On Periodic Driven Fermi-Hubbard Model

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We've been studying the problem of periodic driven Fermi-Hubbard model for a while. In this manuscript I write down the model, the method, the analysis and some numerical results of this problem.

MODEL

Consider an optical lattice along the x direction. Shaking the lattice along the x direction results in the following periodically modulated lattice potential:

$$V(x + A\cos(\omega t)). \tag{1}$$

Here A is the shaking amplitude and ω is the shaking frequency. The Hamiltonian for an atom of mass m in such a lattice is given by

$$\hat{H}_{atom}(t) = \frac{\hat{p}^2}{2m} + \hat{V}(\hat{x} + A\cos(\omega t)) \tag{2}$$

in the laboratory frame, where $\hat{x}$ ($\hat{p}$) is the position (momentum) operator of the atom. In the comoving frame, the Hamiltonian is written as

$$\hat{H}_{co}(t) = \frac{\hat{p}^2}{2m} + \hat{V}(x) + m\omega^2 A\cos(\omega t)\hat{x}, \tag{3}$$

related to the original Hamiltonian in Eq.(2) via a unitary transformation [1]

$$\hat{U}_{co}(t) = \exp(-i\hat{x}p_0(t))\exp(i\hat{p}x_0(t)), \tag{4}$$

where $x_0(t) = -A\cos(\omega t)$, and $p_0(t) = m\omega A\sin(\omega t)$.

Now consider a single-band tight-binding model with only nearest neighboring tunneling and on-site Hubbard interaction in such a lattice. Since the interaction term involves only on-site particle density operators and is translational invariant, it commutes with $\hat{U}_{co}$, and remains the same after the unitary transformation. Hence, the Hamiltonian is written in second quantization notation as

$$\hat{H}(t) = -\sum_{\substack{\langle i,j\rangle \\ \sigma=\uparrow,\downarrow}} t\hat{c}_i^\dagger \hat{c}_j + U\sum_i \hat{n}_{i\uparrow}\hat{n}_{i\downarrow} + \sum_{i,\sigma} f_i(t)\hat{n}_{i\sigma}. \tag{5}$$

where $\hat{c}_{i\sigma}$ ($\hat{c}_{i\sigma}^\dagger$) is the Fermionic annihilation (creation) operator on site i with spin σ, $\hat{n}_{i\sigma}$ is the density operator on site i with spin σ, $\langle\ldots\rangle$ denotes the nearest neighboring sites, and $f_i(t) = m\omega^2 A\cos(\omega t)x_i$ with x_i the position of the ith lattice site. Such a Hamiltonian appears in a recent experimental work[2] alongside with the harmonic trap therein.

In the following, we consider this Hamiltonian in a two-dimensional (2D) square lattice. The Hamiltonian is the same as above except for the lattice index i is a 2D vector: $i = (x_i, y_i)$, where $x_i(y_i)$ is the x(y)-axis coordinate of the i site. Hence, the time-dependent term is $f_i(t) = m\omega^2 A\cos(\omega t)(x_i + y_i)$. This is the very beginning Hamiltonian of our work. And so far, no further approximation has been made except for the single-band tight-binding and on-site interaction assumptions.

Rotation.— Now we introduce another unitary transformation to do, similar to [2] therein,

$$\hat{R}(t) = \exp(i\sum_j l\omega t\hat{n}_{j\uparrow}\hat{n}_{j\downarrow})\exp(i\sum_{j\sigma} F_j(t)\hat{n}_{j\sigma}), \tag{6}$$

where $F_j(t) = \int^\tau f_j(\tau)d\tau$. The value of l is determined by the comparison of shaking frequency with the interaction strength U and other energy scales of the system. When the shaking frequency is much larger than all other energy scales of the system, then $l = 0$ and the unitary transformation $R_0(t) = \exp(i\sum_{j\sigma} F_j(t)\hat{n}_{j\sigma})$ is done on the Hamiltonian in Eq.(5), resulting in a time-dependent Hamiltonian, the lowest order of the high-frequency expansion[3] of which is simply a usual Fermi-Hubbard Hamiltonian with only the tunneling coefficients renormalized as $\tilde{t} = t\mathcal{J}_0(\mathcal{A})$, where we use $\mathcal{J}_l$ to denote the lth Bessel function and $\mathcal{A} = mA\omega d$ is the renormalized shaking amplitude hereinafter. Here d is the distance of two Wannier wave packets in the nearest neighboring sites on the lattice.

When the shaking frequency, or l multiple of it, is comparable to the Hubbard interaction strength U, that is, $l\hbar\omega \approx U$, we say a l-photon resonance occurs. The unitary transformation $R(t)$ in Eq.(6) with $l \neq 0$ is then done to the Hamiltonian in Eq.(5). The time-dependent Hamiltonian resulting from this transformation is obtained as

$$\hat{H}_{\rm rot}(t) = -\sum_{\langle i,j\rangle,\sigma} t\exp[il\omega t(\hat{n}_{i\bar{\sigma}} - \hat{n}_{j\bar{\sigma}}) - i\mathcal{A}\eta_{ij}\sin(\omega t)]\hat{c}_{i,\sigma}^\dagger\hat{c}_{j,\sigma} + (U - l\omega)\sum_i \hat{n}_{i\uparrow}\hat{n}_{i\downarrow}. \tag{7}$$

Here we have defined η_{ij} to be 1 if $i = j + \mathbf{d}_x$ or $i = j + \mathbf{d}_y$, and -1 if $i = j - \mathbf{d}_x$ or $i = j - \mathbf{d}_y$, where $\mathbf{d}_\mu$ is the lattice vector along μ direction.

In the case the shaking frequency is much larger than the tunneling coefficient t and effective interaction strength $g = U - l\omega$, high-frequency expansion[3] is valid to derive an effective static Hamiltonian in a stroboscopic sense. To the lowest order,

$$\begin{aligned}\hat{H}_{\text{eff}} &= \frac{1}{T}\int_{t_0}^{t_0+T} \hat{H}_{\text{rot}}(t)dt \\ &= \sum_{\langle i,j\rangle,\sigma} -t\left[\mathcal{J}_0(\mathcal{A})\hat{a}_{ij\bar{\sigma}} + \mathcal{J}_l(\eta_{ij}\mathcal{A})\hat{b}^l_{ij\bar{\sigma}}\right]\hat{c}^\dagger_{i\sigma}\hat{c}_{j\sigma} + g\sum_i \hat{n}_{i\uparrow}\end{aligned} \tag{8}$$

where $\bar{\sigma}$ denotes the complement of σ, and

$$\hat{a}_{ij\sigma} = (1-\hat{n}_{i\sigma})(1-\hat{n}_{j\sigma}) + \hat{n}_{i\sigma}\hat{n}_{j\sigma}, \tag{9}$$

$$\hat{b}^l_{ij\sigma} = (-1)^l(1-\hat{n}_{i\sigma})\hat{n}_{j\sigma} + \hat{n}_{i\sigma}(1-\hat{n}_{j\sigma}) \tag{10}$$

are projection operators of the nearest neighboring sites. Denoting $t_0 = t\mathcal{J}_0(\mathcal{A})$ and $t_1 = \mathcal{J}_l(\eta_{ij}\mathcal{A})$, the effective Hamiltonian is rewritten as

$$\hat{H}_{\text{eff}} = \sum_{\langle i,j\rangle,\sigma}\left(-t_0\hat{a}_{ij\bar{\sigma}} - t_1\hat{b}^l_{ij\bar{\sigma}}\right)\hat{c}^\dagger_{i\sigma}\hat{c}_{j\sigma} + g\sum_i \hat{n}_{i\uparrow}\hat{n}_{i\downarrow}. \tag{11}$$

Here the site dependence of t_1 is made implicitly. Note, however, that for even l the Bessel function J_l is an even function, in which case η_{ij} can be simply dropped and t_1 becomes a constant. In the following, we study this effective Hamiltonian carefully.

Correlated tunneling.— The tunneling term in $\hat{H}_{\text{eff}}$ is density-dependent thus termed correlated tunneling. The effect of correlated tunneling is illustrated in FIG. 1.

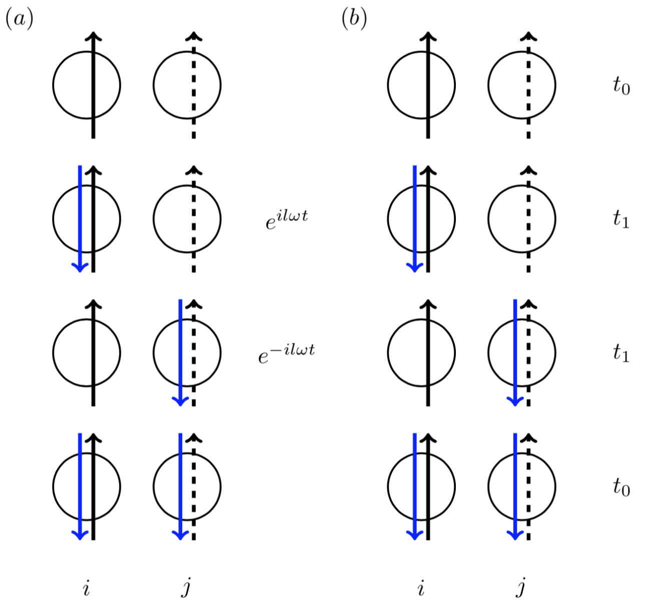

FIG. 1: Schematic diagram of correlated tunneling process of spin-up state from j site (dashed up arrow) to i site (solid up arrow). The tunneling process of spin-up state depends on the projection of spin-down state (solid down arrow) in both the time-dependent model and the time-averaged effective static model. (a) Extra time-dependent phase of the tunneling term of the time-dependent $\hat{H}_{\text{rot}}$ in Eq.(7). (b) Tunneling strength of different tunneling process of the effective static Hamiltonian $\hat{H}_{\text{eff}}$ in Eq.(11).

SYMMETRY

SO(4) symmetry.— A usual bipartite Fermi-Hubbard model, written as $\hat{H} = -t\sum_{\langle i,j\rangle,\sigma}\hat{c}^\dagger_{i\sigma}\hat{c}_{j\sigma} + g\sum_i(\hat{n}_{i\uparrow}-1/2)(\hat{n}_{i\downarrow}-1/2)$, possesses SO(4) symmetry [4]. The SO(4) symmetry is resolved into two SU(2) symmetries, the spin SU(2) and the charge SU(2). For current models, the spin SU(2) is pervasive. Defining single-particle operators [4]

$$\hat{S}_z = \frac{1}{2}\sum_i \hat{c}^\dagger_{i\uparrow}\hat{c}_{i\uparrow} - \hat{c}_{i\downarrow}\hat{c}_{i\downarrow}, \quad \hat{S}_+ = \sum_i \hat{c}^\dagger_{i\uparrow}\hat{c}_{i\downarrow}, \tag{12}$$

and

$$\hat{S}_- = \hat{S}^\dagger_+, \quad \hat{S}_x = \frac{\hat{S}_+ + \hat{S}_-}{2}, \quad \hat{S}_y = \frac{\hat{S}_+ - \hat{S}_-}{2i}, \tag{13}$$

easy to check that $\{\hat{S}_z, \hat{S}_x, \hat{S}_y\}$ generates a $su(2)$ algebra. The very beginning time-dependent Hamiltonian $\hat{H}(t)$ in Eq.(5) is invariant under this SU(2) symmetry operation. Time average does not alter this attribute. It is straightforward to verify that the effective static Hamiltonian $\hat{H}_{\text{eff}}$ in Eq.(11) also possess this SU(2) symmetry no matter l even or odd.

As for the charge SU(2), the time-dependent Hamiltonian lacks if for the appearance of the time-dependent onsite energy term (3rd term in Eq.(5)). Nevertheless, it reappears in the effective Hamiltonian with l even. We introduce the charge $su(2)$ as follows [4].

$$\hat{L}_z = -\frac{1}{2}\sum_i \hat{c}^\dagger_{i\uparrow}\hat{c}_{i\uparrow} + \hat{c}^\dagger_{i\downarrow}\hat{c}_{i\downarrow} + \frac{1}{2}N, \tag{14}$$

$$\hat{L}_+ = \sum_i (-1)^i\hat{c}_{i\uparrow}\hat{c}_{i\downarrow} = \sum_i \exp(i\mathbf{Q}\cdot\mathbf{x}_i)\hat{c}_{i\uparrow}\hat{c}_{i\downarrow} \tag{15}$$

where $\mathbf{Q} = (\pi,\pi)$, N is the total number of lattice sites hereinafter, and

$$\hat{L}_- = \hat{L}^\dagger_+, \quad \hat{L}_x = \frac{\hat{L}_+ + \hat{L}_-}{2}, \quad \hat{L}_y = \frac{\hat{L}_- - \hat{L}_-}{2i}. \tag{16}$$

$\{\hat{L}_z, \hat{L}_x, \hat{L}_y\}$ forms a $su(2)$ algebra. It can be proved that the effective Hamiltonian $\hat{H}_{\text{eff}}$ with l even also has this symmetry. However, this symmetry doesn't shown up in the l odd cases.

Particle-hole symmetry.— The half-filled effective Hamiltonian in Eq.(11) have particle-hole symmetry no matter l even or odd. (i) When l is even, define particle-hole transformation $\hat{C} : \hat{c}_{i\sigma} \to (-1)^i\hat{c}^\dagger_{i\sigma}$. The Hamiltonian is invariant under $\hat{C}$. $[\hat{C}, \hat{H}_{\text{eff}}] = 0$. (ii) When

l is odd, define particle-hole transformation $\hat{C}$: $\hat{c}_{i\sigma} \rightarrow (-1)^i \hat{c}_{i\sigma}^\dagger$, and bipartite transformation $\hat{S}$, which switches the A/B sublattices. The Hamiltonian is invariant under the combination of $\hat{C}$ and $\hat{S}$. $[\hat{C}\hat{S}, \hat{H}_{\text{eff}}] = 0$.

MEAN-FIELD TREATMENT

The effective Hamiltonian $\hat{H}_{\text{eff}}$ with l even possesses SO(4) symmetry, as discussed in the above section, and particle-hole symmetry at half filling. These symmetries make the even l cases even more interesting. Take all these into consideration, we provide a mean-field treatment first to the even l half-filling cases, in which the total charge density and total magnetic momentum is conserved.

$$\langle \hat{n} \rangle = 1, \quad \langle \hat{S} \rangle = 0. \tag{17}$$

Order parameters.— The bipartite lattice is divided into A/B sublattices. The charge density wave (CDW) order and spin density wave (SDW) order, denoted as c and s respectively, are introduced as

$$\langle \hat{n}_A \rangle = 1 + c, \quad \langle \hat{S}_A^z \rangle = s, \tag{18}$$

where $\hat{n}_A = \frac{1}{N/2} \sum_{i \in A} \hat{n}_i$, and $\hat{S}_A^z$ is the same as defined in Eq.(12) except the summation is over A sublattice. We do not explicitly introduce the super-conductor (SC)[6] and out-of-z-direction SDW order parameters here, because the model possess the SO(4) symmetry as a usual Fermi-Hubbard model does. These orders are equivalent to their z-direction counterparts and can be rotated by the two SU(2) groups.

Mean-field Hamiltonian.— We establish the mean-filed theory using a standard path integral approach. See supplemental materials for the derivation in details. Also see [5] for further reference. After the establishment of the mean-field theory, the mean-filed Hamiltonian is given as

$$\begin{aligned}\hat{H}_{\text{mf}} = \sum_{\langle i,j \rangle, \sigma} &\{-t_0[(1-n_{A\bar{\sigma}})(1-n_{B\bar{\sigma}}) + n_{A\bar{\sigma}} n_{B\bar{\sigma}}] - t_1[(1-n_{A\bar{\sigma}}) n_{B\bar{\sigma}} + n_{A\bar{\sigma}}(1-n_{B\bar{\sigma}})]\} \hat{c}_{i\sigma}^\dagger \hat{c}_{j\sigma} + \frac{gN}{2}(n_{A\uparrow} n_{A\downarrow} + n_{B\uparrow} n_{B\downarrow}) \\ &+ \eta_c \left[c - \frac{(\hat{n}_{A\uparrow} + \hat{n}_{A\downarrow}) - (\hat{n}_{B\uparrow} + \hat{n}_{B\downarrow})}{2} \right] + \eta_s \left[s - \frac{(\hat{n}_{A\uparrow} - \hat{n}_{A\downarrow}) - (\hat{n}_{B\uparrow} - \hat{n}_{B\downarrow})}{2} \right]\end{aligned} \tag{19}$$

where $\hat{n}_{\mu\sigma} = \frac{1}{N/2} \sum_{i \in \mu} \hat{n}_{i\sigma}$, $\mu = A$ or B, and $n_{\mu\sigma} = \langle \hat{n}_{\mu\sigma} \rangle$. Here η_c and η_s are Lagrangian multipliers of the CDW and SDW orders, respectively, who are also parameters that need to be varied to optimize the ground state energy of $\hat{H}_{\text{mf}}$ such that we get a mean-field solution.

The particle density parameters $n_{\mu\sigma}$ as the function of c, s can be derived with the help of their definitions. After some substitution and simplification, the mean-filed Hamiltonian in momentum space is written as

$$\begin{aligned}\hat{H}_{\text{mf}} = &\sum_{\mathbf{k}\sigma} -P_\sigma(c,s) Q(\mathbf{k}) \hat{a}_{\mathbf{k}\sigma}^\dagger \hat{b}_{\mathbf{k}\sigma} + \text{H.c.} \\ &+ \frac{gN}{2}(1 + c^2 - s^2) \\ &+ \eta_c \left[c - \frac{(\hat{n}_{A\uparrow} + \hat{n}_{A\downarrow}) - (\hat{n}_{B\uparrow} + \hat{n}_{B\downarrow})}{2} \right] \\ &+ \eta_s \left[s - \frac{(\hat{n}_{A\uparrow} - \hat{n}_{A\downarrow}) - (\hat{n}_{B\uparrow} - \hat{n}_{B\downarrow})}{2} \right]\end{aligned} \tag{20}$$

where $\hat{a}_{\mathbf{k}\sigma}(\hat{b}_{\mathbf{k}\sigma})$ are the annihilation operators on $A(B)$ sublattice of quasi-momentum $\mathbf{k}$ and spin σ, $Q(\mathbf{k}) = \sum_i \exp(i\mathbf{k} \cdot \mathbf{d}_i)$, $\mathbf{d}_i$ is the lattice vectors along all directions, and

$$P_\uparrow(c,s) = \frac{t_0}{2}\left[1 - (c-s)^2\right] + \frac{t_1}{2}\left[1 + (c-s)^2\right] \tag{21}$$

$$P_\downarrow(c,s) = \frac{t_0}{2}\left[1 - (c+s)^2\right] + \frac{t_1}{2}\left[1 + (c+s)^2\right] \tag{22}$$

The summation of $\mathbf{k}$ in Eq.(20) is over the first Brillioun zone.

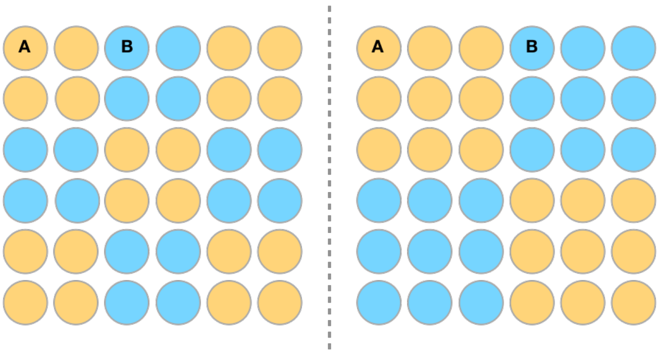

FIG. 2: Schematic showing the lattice of enlarged unit cell. Left panel: unit cell enlarged to 2×2. Right panel: unit cell enlarged to 3×3.

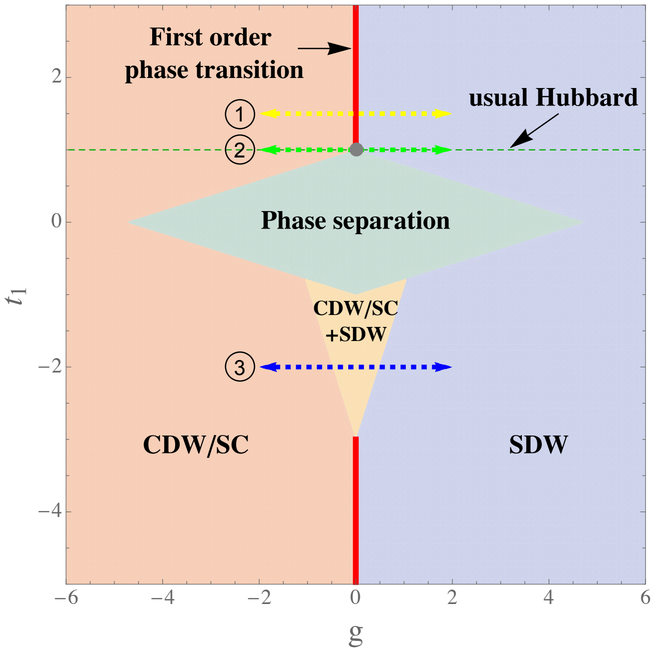

FIG. 3: Mean-filed phase diagram with fixed t_0 chosen as unit. The CDW/SC and SDW order phase are marked straightforwardly in the diagram. In addition, a CDW/SC and SDW coexistence phase emerged, marked as CDW/SC + SDW in the figure. Also, the green region is a phase separation region. When $t_0 = t_1$, the original effective Hamiltonian $\hat{H}_{\rm eff}$ reduces to a usual Hubbard model, as indicated in the diagram by green dashed line. Solid red lines denote first order phase transition. Gray point located at (0, 1) is the second-order phase transition point in a usual Hubbard model. Numbers ①, ②, and ③ marked three lines in the parameter space for the calculation shown in the main text.

Mean-field phase diagram.— Solving the mean-field Hamiltonian $\hat{H}_{\rm mf}$ above returns us a group of mean-field solutions of the system, which are essentially saddle point approximation of the interacting Hamiltonian $\hat{H}_{\rm eff}$. The mean-filed phase diagram is determined by comparing the ground-state energy of these solutions at each region of the parameter space and choosing the lowest one as the corresponding phase. We find that, for different parameter regime, the system supports CDW/SC phase, SDW phase, and coexistence of CDW/SC and SDW phase. Here the CDW/SC and SDW orders are all of momentum $\mathbf{Q} = (\pi, \pi)$. Meanwhile, we also check the cases of enlarged unit cell, illustrated in FIG. 5, after which there is in addition a phase separation regime. The mean-filed diagram is shown in FIG. 3.

The physics for large g is the same as that in a usual Hubbard model. Take repulsive interaction (positive g) as an example, one could derive the effective spin-exchange model in the Mott limit in a standard manner:

$$\hat{H}_{\rm ex} = \frac{4t_1}{g} \sum_{\langle i,j \rangle} \hat{\mathbf{S}}_i \cdot \hat{\mathbf{S}}_j, \tag{23}$$

which is responsible for the SDW in large positive g limit regardless of t_0 and t_1. However, when g is small, the physics is much more rich and shows a t_1 dependence. From the effective Hamiltonian $\hat{H}_{\rm eff}$ in Eq.(11) and the definition of $\hat{a}_{ij\sigma}$ and $\hat{b}^l_{ij\sigma}$, it is seen that the system will get an energy gain from block of domains of very large, or small, particle density, which indicates the possibility of phase separation.

Moreover, the phase transition from CDW/SC phase to SDW phase is a first-order phase transition (red lines in FIG.3), contrast to that in a usual Hubbard model, which is a second-order one (gray point in FIG.3). Whereas, the coexistence phase sustain also a step jump of the CDW/SC and SDW orders on the g goes to 0 crossover line, which mimic what happens on the first-order transition line. The emergence of the novel phase, including the coexistence phase and phase separation region, and the first-order phase transition, is resulting fundamentally from the correlated tunneling essence. It indicates that the orders are favored by the correlated tunneling at large $|t_1|$, consistent with our previous observations. To demonstrate the essence of the orders and phase transitions, we calculated the magnitude of the orders along three lines in the parameter space, the dashed arrow lines and marked with ①, ②, ③ in FIG. 3. The results are shown in FIG. 4. From the results we see that xxx, and in the special regime $t_1 = t_0$ it reproduces that in a usual Hubbard model.

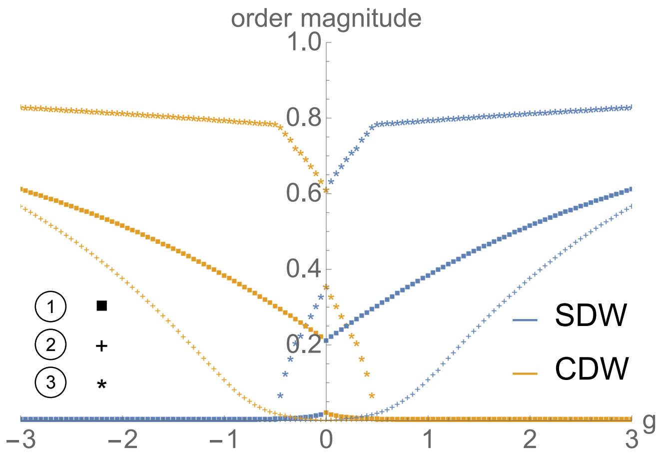

FIG. 4: Magnitude of order parameters along ① $t_1 = 1.5$, ② $t_1 = 1.0$, ③ $t_1 = -2.0$ in FIG. 3.

As for the phase separation regime, the previous mean-field Hamiltonian yields barely trivial order. However, with the unit cell enlarged, as shown in FIG. 5 for two typical cases of unit cell enlarged to 2×2 and 3×3, non-trivial orders emerge. In such cases we still use the mean-filed Hamiltonian (19). The only difference is the partition of A and B. Numerical results in FIG. 5 show the larger the unit cell is enlarged the more favorable the ground state energy, which leads definitely to the phase

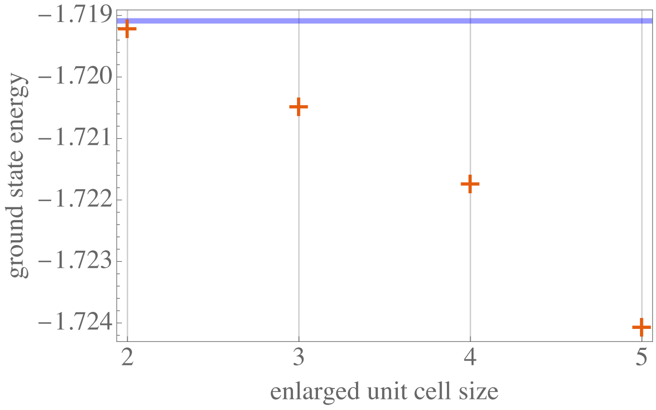

FIG. 5: Ground state energy of the mean-field Hamiltonian with unit cell enlarged to different size at $t_1 = 0$, $g = -0.2$. Blue solid line denotes the energy of the free solution.

separation. To see more clearly of the phase separation essentials, we compute the ground state energy as a function of particle density with no interaction. The results are shown in FIG. 6 for two different sets of parameters. While in the coexistence phase it is a stable extremum (local minimum), at half filling, presented in FIG. 6(b), it is a non-stable extremum (local maximum) in the phase separation regime, shown in FIG. 6(a). This justifies the phase separation, which, manifested by this calculation, is deeply related to the correlated tunneling process.

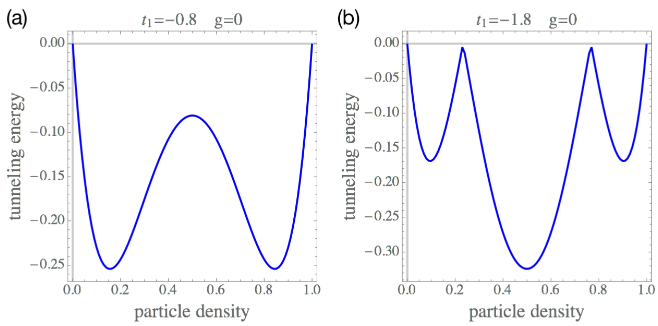

FIG. 6: The energy (of tunneling term) vs. particle density of only one spin species with two different sets of parameters: (a) $t_1 = -0.8$ and $g = 0$; and (b) $t_1 = -1.8$ and $g = 0$. When particle density is half a unit, the energy value is a non-stable extremum (local maximum) in (a) while stable extremum (local minimum) in (b).

SUMMARY AND OUTLOOK

In this work, we consider periodically modulating a 2D Fermi-Hubbard model. On resonance driving results in a correlated tunneling interacting model which we name the Floquet-Hubbard model. It is derived as an effective static Hamiltonian in the high-frequency expansion sense, which possesses certain symmetry yet depending on the original diving details. We discuss the symmetry of the system in detail in the main text. The Floquet-Hubbard Hamiltonian contains 6-fermion tunneling terms and 4-fermion interaction terms, prohibited of analytical diagonalization directly. Using the path integral approach, we establish a mean-field theory of this model, especially with even-photon resonance, which returns us a mean-filed phase diagram of the system, perfectly consistent, in certain limits, with the usual Hubbard model and perturbation theory, that there are CDW/SC or SDW phases in either attractive or repulsive interaction regime. Moreover, there are new phases emergent. When the interaction strength is small, or not so large, and t_1 is smaller or negative comparable to t_0, there emerges a coexistence phase of CDW/SC and SDW, and a phase separation phase. In aid of detailed analysis, as well as numerical calculation, we attribute the emergence of these new phases to the correlated tunneling essence. This differs from the usual Hubbard model in a fundamental way, which would be assistant in helping people understanding problems like high T_c.

If the shaking resonance is an odd-photon one, the symmetry of the system would be quite different. How does this mean-field treatment work on odd-resonance shaking cases is an interesting subject which we defer to future study.

[1] References for quantum mechanics unitary transformation here. For example, Shankar, Principle of Quantum Mechanics.
[2] Frederik Görg, Michael Messer, Kilian Sandholzer, Gregor jotzu, rémi Desbuquois and Tilman Esslinger, *Enhancement and sign change of magnetic correlations in a driven quantum many-body system*, Nature **553**, 481–485 (2018).
[3] References for high-frequency expansion here.
[4] C. N. Yang and S. C. Zhang, SO_4 symmetry in a Hubbard model, Mod. Phys. Lett. B **4**, 759 (1990).
[5] Nagaosa, Quantum Field Theory in Condensed Matter Physics. Springer 1999.
[6] Note that the Fermi surface is nesting at half filling. As a result, the SC (or Cooper pair) order is (π, π) pairing, which is coincidentally the expectation value of $\hat{L}_+$.

Supplemental materials: Path integral derivation of the mean-field theory

In this appendix, we give a derivation of the mean-field Hamiltonian Eq. (19) by using the path integral. In this language, the partition function of the system in real-time is written as:

$$\mathcal{Z} = \int \mathcal{D}\psi D\bar{\psi} \exp(i \int dt L) \tag{24}$$

$$L = \sum_{i,\sigma} i\bar{\psi}_{i\sigma}\partial_t \psi_{i\sigma} + \sum_{\langle i,j\rangle,\sigma} \left(t_0 a_{ij\bar{\sigma}}(\bar{\psi}\psi) + t_1 b^l_{ij\bar{\sigma}}(\bar{\psi}\psi)\right) \bar{\psi}_{i\sigma}\psi_{j\sigma}$$
$$- g\sum_i \bar{\psi}_{i\uparrow}\psi_{i\uparrow}\bar{\psi}_{i\downarrow}\psi_{i\downarrow} \tag{25}$$

Where ψ correspond to the fermion field. $a_{ij\bar{\sigma}}$ and $b^l_{ij\bar{\sigma}}$ are defined by replacing operators in $\hat{a}_{ij\bar{\sigma}}$ and $\hat{b}^l_{ij\bar{\sigma}}$ by fields. Since the Hamiltonian Eq. (11) contains six fermion terms, the traditional decoupling based on Hubbard-Stratonovich transformation does not work. Instead, we directly introduce auxiliary bosonic field by inserting a delta function:

$$\mathcal{Z} = \int \mathcal{D}\psi D\bar{\psi} Dn \prod_{i\sigma} \delta(n_{i,\sigma} - \bar{\psi}_{i,\sigma}\psi_{i,\sigma}) \exp(i \int dt L) \tag{26}$$

$$L = \sum_{i,\sigma} i\bar{\psi}_{i\sigma}\partial_t \psi_{i\sigma} + \sum_{\langle i,j\rangle,\sigma} \left(t_0 a_{ij\bar{\sigma}}(n) + t_1 b^l_{ij\bar{\sigma}}(n)\right) \bar{\psi}_{i\sigma}\psi_{j\sigma}$$
$$- g\sum_i n_{i\uparrow}n_{i\downarrow} \tag{27}$$

Due to the delta function, one could replace all $\psi^\dagger\psi$ by n in the action. If we integrate out n fields first, we get our original action back. Now we introduce another field to write the delta function into an integral:

$$\mathcal{Z} = \int \mathcal{D}\psi D\bar{\psi} Dn D\eta \exp(i \int dt L) \tag{28}$$

$$L = \sum_{i,\sigma} i\bar{\psi}_{i\sigma}\partial_t \psi_{i\sigma} + \sum_{\langle i,j\rangle,\sigma} \left(t_0 a_{ij\bar{\sigma}}(n) + t_1 b^l_{ij\bar{\sigma}}(n)\right) \bar{\psi}_{i\sigma}\psi_{j\sigma}$$
$$- g\sum_i n_{i\uparrow}n_{i\downarrow} + \sum_{i\sigma} \eta_{i\sigma}(n_{i,\sigma} - \bar{\psi}_{i,\sigma}\psi_{i,\sigma}) \tag{29}$$

As a result the fermion becomes quadratic. In general one could integrate out all the fermions to get an effective action of bosonic degree of freedom. The mean-field approximation is to say all the bosonic fields will be replaced by its saddle point solution. Here we restrict to the case with n and η only depend on sublattices. By changing the variable from $(n_{i\sigma}, \eta_{i\sigma})$ to (n, s_{total}, c, s) and focusing on the spin-balanced unit filling case, one find this is equivalent to using the Hamiltonian Eq. (19) and varying all the bosonic fields to optimize the ground-state energy. It is also straightforward to show our formalism reduce to the Hubbard-Stratonovich type mean-field theory when we set $t_1 = t_0$ when the model reduces to the original Hubbard model.