Note on shaking square lattice

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I. FLOQUET THEORY AND EFFECTIVE HAMILTONIAN

For time periodic Hamiltonian $H(t) = H(t+T)$, an effective Hamiltonian is defined by Floquet operator $U(T) = \mathcal{T}e^{-i\int_{\eta T}^{\eta T+T} H(t)\mathrm{d}t} \equiv e^{-iH_{eff}T}$. Expanding $U(T)$ to T order,

$$\begin{aligned}
U(T) &= \lim_{N\to\infty}\prod_{j=0}^{N} e^{-iH(t_j)\Delta t} \\
&= \lim_{N\to\infty}\prod_{j=0}^{N}(1-iH(t_j)\Delta t) \\
&= \lim_{N\to\infty}\left(1-i\Delta t\sum_{j=0}^{N}H(t_j)+(-i\Delta t)^2\sum_{j>k}H(t_j)H(t_k)+\cdots\right) \\
&= 1-i\int_0^T \mathrm{d}tH(t)+(-i)^2\int_0^T\mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2H(t_1)H(t_2)+\cdots
\end{aligned}$$

Comparing coefficients, we obtain

$$H_{eff} = H_0+\frac{1}{\hbar\omega}\sum_{n>0}\frac{1}{n}\left(e^{i\eta n\pi}[H_0,H_n]-e^{-i\eta n\pi}[H_0,H_{-n}]+[H_n,H_{-n}]\right)+O(\frac{1}{\omega^2})$$

II. SINGLE PHONON RESONANCE

A. Tight binding model

Single particle Hamiltonian in 2D shaken square lattice reads

$$h(t)=\frac{p^2}{2m}-V\cos(2k_0x+f\cos\omega t)-V\cos(2k_0y+f\cos(\omega t+\alpha))$$

Schrodinger equation reads

$$(h(t)-i\hbar\partial_t)\Psi=0$$

With translational transformation, we obtain

$$T_{\boldsymbol{\xi}}(h(t)-i\hbar\partial_t)T_{\boldsymbol{\xi}}^\dagger T_{\boldsymbol{\xi}}\Psi=\left[T_{\boldsymbol{\xi}}(h(t)-i\hbar\partial_t)T_{\boldsymbol{\xi}}^\dagger-i\hbar\partial_t\right]T_{\boldsymbol{\xi}}\Psi=0,$$

with

$$T_{\boldsymbol{\xi}}=e^{-\frac{i}{\hbar}\boldsymbol{\xi}\cdot\boldsymbol{p}},\quad \boldsymbol{\xi}(t)=\frac{f}{2k_0}(\cos\omega t\,\hat{x}+\cos(\omega t+\alpha)\,\hat{y}).$$

In co-moving frame of reference, Hamiltonian reads

$$\begin{aligned}
H(t) &= T_{\boldsymbol{\xi}}(h(t)-i\hbar\partial_t)T_{\boldsymbol{\xi}}^\dagger \\
&= \frac{p^2}{2m}-V\cos 2k_0x-V\cos 2k_0y+\dot{\boldsymbol{\xi}}(t)\cdot\boldsymbol{p} \\
&= \frac{p^2}{2m}-V\cos 2k_0x-V\cos 2k_0y-\frac{\omega f}{2k_0}(\sin\omega t\,p_x+\sin(\omega t+\alpha)\,p_y)\equiv\sum_{n=-1}^{1}H_ne^{in\omega t},
\end{aligned}$$

with

$$\begin{aligned}
H_0 &= \frac{p^2}{2m} - V\cos 2k_0 x - V\cos 2k_0 y \\
H_1 &= i\frac{\omega f}{4k_0}(p_x + e^{i\alpha}p_y) = \frac{\hbar\omega f}{4k_0}(\partial_x + e^{i\alpha}\partial_y) \\
H_{-1} &= H_1^\dagger = -i\frac{\omega f}{4k_0}(p_x + e^{-i\alpha}p_y) = -\frac{\hbar\omega f}{4k_0}(\partial_x + e^{-i\alpha}\partial_y).
\end{aligned}$$

Second quantized Hamiltonian reads

$$\hat{H}(t) = \int \mathrm{d}^2\boldsymbol{r}\hat{\Psi}^\dagger(\boldsymbol{r})H(t)\hat{\Psi}(\boldsymbol{r}) = \sum_{\substack{\lambda'\lambda \\ \boldsymbol{mn}}} H^{\lambda'\lambda}_{\boldsymbol{mn}}(t)\hat{a}^\dagger_{\lambda'\boldsymbol{m}}\hat{a}_{\lambda\boldsymbol{n}}$$

with

$$\hat{\Psi}(\boldsymbol{r}) = \sum_{\lambda\boldsymbol{n}} \hat{a}_{\lambda\boldsymbol{n}} w_\lambda(\boldsymbol{r}-\boldsymbol{n}), \quad H^{\lambda'\lambda}_{\boldsymbol{mn}}(t) = \int \mathrm{d}^2\boldsymbol{r} w^*_{\lambda'}(\boldsymbol{r}-\boldsymbol{m})H(t)w_\lambda(\boldsymbol{r}-\boldsymbol{n})$$

where $\boldsymbol{m}$ and $\boldsymbol{n}$ denote the lattice site, λ and λ' denote the different orbitals. Here we simply take the lattice constant to be 1. In tight binding limit, we can approximate Wannier's functions by harmonic oscillator states,

$$\begin{aligned}
w_s(x,y) &= \varphi_s(x)\varphi_s(y) \\
w_{p_x}(x,y) &= \varphi_p(x)\varphi_s(y) \\
w_{p_y}(x,y) &= \varphi_s(x)\varphi_p(y)
\end{aligned}$$

with dimensionless $\varphi_s(x) = (\frac{a}{\pi})^{1/4}e^{-ax^2/2}, \varphi_p(x) = (\frac{a}{\pi})^{1/4}\sqrt{2a}xe^{-ax^2/2}, a = \pi^2\sqrt{2V/E_r}$.
Let's take the Fourier transform with respect to $\boldsymbol{r}$,

$$\hat{a}_{\lambda\boldsymbol{n}} = \frac{1}{\sqrt{L}}\sum_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{n}}\hat{a}_{\lambda\boldsymbol{k}}, \quad \hat{a}^\dagger_{\lambda'\boldsymbol{m}} = \frac{1}{\sqrt{L}}\sum_{\boldsymbol{k}} e^{-i\boldsymbol{k}\cdot\boldsymbol{m}}\hat{a}^\dagger_{\lambda'\boldsymbol{k}}.$$

The Hamiltonian has translational symmetry, which means $H^{\lambda'\lambda}_{\boldsymbol{mn}} = H^{\lambda'\lambda}_\delta$ with $\delta = \boldsymbol{n}-\boldsymbol{m}$.

$$\hat{H}(t) = \sum_{\boldsymbol{k}} \hat{H}^{\lambda'\lambda}_{\boldsymbol{k}}, \quad \hat{H}^{\lambda'\lambda}_{\boldsymbol{k}} = \sum_{\lambda'\lambda,\delta} e^{i\boldsymbol{k}\cdot\delta}H^{\lambda'\lambda}_\delta \hat{a}^\dagger_{\lambda'\boldsymbol{k}}\hat{a}_{\lambda\boldsymbol{k}}$$

$$\begin{aligned}
\langle w_{\lambda'}(\boldsymbol{r})|H_0(\boldsymbol{r})|w_\lambda(\boldsymbol{r}-\delta)\rangle &= \langle w_{\lambda'}(-\boldsymbol{r})|H_0(-\boldsymbol{r})|w_\lambda(-\boldsymbol{r}-\delta)\rangle = (-)^{\lambda'+\lambda}\langle w_{\lambda'}(\boldsymbol{r})|H_0(\boldsymbol{r})|w_\lambda(\boldsymbol{r}+\delta)\rangle \\
\langle w_{\lambda'}(\boldsymbol{r})|H_1(\boldsymbol{r},t)|w_\lambda(\boldsymbol{r}-\delta)\rangle &= (-)^{\lambda'+\lambda+1}\langle w_{\lambda'}(\boldsymbol{r},t)|H_1(\boldsymbol{r},t)|w_\lambda(\boldsymbol{r}+\delta)\rangle
\end{aligned}$$

So Hamiltonian in momentum space reads

$$\begin{aligned}
\hat{H}(t) &= \sum_{\boldsymbol{k}} \begin{pmatrix} \hat{a}^\dagger_{s\boldsymbol{k}} & \hat{a}^\dagger_{p_x\boldsymbol{k}} & \hat{a}^\dagger_{p_y\boldsymbol{k}} \end{pmatrix} H(\boldsymbol{k},t) \begin{pmatrix} \hat{a}_{s\boldsymbol{k}} \\ \hat{a}_{p_x\boldsymbol{k}} \\ \hat{a}_{p_y\boldsymbol{k}} \end{pmatrix}, \\
H(\boldsymbol{k},t) &= H_0(\boldsymbol{k}) + H_1(\boldsymbol{k})e^{i\omega t} + H_{-1}(\boldsymbol{k})e^{-i\omega t}, \\
H_0(\boldsymbol{k}) &= \begin{pmatrix} E_s & iK_x & iK_y \\ -iK_x & E_{p_x} & 0 \\ -iK_y & 0 & E_{p_y} \end{pmatrix}, \quad H_1(\boldsymbol{k}) = \begin{pmatrix} \Lambda_s & \Omega_x & e^{i\alpha}\Omega_y \\ -\Omega_x & \Lambda_{p_x} & 0 \\ -e^{i\alpha}\Omega_y & 0 & \Lambda_{p_y} \end{pmatrix}, \quad H_{-1}(\boldsymbol{k}) = \begin{pmatrix} \Lambda^*_s & -\Omega_x & -e^{-i\alpha}\Omega_y \\ \Omega_x & \Lambda^*_{p_x} & 0 \\ e^{-i\alpha}\Omega_y & 0 & \Lambda^*_{p_y} \end{pmatrix}.
\end{aligned}$$

with

$$\begin{aligned}
E_s &= \epsilon_s + 2t_s(\cos k_x + \cos k_y) \\
E_{p_x} &= \epsilon_p + 2t_{p\sigma}\cos k_x + 2t_{p\pi}\cos k_y, \quad E_{p_y} = \epsilon_p + 2t_{p\sigma}\cos k_y + 2t_{p\pi}\cos k_x \\
K_x &= 2t_{sp}\sin k_x, \qquad K_y = 2t_{sp}\sin k_y
\end{aligned}$$

$$
\begin{aligned}
\Lambda_s &= i2h_s(\sin k_x + e^{i\alpha}\sin k_y) \\
\Lambda_{p_x} &= i2(h_p \sin k_x + e^{i\alpha} h_s \sin k_y), \quad \Lambda_{p_y} = i2(h_s \sin k_x + e^{i\alpha} h_p \sin k_y) \\
\Omega_x &= h_{sp} + 2h_{sp1}\cos k_x, \qquad\qquad \Omega_y = h_{sp} + 2h_{sp1}\cos k_y \\
\epsilon_s &= \langle w_s(x,y)|H_0(x,y)|w_s(x,y)\rangle \\
\epsilon_p &= \langle w_{p_x}(x,y)|H_0(x,y)|w_{p_x}(x,y)\rangle = \langle w_{p_y}(x,y)|H_0(x,y)|w_{p_y}(x,y)\rangle \\
t_s &= \langle w_s(x,y)|H_0(x,y)|w_s(x\pm 1,y)\rangle = \langle w_s(x,y)|H_0(x,y)|w_s(x,y\pm 1)\rangle \\
t_{p\sigma} &= \langle w_{p_x}(x,y)|H_0(x,y)|w_{p_x}(x\pm 1,y)\rangle = \langle w_{p_y}(x,y)|H_0(x,y)|w_{p_y}(x,y\pm 1)\rangle \\
t_{p\pi} &= \langle w_{p_x}(x,y)|H_0(x,y)|w_{p_x}(x,y\pm 1)\rangle = \langle w_{p_y}(x,y)|H_0(x,y)|w_{p_y}(x\pm 1,y)\rangle \\
t_{sp} &= \langle w_s(x,y)|H_0(x,y)|w_{p_x}(x-1,y)\rangle = -\langle w_s(x,y)|H_0(x,y)|w_{p_x}(x+1,y)\rangle \\
&= \langle w_s(x,y)|H_0(x,y)|w_{p_y}(x,y-1)\rangle = -\langle w_s(x,y)|H_0(x,y)|w_{p_y}(x,y+1)\rangle \\
h_s &= \langle w_s(x,y)|\frac{\hbar\omega f}{4k_0}\partial_x|w_s(x-1,y)\rangle = -\langle w_s(x,y)|\frac{\hbar\omega f}{4k_0}\partial_x|w_s(x+1,y)\rangle \\
&= \langle w_s(x,y)|\frac{\hbar\omega f}{4k_0}\partial_y|w_s(x,y-1)\rangle = -\langle w_s(x,y)|\frac{\hbar\omega f}{4k_0}\partial_y|w_s(x,y+1)\rangle \\
&= \langle w_{p_x}(x,y)|\frac{\hbar\omega f}{4k_0}\partial_y|w_{p_x}(x,y-1)\rangle = -\langle w_{p_x}(x,y)|\frac{\hbar\omega f}{4k_0}\partial_y|w_{p_x}(x,y+1)\rangle \\
&= \langle w_{p_y}(x,y)|\frac{\hbar\omega f}{4k_0}\partial_x|w_{p_y}(x-1,y)\rangle = -\langle w_{p_y}(x,y)|\frac{\hbar\omega f}{4k_0}\partial_x|w_{p_y}(x+1,y)\rangle \\
h_p &= \langle w_{p_x}(x,y)|\frac{\hbar\omega f}{4k_0}\partial_x|w_{p_x}(x-1,y)\rangle = -\langle w_{p_x}(x,y)|\frac{\hbar\omega f}{4k_0}\partial_x|w_{p_x}(x+1,y)\rangle \\
&= \langle w_{p_y}(x,y)|\frac{\hbar\omega f}{4k_0}\partial_y|w_{p_y}(x,y-1)\rangle = -\langle w_{p_y}(x,y)|\frac{\hbar\omega f}{4k_0}\partial_y|w_{p_y}(x,y+1)\rangle \\
h_{sp} &= \langle w_s(x,y)|\frac{\hbar\omega f}{4k_0}\partial_x|w_{p_x}(x,y)\rangle = \langle w_s(x,y)|\frac{\hbar\omega f}{4k_0}\partial_y|w_{p_y}(x,y)\rangle \\
h_{sp1} &= \langle w_s(x,y)|\frac{\hbar\omega f}{4k_0}\partial_x|w_{p_x}(x\pm 1,y)\rangle = \langle w_s(x,y)|\frac{\hbar\omega f}{4k_0}\partial_y|w_{p_y}(x,y\pm 1)\rangle
\end{aligned}
$$

Make a unitary transformation, $H(\boldsymbol{k},t)$ becomes

$$
\tilde{H}(\boldsymbol{k},t) = U(t)(H(\boldsymbol{k},t) - i\partial_t)U^\dagger(t) = \sum_n \tilde{H}_n e^{in\omega t}
$$

with

$$
\begin{aligned}
U(t) &= \begin{pmatrix} e^{-i\omega t} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
\tilde{H}_0 &= \begin{pmatrix} E_s + \omega & \Omega_x & e^{i\alpha}\Omega_y \\ \Omega_x & E_{p_x} & 0 \\ e^{-i\alpha}\Omega_y & 0 & E_{p_y} \end{pmatrix} \\
\tilde{H}_1 &= \begin{pmatrix} \Lambda_s & 0 & 0 \\ -iK_x & \Lambda_{p_x} & 0 \\ -iK_y & 0 & \Lambda_{p_y} \end{pmatrix}, \quad \tilde{H}_{-1} = \begin{pmatrix} \Lambda_s^* & iK_x & iK_y \\ 0 & \Lambda_{p_x}^* & 0 \\ 0 & 0 & \Lambda_{p_y}^* \end{pmatrix} \\
\tilde{H}_2 &= \begin{pmatrix} 0 & 0 & 0 \\ -\Omega_x & 0 & 0 \\ -e^{i\alpha}\Omega_y & 0 & 0 \end{pmatrix}, \quad \tilde{H}_{-2} = \begin{pmatrix} 0 & -\Omega_x & -e^{-i\alpha}\Omega_y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\end{aligned}
$$

We obatin the Hamiltonian with rotating-wave approximation $H_{RWA} = \tilde{H}_0$. Interaction energy reads

$$
\begin{aligned}
\hat{H}_{int} &= \frac{g}{2}\int \mathrm{d}\boldsymbol{r}\hat{\Psi}^\dagger(\boldsymbol{r})\hat{\Psi}^\dagger(\boldsymbol{r})\hat{\Psi}(\boldsymbol{r})\hat{\Psi}(\boldsymbol{r}) \\
E_{int} &= \frac{gN^2}{2}\int \mathrm{d}\boldsymbol{r}|\Psi_{\boldsymbol{k}}(\boldsymbol{r})|^4
\end{aligned}
$$

where $\hat{\Psi}(\boldsymbol{r}) = \sqrt{N}\Psi_{\boldsymbol{k}}(\boldsymbol{r})$, N is partical number, $\Psi_{\boldsymbol{k}}(\boldsymbol{r})$ is condensate wave function of H_{RWA}.
After shaking, dressed s band is blue-detuned, whose eigenstate is $\Psi_{\boldsymbol{k}}(\boldsymbol{r},t) = a_{\boldsymbol{k}} e^{i\omega t}\Psi_{s\boldsymbol{k}}(\boldsymbol{r}) + b_{\boldsymbol{k}}\Psi_{p_x\boldsymbol{k}}(\boldsymbol{r}) + c_{\boldsymbol{k}}\Psi_{p_y\boldsymbol{k}}(\boldsymbol{r})$. onsite interaction energy per partical is

$$\begin{aligned}
\frac{E_{int}}{N} &= \frac{gN}{2}\frac{1}{T}\int_0^T \mathrm{d}t \int \mathrm{d}\boldsymbol{r}|\Psi_{\boldsymbol{k}}(\boldsymbol{r},t)|^4 \\
&= \frac{gN}{2L}\frac{1}{T}\int_0^T \mathrm{d}t \int \mathrm{d}\boldsymbol{r}|a_{\boldsymbol{k}} e^{i\omega t}\phi_s(\boldsymbol{r}) + b_{\boldsymbol{k}}\phi_{p_x}(\boldsymbol{r}) + c_{\boldsymbol{k}}\phi_{p_y}(\boldsymbol{r})|^4 \\
&= \frac{g\nu}{2}\int \mathrm{d}\boldsymbol{r}|a_{\boldsymbol{k}}|^4|\phi_s(\boldsymbol{r})|^4 + 4|a_{\boldsymbol{k}}|^2|\phi_s(\boldsymbol{r})|^2|b_{\boldsymbol{k}}\phi_{p_x}(\boldsymbol{r}) + c_{\boldsymbol{k}}\phi_{p_y}(\boldsymbol{r})|^2 + |b_{\boldsymbol{k}}\phi_{p_x}(\boldsymbol{r}) + c_{\boldsymbol{k}}\phi_{p_y}(\boldsymbol{r})|^4 \\
&= \frac{g\nu}{2}\Big[|a_{\boldsymbol{k}}|^4 u_s^2 + 4|a_{\boldsymbol{k}}|^2\left(|b_{\boldsymbol{k}}|^2 + |c_{\boldsymbol{k}}|^2\right)u_s u_{sp} + \left(|b_{\boldsymbol{k}}|^4 + |c_{\boldsymbol{k}})|^4\right)u_s u_p \\
&\quad + \left(b_{\boldsymbol{k}}^2 c_{\boldsymbol{k}}^{*2} + b_{\boldsymbol{k}}^{*2} c_{\boldsymbol{k}}^2 + 4|b_{\boldsymbol{k}} c_{\boldsymbol{k}}|^2\right)u_{sp}^2\Big],
\end{aligned}$$

where $u_s = \int \mathrm{d}x|\varphi_s(x)|^4$, $u_p = \int \mathrm{d}x|\varphi_p(x)|^4$, $u_{sp} = \int \mathrm{d}x|\varphi_s(x)\varphi_p(x)|^2$.

B. Transition

When shaking amplitude increases, ground state degeneracy in the dressed s band (blue detuned) changes from one to four, i.e. $\Psi_{\boldsymbol{k}} = \Psi_{(0,0)} \to \Psi_{(\pm k_m, \pm k_m)}, k_m \neq 0$. Fig.1 shows 4 energy minima emerge in the dressed s band as shaking amplitude increases. Fig.2 shows interaction will delay the emergence under tight binding approximation. When changing shaking types, non-interacting transition curve does not change and interacting transition curve changes in an interval bounded by curve $\alpha = 0$ or π (solid red line in Fig.2) and curve $\alpha = \pi/2$ or $3\pi/2$ (dashed red line in Fig.2). This is because α does not appear in H_{RWA} after a time independent unitary transformation, but always appears in eigenstates.

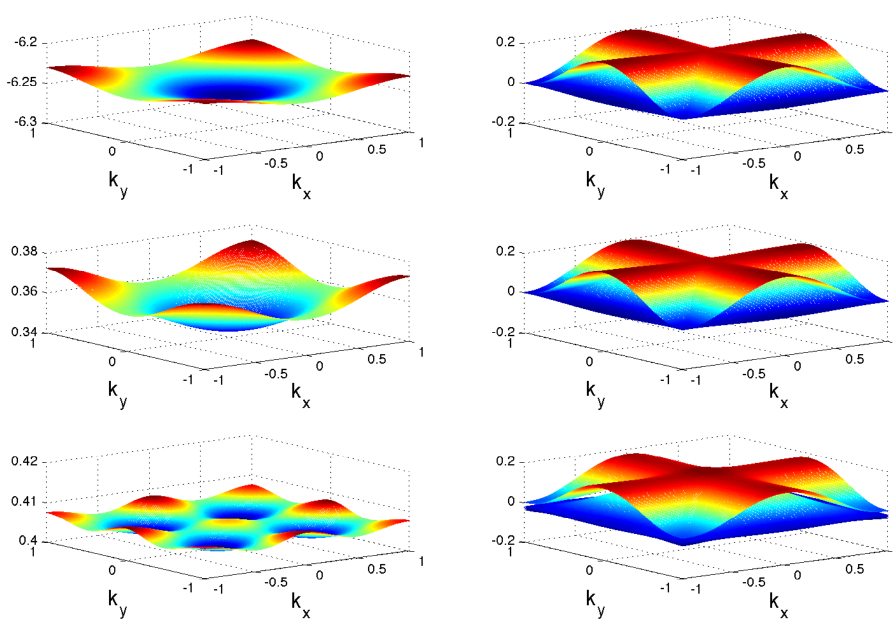

FIG. 1: Energy dispersion for parameters $V = 6.5E_r, \omega = 6.6E_r$. Top: before shaking; middle: small shaking amplitude f=0.01; bottom: large shaking amplitude f=0.04.

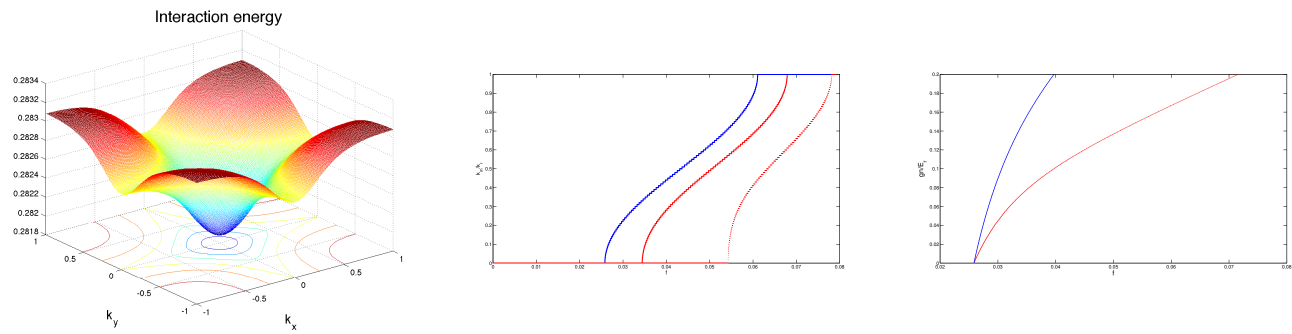

FIG. 2: Left: interaction energy. Middle: condensate momentum as a function of f. Blue : non-interacting case; red : interacting case; solid: $\alpha = 0$; dashed: $\alpha = \pi/2$. Right: Interaction-shaking amplitude phase diagram for a fixed frequency . Blue: $\alpha = 0$; red: $\alpha = \pi/2$

C. Stripe phase

When shaking amplitude is sufficiently large, there will be four degenerate ground state in dressed s band. Because Hamiltonian $H_{RWA}(\boldsymbol{k})$ has C_4 symmetry,

$$\begin{aligned}
\Psi_{\boldsymbol{k}_1}(\boldsymbol{r},t) &= a_{k_m} e^{i\omega t}\Psi_{s\boldsymbol{k}_1}(\boldsymbol{r}) + b_{k_m}\Psi_{p_x\boldsymbol{k}_1}(\boldsymbol{r}) + c_{k_m}\Psi_{p_y\boldsymbol{k}_1}(\boldsymbol{r}) \\
\Psi_{\boldsymbol{k}_2}(\boldsymbol{r},t) &= a_{k_m} e^{i\omega t}\Psi_{s\boldsymbol{k}_2}(\boldsymbol{r}) + b_{k_m}\Psi_{p_x\boldsymbol{k}_2}(\boldsymbol{r}) + c_{k_m}\Psi_{p_y\boldsymbol{k}_2}(\boldsymbol{r}) \\
\Psi_{\boldsymbol{k}_3}(\boldsymbol{r},t) &= a_{k_m} e^{i\omega t}\Psi_{s\boldsymbol{k}_3}(\boldsymbol{r}) + b_{k_m}\Psi_{p_x\boldsymbol{k}_3}(\boldsymbol{r}) + c_{k_m}\Psi_{p_y\boldsymbol{k}_3}(\boldsymbol{r}) \\
\Psi_{\boldsymbol{k}_4}(\boldsymbol{r},t) &= a_{k_m} e^{i\omega t}\Psi_{s\boldsymbol{k}_4}(\boldsymbol{r}) + b_{k_m}\Psi_{p_x\boldsymbol{k}_4}(\boldsymbol{r}) + c_{k_m}\Psi_{p_y\boldsymbol{k}_4}(\boldsymbol{r})
\end{aligned}$$

where $\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3, \boldsymbol{k}_4 = (\pm k_m, \pm k_m)$.

We suppose the condensate wave function is a superposition of these four wave functions for a given k_m,

$$\begin{aligned}
\Psi_{k_m}(\boldsymbol{r},t) &= \beta_1\Psi_{\boldsymbol{k}_1}(\boldsymbol{r},t) + \beta_2\Psi_{\boldsymbol{k}_2}(\boldsymbol{r},t) + \beta_3\Psi_{\boldsymbol{k}_3}(\boldsymbol{r},t) + \beta_4\Psi_{\boldsymbol{k}_4}(\boldsymbol{r},t) \\
&= \beta_1\Big[a_{k_m} e^{i\omega t}\Psi_{s\boldsymbol{k}_1}(\boldsymbol{r}) + b_{k_m}\Psi_{p_x\boldsymbol{k}_1}(\boldsymbol{r}) + c_{k_m}\Psi_{p_y\boldsymbol{k}_1}(\boldsymbol{r})\Big] + \beta_2\Big[a_{k_m} e^{i\omega t}\Psi_{s\boldsymbol{k}_2}(\boldsymbol{r}) + b_{k_m}\Psi_{p_x\boldsymbol{k}_2}(\boldsymbol{r}) + c_{k_m}\Psi_{p_y\boldsymbol{k}_2}(\boldsymbol{r})\Big] \\
&\quad + \beta_3\Big[a_{k_m} e^{i\omega t}\Psi_{s\boldsymbol{k}_3}(\boldsymbol{r}) + b_{k_m}\Psi_{p_x\boldsymbol{k}_3}(\boldsymbol{r}) + c_{k_m}\Psi_{p_y\boldsymbol{k}_3}(\boldsymbol{r})\Big] + \beta_4\Big[a_{k_m} e^{i\omega t}\Psi_{s\boldsymbol{k}_4}(\boldsymbol{r}) + b_{k_m}\Psi_{p_x\boldsymbol{k}_4}(\boldsymbol{r}) + c_{k_m}\Psi_{p_y\boldsymbol{k}_4}(\boldsymbol{r})\Big] \\
&= \frac{1}{\sqrt{L}}\sum_{\boldsymbol{l}} \beta_1\Big[a_{k_m} e^{i\omega t} e^{i\boldsymbol{k}_1\cdot\boldsymbol{l}}\phi_s(\boldsymbol{r}-\boldsymbol{l}) + b_{k_m} e^{i\boldsymbol{k}_1\cdot\boldsymbol{l}}\phi_{p_x}(\boldsymbol{r}-\boldsymbol{l}) + c_{k_m} e^{i\boldsymbol{k}_1\cdot\boldsymbol{l}}\phi_{p_y}(\boldsymbol{r}-\boldsymbol{l})\Big] \\
&\quad + \beta_2\Big[a_{k_m} e^{i\omega t} e^{i\boldsymbol{k}_2\cdot\boldsymbol{l}}\phi_s(\boldsymbol{r}-\boldsymbol{l}) + b_{k_m} e^{i\boldsymbol{k}_2\cdot\boldsymbol{l}}\phi_{p_x}(\boldsymbol{r}-\boldsymbol{l}) + c_{k_m} e^{i\boldsymbol{k}_2\cdot\boldsymbol{l}}\phi_{p_y}(\boldsymbol{r}-\boldsymbol{l})\Big] \\
&\quad + \beta_3\Big[a_{k_m} e^{i\omega t} e^{i\boldsymbol{k}_3\cdot\boldsymbol{l}}\phi_s(\boldsymbol{r}-\boldsymbol{l}) + b_{k_m} e^{i\boldsymbol{k}_3\cdot\boldsymbol{l}}\phi_{p_x}(\boldsymbol{r}-\boldsymbol{l}) + c_{k_m} e^{i\boldsymbol{k}_3\cdot\boldsymbol{l}}\phi_{p_y}(\boldsymbol{r}-\boldsymbol{l})\Big] \\
&\quad + \beta_4\Big[a_{k_m} e^{i\omega t} e^{i\boldsymbol{k}_4\cdot\boldsymbol{l}}\phi_s(\boldsymbol{r}-\boldsymbol{l}) + b_{k_m} e^{i\boldsymbol{k}_4\cdot\boldsymbol{l}}\phi_{p_x}(\boldsymbol{r}-\boldsymbol{l}) + c_{k_m} e^{i\boldsymbol{k}_4\cdot\boldsymbol{l}}\phi_{p_y}(\boldsymbol{r}-\boldsymbol{l})\Big] \\
&= \frac{1}{\sqrt{L}}\sum_{\boldsymbol{l}}(\beta_1 e^{i\boldsymbol{k}_1\cdot l} + \beta_2 e^{i\boldsymbol{k}_2\cdot l} + \beta_3 e^{i\boldsymbol{k}_3\cdot l} + \beta_4 e^{i\boldsymbol{k}_4\cdot l}) \times \Big[a_{k_m} e^{i\omega t}\phi_s(\boldsymbol{r}-\boldsymbol{l}) + b_{k_m}\phi_{p_x}(\boldsymbol{r}-\boldsymbol{l}) + c_{k_m}\phi_{p_y}(\boldsymbol{r}-\boldsymbol{l})\Big],
\end{aligned}$$

where $\beta_1^2 + \beta_2^2 + \beta_3^2 + \beta_4^2 = 1$ and we assume $\beta_1, \beta_2, \beta_3, \beta_4$ are all real. Using this ansatz to minimize the interaction energy numerically,

$$\begin{aligned}
\frac{\tilde{E}_{int}}{N} &= \frac{gN}{2}\frac{1}{T}\int_0^T \mathrm{d}t \int \mathrm{d}\boldsymbol{r}|\Psi_{k_m}(\boldsymbol{r},t)|^4 \\
&= \frac{gN}{2L^2}\sum_{\boldsymbol{l}} |\beta_1 e^{i\boldsymbol{k}_1\cdot l} + \beta_2 e^{i\boldsymbol{k}_2\cdot l} + \beta_3 e^{i\boldsymbol{k}_3\cdot l} + \beta_4 e^{i\boldsymbol{k}_4\cdot l}|^4 \times \\
&\quad \frac{1}{T}\int_0^T \mathrm{d}t \int \mathrm{d}\boldsymbol{r}|a_{k_m} e^{i\omega t}\phi_s(\boldsymbol{r}-\boldsymbol{l}) + b_{k_m}\phi_{p_x}(\boldsymbol{r}-\boldsymbol{l}) + c_{k_m}\phi_{p_y}(\boldsymbol{r}-\boldsymbol{l})|^4
\end{aligned}$$

$$
\begin{aligned}
&= \gamma \frac{g\nu}{2}\frac{1}{T}\int_0^T \mathrm{d}t \int \mathrm{d}\boldsymbol{r} |a_{k_m} e^{i\omega t}\phi_s(\boldsymbol{r}) + b_{k_m}\phi_{p_x}(\boldsymbol{r}) + c_{k_m}\phi_{p_y}(\boldsymbol{r})|^4 \\
&= \gamma \frac{E_{int}}{N},
\end{aligned}
$$

where E_{int} is interaction energy of any of the four degenerate ground state and $\gamma = \frac{1}{L}\sum_{\boldsymbol{l}} |\beta_1 e^{i\boldsymbol{k}_1\cdot \boldsymbol{l}} + \beta_2 e^{i\boldsymbol{k}_2\cdot \boldsymbol{l}} + \beta_3 e^{i\boldsymbol{k}_3\cdot \boldsymbol{l}} + \beta_4 e^{i\boldsymbol{k}_4\cdot \boldsymbol{l}}|^4$, we obtain $\beta_1 = \pm 1$, or $\beta_2 = \pm 1$, or $\beta_3 = \pm 1$, or $\beta_4 = \pm 1$, which means BEC condenses at certain $\boldsymbol{k}$ and there is no stripe phase.

III. TWO PHONON RESONANCE

A. Effective Hamiltonian

Under $s, p_+ = -\frac{1}{2}(p_x + ip_y), p_- = \frac{1}{2}(p_x - ip_y)$ basis and $\alpha = -\pi/2$,

$$
H_0(\boldsymbol{k}) = \begin{pmatrix} E_s & -\frac{iK_x - K_y}{\sqrt{2}} & \frac{iK_x + K_y}{\sqrt{2}} \\ \frac{iK_x + K_y}{\sqrt{2}} & \frac{E_{p_x} + E_{p_y}}{2} & \frac{E_{p_y} - E_{p_x}}{2} \\ -\frac{iK_x - K_y}{\sqrt{2}} & \frac{E_{p_y} - E_{p_x}}{2} & \frac{E_{p_x} + E_{p_y}}{2} \end{pmatrix}
$$

$$
H_1(\boldsymbol{k}) = \begin{pmatrix} \Lambda_s & -\frac{\Omega_x + \Omega_y}{\sqrt{2}} & \frac{\Omega_x - \Omega_y}{\sqrt{2}} \\ \frac{\Omega_x - \Omega_y}{\sqrt{2}} & \frac{\Lambda_{p_x} + \Lambda_{p_y}}{2} & \frac{\Lambda_{p_y} - \Lambda_{p_x}}{2} \\ -\frac{\Omega_x + \Omega_y}{\sqrt{2}} & \frac{\Lambda_{p_y} - \Lambda_{p_x}}{2} & \frac{\Lambda_{p_x} + \Lambda_{p_y}}{2} \end{pmatrix}, \quad H_{-1}(\boldsymbol{k}) = \begin{pmatrix} \Lambda_s^* & \frac{\Omega_x - \Omega_y}{\sqrt{2}} & -\frac{\Omega_x + \Omega_y}{\sqrt{2}} \\ -\frac{\Omega_x + \Omega_y}{\sqrt{2}} & \frac{\Lambda_{p_x}^* + \Lambda_{p_y}^*}{2} & \frac{\Lambda_{p_y}^* - \Lambda_{p_x}^*}{2} \\ \frac{\Omega_x - \Omega_y}{\sqrt{2}} & \frac{\Lambda_{p_y}^* - \Lambda_{p_x}^*}{2} & \frac{\Lambda_{p_x}^* + \Lambda_{p_y}^*}{2} \end{pmatrix}
$$

Make a unitary transformation,

$$
\tilde{H}(\boldsymbol{k}, t) = U(t)(H(\boldsymbol{k}, t) - i\partial_t)U^\dagger(t)
$$

where

$$
U(t) = \begin{pmatrix} e^{-i2\omega t} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
$$

$$
\tilde{H}_0(\boldsymbol{k}) = \begin{pmatrix} E_s + 2\omega & 0 & 0 \\ 0 & \frac{E_{p_x} + E_{p_y}}{2} & \frac{E_{p_y} - E_{p_x}}{2} \\ 0 & \frac{E_{p_y} - E_{p_x}}{2} & \frac{E_{p_x} + E_{p_y}}{2} \end{pmatrix}
$$

$$
\tilde{H}_1(\boldsymbol{k}) = \begin{pmatrix} \Lambda_s & 0 & 0 \\ -\frac{\Omega_x + \Omega_y}{\sqrt{2}} & \frac{\Lambda_{p_x} + \Lambda_{p_y}}{2} & \frac{\Lambda_{p_y} - \Lambda_{p_x}}{2} \\ \frac{\Omega_x - \Omega_y}{\sqrt{2}} & \frac{\Lambda_{p_y} - \Lambda_{p_x}}{2} & \frac{\Lambda_{p_x} + \Lambda_{p_y}}{2} \end{pmatrix}, \quad \tilde{H}_{-1}(\boldsymbol{k}) = \begin{pmatrix} \Lambda_s^* & -\frac{\Omega_x + \Omega_y}{\sqrt{2}} & \frac{\Omega_x - \Omega_y}{\sqrt{2}} \\ 0 & \frac{\Lambda_{p_x}^* + \Lambda_{p_y}^*}{2} & \frac{\Lambda_{p_y}^* - \Lambda_{p_x}^*}{2} \\ 0 & \frac{\Lambda_{p_y}^* - \Lambda_{p_x}^*}{2} & \frac{\Lambda_{p_x}^* + \Lambda_{p_y}^*}{2} \end{pmatrix}
$$

$$
\tilde{H}_2(\boldsymbol{k}) = \begin{pmatrix} 0 & 0 & 0 \\ \frac{iK_x + K_y}{\sqrt{2}} & 0 & 0 \\ -\frac{iK_x - K_y}{\sqrt{2}} & 0 & 0 \end{pmatrix}, \quad \tilde{H}_{-2}(\boldsymbol{k}) = \begin{pmatrix} 0 & -\frac{iK_x - K_y}{\sqrt{2}} & \frac{iK_x + K_y}{\sqrt{2}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
$$

$$
\tilde{H}_3(\boldsymbol{k}) = \begin{pmatrix} 0 & 0 & 0 \\ \frac{\Omega_x - \Omega_y}{\sqrt{2}} & 0 & 0 \\ -\frac{\Omega_x + \Omega_y}{\sqrt{2}} & 0 & 0 \end{pmatrix}, \quad \tilde{H}_{-3}(\boldsymbol{k}) = \begin{pmatrix} 0 & \frac{\Omega_x - \Omega_y}{\sqrt{2}} & -\frac{\Omega_x + \Omega_y}{\sqrt{2}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
$$

$$
H_{eff} = \tilde{H}_0 + \frac{1}{\hbar\omega}\sum_{n=1}^{3}\frac{1}{n}\Big([\tilde{H}_0, \tilde{H}_n] - [\tilde{H}_0, \tilde{H}_{-n}] + [\tilde{H}_n, \tilde{H}_{-n}]\Big)
$$

p_+- and p_-- band has band gap $\frac{4\Omega_x\Omega_y}{3} \sim \frac{4hsp^2}{3}$, which can be larger than band width of $p_\pm$- band by adjusting lattice depth and shanking amplitude. So we can project out p_+- or p_-- band depending on resonance parameters. We take two phonon resonance between s- and p_+- band for example ($\alpha = -\pi/2$).

$$
\begin{aligned}
\tilde{H}_{eff} &= \begin{pmatrix} H_B & H_{BC} \\ H_{BC}^\dagger & H_C \end{pmatrix} \\
H_{B,eff} &= H_B + H_{BC}(E-H_C)^{-1}H_{BC}^\dagger
\end{aligned}
$$

One can approximate $(E-H_C)^{-1} \simeq (E_{p_+}-E_{p_-})^{-1} \simeq \frac{3\omega}{4h_{sp}^2}$ and obtain $H_{B,eff} = (E_s(\boldsymbol{k})+2\omega)\mathbf{I} + \boldsymbol{d}(\boldsymbol{k})\cdot\boldsymbol{\sigma}$. Assuming $t_s, t_p, h_{sp1}, h_s, h_p \ll h_{sp}$ and $\omega/E_r = 10^0 \sim 10^1$, we obtain

$$
\begin{aligned}
\boldsymbol{d}(\boldsymbol{k}) &= (A + B(2\cos k_x + \cos k_y) - C\sin k_y,\ A + B(\cos k_x + 2\cos k_y) + C\sin k_x,\ D + E(\cos k_x + \cos k_y)) \\
A &= \frac{\delta h_{sp}}{\omega}, \quad \delta = \epsilon_s + 2\omega - \epsilon_p \\
B &= \frac{2(t_s - t_{p\sigma})h_{sp}}{3\omega} \\
C &= \frac{\sqrt{2}(h_s - h_p)h_{sp}}{\omega} \\
D &= \frac{\delta}{2} - \frac{7h_{sp}^2}{3\omega} \\
E &= \frac{t_s - t_{p\sigma}}{2}
\end{aligned}
$$

We can not get non-zero Chern number by parameters connected with edge states. So the effective Hamiltonian is not reliable. Maybe we can obtain more precision effective Hamiltonian through higer order perturbation.
In single phonon resonance case,

$$
\boldsymbol{d}(\boldsymbol{k}) = \left(-\sqrt{2}h_{sp},\ -\frac{t_{sp}\delta}{\sqrt{2}\omega}\sin k_x,\ -\frac{\delta^2}{4\omega} + \frac{t}{2}(\cos k_x - \cos k_y) - \frac{3h_{sp}t_{sp}}{\omega}\sin k_y\right), \quad \delta = \epsilon_s + \omega - \epsilon_p.
$$

$\hat{\boldsymbol{d}}(\boldsymbol{k})$ can not cover the whole Bloch sphere, so Chern number is $\frac{1}{4\pi}\int\int \mathrm{d}k_x \mathrm{d}k_y \hat{\boldsymbol{d}}\cdot\left(\partial_{k_x}\hat{\boldsymbol{d}} \times \partial_{k_y}\hat{\boldsymbol{d}}\right) = 0$.

B. Edge State

Now consider 2D square lattice is periodic along $\hat{x}$ direction and finite along $\hat{y}$ direction.

$$
H(\boldsymbol{r},t) = H_0(\boldsymbol{r}) + 2i\left(\sin\omega t H_1(x) + \sin(\omega t + \alpha)H_1(y)\right)
$$

Let's take the Fourier transform with respect to x,

$$
\hat{a}_{\lambda\boldsymbol{n}} = \frac{1}{\sqrt{L_x}}\sum_{k_x} e^{ik_x n_x}\hat{a}_{\lambda k_x n_y}, \quad \hat{a}^\dagger_{\lambda'\boldsymbol{m}} = \frac{1}{\sqrt{L_x}}\sum_{k_x} e^{-ik_x m_x}\hat{a}^\dagger_{\lambda' k_x m_y}.
$$

$$
\begin{aligned}
\hat{H}(t) &= \frac{1}{L_x}\sum_{\substack{\lambda'\lambda \\ \boldsymbol{mn}}} H^{\lambda'\lambda}_{\boldsymbol{mn}}(t) e^{i(-k_x' m_x + k_x n_x)}\hat{a}^\dagger_{\lambda' k_x' m_y}\hat{a}_{\lambda k_x n_y} \\
&= \frac{1}{L_x}\sum_{\substack{\lambda'\lambda \\ m_y n_y}}\sum_{m_x\delta_x}\sum_{k_x' k_x} H^{\lambda'\lambda}_{\boldsymbol{\delta}}(t) e^{i(-k_x'+k_x)m_x} e^{ik_x\delta_x}\hat{a}^\dagger_{\lambda' k_x' m_y}\hat{a}_{\lambda k_x n_y} \\
&= \sum_{\substack{\lambda'\lambda \\ m_y n_y}}\sum_{k_x,\delta_x} H^{\lambda'\lambda}_{\boldsymbol{\delta}}(t) e^{ik_x\delta_x}\hat{a}^\dagger_{\lambda' k_x m_y}\hat{a}_{\lambda k_x n_y} = \sum_{k_x}\sum_{\substack{\lambda'\lambda \\ m_y n_y}} H^{\lambda'\lambda}_{m_y n_y}(k_x,t)\hat{a}^\dagger_{\lambda' k_x m_y}\hat{a}_{\lambda k_x n_y}
\end{aligned}
$$

with $H^{\lambda'\lambda}_{m_y n_y}(k_x,t) = \sum_{\delta_x} H^{\lambda'\lambda}_{\delta_x,\delta_y}(t)e^{ik_x\delta_x}$, $\boldsymbol{\delta} = \delta_x\hat{x} + \delta_y\hat{y} = \boldsymbol{n} - \boldsymbol{m}$.
Hamiltonian reads

$$
\hat{H}(t) = \sum_n \begin{pmatrix} \hat{a}^\dagger_{sn} & \hat{a}^\dagger_{p_x n} & \hat{a}^\dagger_{p_y n} \end{pmatrix} M \begin{pmatrix} \hat{a}_{sn} \\ \hat{a}_{p_x n} \\ \hat{a}_{p_y n} \end{pmatrix} + \begin{pmatrix} \hat{a}^\dagger_{sn} & \hat{a}^\dagger_{p_x n} & \hat{a}^\dagger_{p_y n} \end{pmatrix} T \begin{pmatrix} \hat{a}_{sn+1} \\ \hat{a}_{p_x n+1} \\ \hat{a}_{p_y n+1} \end{pmatrix} + \begin{pmatrix} \hat{a}^\dagger_{sn+1} & \hat{a}^\dagger_{p_x n+1} & \hat{a}^\dagger_{p_y n+1} \end{pmatrix} T^\dagger \begin{pmatrix} \hat{a}_{sn} \\ \hat{a}_{p_x n} \\ \hat{a}_{p_y n} \end{pmatrix}
$$

with

$$
M = \begin{pmatrix} \epsilon_s + 2t_s \cos k_x - 4h_s \sin k_x \sin \omega t & i\,(K_x + 2\Omega_x \sin \omega t) & i2h_{sp} \sin(\omega t + \alpha) \\ -i\,(K_x + 2\Omega_x \sin \omega t) & \epsilon_p + 2t_{p\sigma} \cos k_x - 4h_p \sin k_x \sin \omega t & 0 \\ -i2h_{sp} \sin(\omega t + \alpha) & 0 & \epsilon_p + 2t_{p\pi} \cos k_x - 4h_s \sin k_x \sin \omega t \end{pmatrix}
$$

$$
T = \begin{pmatrix} t_s + i2h_s \sin(\omega t + \alpha) & 0 & t_{sp} + i2h_{sp1} \sin(\omega t + \alpha) \\ 0 & t_{p\pi} + i2h_s \sin(\omega t + \alpha) & 0 \\ -t_{sp} - i2h_{sp1} \sin(\omega t + \alpha) & 0 & t_{p\sigma} + i2h_p \sin(\omega t + \alpha) \end{pmatrix}
$$

Diagonalizing Floquet operator, we obtain quasi-energy dispersion. Fig.3 shows edge states in the quasi-energy spectrum.

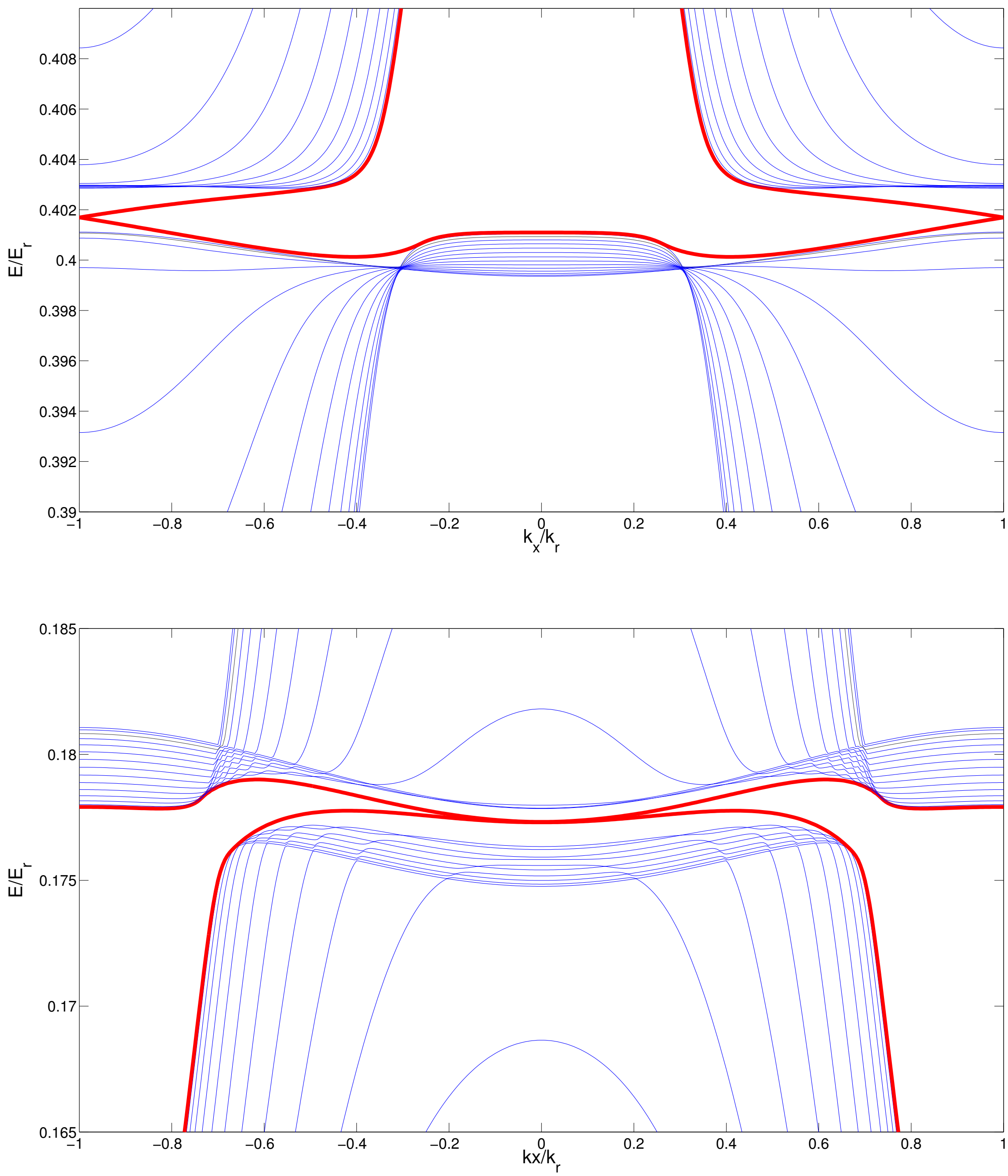

FIG. 3: Edge states in the quasi-energy spectrum. Top: $V = 6.5E_r, \omega = 3.53E_r, f = 0.6$; botom: $V = 6.5E_r, \omega = 3.4E_r, f = 0.6$.