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Note on dynamical symmetry

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ABSTRACT:

In this updated version, numeric results are added of interacting two-particle problems in AA model and flux ladder (square and triangle) to support our arguments.

It is shown that, by merely a two-body calculation, the dynamical symmetry of the order parameter (that is the imbalance) is emerged statistically in the AA model. And the two-body calculation is already good enough to essentially explain the experimental results in Ref[2].

As for the bosonic flux ladder, triangle one lacks some dynamical symmetries (the dynamical symmetry of $+U/-U$) presented in the square one by a straightforward two-body ED.

Contents

1 Outline of ideas

Consider a system describe by the following Hamiltonian

$$H = H_0 + H_I \tag{1.1}$$

where H_0 is the single-particle Hamiltonian and H_I describe the part of interaction. Mostly, the interaction part could be characterized by an interacting parameter U, for example, written as a tight-binding model, the onsite Hubbard interaction strength U for fermionic system $H_I = U \sum_j n_{j\uparrow} n_{j\downarrow}$ or spinless bosonic system $H_I = U \sum_j n_j(n_j - 1)/2$. Sometimes the system holds some symmetries. Sometimes symmetry operations performmed on the system relates only the the values of parameters of Hamiltonian while not change its form. Then some dynamical symmetry emerges when we consider the evolution of the system under the Hamiltonian of different parameters. Many generic symmetries (e.g., time-reversal symmetry) are held by the interaction part H_I, that is H_I is invariant under such symmetry operation. Then it lead us to see the relation between symmetry operations on single-particle Hamiltonian H_0 and the emergent dynamical symmetries.

As a first try, we take a look into the emergent $+U/-U$ dynamical symmetry of Hubbard interaction.

Suppose the system is characterized by Hamiltonian $H = H_0(J, \Delta, \phi, \dots) + H_I(U)$, where H_0 is the single particle Hamiltonian and H_I describes Hubbard interaction. J, Δ, ϕ, U, $\dots$ are parameters. Time-reversal symmetry operation is defined as some anti-unitary operation R_t such that $R_t \mathrm{i} R_t = -\mathrm{i}$. In most generic case time-reversal (when locally acts on the system) leaves interaction part invariant, that is $R_t H_I R_t^{-1} = H_I$.

Theorem 1 *If there is some unity transformation W such that $SH_0S^{-1} = -H_0$ and $SH_IS^{-1} = H_I$, where the antiunitary operation $S = R_tW$ is the combination of time-reversal operation and W, then there are some emergent dynamical symmetries with respect* *to a set of S-symmetric observables $\{\hat{O}\}$ and S-invariant initial states $\{|\psi_i\rangle\}$. By S-symmetric we mean an operator O is even/odd under symmetry operation W: $SOS^{-1} = \pm O$. By S-invariant we mean an initial state invariant under W up to a $U(1)$ phase: $S|\psi_i\rangle = e^{\mathrm{i}\varphi}|\psi_i\rangle$.*

Formally, for a single initial state,

$$\begin{aligned}
\langle O(t)\rangle_{+U} &= \langle\psi_i|e^{\mathrm{i}Ht}Oe^{-\mathrm{i}Ht}|\psi_i\rangle \\
&= \langle\psi_i|e^{\mathrm{i}(H_0+H_I[U])t}Oe^{-\mathrm{i}(H_0+H_I[U])t}|\psi_i\rangle \\
&= \langle\psi_i|S^{-1}Se^{\mathrm{i}(H_0+H_I[U])t}S^{-1}SOS^{-1}Se^{-\mathrm{i}(H_0+H_I[U])t}S^{-1}S|\psi_i\rangle \\
&= \langle\psi_i|e^{-\mathrm{i}\varphi}e^{-\mathrm{i}(-H_0+H_I[U])t}(\pm O)e^{\mathrm{i}(-H_0+H_I[U])t}e^{\mathrm{i}\varphi}|\psi_i\rangle \\
&= \pm\langle\psi_i|e^{\mathrm{i}(H_0-H_I[U])t}Oe^{-\mathrm{i}(H_0-H_I[U])t}|\psi_i\rangle \\
&= \pm\langle\psi_i|e^{\mathrm{i}(H_0+H_I[-U])t}Oe^{-\mathrm{i}(H_0+H_I[-U])t}|\psi_i\rangle \\
&= \pm\langle O(t)\rangle_{-U}
\end{aligned}$$

$+/-$ in front of r.h.s. of last line corresponds to even/odd behavior of observable O under the transformation of S.

The theorem also holds for a mixed initial state $\rho_i = \sum_j p_j|\psi_j\rangle\langle\psi_j|$ where each mixed single state $|\psi_j\rangle$ fulfills the requirement of *theorem 1*.

$$\begin{aligned}
\langle O(t)\rangle_{\rho_i,+U} &= Tr(\rho_i O_{+U}(t)) \\
&= \sum_j p_j\langle O(t)\rangle_{j,+U} \\
&= \sum_j p_j(\pm)\langle O(t)\rangle_{j,-U} \\
&= Tr(\pm\rho_i O_{-U}(t)) \\
&= \pm\langle O(t)\rangle_{\rho_i,-U}
\end{aligned}$$

2 2D Fermionic Hubbard model

Hamiltonian writes

$$\begin{aligned}
H &= -t\sum_{\langle l,m\rangle,\sigma} c^\dagger_{l\sigma}c_{m\sigma} + h.c. + U\sum_l n_{l\uparrow}n_{l\downarrow} && (2.1)\\
&= H_0(t) + H_I(U) && (2.2)
\end{aligned}$$

For bipartite lattice, there exist an unitary transformation $W : c_{l\sigma} \to (-1)^l c_{l\sigma}$, where l is even/odd if l-site belongs to A/B sublattice, such that $WH_0W^{-1} = -H_0$, $WH_IW^{-1} = H_I$.[1] Time-reversal operation defined as (1) $R_t = K : \mathrm{i} \to -\mathrm{i}$ acting on two species the

[1]This is related to the chiral symmetry of single-particle Hamiltonian of bipartite lattice. That is, $\{\Gamma, H_0\} = 0$ where $\Gamma = P_A - P_B$ is the chiral operator of bipartite lattice. This leaves the energy spectrum symmetric about the positive and negative part. An simplest example is the SSH chain. The same story could not be produced in, say, a triangle lattice in 2D, which is not bipartite.

same way as for the spinless case, or (2) $R_t = \mathrm{i}\sigma_y K$ for ordinary spin-1/2 case. Either way makes it possible to leave H invariant under its transformation. Thus $S = R_t W$ fulfill our requirement in *theorem 1*.

Consider a set of observables

$$\mathcal{O} = \{n_l = n_{l\uparrow} + n_{l\downarrow} \mid l \in \text{number of all lattice sites}\}$$

and a set of many-body initial states

$$ini = \{\prod_{\substack{m\in D\\ \sigma\in S}} c^\dagger_{m,\sigma}|0\rangle \mid D \in \text{a set of initial area of lattice sites}; S \in \text{subspaces of spin-1/2}\}$$

Here the many-body initial states are considered as many atom wavepackets localised at different single lattice sites decoherent from each other. Such a set of operators and initial states fulfils the requirement of *theorem 1*, and $\forall O \in \mathcal{O}$, O is even under S, $SOS^{-1} = O$. Hence we expect an even dynamical symmetry of $O(t)$ under $+U/-U$ alternation:

$$\langle O(t)\rangle_{+U} = \langle O(t)\rangle_{-U}$$

This dynamical symmetry has been observed in Ref[1].

3 1D fermionic Aubry-André model

Hamiltonian writes

$$H = -J\sum_{j,\sigma} c^\dagger_{j,\sigma}c_{j+1,\sigma} + h.c. + \Delta\sum_{j,\sigma}\cos(2\pi\beta j + \phi)c^\dagger_{j,\sigma}c_{j,\sigma} + U\sum_j n_{j\uparrow}n_{j\downarrow} \tag{3.1}$$

where β is in general incommesurate with the lattice and ϕ denotes the disorder phase. In brief, $H = H_0(J,\Delta,\phi) + H_I(U)$. The model displays thermalized/localization transition at some critical quasi-disorder strength Δ_c, which depends on interacting parameter U. As reported in Ref[2], it is chosen the imbalace $\mathcal{I}$ between the respective atom numbers on even (N_e) and odd (N_o) sites to be the order parameter signals localization with respect to a set of "CDW" initial states.

$$\mathcal{I} = \frac{N_e - N_o}{N_e + N_o} \tag{3.2}$$

$$CDW = \{\prod_{j\in even} (c^\dagger_{j\uparrow})^{\alpha_{j\uparrow}}(c^\dagger_{j\downarrow})^{\alpha_{j\downarrow}}|0\rangle \mid \alpha_{j\sigma} = 0 \text{ or } 1\} \tag{3.3}$$

That is, the initial state in CDW set is with zero, one, or two atoms one even sites while zero on odd ones.

In MBL phase, the order parameter $\mathcal{I}$ is non-zero in a statistical average meaning – average over disorder phase ϕ (and long-time evolution time τ).

Define unitary transformation, $W : c_{l\sigma} \to (-1)^l c_{l\sigma}$, and time-reversal operation R_t in either two ways, as in the above section. $R_t H R_t^{-1} = H$. $WH(J,\Delta,U)W^{-1} = H(-J,\Delta,U)$. As for observables and initial states, $\mathcal{I}$ is invariant under $R_t W$ combination operation and states of CDW invariant up to a global phase.

argument 1. Since $H(\Delta, \phi) = H(-\Delta, \phi + \pi)$,

$$\begin{aligned}(R_t W)H(R_t W)^{-1} &= (R_t W)(H_0(J, \Delta, \phi) + H_I(U))(R_t W)^{-1} \\ &= H_0(-J, -\Delta, \phi + \pi) + H_I(U) \\ &= -H_0(J, \Delta, \phi + \pi) + H_I(U)\end{aligned}$$

Since $\langle \mathcal{I} \rangle$ is averaged over several different disorder phase configuration $\{\phi_d\}$, we make a hypothesis that each average is taken over both ϕ and $\phi + \pi$ and these configuration yields the same averaged imbalance $\mathcal{I}$ for the same initial state over long time evolution. Therefore

$$\begin{aligned}\langle \mathcal{I}(t) \rangle_{+U,\{\phi_d\}} &= \sum_d \langle \mathcal{I}(t) \rangle_{+U,\phi_d} \\ &= \sum_d \langle \mathcal{I}(t) \rangle_{-U,\phi_d+\pi} \\ &= \sum_d \langle \mathcal{I}(t) \rangle_{-U,\phi_d} \\ &= \langle \mathcal{I}(t) \rangle_{-U,\{\phi_d\}}\end{aligned}$$

argument 2. We seek out an translation operation which "approximately" reproduces the translational symmetry of the system:

$$2\pi\beta|j - i| = (2l + 1)\pi \quad \Longrightarrow \quad |j - i| = (2l + 1)/2\beta$$
$$\text{(most closely to an integer and even)}$$

Thus we make a translational transformation $j \to j + (2l + 1)/2\beta$ which in our example[2] $\beta = 0.721$ and $(2l + 1)/2\beta$ could be $2, 34, 52, \ldots$ That is, say, $T : c_j \to c_{j+2}$. Under such transformation, combined with R_t and W defined above, $H_0 \to -H_0$, $H_I \to H_I$, $\mathcal{I} \to \mathcal{I}$. If we make another hypothesis that the initial states are translational invariant in a statistical average mean, then $\langle \mathcal{I} \rangle$ is invariant over long time evolution, thus displaying $+U/-U$ symmetry.

Above result could be checked using DMRG. (see Ref[2])

ED is hard to perform for the full Hilbert space. But for a few fermions? (fixed particle number) Are there emergent dynamical symmetries in localized/disordered system?

3.1 numerics

We do numerics to provide evidence supporting argument 1.

Firstly we calculate the case for two interacting fermions (with one spin-up and the other spin-down) on a lattice of 30 sites, with periodic boundary conditions.

We try to extract out the information of the thermalized/localized phase by doing **MERELY** two-body calculation — cause it's hard do ED for large particle number. We calculate dynamical evolution of all possible two-body initial states of prepared on odd sites (1/4 in total) for different sets of parameters $\{U, \Delta\}$. All the local density operators and the summation or difference of them are good observables under $(R_t W)$-transformation defined in argument 1, as well as all the initial states.

Firstly, we show the dynamical symmetry of the system between $\{U, \phi\}$ and $\{-U, \phi+\pi\}$ by performming an exact time evolution of the total density operators of odd and even sites of a certain initial state $|\psi_i\rangle = c^\dagger_{1\uparrow} c^\dagger_{1\downarrow}|0\rangle$, namely, two fermions with one spin-up and one spin-down prepared in the same 1st site. (such states are called doublon in Ref[2].)

Results shown in Figure 1. See that the curves in (a) and (b) are exactly the same.

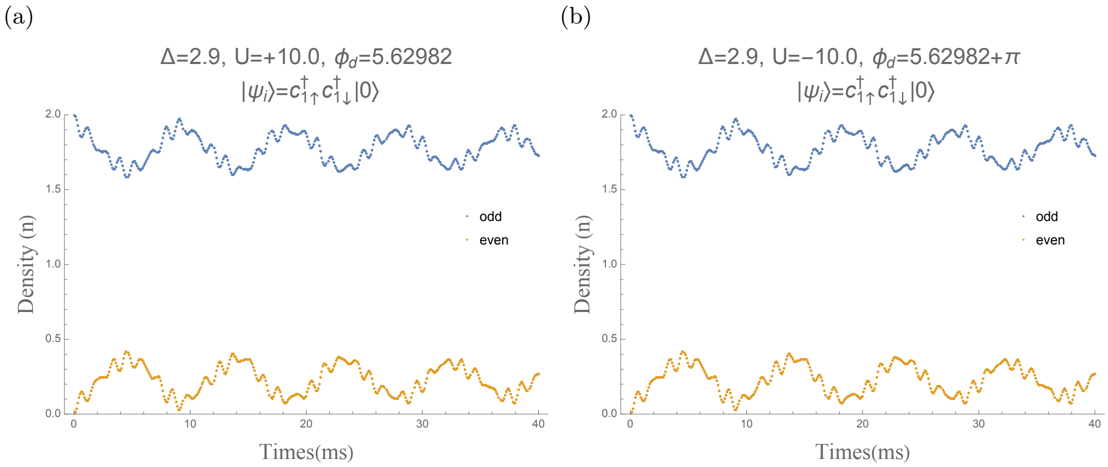


Figure 1: Time evolution of n_{odd} and n_{even} for a certain initial state, with $\{U, \phi\}$ vs. $\{-U, \phi + \pi\}$ as parameters. The values of these observables are exactly the same, even the states evolved with time differs from each other a lot.

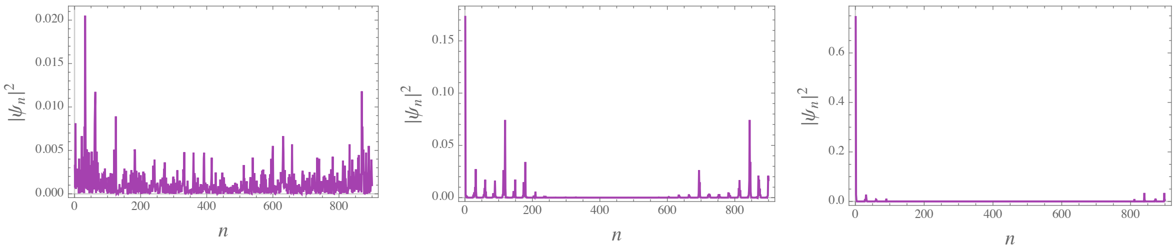
We then do time evolution for all 225 (1/4 of 900) initial states under different parameter settings, and calculate the average odd-even imbalance $\mathcal{I}$ with up to 1000 disorder sample. Here the imbalace operator is defined as

$$\hat{\mathcal{I}} = \sum_{j \in odd} \hat{n}_j - \sum_{j \in even} \hat{n}_j$$

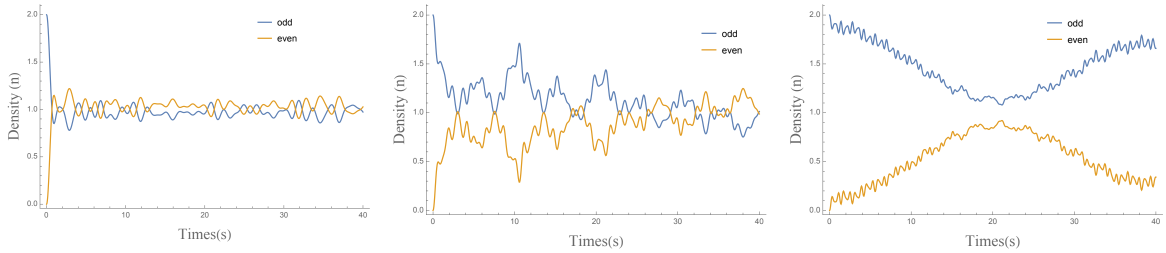
It shows that for large quasi-disorder strength Δ and proper U the imbalance remains non-zero, even far from zero, for all two-body initial states, manifesting the localization phase. While for some small Δ, the imbalance value relaxes to nearly zero, a signal of the scrambling of information initially prepared in the CDW-order states.

$$CDW = \{ \prod_{j=1,3,5,\ldots,29} (c^\dagger_{j\uparrow})^{\alpha_{j\uparrow}} (c^\dagger_{j\downarrow})^{\alpha_{j\downarrow}} |0\rangle \mid \alpha_{j\sigma} = 0 \text{ or } 1\}$$

(a) $\Delta/J = 1.0, U/J = 2.0$ (b) $\Delta/J = 2.9, U/J = 5.0$ (c) $\Delta/J = 8.0, U/J = 10.3$



(d) time evolution: $0 \sim 40$



(g) at final time $t_f = 40$, averging over 1000 disorder samples

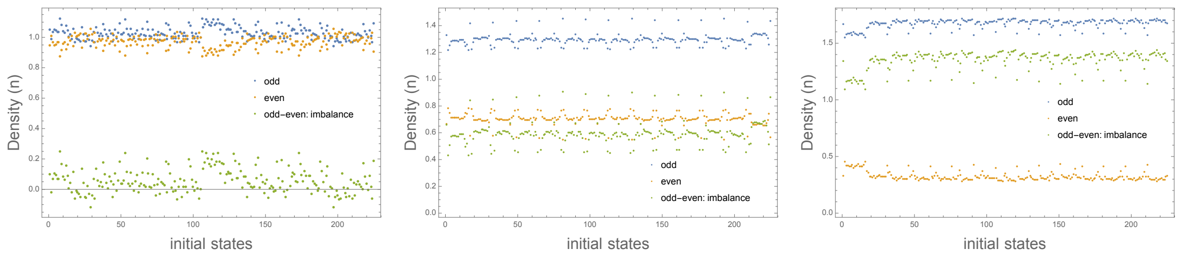
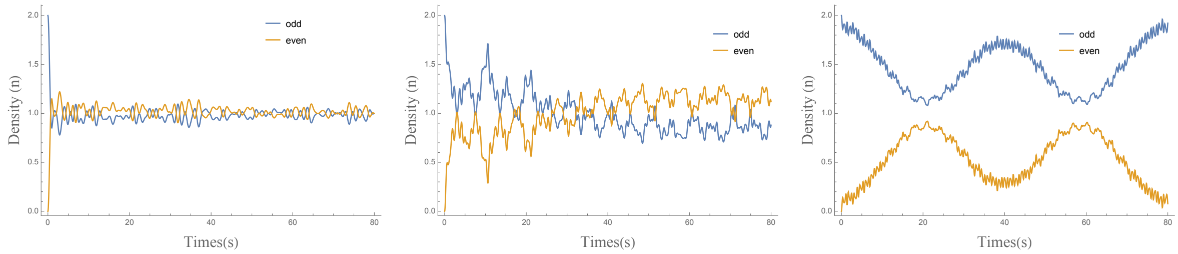


Figure 2: Thermalization vs. localization. initial state: $c^\dagger_{1\uparrow}c^\dagger_{1\downarrow}|0\rangle$. Three columns correspond to three different sets of parameter settings, system under which behaves as thermalized(a), localized(c), and in between(b). The first row are final states evolved after $t = 40$. The second row are dynamical evolution of total density operators (odd and even) in the time interval $0 \sim 40$. The third row are densities and imbalance values after evolution from all CDW initial states (defined in main text), at final time $t_f = 40$, averaging over 1000 disorder samples. Below: long time evolution ($0 \sim 80$) of observables. see distinguishable patterns.

(a) long time: $0 \sim 80$



From Figure 2 we see that, the imbalace is a good quantity that may serve well as the order parameter of thermalization-localization transition. Good as its order parameter attribute, its invariance under $R_t W$ transformation makes it a good observable to demonstrate the dynamical symmetry of $+U/-U$ interaction even in a disorder system in the statistical sense. And this can be shown clearly even within merely a two-body framework.

To see this clearly, we generalize the calculation of those gives Figure 2(d-f), and analyse the data produced. The evolution time is still set as $t_f = 40$. We find that the under same Δ, the imbalance value at t_f for U and $-U$ for all two-body CDW initial states are almost the same for averaging over large disorder samples. We define a quantity, namely the relative imbalance, to characterize the fineness of statistical averaging quantitatively. The relative imbalance for each initial state is defined as the absolute difference of $\mathcal{I}_{+U}$ and $\mathcal{I}_{-U}$ divides two times of their mean value, averaged over certain disorder samples. That is,

$$d\mathcal{I} = \frac{|\mathcal{I}_{+U} - \mathcal{I}_{-U}|}{\mathcal{I}_{+U} + \mathcal{I}_{-U}}$$

The relative imbalance of the system is averaged again over an equal ensemble, namely an ensemble of all initial states with equal possibilities. (such an ensemble may not be ideal but good enough to keep the essense of the story.) Plotted in Figure 3 for two sets of parameters.

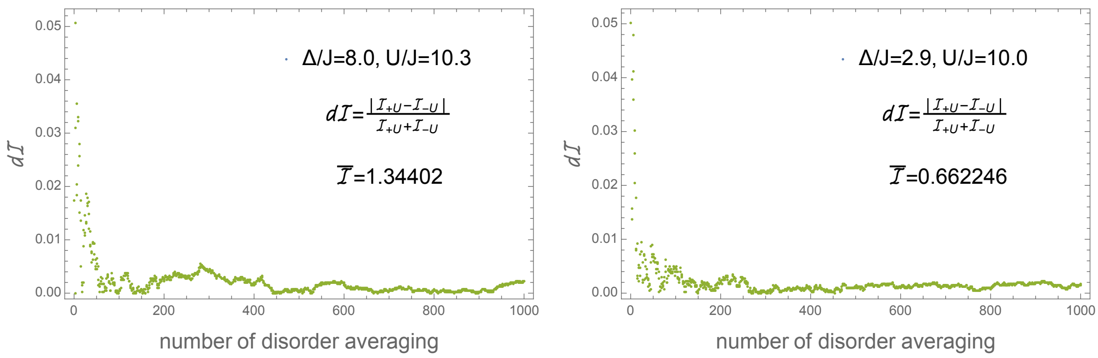


Figure 3: relative imbalance variation vs. number of disorder sample. These two cases are both in localized phase. The imbalance value, averaged over 1000 disorder samples and over an equal ensemble, is $\bar{\mathcal{I}} = 1.344, 0.662$ repectively. In both cases, the error of $+U/-U$ to their mean value is less than 0.01 for such an ensemble.

We see that in a statitical meaning the imbalance is truly a good quantity to serves as the order parameter of thermalization-localization transition in a quasi-disorder system and demonstrate the (statistically) emergent dynamical symmetry of $+U/-U$ in such a system, manifested already in just a two-body calculation.

Q: Does the two-body calculation essential enough to understand the many-body physics in such a interacting system? We want to do more calculations with different $\{\Delta, U\}$ to show curves like in Fig.4(b) of Ref[2]. Even good, a color figure like Fig.4(a). Worth it?

4 Bosonic Harper-Hubbard model

For 2D bipartite lattice, the Hamiltonian writes

$$H = -\sum_{i,j} J_X e^{-\mathrm{i}\phi_{i,j}} a^\dagger_{i+1,j} a_{i,j} + J_Y a^\dagger_{i,j+1} a_{i,j} + h.c. + \frac{U}{2}\sum_{i,j} \hat{n}_{i,j}(\hat{n}_{i,j} - 1) \tag{4.1}$$

Here $a_j^{(\dagger)}$ are bosonic annhilation(creation) operation on $j$site, and the index $i(j)$ runs over $x(y)$ direction. A gauge can be chosen for homogeneous flux pattern: $\phi_{ij} = j\phi$ where $\phi \in [0, 2\pi)$ is a constanct. The Hamiltonian reduces to

$$H = -\sum_{i,j} J_X e^{-\mathrm{i}j\phi} a^\dagger_{i+1,j} a_{i,j} + J_Y a^\dagger_{i,j+1} a_{i,j} + h.c. + \frac{U}{2}\sum_{i,j} \hat{n}_{i,j}(\hat{n}_{i,j} - 1) \tag{4.2}$$

Define symmetry operations:

(1) time-reversal (antiunitary) operation $R_t : \mathrm{i} \to -\mathrm{i}$

(2) mirror operation (reflection across $y = 0$) $M_1 : a_{i,j} \to a_{i,-j}$

(3) mirror operation (reflection across $x = 0$) $M_2 : a_{i,j} \to a_{-i,j}$

(4) sublattice chiral operation $W : a_{i,j} \to (-)^{i+j} a_{i,j}$

Easy to see that the combination of R_t, W, and one of M, say, $S_1 = R_t M_1 W$ fulfills the requisite in *theorem 1*

$$S_1 H_0 S_1^{-1} = -H_0$$
$$S_1 H_I S_1^{-1} = H_I$$

that relates the symmetric dynamical behaviour of $+/-U$.

Moreover, two mirror symmetric operation M_1, M_2 relates $\phi \leftrightarrow -\phi$. While R_t relates $+U \leftrightarrow -U$ in dynamical consideration. Bipartite W make the sign of J_y, J_y doesn't matter. The interplay of R_t, M_1, M_2 relates the symmetrical dynamical behaviour of $\pm U$ to mirror reflection, or $\pm\phi$.

4.1 1D square flux ladder

$$H = -\sum_j J_X(e^{-\mathrm{i}\phi/2} a^\dagger_{j+1} a_j + e^{\mathrm{i}\phi/2} b^\dagger_{j+1} b_j) + J_Y a^\dagger_j b_j + h.c. + \frac{U}{2}\sum_j n_j^{(a)}(n_j^{(a)} - 1) + n_j^{(b)}(n_j^{(b)} - 1)$$

Here, for simplicity, we denote the upper(lower) leg as $A(B)$ as manifested in the expression of the above Hamiltonian [$a(b)$ for bosonic field operators] and drop the y-index.

Observables:

(1) odd.

$y_{\mathrm{CoM}}^{L(R)} = \dfrac{\sum\limits_{j<0(>0)} n_j^{(a)} - n_j^{(b)}}{\sum\limits_{j<0(>0)} n_j^{(a)} + n_j^{(b)}}$ is odd under M_1. Reveals the $+\phi/-\phi$ symmetry. see Figure 4(a-e)

$y^{L(R)}$ is odd under $S_1 = R_t M_1 W$. Reveals the $+U/-U$ symmetry.

Moreover, $y_{\mathrm{CoM}}^{L(R)}$ is odd under $M_1 M_2$, transformed to each other. That is, $(M_1 M_2) y^L (M_1 M_2)^{-1} = -y^R$. Plus $(M_1 M_2) H (M_1 M_2)^{-1} = H$. Therefore $y^L(t) = -y^R(t)$ for $M_1 M_2$ invariant initial states.

(2) even. $y^{L(R)}$ under M_2 for $\phi = \pi$. see Figure 4(e), 5(d,e)

(3) non-symmetric. $y^{L(R)}$ under M_2 for $\phi \neq \pi$. see Figure 5(a,b,c)

See Ref[3] for reference. Should we solve the 2-body bound state problem in a analytical way?

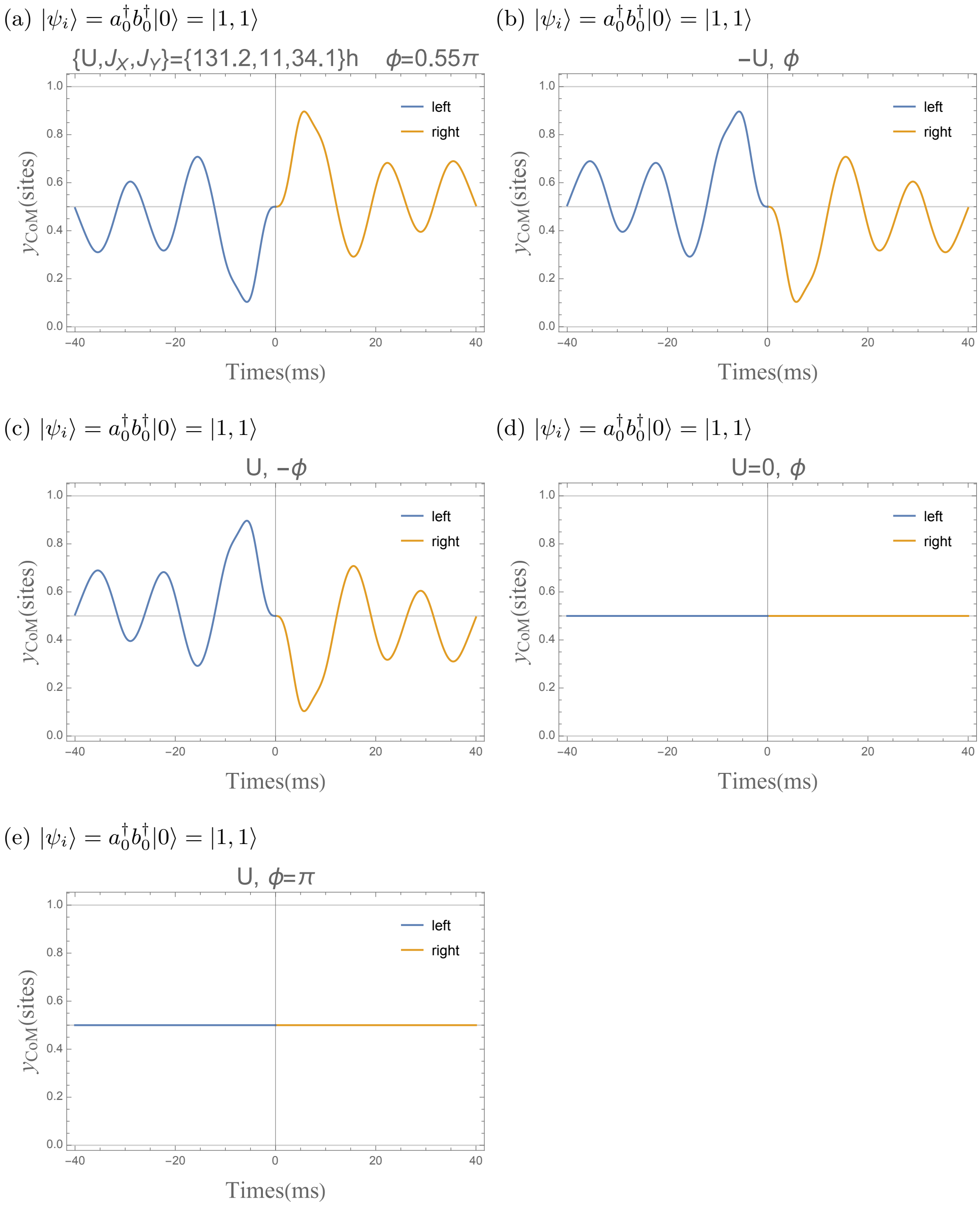


Figure 4: 1D flux ladder: dynamical behaviour of observable $y_{\mathrm{CoM}}^{L(R)}$ for different U, ϕ, and initial state prepared in $|\psi_i\rangle = a_0^\dagger b_0^\dagger |0\rangle = |1,1\rangle$, with hopping strength $J_X = 11.0h$, $J_Y = 34.1h$ fixed. The transformation of initial states as well as the Hamiltonian and y_{CoM} under certain operation should be examined case by case.

4.2 1D triangle flux ladder

$$H = -\sum_j J_X(e^{-\mathrm{i}\phi} a_{j+1}^\dagger a_j + e^{\mathrm{i}\phi} b_{j+1}^\dagger b_j) + J_Y(a_j^\dagger b_j + a_j^\dagger b_{j+1}) + h.c. + \frac{U}{2}\sum_{j,l} n_j^{(l)}(n_j^{(l)} - 1)$$

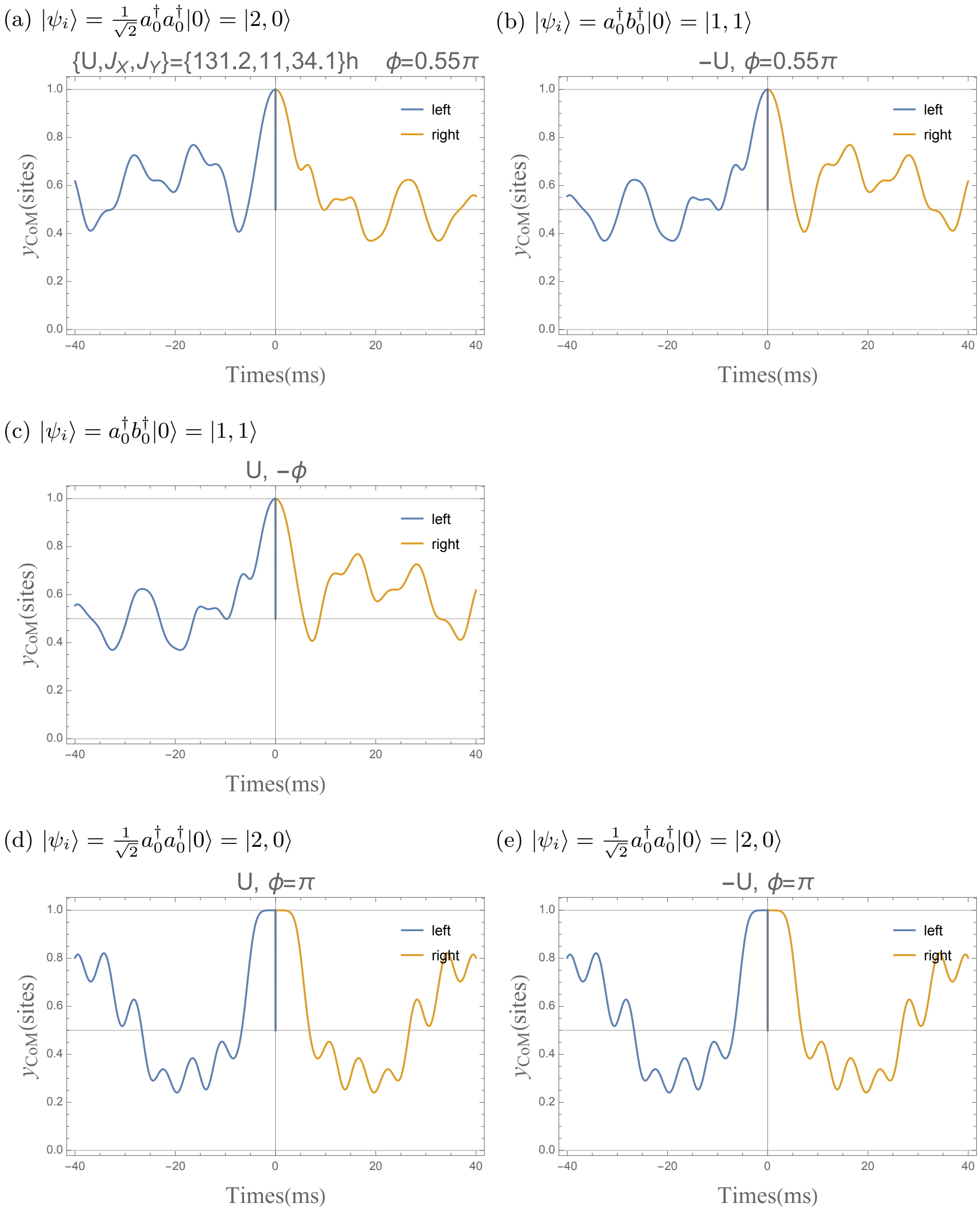


Figure 5: initial state prepared in $|\psi_i\rangle = \frac{1}{\sqrt{2}} a_0^\dagger a_0^\dagger |0\rangle = |2,0\rangle$.

Here ϕ denotes the flux penetrating a smallest triangle hopping loop (different from that in square ladder case).

Triangle lattice lacks the (single-particle) chiral symmetry held by those bipartite lattices. This leads to a difficulty to relate the dynamical behavior of observables at U to those at $-U$. The system therefore no longer performs dynamical symmetry that emerges from the interaction part. Some other geometric symmetries might still be held, like the mirror symmetry. We calculate the behavior of $y_{\mathrm{CoM}^{L(R)}}$ of a triangle flux ladder in periodic boundary conditions. To compare with those cases in square ladder in Figure 4 and Figure 5, two kinds of initial states are both calculated.

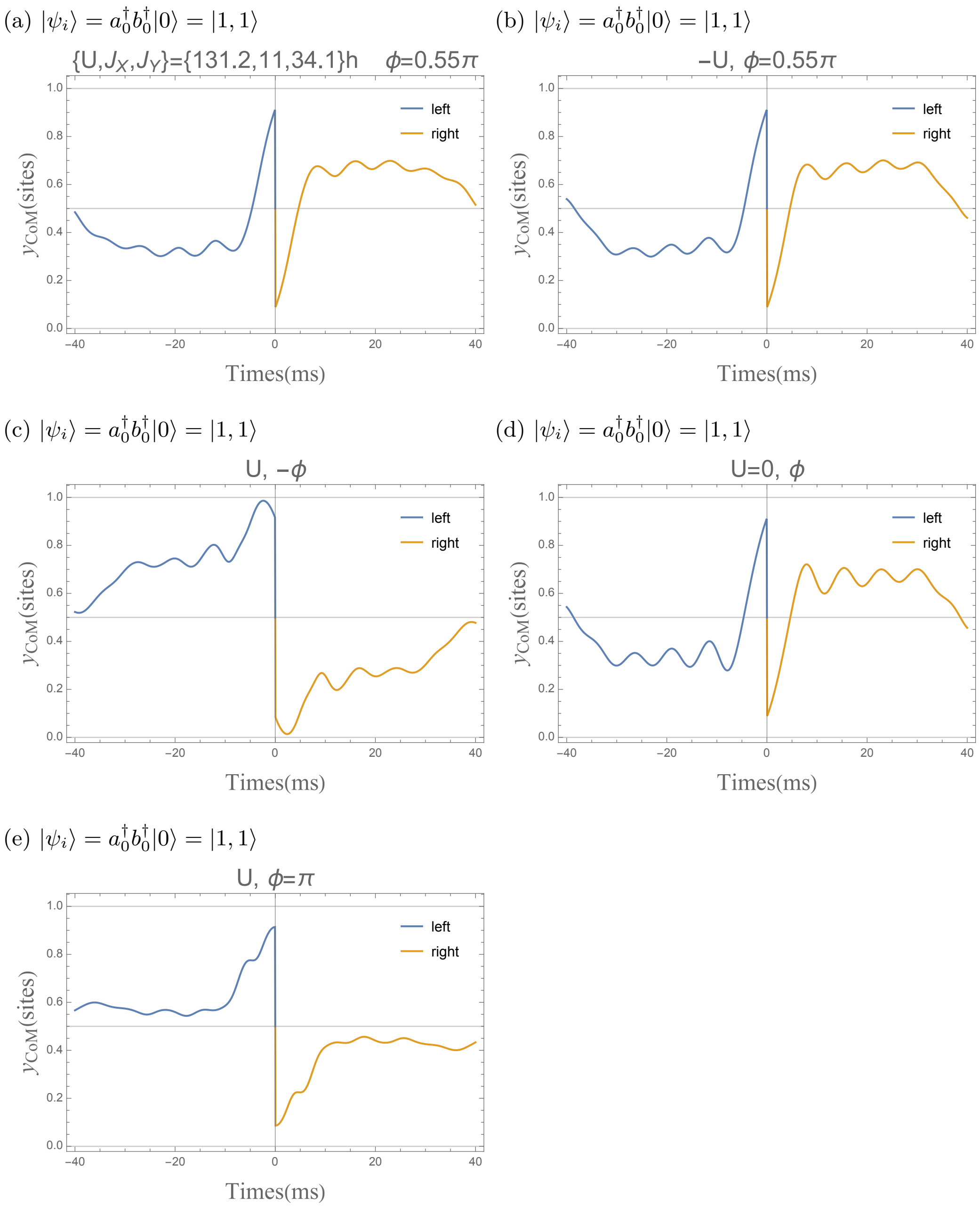


Figure 6: 1D triangle flux ladder: with initial state prepared in $|\psi_i\rangle = a_0^\dagger b_0^\dagger|0\rangle = |1,1\rangle$.

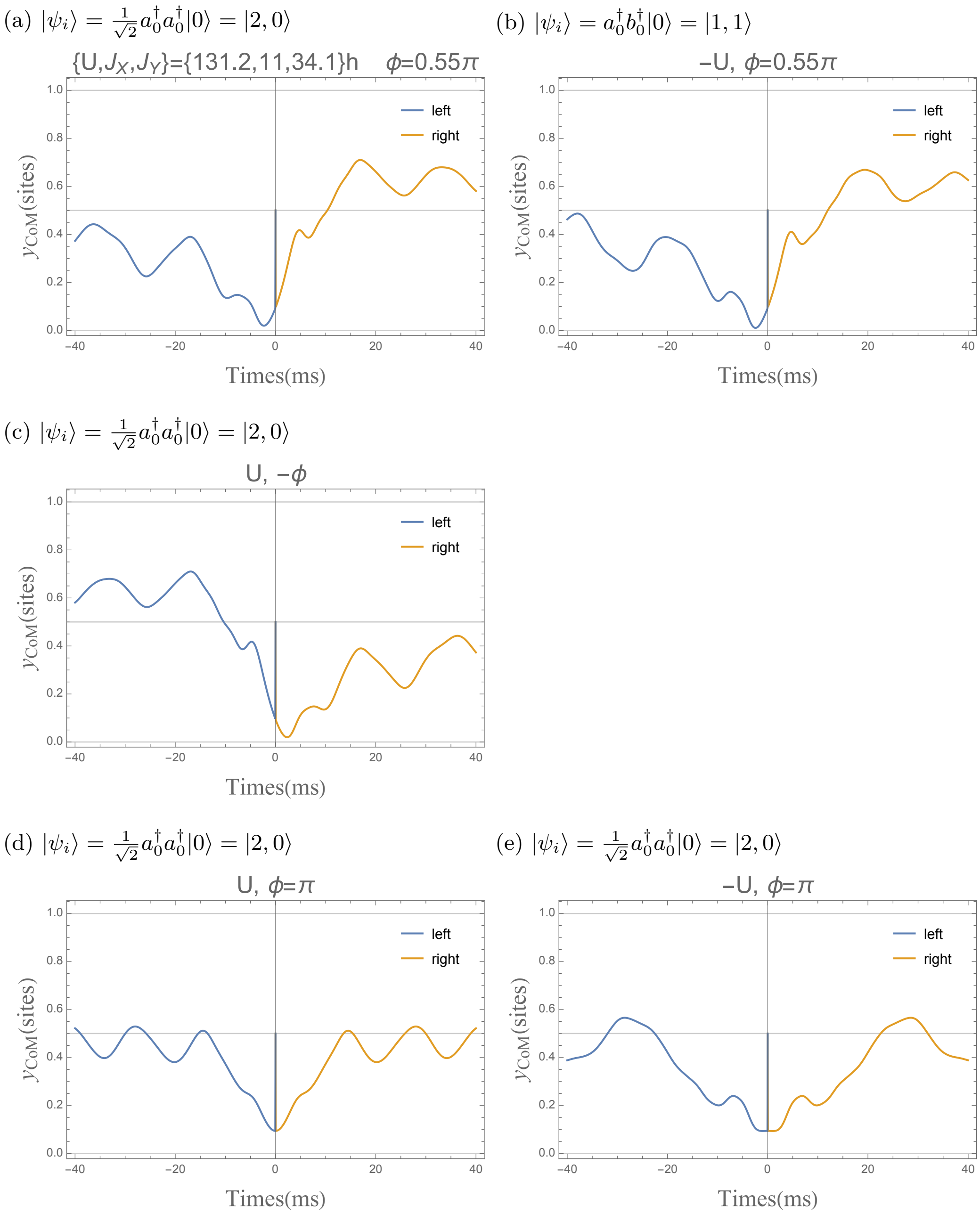


Figure 7: initial state prepared in $|\psi_i\rangle = \frac{1}{\sqrt{2}} a_0^\dagger a_0^\dagger |0\rangle = |2,0\rangle$. Here renumbered the indices of chain A and B. A: -7;;+7 B: 0,-7;;-1,+1;;+7 . (d)(e) shows symmetric evolving of left and right part, which is a reflective of the preserved spacial mirror symmetry across the origin of x-axis ($x = 0$) when $\phi = 0, \pi$.

(a) $|\psi_i\rangle = \frac{1}{\sqrt{2}} a_0^\dagger a_0^\dagger |0\rangle = |2, 0\rangle$

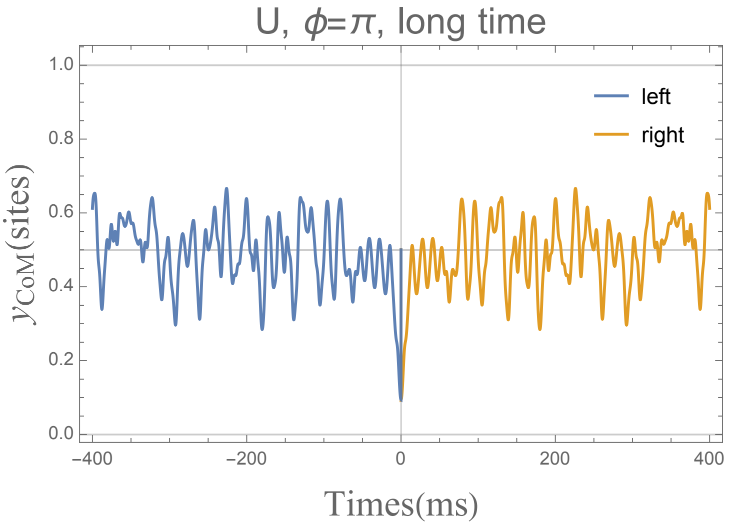


Figure 8: triangle: long time behavior. oscillates around 0.5

Acknowledgments

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