Data Structures

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Today's material

- Built-in data structures and their applications
- Augmenting a data structure
- Union-Find
- Precomputations like prefix sums
- Square root decomposition
- Segment trees
- Sparse tables

Applications of Arrays and Linked Lists

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- Most problems require storing data, usually in an array
- On modern machines, arrays are almost always a better choice than a linked list
- There are however a few cases where linked lists are better

Example problem: Broken Keyboard

- http://uva.onlinejudge.org/external/119/11988.html

Applications of Stacks

- Processing events in a last-in first-out order

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- Simulating recursion

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- And a lot more

Applications of Queues

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Applications of Priority Queues

- Processing events in order of priority

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- Finding a shortest path in a graph

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Applications of Maps

- Associating a value with a key

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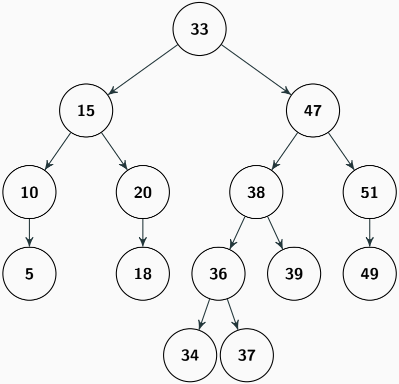
- Example: Augmenting binary search trees

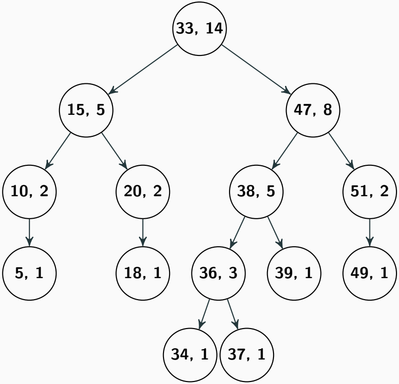
Augmenting Binary Search Trees

- We have a binary search tree and want to efficiently:
    - Count number of elements $< x$
    - Find the kth smallest element
- Naive method is to go through all vertices, but that is slow: $O(n)$



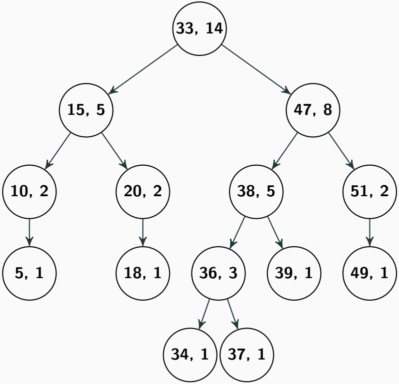
Augmenting Binary Search Trees

- Idea: In each vertex store the size of the subtree
- This information can be maintained when we insert/delete elements without increasing time complexity



Augmenting Binary Search Trees

- Count number of elements < 38
  - Search for 38 in the tree
  - Count the vertices that we pass by that are less than x
  - When we are at a vertex where we should go right, get the size of the left subtree and add it to our count



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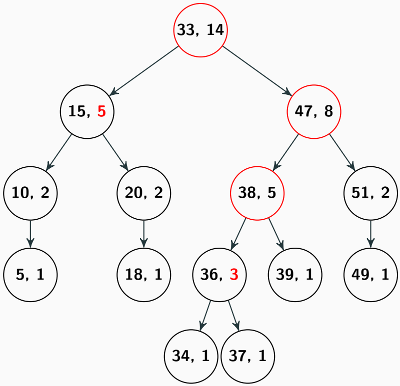
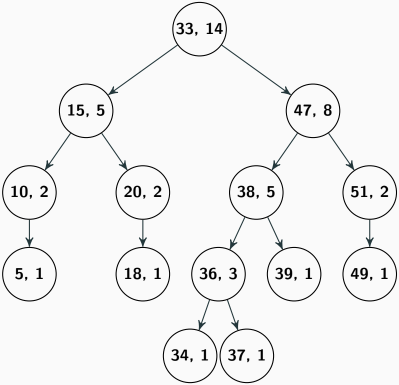


Augmenting Binary Search Trees

- Find kth smallest element
  - We're on a vertex whose left subtree is of size m
  - If $k = m + 1$, we found it
  - If $k \leq m$, look for the kth smallest element in the left subtree
  - If $k > m + 1$, look for the $k - m - 1$st smallest element in the right subtree



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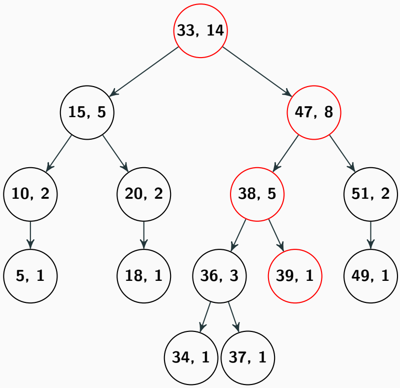
- Example: $k = 11$



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- Operation `union(x, y)` unions the sets of which x and y are members.

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- At any given point `find(x)` returns some value in the same set as x.
- The important bit is that `find(x)` returns the same value for all elements of the same set, the representative.

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- To get the representative of x we go to the parent of our current item (starting at x) until the item has no parent.
- Then to unite x, y we simply make the representative of x the parent of the representative of y.

Naïve Union-Find implementation

```
struct union_find {
    vector<int> parent;
    union_find(int n) {
        parent = vector<int>(n);
        for(int i = 0; i < n; i++) {
            parent[i] = i;
        }
    }
    int find(int x) {
        return parent[x] == x ? x : find(parent[x]);
    }
    void unite(int x, int y) {
        parent[find(x)] = find(y);
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- The key to making this more efficient is making those chains shorter.
- We do this by flattening the chain each time we query `find`, so the amortized complexity becomes good.
- The worst case is still $\mathcal{O}(n)$ but the amortized complexity is $\mathcal{O}(\alpha(n))$ which may as well be a constant, as it is < 5 for n equal to the number of atoms in the observable universe.

Path compressed Union-Find implementation

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        }
    }
    int find(int x) {
        if(parent[x] == x) return x;
        return parent[x] = find(parent[x]);
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    void unite(int x, int y) {
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- When tracking size you can use it to always perform small-to-large merges for $\mathcal{O}(\log n)$ time complexity.

Example problem: Skolavslutningen

- https://open.kattis.com/problems/skolavslutningen

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- We want to answer these queries efficiently, or in other words, without looking through all elements.
- Sometimes we also want to update elements.

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- Simplification: only support queries of the form $\mathrm{sum}(0, j)$
- Notice that $\mathrm{sum}(i, j) = \mathrm{sum}(0, j) - \mathrm{sum}(0, i-1)$

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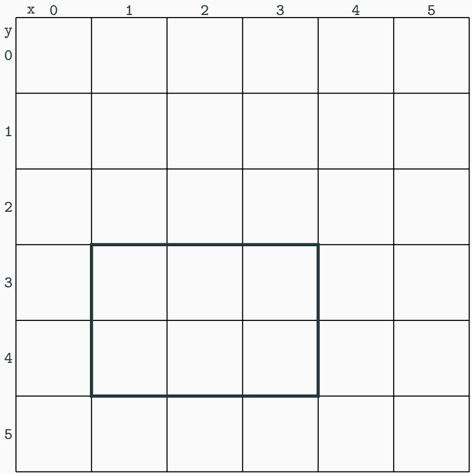
- We let $\mathrm{sum}(x_i, x_j, y_i, y_j)$ denote the query for the sum from x_i to x_j along the x-dimension, and the same for y.
- Then the formula becomes

$$\begin{aligned}\mathrm{sum}(x_i, x_j, y_i, y_j) &= \mathrm{sum}(0, x_j, 0, y_j) \\ &- \mathrm{sum}(0, x_{i-1}, 0, y_j) \\ &- \mathrm{sum}(0, x_j, 0, y_{i-1}) \\ &+ \mathrm{sum}(0, x_{i-1}, 0, y_{i-1})\end{aligned}$$

2D sum



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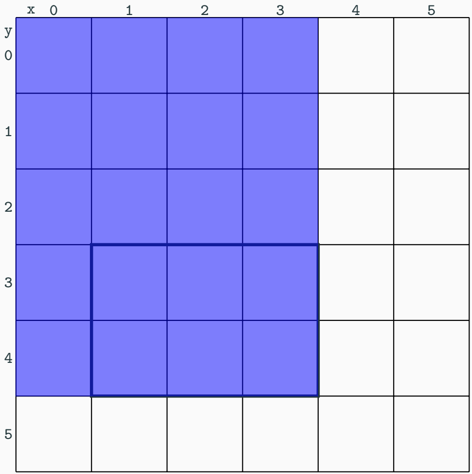

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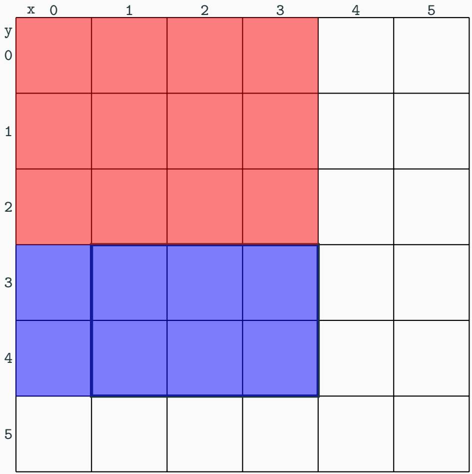
2D sum



`query(1, 3, 3, 4)`

`query(0, 3, 0, 4)`

`query(0, 4, 0, 2)`

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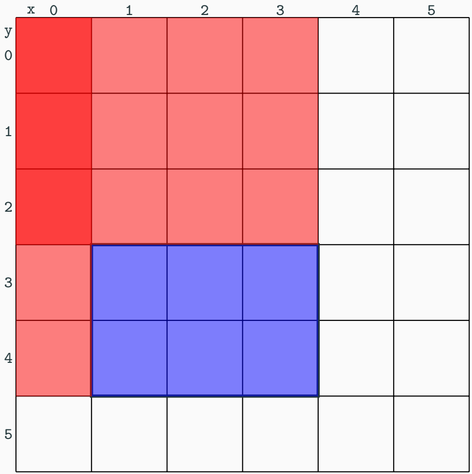

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query(0, 4, 0, 2)

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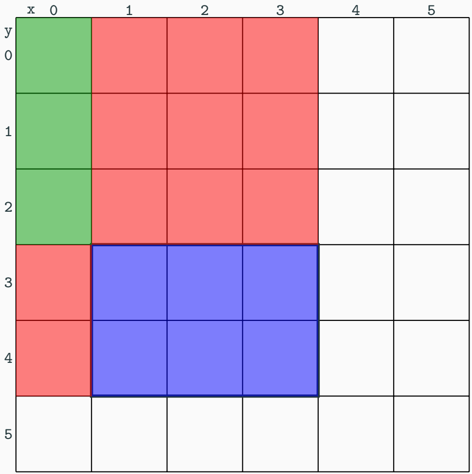


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```

Range sum on a mutable array

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    - updating an element

Range sum on a mutable array

- What if we want to support:
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- How do we support these queries efficiently?

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- Time complexity is minimized for $k = \sqrt{n}$:
  - Updating in $O(\sqrt{n})$
  - Querying in $O(n/\sqrt{n} + \sqrt{n}) = O(\sqrt{n})$
- Also known as square root decomposition, and is a very powerful technique

Example problem: Supercomputer

- https://open.kattis.com/problems/supercomputer

Range queries

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Range queries

- Now we know how to do these queries in $O(\sqrt{n})$
- May be too slow if n is large and many queries
- Can we do better?

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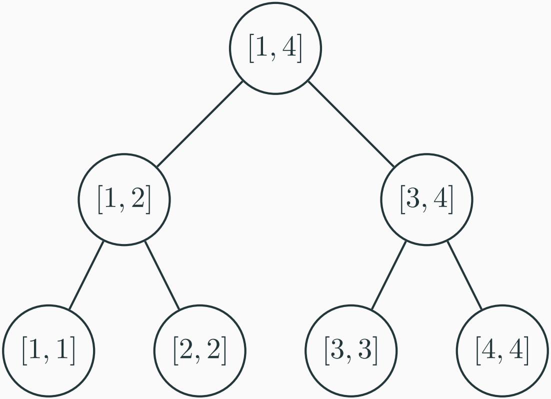
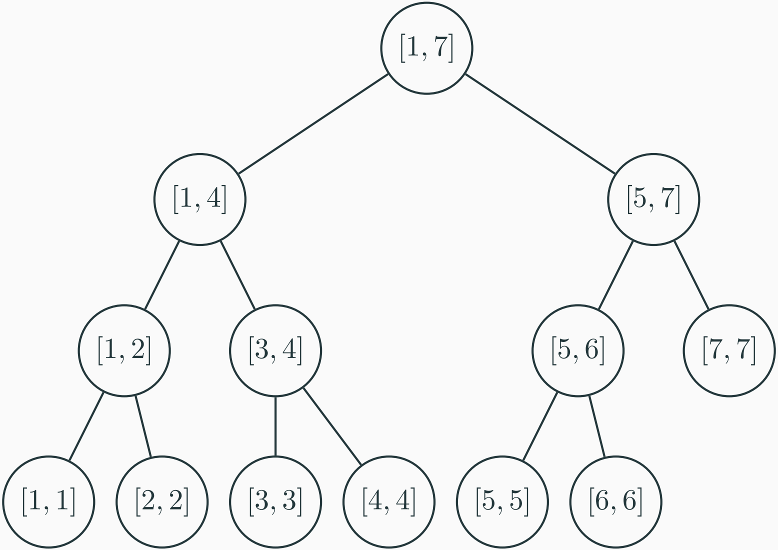
- Then we have $\mathcal{O}(n)$ nodes and each query can be pieced together from $\mathcal{O}(\log(n))$ node values.
- We travel down the tree looking for the left and right end points, adding intervals that are completely inside our query range.
- When we update a value we only need to update the parents of that node up to the root, at most $\mathcal{O}(\log(n))$ nodes.

Drawn Segment Tree, $n = 4$



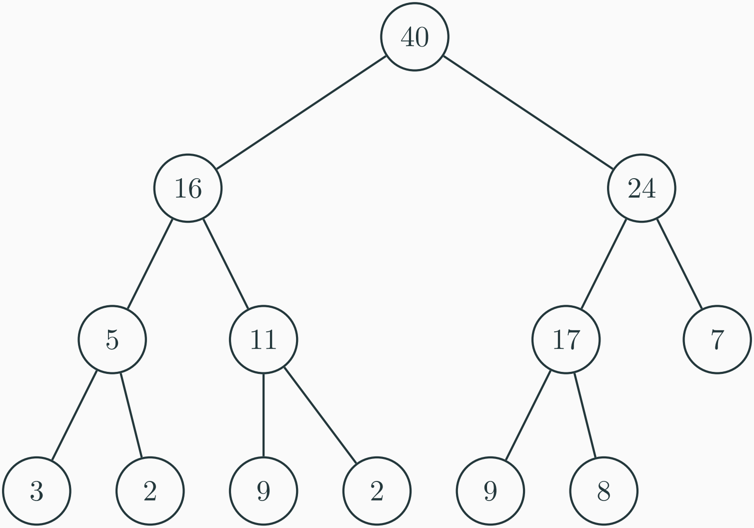
Drawn Segment Tree, $n = 7$



Segment Tree - Code

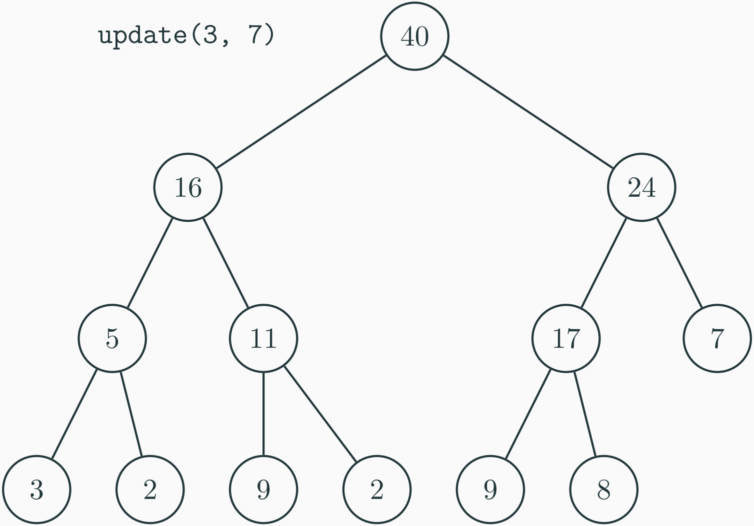
```
struct segment_tree {
    segment_tree *left, *right;
    int from, to, value;
    segment_tree(int from, int to)
        : from(from), to(to), left(NULL), right(NULL), value(0) { }
};

segment_tree* build(const vector<int> &arr, int l, int r) {
    if (l > r) return NULL;
    segment_tree *res = new segment_tree(l, r);
    if (l == r) {
        res->value = arr[l];
    } else {
        int m = (l + r) / 2;
        res->left = build(arr, l, m);
        res->right = build(arr, m + 1, r);
        if (res->left != NULL) res->value += res->left->value;
        if (res->right != NULL) res->value += res->right->value;
    }
    return res;
}
```

Updates



Updates

`update(3, 7)`



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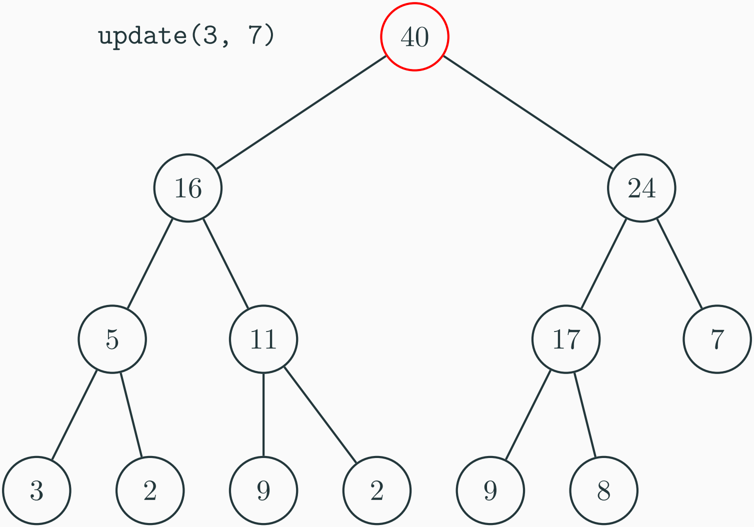
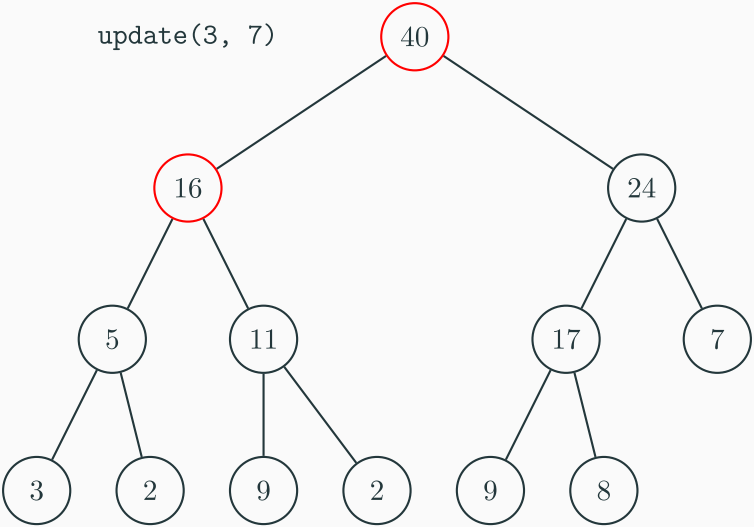


Updates



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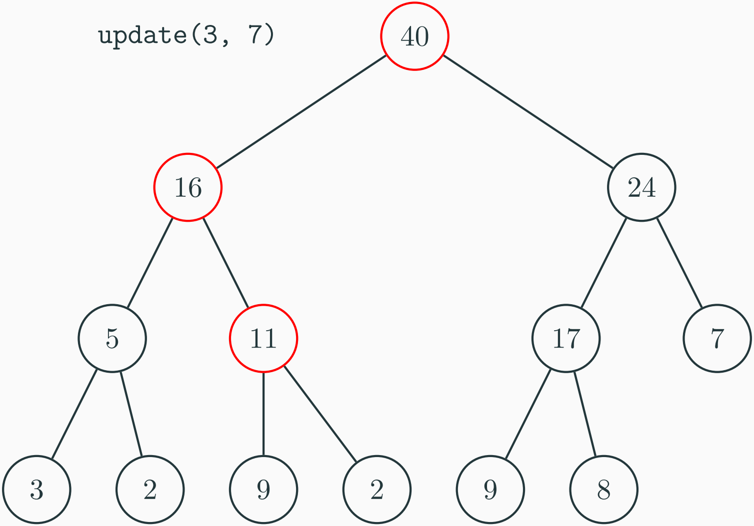
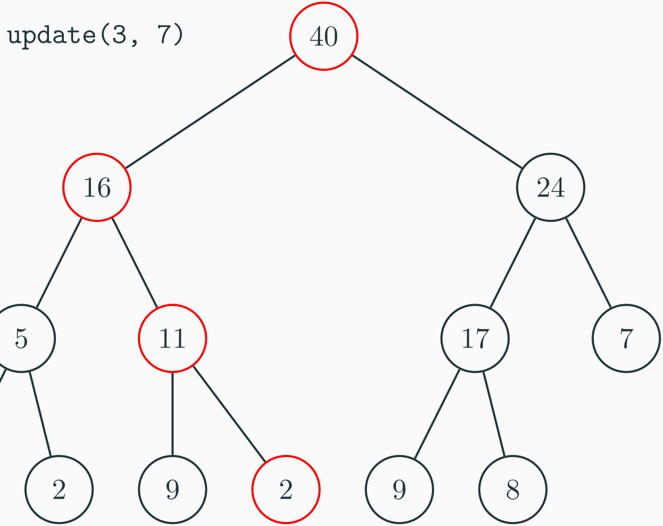


Updates



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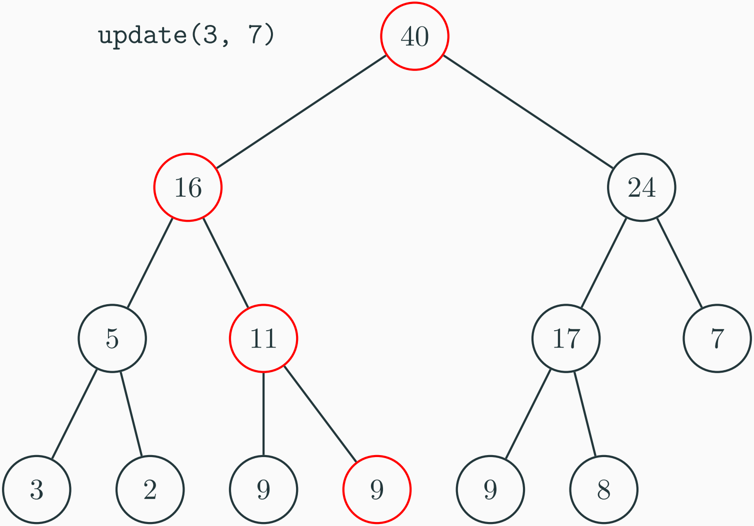

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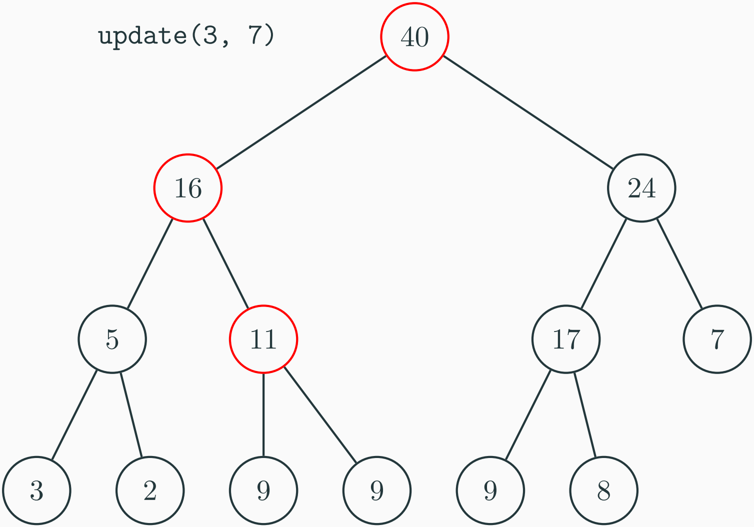

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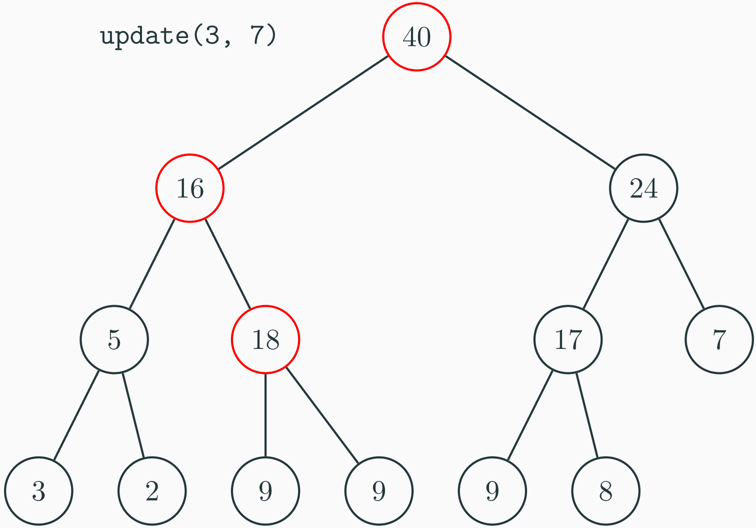

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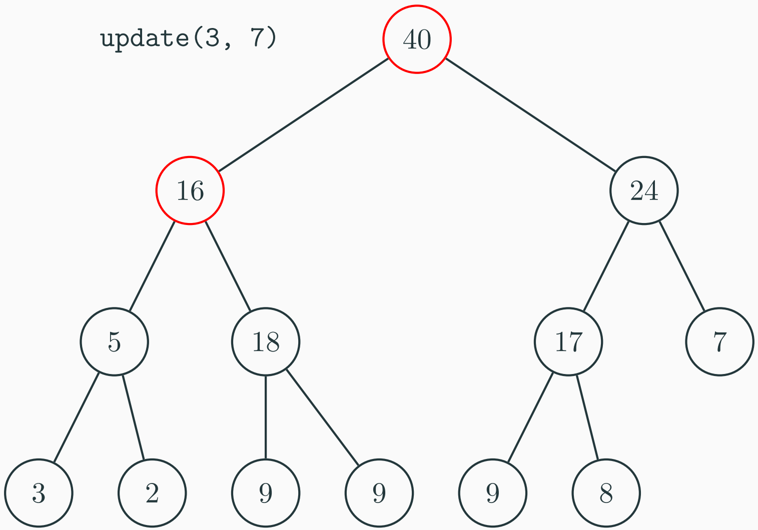

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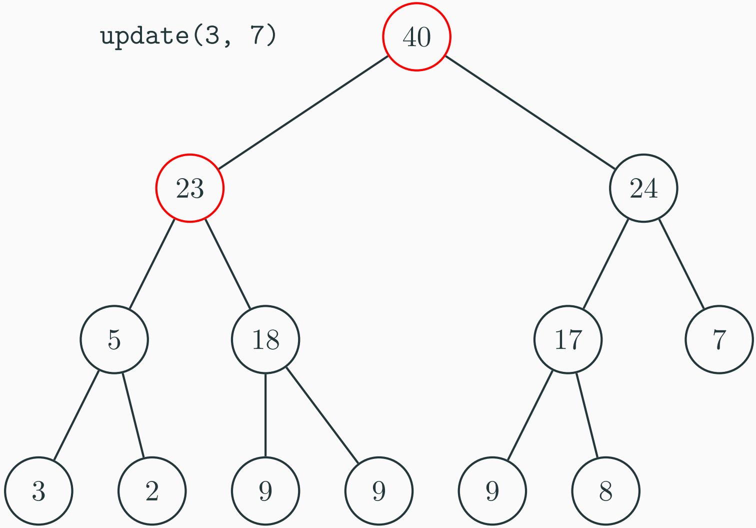

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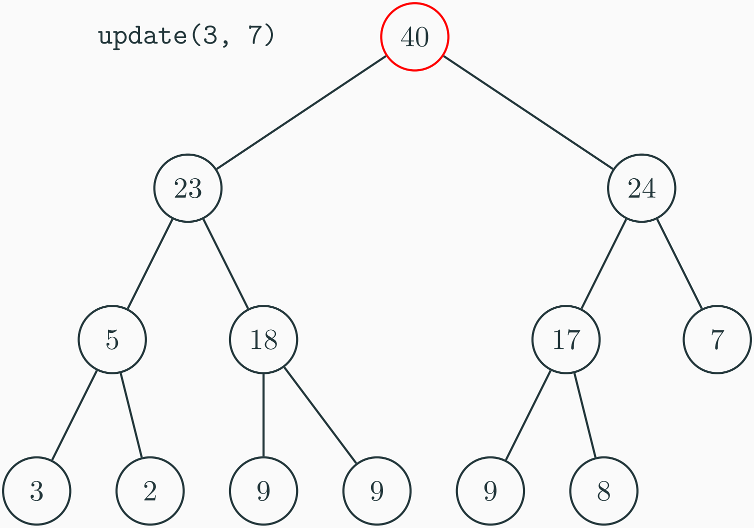

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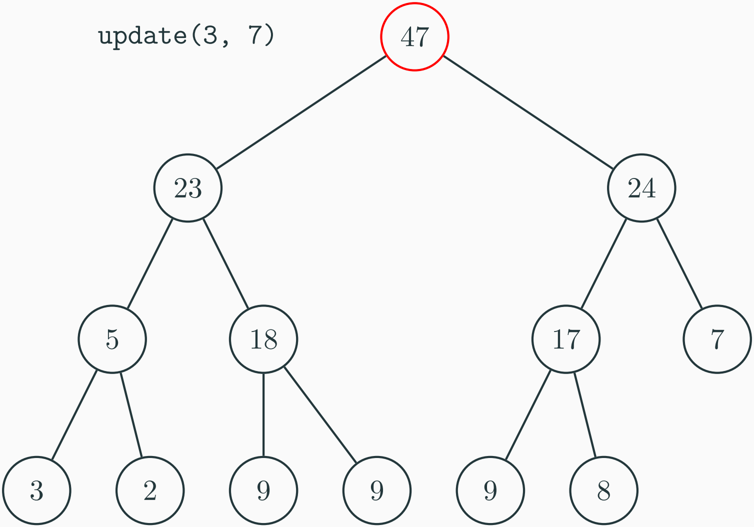

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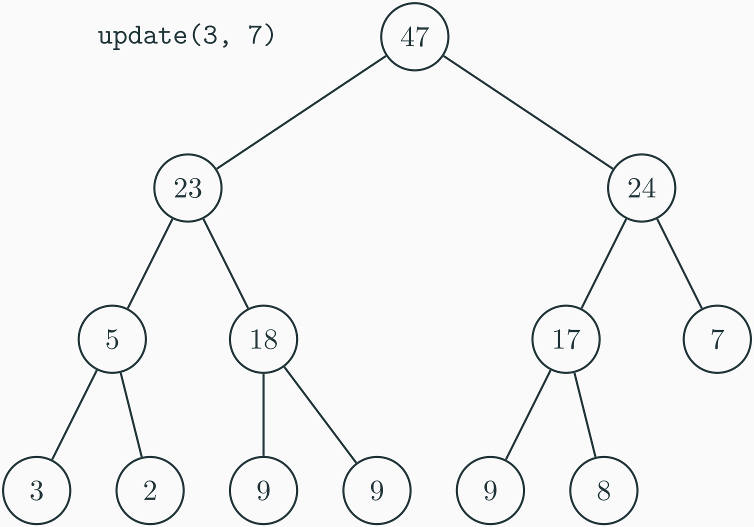

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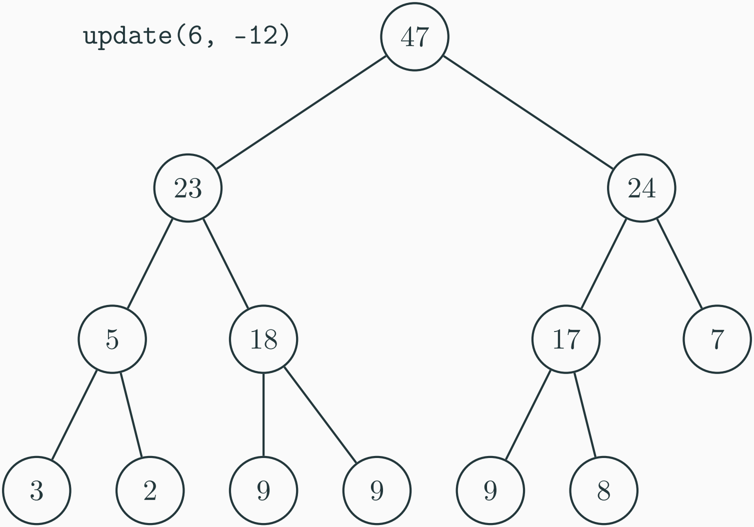

Updates

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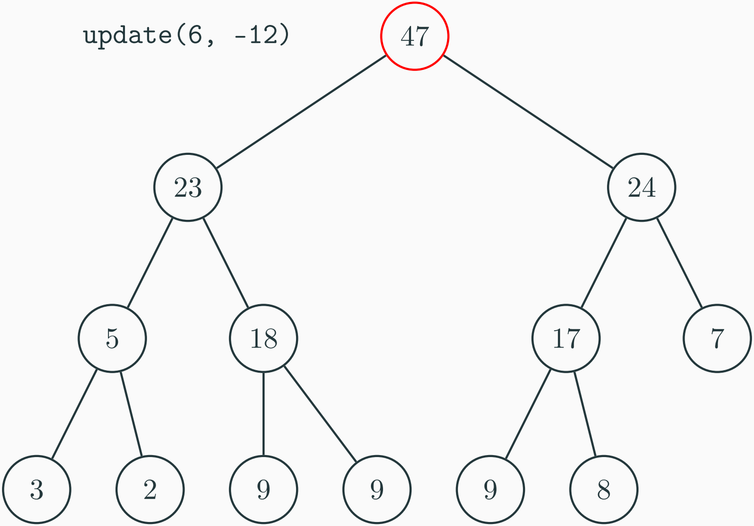

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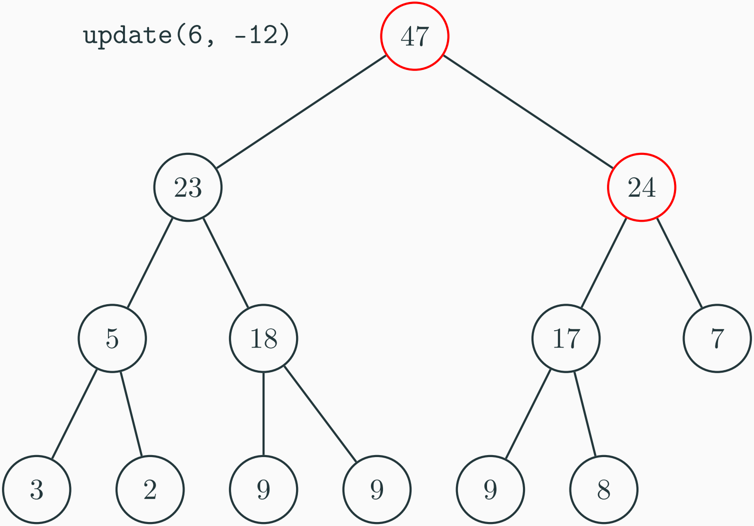

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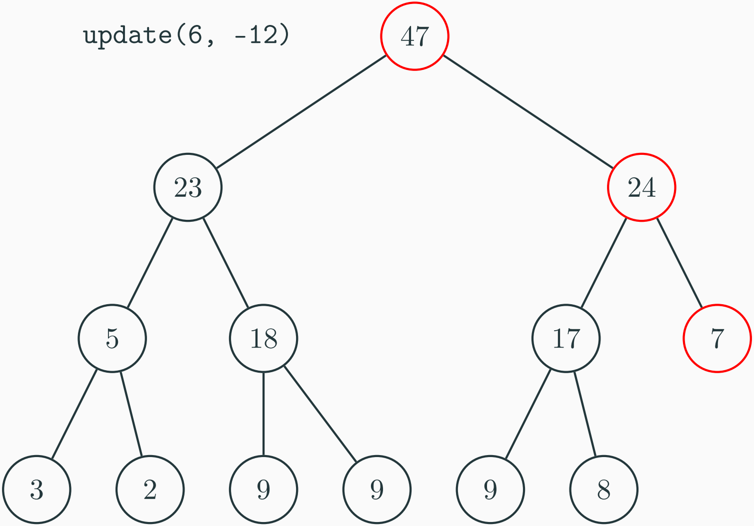

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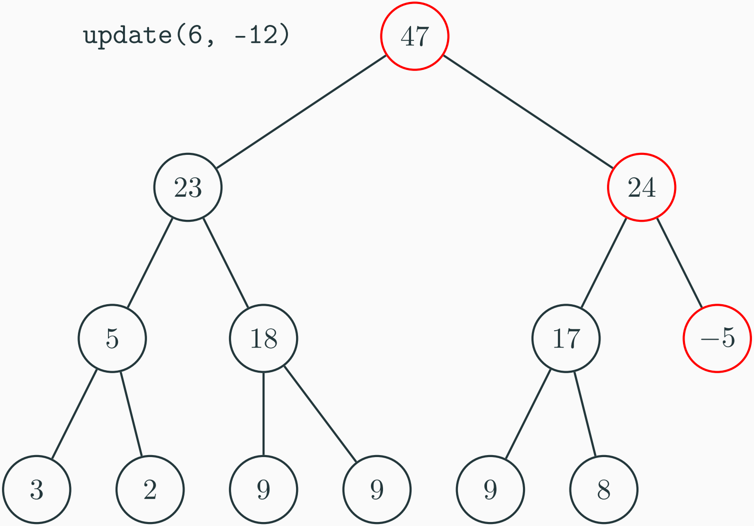
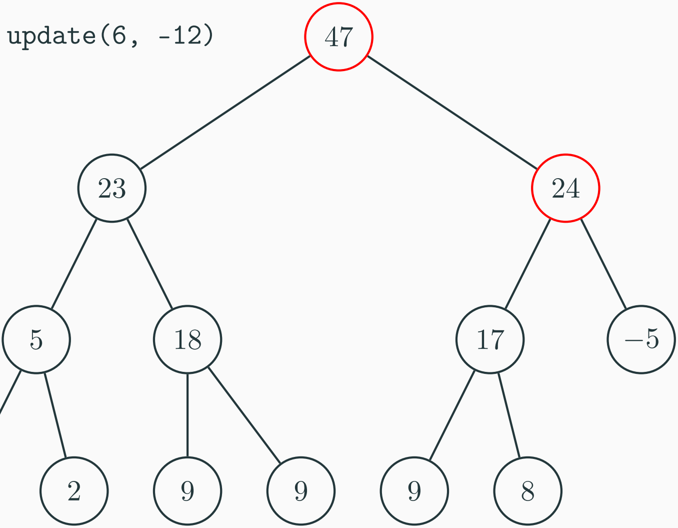


Updates



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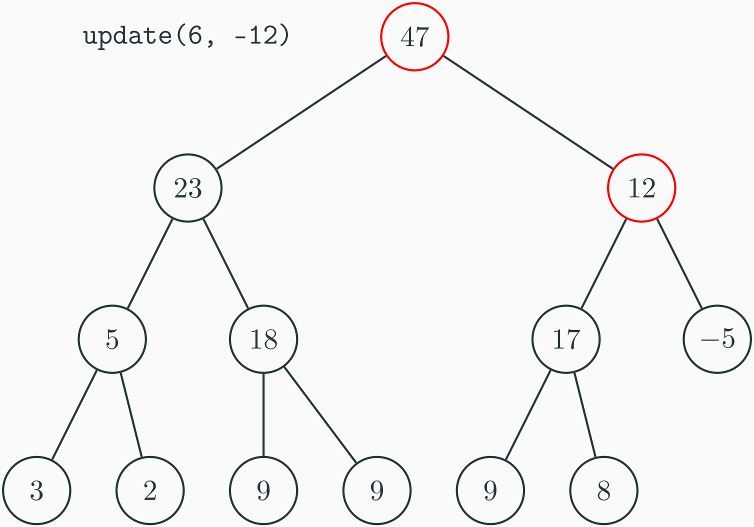

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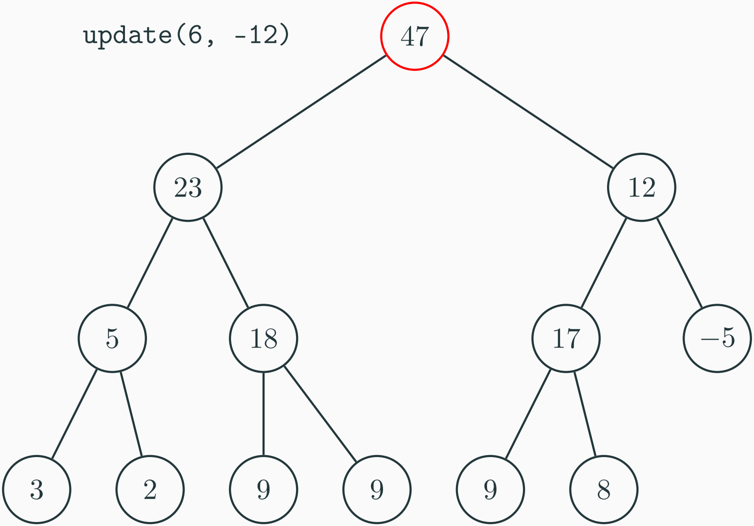

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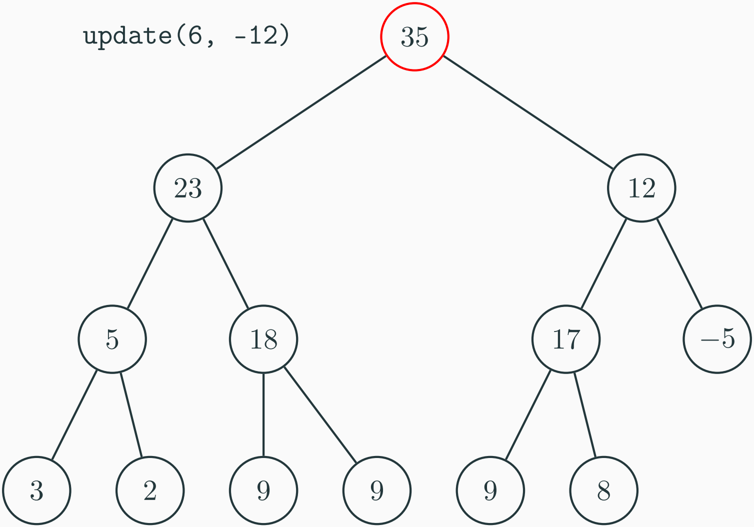

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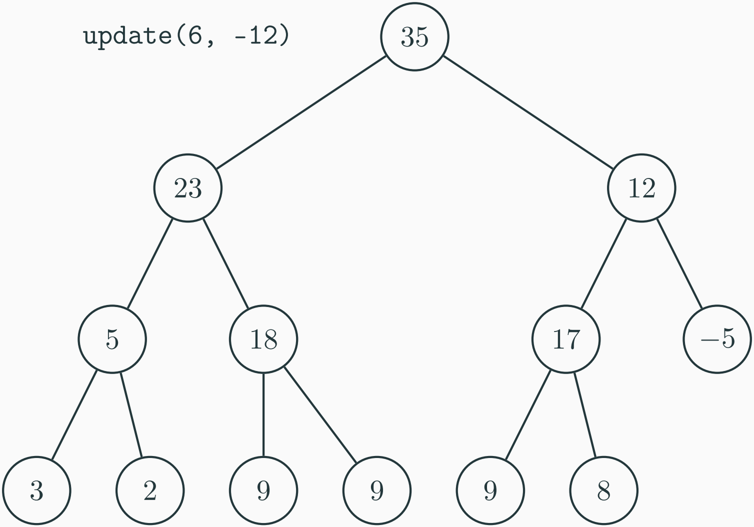

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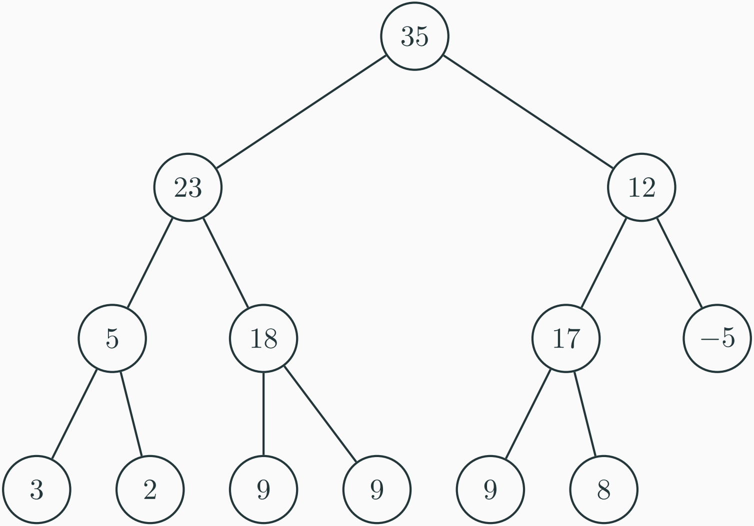
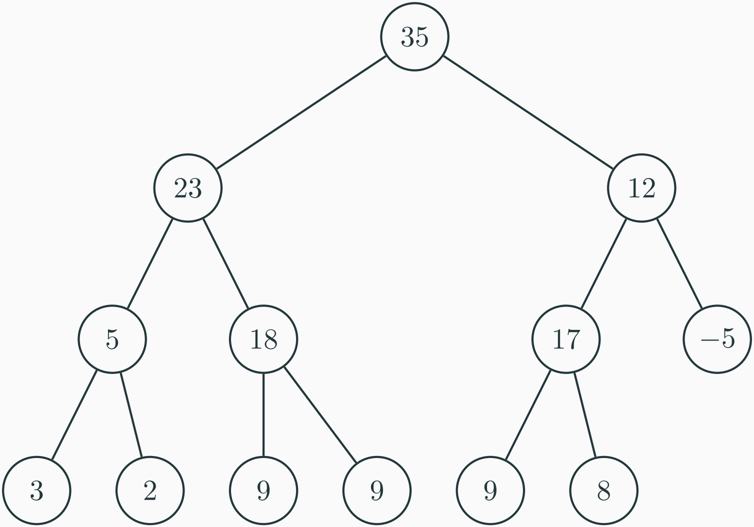


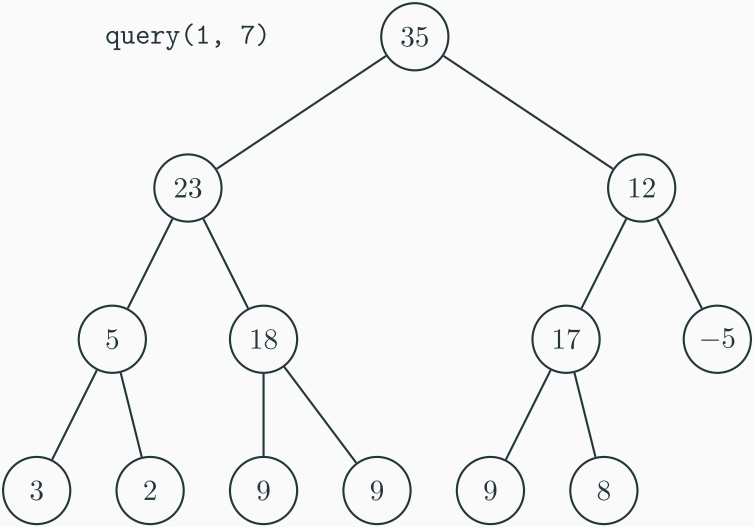
Updating a Segment Tree - Code

```
int update(segment_tree *tree, int i, int val) {
    if (tree == NULL) return 0;
    if (tree->to < i) return tree->value;
    if (i < tree->from) return tree->value;
    if (tree->from == tree->to && tree->from == i) {
        tree->value = val;
    } else {
        tree->value = update(tree->left, i, val) + update(tree->right, i, val);
    }
    return tree->value;
}
```

Querying

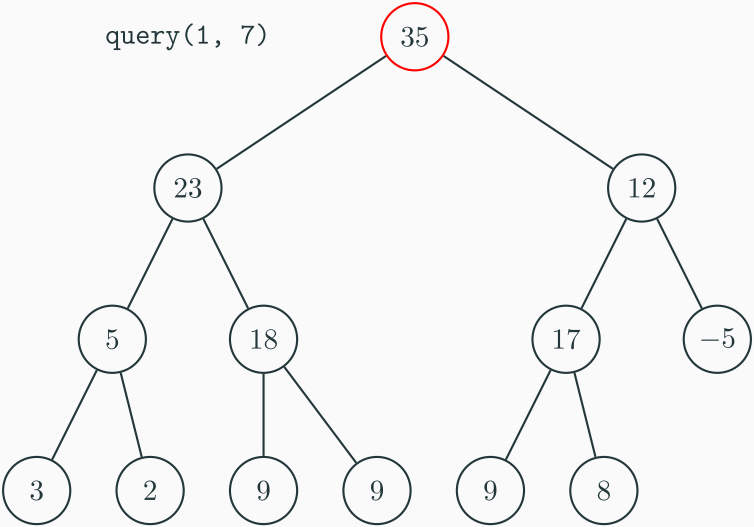


Querying

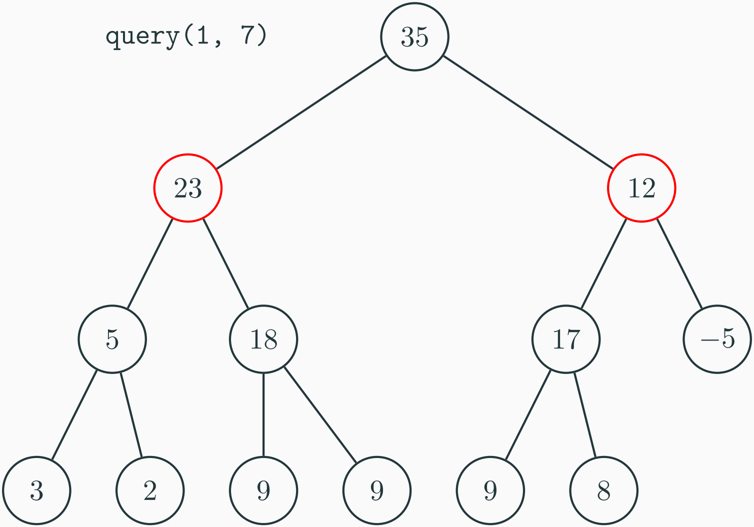


Querying

`query(1, 7)`



Querying



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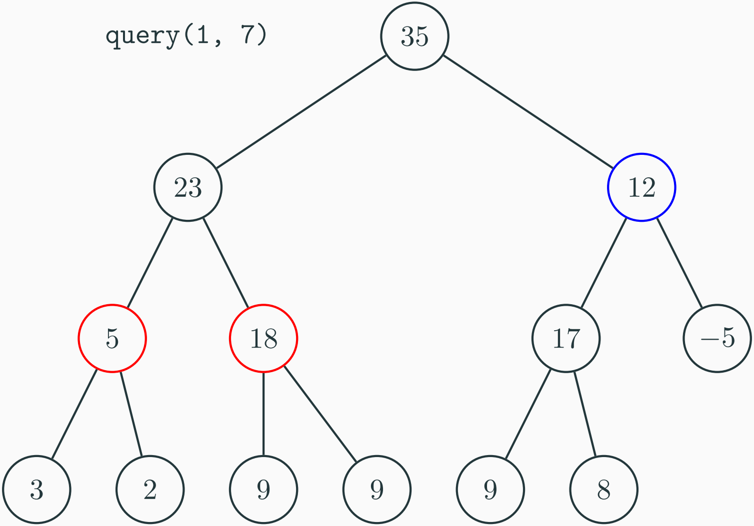
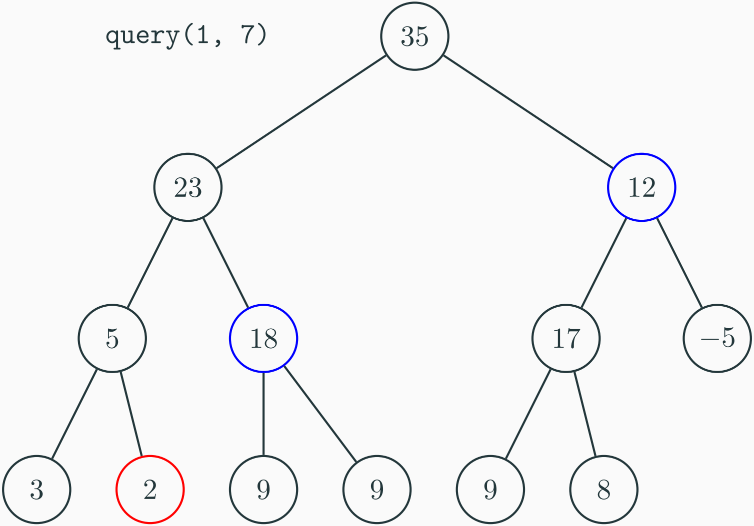
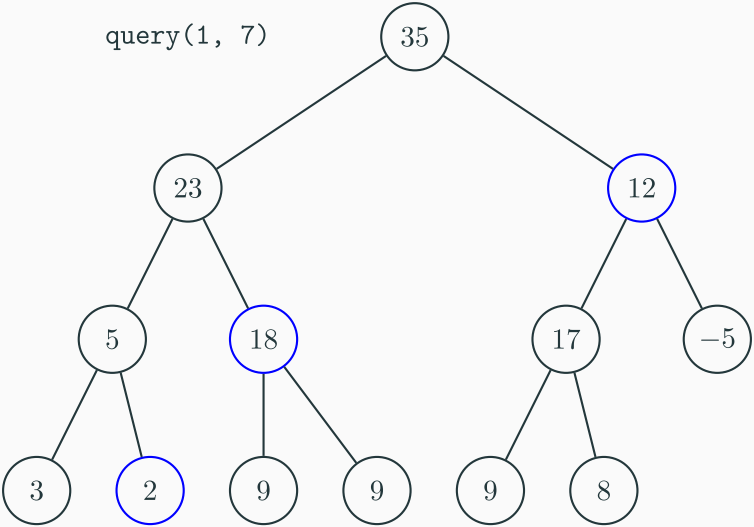


Querying

`query(1, 7)`

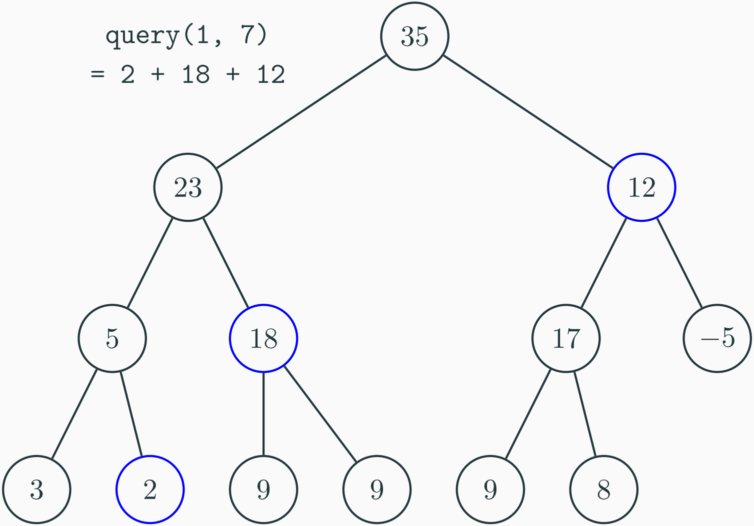


Querying

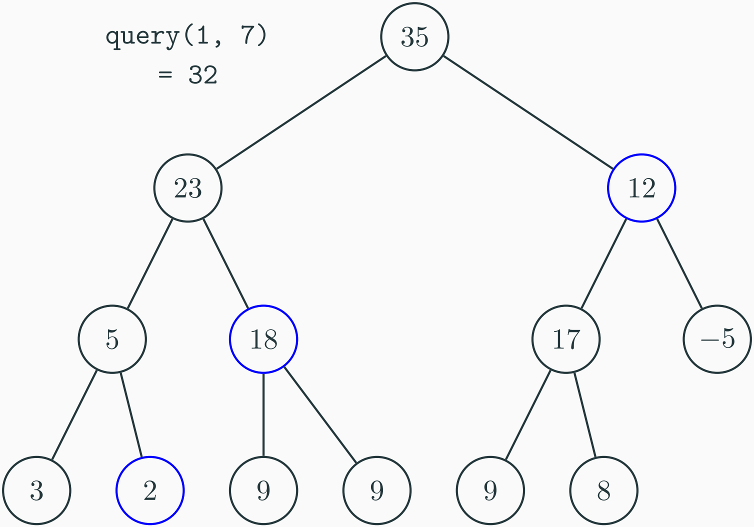


Querying

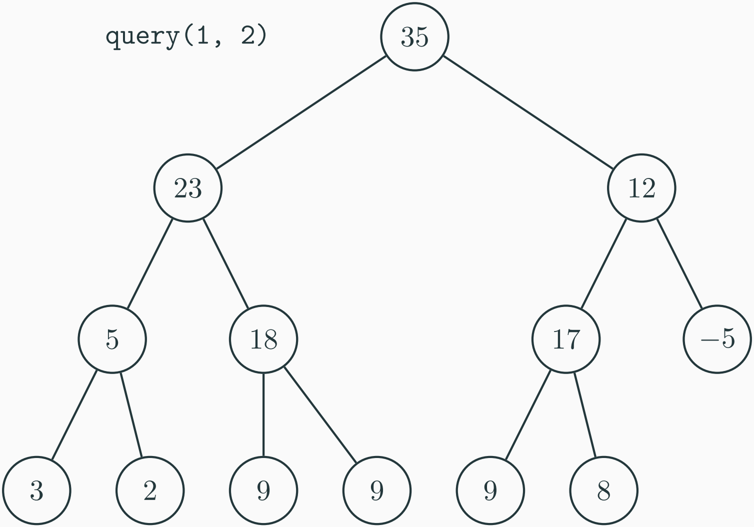
```
query(1, 7)
= 2 + 18 + 12
```



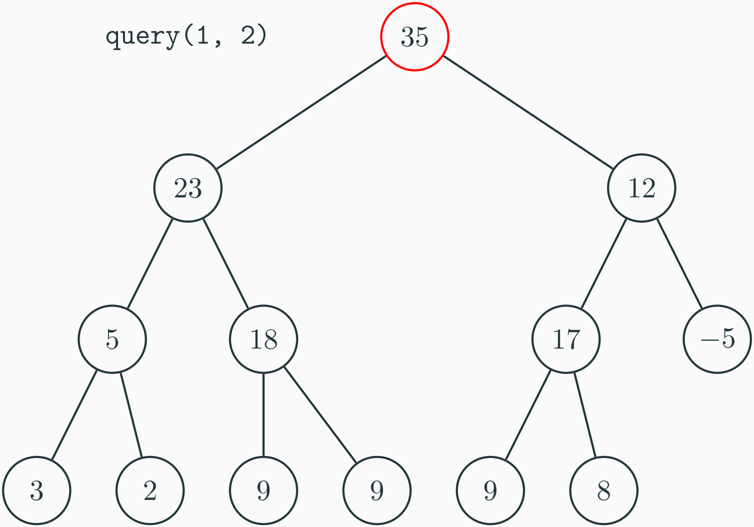
Querying



Querying

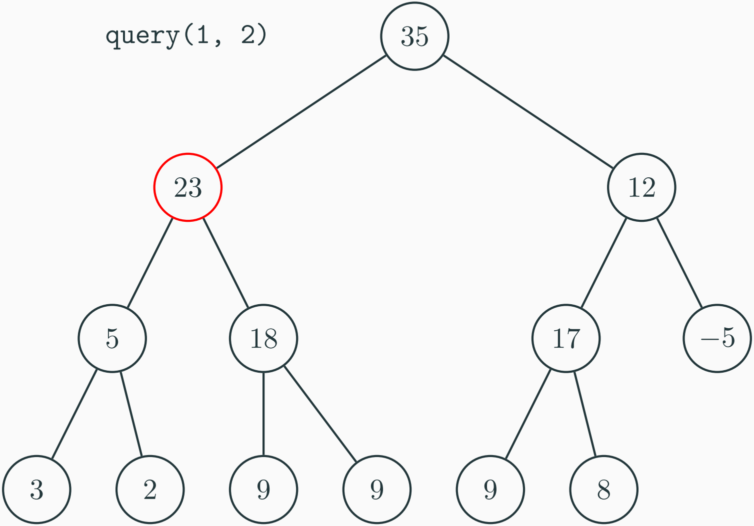


Querying

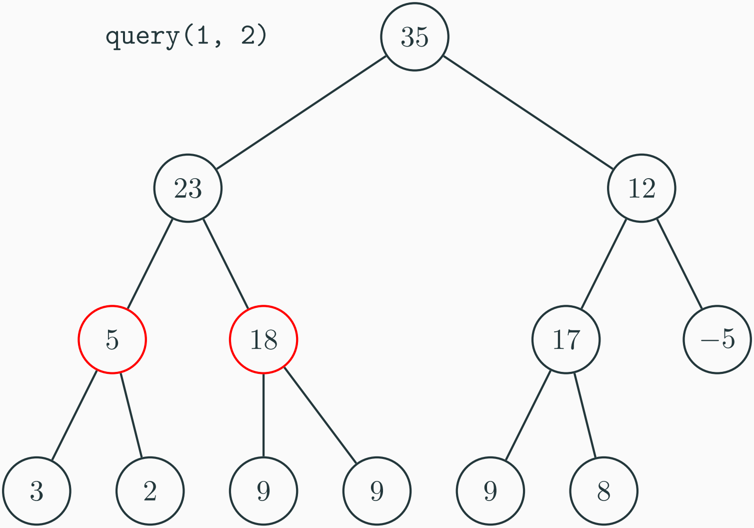


Querying

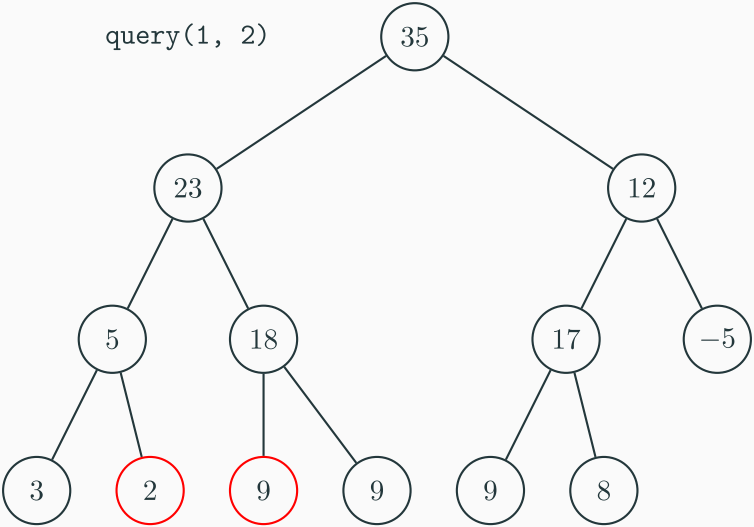
`query(1, 2)`



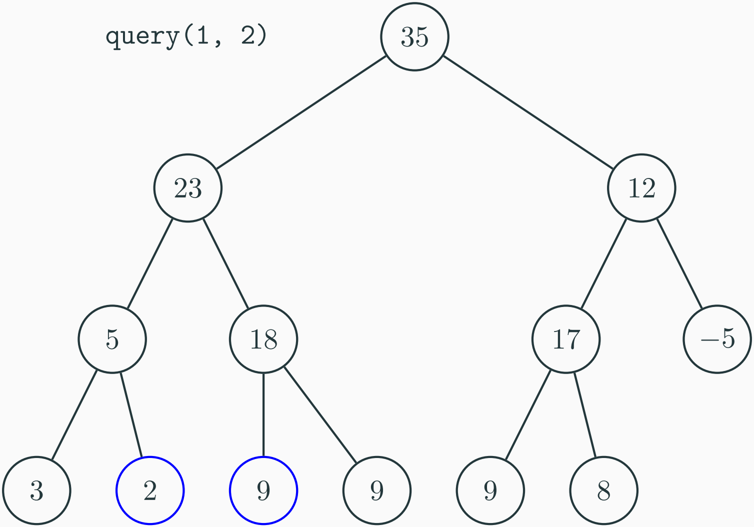
Querying



Querying

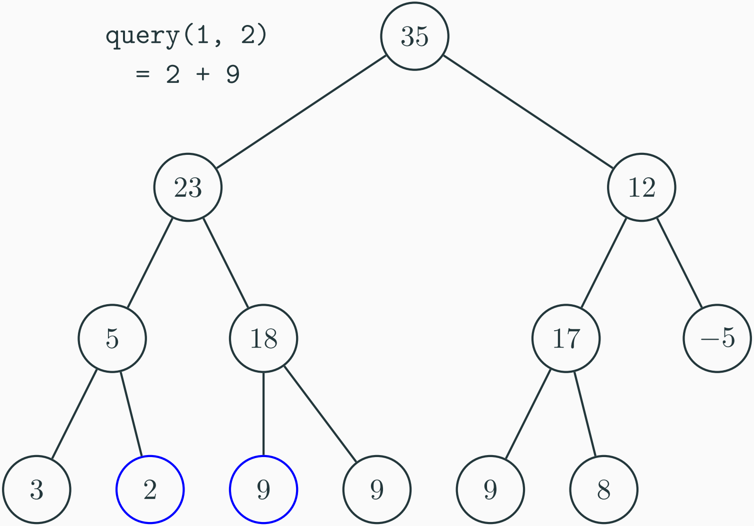


Querying



Querying

```
query(1, 2)
  = 2 + 9
```



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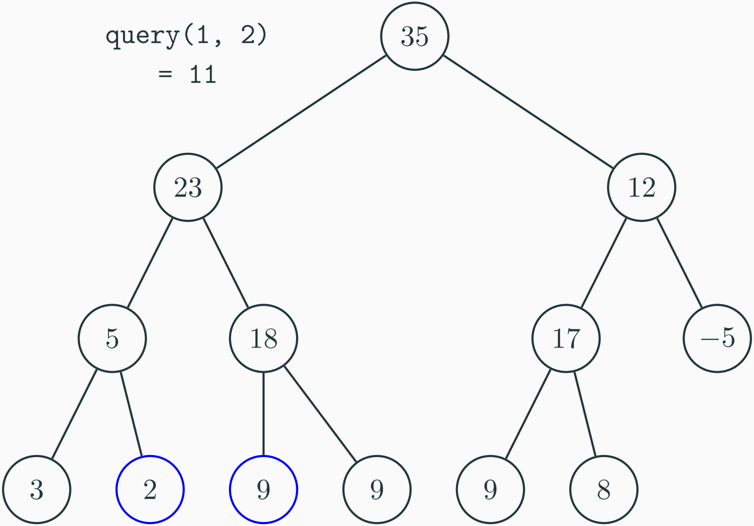

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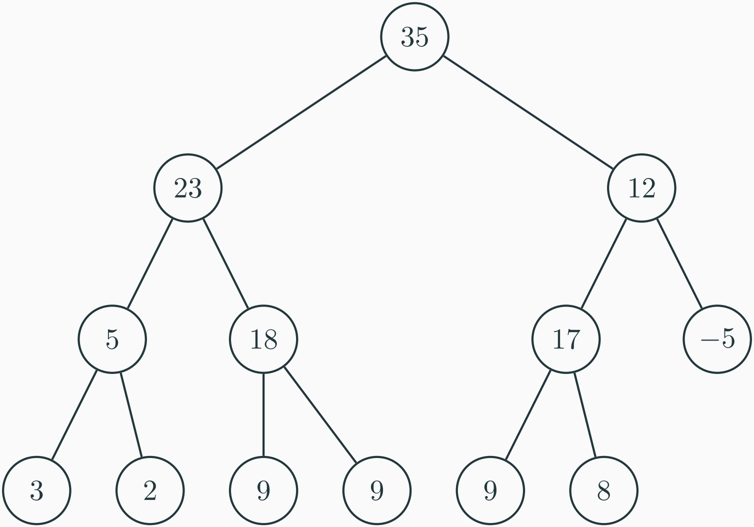

Querying a Segment Tree - Code

```
int query(segment_tree *tree, int l, int r) {
    if (tree == NULL) return 0;
    if (l <= tree->from && tree->to <= r) return tree->value;
    if (tree->to < l) return 0;
    if (r < tree->from) return 0;
    return query(tree->left, l, r) + query(tree->right, l, r);
}
```

Segment Tree

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- Any associative operator will work.
- So any operator f such that $f(a, f(b, c)) = f(f(a, b), c)$ for all a, b, c.
- Also possible to update a range of values in $O(\log n)$, which will be covered in bonus slides.

Example problem: Movie Collection

- https://open.kattis.com/problems/moviecollection

Another $\log(n)$ idea

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- Then to retrieve a sum from i to j we always take the biggest chunk we can that's stored at i, which will always be at least half.
- Then we continue until we reach j, moving i along and collecting the results.
- This is what is known as a sparse table.

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- Querying takes $\mathcal{O}(\log(n))$, however updating is slow and difficult.
- Why would we then ever use this instead of segment trees?

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- The naïve solution is to calculate it every time, giving a time complexity of $\mathcal{O}(qm\mathcal{O}(f))$.
- How might we use sparse tables to do better?

Binary lifting ctd.

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Binary lifting ctd.

- Let $f^{[y]}(x)$ denote the result of applying f exactly y times to x
- For each i we store $f^{[2^j]}(i)$ as a sparse table
- Then we can compute these in increasing order of j, calculating $j = 1$ using f itself and then for larger j letting $f^{[2^j]}(x) = f^{[2^{j-1}]}(f^{[2^{j-1}]}(x))$
- Thus we can precompute the table in $\mathcal{O}(n(\mathcal{O}(f) + \log(n)))$ and each query takes $\mathcal{O}(\log(m))$, a much better time complexity

Sparse table example

7	1	6	4	8	0	9	2	2	7	1	6	$j = 0$

Sparse table example

												$j = 1$
7	1	6	4	8	0	9	2	2	7	1	6	$j = 0$

Sparse table example

8												$j = 1$
7	1	6	4	8	0	9	2	2	7	1	6	$j = 0$

Sparse table example

8	7											$j = 1$
7	1	6	4	8	0	9	2	2	7	1	6	$j = 0$

Sparse table example

8	7	10										$j = 1$
7	1	6	4	8	0	9	2	2	7	1	6	$j = 0$

Sparse table example

8	7	10	12									$j = 1$
7	1	6	4	8	0	9	2	2	7	1	6	$j = 0$

Sparse table example

8	7	10	12	8								$j = 1$
7	1	6	4	8	0	9	2	2	7	1	6	$j = 0$

Sparse table example

8	7	10	12	8	9							$j = 1$
7	1	6	4	8	0	9	2	2	7	1	6	$j = 0$

Sparse table example

8	7	10	12	8	9	11						$j = 1$
7	1	6	4	8	0	9	2	2	7	1	6	$j = 0$

Sparse table example

8	7	10	12	8	9	11	4					$j = 1$
7	1	6	4	8	0	9	2	2	7	1	6	$j = 0$

Sparse table example

8	7	10	12	8	9	11	4	9				$j = 1$
7	1	6	4	8	0	9	2	2	7	1	6	$j = 0$

Sparse table example

8	7	10	12	8	9	11	4	9	8			$j = 1$
7	1	6	4	8	0	9	2	2	7	1	6	$j = 0$

Sparse table example

8	7	10	12	8	9	11	4	9	8	7		$j = 1$
7	1	6	4	8	0	9	2	2	7	1	6	$j = 0$

Sparse table example

8	7	10	12	8	9	11	4	9	8	7	6	$j = 1$
7	1	6	4	8	0	9	2	2	7	1	6	$j = 0$

Sparse table example

18	19	18	21	19	13	20	12	16	14	7	6	$j = 2$
8	7	10	12	8	9	11	4	9	8	7	6	$j = 1$
7	1	6	4	8	0	9	2	2	7	1	6	$j = 0$

Sparse table example

37	32	38	33	35	27	27	18	16	14	7	6	$j = 3$
18	19	18	21	19	13	20	12	16	14	7	6	$j = 2$
8	7	10	12	8	9	11	4	9	8	7	6	$j = 1$
7	1	6	4	8	0	9	2	2	7	1	6	$j = 0$

Sparse table example

`query(1, 8) = 19 + 9 + 2`

37	32	38	33	35	27	27	18	16	14	7	6	$j = 3$
18	19	18	21	19	13	20	12	16	14	7	6	$j = 2$
8	7	10	12	8	9	11	4	9	8	7	6	$j = 1$
7	1	6	4	8	0	9	2	2	7	1	6	$j = 0$

Sparse table example

`query(0, 9) = 37 + 9`

$\boxed{37}$	32	38	33	35	27	27	18	16	14	7	6	$j = 3$
18	19	18	21	19	13	20	12	16	14	7	6	$j = 2$
8	7	10	12	8	9	11	4	$\boxed{9}$	8	7	6	$j = 1$
7	1	6	4	8	0	9	2	2	7	1	6	$j = 0$

Example problem: Stikl

- https://open.kattis.com/problems/stikl