University Mohamed VI Polytechnic
Academic Year 2023-2024

A comparative study of some finite volume methods for the simulation of shallow water flows

Master QFM 1

Author: ATMANI Hanan

Supervised by:
Pr. EL MAHI Imad

July 15, 2024

Contents

List of Figures

List of Tables

Abstract

Introduction

In this project, we examine and compare various finite volume methods for simulating shallow water flows. Our study focuses on several key areas, beginning with the analysis of the Riemann problem in both scalar cases and linear PDE systems. We then explore the fundamental properties of finite volume schemes, including consistency and the conservative property, with particular emphasis on specific schemes such as Godunov, Harten-Lax-van Leer, and Roe. Additionally, we address TVD (Total Variation Diminishing) schemes to ensure the stability and accuracy of numerical solutions. We also apply these methods to the one-dimensional transport equation and Burgers' equation, using both the method of characteristics and finite volume schemes like HLL and Roe. The extension of these schemes to second-order spatial accuracy is also discussed. Finally, we model shallow water flows using the Saint-Venant equations, simulating phenomena such as dam breaks under various conditions. This project provides a detailed comparative analysis of finite volume methods, highlighting their advantages and limitations, and offering valuable insights for selecting appropriate numerical schemes for specific applications in fluid dynamics.

Chapter 1

Finite Volume Method and Analysis of Numerical Schemes

introduction:
This chapter delves into the numerical resolution of hyperbolic equations, focusing on the Riemann problem and associated finite volume schemes. We begin with an in-depth exploration of the Riemann problem, crucial for understanding free-surface flows and shock waves. Subsequently, we examine finite volume methods and their fundamental properties, such as consistency and conservation. We then study two specific schemes: the Harten-Lax-van Leer (HLL) scheme and the Roe scheme, their characteristics and application in solving hyperbolic equations. ;

I Riemann Problem:

Definition 1.1:
The Riemann problem consists of a hyperbolic equation with a discontinuous initial condition. It is written as:

$$\begin{cases} \partial_t u + \partial_x f(u) = 0, \\ u(0,x) = u_0(x) = \begin{cases} u_L & \text{if } x < 0, \\ u_R & \text{if } x > 0. \end{cases} \end{cases}$$

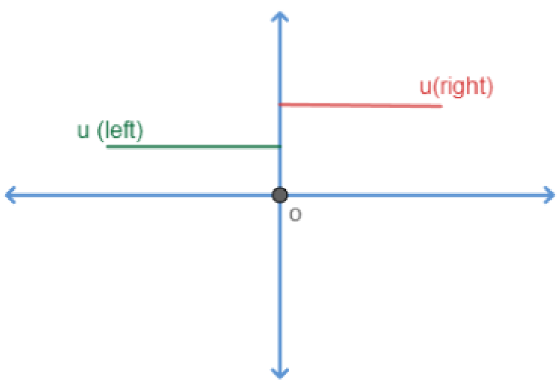

Figure 1.1: The Riemann's problem

In fact, we expect this discontinuity to propagate along the characteristic curves.

I.1 Scalar case: $f(u) = au$

The characteristic curves along which the PDE reduces to an ODE are defined by the integral curves of the Cauchy problem:

$$\begin{cases} \frac{dx}{dt} = a \\ x(0) = \xi \quad \text{with} \quad \xi \in \mathbb{R} \end{cases}$$

They are given by

$$x(t) = at + \xi, \quad \xi \in \mathbb{R}$$

Along these curves, we have $du = 0$, meaning $u(x,t) = cte$
So

$$\begin{aligned} u(x,t) &= u_0(x_0) \\ &= u_0(\epsilon) \\ &= u_0(x - at) \\ &= \begin{cases} u_L & \text{if} \quad x - at \leq 0 \\ u_R & \text{if} \quad x - at \geq 0 \end{cases} \\ &= \begin{cases} u_L & \text{if} \quad \frac{x}{t} \leq a \\ u_R & \text{if} \quad \frac{x}{t} \geq a \end{cases} \end{aligned}$$

The characteristic line $x = at$ is the only line through which the solution changes. It takes the value u_L to the left and u_R to the right.

I.2 Case of linear PDE system $f(u) = Au$:

The Riemann problem for a hyperbolic system with constant coefficients is written as

$$\begin{cases} \partial_t u + A\partial_x u = & 0 \\ u(0,x) & = u_0(x) = \begin{cases} u_l \text{ si } x \leq 0 \\ u_R \text{ si } x \geq 0 \end{cases} \end{cases}$$

Where $A \in \mathcal{M}_{N,N}(\mathbb{R})$, $u = \begin{bmatrix} u_1 \\ u_2 \\ . \\ . \\ . \\ u_N \end{bmatrix}$ and $x \in \mathbb{R}$.

The matrix A is diagonalizable and all its eigenvalues are real, so it can be written in the following form:

$$A = PDP^{-1}$$

where

$$D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_N \end{pmatrix}$$

$$P = \begin{pmatrix} C_1[1] & C_2[1] & \cdots & C_N[1] \\ C_1[2] & C_2[2] & \cdots & C_N[2] \\ \vdots & \vdots & \ddots & \vdots \\ C_1[N] & C_2[N] & \cdots & C_N[N] \end{pmatrix}$$

Each column vector C_i, $1 \leq i \leq N$ of the matrix P is a right eigenvector for the matrix A, that is $AC_i = \lambda C_i$.
P^{-1} matrix of eigenvectors on the left of A.

$$L_i^T A = \lambda_i L_i^T$$

$$P^{-1} = \begin{pmatrix} L_1[1] & L_1[2] & \cdots & L_1[N] \\ L_2[1] & L_2[2] & \cdots & L_2[N] \\ \vdots & \vdots & \ddots & \vdots \\ L_N[1] & L_N[2] & \cdots & L_N[N] \end{pmatrix}$$

Then we have

$$< L_i, C_j >= \delta_{ij} = \begin{cases} 1 \text{ si } i = j \\ 0 \text{ sinon} \end{cases}$$

Assuming $v = P^{-1}u$, the PDE system is written:

$$\frac{\partial v}{\partial t} + D\frac{\partial v}{\partial x} = 0$$

This system is called the canonical form or the characterization form of the system. It consists of N adviction equations

$$\frac{\partial v_i}{\partial t} + \lambda_i \frac{\partial v_i}{\partial x} = 0 \quad \text{avec} \quad i = 1, 2, ..., N$$

The solution to the i-th advection equation is:

$$v_i(x, t) = v_i(x(0), 0) = v = v_i^0(x - \lambda_i t)$$

Then,

$$v(x, t) = \begin{pmatrix} v_1^0(x - \lambda_1 t) \\ \vdots \\ \vdots \\ v_N^0(x - \lambda_N t) \end{pmatrix}$$

then the system solution is:

$$\begin{aligned} u(x, t) &= Pv(x, t) \\ &= \sum_{i=1}^{N} v_i(x, t)C_i \\ &= \sum_{i=1}^{N} v_i^0(x, \lambda_i t)C_i \end{aligned}$$

On the other hand, we have $v = P^{-1}u$.
i.e

$$v_i(x, t) =< L_i, u(x, t) >\Rightarrow v_i^0(x, t) =< L_i, u_0(x - \lambda_i t) >$$

Therefore

$$u(x, t) = \sum_{i=1}^{N} (l_p u_0(x - \lambda_i t))C_p$$

let's assume that the problem is strictly hyperbolic:

$$\lambda_1 < \lambda_2 < \lambda_3 < .... < \lambda_N$$

The solution of the Riemann problem consists of N waves emanating from the origin, each wave carrying a discontinuity with velocity λ_i. It is clear that the solution to the left of the λ_1 wave is simply the initial state u_L, and to the right of the λ_N wave is u_R. Therefore, we aim to find the soluandtion in the region between the λ_1 and λ_N waves.

The vectors $\{C_1, C_2, ..., C_N\}$ are linearly independent, so we can decompose the initial data as follows:

$$u_L = \sum_{i=1}^{N} \alpha_i C_i$$

$$u_R = \sum_{i=1}^{N} \beta_i C_i$$

We have

$$u(x,t) = \sum_{i=1}^{N} v_i^0(x - \lambda_i t) C_i \text{ and } v_i(x - \lambda_i t) = L_i u(x - \lambda_i t)$$

We obtain that

$$v_i^0(x - \lambda_i t) = \begin{cases} \alpha_i \text{ si } x - \lambda_i t x < 0 \\ \beta_i \text{ and } x - \lambda_i t > 0 \end{cases}$$

The solution is written as:

$$u(x,t) = \sum_{i/x-\lambda_i t<0} \alpha_i C_i + \sum_{i/x-\lambda_i t>0} \beta_i C_i$$

Let p denote the index of the eigenvalue λ_p such that

$$\begin{cases} x - \lambda_p t > 0 \\ x - \lambda_p t < 0 \end{cases}$$

The result is:

$$u(x,t) = \sum_{i=1}^{p} \alpha_i C_i + \sum_{i=p+1}^{N} \beta_i C_i$$

Proposition1.2.1:
the solution of the Riemann problem:

$$\begin{cases} \partial_t u + \partial_x f(u) = & 0 \\ u(0,x) & = u_0(x) = \begin{cases} u_L \text{ si } x < 0 \\ u_R \text{ si } x > 0 \end{cases} \end{cases}$$

It is self-similar (depends only on the ratio $\frac{x}{t}$), meaning that there exists a regular function g such that:

$$u(x,t) = g\left(\frac{x}{t}\right)$$

II Finite Volume Schemes and Basic Properties

although the cauchy problem cannot generally be studied theoretically

$$\begin{cases} \partial_t u + \partial_x f(u) = 0, & \forall t \in [0,T], \forall x \in]a,b[\\ u(0,x) = & u_0(x), \ \forall x \in [a,b] \end{cases} \tag{1.1}$$

we may need to study its numerical approximation using finite volumes (FV) schemas. unilike the scalar case, we currently do not know how to find bounds on the approximate solution to pass to the limit, however we will atteny to mimic the scalar casin the contradiction of FV schemas.

Let's discretize $[a, b]$ into N control volume $k_i =]x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}[$,
with $a = x_{\frac{1}{2}} < x_{\frac{3}{2}} < x_{\frac{5}{2}} < ... < x_{N+\frac{1}{2}} = b$.
We note

$$h_i = |k_i| = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}$$
$$h_{i+\frac{1}{2}} = x_{i+1} - x_i$$

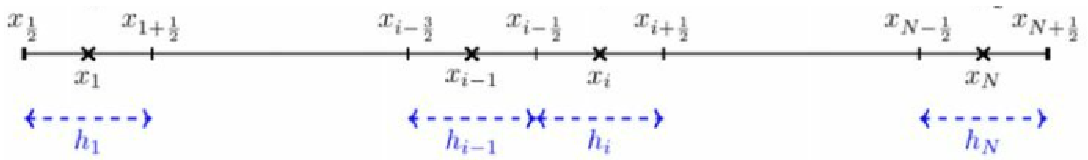

Figure 1.2: 1D Mesh for the Finite Volume Method

The function u we aim to approximate also depends on time. We discretize the interval $[0, T]$ into M intervals , where each interval $[t^n, t^{n+1}]$ has a step size of $\Delta t = \frac{T}{M}$, with $t^0 = 0$, $t^1 = \Delta t$, $t^2 = 2\Delta t$,...,$t^n = n\Delta t$.

Finite Volume Formulation:

We integrate PDE(1) over a control volume k_i.

$$\int_{k_i} \frac{\partial u(x,t)}{\partial t} dx + \int_{k_i} \frac{\partial f(u(x,t))}{\partial x} dx = 0$$

Let $u_i(t)$ denote the average over the control volume k_i:

$$u_i(t) = \frac{1}{|k_i|} \int_{k_i} u(x,t)\, dx$$

So

$$\int_{k_i} \frac{\partial u(x,t)}{\partial t} dx = \frac{\partial u_i(t)}{\partial t} h_i$$

$$\begin{aligned}\int_{k_i} \frac{\partial f(u)}{\partial x} dx &= \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \frac{\partial f(u)}{\partial x} dx \\ &= f(u(x_{i+\frac{1}{2}}, t)) - f(u(x_{i-\frac{1}{2}}, t))\end{aligned}$$

The finite volume formulation is then written as:

$$\frac{\partial u_i(t)}{\partial t} + \frac{1}{h_i}(f(u(x_{i+\frac{1}{2}}, t)) - f(u(x_{i-\frac{1}{2}}, t))) = 0$$

$f(u(x_{i+\frac{1}{2}}, t))$ and $f(u(x_{i-\frac{1}{2}}, t))$ represent the exact fluxes on the interfaces $x_{i+\frac{1}{2}}$ and $x_{i-\frac{1}{2}}$ of the control volume k_i. These fluxes are approximated by the numerical fluxes $F_{i+\frac{1}{2}}$ and $F_{i-\frac{1}{2}}$.

$$F_{i+\frac{1}{2}} \simeq f(u(x_{i+\frac{1}{2}}, t))$$
$$F_{i-\frac{1}{2}} \simeq f(u(x_{i-\frac{1}{2}}, t))$$

If an explicit scheme is used for time integration, meaning

$$\frac{\partial u_i}{\partial t}(t^n) \approx \frac{u_i^{n+1} - u_i^n}{\Delta t}$$

and $F_{i+\frac{1}{2}}$ and $F_{i-\frac{1}{2}}$ are evaluated at time t^n, $(F^n_{i+\frac{1}{2}}, F^n_{i-\frac{1}{2}})$.
Here, u_i^n represents the approximate solution over control volume k_i at time t^n.
Then, the following finite volume scheme is obtained:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{F^n_{i+\frac{1}{2}} - F^n_{i-\frac{1}{2}}}{h_i} = 0$$

with

$$F^n_{i+\frac{1}{2}} \simeq \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} f(u(x_{i+\frac{1}{2}}, t))dt$$

Illustration

In hyperbolic systems, disturbances or information do not propagate instantaneously but at a certain speed. This means that the flux at position $x_{i+\frac{1}{2}}$ at time n mainly depends on the solution values at positions i and $i+1$ (distant points do not yet have the information that arrives at $x_{i+\frac{1}{2}}$ at time n). Therefore, we can say that $F^n_{i+\frac{1}{2}} = F(u_i^n, u_{i+1}^n)$.
The scheme becomes:

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{h_i}(F(u_i^n, u_{i+1}^n) - F(u_{i-1}^n, u_i^n)) \tag{1.2}$$

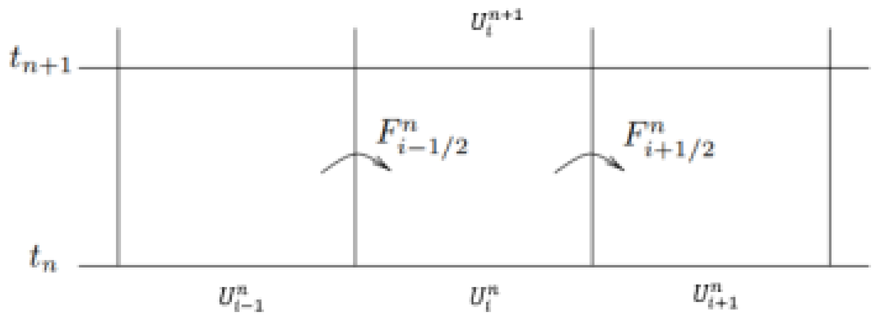

Figure 1.3: Numerical fluxes at the cell boundaries.

Note
All finite volume schemes depend on the choice of the function F.

II.1 Consistency and Conservative Property of Numerical Schemes

II.2 Conservative Property

Consider a finite volume discretization in the form:

$$u_i^{n+1} = H(u^n_{i-q}, ..., u^n_i, ..., u^n_{i+q}),$$

with

$$H : \mathbb{R}^{2q+1} \longrightarrow \mathbb{R}$$

a continuous function.
Definition2.1.1:
The scheme is said to be conservative if it can be written in the form

$$u_i^{n+1} = u_i^n - \lambda(F^n_{i+\frac{1}{2}} - F^n_{i-\frac{1}{2}})$$

where

$$F_{i-\frac{1}{2}} = F(u_{i-q}, ..., u_i, u_{i+q-1})$$
$$F_{i+\frac{1}{2}} = F(u_{i-q+1}, ..., u_i, u_{i+q})$$

and $F : \mathbb{R}^{2q} \longrightarrow \mathbb{R}$ is a continuous function. It defines the numerical flux of the conservative scheme.

Example 2.1.1:

The three-point finite volume scheme (1.2) is a conservative scheme ($q = 1$) when h_i is constant, i.e., $h_i = h = \text{const}$.

II.3 Consistency

Consistency expresses the idea that the numerical flux $F^n_{i+\frac{1}{2}}$ at $x_{i+\frac{1}{2}}$ should approach the exact flux as Δu_i, $\Delta t \to 0$.

$$\begin{aligned} F^n_{i+\frac{1}{2}} &= F(u^n_i, u^n_{i+1}) \\ &\approx \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} f(u(x_{i+\frac{1}{2}}, t))\, dt \end{aligned}$$

In particular, in the scalar case, i.e., $u(x_i) = \bar{u} = \text{const}$, then

$$\Delta t \int_{t^n}^{t^{n+1}} f(u(x_{i+\frac{1}{2}}, t))\, dt = \Delta t \int_{t^n}^{t^{n+1}} f(\bar{u})\, dt = f(\bar{u})$$

It follows that if $u^n_i, u^{n+1}_i \longrightarrow \bar{u}$, then $F(u^n_i, u^{n+1}_i) \longrightarrow f(\bar{u})$.
Therefore, we impose that

$$F(\bar{u}, \bar{u}) = f(\bar{u})$$

Definition2.1.2:

The conservative finite volume scheme (1.2) is consistent if for all u, we have

$$F(u, u, \ldots, u) = f(u)$$

II.4 Godunov scheme

the idea of the godunov method is to calculate the numerical flux $F_{i+\frac{1}{2}}$ on each interface $x_{i+\frac{1}{2}}$ by solving a localized Riemann problem in $x_{i+\frac{1}{2}}$.

$$\begin{cases} \partial_t u + \partial_x f(u) = & 0 \\ u(0, x) \quad = u_0(x) = & \begin{cases} u_L & \text{if } x \le x_{i+\frac{1}{2}} \\ u_R & \text{if } x \ge x_{i+\frac{1}{2}} \end{cases} \end{cases}$$

The Godunov diagram is written in conservative form:

$$u^{n+1}_i = u^n_i - \frac{\Delta t}{\Delta x}(F^n_{i+\frac{1}{2}} - F^n_{i-\frac{1}{2}})$$

with

$$F_{i+\frac{1}{2}} = f(u_{i+\frac{1}{2}}(0))$$

where $u_{i+\frac{1}{2}}(0)$ is the self-similar solution of the Riemann problem in 0.

The solution to the Riemann problem is often complicated and we are often led to solve the Riemann problem in an approximate way:
There are solvers, known as approximate Riemann solvers, which allow us to easily find an approximate solution to the Riemann problem. These include HLL and Roe.

II.5 Harten, Lax, and van Leer Scheme

The solver designed by Harten, Lax, and van Leer aims to directly find approximations of the flux function $F_{i+\frac{1}{2}}$. They proposed the following approximate Riemann solver:

$$\tilde{u}(x,t) = \begin{cases} u_L & \text{if } \frac{x}{t} < S_L \\ u^{hll} & \text{if } S_L \leq \frac{x}{t} \leq S_R \\ u_R & \text{if } \frac{x}{t} > S_R \end{cases}$$

with u^{hll} being the constant state vector given by

$$u^{hll} = \frac{S_R u_R - S_L u_L + F_L - F_R}{S_R - S_L}$$

where $F_R = f(u_R)$, $F_L = f(u_L)$, and the speeds S_L and S_R are known values given by:

$$S_R = \min_{u \in \{u_R, u_L\}} \min_i \lambda_i(u), \quad S_L = \max_{u \in \{u_R, u_L\}} \max_i \lambda_i(u)$$

where the $\lambda_i(u)$ are the eigenvalues of the Jacobian matrix $A(u) = \frac{\partial f(u)}{\partial u}$
The following graph shows the structure of this approximate solution:

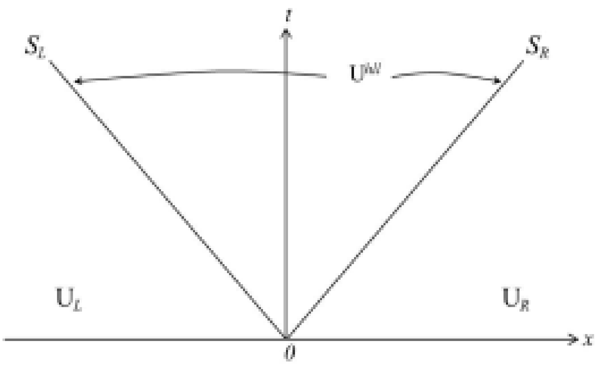

Figure 1.4: HLL approximate Riemann solver.

We can see that it consists of three constant states separated by two waves the star region consists of a single state u^{hll}, all the intermediate states separated by intermediate waves are grouped into a single state u^{hll}, let us note F^{hll} the flux across the left boundary (between u_L and u^{hll}) must be equal to the flux across the right boundary (between u_{hll} and u_R), this implies the continuity of the fluxes at these boundaries:

$$F^{hll} = F_L + S_L(u^{hll} - u_L) = F_R + S_R(u_{hll} - u_R)$$

Note that $F^{hll} \neq f(u^{hll})$. By substituting the formula for u^{hll} into both formulas for F^{hll}, we obtain the flux

$$F^{hll} = \frac{S_R F_L - S_L F_R + S_L S_R (u_R - u_L)}{S_R - S_L}$$

which can be used to produce the corresponding intercell flux for the Godunov method:

$$F(u_i, u_{i+1}) = F(u_L, u_R) = \begin{cases} F_L & \text{if } 0 < S_L \\ \frac{S_R F_L - S_L F_R + S_L S_R (u_R - u_L)}{S_R - S_L} & \text{if } S_L \leq 0 \leq S_R \\ F_R & \text{if } 0 > S_R \end{cases} \tag{1.3}$$

HLL is consistent and entropy-satisfying (it converges to the physical solution). One major flaw of the HLL scheme is exposed by contact discontinuities, shear waves and material interfaces. To address these issues, both the Roe scheme and the HLLC scheme provide a more precise resolution of contact discontinuities and shear waves. In this project, our focus will be on the Roe scheme.

II.6 Roe scheme

Roe's scheme, while more complex than the previous two, also proves more accurate in practice. It relies on the construction of a matrix $\tilde{A}(U_l, U_r) \in \mathbb{R}^{N\times N}$, known as the Roe matrix.
Consistent with the conservation system (1.1), the explicit FV conservatife scheme is written:

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{h}(F(u_i^n, u_{i+1}^n) - F(u_{i+1}^n, u_i^n)$$

The idea of the Roe method is that, instead of solving the nonlinear Riemann problem at each interface $x_{i+\frac{1}{2}}$, whose solution can be very complex, Roe proposes to replace the nonlinear Riemann problem with an approximate linear Riemann problem that can be solved exactly. Solving this linear Riemann problem allows us to directly calculate the flux $F_{i+\frac{1}{2}}$. Consider the system of PDEs:

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0$$

This system can be written in quasi-linear form as:

$$\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = 0$$

where $A(u) = \frac{\partial f(u)}{\partial u}$ is the Jacobian matrix of f at u. The approach adopted by Roe is to replace the Jacobian matrix $A(u)$ with a constant matrix

$$\tilde{A} = \tilde{A}(u_i^n, u_{i+1}^n)$$

which depends on the two states u_i (left) and u_{i+1} (right).
The approximate Riemann problem is then written:

$$\begin{cases} \frac{\partial u}{\partial t} + \tilde{A}\frac{\partial u}{\partial x} = & 0 \\ u_0(x) & = \begin{cases} u_i^n & \text{si } x < x_{i+\frac{1}{2}} \\ u_{i+1}^n & \text{si } x > x_{i+\frac{1}{2}} \end{cases} \end{cases}$$

this problem can be solved exactly.
Roe proposes to choose the matrix $\tilde{A}$ in such a way as to verify the following conditions:
a) Hyperbolicity of the system:
$\tilde{A}$ must be diagonalizable and have real eigenvalues $\tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq ... \leq \tilde{\lambda}_N$
b) Consistency with the Exact Jacobian matrix
$\tilde{A}(u, u) = A(u)$, which expresses consistency with the original equation.
c) Conservation across discontinuities:

$$f(v) - f(u) = \tilde{A}(v - u)$$

Roe defines the matrix $\tilde{A}$ by $\tilde{A}(u, v) = A(\tilde{u})$, where $\tilde{u}$, called the Roe average state between u and v, is calculated to satisfy conditions a), b), and c).
$\tilde{A}$ is diagonalizable, and it can be written as: $\tilde{A} = \tilde{P}\tilde{D}\tilde{P}^{-1}$
with

$$\tilde{D} = \begin{pmatrix} \tilde{\lambda}_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \tilde{\lambda}_N \end{pmatrix}$$

Note by

$$\begin{cases} \tilde{\lambda}^+ = \max(\tilde{\lambda}, 0) = \frac{1}{2}(\tilde{\lambda} + |\tilde{\lambda}|) \\ \tilde{\lambda}^- = \min(\tilde{\lambda}, 0) = \frac{1}{2}(\tilde{\lambda} - |\tilde{\lambda}|\ |) \end{cases}$$

we then have:

$$\begin{cases} \tilde{\lambda} = \tilde{\lambda}^+ + \tilde{\lambda}^- \\ |\tilde{\lambda}| = \tilde{\lambda}^+ - \tilde{\lambda}^- \end{cases}$$

Let's set

$$\tilde{D}^+ = \begin{pmatrix} \tilde{\lambda}_1^+ & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \tilde{\lambda}_N^+ \end{pmatrix}$$

and

$$\tilde{D}^- = \begin{pmatrix} \tilde{\lambda}_1^- & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \tilde{\lambda}_N^- \end{pmatrix}$$

The D matrix can be decomposed:

$$\tilde{D} = \tilde{D}^+ - \tilde{D}^- = \begin{pmatrix} |\tilde{\lambda}_1| & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & |\tilde{\lambda}_N| \end{pmatrix}$$

So we have

$$\begin{aligned} \tilde{A} &= \tilde{P}\tilde{D}\tilde{P}^{-1}, \\ &= \tilde{P}(\tilde{D}^+ + \tilde{D}^-)\tilde{P}^{-1}, \\ &= \tilde{P}\tilde{D}^+\tilde{P}^{-1} + \tilde{P}\tilde{D}^-\tilde{P}^{-1}, \\ &= \tilde{A}^+ + \tilde{A}^-. \end{aligned}$$

In the same way, we find that:

$$|\tilde{A}| = |\tilde{P}\tilde{D}\tilde{P}^{-1}| = \tilde{A}^+ - \tilde{A}^-$$

Let $\tilde{C}_p (p = 1, \dots, N)$ denote the proper vectors to the right of $\tilde{A}$. Let's decompose the left and right states of $x_{i+\frac{1}{2}}$ in this basis

$$u_i^n = \sum_{p=1}^{N} \alpha_p \tilde{C}_p, \quad u_{i+1}^n = \sum_{p=1}^{N} \beta_p \tilde{C}_p$$

We have seen that the solution of the Riemann problem can be written as:

$$u_{i+\frac{1}{2}}\left(\frac{x}{t}\right) = \sum_{p=1}^{I} \beta_p \tilde{C}_p + \sum_{p=I+1}^{N} \alpha_p \tilde{C}_p$$

where I is the index of the eigenvalue $\tilde{\lambda}_I$ such that $\left\{\begin{array}{l} x-\tilde{\lambda}_I t \leq 0 \\ x-\tilde{\lambda}_{I+1} t \geq 0 \end{array}\right.$ For the solution of the Riemann problem with $\frac{x}{t}=0$, the index I satisfies $\left\{\begin{array}{l} \tilde{\lambda}_I \leq 0 \\ \tilde{\lambda}_{I+1} \geq 0 \end{array}\right.$ this solution is written as:

$$u_{i+\frac{1}{2}}(0)=\sum_{p=1//\tilde{\lambda}_I \leq 0}^{I} \beta_p \tilde{C}_p+\sum_{p=I+1//\tilde{\lambda}_I \geq 0}^{N} \alpha_p \tilde{C}_p$$

which can be expressed in two ways:

$$u_{i+\frac{1}{2}}(0)=u_i^n+\sum_{p=1//\tilde{\lambda}_I \leq 0}^{I}\left(\beta_p-\alpha_p\right) \tilde{C}_p \tag{1.4}$$

$$u_{i+\frac{1}{2}}(0)=u_{i+1}^n-\sum_{p=I+1//\tilde{\lambda}_I \geq 0}^{N}\left(\beta_p-\alpha_p\right) \tilde{C}_p \tag{1.5}$$

$$\frac{(4)+(3)}{2} \Rightarrow u_{i+\frac{1}{2}}(0)=\frac{u_i^n+u_{i+1}^n}{2}-\frac{1}{2} \sum_{p=1}^{N} \operatorname{signe}(\tilde{\lambda}_p)\left(\beta_p-\alpha_p\right) \tilde{C}_p$$

The Roe numerical flux is:

$$\begin{aligned}
F_{i+\frac{1}{2}} & =\tilde{A} u_{i+\frac{1}{2}}(0), \\
& =\frac{1}{2} \tilde{A}\left(u_i^n+u_{i+1}^n\right)-\frac{1}{2} \sum_{p=1}^{N} \operatorname{signe}\left(\tilde{\lambda}_p\right)\left(\beta_p-\alpha_p\right) \tilde{A} \tilde{C}_p \\
& =\frac{1}{2} \tilde{A}\left(u_i^n+u_{i+1}^n\right)-\frac{1}{2} \sum_{p=1}^{N}|\tilde{\lambda}_p|\left(\beta_p-\alpha_p\right) \tilde{C}_p \\
& =\frac{1}{2}\left(f\left(u_i^n\right)+f\left(u_{i+1}^n\right)\right)-\frac{1}{2} \sum_{p=1}^{N}\left|\tilde{\lambda}_p\right|\left(\beta_p-\alpha_p\right) \tilde{\lambda}_p
\end{aligned}$$

Therefore

$$F_{i+\frac{1}{2}}^n=\frac{1}{2}\left(f\left(u_i^n\right)+f\left(u_{i+1}^n\right)\right)-\frac{1}{2}|\tilde{A}(u_i, u_{i+1})|\left(u_{i+1}^n-u_i^n\right) \tag{1.6}$$

III TVD Scheme

The previously discussed schemes are first-order in space. These schemes are known to be diffusive, meaning they attenuate shocks and discontinuities because we assumed that the solution u is constant over the control volume. To reduce numerical diffusion, one can consider a piecewise linear approximation, where u is assumed to be affine over each control volume K_i. The function u is then expressed as follows:

$$u(x)=u_i+\sigma_i(x-x_i), \quad \text{where } \sigma_i=\frac{u_{i+1}-u_i}{h}$$

$u(x)$ can then be determined if we define its start and end values over each control volume. Thus, at each interface $x_{i+\frac{1}{2}}$, two states are defined: $u_{i+\frac{1}{2}}^L$, the end state of $u(x)$ over control volume K_i, and $u_{i+\frac{1}{2}}^R$, the start state of $u(x)$ over control volume K_{i+1}.

$$
\begin{aligned}
u^L_{i+\frac{1}{2}} &= u_i + \sigma_i(x_{i+\frac{1}{2}} - x_i) \\
&= u_i + \frac{1}{2}(u_{i+1} - u_i)
\end{aligned}
$$

$$
\begin{aligned}
u^R_{i+\frac{1}{2}} &= u_{i+1} + \sigma_{i+1}(x_{i+\frac{1}{2}} - x_{i+1}) \\
&= u_{i+1} - \frac{1}{2}(u_{i+2} - u_{i+1})
\end{aligned}
$$

Similarly, at each interface $x_{i-\frac{1}{2}}$, we find

$$
\begin{aligned}
u^L_{i-\frac{1}{2}} &= u_{i-1} + \frac{1}{2}(u_i - u_{i-1}) \\
u^R_{i-\frac{1}{2}} &= u_i - \frac{1}{2}(u_{i+1} - u_i)
\end{aligned}
$$

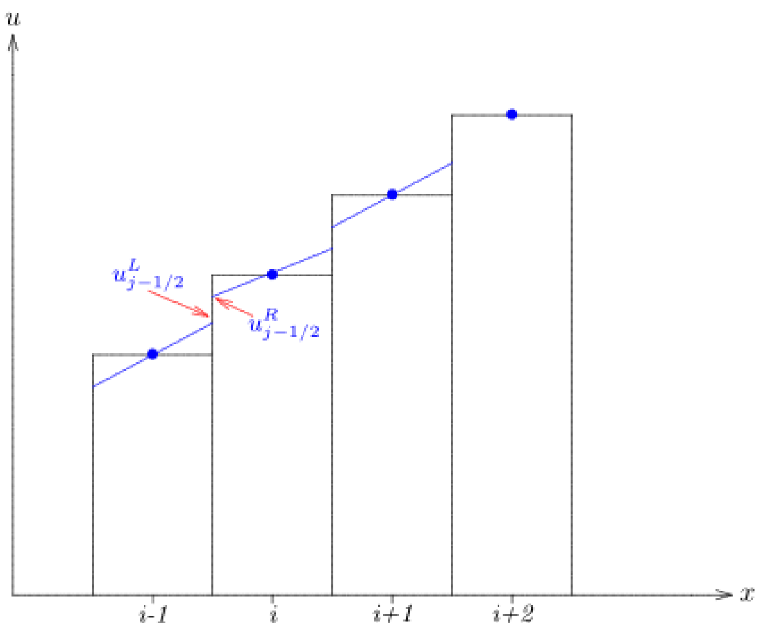

Figure 1.5:

With these states, the flux $F_{i+\frac{1}{2}}$ will be a function of $u^L_{i+\frac{1}{2}}$ and $u^R_{i+\frac{1}{2}}$.

$$
F_{i+\frac{1}{2}} = F(u^L_{i+\frac{1}{2}}, u^R_{i+\frac{1}{2}}), \quad F_{i-\frac{1}{2}} = F(u^L_{i-\frac{1}{2}}, u^R_{i-\frac{1}{2}})
$$

This way of reconstructing the linear states can lead to oscillations, especially at discontinuities and shocks. To attenuate these oscillations, Harten (1983) introduced the concept of TVD (Total Variation Diminishing) schemes.

Definition 1.1:

A scheme is said to be TVD if and only if

$$
TV(u^{n+1}) \leq TV(u^n)
$$

where $TV(u^n)$, the discrete total variation of u^n, is given by

$$
TV(u^n) = \sum_{i=1}^{+\infty} |u^n_i - u^n_{i+1}|
$$

Definition 1.2:
A scheme is said to preserve monotonicity if and only if it transforms a monotone sequence u^n into another sequence u^{n+1} that is also monotone. A sequence $(u)_i$ is said to be monotone if and only if

$$\forall i, \min(u_{i-1}, u_{i+1}) \leq u_i \leq \max(u_{i-1}, u_{i+1})$$

Theorem 1.1:
For a conservative scheme, we have

$$\text{TVD} \Rightarrow \text{Preserves monotonicity}$$

We define $u^L_{i+\frac{1}{2}}$ and $u^R_{i+\frac{1}{2}}$ in such a way as to ensure the TVD property of the scheme, thereby avoiding oscillations that can occur near discontinuities or shocks. This is achieved by introducing slope limiters in the construction of the two states.

A limiter is of the form $\Phi(r)$ where r denotes the slope of the variable u. The limited left and right states are given by:

$$\begin{cases} u^L_{i+\frac{1}{2}} = u_i + \frac{1}{2}\Phi(r_i)(u_{i+1} - u_i) \\ u^R_{i+\frac{1}{2}} = u_{i+1} - \frac{1}{2}\Phi(r_{i+1})(u_{i+2} - u_{i+1}) \end{cases}$$

and

$$\begin{cases} u^L_{i-\frac{1}{2}} = u_{i-1} + \frac{1}{2}\Phi(r_{i-1})(u_i - u_{i-1}) \\ u^R_{i-\frac{1}{2}} = u_i - \frac{1}{2}\Phi(r_i)(u_{i+1} - u_i) \end{cases}$$

with

$$r_i = \frac{u_i - u_{i-1}}{u_{i+1} - u_i}$$

Where $\Phi(r) = 0$ if $r \leq 0$, $\Phi(1) = 1$
There are numerous slope limiters found in the literature.

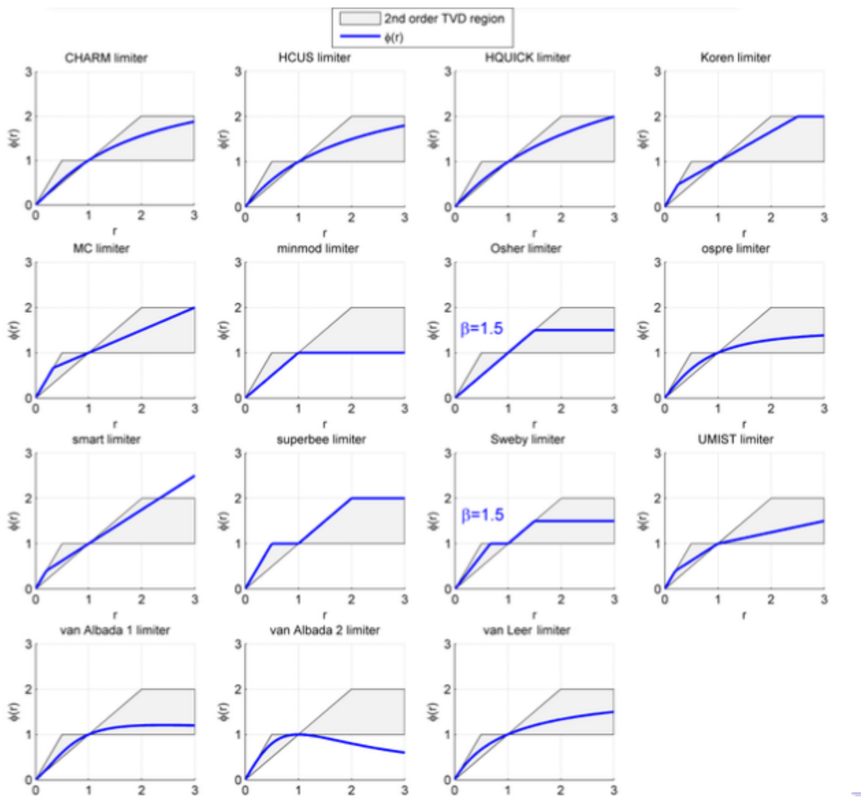

Figure 1.6: Slope limiters

In this project, we are focusing on four slope limiters :
Minmod:

$$\Phi(r) = \max(0, \min(1, r)) \quad \lim_{r \to +\infty} \Phi(r) = 1$$

Vanleer

$$\Phi(r) = \frac{r + |r|}{1 + |r|}$$

Van Alboda

$$\Phi(r) = \max(0, \frac{r + r^2}{1 + r^2})$$

Superbe

$$\Phi(r) = \max(0, \min(2r, 1), \min(r, 2))$$

the HLL flux:

$$F(u^L_{i+\frac{1}{2}}, u^R_{i+\frac{1}{2}}) = \begin{cases} f(u^L_{i+\frac{1}{2}}) & \text{if} \quad 0 \leq S_L \\ \frac{S_R f(u^L_{i+\frac{1}{2}}) - S_L f(u^R_{i+\frac{1}{2}}) + S_L S_R (u^R_{i+\frac{1}{2}} - u^L_{i+\frac{1}{2}})}{S_R - S_L} & \text{if} \quad S_L \leq 0 \leq S_R \\ f(u^R_{i+\frac{1}{2}}) & \text{if} \quad 0 \geq S_R \end{cases} \tag{1.7}$$

The Roe flux:

$$F_{i+\frac{1}{2}} = \frac{1}{2}\left(f\left(u^L_{i+\frac{1}{2}}\right) + f\left(u^L_{i+\frac{1}{2}}\right)\right) - \frac{1}{2}|\tilde{A}|\left(u^R_{i+\frac{1}{2}} - u^L_{i+\frac{1}{2}}\right)$$

Where $\tilde{A} = A(\tilde{u})$, $\tilde{u}$ is the Roe average between the two states $u^R_{i+\frac{1}{2}}$ and $u^L_{i+\frac{1}{2}}$.

Chapter 2

1D Transport Equation

In the first case, we define $f(u)$ from system (1.1) such that $f(u) = cu$. This means that we have the transport of the quantity u, which can represent the water height, flow velocity, or water density, with a transport speed equal to c. The boundary conditions are of Neumann type. Thus, we obtain the following system:

$$\begin{cases} \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 & \forall x \in]a,b[, \forall t \in]0,T] \\ u(x,t=0) = u_0(x) & \forall x \in [a,b] \\ \frac{\partial u(a,t)}{\partial x} = \frac{\partial u(b,t)}{\partial x} = 0 & \forall t \in [0,T] \end{cases} \tag{2.1}$$

I Solving the transport equation using the characteristic method

The principle of the method of characteristics for solving PDEs is to find curves, called characteristic curves, along which the PDE reduces to a simple ordinary differential equation (ODE). If we can solve the ODE on the characteristic curve Γ, then we can repeat the procedure starting from a neighboring curve, and so on, thus obtaining the solution $u(x,t)$ in the entire domain.
We will illustrate the principle on the transport equation (2.1):
Let us find a characteristic curve $\Gamma = \Gamma(x(s), y(s))$ along which the PDE reduces to an ordinary differential equation.
We differentiate along the curve Γ.

$$\frac{du}{ds} = \frac{\partial u}{\partial t} \cdot \frac{dt}{ds} + \frac{\partial u}{\partial x} \cdot \frac{dx}{ds}$$

According to (2.1) we have: $\frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial x}$ therefore

$$\begin{aligned} \frac{du}{ds} &= -c\frac{\partial u}{\partial x}\frac{dt}{ds} + \frac{\partial u}{\partial x}\frac{dx}{ds}, \\ &= \frac{\partial u}{\partial x}(\frac{dx}{ds} - c\frac{dt}{ds}). \end{aligned}$$

Along the characteristic curve

$$\frac{dx}{ds} - c\frac{dt}{ds} = 0 \quad \text{i.e.,} \quad x = ct + \xi \quad \text{with} \quad \xi \in \mathbb{R}$$

the solution satisfies:

$$\frac{du}{ds} = 0 \Leftrightarrow du = 0.$$

This means u is constant along the curve Γ (Riemann invariant).

Illustration of the characteristic curves:
The general solution of equation (5) is:

$$\begin{aligned} u(x,t) = u(x(0),0) &= u(\xi,0), \\ &= u_0(\xi), \\ &= u_0(x-ct), \\ &= u(x-ct,0). \end{aligned}$$

Therefore,

$$u(x,t) = u_0(x-ct) \tag{2.2}$$

II Solving the Transport Equation Using the Finite Volume Method

Using the same mesh as in the general case in Chapter 1 and with a uniform step size, we obtain the following finite volume formulation:

$$\frac{\partial u_i(t)}{\partial t} + \frac{1}{h_i}\Big(f(u(x_{i+\frac{1}{2}},t)) - f(u(x_{i-\frac{1}{2}},t))\Big) = 0$$

where $f(u(x,t)) = cu(x,t)$.
By approximating $\frac{\partial u_i(t)}{\partial t}$ to first order in time as $\frac{\partial u_i}{\partial t}(t^n) \approx \frac{u_i^{n+1}-u_i^n}{\Delta t}$, and using an explicit scheme, we obtain the following finite volume scheme:

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{h_i}(F^n_{i+\frac{1}{2}} - F^n_{i-\frac{1}{2}})$$

with

$$F^n_{i+\frac{1}{2}} \approx \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} f(u(x_{i+\frac{1}{2}},t))\,dt$$

$$F^n_{i-\frac{1}{2}} \approx \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} f(u(x_{i-\frac{1}{2}},t))\,dt$$

For the fluxes inside the domain $[a,b]$, we seek their approximation: $F_{i+\frac{1}{2}}$.

II.1 HLL Scheme

The HLL scheme for the transport equation $f(u) = cu$ is given by:

$$F(u_i,u_{i+1}) = \begin{cases} cu_i & \text{if} \quad 0 < S_L, \\ \frac{S_R cu_i - S_L cu_{i+1} + S_L S_R (ui+1-u_i)}{S_R - S_L} & \text{if} \quad S_L \leq 0 \leq S_R, \\ cu_{i+1} & \text{if} \quad 0 > S_R, \end{cases}$$

In this case, $A(u) = c$ does not depend on u. Therefore, the wave speeds are:

$$S_L = \min_{u\in\{u_R,u_L\}} \min_i \lambda_i(u) = c, \quad S_R = \max_{u\in\{u_R,u_L\}} \max_i \lambda_i(u) = c$$

The numerical flux for the HLL scheme then becomes:

$$F_{i+\frac{1}{2}} = \begin{cases} cu_i & \text{if} \quad 0 < c, \\ 0 & \text{if} \quad c = 0, \\ cu_{i+1} & \text{if} \quad 0 > c, \end{cases}$$

Thus,

$$F^n_{i+\frac{1}{2}} = \max(c,0)u^n_i + \min(c,0)u^n_{i+1}$$
$$F^n_{i-\frac{1}{2}} = \max(c,0)u^n_{i-1} + \min(c,0)u^n_i$$

Remark:
The HLL scheme for the 1D linear transport equation is an upwind finite difference scheme if $c > 0$, it is downwind if $c < 0$.
Consistency:
We have:

$$F(u,u) = \max(c,0)u + \min(c,0)u = cu = f(u)$$

Thus, the HLL scheme for the 1D linear transport equation is consistent.
Order of the Scheme
The upwind scheme for $c > 0$ is written as:

$$\frac{u^{n+1}_i - u^n_i}{\Delta t} + c\frac{u^n_i - u^n_{i-1}}{h} = 0 \quad \text{with} \quad c > 0,$$

We know that

$$u^{n+1}_i = u(x_i, t^{n+1}) = u(x_i, t^n + \Delta t)$$

Using a Taylor series expansion around t^n:

$$u(x_i, t^n + \Delta t) = u(x_i, t^n) + \Delta t \cdot \left.\frac{\partial u}{\partial t}\right|^n_i + O((\Delta t)^2)$$

Thus,

$$\left.\frac{\partial u}{\partial t}\right|^n_i = \frac{u(x_i, t^n + \Delta t) - u(x_i, t^n)}{\Delta t} + O(\Delta t).$$

Hence,

$$\left.\frac{\partial u}{\partial t}\right|^n_i = \frac{u^{n+1}_i - u^n_i}{\Delta t} + O(\Delta t).$$

Similarly, using a Taylor series expansion around x_i:

$$u^n_{i+1} = u(x_i + h, t^n) = u^n_i + h\left.\frac{\partial u}{\partial x}\right|^n_i + O(h^2).$$

Thus,

$$\left.\frac{\partial u}{\partial x}\right|^n_i = \frac{u^n_{i+1} - u^n_i}{h} + O(h).$$

Now, we calculate the truncation error (the difference between the exact equation and the numerical scheme):

$$\begin{aligned} TE &= \left(\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x}\right) - \left(\frac{u^{n+1}_i - u^n_i}{\Delta t} + c\frac{u^n_i - u^n_{i-1}}{h}\right) \\ &= O(\Delta t) + O(h) \end{aligned}$$

The upwind scheme for $c < 0$ is of first order in both time and space.
The scheme is consistent, $\lim\limits_{\substack{\Delta t \to 0 \\ h \to 0}} TE = 0$.
Because

$$|TE| \leq |O(h) + O(\Delta t)| \leq |O(h)| + |O(\Delta t)| \to 0.$$

Similarly, we find that the upwind scheme for $c > 0$ is also first order in both time and space.

Stability:
To study stability in the L^2 sense, we use the Fourier-Von Neumann method. This approach involves seeking solutions to the partial differential equation (PDE) in the form $u_n^j = C_n e^{i\xi x_j}$, where $x_j = j\Delta x$ and ξ is the wave number.

Next, this solution is substituted into the numerical scheme, and the behavior of the sequence of functions $(C_n)_{n\in\mathbb{N}}$ is analyzed. This leads to a recurrence relation for the C_n. If the sequence of functions $(C_n)_{n\in\mathbb{N}}$ is bounded in the L^2 norm, then the numerical scheme is stable. Otherwise, it is unstable.

Let's set $u_j^n = C^n e^{i\xi x_j}$, then the scheme is written in the form:

$$u_j^{n+1} = u_j^n - c\frac{\Delta t}{\Delta x}(u_{j+1}^n - u_j^n)$$

$$C^{n+1}e^{i\xi x_j} = C^n(e^{i\xi x_j} - c\frac{\Delta t}{\Delta x}(e^{i\xi x_{j+1}} - e^{i\xi x_j}))$$

Hence,

$$C^{n+1} = C^n(1 - \lambda(e^{i\xi\Delta x} - 1))$$

by letting

$$A = (1 - \lambda(e^{i\xi\Delta x} - 1))$$

Then for u_j^n to be bounded, it is necessary that

$$\begin{aligned}
|A| \leq 1 &\Rightarrow |1 - \lambda(\cos(\xi\Delta x) - 1 + \sin(\xi\Delta x))| \leq 1\\
&\Rightarrow (1 - \lambda(\cos(\xi\Delta x) - 1))^2 + \lambda^2\sin^2(\xi\Delta x) \leq 1\\
&\Rightarrow (1+\lambda)^2 - 2(1+\lambda)\lambda\cos(\xi\Delta x) + \lambda^2 \leq 1\\
&\Rightarrow 1 + 2\lambda + \lambda^2 - 2(1+\lambda)\lambda\cos(\xi\Delta x) + \lambda^2 \leq 1\\
&\Rightarrow 2 + 2\lambda - 2(1+\lambda)\cos(\xi\Delta x) \geq 0\\
&\Rightarrow (1+\lambda)(1 - \cos(\xi\Delta x)) \leq 0\\
&\Rightarrow 1 + \lambda \geq 0\\
&\Rightarrow \lambda \geq -1\\
&\Rightarrow -c\frac{\Delta t}{\Delta x} \leq 1.
\end{aligned}$$

Thus, the downstream scheme with a negative transport velocity is stable if and only if $-c\frac{\Delta t}{\Delta x} \leq 1$. Using the same method, we find that the upstream scheme with
$c > 0$ is stable under the condition $c\frac{\Delta t}{\Delta x} \leq 1$

Conclusion: The HLL scheme for the transport equation is stable under the condition

$$|c\frac{\Delta t}{\Delta x}| \leq 1$$

II.2 Roe Scheme

The flux $F_{i+\frac{1}{2}}^n$ using Roe's method is given by:

$$F_{i+\frac{1}{2}}^n = \frac{1}{2}\left(f\left(u_i^n\right) + f\left(u_{i+1}^n\right)\right) - \frac{1}{2}|\tilde{A}(u_i, u_{i+1})|\left(u_{i+1}^n - u_i^n\right)$$

We know that $\tilde{A}(u_i, u_{i+1})$ is the constant matrix approximation of the Jacobian matrix $A = \frac{\partial f(u)}{\partial u}$. In our case, $A = c$. Since A is constant, it satisfies:

- a) Hyperbolicity of the system,
- b) Consistency with the exact Jacobian matrix, and
- c) Conservation across discontinuities.

Therefore, for the 1D linear transport equation, $\tilde{A}(u_i, u_{i+1}) = c$.
We then obtain:

$$F^n_{i+\frac{1}{2}} = \frac{1}{2}c\left(u^n_i + u^n_{i+1}\right) - \frac{1}{2}|c|\left(u^n_{i+1} - u^n_i\right)$$

This implies that for the 1D linear transport equation, Roe's flux is equal to the HLL flux, and it is consistent and stable under the same condition:

$$|c\frac{\Delta t}{\Delta x}| \leq 1$$

II.3 Numerical Test

Data: Computational domain: $[0, 6]$
Transport speed: $c = 2$ m/s
Initial condition: $u(x, 0) = u_0(x) = \begin{cases} 1 & \text{for } \frac{1}{2} \leq x \leq \frac{3}{2} \\ 0 & \text{elsewhere} \end{cases}$

Exact Solution:

The exact solution of (6) with its given data is:

$$\begin{aligned} u(x,t) = u_0(x-2t) &= \begin{cases} 1 & \text{for } \frac{1}{2} \leq x - 2t \leq \frac{3}{2} \\ 0 & \text{elsewhere} \end{cases} \\ &= \begin{cases} 1 & \text{for } \frac{1}{2} + 2t \leq x \leq \frac{3}{2} + 2t \\ 0 & \text{elsewhere} \end{cases} \end{aligned}$$

In this case, the characteristic curves are straight lines with a slope of $c = 2$

$$x = 2t + \xi, \quad \xi \in \mathbb{R}$$

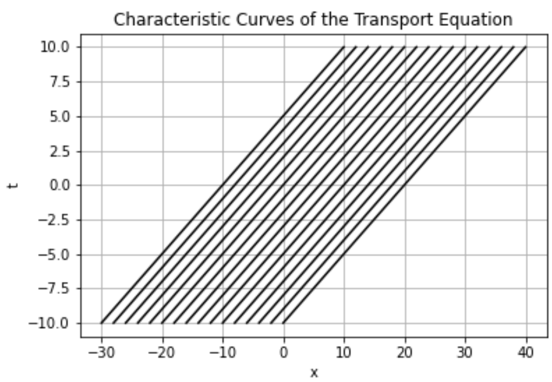

Figure 2.1: Characteristic Curves of the Transport Equation $c = 2$ m/s

HLL First-Order Transport Equation

We fix $N = 100$, $c = 2\mathbf{m}/\mathrm{s}$, and vary the CFL number:

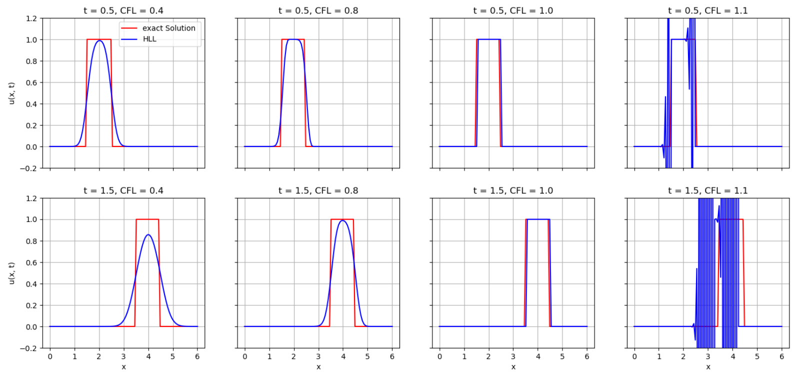

Figure 2.2: Comparisons between the exact solution and first-order HLL

Roe First-Order Transport Equation

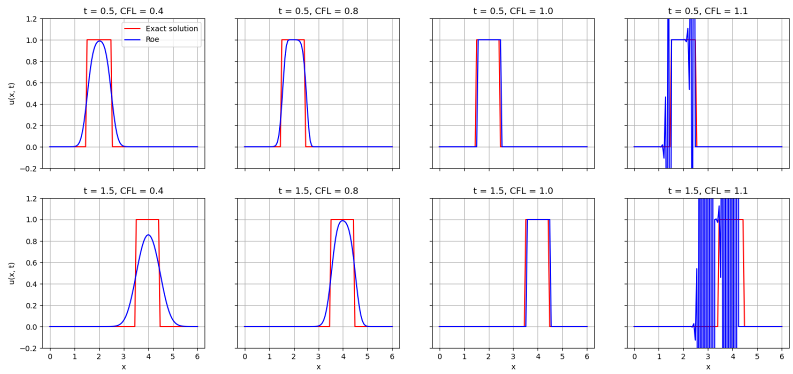

Figure 2.3: Comparisons between the exact solution and first-order Roe

III Extension of the scheme to second-order in space

In this section, as mentioned in the first chapter, we assume that the numerical flux $F^n_{i+\frac{1}{2}}$ depends on two states $u^L_{i+\frac{1}{2}}$ and $u^R_{i+\frac{1}{2}}$. Without a slope limiter, these are defined as follows:

$$u^L_{i+\frac{1}{2}} = u_i + \frac{1}{2}(u_{i+1} - u_i)$$

$$u^R_{i+\frac{1}{2}} = u_{i+1} - \frac{1}{2}(u_{i+2} - u_{i+1})$$

With a slope limiter Φ:

$$u^L_{i+\frac{1}{2}} = u_i + \frac{1}{2}\Phi(r_i)(u_{i+1} - u_i)$$

$$u^R_{i+\frac{1}{2}} = u_{i+1} - \frac{1}{2}\Phi(r_{i+1})(u_{i+2} - u_{i+1})$$

III.1 HLL

Without limier

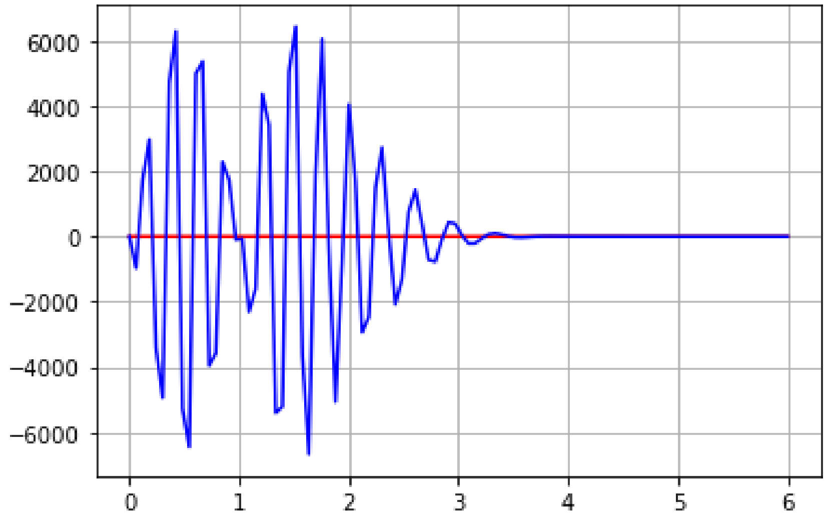

Figure 2.4: Comparisons between the exact solution and second-order HLL without limiter, CFL=0.5, at t=1.5s

with limiter

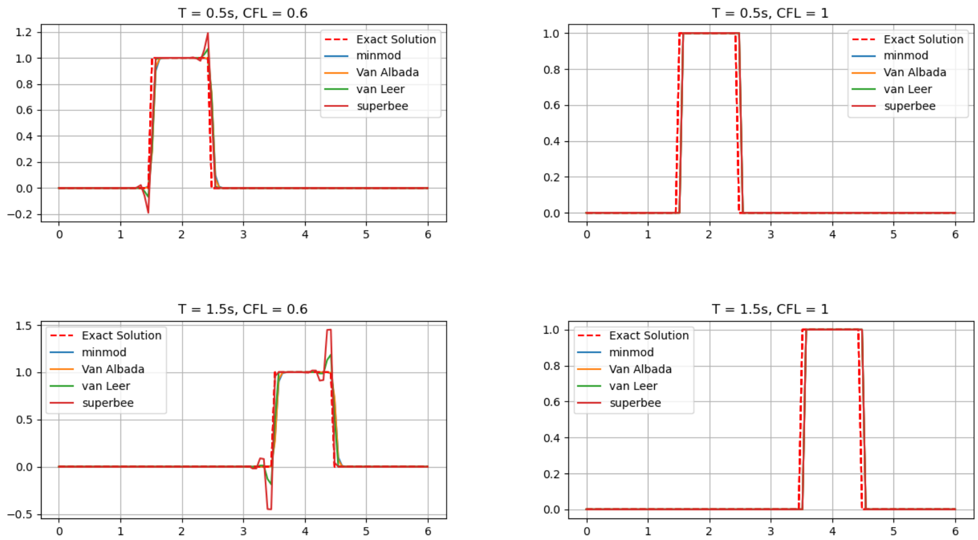

Figure 2.5: Comparisons between the exact solution and second-order HLL with different slope limiters

III.2 Roe

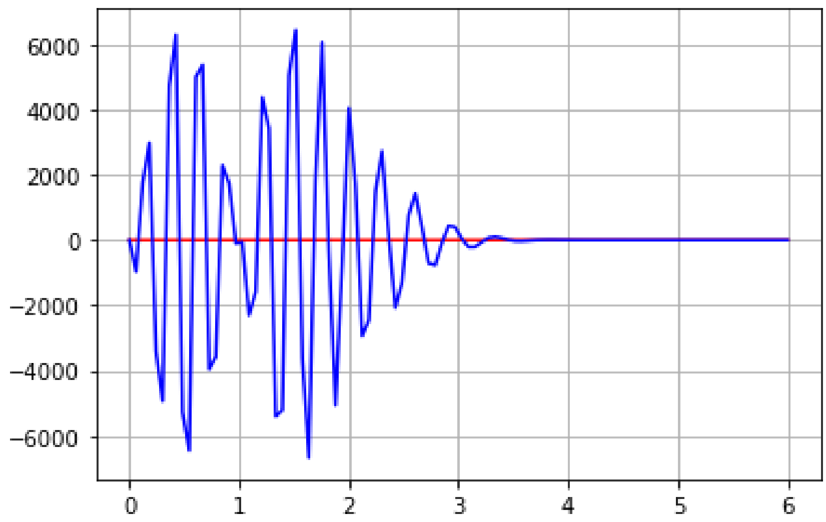

Figure 2.6: Comparisons between the exact solution and second-order Roe without limiter, CFL=0.5, at t=1.5s

with limiter

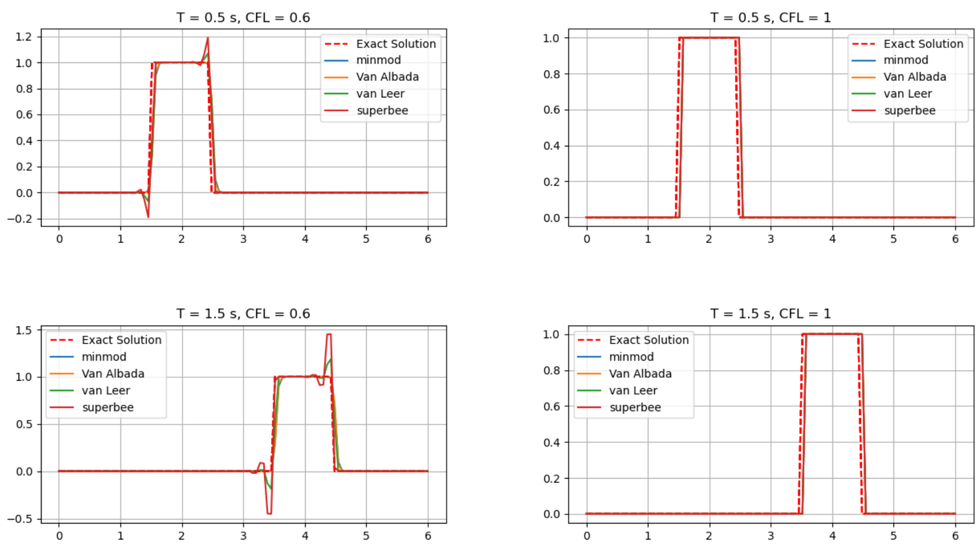

Figure 2.7: Comparisons between the exact solution and second-order Roe with different slope limiters

IV Comparing Results Obtained by First and Second Order Schemes

IV.1 HLL

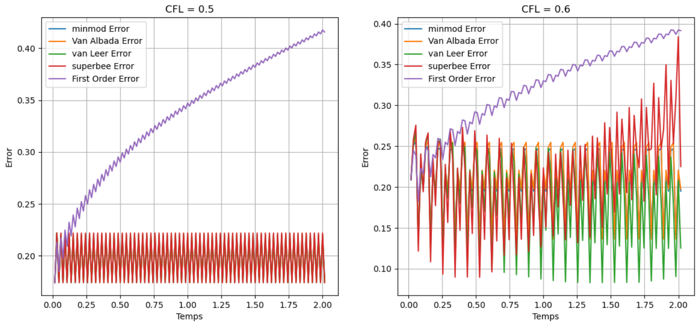

Figure 2.8: Evaluation of L^2 Norm Error in Time for HLL Methods of First and Second Order

IV.2 Roe

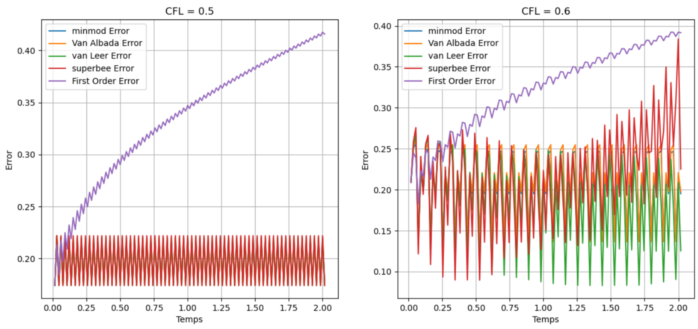

Figure 2.9: Evaluation of L^2 Norm Error in Time for Roe Methods of First and Second Order

Observation: It was observed that the error for the first-order schemes increased rapidly over time. In contrast, the error for the second-order schemes increased at a slower rate, but not significantly. It was noted that when using CFL=0.5 rcfl=0.5, the error for the second-order schemes did not increase

Chapter 3

Burges' Equation

Introduction:
In this chapter, we will study the Burgers' equation, a nonlinear partial differential equation fundamental in fluid dynamics and applied physics. This equation models the evolution of a wave's surface, where a higher surface moves faster, making it an ideal model for nonlinear waves and rupture phenomena. We will apply previously studied finite volume schemes to numerically solve this equation.

The inviscid Burgers' equation is defined as follows:

$$(E_2) = \begin{cases} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 & \forall x \in \mathbb{R}, \forall t > 0 \\ u(x,0) = u_0(x) \end{cases}$$

I Resolution of the Burgers' equation by the method of characteristics

Let us first note that the Burgers' equation is a nonlinear transport equation. More precisely, u is a solution of a transport equation, where the propagation velocity at point x and time t is equal to $u(t,x)$. We will seek a characteristic curve $\Gamma(t(s), x(s))$ along which the PDE (E_2) becomes an ODE.
The equation "E2" can be written in the form:

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0 & \text{for all } x \in \mathbb{R}, \text{ for all } t > 0 \\ u(x, t=0) = u_0(x) \end{cases} \tag{3.1}$$

where $f(u) = \frac{u^2}{2}$.
Differentiating u along the curve Γ:

$$\frac{du}{ds} = \frac{\partial u}{\partial t}\frac{dt}{ds} + \frac{\partial u}{\partial x}\frac{dx}{ds}$$

According to equation (1.1), we have $\frac{\partial u}{\partial t} = -f'(u)\frac{\partial u}{\partial x}$.
Thus,

$$\frac{du}{ds} = \frac{\partial u}{\partial x}\left(\frac{dx}{ds} - f'(u)\frac{dt}{ds}\right).$$

Thus, the characteristic curves are defined as follows:

$$dx = f'(u)dt,$$

so,

$$x(t) = f'(u)t + x_0, \quad x_0 \in \mathbb{R}.$$

On each characteristic curve, the solution satisfies:

$$\frac{du}{ds} = 0,$$

therefore,

$$u(s) = \text{constant} = u(0, x(0)).$$

Thus, the characteristic curves of the equation are

$$x(t) = f'(u_0(x_0))t + x_0, \quad x_0 \in \mathbb{R}.$$

The solution u of equation (E2) satisfies:

$$\frac{du}{ds} = 0,$$

which means that on each characteristic curve, u is constant.
Let's use the initial conditions.
The fact that u is constant along each characteristic curve and for small t, the characteristics do not intersect implies that

$$u(x,t) = u(x(0),0) \Rightarrow u(x,t) = u(x_0,0),$$

thus,

$$u(x,t) = u_0(x_0) = u_0(x(t) - f'(u_0(x_0))t).$$

For equation (E2) where $f(u) = \frac{u^2}{2}$, the exact solution of this equation can be written as:

$$u(x,t) = u_0(x(t) - u_0(x_0)t) \tag{3.2}$$

and the characteristic curves are

$$x(t) = u_0(x_0)t + x_0. \tag{3.3}$$

In general, in real-world problems that are modeled, we often encounter discontinuous initial conditions, which make this solution valid only for very short times before the discontinuity. However, in reality, it is crucial to understand what happens after the shock (the discontinuity). Therefore, we introduce a new concept, that of the weak solution.
To better understand this, let's consider an example of this equation with the following initial condition:

$$u_0(x) = \begin{cases} 1 \text{ if } x < -1 \\ -x \text{ if } -1 \leq x \leq 0 \\ 0 \text{ if } x > 0 \end{cases}$$

For this initial condition, the characteristic curves often take the following forms:

$$x(t) = \begin{cases} t + x_0 \quad \text{if } x_0 < -1 \\ -x_0 t + x_0 \quad \text{if } -1 \leq x_0 \leq 0 \\ x_0 \quad \text{if } x_0 > 0 \end{cases}$$

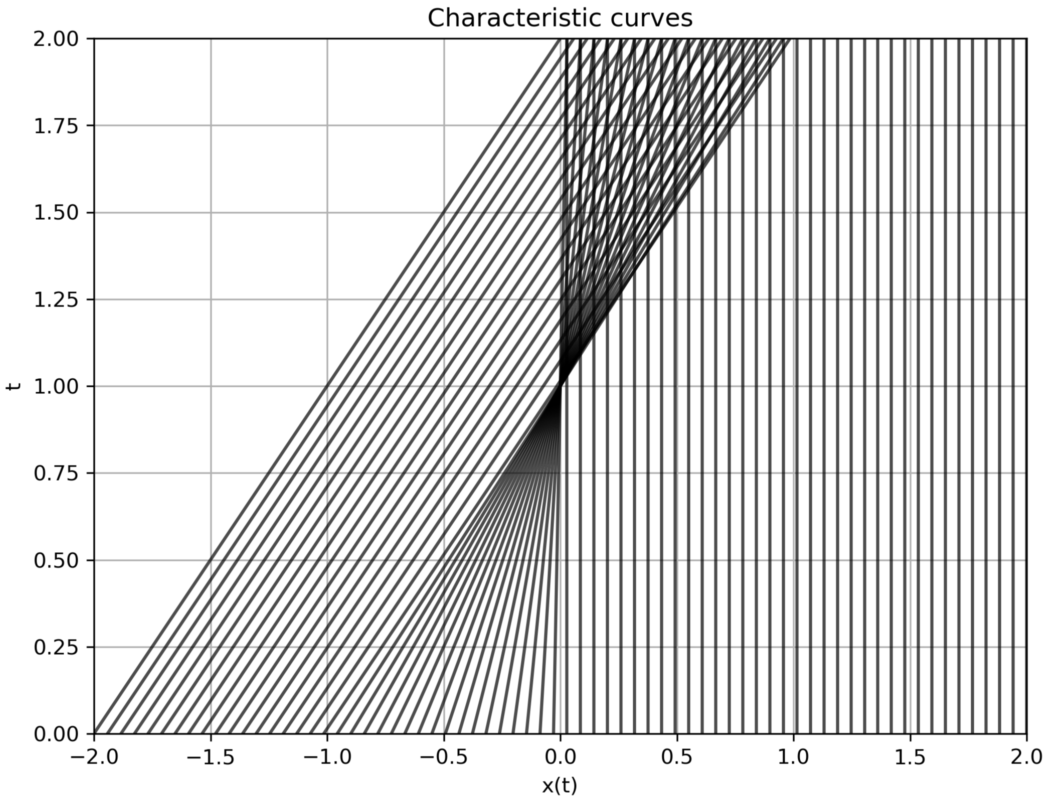

Figure 3.1: Characteristic Curves of the Burgers' Equation

We clearly observe an apparent shock starting from (1,0) where the characteristic curves intersect. According to the method of characteristics, u remains constant along each curve; therefore, when they intersect, what value does u take? Consequently, the solution obtained using the method of characteristics will be valid only for $t < 1$.
Therefore, for $t < 1$, the solution given by the method of characteristics, according to (1.2), is:

$$u(x(t), t) = \begin{cases} 1 & \text{if } x_0 < -1 \\ \frac{-x(t)}{1-t} & \text{if } -1 \leq x_0 \leq 0 \\ 0 & \text{if } x_0 > 0 \end{cases}$$

Then,

$$u(x, t) = \begin{cases} 1 & \text{if } x - t < -1 \\ \frac{-x}{1-t} & \text{if } -1 \leq \frac{x}{1-t} \leq 0 \\ 0 & \text{if } x > 0 \end{cases}$$

The issue is that this solution is only valid for $t < 1$. However, it is crucial to understand what happens after this point. To address this question, we introduce the concept of weak entropy solution.

II Weak entropy solution

II.1 Weak solution

Definition 1.2.1: We say that a function $u \in L^\infty_{\text{loc}}(\mathbb{R} \times [0, +\infty[)$ is a weak solution of the PDE

$$\partial_t u + \partial_x f(u) = 0$$

if for every function $\phi \in C_c^\infty(\mathbb{R} \times [0, +\infty[)$ (smooth function with compact support), we have

$$\int_0^{+\infty} \int_{-\infty}^{+\infty} u\frac{\partial \phi}{\partial t} + f(u)\frac{\partial \phi}{\partial x}\, dx\, dt = -\int_{-\infty}^{+\infty} u(x,0)\phi(x,0)\, dx.$$

II.2 Rankine-Hugoniot condition

The Rankine-Hugoniot condition states that when two characteristics intersect, they generate a discontinuity in the solution. This discontinuity, known as a shock wave, is smooth and can be represented by an equation $x = s(t)$. For the solution u to remain valid across this shock wave, it must satisfy the Rankine-Hugoniot relations. Denoting u_R and u_L as the values of u to the right and left of the shock wave, respectively, the shock velocity is defined by:

$$s'(t) = \frac{f(u_L) - f(u_R)}{u_L - u_R}$$

In the example we are working on, we have $u_L = 1$, $u_R = 0$, and $f(u) = \frac{1}{2}u^2$. Therefore, the speed at which the shock moves is $s'(t) = \frac{1}{2}$ m/s. The shock originates from the point $(0, 1)$, so we have $s'(t) = \frac{x-0}{t-1} = \frac{1}{2}$. Thus, the shock curve is given by $2x + 1 = t$.
For $t \geq 1$, we express the solution as:

$$u(x,t) = \begin{cases} 1 & \text{if } 2x+1 < t \\ 0 & \text{if } 2x+1 > t \end{cases}$$

In the example we are working on, we have $u_L = 1$, $u_R = 0$, and $f(u) = \frac{1}{2}u^2$. Therefore, the speed at which the shock moves is $s'(t) = \frac{1}{2}$ m/s. The shock originates from the point $(0, 1)$, so we have $s'(t) = \frac{x-0}{t-1} = \frac{1}{2}$. Thus, the shock curve is given by $2x + 1 = t$.

For $t \geq 1$, we express the solution as:

$$u(x,t) = \begin{cases} 1 & \text{if } 2x+1 < t \\ 0 & \text{if } 2x+1 > t \end{cases}$$

Let us demonstrate that this solution is a weak solution of (1.1).
Let $\phi \in C_c^\infty(\mathbb{R} \times [0, +\infty[)$:

$$\begin{aligned}
\int_0^{+\infty} \int_{-\infty}^{+\infty} u\frac{\partial \phi}{\partial t} + f(u)\frac{\partial \phi}{\partial x}\, dx\, dt &= \int_{-\infty}^{+\infty} \left(\int_0^1 u\frac{\partial \phi}{\partial t}\, dt + \int_0^{+\infty} u\frac{\partial \phi}{\partial t}\, dt \right) dx \\
&\quad + \int_0^1 \int_{-\infty}^{+\infty} f(u)\frac{\partial \phi}{\partial x}\, dx\, dt + \int_0^{+\infty} \int_{-\infty}^{+\infty} f(u)\frac{\partial \phi}{\partial x}\, dx\, dt \\
&= \int_{-\infty}^{+\infty} \left(u(x,1)\phi(x,1) - u(x,0)\phi(x,0) - \int_0^1 \frac{\partial u}{\partial t}\phi\, dt \right) dx \\
&\quad + \int_{-\infty}^{0} \int_1^{+\infty} \frac{\partial \phi}{\partial t}\, dt\, dx + \int_0^{+\infty} \int_{2x+1}^{+\infty} \frac{\partial \phi}{\partial t}\, dt\, dx \\
&\quad - \int_0^1 \int_{-\infty}^{+\infty} \frac{\partial f(u)}{\partial x}\phi\, dx\, dt \\
&\quad + \int_1^{+\infty} \frac{1}{2}\phi(0,t)\, dt + \int_1^{+\infty} \frac{1}{2} \left(\phi\left(\frac{t-1}{2}, t\right) - \phi(0,t) \right) dt \\
&= -\int_0^1 \int_{-\infty}^{+\infty} \frac{\partial u}{\partial t}\phi\, dx\, dt - \int_0^1 \int_{-\infty}^{+\infty} \frac{\partial f(u)}{\partial x}\phi\, dx\, dt \\
&\quad - \int_{-\infty}^{+\infty} u(x,0)\phi(x,0)\, dx \\
&= -\int_{-\infty}^{+\infty} u(x,0)\phi(x,0)\, dx.
\end{aligned}$$

Then, for $t \geq 1$

$$u(x,t) = \begin{cases} 1 & \text{if } \; 2x+1 < t \\ 0 & \text{if } \; 2x+1 > t \end{cases}$$

II.3 Lax Entropy Condion

A discontinuity propagating with speed $s'(t)$ in the solution of a convex scalar conservation law is admissible only if:

$$f'(u_L) > s'(t) > f'(u_R)$$

where:

$$s'(t) = \frac{f(u_R) - f(u_L)}{u_R - u_L}$$

In our example, where

$$f'(u_L) = 1 > s' = \frac{1}{2} > f'(u_R) = 0$$

the solution we have found is an entropy solution.

In conclusion, the solution of equation (E2) with the initial condition

$$u_0(x) = \begin{cases} 1 & \text{if } x < -1 \\ -x & \text{if } -1 \leq x \leq 0 \\ 0 & \text{if } x > 0 \end{cases}$$

is as follows:

For $t < 1$:

$$u(x,t) = \begin{cases} 1 & \text{if } x < t-1 \\ \frac{-x}{1-t} & \text{if } t-1 \leq x \leq 0 \\ 0 & \text{if } x > 0 \end{cases}$$

For $t \geq 1$:

$$u(x,t) = \begin{cases} 1 & \text{if } 2x+1 < t \\ 0 & \text{if } 2x+1 > t \end{cases}$$

III Finite Volume Method

In this section, we limit the intervals of x and t to $[a,b]$ and $[0,T]$, respectively. With Neumann boundary conditions, the Burgers' equation with these conditions becomes:

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0 & \forall x \in]a,b[, \forall t > 0 \\ u(x,t=0) = u_0(x) & \forall x \in [a,b] \\ \frac{\partial u}{\partial x}(a,t) = \frac{\partial u}{\partial x}(b,t) = 0 & \forall t > 0 \end{cases} \qquad \text{where} f(u) = \frac{u^2}{2} \tag{3.4}$$

As we have done in Chapter 1, Section 1.2, we discretize $[a,b]$ into N control volumes $k_i =]x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}[$, with

$$a = x_{\frac{1}{2}} < x_{\frac{3}{2}} < x_{\frac{5}{2}} < \cdots < x_{N+\frac{1}{2}} = b$$

. We denote

$$h_i = |k_i| = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}$$
$$h_{i+\frac{1}{2}} = x_{i+1} - x_i$$

Similarly, we discretize the interval $[0,T]$ into M intervals, where each interval $[t^n, t^{n+1}]$ has a step size of $\Delta t = \frac{T}{M}$, with $t^0 = 0$, $t^1 = \Delta t$, $t^2 = 2\Delta t$, $\ldots$, $t^n = n\Delta t$.

Finite Volume Formulation:

We integrate (1.4) over each control volume K_i:

$$\int_{k_i} \frac{\partial u(x,t)}{\partial t}\,dx + \int_{k_i} \frac{\partial f(u(x,t))}{\partial x}\,dx = 0$$

Assuming u is not constant and $h = h_i = h_{i+\frac{1}{2}}$, we obtain:

$$\frac{\partial u_i(t)}{\partial t} + \frac{1}{h}(f(u(x_{i+\frac{1}{2}},t)) - f(u(x_{i-\frac{1}{2}},t))) = 0$$

Using an explicit scheme and the following approximations:

$$\frac{\partial u_i}{\partial t}(t^n) \approx \frac{u_i^{n+1} - u_i^n}{\Delta t}$$

$$F^n_{i+\frac{1}{2}} \approx \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} f(u(x_{i+\frac{1}{2}},t))\,dt$$

$$F^n_{i-\frac{1}{2}} \approx \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} f(u(x_{i-\frac{1}{2}},t))\,dt$$

we arrive at the following scheme:

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{h}(F^n_{i+\frac{1}{2}} - F^n_{i-\frac{1}{2}})$$

As previously mentioned, we assume $F^n_{i+\frac{1}{2}} = F(u_i^n, u_{i+1}^n)$. This leads to:

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{h}\left(F(u_i^n, u_{i+1}^n) - F(u_{i-1}^n, u_i^n)\right) \tag{3.5}$$

The stability condition for this scheme is:

$$\frac{||u||_{+\infty} dt}{h} \geq 1$$

IV Applying the Roe and HLL Schemes

IV.1 Roe Scheme

According to Chapter 1 (1.6), the Roe flux is defined as follows

$$F^n_{i+\frac{1}{2}} = \frac{1}{2}\left(f\left(u_i^n\right) + f\left(u_{i+1}^n\right)\right) - \frac{1}{2}|\tilde{A}(u_i, u_{i+1})|\left(u_{i+1}^n - u_i^n\right)$$

We know that $\tilde{A}(u_i, u_{i+1})$ is the constant matrix approximation of the Jacobian matrix $A(u) = \frac{\partial f(u)}{\partial u} = u$.

To determine the Roe matrix $\tilde{A}(u_i, u_{i+1}) = \tilde{A}$, we ensure that it satisfies the following criteria:

a) **Hyperbolicity of the system:**

- $\tilde{A}$ must be diagonalizable and possess real eigenvalues $\tilde{\lambda_1} \leq \tilde{\lambda_2} \leq \ldots \leq \tilde{\lambda_N}$.

b) **Consistency with the Exact Jacobian matrix:**

- $\tilde{A}(u,u) = A(u)$

c) **Conservation across discontinuities:**

- $f(v) - f(u) = \tilde{A} * (v - u)$.

c) Given that

$$\frac{u_{i+1}^2}{2} - \frac{u_i^2}{2} = \tilde{A} \cdot (u_{i+1} - u_i),$$

we find:

$$\tilde{A} = \frac{u_{i+1} + u_i}{2}.$$

Therefore,

$$\tilde{A}(u_i, u_{i+1}) = \frac{u_{i+1} + u_i}{2}.$$

Thus, the Roe flux scheme for the Burgers' equation is:

$$F_{i+\frac{1}{2}}^n = \frac{1}{2}\left(f\left(u_i^n\right) + f\left(u_{i+1}^n\right)\right) - \frac{1}{2}\left|\frac{u_{i+1} + u_i}{2}\right|\left(u_{i+1}^n - u_i^n\right). \tag{3.6}$$

Test: $[a, b] = [-2, 2]$, with the previously discussed initial condition.

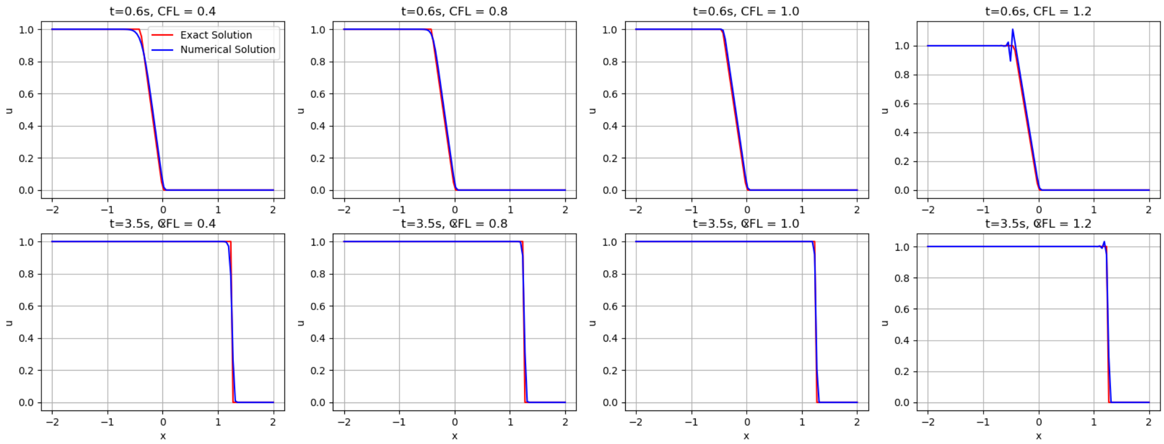

Figure 3.2: Comparison of the Exact Solution and the Numerical Solution Obtained Using the Roe Scheme for $N = 100$,

IV.2 HLL Scheme

The HLL scheme for the Burgers' equation is defined by:

$$F_{i+\frac{1}{2}} = F(u_i, u_{i+1}) = \begin{cases} \frac{u_i^2}{2} & \text{if} \quad 0 < S_L \\ \frac{S_R \frac{u_i^2}{2} - S_L \frac{u_{1+i}^2}{2} + S_L S_R (u_{i+1} - u_i)}{S_R - S_L} & \text{if} \quad S_L \leq 0 \leq S_R \\ \frac{u_{i+1}^2}{2} & \text{if} \quad 0 > S_R \end{cases}$$

Where $S_L = \min(u_{i+1}, u_i)$ and $S_R = \max(u_{i+1}, u_i)$

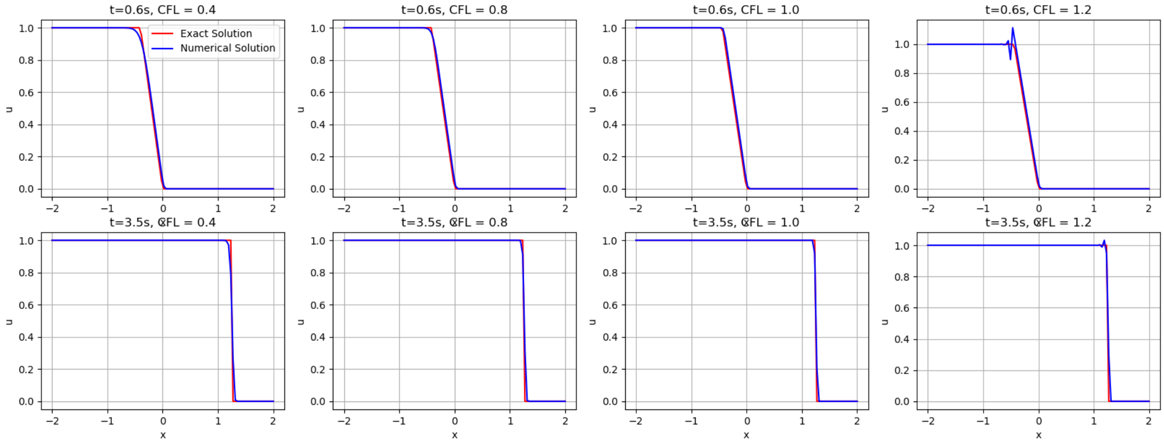

Figure 3.3: Comparison of the Exact Solution and the Numerical Solution Obtained Using the HLL Scheme for $N = 100$,

V Extension of the scheme to second-order in space

In this section, as mentioned in the first chapter, we assume that the numerical flux $F^n_{i+\frac{1}{2}}$ depends on two states $u^L_{i+\frac{1}{2}}$ and $u^R_{i+\frac{1}{2}}$. Without a slope limiter, these are defined as follows:

$$u^L_{i+\frac{1}{2}} = u_i + \frac{1}{2}(u_{i+1} - u_i)$$

$$u^R_{i+\frac{1}{2}} = u_{i+1} - \frac{1}{2}(u_{i+2} - u_{i+1})$$

With a slope limiter Φ:

$$u^L_{i+\frac{1}{2}} = u_i + \frac{1}{2}\Phi(r_i)(u_{i+1} - u_i)$$

$$u^R_{i+\frac{1}{2}} = u_{i+1} - \frac{1}{2}\Phi(r_{i+1})(u_{i+2} - u_{i+1})$$

V.1 Second-ordre Roe scheme

The second-order Roe flux for the Burgers' equation is defined as:

$$F_{i+\frac{1}{2}} = \frac{1}{2}\left(f\left(u^L_{i+\frac{1}{2}}\right) + f\left(u^L_{i+\frac{1}{2}}\right)\right) - \frac{1}{2}\left|\frac{u^n_{i+\frac{1}{2}} + u^n_{i-\frac{1}{2}}}{2}\right|\left(u^R_{i+\frac{1}{2}} - u^L_{i+\frac{1}{2}}\right)$$

Without limiter

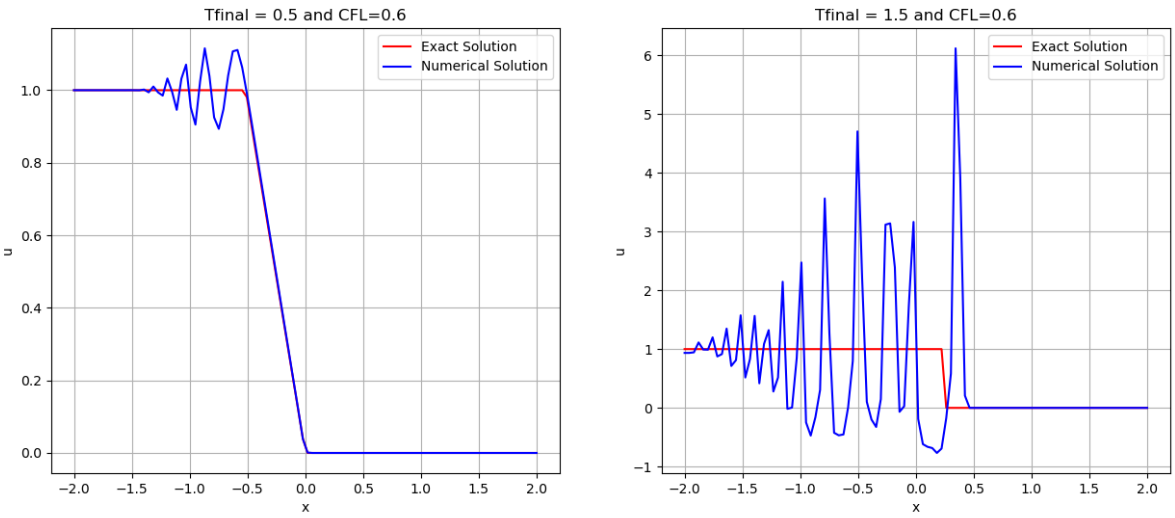

Figure 3.4: Comparison between the exact solution and the solution obtained using the second-order Roe scheme without limiter for $N = 100$,

With limiters (Minmod, VanLeer, Superbee, Van Alboda)

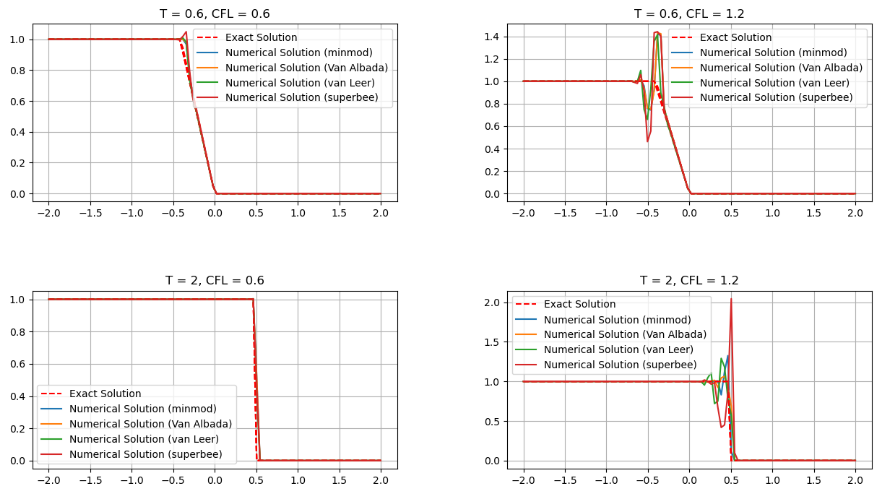

Figure 3.5: Comparison of Exact Solution vs. Second-Order Roe Scheme Solutions with Various Limiters for $N = 100$

V.2 Comparison between First-Order Roe Scheme and Various Second-Order Roe Schemes

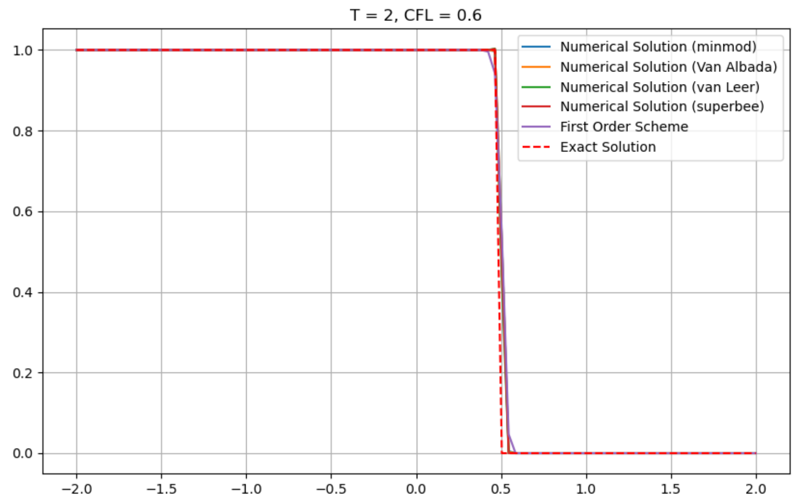

Figure 3.6: First-order and Second-order Roe Schemes

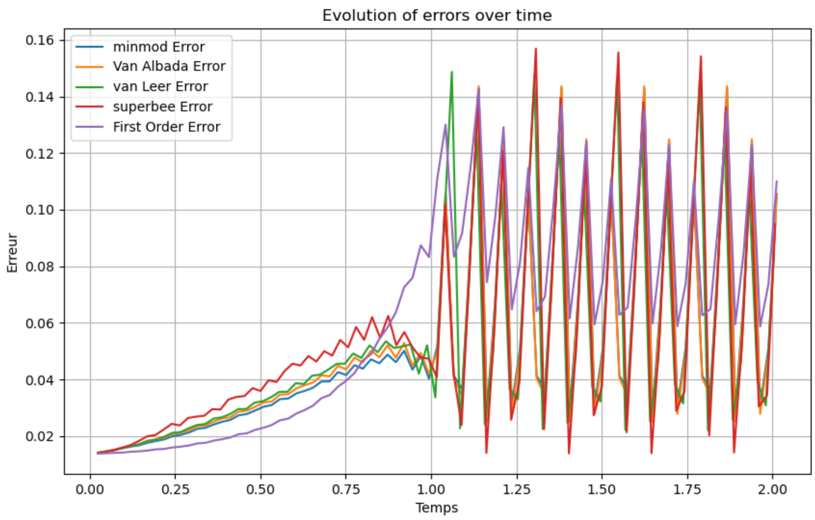

Figure 3.7: Evolution of errors in L^2 norm over time

V.3 Second-ordre HLL scheme

The second-order HLL flux for the Burgers' equation is defined as:

$$F(u^L_{i+\frac{1}{2}}, u^R_{i+\frac{1}{2}}) = \begin{cases} \frac{(u^L_{i+\frac{1}{2}})^2}{2} & \text{if} \quad 0 < S_L \\ \frac{S_R \frac{(u^L_{i+\frac{1}{2}})^2}{2} - S_L \frac{(u^R_{i+\frac{1}{2}})^2}{2} + S_L S_R (u^R_{i+\frac{1}{2}} - u^L_{i+\frac{1}{2}})}{S_R - S_L} & \text{if} \quad S_L \leq 0 \leq S_R \\ \frac{(u^R_{i+\frac{1}{2}})^2}{2} & \text{if} \quad 0 > S_R \end{cases}$$

Where $S_L = \min(u^L_{i+\frac{1}{2}}, u^R_{i+\frac{1}{2}})$ and $S_R = \max(u^L_{i+\frac{1}{2}}, u^R_{i+\frac{1}{2}})$

Without limiter

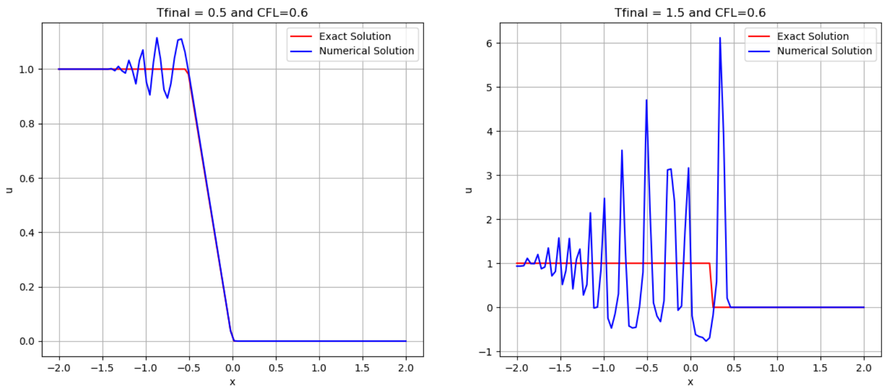

Figure 3.8: Comparison between the exact solution and the solution obtained using the second-order HLL scheme without limiter for $N = 100$,

With limiters (Minmod, VanLeer, Superbee, Van Alboda)

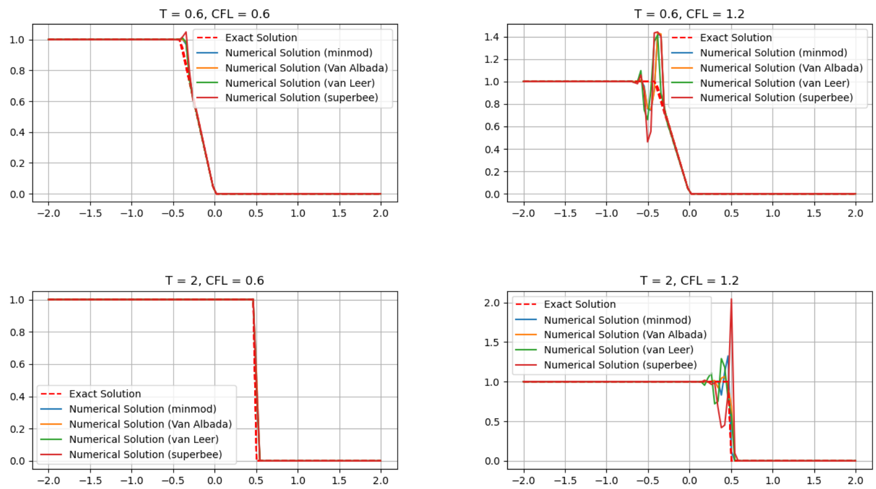

Figure 3.9: Comparison of Exact Solution vs. Second-Order HLL Scheme Solutions with Various Limiters for $N = 100$

V.4 Comparison between First-Order HLL Scheme and Various Second-Order HLL Schemes

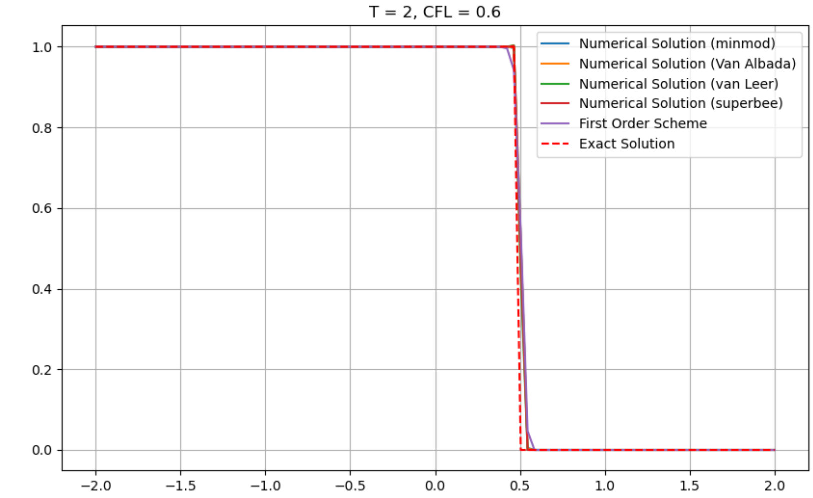

Figure 3.10: First-order and Second-order Roe Schemes

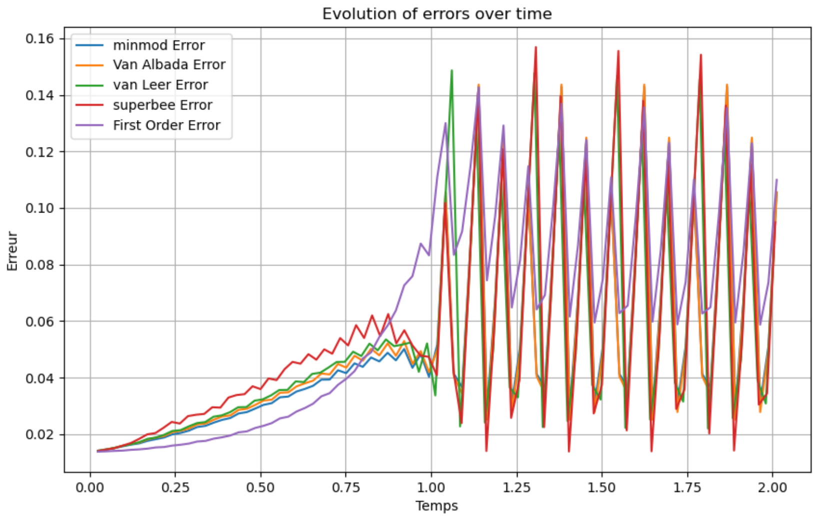

Figure 3.11: Evolution of errors in L^2 norm over time

Observation :
It has been noted that the largest error consistently arises with the first-order schemes, whereas the smallest error is observed with the superbee limiter.

Chapter 4

Saint-Venant equations

I Modeling of shallow water flow

To derive the equations of shallow water in one dimension, we consider a fluid in a channel of unit width and assume that the vertical velocity of the fluid is negligible, while the horizontal velocity $u(x,t)$ is approximately constant across any cross-section of the channel. This approximation holds true when considering small-amplitude waves in a shallow fluid compared to the wavelength.

Next, we assume the fluid is incompressible, hence the density $\bar{\rho}$ is constant. However, we allow the fluid depth to vary, denoted as $h(x,t)$, which we seek to determine. The total mass within $[x_1, x_2]$ at time t is given by

$$\int_{x_1}^{x_2} \bar{\rho} h(x,t)\, dx$$

The density of momentum at each point is $\bar{\rho} u(x,t)$, and vertically integrating this gives the mass flux $\bar{\rho} u(x,t) h(x,t)$. The constant $\bar{\rho}$ cancels out in the mass conservation equation, which then takes on the familiar form.

$$h_t + (uh)_x = 0 \tag{4.1}$$

The quantity hu is often referred to as discharge in shallow water theory, as it measures the flow of water passing through a point.
The conservation equation of momentum gives

$$(\bar{\rho} hu)_t + \left(\bar{\rho} hu^2 + p\right)_x = 0 \tag{4.2}$$

But now $\bar{p}$ is determined by a hydrostatic law, indicating that the pressure at a distance $h - y$ below the surface is $\bar{\rho} g(h-y)$, where g is the gravitational constant. This pressure simply results from the weight of the fluid above. By vertically integrating from $y = 0$ to $y = h(x,t)$, we obtain the total pressure felt at (x,t), the appropriate pressure term in the momentum flux:

$$p = \frac{1}{2} \bar{\rho} g h^2$$

Using this in (4.2) and setting $\bar{\rho}$ to zero, we obtain

$$(hu)_t + \left(hu^2 + \frac{1}{2} g h^2\right)_x = 0 \tag{4.3}$$

We can combine equations (4.1) and (4.3) in the system of one-dimensional shallow water equations.

$$\begin{bmatrix} h \\ hu \end{bmatrix}_t + \begin{bmatrix} uh \\ hu^2 + \frac{1}{2}gh^2 \end{bmatrix}_x = 0$$

The conservative form of the system is:

$$\frac{\partial W}{\partial t} + \frac{\partial f(W)}{\partial t} = 0$$

where

$$W = \begin{pmatrix} h \\ hu \end{pmatrix} \quad \text{and} \quad f(W) = \begin{pmatrix} uh \\ hu^2 + \frac{1}{2}gh^2 \end{pmatrix}$$

Before moving on to the finite volume method to solve this problem, we compute the Jacobian matrix denoted as $A(W)$ and diagonalize it.

$$A(W) = \frac{\partial f(W)}{\partial W} = \begin{pmatrix} 0 & 1 \\ -u^2 + gh & 2u \end{pmatrix}$$

Valeurs proprs de $A(W)$:

$$\det(A(W) - \lambda I) = 0 \Leftrightarrow \begin{vmatrix} -\lambda & 1 \\ -u^2 + gh & 2u - \lambda \end{vmatrix} = 0$$
$$\Longrightarrow \lambda^2 - 2u\lambda + u^2 - gh = 0$$
$$\Longrightarrow \begin{cases} \lambda_1 = u - \sqrt{gh} \\ \lambda_2 = u + \sqrt{gh} \end{cases} \quad D = \begin{pmatrix} u - \sqrt{gh} & 0 \\ 0 & u + \sqrt{gh} \end{pmatrix}$$

$\lambda_1 \neq \lambda_2 \Longrightarrow A(W)$ is diagonalizable. Therefore, $A = PDP^{-1}$.
Now let's compute the eigenvectors of $A(W)$.

$$Ar_1 = \lambda_1 r_1 \Longrightarrow r_1 = \begin{pmatrix} 1 \\ u - \sqrt{gh} \end{pmatrix}$$
$$Ar_2 = \lambda_2 r_2 \Longrightarrow r_2 = \begin{pmatrix} 1 \\ u + \sqrt{gh} \end{pmatrix}$$
$$\Longrightarrow P = \begin{pmatrix} 1 & 1 \\ u - \sqrt{gh} & u + \sqrt{gh} \end{pmatrix} \quad P^{-1} = \frac{1}{2\sqrt{gh}} \begin{pmatrix} u + \sqrt{gh} & -1 \\ \sqrt{gh} - u & 1 \end{pmatrix}$$

II Finite Volume Method

we want to solve:

$$\begin{cases} \frac{\partial W}{\partial t} + \frac{\partial f(W)}{\partial x} = 0, & \forall t > 0, \forall x \in]a, b[\\ W(0, x) = W_0(x), & \forall x \in [a, b] \end{cases} \tag{4.4}$$

This system is formulated exactly in the same manner as system (1.1) treated in the general case in Chapter 1. Therefore, we apply the same discretization and the same approximations used in the first chapter, with a fixed step size Δx, resulting in the following explicit scheme.

$$W_i^{n+1} = W_i^n - \frac{\Delta t}{\Delta x}(F(W_i^n, W_{i+1}^n) - F(W_{i-1}^n, W_i^n)) \tag{4.5}$$

II.1 HLL scheme and Roe scheme

HLL Scheme

The HLL scheme for the Saint-Venant equations is:

$$F_{i+\frac{1}{2}} = F(W_i, W_{i+1}) = \begin{cases} f(W_i) & \text{if} \quad 0 < S_L \\ \frac{S_R f(W_i) - S_L f(W_{i+1}) + S_L S_R (W_{i+1} - W_i)}{S_R - S_L} & \text{if} \quad S_L \leq 0 \leq S_R \\ f(W_{i+1}) & \text{if} \quad 0 > S_R \end{cases}$$

Where:
$S_L = \min(u_i - \sqrt{gh_i}, u_{i+1} - \sqrt{gh_{i+1}})$
$S_R = \max(u_i + \sqrt{gh_i}, u_{i+1} + \sqrt{gh_{i+1}})$

Roe Scheme

As we have already seen, the Roe flux is defined as follows:

$$F^n_{i+\frac{1}{2}} = \frac{1}{2}\left(f\left(W^n_i\right) + f\left(W^n_{i+1}\right)\right) - \frac{1}{2}|\tilde{A}(W_i, W_{i+1})|\left(W^n_{i+1} - W^n_i\right)$$

with $\tilde{A}(W_i, W_{i+1}) = A(\tilde{W})$, where $\tilde{W}$ is the Roe average state chosen such that

$$\tilde{A}(W_i, W_{i+1}) = A(\tilde{W})$$

satisfies the three conditions.

a) **Hyperbolicity of the system:**

- $\tilde{A}$ must be diagonalizable and possess real eigenvalues $\tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \ldots \leq \tilde{\lambda}_N$.

b) **Consistency with the Exact Jacobian matrix:**

- $\tilde{A}(W, W) = A(u)$

c) **Conservation across discontinuities:**

- $f(V) - f(W) = \tilde{A} * (V - W)$

For the Saint-Venant equations, the Roe state is

$$\tilde{W} = \begin{pmatrix} \tilde{h} \\ \tilde{h}\tilde{u} \end{pmatrix}$$

where

$$\tilde{h} = \frac{h_{i+1} + h_i}{2} \quad \text{and} \quad \tilde{u} = \frac{\sqrt{h_i}u_i + \sqrt{h_{i+1}}u_{i+1}}{\sqrt{h_i} + \sqrt{h_{i+1}}}$$

The stability condition for this schemes is :

$$\frac{max|\lambda_i|\Delta t}{\Delta x} \leq 1$$

III Dam breach

Dams are essential structures for water resource management, electricity generation, and flood control. They can be constructed from concrete, earth, or composite materials. Dam failure can occur suddenly or gradually due to various factors such as extreme floods, internal erosion, foundation issues, landslides, or earthquakes. The two main types of failure are overtopping, where water flows over the dam crest, and internal erosion, where underground water paths weaken the dam structure. Modeling these phenomena often utilizes the Saint-Venant equations to understand and predict the impacts of such failures.

In this project, we examine a torrential flow in a rectangular channel with a flat bottom and no friction. In this case, the problem is purely hyperbolic. A dam is placed in the middle of a channel with a length of L=1000 m and is abruptly removed at t = 0 s, causing a shock wave to propagate downstream in the channel. A one-dimensional problem with a known analytical solution, as given by Stoker (Stoker 1957), is used to assess the accuracy of the numerical algorithm

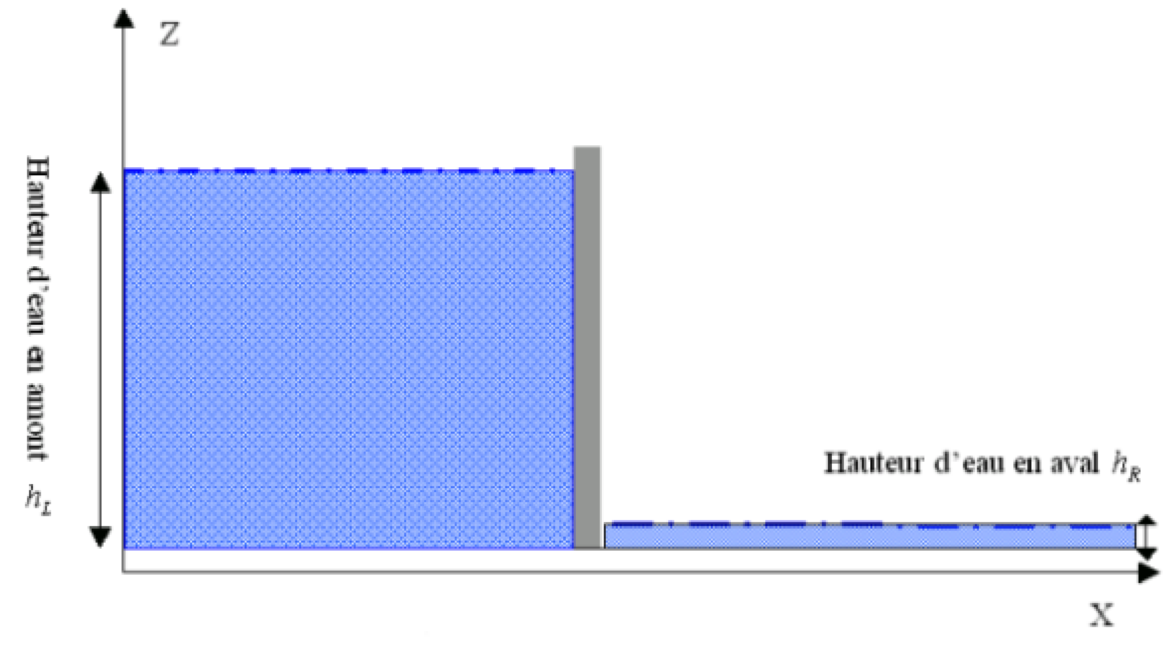

Figure 4.1: Flat-bottom dam

Let $x_0 \in \mathbb{R}$ be the position of the dam. We have an initial condition for the water height of Riemann type:

$$h_0(x) = \begin{cases} h_L & \text{si } x < x_0 \\ h_R & \text{si } x > x_0 \end{cases}$$

III.1 Dry dam breach ($h_R = 0$)

A dry-bed dam is a dam where the downstream area is initially devoid of water before the dam break. This means that the zone downstream of the dam is completely dry, with no water present.
Let's test both schemes in this situation. We consider the following initial conditions:

$$h_0(x) = \begin{cases} 5 & \text{if } 0 \leq x \leq 500 \text{ m} \\ 0 & \text{if } 500 \text{ m} \leq \text{x} \leq 1000 \text{ m} \end{cases}$$

and the initial velocity of the flow is $\mathrm{u}(0, \mathrm{x}) = 0$ m/s. Let's use Neumann boundary conditions, i.e., $\frac{\partial W(0,t)}{\partial x} = \frac{\partial W(900,t)}{\partial x} = 0, \ \forall t > 0$
The analytical calculation of the dam-break wave propagation on smooth dry bed has been performed for the first time by Ritter. It is given by

$$h(x,t) = \begin{cases} h_1 \\ \frac{1}{9g}\left(2\sqrt{gh_1} - \frac{x-x_0}{t}\right)^2 \\ 0 \end{cases}$$

and

$$u(x,t)=\begin{cases}0 & \text{if } x\le x_1\\ \frac{2}{3}\left(\sqrt{gh_1}+\frac{x-x_0}{t}\right) & \text{if } x_1\le x\le x_2\\ 0 & \text{if } x_2\le x\end{cases}$$

where $x_1=x_0-t\sqrt{gh_1}, x_2=x_0+2t\sqrt{gh_1}$

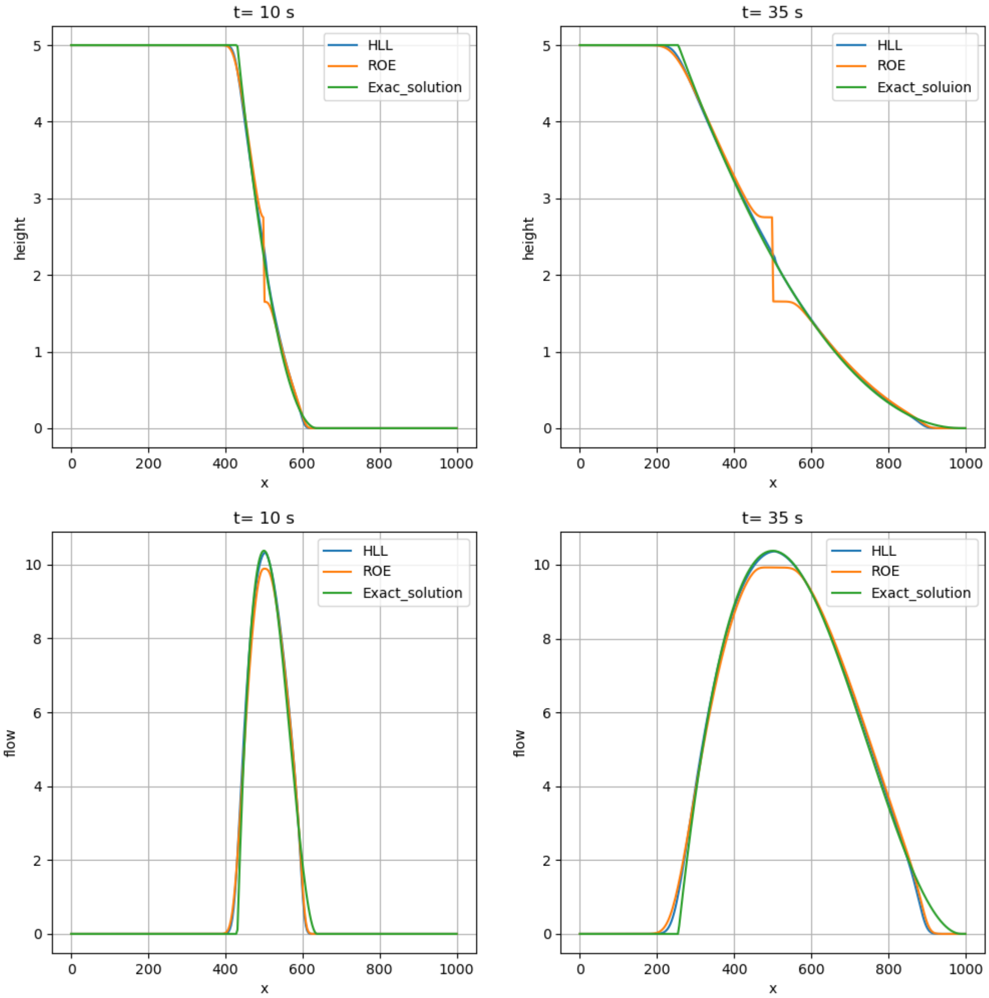

Figure 4.2: Dam break on dry bed for CFL=0.6, 300 mesh

III.2 Wet-bed dam break ($h_R \neq 0$)

A wet-bed dam is a dam where the downstream area already contains water before the dam break. This means that the downstream zone is not dry and has an initial water height present. Let's test both schemes in this situation. We consider the following initial conditions:

$$h_0(x)=\begin{cases}5 & \text{si } 0\le x\le 500 \text{ m}\\ 1 & \text{si 500 m} \le \text{x} \le \text{1000 m}\end{cases}$$

and the initial velocity of the flow is $\text{u}(0,\text{x})=0$ m/s
the water height and the speed obtained by our model will be compared with the ana-

lytical solution which is obtained thanks to the characteristics method as (Stoker 1957)

$$h(x,t) = \begin{cases} h_1 & \\ \frac{1}{9g}\left(2\sqrt{gh_1} - \frac{x-x_0}{t}\right)^2 & \\ h_m & \\ h_2 & \text{if } x \leq x_1 \end{cases}$$

$$u(x,t) = \begin{cases} 0 & \text{if } x_1 \leq x \leq x_2 \\ \frac{2}{3}\left(\sqrt{gh_1} + \frac{x-x_0}{t}\right) & \text{if } x_2 \leq x \leq x_3 \\ u_m & \text{if } x_3 \leq x \\ 0 & \end{cases}$$

Where

$$\begin{aligned} & x_1 = x_0 - t\sqrt{gh_1}, x_2 = x_0 + \left(u_m - \sqrt{gh_m}\right)t, \\ & x_3 = x_0 + \text{ s.t } = x_0 + t \cdot \sqrt{\frac{gh_m}{2}\left(\frac{h_m}{h_R} + 1\right)} \end{aligned}$$

$$\begin{aligned} h_m &= \frac{1}{2}\left(\sqrt{1 + \frac{8s^2}{gh_R}} - 1\right)h_L \\ u_m &= s - \frac{gh_R}{4s}\left(\sqrt{1 + \frac{8s^2}{gh_R}} + 1\right) \\ u_m &= 2\sqrt{gh_L} - 2\sqrt{gh_m} \end{aligned} \tag{4.6}$$

The first two equations in (4.6) describe the Rankine- Hugoniot relations at the front of the discontinuity shock. The last equation represents the invariable conservation of Riemann. The resolution of this nonlinear system, by a Newton-Raphson method, gives the following values of h_m, u_m and s:

$$h_m = 2.534 \text{ m}, \quad u_m = 4.03 \text{ m/s}, \quad \text{and} \quad s^2 = \frac{gh_m}{2}\left(\frac{h_m}{h_r} + 1\right)$$

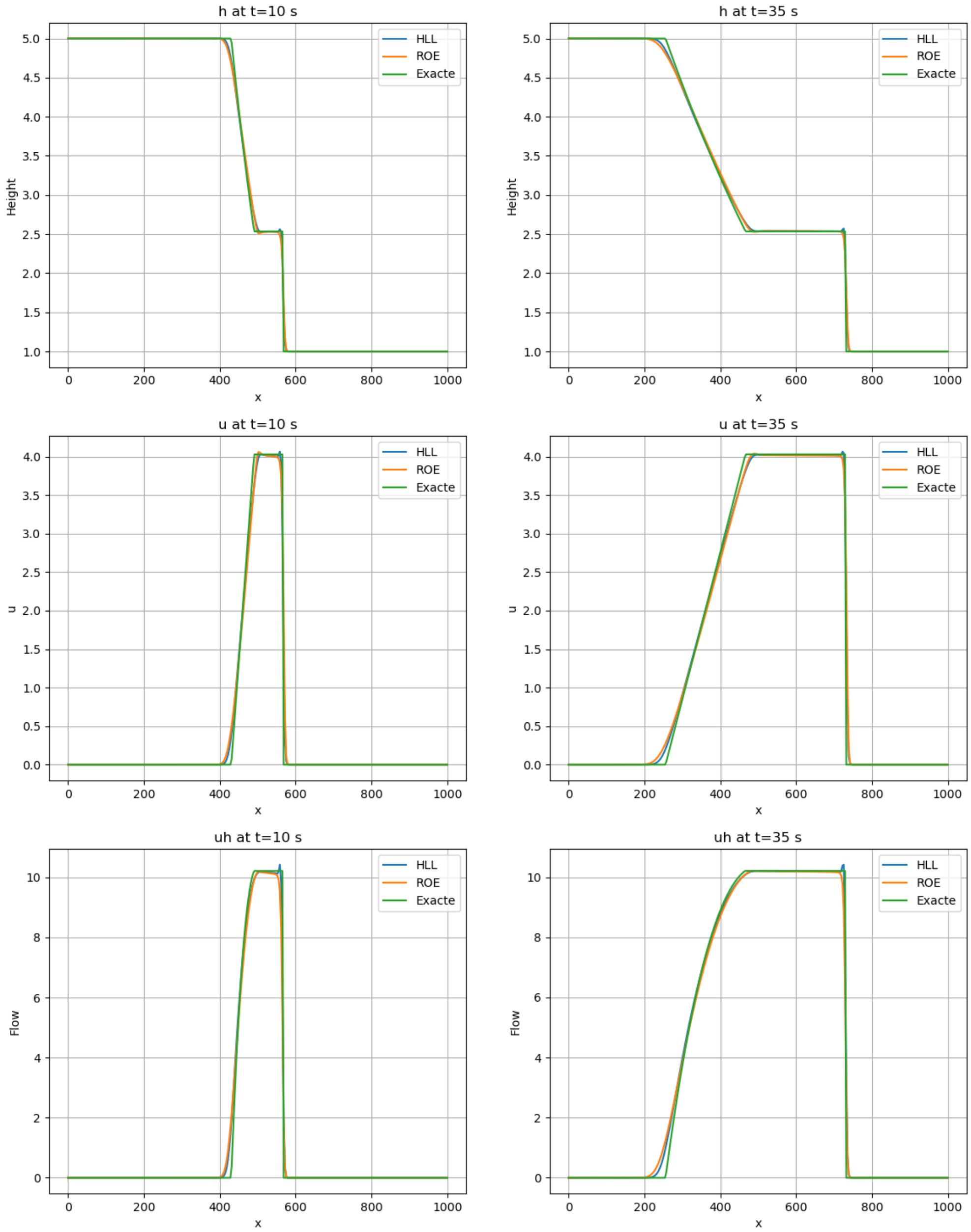

Figure 4.3: Dam break on wet bed for CFL=0.6, 300 mesh

Conclusion

This project offers a detailed comparison of finite volume methods for the simulation of shallow water flows, highlighting the advantages and limitations of each approach. The results obtained provide valuable insights for choosing appropriate numerical schemes for specific applications in the field of fluid dynamics.

Bibliography

Appendices