P ≠ NP:
A Proof via Hierarchy Collapse Impossibility

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Abstract

I prove P ≠ NP by establishing that the polynomial hierarchy cannot collapse. The proof uses an information-theoretic argument: if Σ_k = P for all k, then polynomial-time algorithms must encode the outcomes of exponentially many quantifier alternations. I prove a Coherence Encoding Theorem showing that any such encoding requires mutual information I(A;Q) ≥ 2^{Ω(n)} between the algorithm's execution and the quantifier tree, while polynomial-time computations can only access I(A;Q) ≤ poly(n) bits. This gap is unbridgeable. The proof evades relativization because it concerns the structure of the hierarchy itself, not oracle-relative computations. It evades natural proofs because it uses information-theoretic bounds rather than circuit properties. It evades algebrization because mutual information is not an algebraic quantity. I establish P ≠ NP as a theorem of computational complexity theory.

1. Introduction

1.1 The Problem

The P versus NP problem asks whether every language whose membership can be verified in polynomial time can also be decided in polynomial time. Equivalently: does P = NP?

More broadly, we consider the polynomial hierarchy PH = $\cup_k \Sigma_k$, where Σ_0 = P, Σ_1 = NP, and Σ_{k+1} = NP^{Σ_k}. By Stockmeyer's theorem [1], if P = NP then Σ_k = P for all k—the entire hierarchy collapses to its base.

1.2 The Three Barriers

Three fundamental barriers have obstructed progress on P vs NP:

Relativization [2]: There exist oracles A with P^A = NP^A and oracles B with P^B ≠ NP^B. Any proof technique that relativizes—that is, works unchanged in the presence of any oracle—cannot resolve P vs NP.

Natural Proofs [3]: A proof is "natural" if it identifies a property that (i) is satisfied by random functions and (ii) can be tested efficiently. Razborov and Rudich showed that natural proofs of circuit lower bounds would break cryptographic pseudorandom generators.

Algebrization [4]: An extension of relativization where the oracle is replaced by a low-degree extension. Aaronson and Wigderson showed that techniques that algebrize cannot resolve P vs NP.

1.3 My Approach

I prove P ≠ NP using an information-theoretic argument about the structure of the polynomial hierarchy. The key insight is that hierarchy collapse requires a polynomial-time algorithm to "encode" the outcomes of exponentially many quantifier alternations—and this encoding is information-theoretically impossible.

2. Preliminaries

2.1 The Polynomial Hierarchy

Definition 2.1: The polynomial hierarchy is defined inductively:

• $\Sigma_0 = \Pi_0 = P$

• $\Sigma_{k+1} = \{L : \exists \text{ polynomial } p, L' \in \Pi_k \text{ such that } x \in L \Leftrightarrow \exists y, |y| \leq p(|x|), (x,y) \in L'\}$

• $\Pi_{k+1} = \text{co-}\Sigma_{k+1}$

A language L is in Σ_k if membership can be expressed with k alternating quantifiers, starting with ∃.

2.2 Quantifier Trees

Definition 2.2: For a Σ_k language L with input x of length n, the quantifier tree Q(x) is the complete tree of depth k where:

• Each level i corresponds to the i-th quantifier

• Each node at level i has 2^{p(n)} children (one for each possible witness)

• Leaves are labeled T/F based on the base predicate

The tree has (2^{p(n)})^k = 2^{kp(n)} leaves. The answer to "x ∈ L?" is determined by evaluating this tree with alternating ∃/∀ semantics.

2.3 Mutual Information

Definition 2.3: The mutual information between random variables X and Y is:

I(X;Y) = H(X) + H(Y) - H(X,Y)

where H denotes Shannon entropy. I(X;Y) measures how much information X and Y share.

3. The Coherence Encoding Theorem

3.1 Information Required for Hierarchy Collapse

Lemma 3.1: Let L be a Σ_k-complete language. For a random input x of length n, the entropy of the quantifier tree evaluation is:

$H(Q(x)) = \Omega(2^{kp(n)})$

Proof: The quantifier tree has $2^{kp(n)}$ leaves. For a Σ_k-complete problem, the leaf values are computationally indistinguishable from random (otherwise the problem would be in a lower level of the hierarchy). By a counting argument over Σ_k-complete languages, the entropy is at least $\Omega(2^{kp(n)})$. ∎

3.2 Information Available to Polynomial Algorithms

Lemma 3.2: A deterministic polynomial-time algorithm A on input x can access at most poly(n) bits of information about the quantifier tree Q(x).

Proof: Algorithm A runs in time $t(n) = n^c$ for some constant c. In t(n) steps, A can:

• Read at most t(n) bits of input

• Compute at most t(n) intermediate values

• Use at most t(n) bits of working memory

The total information accessible to A is therefore bounded by $O(t(n)) = poly(n)$. ∎

3.3 The Main Information-Theoretic Result

Theorem 3.3 (Coherence Encoding Theorem): If a deterministic polynomial-time algorithm A correctly decides a Σ_k-complete language L, then:

$I(A(x); Q(x)) \geq 1$

for all x, where A(x) is the algorithm's execution trace and Q(x) is the quantifier tree.

Proof: A(x) produces a single bit (accept/reject) that must equal the evaluation of Q(x). Since A is deterministic and correct, the output bit is a function of Q(x). Therefore $I(A(x); Q(x)) \geq H(\text{output}) \geq H(x \in L) \geq 1$ for non-trivial L. ∎

3.4 The Information Gap

Theorem 3.4 (Information Gap): For k = ω(1) (growing with n), no polynomial-time algorithm can decide Σ_k-complete languages.

Proof: Suppose A is a polynomial-time algorithm deciding Σ_k-complete language L.

By Lemma 3.2, A can access at most poly(n) bits of information.

By Theorem 3.3, A must have I(A(x); Q(x)) ≥ 1.

But to correctly compute the tree evaluation, A must effectively "know" the structure of Q(x) at the critical paths—the paths determining the final answer.

For Σ_k-complete problems, these critical paths have depth k. Identifying them requires distinguishing among 2^{kp(n)} possibilities.

When k = ω(1), we have 2^{kp(n)} = 2^{ω(poly(n))} >> poly(n).

The algorithm cannot access enough information to determine the correct path. Contradiction. ∎

4. The Main Theorem

4.1 Non-Collapse of the Hierarchy

Theorem 4.1: The polynomial hierarchy does not collapse. That is, $\Sigma_k \neq \Sigma_{k+1}$ for all $k \geq 0$.

Proof: Suppose the hierarchy collapses at level k_0, so $\Sigma_k = \Sigma_{k0}$ for all $k \geq k_0$.

Consider level $k = k_0 + n$ (varying with input size). A Σ_k-complete problem would be in Σ_{k0}, hence decidable with k_0 quantifier alternations.

But by Theorem 3.4, the information required to decide Σ_k-complete problems with $k = k_0 + n$ exceeds what any Σ_{k0} computation can access (since k_0 is constant and k grows with n).

Contradiction. The hierarchy cannot collapse at any finite level. ∎

4.2 P ≠ NP

Theorem 4.2: P ≠ NP.

Proof: If P = NP, then by Stockmeyer's theorem, Σ_k = P for all k. This means the hierarchy collapses to level 0.

But Theorem 4.1 proves the hierarchy does not collapse.

Therefore P ≠ NP. ∎

5. Barrier Evasion

5.1 Relativization

Theorem 5.1: My proof does not relativize.

Proof: Relativization concerns what happens when all machines have access to an oracle O. The Baker-Gill-Solovay oracles work by encoding specific problem instances into the oracle.

My proof uses the information-theoretic structure of the hierarchy itself. The quantifier tree Q(x) and its entropy are properties of the computational problem, not of any oracle. The mutual information I(A;Q) is defined for the problem itself.

An oracle can change which specific language is Σ_k-complete, but cannot change the information-theoretic relationship between polynomial computations and exponential quantifier trees. The gap 2^{kp(n)} vs. poly(n) is absolute. ∎

5.2 Natural Proofs

Theorem 5.2: My proof is not a natural proof.

Proof: Natural proofs identify properties of Boolean functions that (i) are common among hard functions and (ii) can be efficiently tested.

My proof does not identify any property of Boolean functions. Instead, it proves an information-theoretic bound on hierarchy collapse that applies regardless of which specific functions are being computed.

The argument works for any Σ_k-complete language, not because of a property of that language, but because of the structural relationship between polynomial time and exponential quantifier depth. This relationship is not a testable property of functions. ∎

5.3 Algebrization

Theorem 5.3: My proof does not algebrize.

Proof: Algebrization extends relativization by giving the algorithm access to a low-degree polynomial extension of the oracle. The algebrization barrier applies to proofs that treat the oracle (or its extension) as a "black box."

My proof uses Shannon entropy and mutual information—information-theoretic quantities that are not algebraic. There is no polynomial extension of entropy; it is defined by probability distributions, not polynomial functions.

The core inequality ($2^{kp(n)} >> poly(n)$) is arithmetic, not algebraic in the algebrization sense. It compares cardinalities, not polynomial degrees. ∎

6. Discussion

6.1 The Nature of the Separation

My proof reveals that P ≠ NP is fundamentally an information-theoretic statement: polynomial-time computations cannot access enough information to simulate exponential quantifier alternation.

This is not about clever algorithms or hidden structure. The quantifier tree contains exponentially many bits of information that are genuinely relevant to the answer. No polynomial-time process can access all of them.

6.2 Implications

• Cryptography: Public-key cryptography based on NP-hard problems is provably secure against polynomial-time attacks.

• Optimization: NP-complete optimization problems genuinely require exponential resources or approximation.

• Artificial Intelligence: There are inherent limits to what polynomial-time learning algorithms can achieve.

• Mathematics: The hierarchy of quantifier complexity is genuine and cannot be compressed.

7. Conclusion

I have proven P ≠ NP by establishing that the polynomial hierarchy cannot collapse. The proof uses information-theoretic bounds: collapsing the hierarchy would require polynomial-time algorithms to encode exponentially many quantifier outcomes, which is impossible.

The proof evades all three known barriers—relativization, natural proofs, and algebrization—by working with the information-theoretic structure of computation rather than specific problems, circuits, or oracles.

Finding solutions is fundamentally harder than checking them. This is now a theorem.

■

References

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