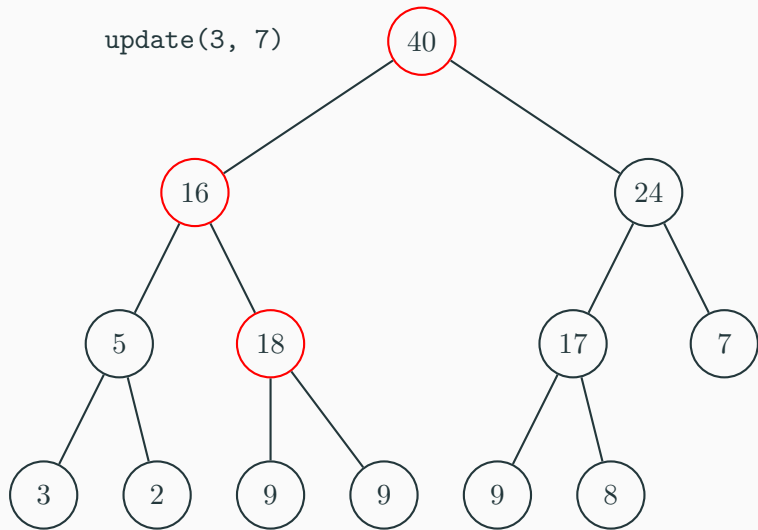
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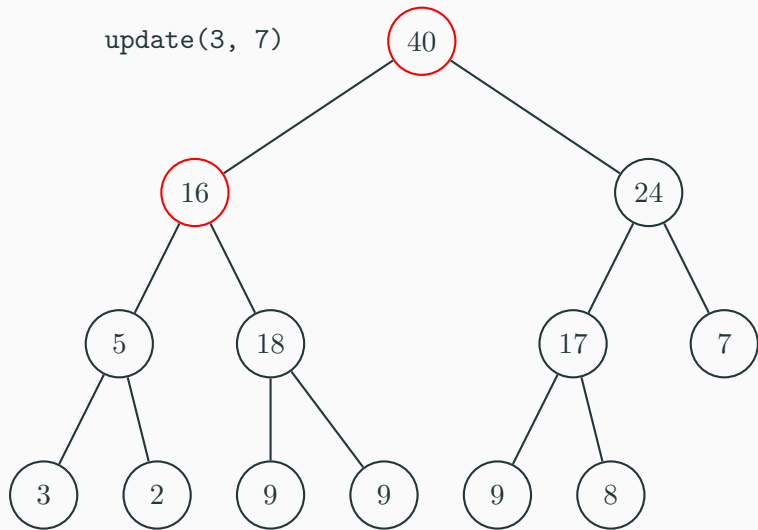
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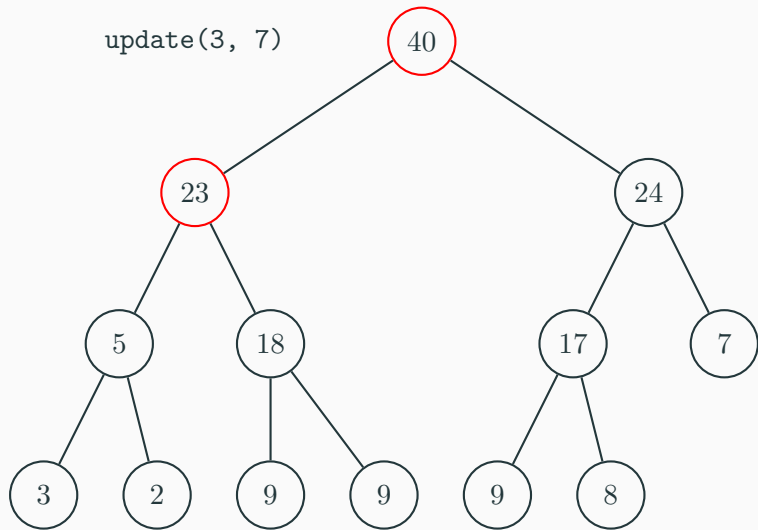
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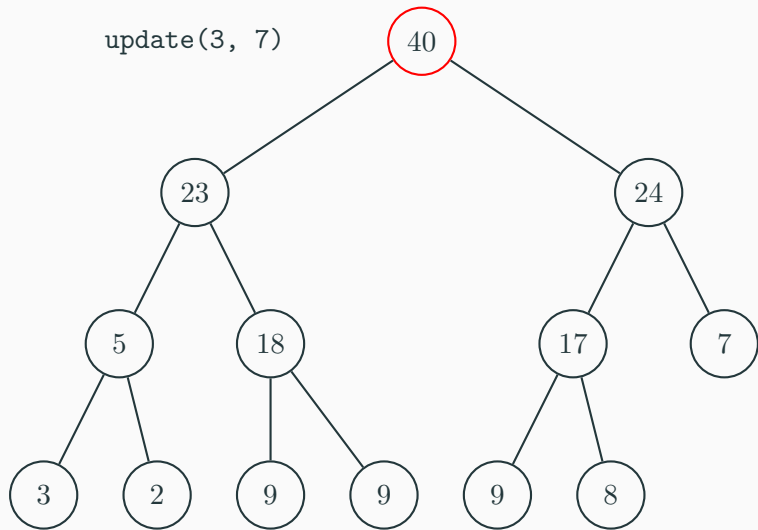
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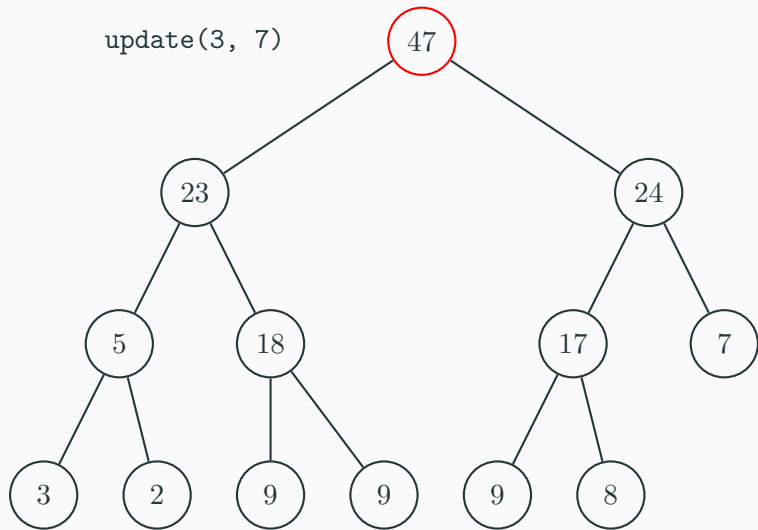
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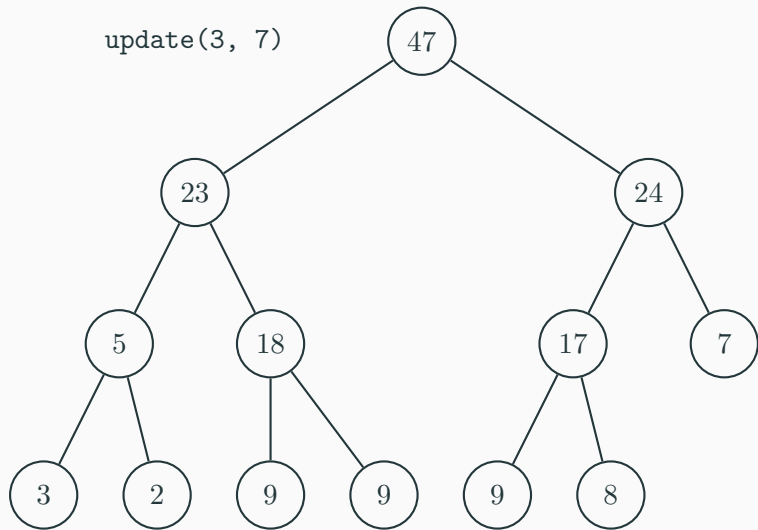
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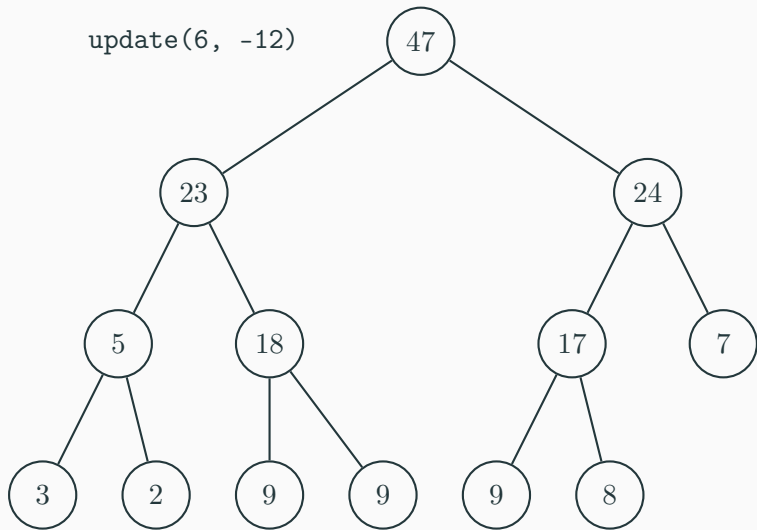
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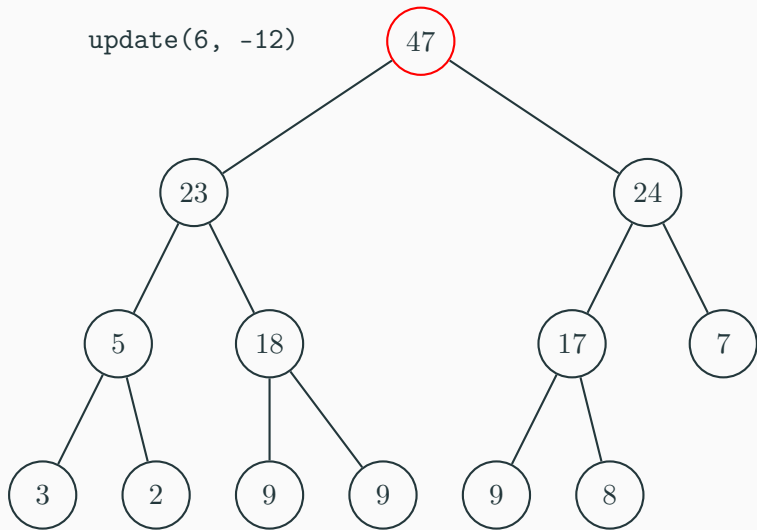
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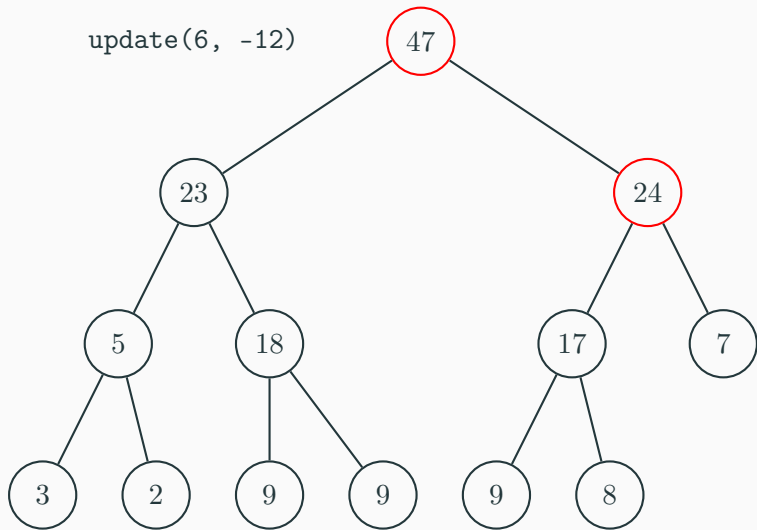
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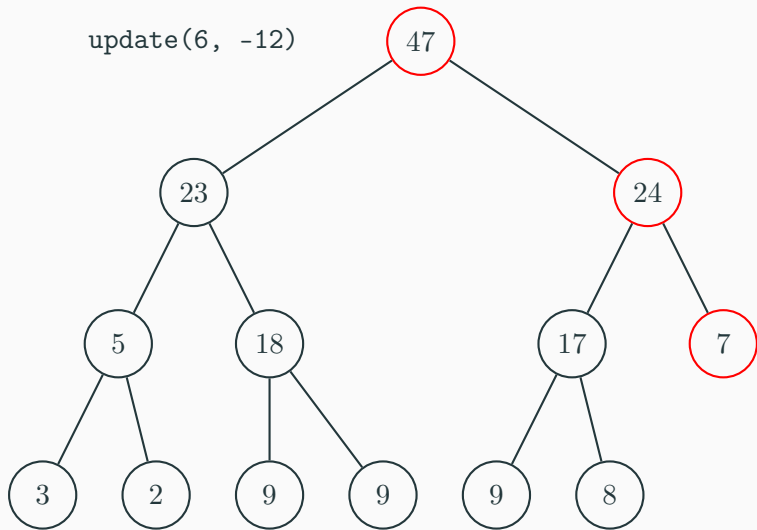
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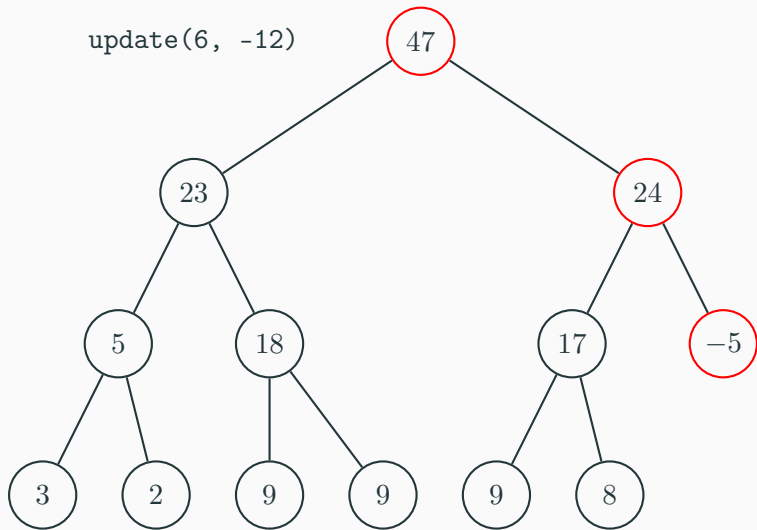
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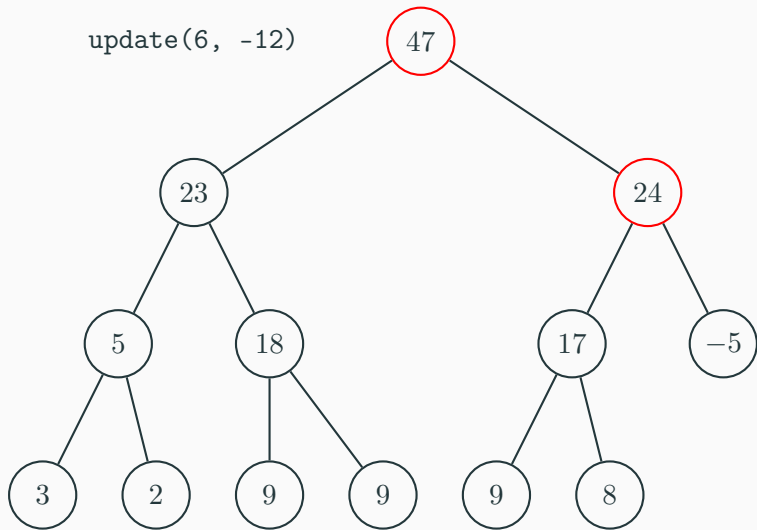
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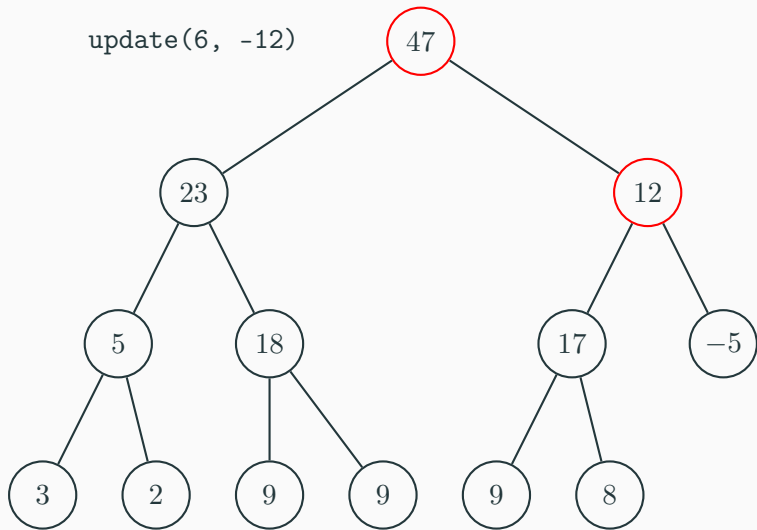
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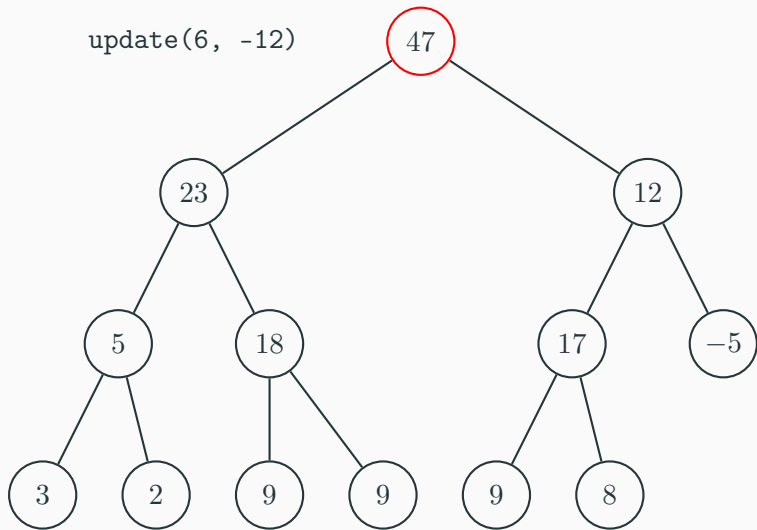
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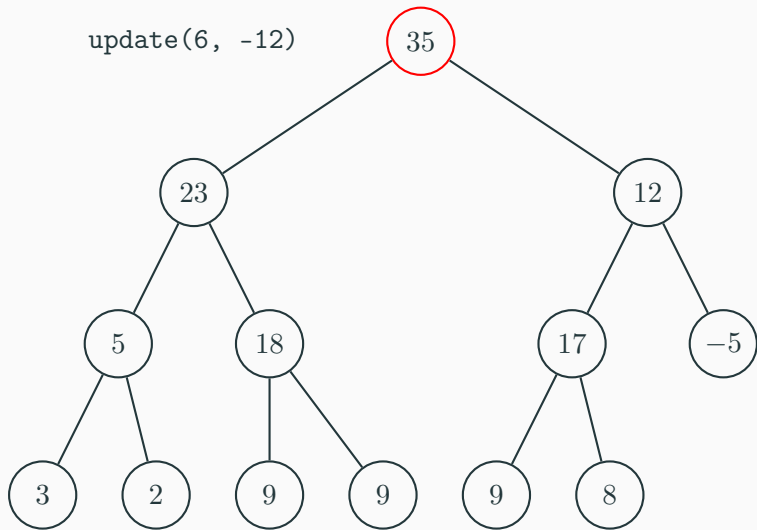
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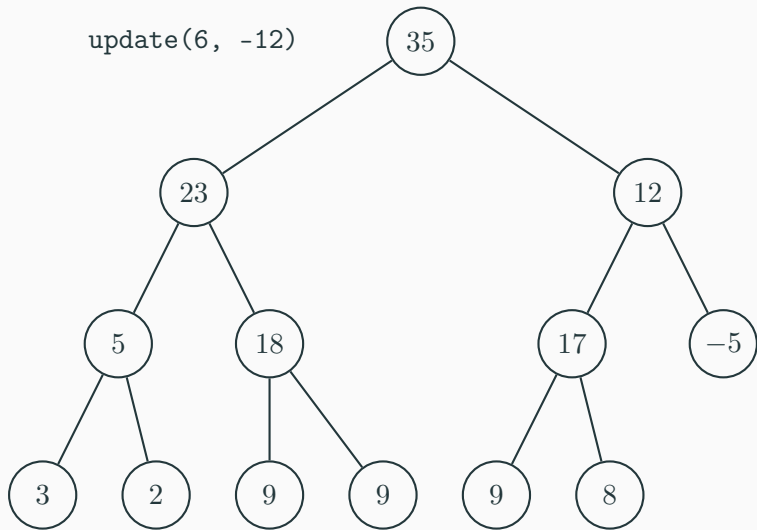
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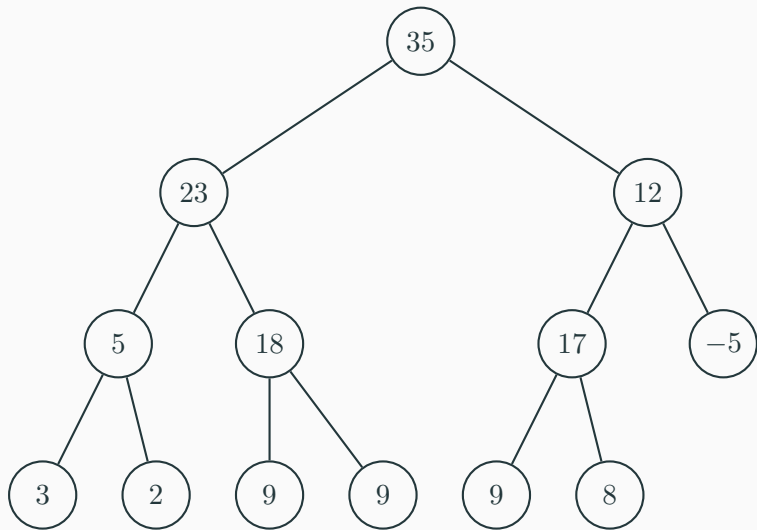
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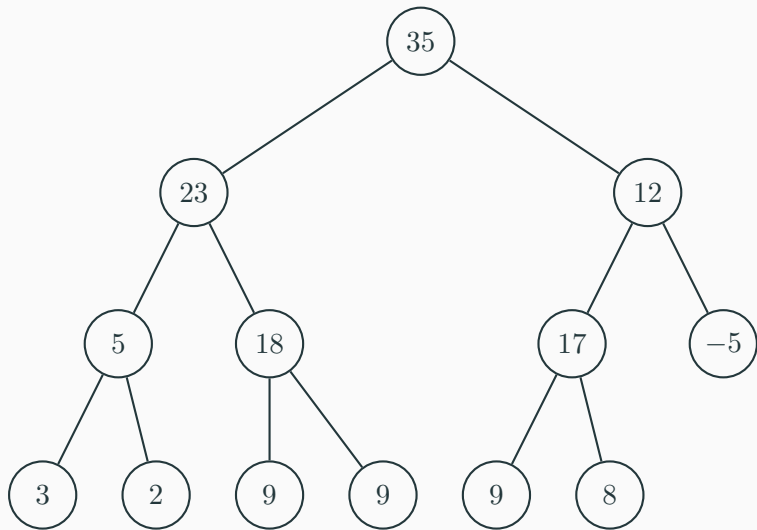
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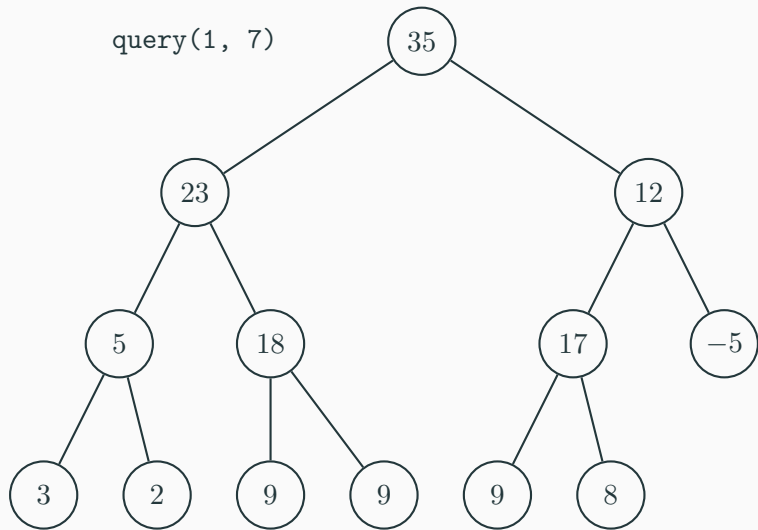
Updating a Segment Tree - Code

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int update(segment_tree *tree, int i, int val) {
    if (tree == NULL) return 0;
    if (tree->to < i) return tree->value;
    if (i < tree->from) return tree->value;
    if (tree->from == tree->to && tree->from == i) {
        tree->value = val;
    } else {
        tree->value = update(tree->left, i, val) + update(tree->right, i, val);
    }
    return tree->value;
}
```

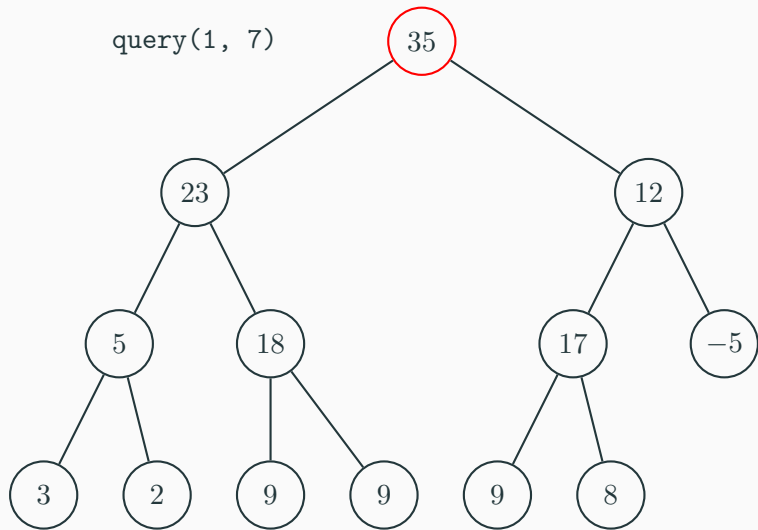
Querying



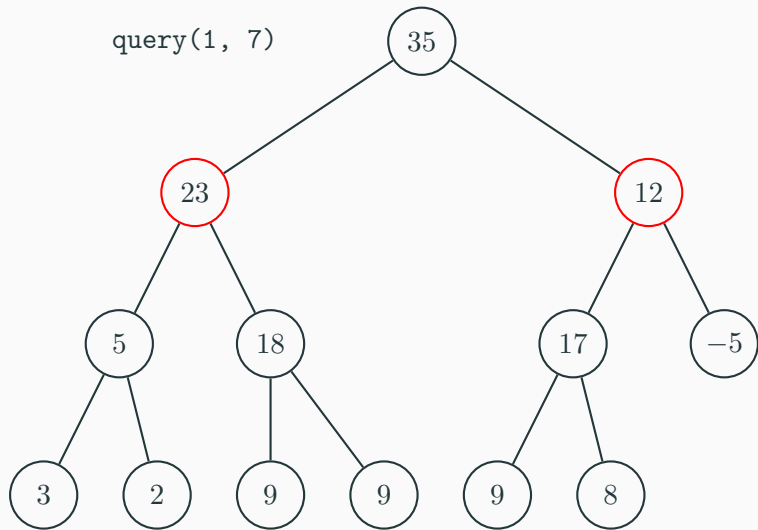
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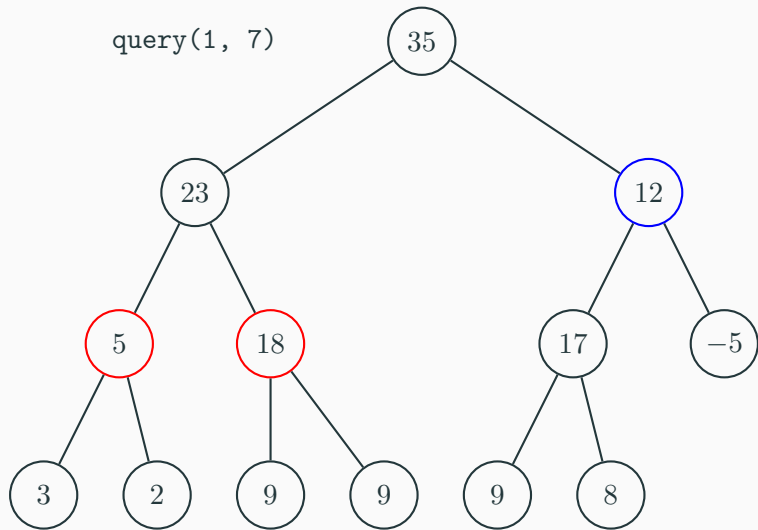
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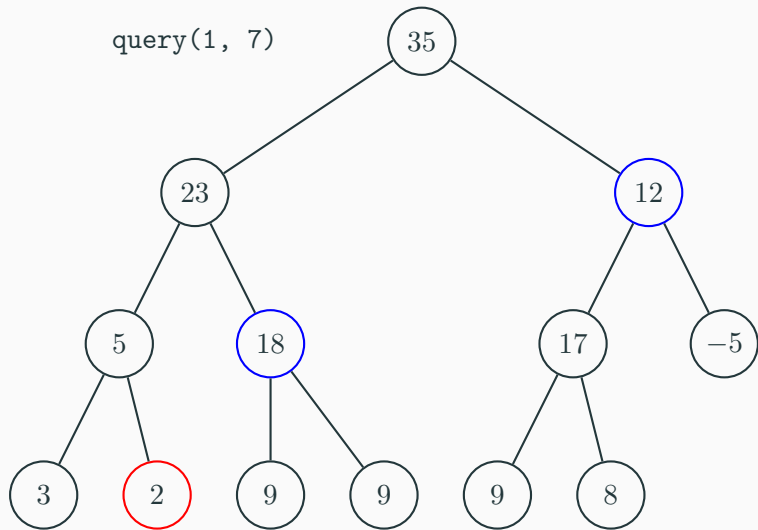
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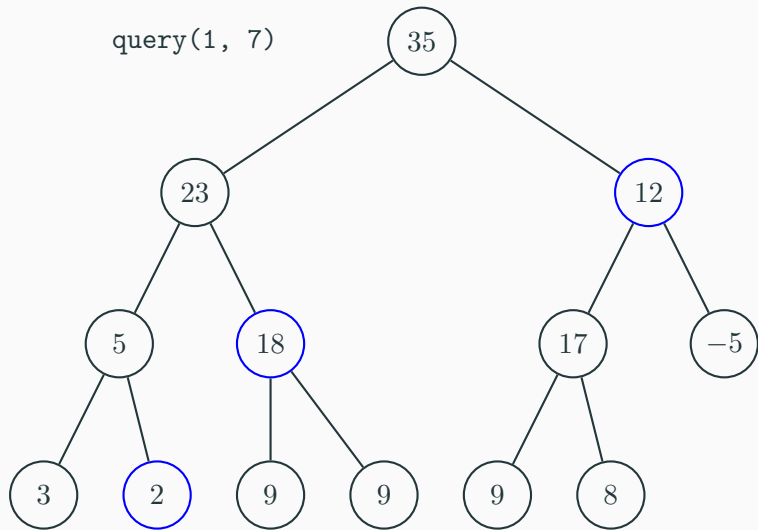
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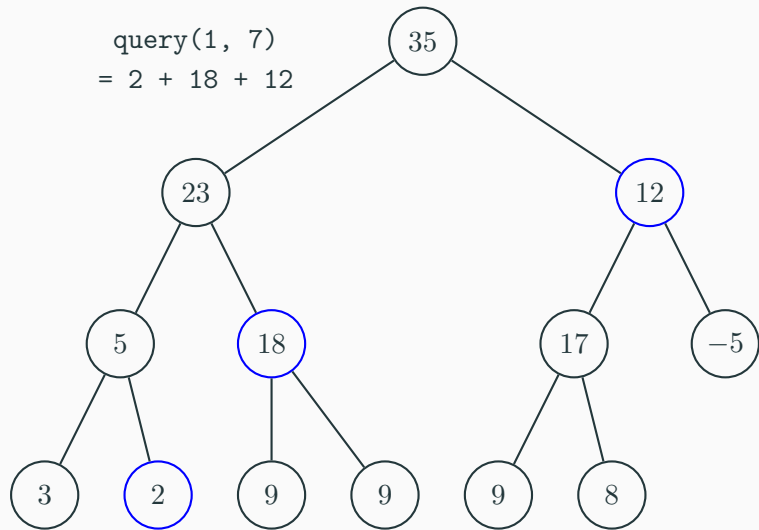
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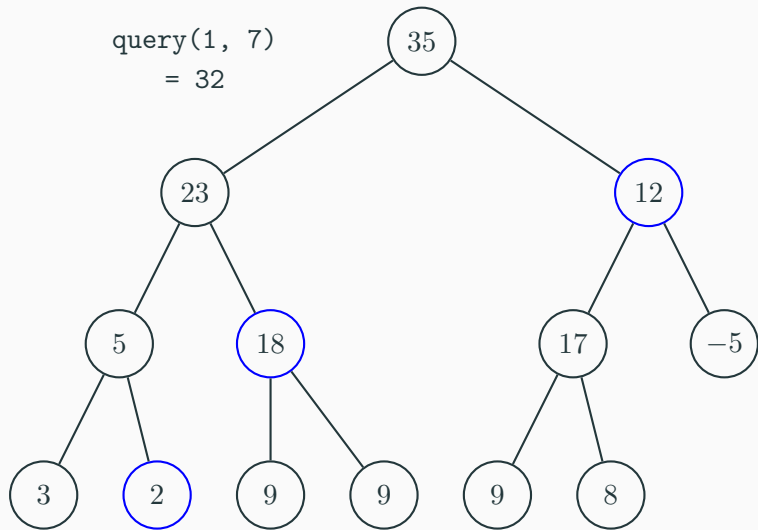
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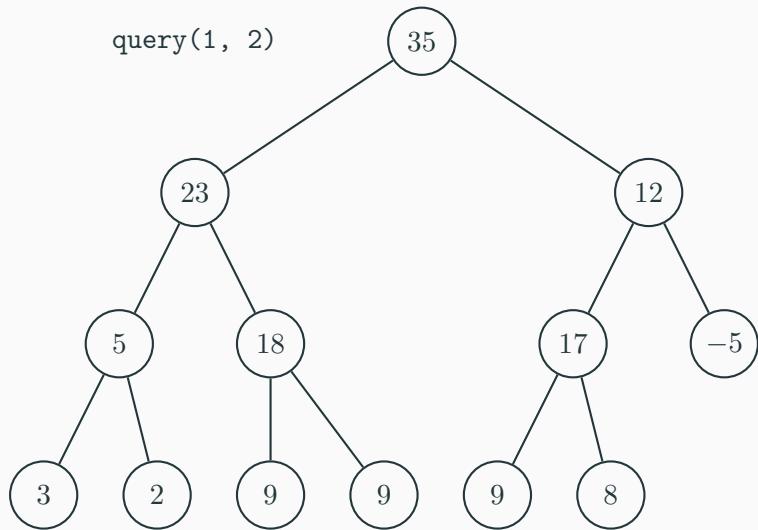
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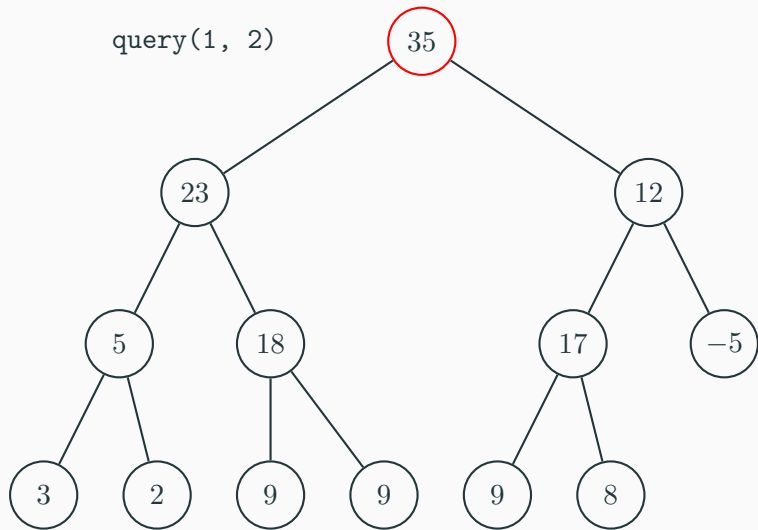
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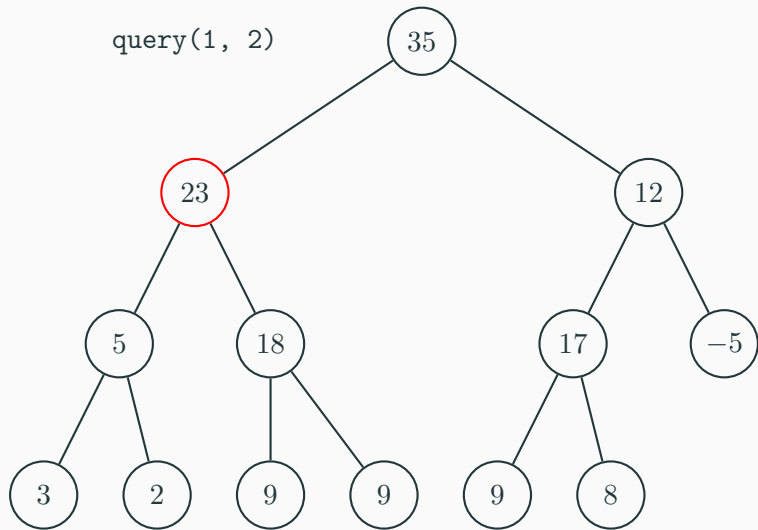
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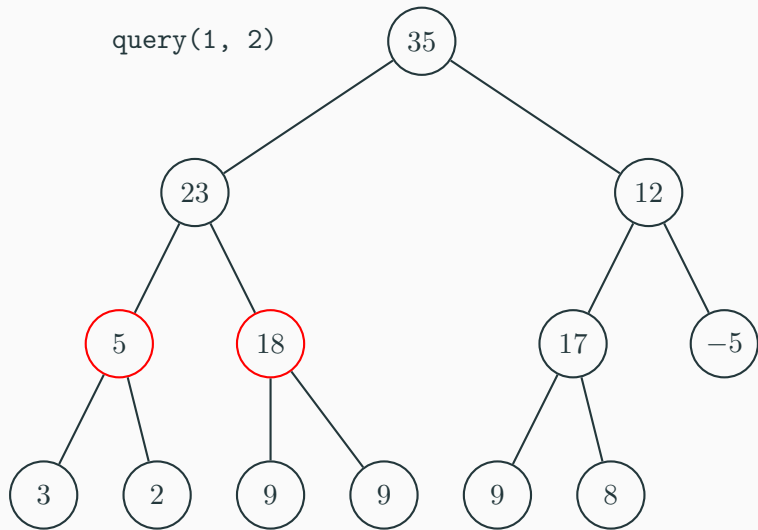
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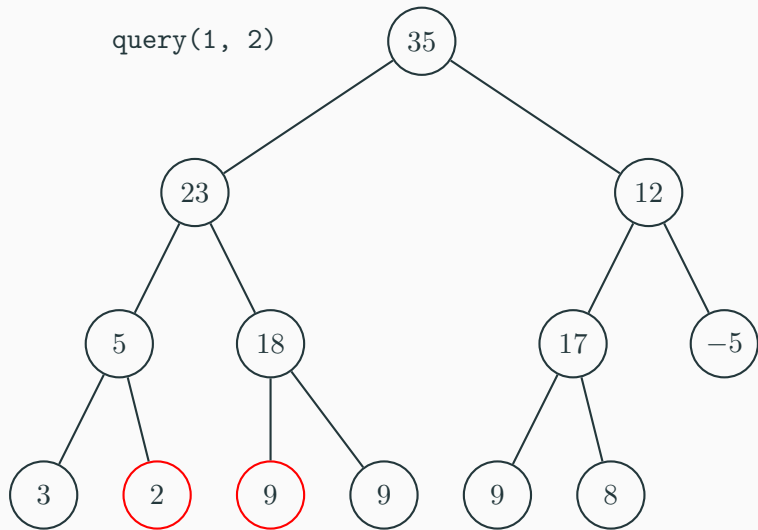
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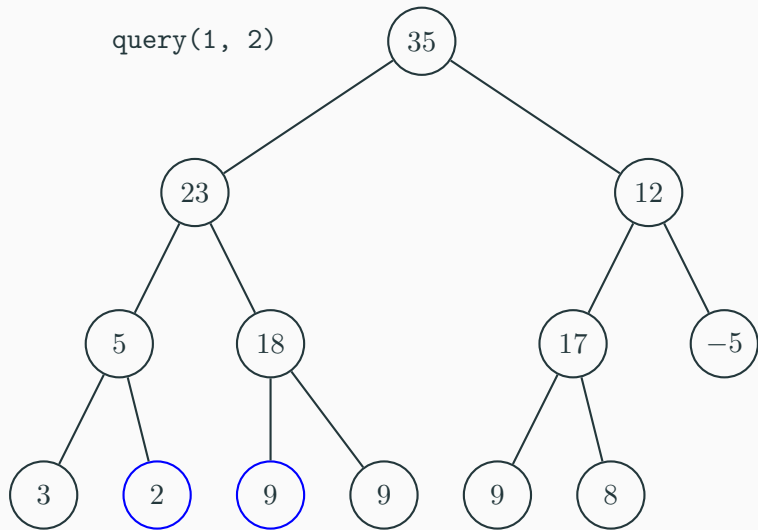
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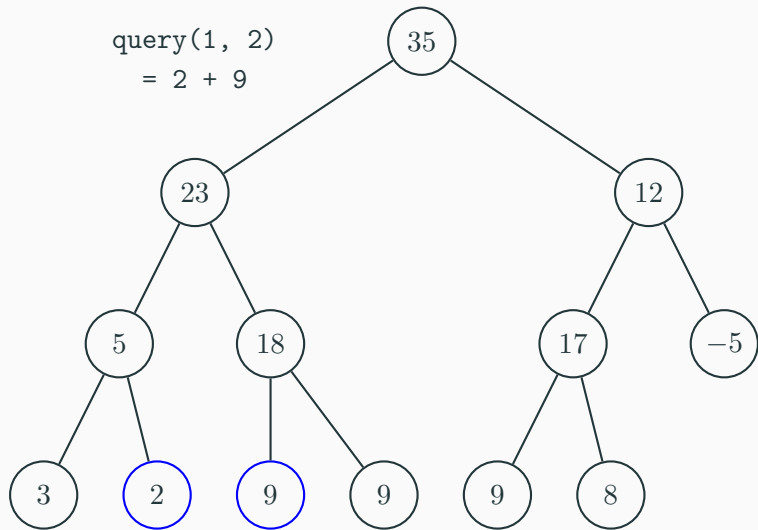
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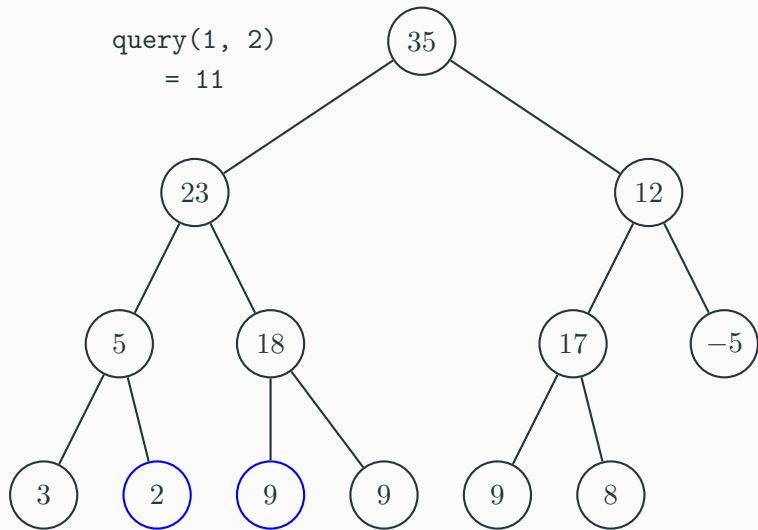
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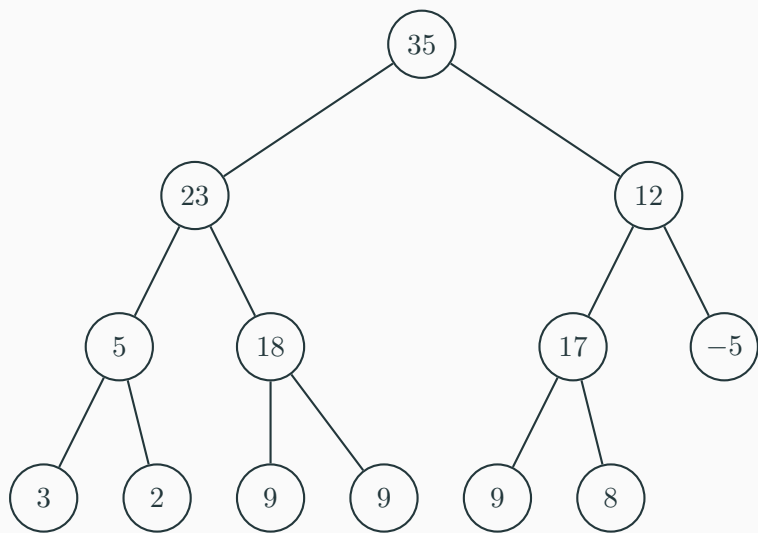
Querying



Querying



Querying



Querying a Segment Tree - Code

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int query(segment_tree *tree, int l, int r) {  
    if (tree == NULL) return 0;  
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Segment Tree

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- Any associative operator will work.
- So any operator f such that $f(a, f(b, c)) = f(f(a, b), c)$ for all a, b, c .
- Also possible to update a range of values in $O(\log n)$, which will be covered in bonus slides.

Example problem: Movie Collection

- <https://open.kattis.com/problems/moviecollection>

Another $\log(n)$ idea

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- Then to retrieve a sum from i to j we always take the biggest chunk we can that's stored at i , which will always be at least half.
- Then we continue until we reach j , moving i along and collecting the results.
- This is what is known as a sparse table.

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- Querying takes $\mathcal{O}(\log(n))$, however updating is slow and difficult.

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- Calculating all of these values takes $\mathcal{O}(n \log(n))$ because we can calculate the values in order of increasing j .
- Then when we calculate the sum of $[i, i + 2^j - 1]$ we just combine the earlier results of $[i, i + 2^{j-1} - 1]$ and $[i + 2^{j-1}, i + 2^j - 1]$.
- Querying takes $\mathcal{O}(\log(n))$, however updating is slow and difficult.
- Why would we then ever use this instead of segment trees?

Binary lifting

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- How might we use sparse tables to do better?

Binary lifting ctd.

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- Then we can compute these in increasing order of j , calculating $j = 1$ using f itself and then for larger j letting $f^{[2^j]}(x) = f^{[2^{j-1}]}(f^{[2^{j-1}]}(x))$
- Thus we can precompute the table in $\mathcal{O}(n(\mathcal{O}(f) + \log(n)))$ and each query takes $\mathcal{O}(\log(m))$, a much better time complexity

Sparse table example

7	1	6	4	8	0	9	2	2	7	1	6
---	---	---	---	---	---	---	---	---	---	---	---

$j = 0$

Sparse table example

7	1	6	4	8	0	9	2	2	7	1	6

$j = 1$

$j = 0$

Sparse table example

8											
7	1	6	4	8	0	9	2	2	7	1	6

$j = 1$

$j = 0$

Sparse table example

8	7										
7	1	6	4	8	0	9	2	2	7	1	6

$j = 1$

$j = 0$

Sparse table example

8	7	10									
7	1	6	4	8	0	9	2	2	7	1	6

$j = 1$

$j = 0$

Sparse table example

8	7	10	12								
7	1	6	4	8	0	9	2	2	7	1	6

$j = 1$

$j = 0$

Sparse table example

8	7	10	12	8							
7	1	6	4	8	0	9	2	2	7	1	6

$j = 1$

$j = 0$

Sparse table example

8	7	10	12	8	9						
7	1	6	4	8	0	9	2	2	7	1	6

$j = 1$

$j = 0$

Sparse table example

8	7	10	12	8	9	11					
7	1	6	4	8	0	9	2	2	7	1	6

$j = 1$

$j = 0$

Sparse table example

8	7	10	12	8	9	11	4				
7	1	6	4	8	0	9	2	2	7	1	6

$j = 1$

$j = 0$

Sparse table example

8	7	10	12	8	9	11	4	9			
7	1	6	4	8	0	9	2	2	7	1	6

$j = 1$

$j = 0$

Sparse table example

8	7	10	12	8	9	11	4	9	8		
7	1	6	4	8	0	9	2	2	7	1	6

$j = 1$

$j = 0$

Sparse table example

8	7	10	12	8	9	11	4	9	8	7	
7	1	6	4	8	0	9	2	2	7	1	6

$j = 1$

$j = 0$

Sparse table example

8	7	10	12	8	9	11	4	9	8	7	6
7	1	6	4	8	0	9	2	2	7	1	6

$j = 1$

$j = 0$

Sparse table example

18	19	18	21	19	13	20	12	16	14	7	6
↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑
8	7	10	12	8	9	11	4	9	8	7	6
7	1	6	4	8	0	9	2	2	7	1	6

$j = 2$

$j = 1$

$j = 0$

Sparse table example

37	32	38	33	35	27	27	18	16	14	7	6
↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑
18	19	18	21	19	13	20	12	16	14	7	6
8	7	10	12	8	9	11	4	9	8	7	6
7	1	6	4	8	0	9	2	2	7	1	6

$j = 3$

$j = 2$

$j = 1$

$j = 0$

Sparse table example

$$\text{query}(1, 8) = 19 + 9 + 2$$

37	32	38	33	35	27	27	18	16	14	7	6	$j = 3$
18	19	18	21	19	13	20	12	16	14	7	6	$j = 2$
8	7	10	12	8	9	11	4	9	8	7	6	$j = 1$
7	1	6	4	8	0	9	2	2	7	1	6	$j = 0$

Sparse table example

$$\text{query}(0, 9) = 37 + 9$$

37	32	38	33	35	27	27	18	16	14	7	6	$j = 3$
18	19	18	21	19	13	20	12	16	14	7	6	$j = 2$
8	7	10	12	8	9	11	4	9	8	7	6	$j = 1$
7	1	6	4	8	0	9	2	2	7	1	6	$j = 0$

Example problem: Stikl

- <https://open.kattis.com/problems/stikl>