Dated: June 16 2017 — June 17 2017

Note on dynamical symmetry (II)

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Abstract:

Contents

1 Proposal of the concept

Consider a system describe by the following Hamiltonian

$$H = H_0 + H_I \tag{1.1}$$

where H_0 is the single-particle Hamiltonian and H_I describe the part of interaction. Mostly, the interaction part could be characterized by an interacting parameter U, for example, written as a tight-binding model, the onsite Hubbard interaction strength U for fermionic system $H_I = U \sum_j n_{j\uparrow} n_{j\downarrow}$ or spinless bosonic system $H_I = U \sum_j n_j(n_j - 1)/2$. Sometimes the system holds some symmetries. Sometimes symmetry operations performmed on the system relates only the the values of parameters of Hamiltonian while not change its form. Then some dynamical symmetry emerges when we consider the evolution of the system under the Hamiltonian of different parameters. Many generic symmetries (e.g., time-reversal symmetry) are held by the interaction part H_I, that is H_I is invariant under such symmetry operation. Then it lead us to see the relation between symmetry operations on single-particle Hamiltonian H_0 and the emergent dynamical symmetries.

As a first try, we take a look into the emergent $+U/-U$ dynamical symmetry of Hubbard interaction.

Suppose the system is characterized by Hamiltonian $H = H_0(J, \Delta, \phi, \ldots) + H_I(U)$, where H_0 is the single particle Hamiltonian and H_I describes Hubbard interaction. J, Δ, ϕ, U, $\ldots$ are parameters. Time-reversal symmetry operation is defined as some anti-unitary operation R_t such that $R_t \mathrm{i} R_t = -\mathrm{i}$. In most generic case time-reversal (when locally acts on the system) leaves interaction part invariant, that is $R_t H_I R_t^{-1} = H_I$.

Theorem 1 *If there is some unity transformation W such that $SH_0S^{-1} = -H_0$ and $SH_IS^{-1} = H_I$, where the antiunitary operation $S = R_tW$ is the combination of time-reversal operation and W, then there are some emergent dynamical symmetries with respect to a set of S-symmetric observables $\{\hat{O}\}$ and S-invariant initial states $\{|\psi_i\rangle\}$. By S-symmetric we mean an operator O is even/odd under symmetry operation W: $SOS^{-1} = \pm O$. By S-invariant we mean an initial state invariant under W up to a $U(1)$ phase: $S|\psi_i\rangle = e^{i\varphi}|\psi_i\rangle$.*

Formally, for a single initial state,

$$\begin{aligned}
\langle O(t)\rangle_{+U} &= \langle\psi_i|e^{iHt}Oe^{-iHt}|\psi_i\rangle \\
&= \langle\psi_i|e^{i(H_0+H_I[U])t}Oe^{-i(H_0+H_I[U])t}|\psi_i\rangle \\
&= \langle\psi_i|S^{-1}Se^{i(H_0+H_I[U])t}S^{-1}SOS^{-1}Se^{-i(H_0+H_I[U])t}S^{-1}S|\psi_i\rangle \\
&= \langle\psi_i|e^{-i\varphi}e^{-i(-H_0+H_I[U])t}(\pm O)e^{i(-H_0+H_I[U])t}e^{i\varphi}|\psi_i\rangle \\
&= \pm\langle\psi_i|e^{i(H_0-H_I[U])t}Oe^{-i(H_0-H_I[U])t}|\psi_i\rangle \\
&= \pm\langle\psi_i|e^{i(H_0+H_I[-U])t}Oe^{-i(H_0+H_I[-U])t}|\psi_i\rangle \\
&= \pm\langle O(t)\rangle_{-U}
\end{aligned}$$

$+/-$ in front of r.h.s. of last line corresponds to even/odd behavior of observable O under the transformation of S.

The theorem also holds for a mixed initial state $\rho_i = \sum_j p_j|\psi_j\rangle\langle\psi_j|$ where each mixed single state $|\psi_j\rangle$ fulfills the requirement of *theorem 1*.

$$\begin{aligned}
\langle O(t)\rangle_{\rho_i,+U} &= Tr(\rho_i O_{+U}(t)) \\
&= \sum_j p_j\langle O(t)\rangle_{j,+U} \\
&= \sum_j p_j(\pm)\langle O(t)\rangle_{j,-U} \\
&= Tr(\pm\rho_i O_{-U}(t)) \\
&= \pm\langle O(t)\rangle_{\rho_i,-U}
\end{aligned}$$

2 Fermi-Hubbard model - square vs. triangle: different lattices

A Fermi-Hubbard Hamiltonian generally writes

$$H = -\sum_{\langle i,j\rangle,\sigma} t_{ij}c^\dagger_{i\sigma}c_{j\sigma} + h.c. + \sum_i Un_{i\uparrow}n_{i\downarrow}$$

We study the two-body problem (one spin-up and the other spin-down) on a ladder of square lattice vs. triangle lattice, separately.

2.1 Square ladder

Hamiltonian writes

$$H = -\sum_{i,j,\sigma} J_X c^\dagger_{i,j+1,\sigma} c_{i,j,\sigma} + J_Y c^\dagger_{1,j,\sigma} c_{0,j,\sigma} + H.c. + \sum_i U n_{i\uparrow} n_{i\downarrow} \tag{2.1}$$

J_X needs not equal J_Y generally but for simplicity we take them to be equal in the following.

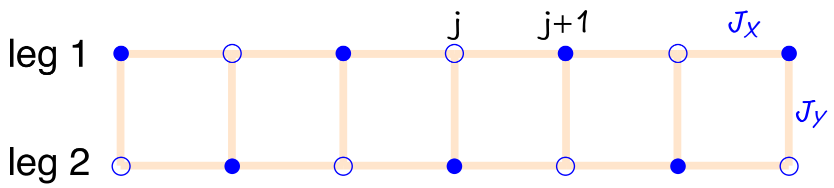

Figure 1: Fermi-Hubbard on a square ladder. Solid(hollow) points ∈ A(B) sublattice.

Notice that the lattice is bipartite, which means it can be devided into two sublattices. A simple way to do this is that to decide (i, j) site belong to $A(B)$ sublattice if $i + j \in$ odd(even).

Define an unitary transformation $W : c_{l\sigma} \to (-1)^l c_{l\sigma}$, where l is even/odd if l-site belongs to A/B sublattice, such that $W H_0 W^{-1} = -H_0$, $W H_I W^{-1} = H_I$.[1] Time-reversal operation defined as (1) $R_t = K : \mathrm{i} \to -\mathrm{i}$ acting on two species the same way as for the spinless case, or (2) $R_t = \mathrm{i}\sigma_y K$ for ordinary spin-1/2 case. Either way makes it possible to leave H invariant under its transformation. Thus $S = R_t W$ fulfill our requirement in *theorem 1* and there is an emergent dynamical symmetry about U in this system.

To reveal the dynamical symmetry of interaction strength U, we solve the two-body problem on a ladder of $2 \times 15 = 30$ sites under periodic boundary conditions. The initial state is prepared in an uniform way

$$|\psi_i\rangle = \sum_j (-1)^j c^\dagger_{j\uparrow} c^\dagger_{j\downarrow} |0\rangle \tag{2.2}$$

where the summation index j runs over all sites and the phase factor before each superposed state equals $1(-1)$ if the site j belongs to $A(B)$ sublattice. Such a two-body state respect the translational symmetry of the system and is invariant under $R_t W$. Noted also any local density operator is invariant under $R_t W$.

[1]This is related to the chiral symmetry of single-particle Hamiltonian of bipartite lattice. That is, $\{\Gamma, H_0\} = 0$ where $\Gamma = P_A - P_B$ is the chiral operator of bipartite lattice. This leaves the energy spectrum symmetric about the positive and negative part. An simplest example is the SSH chain. The same story could not be produced in, say, a triangle lattice in 2D, which is not bipartite.

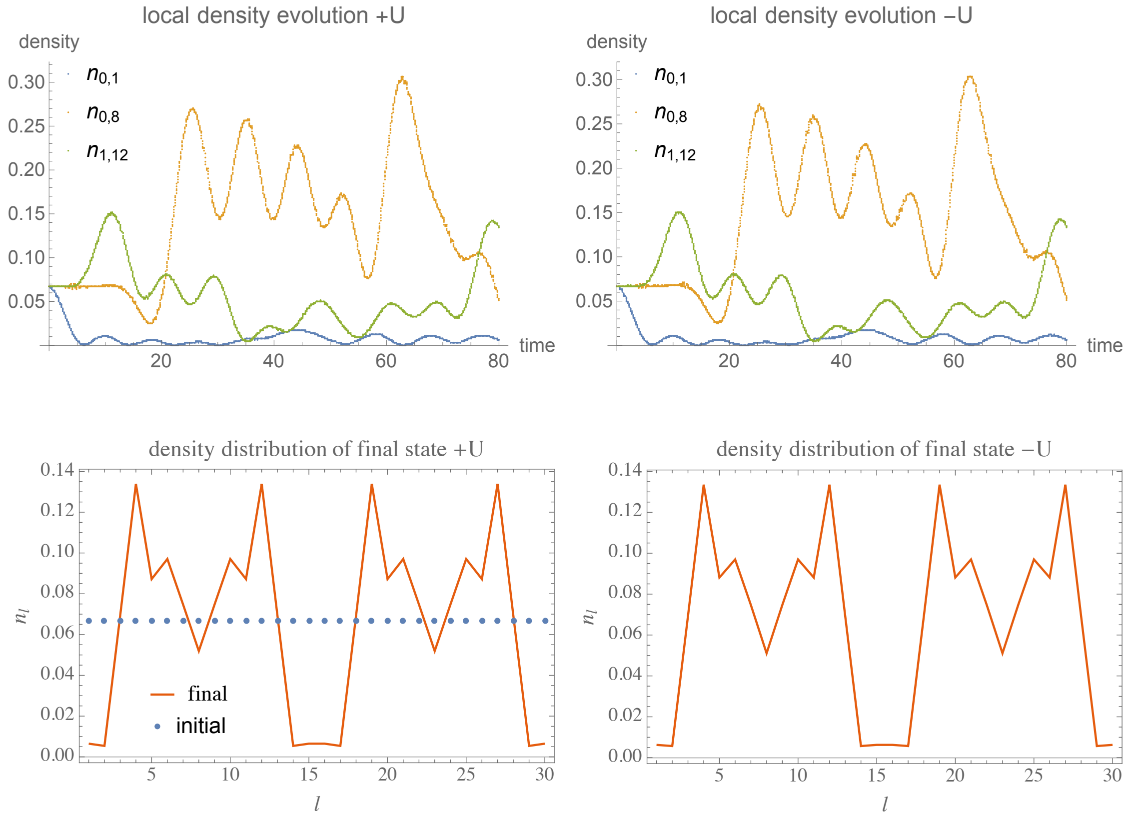

Figure 2: parameters: $J_Y/J_X = 1$, $U/J_X = \pm 10$. The evolution of local density operators are exactly the same for $+U$ and $-U$.

2.2 Triangle ladder

Hamiltonian writes

$$H = -\sum_{i,j,\sigma} J_X c^\dagger_{i,j+1,\sigma} c_{i,j,\sigma} + J_Y c^\dagger_{1,j,\sigma} c_{0,j,\sigma} + J_Y c^\dagger_{1,j+1,\sigma} c_{0,j,\sigma} + H.c. + \sum_i U n_{i\uparrow} n_{i\downarrow} \quad (2.3)$$

Triangle lattice is not bipartite. The W transformation cannot be defined here. And it lacks dynamical symmetry shown above in the square one.

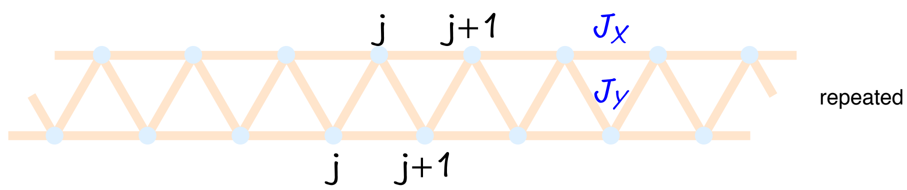

Figure 3: Fermi-Hubbard on a triangle ladder. Here no division can be made of A/B sublattices. Periodic boundary conditions are assumed in the calculation.

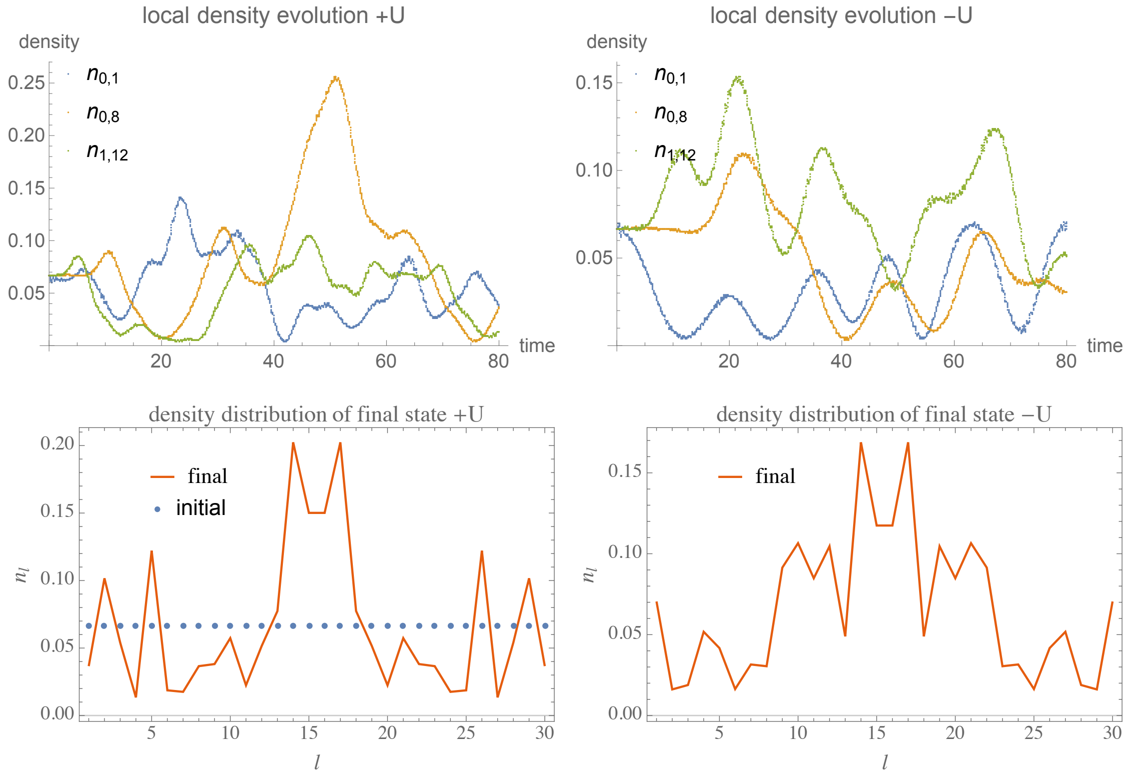

Figure 4: same parameters: $J_Y/J_X = 1$, $U/J_X = \pm 10$. The evolution of local density operators are not the same for $+U$ and $-U$.

3 Bosonic flux ladder: different initial states

$$H = -\sum_j J_X(e^{-\mathrm{i}\phi/2}a^\dagger_{j+1}a_j + e^{\mathrm{i}\phi/2}b^\dagger_{j+1}b_j) + J_Y a^\dagger_j b_j + h.c. + \frac{U}{2}\sum_j n^{(a)}_j(n^{(a)}_j - 1) + n^{(b)}_j(n^{(b)}_j - 1) \tag{3.1}$$

where bosonic field operators $a^{(\dagger)}_j$ on leg 1 and $b^{(\dagger)}_j$ on leg 2.

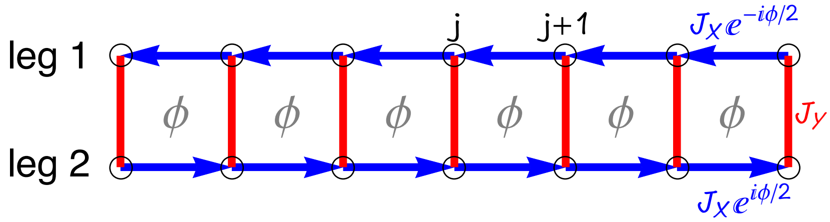

Figure 5: flux ladder: arrow denotes direction of hopping with phase gain ϕ.

We've discussed a lot about this model. The operators we consider here are the center of mass on y-direction, defined as

$$y_{\mathrm{CoM}}^{(L)} = \frac{\sum\limits_{j<0} n_j^{(a)}}{\sum\limits_{j<0} n_j^{(a)} + n_j^{(b)}} \tag{3.2}$$

$$y_{\mathrm{CoM}}^{(R)} = \frac{\sum\limits_{j>0} n_j^{(a)}}{\sum\limits_{j>0} n_j^{(a)} + n_j^{(b)}} \tag{3.3}$$

We do the numerics of the two-body problem on a ladder of 15 rungs under open boundary conditions.[2] Parameters are chosen: $\{U, J_X, J_Y\} = \{131.2, 11, 34.1\} \times 2\pi\hbar$. With different initial states. Here come the results.

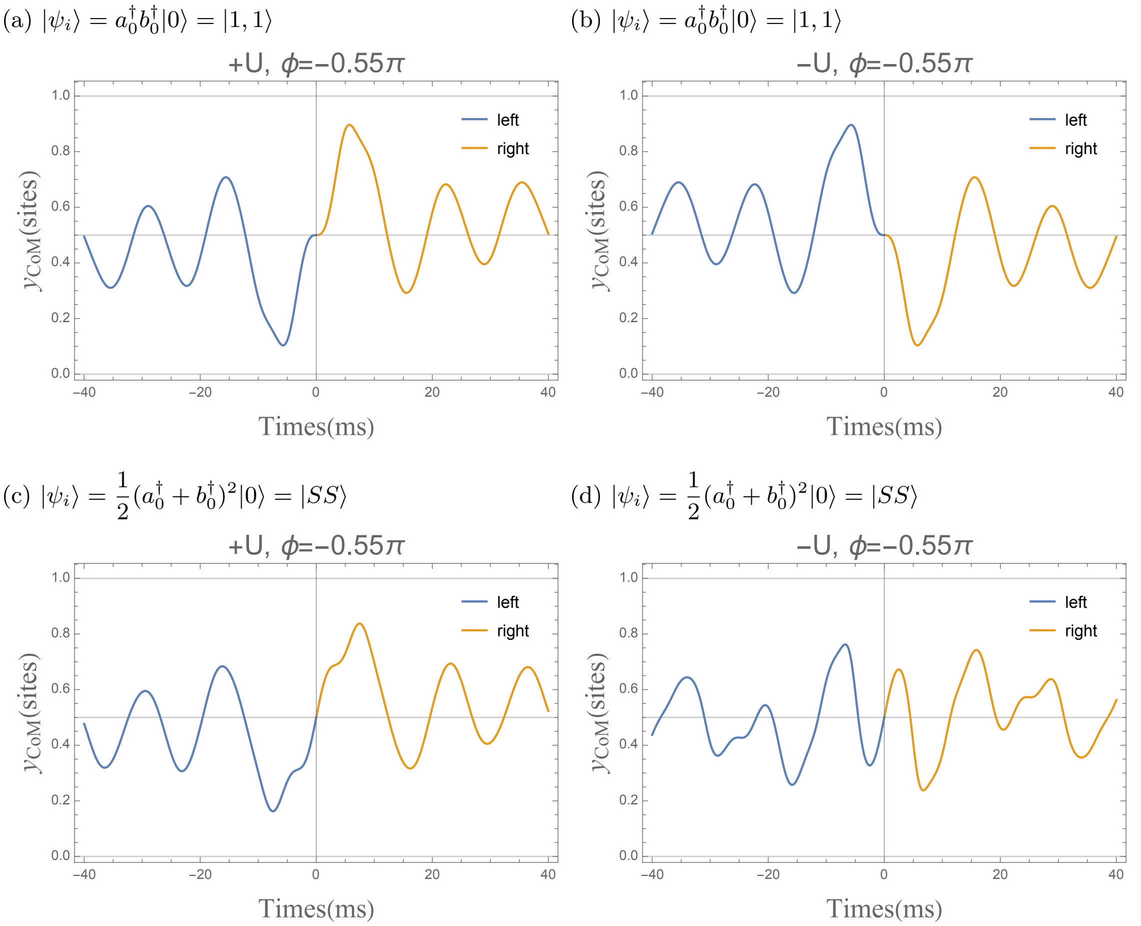

Figure 6: Flux ladder: time evolution of $y_{\mathrm{CoM}^{L(R)}}$ for different initial states and $\pm U$. See that for initial state $|\psi_i\rangle = a_0^\dagger b_0^\dagger |0\rangle$ there is odd symmetry between the dynamical evolution of operators under $+U/-U$ while for $|\psi_i\rangle = \frac{1}{2}(a_0^\dagger + b_0^\dagger)^2 |0\rangle$ there isn't.

[2] To compare with experimental results in Ref[3].

4 AA model: different operators

Consider two interacting fermions on 1D chains with different settings of onsite energy.

$$H = -\sum_{\langle i,j\rangle,\sigma} Jc^{\dagger}_{i\sigma}c_{j\sigma} + h.c. + \sum_{j\sigma}(-1)^j \Delta c^{\dagger}_{j\sigma}c_{j\sigma} + U\sum_j n_{j\uparrow}n_{j\downarrow} \tag{4.1}$$

$$H = -\sum_{\langle i,j\rangle,\sigma} Jc^{\dagger}_{i\sigma}c_{j\sigma} + h.c. + \sum_{l\sigma}(-1)^l (\Delta_1 c^{\dagger}_{2l,\sigma}c_{2l,\sigma} + \Delta_2 c^{\dagger}_{2l+1,\sigma}c_{2l+1,\sigma}) + U\sum_{j\sigma} n_{j\uparrow}n_{j\downarrow} \tag{4.2}$$

We solve the two-body problem separately.

4.1 Δ, $-\Delta$

Define unitary operatos: sublattice chiral operator $W : c_l \to (-)^l c_l$; translation operator $T_1 : c_l \to c_{l+1}$. The combination of transformations $S_1 = R_t W T_1$ fulfils the requirement in *theorem* 1.

Define observables:

$$n^{(A)} = \sum_j n_{2j-1} \tag{4.3}$$

$$n^{(B)} = \sum_j n_{2j} \tag{4.4}$$

$$n^{(A)} - n^{(B)} = \sum_j n_{2j-1} - n_{2j} \tag{4.5}$$

$n^{(A)} - n^{(B)}$ is odd under T_1 (hence $R_t W T_1$). Chose uniform initial state to evolve. Results show

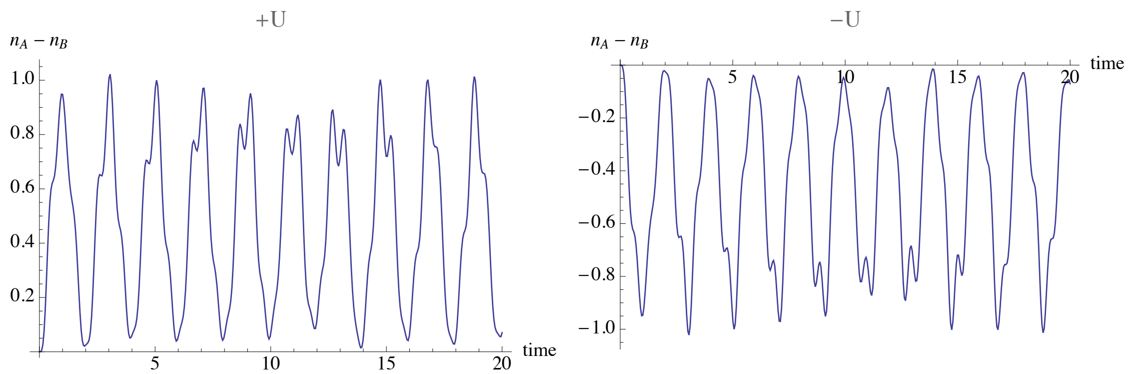

Figure 7: AA chain 1: the dynamical behavior of $n^{(A)} - n^{(B)}$ is odd with $\pm U$. Parameters: $\Delta/J = 0.8$, $U/J = 10$. Initial state: $|\psi_i\rangle = \frac{1}{\sqrt{N}}\sum_l c^{\dagger}_{l\uparrow}c^{\dagger}_{l\downarrow}|0\rangle$

4.2 Δ_1, Δ_2, $-\Delta_1$, $-\Delta_2$

Instead of T_1, here we define $T_2 : c_l \to c_{l+2}$. And a series of operators:

$$n^{(A)} = \sum_j n_{4j+1} \tag{4.6}$$

$$n^{(B)} = \sum_j n_{4j+2} \tag{4.7}$$

$$n^{(C)} = \sum_j n_{4j+3} \tag{4.8}$$

$$n^{(D)} = \sum_j n_{4j} \tag{4.9}$$

$$n^{(A)} - n^{(B)} + n^{(C)} - n^{(D)} = \sum_j n_{4j+1} - n_{4j+2} + n_{4j+3} - n_{4j} \tag{4.10}$$

$$n^{(A)} + n^{(B)} - n^{(C)} - n^{(D)} = \sum_j n_{4j+1} + n_{4j+2} - n_{4j+3} - n_{4j} \tag{4.11}$$

See that $n^{(A)} - n^{(B)} + n^{(C)} - n^{(D)}$ is even under T_2 while $n^{(A)} + n^{(B)} - n^{(C)} - n^{(D)}$ is odd.

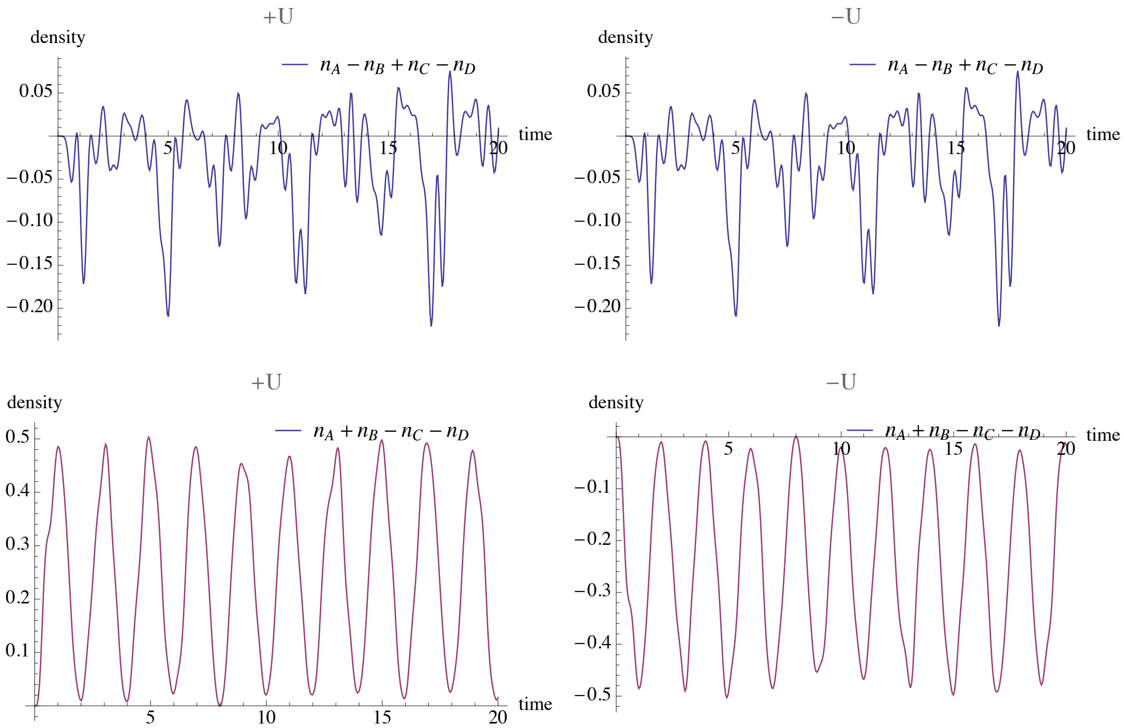

Figure 8: AA chain 2: the dynamical behavior of different operators. $n^{(A)} - n^{(B)} + n^{(C)} - n^{(D)}$ is even while $n^{(A)} + n^{(B)} - n^{(C)} - n^{(D)}$ odd with $\pm U$. Parameters: $\Delta_1/J = 0.3$, $\Delta_2/J = 1.3$, $U/J = 10$. Initial state: $|\psi_i\rangle = \frac{1}{\sqrt{N}} \sum_l c^\dagger_{l\uparrow} c^\dagger_{l\downarrow} |0\rangle$

Acknowledgments

I thank helpful discussion with Jinlong, Pengfei and Bingtian.

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