6. QUANTUM COMPUTING

Unpaired Majorana fermions in quantum wires

given in Eqn (2) and allows to optimize this process of creating entangled electrons.

In Section 3.1 we have discussed the propagation of entangled electrons in Fermi leads, i.e. in the presence of many other (identical) electrons interacting with the electrons belonging to the entangled pair. We find that the entanglement becomes reduced by a factor z_F^2 due to the transport through such an environment, where z_F denotes the quasiparticle weight factor of the host material. For a two-dimensional electron gas, we explicitly calculate z_F, see Eqn (6). Then, in Section 3.2, we discuss a method for detecting entangled electrons which were produced, e.g., using the method from Section 2. We consider a scattering setup with a beam splitter, where electrons to be tested are injected in the two ingoing arms, and the current noise is measured in one of the outgoing arms. For the maximally entangled singlet and triplet states of electrons with equal energies (such as those produced by the method presented in Section 2) we find the resultant Eqn (11), predicting an enhancement by a factor of two of noise for the singlet, and a complete reduction for the three triplets. We conclude that the enhancement of noise unambiguously indicates an entangled state (the spin singlet).

Finally, in Section 4 we analyze a different situation, in which the entanglement of the *ground state* of a double dot is probed. This is done by measuring the Aharonov–Bohm oscillations in the co-tunneling current which are predicted in Eqns (16) and (17). It is found that the phase-coherent part (17) which distinguishes spin singlets from triplet factorizes in the expression (16) for the co-tunneling current.

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Unpaired Majorana fermions in quantum wires

A Yu Kitaev

Abstract. Certain one-dimensional Fermi systems have an energy gap in the bulk spectrum while boundary states are described by one Majorana operator per boundary point. A finite system of length L possesses two ground states with an energy difference proportional to $\exp(-L/l_0)$ and different fermionic parities. Such systems can be used as qubits since they are intrinsically immune to decoherence. The property of a system to have boundary Majorana fermions is expressed as a condition on the bulk electron spectrum. The condition is satisfied in the presence of an arbitrary small energy gap induced by proximity of a three-dimensional p-wave superconductor, provided that the normal spectrum has an odd number of Fermi points in each half of the Brillouin zone (each spin component counts separately).

1. Introduction

Implementing a full-scale quantum computer is a major challenge to modern physics and engineering. Theoretically, this goal should be achievable due to the possibility of fault-tolerant quantum computation [1]. Unlimited quantum computation is possible if errors in the implementation of each gate are below a certain threshold [2–5]. Unfortunately, for conventional fault-tolerance schemes the threshold appears to be about 10^{-4}, which is beyond the reach of current technologies. It has been also suggested that fault-tolerance can be achieved at the physical level (instead of using quantum error-correcting codes). The first proposal of these kind [6] was based on non-Abelian anyons in two-dimensional systems. A mathematical result concerning universal quantum computation with certain type of anyons has been recently obtained [7], but, generally, this approach is

A Yu Kitaev Microsoft Research, Microsoft, Redmond, WA 98052, USA
L D Landau Institute for Theoretical Physics, Russian Academy of Sciences, ul. Kosygina 2, 117940 Moscow, Russian Federation

still undeveloped. In these paper we describe another (theoretically, much simpler) way to construct decoherence-protected degrees of freedom in one-dimensional systems ('quantum wires'). Although it does not automatically provide fault-tolerance for *quantum gates*, it should allow, when implemented, to build a reliable *quantum memory*.

The reason why quantum states are so fragile is that they are sensitive to errors of two kinds. A classical error, represented by an operator σ_j^x, flips the jth qubit changing $|0\rangle$ to $|1\rangle$ and vice versa. A phase error σ_j^z changes the sign of all states with the jth qubit equal to 1 (i. e. jth spin down, if the qubits are spins) relative to the states with the jth qubit equal to 0. It is generally easy to get rid of one type of errors, but not of both. However, the following method of eliminating the classical errors is worth considering. Let each qubit be a site that can be either empty or occupied by an electron (with spin up, say, the other spin direction being forbidden). Let us denote the empty and the occupied states by $|0\rangle$ and $|1\rangle$, respectively. (Such sites are not exactly qubits because electrons are fermions, but they can be also used for quantum computation [8].) Now single classical errors become impossible because the electric charge is conserved. Even in superconducting systems, the fermionic parity (i.e. the electric charge (mod 2)) is conserved. Two classical errors can still happen at two sites simultaneously, but this would require that an electron jumps from one site to the other. Such jumps can be avoided by placing the 'fermionic sites' far apart from each other, provided the medium between them has an energy gap in the excitation spectrum.

Obviously, this method does not protect from phase errors which are now described by operators $a_j^\dagger a_j$. To the contrary, different electron configurations will have different energies and thus will pick up different phases over time. Even without actual inelastic processes, this will produce the same effect as decoherence. However, a simple mathematical observation suggests that the situation could be improved. Each fermionic site is described by a pair of annihilation and creation operators a_j, $a_j^\dagger$. One can formally define *Majorana operators*

$$c_{2j-1} = a_j + a_j^\dagger, \qquad c_{2j} = \frac{a_j - a_j^\dagger}{\mathrm{i}}, \qquad j = 1, \ldots, N \tag{1}$$

which satisfy the relations

$$c_m^\dagger = c_m, \qquad c_l c_m + c_m c_l = 2\delta_{lm}, \qquad l, m = 1, \ldots, 2N. \tag{2}$$

If the operators c_{2j-1} and c_{2j} belonged to different sites then the phase error

$$a_j^\dagger a_j = \frac{1}{2}(1 + \mathrm{i} c_{2j-1} c_{2j})$$

would be unlikely to occur. Indeed, it would require interaction between the two 'Majorana sites' which could be possibly avoided. Note that a single Majorana operator c_{2j-1} or c_j can not appear as a term in any reasonable Hamiltonian because it does not preserve the fermionic parity. Thus an isolated Majorana site (usually called a *Majorana fermion*) is immune to any kind of error!

Unfortunately, Majorana fermions are not readily available in solid state systems. The goal of this paper is to construct Hamiltonians which would give rise to Majorana fermions as effective low-energy degrees of freedom. Surprisingly, this can be done even with non-interacting electrons. (Some interaction is actually needed to create superconductivity, but it can be effectively described by terms like $\Delta a_j a_k$.) The general idea is quite simple. An arbitrary quadratic Hamiltonian can be written in the form

$$H = \frac{\mathrm{i}}{4} \sum_{l,m} A_{lm} c_l c_m, \qquad A_{lm}^* = A_{lm} = -A_{ml}. \tag{3}$$

Its ground state can be described as 'pairing' of Majorana operators: normal mode creation and annihilation operators $\tilde{a}_m^\dagger, \tilde{a}_m$ which are certain linear combinations of c_l, come in pairs. (In this sense, an insulator and a superconductor represent different types of pairing.) In some cases, most Majorana operators are paired up with an energy gap while few ones (localized at the boundary or defects) remain 'free'. For example, unpaired Majorana fermions exist on vortices in chiral two-dimensional p-wave superconductors [9, 10]. We will show that Majorana fermions can also occur at the ends of quantum wires.

2. A toy model and the qualitative picture

We are going to describe a simple but rather unrealistic model which exhibits unpaired Majorana fermions. It attempts to catch two important properties which seem necessary for the phenomenon to occur. Firstly, the U(1) symmetry $a_j \mapsto \mathrm{e}^{\mathrm{i}\phi} a_j$, corresponding to the electric charge conservation, must be broken down to a $\mathbf{Z}_2$ symmetry, $a_j \mapsto -a_j$. Indeed, if a single Majorana operator can be localized, symmetry transformation should not mix it with other operators. So we should consider superconductive systems. The particular mechanism of superconductivity is not important; we may just think that our quantum wire lies on the surface of three-dimensional superconductor (Fig. 1). The second property is less obvious and will be fully explained in Section 2. Roughly speaking, the electron spectrum must strongly depend on the spin. Here we will simply assume that only one spin component (say, $\uparrow$) is present†.

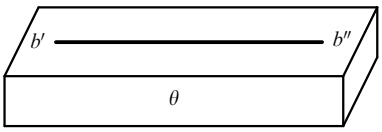

Figure 1. A piece of 'quantum wire' on the surface of three-dimensional superconductor.

Consider a chain consisting of $L \gg 1$ sites. Each site can be either empty or occupied by an electron (with a fixed spin direction). The Hamiltonian is

$$H_1 = \sum_j \left[-w(a_j^\dagger a_{j+1} + a_{j+1}^\dagger a_j) - \mu\left(a_j^\dagger a_j - \frac{1}{2}\right) + \Delta a_j a_{j+1} + \Delta^* a_{j+1}^\dagger a_j^\dagger \right]. \tag{4}$$

Here w is a hopping amplitude, μ a chemical potential, and $\Delta = |\Delta| \mathrm{e}^{\mathrm{i}\theta}$ the induced superconducting gap. It is convenient

† It appears that only a triplet (p-wave) superconductivity in the three-dimensional substrate can effectively induce the desired pairing between electrons with the same spin direction — at least, this is true in the absence of spin-orbit interaction.

to hide the dependence on the phase parameter θ into the definition of Majorana operators:

$$c_{2j-1} = \exp\left(\mathrm{i}\frac{\theta}{2}\right) a_j + \exp\left(-i\frac{\theta}{2}\right) a_j^\dagger ,$$

$$c_{2j} = -\mathrm{i}\exp\left(\mathrm{i}\frac{\theta}{2}\right) a_j + \mathrm{i}\exp\left(-\mathrm{i}\frac{\theta}{2}\right) a_j^\dagger , \quad j = 1, \ldots, L . \quad (5)$$

In terms of this operators, the Hamiltonian becomes

$$H_1 = \frac{\mathrm{i}}{2}\sum_j \Big[-\mu c_{2j-1}c_{2j} + (w + |\Delta|) c_{2j}c_{2j+1} + (-w + |\Delta|) c_{2j-1}c_{2j+2} \Big] . \quad (6)$$

Let us start with two special cases.

(a) The trivial case: $|\Delta| = w = 0,\ \mu < 0$. Then

$$H_1 = -\mu \sum_j \left(a_j^\dagger a_j - \frac{1}{2} \right) = \frac{\mathrm{i}}{2}(-\mu) \sum_j c_{2j-1}c_{2j} .$$

The Majorana operators c_{2j-1}, c_{2j} from the same site j are paired together to form a ground state with the occupation number 0.

(b) $|\Delta| = w > 0$, $\mu = 0$. In this case

$$H_1 = \mathrm{i}w \sum_j c_{2j}c_{2j+1} . \quad (7)$$

Now the Majorana operators c_{2j}, c_{2j+1} *from different sites* are paired together (Fig. 2). One can define new annihilation and creation operators

$$\tilde{a}_j = \frac{1}{2}(c_{2j} + \mathrm{i}c_{2j+1}) , \quad \tilde{a}_j^\dagger = \frac{1}{2}(c_{2j} - \mathrm{i}c_{2j+1})$$

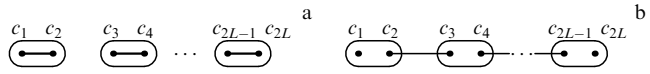

Figure 2. Two types of pairing.

which span the sites j and $j+1$. The Hamiltonian becomes

$$H_1 = 2w \sum_{j=1}^{L-1} \left(\tilde{a}_j^\dagger \tilde{a}_j - \frac{1}{2} \right) .$$

Ground states satisfy the condition $\tilde{a}_j|\psi\rangle = 0$ for $j = 1, \ldots, L-1$. There are two orthogonal states $|\psi_0\rangle$ and $|\psi_1\rangle$ with this property. Indeed, the Majorana operators $b' = c_1$ and $b'' = c_{2L}$ remain unpaired (i. e. do not enter the Hamiltonian), so we can write

$$-\mathrm{i}b'b''|\psi_0\rangle = |\psi_0\rangle, \quad -\mathrm{i}b'b''|\psi_1\rangle = -|\psi_1\rangle . \quad (8)$$

Note that the state $|\psi_0\rangle$ has an even fermionic parity (i.e. it is a superposition of states with even number of electrons) while $|\psi_1\rangle$ has an odd parity. The parity is measured by the operator

$$P = \prod_j (-\mathrm{i}c_{2j-1}c_{2j}) . \quad (9)$$

These two cases represent two phases, or universality classes which exist in the model. A subtle point is that both phases have the same bulk properties. In fact, one phase can be transformed to the other (and vice versa) by mere permutation of Majorana operators,

$$c_m \mapsto c_{m+1} . \quad (10)$$

Such a local transformation (operator algebra automorphism) is usually considered as 'equivalence' in the study of lattice models‡. Yet the boundary properties of the two phases are clearly different: only the phase (b) has unpaired Majorana fermions at the ends of the chain. This is due to the fact that the operators c_{2j-1}, c_{2j} belong to one physical site while c_{2j}, c_{2j+1} do not. We may put it this way: one can not cut a physical site into two halves; if one could, both types of boundary states would be possible in both phases.

Also note that the transformation (10) can not be performed in a continuous fashion, starting from the identity transformation. From the mathematical perspective, it means that one should have different definitions for 'weak' and 'strong' equivalence of lattice models. We will not touch such abstract matters here.

Now we want to study the model at arbitrary values of w, μ and Δ. Let us begin with some generalities. Let N be the total number of fermionic sites in the system, for now $N = L$. The Hamiltonian (6) has the general form (3). Hence it can be reduced to a canonical form

$$H_{\text{canonical}} = \frac{\mathrm{i}}{2}\sum_{m=1}^{N} \epsilon_m b'_m b''_m = \sum_{m=1}^{N} \epsilon \left(\tilde{a}_m^\dagger \tilde{a}_m - \frac{1}{2} \right) , \quad \epsilon_m \geqslant 0 . \quad (11)$$

Here b'_m, b''_m are real linear combinations of c_{2j-1}, c_{2j} with the same commutation relations whereas $\tilde{a}_m = (1/2)(b'_m + \mathrm{i}b''_m)$, $\tilde{a}_m^\dagger = (1/2)(b'_m - \mathrm{i}b''_m)$. More specifically,

$$\begin{pmatrix} b'_1 \\ b''_1 \\ \vdots \\ b'_N \\ b''_N \end{pmatrix} = W \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{2N-1} \\ c_{2N} \end{pmatrix} ,$$

$$WAW^T = \begin{pmatrix} 0 & \epsilon_1 & & & \\ -\epsilon_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & \epsilon_N \\ & & & -\epsilon_N & 0 \end{pmatrix} , \quad (12)$$

where W is a $2N \times 2N$ real orthogonal matrix ($W^T W = W W^T = I$) whose rows are eigenvectors of A. The numbers $\epsilon_m \geqslant 0$ are one-particle excitation energies. However, it is more convenient to deal with a 'double spectrum' $\{\epsilon_m, -\epsilon_m\}$ since the matrix A has eigenvalues $\pm\mathrm{i}\epsilon_m$.

‡ Nonlocal transformations can change the physical properties of the model even more dramatically. The Jordan–Wigner transformation $c_{2j-1} \mapsto \sigma_j^x \prod_{k=1}^{j-1} \sigma_k^z$, $c_{2j} \mapsto \sigma_j^y \prod_{k=1}^{j-1} \sigma_k^z$ transforms our model to a spin chain with xx and yy interactions and a z-directed magnetic field. Unlike Eqn (10), the Jordan–Wigner transformation is well defined at the ends of the chain. However, this mathematical procedure falls apart in the physical context, as far as perturbations are involved. Indeed, the phase (b) has now an order parameter $\langle \sigma^x \rangle \neq 0$. External fields will interact with the order parameter breaking the phase coherence between $|\psi_0\rangle$ and $|\psi_1\rangle$.

The bulk spectrum (energy vs. momentum) is given by

$$\epsilon(q) = \pm\sqrt{(2w\cos q + \mu)^2 + 4|\Delta|^2 \sin^2 q}\,, \quad -\pi \leqslant q \leqslant \pi\,. \tag{13}$$

We may conjecture that the phases (a) and (b) extend to connected domains in the parameter space where the spectrum has a gap. The signs of μ and w seem not to be important, so we actually expect that the phase (a) occurs at $2|w| < |\mu|$ while the phase (b) occupies the domain $2|w| > |\mu|$, $\Delta \neq 0$. (The phase boundary is given by the equation $2|w| = |\mu|$ while $\Delta = 0$, $2|w| > |\mu|$ is a line of normal metal phase inside the domain (b).)

To verify the conjecture, we need to find boundary modes. They correspond to eigenvectors of A localized near the ends of the chain. Due to the spectrum symmetry $\epsilon \mapsto -\epsilon$, zero eigenvalues can occur in a general position. If exist, such zero modes should have the form

$$b' = \sum_j (\alpha'_+ x_+^j + \alpha'_- x_-^j)\, c_{2j-1}\,,$$
$$b'' = \sum_j (\alpha''_+ x_+^{-j} + \alpha''_- x_-^{-j})\, c_{2j}\,,$$
$$x_\pm = \frac{-\mu \pm \sqrt{\mu^2 - 4w^2 + 4|\Delta|^2}}{2(w + |\Delta|)}\,. \tag{14}$$

We will consider two cases corresponding to the expected existence domains of the two phases.

(a) If $2|w| < |\mu|$, we have $|x_+| > 1$, $|x_-| < 1$ or $|x_+| < 1$, $|x_-| > 1$. Therefore, only one of the coefficients α'_+, α'_- (or α''_+, α''_-) can be non-zero, depending on whether the mode is to be localized at the left or at the right end of the chain. This makes it impossible to satisfy boundary conditions. So the supposed zero modes (14) do not exist.

(b) If $2w > |\mu|$, $\Delta \neq 0$, we find that $|x_+|, |x_-| < 1$. Hence b' is localized near $j = 0$ whereas b'' is localized near $j = L$. There are also boundary conditions

$$\alpha'_+ + \alpha'_- = 0\,, \quad \alpha''_+ x_+^{-(L+1)} + \alpha''_- x_-^{-(L+1)} = 0\,,$$

but they can be satisfied too. The zero modes b', b'' are actually the same as the unpaired Majorana fermions discussed above. If $-2w > |\mu|$, $\Delta \neq 0$ then b' and b'' change places. Thus the unpaired Majorana fermions exist in the whole expected domain of the phase (b).

The above analysis is exact in the limit $L \to \infty$. If the chain length L is finite, there is a weak interaction between b' and b''. (For definiteness, we will always assume that b' is at the left end of the chain whereas b'' is at the right end.) This interaction is described by an effective Hamiltonian

$$H_{\text{eff}} = \frac{\mathrm{i}}{2}\, t\, b' b''\,, \quad t \propto \mathrm{e}^{-L/l_0}\,, \tag{15}$$

where l_0^{-1} is the smallest of $\left|\ln|x_+|\right|$ and $\left|\ln|x_-|\right|$ (note that both logarithms have the same sign). Thus the energies of the ground states $|\psi_0\rangle$ and $|\psi_1\rangle$ (see Eqn (8)) differ by t. Note that it is not obvious anymore which state of the two is even and which is odd. In the case $-2w > |\mu|$, the parity is proportional to $(-1)^L$. (This factor is the parity of the bulk part of the chain.)

The effective Hamiltonian (15) still holds if we include small electron–electron interaction (a four-fermion term) into Eqn (4). Indeed, the physical meaning of t is an amplitude for a fermionic quasi-particle to tunnel across the chain. In a long chain, this amplitude vanishes as $\exp(-L/l_0)$ if the bulk spectrum has a gap.

Finally, we will discuss a role of the phase parameter θ ($\Delta = \exp(\mathrm{i}\theta)|\Delta|$). According to Eqn (5), the Majorana operators c_{2j-1}, c_{2j} are multiplied by -1 when θ changes by 2π. The physical parameter Δ is the same at θ and $\theta + 2\pi$, of course, but the ground states should undergo certain transformation as θ changes to $\theta + 2\pi$ adiabatically. Note that the transformation $c_m \mapsto -c_m$ also occurs if one conjugates c_m by the parity operator P. Within the effective Hamiltonian approach, P is the same as $s(L)(-\mathrm{i}b'b'')$ ($s(L) = \pm 1$). Hence the adiabatic change of the superconducting phase by 2π results in the unitary transformation

$$V = s(L)(-\mathrm{i}b'b'')\,:\; V|\psi_0\rangle = |\psi_0\rangle\,,\; V|\psi_1\rangle = -|\psi_1\rangle\,. \tag{16}$$

This is equivalent to transfer of an electron between the ends of the chain. Some physical consequences of this result will be mentioned in Section 4.

3. A general condition for Majorana fermions

Let us consider a general translationally invariant one-dimensional Hamiltonian with short-range interactions. It has been mentioned that the necessary conditions for unpaired Majorana fermions are superconductivity and a gap in the bulk excitation spectrum. The latter is equivalent to the quasi-particle tunneling amplitude vanishing as $\exp(-L/l_0)$. Besides that, it is clear that there should be some parity condition. Indeed, Majorana fermions at the ends of parallel weakly interacting chains may pair up and cancel each other (i.e. the ground state will be non-degenerate). So, provided the energy gap, each one-dimensional Hamiltonian H is characterized by a 'Majorana number' $\mathcal{M} = \mathcal{M}(H) = \pm 1$: the existence of unpaired Majorana fermions is indicated as $\mathcal{M} = -1$. The Majorana number should satisfy $\mathcal{M}(H' \oplus H'') = \mathcal{M}(H')\mathcal{M}(H'')$, where $\oplus$ means taking two non-interacting chains.

Remarkably, the Majorana number reveals itself even if the chain is closed into a loop. This is handy as it eliminates the need to study boundary modes. Let $H(L)$ be the Hamiltonian of a closed chain of length $L \gg l_0$. (H itself is a template which is used to generate $H(L)$ for any L.) We claim that

$$P(H(L_1 + L_2)) = \mathcal{M}(H)\, P(H(L_1))\, P(H(L_2))\,, \tag{17}$$

where $P(X)$ denotes the ground state parity of a Hamiltonian X (assuming that the ground state is unique).

The following argument justifies Eqn (17). An open chain of length L can be described by an effective Hamiltonian which only includes boundary modes. If $\mathcal{M}(H) = -1$, there are Majorana operators b', b'' associated with the ends of the chain. The parity operator P (see Eqn (9)) can be replaced by $s(L)(-\mathrm{i}b'b'')$, where $s(L) = \pm 1$. Thus the fermionic parity of $|\psi_\alpha\rangle$ is $s(L)\,(-1)^\alpha$, $\alpha = 0, 1$. If we close the chain, the effective Hamiltonian is $H_{\text{eff}}(L) = (\mathrm{i}/2)\, u\, b''b'$. (We have chosen to write $b''b'$ in this order because b'' precedes b' in the left-to-right order on the loop, where they are next to each other.) The parameter u represents direct interaction between the chain ends (unlike t from Eqn (15)), so u does not depend on L. The ground state of the closed chain is $|\psi_1\rangle$ if $u > 0$, and $|\psi_0\rangle$ if $u < 0$. Hence

$$P(H(L)) = -s(L)\ \mathrm{sgn}\, u\,.$$

Now let us take two chains, one of length L_1, the other of length L_2. There are two ways to close them up, see Fig. 3. Both cases can be described by effective Hamiltonians:

$$H_{\mathrm{eff}}(L_1) \oplus H_{\mathrm{eff}}(L_2) = \frac{\mathrm{i}}{2}\, u\,(b_1'' b_1' + b_2'' b_2')\,,$$
$$H_{\mathrm{eff}}(L_1 + L_2) = \frac{\mathrm{i}}{2}\, u\,(b_1'' b_2' + b_2'' b_1')\,.$$

Figure 3. Reconnecting closed chains.

It follows that

$$P(H(L_1))\, P(H(L_2)) = s(L_1)\, s(L_2)\,,$$
$$P(H(L_1 + L_2)) = -\, s(L_1)\, s(L_2)\,.$$

So, the equation (17) holds for $\mathcal{M} = -1$. It also obviously holds for $\mathcal{M} = 1$ because in this case there are no boundary modes to worry about.

Computing the Majorana number in general (especially for strongly correlated systems) may be a difficult task. However, the computation can be carried through for any system of non-interacting electrons. Consider a periodic chain of L unit cells with n fermionic sites (i.e. $2n$ Majorana operators) per cell, which totals to $N = nL$ fermionic sites. We will index the Majorana operators as $c_{l\alpha}$, where $l = 1, \ldots, L$, $\alpha = 1, \ldots, 2n$. The Hamiltonian is

$$H = \frac{1}{4} \sum_{l,m} \sum_{\alpha,\beta} B_{\alpha\beta}(m - l)\, c_{l\alpha} c_{m\beta}\,,$$
$$B_{\alpha\beta}(j)^* = B_{\alpha\beta}(j) = -B_{\beta\alpha}(-j)\,. \tag{18}$$

We assume that the chain forms a loop, so $m - l$ should be taken (mod L).

Equation (18) is a special case of Eqn (3), so we will first find $P(H)$ for the general quadratic Hamiltonian (3), assuming that the matrix A is not degenerate. The canonical form of this Hamiltonian (11) has an even ground state $|0\rangle$. The transformation (12) can be represented as conjugation by the parity-preserving unitary operator

$$U = \exp\left(\frac{1}{4} \sum_{l,m} D_{lm} c_l c_m\right)$$

if W has the form $W = \exp D$ for some real skew-symmetric matrix D, i.e. if $\det W = 1$. Otherwise, the transformation (12) changes the parity. Hence

$$P(H) = \mathrm{sgn}\det W = \mathrm{sgn}\,\mathrm{Pf}\, A\,. \tag{19}$$

We remind the reader that the Pfaffian Pf is a function of a skew-symmetric matrix such that $(\mathrm{Pf}\, A)^2 = \det A$. It is defined as follows

$$\mathrm{Pf}\, A = \frac{1}{2^N N!} \sum_{\tau \in S_{2N}} \mathrm{sgn}(\tau)\, A_{\tau(1),\tau(2)} \cdots A_{\tau(2N-1),\tau(2N)}\,. \tag{20}$$

(Here S_{2N} is the set of permutations on $2N$ elements.) For example,

$$\mathrm{Pf} \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix}$$
$$= a_{12}a_{34} + a_{14}a_{23} - a_{13}a_{24}\,.$$

In equation (19) we have used this property of the Pfaffian:

$$\mathrm{Pf}(WAW^T) = \mathrm{Pf}(A)\ \det(W)\,. \tag{21}$$

Now we are to compute the Pfaffian of the matrix B from Eqn (18). First, we use the Fourier transform,

$$\tilde{B}_{\alpha\beta}(q) = \sum_j \exp(\mathrm{i}qj) B_{\alpha\beta}(j), \quad q = 2\pi \frac{k}{L}\ (\mathrm{mod}\ 2\pi),$$
$$k = 0, \ldots, N - 1\,. \tag{22}$$

The matrix $\tilde{B}(q)$ has these symmetries:

$$\tilde{B}^\dagger(q) = -\tilde{B}(q) = \tilde{B}^T(-q)\,. \tag{23}$$

The spectrum $\epsilon(q)$ is a continuous real $2n$-valued function on a circle (real numbers (mod 2π)) given by the eigenvalues of $\mathrm{i}\tilde{B}(q)$. It has the symmetry $\epsilon(-q) = -\epsilon(q)$. The energy gap assumption implies that $\epsilon(q)$ never passes 0. It follows that there are n positive and n negative eigenvalues for any q. Indeed, this is the case for $q = 0$ due to the $\epsilon \mapsto -\epsilon$ symmetry, hence it is true for any q by continuity.

It follows from Eqns (22) and (21) that

$$\mathrm{Pf}\, B = \left[\prod_{q=-q} \mathrm{Pf}\, \tilde{B}(q)\right] \left[\prod_{q \neq -q} \det \tilde{B}(q)\right]. \tag{24}$$

Remember that q is considered (mod 2π), so $q = -q$ when $q = 0$ or $q = \pi$. In the $q \neq -q$ case, each $\{q, -q\}$ pair is counted once. Note that $\det \tilde{B}(q)$ is a positive number since $\mathrm{i}\tilde{B}(q)$ has n positive and n negative eigenvalues. Hence

$$\mathrm{sgn}\,\mathrm{Pf}\, B = \prod_{q=-q} \mathrm{sgn}[\mathrm{Pf}\, \tilde{B}(q)]$$
$$= \begin{cases} \mathrm{sgn}[\mathrm{Pf}\, \tilde{B}(0)]\ \mathrm{sgn}\,[\mathrm{Pf}\tilde{B}(\pi)]\,, & \text{if L is even,} \\ \mathrm{sgn}[\mathrm{Pf}\, \tilde{B}(0)]\,, & \text{if L is odd.} \end{cases} \tag{25}$$

Finally, we get

$$\mathcal{M}(H) = \mathrm{sgn}[\mathrm{Pf}\, \tilde{B}(0)]\ \mathrm{sgn}[\mathrm{Pf}\, \tilde{B}(\pi)]\,. \tag{26}$$

This very general equation can be simplified if superconductivity is a weak effect, i.e. $|\Delta| \ll |\epsilon(0)|, |\epsilon(\pi)|$. Indeed, the right hand side of Eqn (26) makes perfect sense for a U(1)-symmetric Hamiltonian

$$H_0 = \frac{1}{2} \sum_{l,m} \sum_{\alpha,\beta} C_{\alpha\beta}(m - l)\, a_{l\alpha}^\dagger a_{m\beta}\,, \quad C_{\alpha\beta}(j)^* = C_{\beta\alpha}(-j)\,, \tag{27}$$

where $\alpha, \beta = 1, \ldots n$ refer to fermionic sites. The eigenvalues of $\tilde{C}(q)$ (the Fourier transform of C) form a 'single spectrum' $\epsilon_0(q)$. The 'double spectrum' defined above is $\epsilon(q) = \pm\epsilon_0(q)$. It is easy to show that $\mathrm{Pf}\, \tilde{B}(q) = \det \tilde{C}(q)$ for $q = 0, \pi$. Hence

$$\mathcal{M}(H_0) = (-1)^{\nu(\pi)-\nu(0)}\,, \tag{28}$$

where $\nu(q)$ is the number of negative eigenvalues of $\tilde{C}(q)$. Note that $\nu(\pi) - \nu(0)$ equals (mod 2) the number of Fermi points on the interval $[0, \pi]$. (A Fermi point is a point where $\epsilon_0(q)$ passes 0.) In the most interesting case $\nu(\pi) - \nu(0) = 1$ (mod 2), the Hamiltonian H_0 has a gapless spectrum. So, Eqn (28) is only relevant in the presence of superconductivity, i.e. a small symmetry-breaking perturbation which opens an energy gap.

4. Speculations about physical realization

Physical realization of an $\mathcal{M} = -1$ quantum wire is a difficult task because electron spectra are usually degenerate with respect to spin, so $\nu(0)$ and $\nu(\pi)$ are even. The degeneracy at $q = 0$ and $q = \pi$ can be lifted only if the time reversal symmetry is broken. Thus, spin-orbit interaction does not help. External magnetic field could help, but the Zeeman energy $g\mu_B\mathcal{H}$ is usually small compared to other spectrum parameters, so $\nu(0)$ and $\nu(\pi)$ do not change. The situation may be different for charge and spin density waves which add fine features to the electron spectrum. Charge density waves (CDW) tend to occur at the wave vector $q_* = 2q_F$ so that a gap opens at the Fermi level. In the presence of magnetic field, q_F is slightly different for the $\uparrow$ and $\downarrow$ spin components, so it is possible that q_* matches only one of them. The resulting spectrum is shown in Fig. 4 in the $q_*/2\pi$ units. This scenario can be realized if $|\Delta| \lesssim E_{CDW} \lesssim g\mu_B\mathcal{H}$.

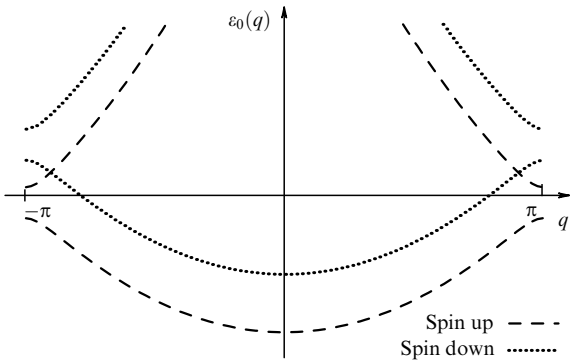

Figure 4. An electron spectrum in the presence of magnetic field and CDW.

Another speculative possibility is to use midgap states at the edge of a two-dimensional p-wave superconductor [11].

A quantum wire bridge between two superconducting leads (see Fig. 5a) could be used as an experimental test for Majorana fermions. When the phase parameter θ_2 in the right piece of superconductor changes by 2π (relative to θ_1), a fermionic quasi-particle is effectively transported to the junction region. At the same time, the Majorana fermions at the ends of the wire switch from $|\psi_0\rangle$ to $|\psi_1\rangle$ or vice versa. If the quasi-particle stays localized, the junction parameters change. They change back when θ_2 changes by another 2π. Thus the Josephson current is 4π-periodic as a function of $\theta = \theta_2 - \theta_1$. In fact, it is more accurate to say that the Josephson energy E_J is 2π-periodic but 2-valued, as shown in Fig. 5b. The two levels may not quite cross at $\theta = \pi$ due to a non-vanishing tunneling amplitude $t \propto \exp(-L/l_0)$, where L is the distance between the junction and the closest end of the wire.

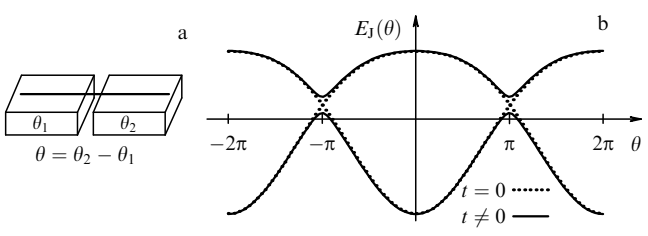

Figure 5. A Josephson junction made of quantum wire.

Interesting phenomena can also take place in the simple layout shown in Fig. 1. Suppose that the superconducting island supporting the quantum wire is connected to a larger piece of superconductor through an ordinary Josephson junction. If the Coulomb energy is comparable to the Josephson energy, spontaneous phase slips can occur. Each 2π phase slip is accompanied by the operator V (see Eqn 16). The phase slips occur by tunneling, so the effective Hamiltonian is

$$H_{\text{eff.1}} = -\lambda V - \lambda^* V^\dagger = \frac{1}{2} s(L)\, t\, b' b''\,, \quad t = 4\,\mathrm{Re}\,\lambda\,, \tag{29}$$

where λ is the amplitude of the $\theta \mapsto \theta + 2\pi$ process while λ^* corresponds to the reverse process. Similarly, if the superconducting island supports two quantum wires, the effective Hamiltonian becomes

$$H_{\text{eff.2}} = -\lambda V_1 V_2 - \lambda^* V_1^\dagger V_2^\dagger = \frac{1}{2} s(L_1) s(L_2)\, t\, b_1' b_1'' b_2' b_2''\,. \tag{30}$$

Turning λ on and off can be possibly used for quantum gates implementation.

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