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CLASSICAL FIFTH-, SIXTH-, SEVENTH-, AND EIGHTH-ORDER RUNGE-KUTTA FORMULAS WITH STEPSIZE CONTROL

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PART III. SEVENTH-ORDER FORMULAS

SECTION XI. THE EQUATIONS OF CONDITION FOR THE RUNGE-KUTTA COEFFICIENTS

44. For a seventh-order Runge-Kutta formula RK7(8) with stepsize control we allow for thirteen evaluations of the differential equations per step:

$$
\left.\begin{aligned}
f_0 &= f(x_0,\ y_0) \\
f_\kappa &= f\Big(x_0 + \alpha_\kappa h,\ y_0 + h \sum_{\lambda=0}^{\kappa-1} \beta_{\kappa\lambda} f_\lambda\Big) \quad (\kappa = 1, 2, 3, \ldots, 12)
\end{aligned}\right\}
$$

$$
\left.\begin{aligned}
y &= y_0 + h \sum_{\kappa=0}^{10} c_\kappa f_\kappa + 0(h^8) \\
\hat{y} &= y_0 + h \sum_{\kappa=0}^{12} \hat{c}_\kappa f_\kappa + 0(h^9)
\end{aligned}\right\}
$$

45. We now need the equations of condition for the eighth-order terms. These are also listed in BUTCHER's paper ([3], Table 1). There are 115 such equations, and we shall refer to them as equations (VIII, 1) through (VIII, 115) - in the same order as in BUTCHER's paper.

46. To reduce these 115 necessary and sufficient conditions to a system of simpler and fewer sufficient conditions, we make - very similar as in Part I and in Part II - the following assumptions:

$$
\left.\begin{aligned}
&\alpha_{10} = \alpha_{12} = 1, \quad \alpha_{11} = 0; \quad \hat{c}_1 = c_1 = 0, \quad \hat{c}_2 = c_2 = 0, \quad \hat{c}_3 = c_3 = 0, \\
&\hat{c}_4 = c_4 = 0, \quad c_5 = \hat{c}_5, \quad c_6 = \hat{c}_6, \quad c_7 = \hat{c}_7, \quad c_8 = \hat{c}_8, \quad c_9 = \hat{c}_9, \\
&\hat{c}_{10} = 0, \quad \hat{c}_{11} = \hat{c}_{12} = c_{10} \\
&\beta_{31} = \beta_{41} = \beta_{51} = \beta_{61} = \beta_{71} = \beta_{81} = \beta_{91} = \beta_{101} = \beta_{111} = \beta_{121} = 0 \\
&\beta_{52} = \beta_{62} = \beta_{72} = \beta_{82} = \beta_{92} = \beta_{102} = \beta_{112} = \beta_{122} = 0
\end{aligned}\right\}
$$

and:

TABLE X. RK 7(8)

$\kappa \backslash \lambda$	α_κ	$\beta_{\kappa\lambda}$ 0	1	2	3	4	5	6	7	8	9	10	11	c_κ	$\hat{c}_\kappa$
0	0	0												$\frac{41}{840}$	0
1	$\frac{2}{27}$	$\frac{2}{27}$												0	
2	$\frac{1}{9}$	$\frac{1}{36}$	$\frac{1}{12}$											0	
3	$\frac{1}{6}$	$\frac{1}{24}$	0	$\frac{1}{8}$										0	
4	$\frac{5}{12}$	$\frac{5}{12}$	0	$-\frac{25}{16}$	$\frac{25}{16}$									0	
5	$\frac{1}{2}$	$\frac{1}{20}$	0	0	$\frac{1}{4}$	$\frac{1}{5}$								$\frac{34}{105}$	
6	$\frac{5}{6}$	$-\frac{25}{108}$	0	0	$\frac{125}{108}$	$-\frac{65}{27}$	$\frac{125}{54}$							$\frac{9}{35}$	
7	$\frac{1}{6}$	$\frac{31}{300}$	0	0	0	$\frac{61}{225}$	$-\frac{2}{9}$	$\frac{13}{900}$						$\frac{9}{35}$	
8	$\frac{2}{3}$	2	0	0	$-\frac{53}{6}$	$\frac{704}{45}$	$-\frac{107}{9}$	$\frac{67}{90}$	3					$\frac{9}{280}$	
9	$\frac{1}{3}$	$-\frac{91}{108}$	0	0	$\frac{23}{108}$	$-\frac{976}{135}$	$\frac{311}{54}$	$-\frac{19}{60}$	$\frac{17}{6}$	$-\frac{1}{12}$				$\frac{9}{280}$	
10	1	$\frac{2383}{4100}$	0	0	$-\frac{341}{164}$	$\frac{4496}{1025}$	$-\frac{301}{82}$	$\frac{2133}{4100}$	$\frac{45}{82}$	$\frac{45}{164}$	$\frac{18}{41}$			$\frac{41}{840}$	0
11	0	$\frac{3}{205}$	0	0	0	0	$-\frac{6}{41}$	$-\frac{3}{205}$	$-\frac{3}{41}$	$\frac{3}{41}$	$\frac{6}{41}$	0			$\frac{41}{840}$
12	1	$-\frac{1777}{4100}$	0	0	$-\frac{341}{164}$	$\frac{4496}{1025}$	$-\frac{289}{82}$	$\frac{2193}{4100}$	$\frac{51}{82}$	$\frac{33}{164}$	$\frac{12}{41}$	0	1		$\frac{41}{840}$

Truncation Error Term: $\text{TE} = \frac{41}{840}(f_0 + f_{10} - f_{11} - f_{12})h$ (134)