Note on Synthetic Spin-Orbit Coupling

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Unitary transformation When we do unitary transformation U, we use the following convention. [1]

$$\psi \to \tilde{\psi} = U^\dagger \psi \tag{1}$$
$$H \to \tilde{H} = U^\dagger H U - U^\dagger \mathrm{i}\partial_t U \tag{2}$$

such that the Shrödinger equation stay the same form, i.e.

$$\text{if} \quad \mathrm{i}\partial_t \psi = H\psi \tag{3}$$
$$\text{then} \quad \mathrm{i}\partial_t \tilde{\psi} = \tilde{H}\tilde{\psi} \tag{4}$$

Light-induced Zeeman energy Apply two counter-propergating laser beams along $\hat{x}$-direction together with a real magnetic field along $\hat{z}$-direction, as shown in Figure 1.

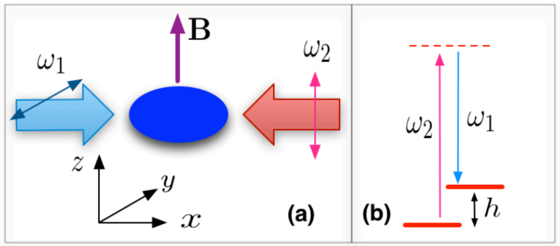

Figure 1: This is the Fig 2.3 on page 21 on Ref[1]. $\delta\omega = \omega_2 - \omega_1$

[1]Note that this convention is the same as in Ref[1] (Page 11), but contrast with Ref[3].

$$\begin{aligned}\boldsymbol{E}^* \times \boldsymbol{E} &= \left(E_1 e^{-\mathrm{i}k_0 x + \mathrm{i}\omega t}\hat{e}_y + E_2 e^{\mathrm{i}k_0 x + \mathrm{i}\omega_2 t}\hat{e}_z\right) \times \left(E_1 e^{\mathrm{i}k_0 x - \mathrm{i}\omega t}\hat{e}_y + E_2 e^{-\mathrm{i}k_0 x - \mathrm{i}\omega_2 t}\hat{e}_z\right) \\ &= E_1 E_2 \left(e^{-\mathrm{i}2k_0 x - \mathrm{i}(\omega_2 - \omega_1)t} - e^{\mathrm{i}2k_0 x + \mathrm{i}(\omega_2 - \omega_1)t}\right)\hat{e}_x \end{aligned} \tag{5}$$

Denote $\delta\omega = \omega_2 - \omega_1$, the Hamiltonian writes

$$\begin{aligned} H_S &= \boldsymbol{B} \cdot \boldsymbol{S} + \mathrm{i}v_S (\boldsymbol{E}^* \times \boldsymbol{E}) \cdot \boldsymbol{S} \\ &= hS_z + \mathrm{i}v_S E_1 E_2 \left(e^{-\mathrm{i}2k_0 x - \mathrm{i}\delta\omega t} - e^{\mathrm{i}2k_0 x + \mathrm{i}\delta\omega t}\right) S_x \end{aligned} \tag{6}$$

U_1: Rotating transformation Do a unitary transformation $U_1 = e^{-\mathrm{i}\delta\omega t S_z}$. This is a time-dependent spin rotation along $\hat{z}$-direction.

$-U_1^\dagger \mathrm{i}\partial_t U_1$ term:

$$\begin{aligned} \mathrm{i}\partial_t U_1 &= \mathrm{i}\partial_t e^{-\mathrm{i}\delta\omega t S_z} = \delta\omega S_z U_1 \\ -U_1^\dagger \mathrm{i}\partial_t U_1 &= -\delta\omega S_z \end{aligned} \tag{7}$$

$U_1^\dagger H_S U_1$ term:

$$U_1^\dagger h S_z U_1 = h S_z \tag{8}$$

$$\begin{aligned} &U_1^\dagger \mathrm{i}v_S E_1 E_2 \left(e^{-\mathrm{i}2k_0 x - \mathrm{i}\delta\omega t} - e^{\mathrm{i}2k_0 x + \mathrm{i}\delta\omega t}\right) S_x U_1 \\ =&\mathrm{i}v_S E_1 E_2 \left(e^{-\mathrm{i}2k_0 x - \mathrm{i}\delta\omega t} - e^{\mathrm{i}2k_0 x + \mathrm{i}\delta\omega t}\right) \cdot \frac{1}{2} \cdot U_1^\dagger \sigma_x U_1 \end{aligned} \tag{9}$$

$$
\begin{aligned}
U_1^\dagger \sigma_x U_1 &= \cos(\delta\omega t)\sigma_x - \sin(\delta\omega t)\sigma_y \\
&= 2\Big(\cos(\delta\omega t)S_x - \sin(\delta\omega t)S_y\Big)
\end{aligned} \tag{10}
$$

To fast calculate rotation along $\hat{z}$-direction, see appendix A as a reference.

$$
\begin{aligned}
\Longrightarrow & \mathrm{i}v_S E_1 E_2 \Big(e^{-\mathrm{i}2k_0x-\mathrm{i}\delta\omega t} - e^{\mathrm{i}2k_0x+\mathrm{i}\delta\omega t}\Big)\cdot\frac{1}{2}\cdot 2\cdot\Big(\cos(\delta\omega t)S_x - \sin(\delta\omega t)S_y\Big) \\
=& \mathrm{i}v_S E_1 E_2 \Big(e^{-\mathrm{i}2k_0x-\mathrm{i}\delta\omega t} - e^{\mathrm{i}2k_0x+\mathrm{i}\delta\omega t}\Big)\cdot\frac{1}{2}\cdot 2\cdot\Big(\frac{e^{\mathrm{i}\delta\omega t}+e^{-\mathrm{i}\delta\omega t}}{2}S_x - \frac{e^{\mathrm{i}\delta\omega t}-e^{-\mathrm{i}\delta\omega t}}{2\mathrm{i}}S_y\Big) \\
=& \mathrm{i}v_S E_1 E_2 \Big(\frac{e^{-\mathrm{i}2k_0x} + \cancel{e^{-\mathrm{i}2k_0x-\mathrm{i}2\delta\omega t}} - \cancel{e^{\mathrm{i}2k_0x+\mathrm{i}\delta\omega t}} - e^{\mathrm{i}2k_0x}}{2}S_x \\
& \qquad + \frac{e^{-\mathrm{i}2k_0x} - \cancel{e^{-\mathrm{i}2k_0x-\mathrm{i}2\delta\omega t}} - \cancel{e^{\mathrm{i}2k_0x+\mathrm{i}\delta\omega t}} + e^{\mathrm{i}2k_0x}}{2\mathrm{i}}S_y\Big) \\
\simeq& \mathrm{i}v_S E_1 E_2 \Big(\frac{e^{-\mathrm{i}2k_0x} - e^{\mathrm{i}2k_0x}}{2}S_x + \frac{e^{-\mathrm{i}2k_0x} + e^{\mathrm{i}2k_0x}}{2\mathrm{i}}S_y\Big) \qquad (11) \\
=& \mathrm{i}v_S E_1 E_2 \Big(-\mathrm{i}\sin(2k_0x)S_x + \mathrm{i}\cos(2k_0x)S_y\Big) \\
=& v_S E_1 E_2 \Big(\sin(2k_0x)S_x - \cos(2k_0x)S_y\Big) \\
=& \Omega\Big[\sin(2k_0x)S_x - \cos(2k_0x)S_y\Big]
\end{aligned}
$$

where we define $\Omega = v_S E_1 E_2$. And the two fast rotating terms are dropped as indicated by the inclined lines in above, which is called "rotating wave approximation".

The Hamiltonian after the first unitary transformation U_1 and rotating wave approximation is written as

$$
\begin{aligned}
H_1 &= U_1^\dagger H_S U_1 - U_1^\dagger \mathrm{i}\partial_t U_1 \\
&= (h-\delta\omega)S_z + \Omega\Big(\sin(2k_0x)S_x - \cos(2k_0x)S_y\Big)
\end{aligned} \tag{12}
$$

U_2: Translation The second unitary transformation we will do is a translation along the $\hat{x}$-direction, $U_2 = e^{-\mathrm{i}\hat{p}\pi/4k_0}$. Here we use natural unit $\hbar = 1$. See that since $[\hat{x},\hat{p}] = \mathrm{i}\hbar$, $\hat{x} = \mathrm{i}\partial_p$ in p-representation. Therefore

$$
U_2^\dagger \hat{x} U_2 = \exp(\mathrm{i}\hat{p}\frac{\pi}{4k_0})\mathrm{i}\partial_p \exp(-\mathrm{i}\hat{p}\frac{\pi}{4k_0}) = \hat{x} + \frac{\pi}{4k_0} \tag{13}
$$

Hence

$$
\begin{aligned}
U_2^\dagger H_1 U_2 &= (h-\delta\omega)S_z + \Omega\Big[\sin(2k_0x+\pi/2)S_x - \cos(2k_0x+\pi/2)S_y\Big] \\
&= (h-\delta\omega)S_z + \Omega\Big(\cos(2k_0x)S_x + \sin(2k_0x)S_y\Big)
\end{aligned} \tag{14}
$$

This is the Hamiltonian for the Zeeman field. An atom moving in such a Zeeman field experience the following Hamiltonian.

$$
H_2 = \frac{\hbar^2\boldsymbol{k}^2}{2m} + (h-\delta\omega)S_z + \Omega\Big(\cos(2k_0x)S_x + \sin(2k_0x)S_y\Big) \tag{15}
$$

U_3: Rotating transformation again We then do another unitary transformation which is a spatial dependent spin rotation $U_3 = e^{-\mathrm{i}2k_0xS_z}$.

$$
U_3 = e^{-\mathrm{i}2k_0xS_z} = e^{-\mathrm{i}k_0x\sigma_z} = \begin{pmatrix} e^{-\mathrm{i}k_0x} & 0 \\ 0 & e^{\mathrm{i}k_0x} \end{pmatrix} \tag{16}
$$

In x-representation $\hat{k}_x = -\mathrm{i}\partial_x$. Therefore

$$
U_3^\dagger \hat{k}_x U_3 = e^{\mathrm{i}k_0x\sigma_z}(-\mathrm{i}\partial_x)e^{-\mathrm{i}k_0x\sigma_z} = -k_0\sigma_z \tag{17}
$$

Hence

$$
U_3^\dagger \frac{\hbar k^2}{2m} U_3 = \frac{\hbar^2}{2m}(k_x - k_0\sigma_z)^2 + \frac{\hbar^2}{2m}\boldsymbol{k}_\perp^2 \tag{18}
$$

And

$$
U_3^\dagger (h-\delta\omega)S_z U_3 = (h-\delta\omega)S_z \tag{19}
$$

and

$$
\begin{aligned}
&U_3^\dagger \Omega \Big(\cos(2k_0x)S_x + \sin(2k_0x)S_y \Big) U_3 \\
=&\frac{\Omega}{2} U_3^\dagger \Big(\cos(2k_0x)\sigma_x + \sin(2k_0x)\sigma_y \Big) U_3
\end{aligned}
$$

With the help of A we show

$$
\begin{aligned}
U_3^\dagger \sigma_x U_3 &= \cos(2k_0x)\sigma_x - \sin(2k_0x)\sigma_y && (20)\\
U_3^\dagger \sigma_y U_3 &= \sin(2k_0x)\sigma_x + \cos(2k_0x)\sigma_y && (21)
\end{aligned}
$$

then

$$
\begin{aligned}
&(\Omega/2)U_3^\dagger \Big(\cos(2k_0x)\sigma_x + \sin(2k_0x)\sigma_y \Big) U_3 \\
=&(\Omega/2)\Big[\cos^2(2k_0x)\sigma_x - \cos(2k_0x)\sin(2k_0x)\sigma_y \\
&\qquad + \sin^2(2k_0x)\sigma_x + \cos(2k_0x)\sin(2k_0x)\sigma_y \Big] \\
=&\frac{\Omega}{2}\sigma_x \\
=&\Omega S_x \qquad (22)
\end{aligned}
$$

Thus the final Hamiltonian is written as

$$
U_3^\dagger H_2 U_3 = \frac{\hbar^2}{2m}(k_x - 2k_0 S_z) + \frac{\hbar^2}{2m}\boldsymbol{k}_\perp^2 + (h - \delta\omega)S_z + \Omega S_x \qquad (23)
$$

Or in other words,

$$
H_{\text{final}} = \frac{\hbar^2 \boldsymbol{k}^2}{2m} + \boldsymbol{h}_{\boldsymbol{k}} \cdot \boldsymbol{S} \qquad (24)
$$

where

$$
\begin{aligned}
\boldsymbol{k} &= (k_y, k_z) && (25)\\
\boldsymbol{h}_{\boldsymbol{k}} &= \Big(\Omega, 0, [h - \delta\omega - \frac{\hbar^2}{2m}2k_0k_x] \Big) && (26)
\end{aligned}
$$

This is the synthetic spin-orbit coupling Hamiltonian.

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References

[1] Hui Zhai, Lecture Note on Cold Atom Physics (2017).

[2] N. Goldman, G. Juzeliunas, P. Ohberg, I. B. Spielman, Light-induced gauge fields for ultracold atoms, Rep. Prog. Phys. **77** 126401 (2014).

[3] Hui Zhai, Microscopy of the interacting Harper-Hofstadter model in the few-body limit, Rep.Prog.Phys. **78**, 026001 (2015).

A

Firstly we show that the Pauli matrices under any rotation along $\hat{z}$-direction will transform as follows. Denote

$$
U_\theta = e^{-\mathrm{i}\theta S_z} = e^{-\mathrm{i}\theta\sigma_z/2} = \cos(\theta/2) - \mathrm{i}\sin(\theta/2)\sigma_z \qquad (27)
$$

then

$$
\begin{aligned}
U_\theta^\dagger \sigma_x U_\theta &= [\cos(\theta/2) + \mathrm{i}\sin(\theta/2)\sigma_z]\sigma_x[\cos(\theta/2) - \mathrm{i}\sin(\theta/2)\sigma_z] \\
&= (\cos^2(\theta/2) - \sin^2(\theta/2))\sigma_x - 2\sin(\theta/2)\cos(\theta/2)\sigma_y \\
&= \cos\theta\sigma_x - \sin\theta\sigma_y && (28)\\
U_\theta^\dagger \sigma_y U_\theta &= [\cos(\theta/2) + \mathrm{i}\sin(\theta/2)\sigma_z]\sigma_y[\cos(\theta/2) - \mathrm{i}\sin(\theta/2)\sigma_z] \\
&= (\cos^2(\theta/2) - \sin^2(\theta/2))\sigma_y + 2\sin(\theta/2)\cos(\theta/2)\sigma_x \\
&= \cos\theta\sigma_y + \sin\theta\sigma_x && (29)\\
U_\theta^\dagger \sigma_z U_\theta &= \sigma_z && (30)
\end{aligned}
$$