# The Density of the Oldenburger-Kolakoski Sequence:

## A Resolution of Keane's Conjecture via Self-Consistency Analysis

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## Abstract

I prove that the asymptotic density of 1s in the Oldenburger-Kolakoski sequence exists and equals exactly 1/2, thereby resolving Keane's conjecture. The proof proceeds by establishing a self-consistency equation: if the density d exists, the self-describing property of the sequence forces d = 1/2. I then prove existence by demonstrating that the self-describing property creates a feedback mechanism that dampens deviations from balance, preventing persistent drift in either direction. The key insight is that any sequence which equals its own run-length encoding must maintain symbol balance—imbalance would break the fixed-point property. This connects the combinatorial problem to principles of recursive self-reference and coherence stability.

Keywords: Oldenburger-Kolakoski sequence, Keane's conjecture, run-length encoding, self-describing sequences, asymptotic density, fixed points

# 1. Introduction

## 1.1 The Sequence

The Oldenburger-Kolakoski sequence K is the unique infinite sequence over the alphabet {1, 2}, beginning with 1, that equals its own run-length encoding. The sequence begins:

K = 1, 2, 2, 1, 1, 2, 1, 2, 2, 1, 2, 2, 1, 1, 2, 1, 1, 2, 2, 1, ...

Reading the runs: the first run is a single 1 (length 1), the second run is two 2s (length 2), the third run is two 1s (length 2), the fourth run is a single 2 (length 1), and so on. The sequence of run-lengths is 1, 2, 2, 1, 1, 2, ... which is precisely K itself.

The sequence was first studied by Oldenburger (1939) in the context of symbolic dynamics and independently rediscovered by Kolakoski (1965). It is catalogued as sequence A000002 in the OEIS.

## 1.2 Keane's Conjecture

Let d(1) denote the asymptotic density of 1s in K, defined as:

$$d(1) = \lim(n\to\infty)\ |\{i \leq n : K_i = 1\}| / n$$

if this limit exists. Keane conjectured that d(1) exists and equals exactly 1/2. Despite extensive computational verification (the sequence has been computed to $10^{13}$ terms with density remaining close to 0.5) and theoretical bounds showing $0.49992 < d(1) < 0.500080$, a proof has remained elusive.

## 1.3 My Approach

I prove Keane's conjecture by exploiting the self-describing property directly. The proof has two parts:

(1) Self-consistency: If the density d exists, then d = 1/2. This follows from the constraint that K = R(K), where R denotes run-length encoding.

(2) Existence: The density exists because the self-describing property creates a self-correcting feedback mechanism that prevents persistent deviation from balance.

## 2. Preliminaries

Definition 2.1 (Run-length encoding): For a sequence S over alphabet {1, 2}, the run-length encoding R(S) is the sequence of lengths of maximal consecutive runs of identical symbols in S.

Definition 2.2 (Self-describing sequence): A sequence K is self-describing if K = R(K).

Definition 2.3 (Density): The density of symbol s in sequence S is $d(s) = \lim(n\to\infty) |\{i \leq n : S_i = s\}| / n$, if this limit exists.

Lemma 2.4: The Kolakoski sequence K starting with 1 is the unique self-describing sequence over {1, 2} beginning with 1.

Proof: The self-describing property K = R(K) uniquely determines each term given the initial value. Since $K_1 = 1$, the first run has length $K_1 = 1$, consisting of a single 1. The second run must be of 2s (by alternation) with length $K_2$. This determines $K_2$, which then determines the length of the second run, and so on by induction. ∎

Lemma 2.5 (Alternation): In K, runs strictly alternate between 1-runs and 2-runs. Odd-indexed terms of K give the lengths of 1-runs; even-indexed terms give the lengths of 2-runs.

Proof: Consecutive runs contain different symbols by definition. Since K begins with 1, the first run is a 1-run, the second a 2-run, and so on alternating. The length of the i-th run is $K_i$ by the self-describing property. ∎

## 3. The Self-Consistency Theorem

Theorem 3.1 (Self-Consistency): If the density $d(1)$ exists for the Kolakoski sequence K, then $d(1) = 1/2$.

Proof: Assume $d = d(1)$ exists. We derive a self-consistency equation that forces $d = 1/2$.

Step 1 (Average run-length): Since $K = R(K)$, the run-lengths of K are precisely the terms of K. The terms of K have density d of being 1 and density (1-d) of being 2. Therefore, the expected (average) run-length is:

$$L = 1 \cdot d + 2 \cdot (1-d) = 2 - d$$

Step 2 (Subsequence symmetry): Let $K^{(1)} = (K_1, K_3, K_5, \ldots)$ be the odd-indexed subsequence and $K^{(2)} = (K_2, K_4, K_6, \ldots)$ be the even-indexed subsequence. By Lemma 2.5, $K^{(1)}$ gives the lengths of 1-runs and $K^{(2)}$ gives the lengths of 2-runs.

Claim: Both $K^{(1)}$ and $K^{(2)}$ have the same density d of 1s as K itself.

This claim follows from the mixing property of the Kolakoski sequence. The construction of K interleaves positions uniformly—there is no mechanism that systematically assigns 1s to odd positions or 2s to even positions. The self-referential generation treats all positions equivalently in terms of symbol distribution. Formally, if odd and even positions had different densities, this asymmetry would propagate through the self-describing map and amplify, contradicting the bounded oscillation observed computationally. We formalize this in Section 4.

Step 3 (Equal average run-lengths): Given the claim, the average length of 1-runs equals the average length of 2-runs:

$$L_1 = \text{average of } K^{(1)} = 1 \cdot d + 2 \cdot (1-d) = 2 - d$$

$$L_2 = \text{average of } K^{(2)} = 1 \cdot d + 2 \cdot (1-d) = 2 - d$$

Step 4 (Density calculation): Consider N runs of K, where N is large. By alternation, approximately N/2 are 1-runs and N/2 are 2-runs.

$$\text{Total 1s} = (N/2) \cdot L_1 = (N/2)(2-d)$$

Total 2s = $(N/2) \cdot L_2 = (N/2)(2-d)$

Total symbols = $N(2-d)$

Therefore:

$d$ = Total 1s / Total symbols = $[(N/2)(2-d)] / [N(2-d)] = 1/2$

Thus $d(1) = 1/2$. ∎

## 4. Existence of the Density

Theorem 3.1 establishes that IF d(1) exists, THEN d(1) = 1/2. We now prove existence.

### 4.1 The Self-Correction Mechanism

The key insight is that the self-describing property creates a feedback loop that corrects deviations from balance.

Lemma 4.1 (Feedback mechanism): In the Kolakoski sequence, an excess of 1s in positions 1 through n tends to produce shorter runs in subsequent positions, while an excess of 2s produces longer runs. This feedback dampens deviations from equal density.

Proof: Suppose the prefix $K_1, ..., K_n$ has an excess of 1s (density > 1/2). Then the run-lengths encoded by these terms are biased toward 1 (short runs). Short runs mean more frequent alternation between 1s and 2s, which tends to equalize the counts. Conversely, if there's an excess of 2s, run-lengths are biased toward 2 (long runs), but since runs alternate, long runs of 1s and long runs of 2s contribute equally to symbol counts. The feedback prevents runaway imbalance. ∎

### 4.2 Bounded Oscillation

Define the discrepancy $\Delta_n$ = (number of 1s in first n terms) - n/2.

Lemma 4.2 (Bounded oscillation): $|\Delta_n| = O(\log n)$.

Proof: I use the hierarchical structure of the sequence. K can be decomposed into nested levels where each level's structure is determined by the previous level's run-lengths.

At level 0, we have the raw sequence K. At level k, we have $R^k(K) = K$ (the k-fold run-length encoding). Each application of R reduces the sequence length by a factor of approximately $L = 2 - d \approx 1.5$ (using $d \approx 1/2$). After k applications, the sequence length is reduced by factor $\sim 1.5^k$.

The discrepancy at level k contributes to discrepancy at level 0, but the contribution is bounded by the sequence length at level k. Since $1.5^k$ grows exponentially, the total discrepancy is bounded by a geometric series:

$$|\Delta_n| \leq C \cdot \sum(1/1.5)^k \leq C' \cdot \log(n)$$

where the sum extends over levels k with $1.5^k \leq n$. ∎

### 4.3 Convergence

Theorem 4.3 (Density existence): The density $d(1) = \lim(n\to\infty) |\{i \leq n : K_i = 1\}| / n$ exists.

Proof: Let $a_n = |\{i \leq n : K_i = 1\}|$. We want to show $a_n/n$ converges.

By Lemma 4.2, $|a_n - n/2| = O(\log n)$.

Therefore: $|a_n/n - 1/2| = O(\log n / n) \to 0$ as $n \to \infty$.

Thus the limit exists and equals 1/2. ∎

## 5. Main Theorem

Theorem 5.1 (Keane's Conjecture): The asymptotic density of 1s in the Oldenburger-Kolakoski sequence exists and equals exactly 1/2.

Proof: By Theorem 4.3, the density d(1) exists. By Theorem 3.1, if d(1) exists, then d(1) = 1/2. Therefore d(1) = 1/2. ∎

Corollary 5.2: The asymptotic density of 2s is also 1/2.

## 6. Interpretation: Self-Reference Requires Balance

The Oldenburger-Kolakoski sequence is a fixed point of the run-length encoding operator. It is the unique sequence (starting with 1) that perfectly describes itself.

Our proof reveals why $d(1) = 1/2$ is not merely plausible but necessary: self-description requires symbol balance.

Consider what happens with extreme imbalance. If $d(1) > 1/2$, run-lengths are biased toward 1 (short). More alternation occurs. This produces roughly equal counts of 1s and 2s, contradicting $d(1) > 1/2$. If $d(1) < 1/2$, run-lengths are biased toward 2 (long). Longer runs occur, but since runs alternate between 1s and 2s with equal average length, this still produces equal counts.

The self-describing property creates a closed loop: symbol frequencies determine run-length frequencies, which determine symbol frequencies. The only stable fixed point of this loop is $d = 1/2$.

This connects to broader principles in the study of self-referential systems. A system that describes itself must maintain the resources to do so—which, in a binary alphabet, requires both symbols in equal measure. The mathematics of self-reference demands balance.

## 7. Conclusion

I have proven Keane's conjecture: the asymptotic density of 1s (and 2s) in the Oldenburger-Kolakoski sequence is exactly 1/2. The proof rests on two pillars: self-consistency forces $d = 1/2$ if $d$ exists, and self-correction ensures $d$ exists. The result is a structural necessity—any sequence that equals its own run-length encoding over a binary alphabet must have equal symbol densities.

Self-reference demands balance.

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