Jacobians

Definition

Forward model:

$$\vec{y} = F(\vec{x}) + \vec{\varepsilon}$$

$\vec{\varepsilon}$ ← Noise

Jacobian:

$$J_{a,b} = \frac{\partial F_a}{\partial x_b}$$

Perhaps i, j would have been a better choice for the indices

In Matlab notation:

$$J(a,b) = \frac{\partial y(a)}{\partial x(b)}$$

- How element a of vector $\vec{y}$ changes if I vary element b of vector $\vec{x}$

Calculation Method A: Perturbation

The naive method that always works, but that is slow.

$$J(i,j) = \frac{\partial y(i)}{\partial x(j)} \approx \frac{\Delta y(i)}{\Delta x(j)}$$

$$= \frac{\vec{F}(\vec{x}+\vec{\delta}_j) - \vec{F}(\vec{x})}{\delta}$$

$$\vec{\delta}_j = \begin{bmatrix} 0 \\ \vdots \\ \delta \\ \vdots \\ 0 \end{bmatrix} \leftarrow j\text{th element}$$

Perhaps $\Delta\vec{x}$ would be better notation for δ_j.)

Why is this relatively inefficient?

I have to run the full model F for every element of $\vec{x}$, perturbing them one at a time.

Analytical method

Integral form of RTE:
(written for only 1 frequency)

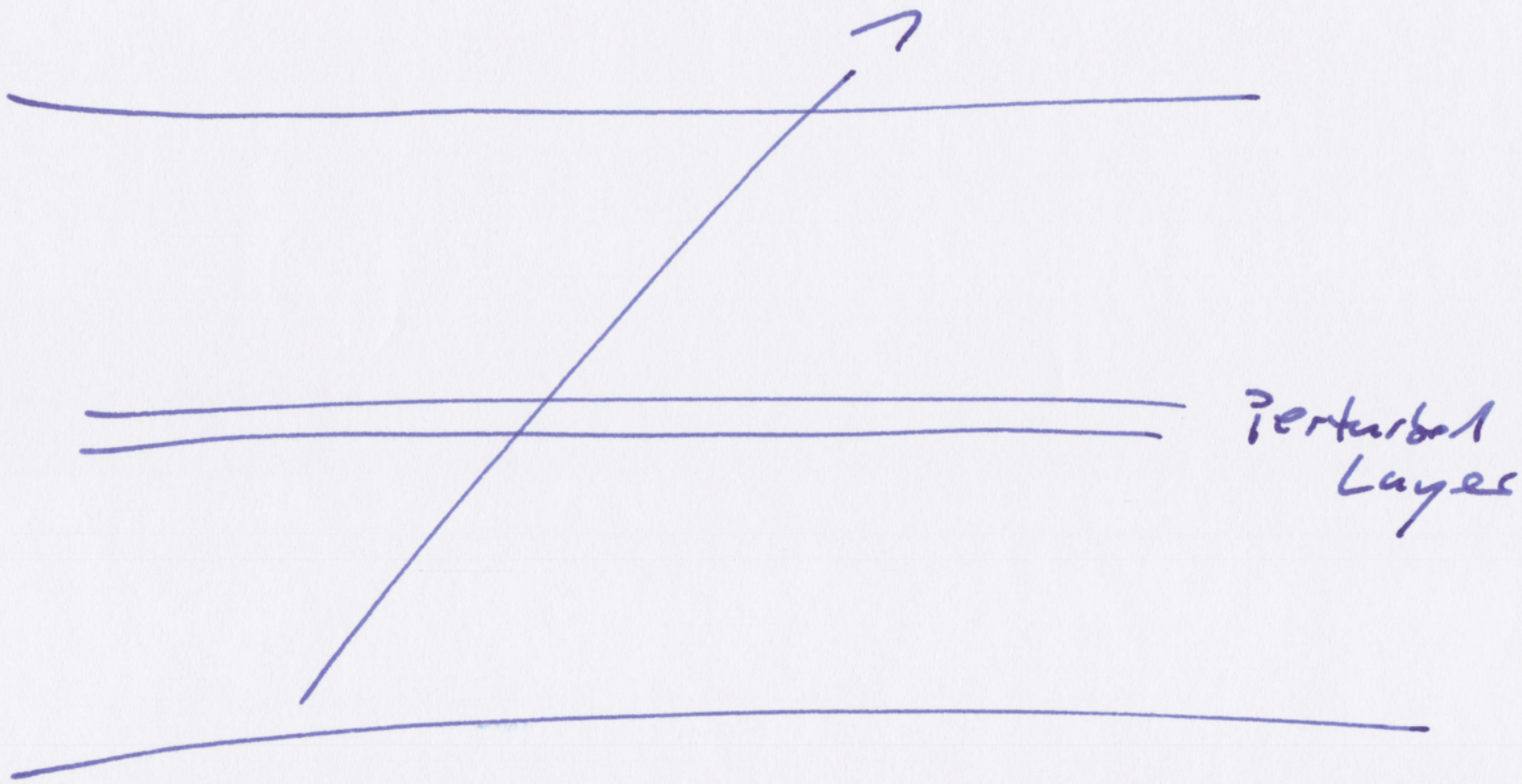
$$y = I = I_0 e^{-\int_0^{\infty} \alpha(s')\,ds'} + \int_0^{\infty} \alpha(s)\, B(T(s))\, e^{-\int_0^{s} \alpha(s')\,ds'}\, ds$$

Disknet:

$$I = I_0 \underbrace{e^{-\sum_0^{N} \alpha_j \Delta s_j}}_{\text{Transmissivity}} + \sum_{i=0}^{N} \alpha_i \Delta s_i B(T_i) \underbrace{e^{-\sum_{j=0}^{i} \alpha_j \Delta s_j}}_{\text{transmissivity}}$$

opacity

emission


Perturbed
Layer

If the atmosphere is perturbed at one position, this will affect the emission at that position, and the transmission of everything behind.

Emission:

$$\frac{\partial}{\partial x_k}\left(\alpha_k \Delta s_k B(\bar{T}_k)\right) = \Delta s_k B(\bar{T}_k) \frac{\partial \alpha_k}{\partial x_k}$$

If x is the number density of some gas(n):

$$\alpha = n \cdot \sigma \quad \rightarrow \quad \frac{\partial \alpha}{\partial n} = \sigma$$

↑ absorption cross-section

Transmission:

$$\tau = \sum \alpha_i \Delta s_i = \alpha_1 \Delta s_1 + \alpha_2 \Delta s_2 + \ldots + \alpha_N \Delta s_N$$

$$\frac{\partial \tau}{\partial x_k} = \Delta s_k \frac{\partial \alpha_k}{\partial x_k}$$

Pros and Cons of different Jacobians

Numerical Perturbation

- (+) Easy to implement
- (+) More "foolproof"
- (+) Works for any parameter in a "black-box" kind of way

Analytical

- (−) Needs a lot of "housekeeping"
- (+) More accurate
- (+) Faster
- (−) Not always feasible

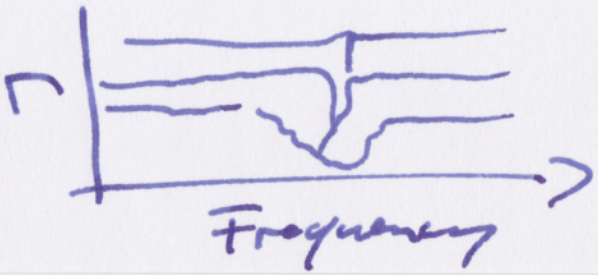
Displaying Jacobians

Since the Jacobian is a matrix, there are three basic ways to display it

1. ~~Bar~~ Contour plot

2. Row plot
   (altitude dependence for selected frequencies)

3. Column plot
   (frequency dependence for selected altitudes)



Opacity Rule

Jacobians tell us, <u>where the information comes from</u>.
(Where do changes in the atmosphere most affect the radiance.)

The opacity rule also tells us this:
Origin of (thermal) radiation is the distance where the opacity, as calculated <u>from</u> the observer, reaches 1.