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Note on dynamical symmetry

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Abstract: In this note, I summarize the key point of emergent dynamical symmetry and list several simple examples might be interesting to consider. They are: 2D fermionic Hubbard model, 1D Aubry-André model, quantum spin chain and 1(2)D bosonic Harper-Hubbard ladder. Despite the triviality in the first sight seeing the so-called emergent dynamical symmetry revealed by these models, it may be interesting to seek out some appropriate observables and initial states to look into. Might be less trivial then, combined with certain models.

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1 Outline of ideas

Consider a system describe by the following Hamiltonian

$$H = H_0 + H_I \tag{1.1}$$

where H_0 is the single-particle Hamiltonian and H_I describe the part of interaction. Mostly, the interaction part could be characterized by an interacting parameter U, for example, written as a tight-binding model, the onsite Hubbard interaction strength U for fermionic system $H_I = U\sum_j n_{j\uparrow}n_{j\downarrow}$ or spinless bosonic system $H_I = U\sum_j n_j(n_j - 1)/2$. Sometimes the system holds some symmetries. Sometimes symmetry operations performmed on the system relates only the the values of parameters of Hamiltonian while not change its form. Then some dynamical symmetry emerges when we consider the evolution of the system under the Hamiltonian of different parameters. Many generic symmetries (e.g., time-reversal symmetry) are held by the interaction part H_I, that is H_I is invariant under such symmetry operation. Then it lead us to see the relation between symmetry operations on single-particle Hamiltonian H_0 and the emergent dynamical symmetries.

As a first try, we look into the emergent $+U/-U$ dynamical symmetry of Hubbard interaction.

Suppose the system is characterized by Hamiltonian $H = H_0(J, \Delta, \phi, \ldots) + H_I(U)$, where H_0 is the single particle Hamiltonian and H_I describes Hubbard interaction. J, Δ, ϕ, U, ... are parameters. Time-reversal symmetry operation is defined as some anti-unitary operation R_t such that $R_t\mathrm{i}R_t = -\mathrm{i}$. In most generic case time-reversal (when locally acts on the system) leaves interaction part invariant, that is $R_t H_I R_t^{-1} = H_I$.

Theorem 1 *If there is some unity transformation W such that $SH_0S^{-1} = -H_0$ and $SH_IS^{-1} = H_I$, where the antiunitary operation $S = R_tW$ is the combination of time-reversal operation and W, then there are some emergent dynamical symmetries with respect* *to a set of S-symmetric observables $\{\hat{O}\}$ and S-invariant initial states $\{|\psi_i\rangle\}$. By S-symmetric we mean an operator O is even/odd under symmetry operation W: $SOS^{-1} = \pm O$. By S-invariant we mean an initial state invariant under W up to a $U(1)$ phase: $S|\psi_i\rangle = e^{i\varphi}|\psi_i\rangle$.*

Formally, for a single initial state,

$$
\begin{aligned}
\langle O(t)\rangle_{+U} &= \langle\psi_i|e^{iHt}Oe^{-iHt}|\psi_i\rangle \\
&= \langle\psi_i|e^{i(H_0+H_I[U])t}Oe^{-i(H_0+H_I[U])t}|\psi_i\rangle \\
&= \langle\psi_i|S^{-1}Se^{i(H_0+H_I[U])t}S^{-1}SOS^{-1}Se^{-i(H_0+H_I[U])t}S^{-1}S|\psi_i\rangle \\
&= \langle\psi_i|e^{-i\varphi}e^{-i(-H_0+H_I[U])t}(\pm O)e^{i(-H_0+H_I[U])t}e^{i\varphi}|\psi_i\rangle \\
&= \pm\langle\psi_i|e^{i(H_0-H_I[U])t}Oe^{-i(H_0-H_I[U])t}|\psi_i\rangle \\
&= \pm\langle\psi_i|e^{i(H_0+H_I[-U])t}Oe^{-i(H_0+H_I[-U])t}|\psi_i\rangle \\
&= \pm\langle O(t)\rangle_{-U}
\end{aligned}
$$

$+/-$ in front of r.h.s. of last line corresponds to even/odd behavior of observable O under the transformation of S.

The theorem also holds for a mixed initial state $\rho_i = \sum_j p_j|\psi_j\rangle\langle\psi_j|$ where each mixed single state $|\psi_j\rangle$ fulfills the requirement of *theorem 1*.

$$
\begin{aligned}
\langle O(t)\rangle_{\rho_i,+U} &= Tr(\rho_i O_{+U}(t)) \\
&= \sum_j p_j\langle O(t)\rangle_{j,+U} \\
&= \sum_j p_j(\pm)\langle O(t)\rangle_{j,-U} \\
&= Tr(\pm\rho_i O_{-U}(t)) \\
&= \pm\langle O(t)\rangle_{\rho_i,-U}
\end{aligned}
$$

2 2D Fermionic Hubbard model

Hamiltonian writes

$$
\begin{aligned}
H &= -t\sum_{\langle l,m\rangle,\sigma} c^\dagger_{l\sigma}c_{m\sigma} + h.c. + U\sum_l n_{l\uparrow}n_{l\downarrow} &(2.1)\\
&= H_0(t) + H_I(U) &(2.2)
\end{aligned}
$$

For bipartite lattice, there exist an unitary transformation $W : c_{l\sigma} \to (-1)^l c_{l\sigma}$, where l is even/odd if l-site belongs to A/B sublattice, such that $WH_0W^{-1} = -H_0$, $WH_IW^{-1} = H_I$.[1] Time-reversal operation defined as (1) $R_t = K : i \to -i$ acting on two species the

[1] This is related to the chiral symmetry of single-particle Hamiltonian of bipartite lattice. That is, $\{\Gamma, H_0\} = 0$ where $\Gamma = P_A - P_B$ is the chiral operator of bipartite lattice. This leaves the energy spectrum symmetric about the positive and negative part. An simplest example is the SSH chain. The same story could not be produced in, say, a triangle lattice in 2D, which is not bipartite.

same way as for the spinless case, or (2) $R_t = \mathrm{i}\sigma_y K$ for ordinary spin-1/2 case. Either way makes it possible to leave H invariant under its transformation. Thus $S = R_t W$ fulfill our requirement in *theorem 1*.

Consider a set of observables

$$\mathcal{O} = \{n_l = n_{l\uparrow} + n_{l\downarrow} \mid l \in \text{number of all lattice sites}\}$$

and a set of many-body initial states

$$ini = \{\prod_{\substack{m\in D\\ \sigma\in S}} c^\dagger_{m,\sigma}|0\rangle \mid D \in \text{a set of initial area of lattice sites}; S \in \text{subspaces of spin-1/2}\}$$

Here the many-body initial states are considered as many atom wavepackets localised at different single lattice sites decoherent from each other. Such a set of operators and initial states fulfils the requirement of *theorem 1*, and $\forall O \in \mathcal{O}$, O is even under S, $SOS^{-1} = O$. Hence we expect an even dynamical symmetry of $O(t)$ under $+U/-U$ alternation:

$$\langle O(t)\rangle_{+U} = \langle O(t)\rangle_{-U}$$

This dynamical symmetry has been observed in Ref[1].

generalization. Consider the ordinary

3 1D fermionic Aubry-André model

Hamiltonian writes

$$H = -J\sum_{j,\sigma} c^\dagger_{j,\sigma}c_{j+1,\sigma} + h.c. + \Delta\sum_{j,\sigma}\cos(2\pi\beta j+\phi)c^\dagger_{j\sigma}c_{j\sigma} + U\sum_j n_{j\uparrow}n_{j\downarrow} \tag{3.1}$$

where β is in general incommesurate with the lattice and ϕ denotes the disorder phase. In brief, $H = H_0(J,\Delta,\phi) + H_I(U)$. The model displays thermalized/localization transition at some critical quasi-disorder strength Δ_c, which depends on interacting parameter U. As reported in Ref[2], it is chosen the imbalace $\mathcal{I}$ between the respective atom numbers on even (N_e) and odd (N_o) sites to be the order parameter signals localization with respect to a set of "CDW" initial states.

$$\mathcal{I} = \frac{N_e - N_o}{N_e + N_o} \tag{3.2}$$

$$CDW = \{\prod_{j\in even}(c^\dagger_{j\uparrow})^{\alpha_{j\uparrow}}(c^\dagger_{j\downarrow})^{\alpha_{j\downarrow}}|0\rangle \mid \alpha_{j\sigma} = 0 \text{ or } 1\} \tag{3.3}$$

That is, the initial state in CDW set is with zero, one, or two atoms one even sites while zero on odd ones.

In MBL phase, the order parameter $\mathcal{I}$ is non-zero in a statistical average meaning – average over disorder phase ϕ (and long-time evolution time τ).

Define unitary transformation, $W: c_{l\sigma} \to (-1)^l c_{l\sigma}$, and time-reversal operation R_t in either two ways, as in the above section. $R_t H R_t^{-1} = H$. $WH(J,\Delta,U)W^{-1} = H(-J,\Delta,U)$. As for observables and initial states, $\mathcal{I}$ is invariant under $R_t W$ combination operation and states of CDW invariant up to a global phase.

argument 1. Since $H(\Delta,\phi) = H(-\Delta,\phi+\pi)$,

$$\begin{aligned}(R_tW)H(R_tW)^{-1} &= (R_tW)(H_0(J,\Delta,\phi)+H_I(U))(R_tW)^{-1}\\ &= H_0(-J,-\Delta,\phi+\pi)+H_I(U)\\ &= -H_0(J,\Delta,\phi+\pi)+H_I(U)\end{aligned}$$

Since $\langle\mathcal{I}\rangle$ is averaged over several different disorder phase configuration $\{\phi_d\}$, we make a hypothesis that each average is taken over both ϕ and $\phi+\pi$ and these configuration yields the same averaged imbalance $\mathcal{I}$ for the same initial state over long time evolution. Therefore

$$\begin{aligned}\langle\mathcal{I}(t)\rangle_{+U,\{\phi_d\}} &= \sum_d \langle\mathcal{I}(t)\rangle_{+U,\phi_d}\\ &= \sum_d \langle\mathcal{I}(t)\rangle_{-U,\phi_d+\pi}\\ &= \sum_d \langle\mathcal{I}(t)\rangle_{-U,\phi_d}\\ &= \langle\mathcal{I}(t)\rangle_{-U,\{\phi_d\}}\end{aligned}$$

argument 2. We seek out an translation operation which "approximately" reproduces the translational symmetry of the system:

$$2\pi\beta|j-i| = (2l+1)\pi \quad\Longrightarrow\quad |j-i| = (2l+1)/2\beta$$
$$\text{(most closely to an integer and even)}$$

Thus we make a translational transformation $j \to j+(2l+1)/2\beta$ which in our example[2] $\beta = 0.721$ and $(2l+1)/2\beta$ could be $2, 34, 52, \ldots$ That is, say, $T : c_j \to c_{j+2}$. Under such transformation, combined with R_t and W defined above, $H_0 \to -H_0$, $H_I \to H_I$, $\mathcal{I} \to \mathcal{I}$. If we make another hypothesis that the initial states are translational invariant in a statistical average mean, then $\langle\mathcal{I}\rangle$ is invariant over long time evolution, thus displaying $+U/-U$ symmetry.

Above result could be checked using DMRG. (see Ref[2])

ED is hard to perform for the full Hilbert space. But for a few fermions? (fixed particle number) Are there emergent dynamical symmetries in localized/disordered system?

generalization 1. Define particle-hole transformation $\mathcal{P} : c_{j\sigma} \to c^\dagger_{j\sigma}$. Then

$$\begin{aligned}&\mathcal{P}(-Jc^\dagger_{j,\sigma}c_{j+1,\sigma})\mathcal{P}^{-1} = Jc^\dagger_{j+1,\sigma}c_{j,\sigma}\\ &\mathcal{P}c^\dagger_{j\sigma}c_{j\sigma}\mathcal{P}^{-1} = 1-c^\dagger_{j\sigma}c_{j\sigma}\\ &\mathcal{P}n_{j\uparrow}n_{j\downarrow}\mathcal{P}^{-1} = 1-(n_{j\uparrow}+n_{j\downarrow})+n_{j\uparrow}n_{j\downarrow}\end{aligned}$$

and

$$\mathcal{P}H\mathcal{P}^{-1} = J^*\sum_{j,\sigma} c^\dagger_{j,\sigma}c_{j+1,\sigma} + h.c. + \Delta\sum_j \cos(2\pi\beta j+\phi)(2-n_j) + U(1-\sum_j n_j + \sum_j n_{j\uparrow}n_{j\downarrow})$$

Suppose

1) J is real.

2) $\sum_j \cos(2\pi\beta j + \phi) = 0$ since β is incommesurate and disorder phase ϕ should be averaged.

3) fix particle number $\sum_j n_j = N$. $U(1-N)$ is a constant that doesn't matter (the dynamical behavior of evolution of wavefunctions).

then the Hamiltonian under P-transformation goes like

$$\mathcal{P}: \quad \begin{array}{ccc} H_0(J,\Delta) + H_I(U) & \longrightarrow & H_0(-J,-\Delta) + H_I(U) \\ = H_0 + H_I & & = -H_0 + H_I \end{array}$$

Time-reversal transformation is defined as $\mathcal{T} = \mathrm{i}\sigma_y K$. Easy to see $\mathcal{T}H\mathcal{T}^{-1} = H$. Now consider observables. The even-odd imbalance $\mathcal{I}$ is odd under $\mathcal{S} = \mathcal{P}\mathcal{T}$.

$$\mathcal{S}\mathcal{I}\mathcal{S}^{-1} = -\mathcal{I}$$

In this case, the system and also observables show nice symmetry. But it's hard find proper initial states.

4 Quantum spin models

It displays ferromagnetic/anti-ferromagnetic dynamical symmetry in quantum spin models like, say, 1D Heisenberg chain.

$$H = \sum_{\langle i,j\rangle} J_{ij} \vec{S}_i \cdot \vec{S}_j$$

or more generally,

$$H = \sum_{\langle i,j\rangle} J_x \sigma_i^x \sigma_j^x + J_y \sigma_i^y \sigma_j^y + J_z \sigma_i^z \sigma_j^z$$

Define symmetry operation $S = \sigma_z K$ where K is taking complex conjugation. Obviously under such transformation $J_x \to -J_x$, $J_y \to -J_y$ while $J_z \to J_z$. This implies some ferromagnetic/anti-ferromaginetic symmetry when we consider the dynamical behaviour of the system under some time evolution.

Question is to seek some interesting observables and initial states to be considered. A trivial case is $S_z = \sum_j S_j^z$ which, even under the transformation above, is an integral of motion of the system.

The dynamical symmetry of Heisenberg spin model can also be investigated in a perspective of fermionic system under Jordan-Wigner transformation. Or even in Majorana language.

5 Bosonic Harper-Hubbard model

For 2D bipartite lattice, the Hamiltonian writes

$$H = -\sum_{i,j} J_X e^{-\mathrm{i}\phi_{i,j}} a^\dagger_{i+1,j} a_{i,j} + J_Y a^\dagger_{i,j+1} a_{i,j} + h.c. + \frac{U}{2} \sum_{i,j} \hat{n}_{i,j}(\hat{n}_{i,j} - 1) \tag{5.1}$$

Here $a_j^{(\dagger)}$ are bosonic annhilation(creation) operation on $j$site, and the index $i(j)$ runs over $x(y)$ direction. A gauge can be chosen for homogeneous flux pattern: $\phi_{ij} = j\phi$ where $\phi \in [0, 2\pi)$ is a constanct. The Hamiltonian reduces to

$$H = -\sum_{i,j} J_X e^{-\mathrm{i}j\phi} a^\dagger_{i+1,j} a_{i,j} + J_Y a^\dagger_{i,j+1} a_{i,j} + h.c. + \frac{U}{2} \sum_{i,j} \hat{n}_{i,j}(\hat{n}_{i,j} - 1) \tag{5.2}$$

Define symmetry operations:

(1) time-reversal (antiunitary) operation $R_t : \mathrm{i} \to -\mathrm{i}$

(2) mirror operation (reflection across $y = 0$) $M_1 : a_{i,j} \to a_{i,-j}$

(3) mirror operation (reflection across $x = 0$) $M_2 : a_{i,j} \to a_{-i,j}$

(4) sublattice chiral operation $W : a_{i,j} \to (-)^{i+j} a_{i,j}$

Easy to see that the combination of R_t, W, and one of M, say, $S_1 = R_t M_1 W$ fulfills the requisite in *theorem 1*

$$S_1 H_0 S_1^{-1} = -H_0$$
$$S_1 H_I S_1^{-1} = H_I$$

that relates the symmetric dynamical behaviour of $+/-U$.

Moreover, two mirror symmetric operation M_1, M_2 relates $\phi \leftrightarrow -\phi$. While R_t relates $+U \leftrightarrow -U$ in dynamical consideration. Bipartite W make the sign of J_y, J_y doesn't matter. The interplay of R_t, M_1, M_2 relates the symmetrical dynamical behaviour of $\pm U$ to mirror reflection, or $\pm\phi$.

5.1 1D bosonic flux ladder

$$H = -\sum_j J_X(e^{-\mathrm{i}\phi/2} a^\dagger_{j+1} a_j + e^{\mathrm{i}\phi/2} b^\dagger_{j+1} b_j) + J_Y a^\dagger_j b_j + h.c. + \frac{U}{2} \sum_j n_j^{(a)}(n_j^{(a)} - 1) + n_j^{(b)}(n_j^{(b)} - 1)$$

Here, for simplicity, we denote the upper(lower) leg as $A(B)$ as manifested in the expression of the above Hamiltonian [$a(b)$ for bosonic field operators] and drop the y-index.

Observables:

(1) odd.

$y_{\rm CoM}^{L(R)} = \dfrac{\sum\limits_{j<0(>0)} n_j^{(a)} - n_j^{(b)}}{\sum\limits_{j<0(>0)} n_j^{(a)} + n_j^{(b)}}$ is odd under M_1. Reveals the $+\phi/-\phi$ symmetry. see Figure 1(a-d)

$y^{L(R)}$ is odd under $S_1 = R_t M_1 W$. Reveals the $+U/-U$ symmetry.

Moreover, $y_{\rm CoM}^{L(R)}$ is odd under $M_1 M_2$, transformed to each other. That is, $(M_1 M_2) y^L (M_1 M_2)^{-1} = -y^R$. Plus $(M_1 M_2) H (M_1 M_2)^{-1} = H$. Therefore $y^L(t) = -y^R(t)$ for $M_1 M_2$ invariant initial states.

(2) even. $y^{L(R)}$ under M_2 for $\phi = \pi$. see Figure 1(e,f)

(3) non-symmetric. $y^{L(R)}$ under M_2 for $\phi \neq \pi$. see Figure 1(h)

See Ref[3] for reference. Should we solve the 2-body bound state problem?

5.2 generalized to 2D case

A direct generalization of $y_{\rm CoM}$ in 2D may be something like this: $n_c = n_{\rm I} - n_{\rm II} - n_{\rm III} + n_{\rm IV}$, where the roman numeral index denotes quadrants. This quantity is odd under either M_1 or M_2 and even under combination thereof. Similar behaviours displaying emergent dynamical symmetries may appear like in the 1D case.

Local density may also be good observables to consider, as also easy to detect in experiment.

Other interesting quantities might include Hall conductance(?) and chiral current.

Acknowledgments

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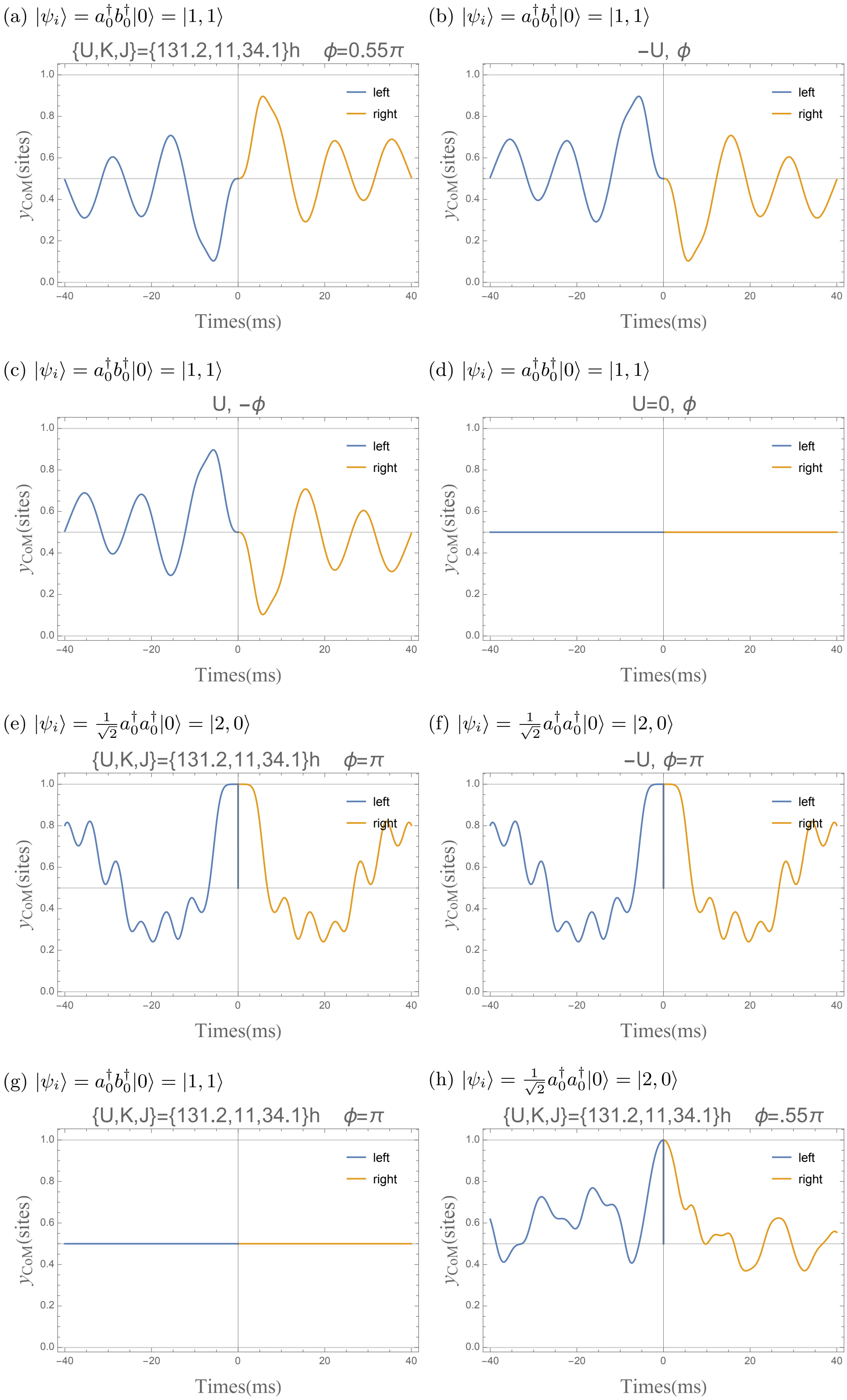

Figure 1: 1D flux ladder: dynamical behaviour of observable $y_{\text{CoM}}^{L(R)}$ for different U, ϕ, and initial states. The transformation of initial states as well as the Hamiltonian and y_{CoM} under certain operation should be examined case by case.