Note on 2-body Hofstadter

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I. MODEL

a. Hofstadter in 2D.— Hofstadter Hamiltonian writes

$$H = \frac{U}{2}\sum_{i,j}\hat{n}_{i,j}(\hat{n}_{i,j}-1) - \sum_{i,j}(Ke^{-\mathrm{i}\phi_{i,j}}a^{\dagger}_{i+1,j}a_{i,j} + Ja^{\dagger}_{i,j+1}a_{i,j} + h.c.) \tag{1}$$

where $i(j)$ denotes index along $\hat{x}(\hat{y})$ direction. It describes a 2D lattice model with (homogeneous) flux penetrating plaquettes.

Keeping only two legs along $\hat{x}$ direction, it becomes a quasi 1D ladder, with $i \in \mathbb{Z}$ and $j \in \{0,1\}$ in Eqn.(1). Or it could be written in $a/b-$sublegs in second-quantization Hamiltonian as

$$H = \frac{U}{2}\sum_{j} n^{(a)}_j(n^{(a)}_j - 1) + n^{(b)}_j(n^{(b)}_j - 1) - \sum_j Ke^{-\mathrm{i}\phi^{(a)}_j}a^{\dagger}_{j+1}a_j + Ke^{-\mathrm{i}\phi^{(b)}_j}b^{\dagger}_{j+1}b_j + Ja^{\dagger}_j b_j + H.c. \tag{2}$$

b. Convention.— To realise $\phi-$flux uniformly in all plaquettes, generally,

$$\left.\begin{aligned} \phi^{(a)}_j &= j\theta^{(a)} + \alpha \\ \phi^{(b)}_j &= j\theta^{(b)} + \beta \end{aligned}\right\} \qquad \phi_j = -\phi^{(b)}_j + \phi^{(a)}_j = j(\theta^{(a)} - \theta^{(b)}) + \alpha - \beta$$

Uniformity requires that ϕ_j independent of j, so that $\theta^{(a)} - \theta^{(b)} = 0, \alpha - \beta = \phi$.

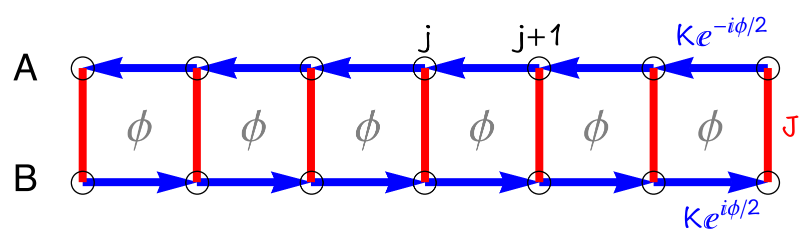
$$\begin{cases} \phi^{(a)}_j & = j\theta + \alpha \\ \phi^{(b)}_j & = j\theta + \beta \\ \alpha - \beta & = \phi \end{cases}$$

It reduces to — one choice of gauge — :

$$\begin{cases} \phi^{(a)}_j = \alpha = \phi/2 \\ \phi^{(b)}_j = \beta = \phi/2 \end{cases} \tag{3}$$

c. Hamiltonian.— In the following, we consider Hamiltonian written as

$$H = \frac{U}{2}\sum_{j} n^{(a)}_j(n^{(a)}_j - 1) + n^{(b)}_j(n^{(b)}_j - 1) - \sum_j Ke^{-\mathrm{i}\phi/2}a^{\dagger}_{j+1}a_j + Ke^{\mathrm{i}\phi/2}b^{\dagger}_{j+1}b_j + Ja^{\dagger}_j b_j + H.c. \tag{4}$$



II. SOLVING 2-BODY PROBLEMS

A. Exact diagonalization of 2-body Hamiltonian

a. Constructing bosonic 2-body Hilbert space.— 2-body states in bosonic states are of form $c^\dagger_{i_1,j_1} c^\dagger_{i_2,j_2} |vac\rangle$ where $c^{(\dagger)}_{i,j}$ is bosonic operator. Gernerally where could be $a^\dagger_{j_1} a^\dagger_{j_2}|0\rangle$, $a^\dagger_{j_1} b^\dagger_{j_2}|0\rangle$, $b^\dagger_{j_1} b^\dagger_{j_2}|0\rangle$.

$$\left\{ a^\dagger_l a^\dagger_m |0\rangle \;,\; a^\dagger_l b^\dagger_m |0\rangle \;,\; b^\dagger_l b^\dagger_m |0\rangle \right\}$$

- $a^\dagger_l a^\dagger_m|0\rangle$ — $\begin{cases} l & = 1;;N \\ m & = 1;;l \end{cases}$ $\quad \dfrac{N(N+1)}{2}$ in total
- $a^\dagger_l b^\dagger_m|0\rangle$ — $\begin{cases} l & = 1;;N \\ m & = 1;;N \end{cases}$ $\quad N^2$
- $b^\dagger_l b^\dagger_m|0\rangle$ — $\begin{cases} l & = 1;;N \\ m & = 1;;N \end{cases}$ $\quad \dfrac{N(N+1)}{2}$

Hence $(2N^2+N)$-dimension the space.

b. Hamiltonian matrix elements within 2-body Hilbert space.—

1. interaction terms

$$\begin{aligned} \frac{U}{2}\sum_j n^{(a)}_j (n^{(a)}_j - 1) a^\dagger_l a^\dagger_l |0\rangle &= U a^\dagger_l a^\dagger_l |0\rangle \\ &\text{else} \quad 0 \\ \frac{U}{2}\sum_j n^{(b)}_j (n^{(b)}_j - 1) b^\dagger_l b^\dagger_l |0\rangle &= U b^\dagger_l b^\dagger_l |0\rangle \\ &\text{else} \quad 0 \end{aligned}$$

2. on-chain hopping terms

$$\begin{aligned} -\sum_j K e^{-\mathrm{i}\phi/2} a^\dagger_{j+1} a_j \quad & a^\dagger_l a^\dagger_m |0\rangle = -K e^{-\mathrm{i}\phi/2} (a^\dagger_{l+1} a^\dagger_m |0\rangle + a^\dagger_l a^\dagger_{m+1} |0\rangle) \\ & a^\dagger_l b^\dagger_m |0\rangle = -K e^{-\mathrm{i}\phi/2} a^\dagger_{l+1} b^\dagger_m |0\rangle \\ & b^\dagger_l b^\dagger_m |0\rangle = 0 \\ -\sum_j K e^{\mathrm{i}\phi/2} b^\dagger_{j+1} b_j \quad & a^\dagger_l a^\dagger_m |0\rangle = 0 \\ & a^\dagger_l b^\dagger_m |0\rangle = -K e^{\mathrm{i}\phi/2} a^\dagger_l b^\dagger_{m+1} |0\rangle \\ & b^\dagger_l b^\dagger_m |0\rangle = -K e^{\mathrm{i}\phi/2} (b^\dagger_{l+1} b^\dagger_m |0\rangle + b^\dagger_l b^\dagger_{m+1} |0\rangle) \end{aligned}$$

* Hermitian conjugate

$$
\begin{aligned}
-\sum_j Ke^{\mathrm{i}\phi/2}a_j^\dagger a_{j+1} \quad & a_l^\dagger a_m^\dagger|0\rangle = -Ke^{\mathrm{i}\phi/2}(a_{l-1}^\dagger a_m^\dagger|0\rangle + a_l^\dagger a_{m-1}^\dagger|0\rangle) \\
& a_l^\dagger b_m^\dagger|0\rangle = -Ke^{\mathrm{i}\phi/2}a_{l-1}^\dagger b_m^\dagger|0\rangle \\
& b_l^\dagger b_m^\dagger|0\rangle = 0 \\
-\sum_j Ke^{-\mathrm{i}\phi/2}b_j^\dagger b_{j+1} \quad & a_l^\dagger a_m^\dagger|0\rangle = 0 \\
& a_l^\dagger b_m^\dagger|0\rangle = -Ke^{-\mathrm{i}\phi/2}a_l^\dagger b_{m-1}^\dagger|0\rangle \\
& b_l^\dagger b_m^\dagger|0\rangle = -Ke^{-\mathrm{i}\phi/2}(b_{l-1}^\dagger b_m^\dagger|0\rangle + b_l^\dagger b_{m-1}^\dagger|0\rangle)
\end{aligned}
$$

3. inter-chain hopping terms

$$
\begin{aligned}
-J\sum_j a_j^\dagger b_j \quad & a_l^\dagger a_m^\dagger|0\rangle = 0 \\
& a_l^\dagger b_m^\dagger|0\rangle = -Ja_l^\dagger a_m^\dagger|0\rangle \\
& b_l^\dagger b_m^\dagger|0\rangle = -J(a_l^\dagger b_m^\dagger|0\rangle + a_m^\dagger b_l^\dagger|0\rangle)
\end{aligned}
$$

* Hermitian conjugate

$$
\begin{aligned}
-J\sum_j b_j^\dagger a_j \quad & a_l^\dagger a_m^\dagger|0\rangle = -J(a_m^\dagger b_l^\dagger|0\rangle + a_l^\dagger b_m^\dagger|0\rangle) \\
& a_l^\dagger b_m^\dagger|0\rangle = -Jb_l^\dagger b_m^\dagger|0\rangle \\
& b_l^\dagger b_m^\dagger|0\rangle = 0
\end{aligned}
$$

PS: Notice that for double-occupied state like $a_l^\dagger a_l^\dagger|0\rangle$, the normalized form writes $\frac{1}{\sqrt{2}}a_l^\dagger a_l^\dagger|0\rangle$, such that elements of interaction term $(U/2 - term)$ between these states times by a factor of 2 while $K - term$ and $J - term$ times by $\sqrt{2}$.

c. Hamiltonian blocks alignment.—

III.

[1] Harvard group, *Microscopy of the interacting Harper-Hofstadter model in the few-body limit*, arXiv: 1612.05631 .