Lectures on Cold Atom Physics
Lecture: Topological States in Optical Lattices

In previous lectures we focus mostly on square or cubic lattices. In recent years, experimentalists have also realized other types of optical lattices, including triangular, honeycomb and kagome lattices. One motivation for studying these lattice geometries is to realize various topological nontrivial phases. Here we shall use honeycomb lattice as an example to demonstrate realizing topological phases in cold atom optical lattices setup.

The Honeycomb Lattice. The unit cell of a honeycomb lattice contains two sites denoted by A and B. The Bravais lattice of the honeycomb lattice is a triangle lattice. We choose the primitive vectors of the Bravais lattice as

$$\mathbf{a}_1 = \left(0, \sqrt{3}\right) a, \quad \mathbf{a}_2 = \left(\frac{3}{2}, -\frac{\sqrt{3}}{2}\right) a, \quad \mathbf{a}_3 = \left(-\frac{3}{2}, -\frac{\sqrt{3}}{2}\right) a, \tag{1}$$

as shown in Fig.1. The reciprocal lattice vectors is given by the relation $\mathbf{a}_i \cdot \mathbf{b}_j = 2\pi\delta_{ij}$ as

$$\mathbf{b}_1 = \frac{2\pi}{a}\left(\frac{1}{3}, \frac{1}{\sqrt{3}}\right), \quad \mathbf{b}_2 = \frac{2\pi}{a}\left(\frac{2}{3}, 0\right) \tag{2}$$

With these reciprocal lattice vectors, the first Brillouin zone can be constructed as shown in Fig.1, where

$$K = \frac{2\pi}{a}\left(0, \frac{2}{3\sqrt{3}}\right), \quad K' = \frac{2\pi}{a}\left(0, -\frac{2}{3\sqrt{3}}\right). \tag{3}$$

Now we consider a tight-binding model in the honeycomb lattice. If we only include the nearest hopping, hopping only occurs between A and B sub-lattices, and the tight-binding Hamiltonian is given by

$$H = -t_1 \sum_{\langle ij \rangle} \left(\hat{c}^\dagger_{\mathrm{B},j} \hat{c}_{\mathrm{A},i} + \mathrm{h.c.}\right) \tag{4}$$

Introducing three displace vectors as

$$\mathbf{d}_1 = (-1, 0)\, a, \quad \mathbf{d}_2 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) a, \quad \mathbf{d}_3 = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) a \tag{5}$$

with $\sum_i \mathbf{d}_i = 0$. This tight-binding Hamiltonian can be written into momentum space as

$$H = \sum_{\mathbf{k}} \left(\hat{c}^\dagger_{\mathrm{A}}(\mathbf{k}), \hat{c}^\dagger_{\mathrm{B}}(\mathbf{k})\right) H(\mathbf{k}) \begin{pmatrix} \hat{c}_{\mathrm{A}}(\mathbf{k}) \\ \hat{c}_{\mathrm{B}}(\mathbf{k}) \end{pmatrix} \tag{6}$$

where the matrix is given by:

$$H(\mathbf{k}) = \begin{pmatrix} 0 & -t_1 \sum_\alpha e^{-i\mathbf{k}\cdot\mathbf{d}_\alpha} \\ -t_1 \sum_\alpha e^{i\mathbf{k}\cdot\mathbf{d}_\alpha} & 0 \end{pmatrix}. \tag{7}$$

$H(\mathbf{k})$ can be expanded in term of the Pauli matrix as

$$H(\mathbf{k}) = \mathbf{B}(\mathbf{k}) \cdot \sigma = \left[-t_1 \sum_\alpha \cos(\mathbf{k}\cdot\mathbf{d}_\alpha)\right] \sigma_x + \left[-t_1 \sum_\alpha \sin(\mathbf{k}\cdot\mathbf{d}_\alpha)\right] \sigma_y. \tag{8}$$

So the band structure can be obtained as

$$E_\pm(\mathbf{k}) = \mp t_1 \sqrt{\left[\sum_\alpha \cos(\mathbf{k}\cdot\mathbf{d}_\alpha)\right]^2 + \left[\sum_\alpha \sin(\mathbf{k}\cdot\mathbf{d}_\alpha)\right]^2} \tag{9}$$

One can see that for K and K' points, $\sum_\alpha e^{i\mathbf{k}\cdot\mathbf{d}_\alpha} = 0$, and the band gap will be closed. Expanding the dispersion nearby K or K' point, one can find a linear dispersion. K and K' points are therefore called Dirac points.

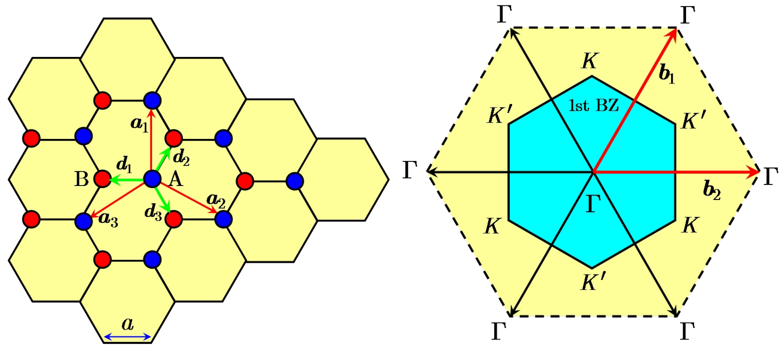

FIG. 1: The honeycomb lattice and its Brillouin zone

The Haldane Model. If there exists only $B_x(\mathbf{k})$ and $B_y(\mathbf{k})$ terms in $H(\mathbf{k})$, generically one can always find solution for $B_x(\mathbf{k}) = 0$ and $B_y(\mathbf{k}) = 0$ for a two-dimensional model, around which the spectrum exhibits Dirac dispersion. That is to say, small distortion of honeycomb lattice can only shifts the position of Dirac point but can not gap it. In order to open up the band gap, one needs to introduce a $B_z(\mathbf{k})\sigma_z$ term in $H(\mathbf{k})$. For this purpose we introduce the Haldane model on the honeycomb lattice. The Haldane model introduces the next-nearest-neighbor hopping with a nontrivial phase factor $\phi_{\mathrm{a}} = -\phi_{\mathrm{b}} = \phi$, as shown in Fig. 2, and an energy offset between A and B sub-lattice denoted by M. The Hamiltonian is written as

$$H = -t_1 \sum_{\langle ij \rangle} \left(\hat{c}^{\dagger}_{\mathrm{B},j} \hat{c}_{\mathrm{A},i} + \mathrm{h.c.} \right) + t_2 \sum_{\langle\langle ij \rangle\rangle} \left(e^{-i\phi_a} \hat{c}^{\dagger}_{\mathrm{A},j} \hat{c}_{\mathrm{A},i} + e^{-i\phi_b} \hat{c}^{\dagger}_{\mathrm{B},j} \hat{c}_{\mathrm{B},i} + \mathrm{h.c.} \right) + M \sum_{i} \left(\hat{c}^{\dagger}_{\mathrm{A},i} \hat{c}_{\mathrm{A},i} - \hat{c}^{\dagger}_{\mathrm{B},i} \hat{c}_{\mathrm{B},i} \right). \tag{10}$$

Now in momentum space $H(\mathbf{k})$ becomes:

$$H(\mathbf{k}) = E_0(\mathbf{k})\,\mathrm{I} + \mathbf{B}(\mathbf{k}) \cdot \sigma \tag{11}$$

where

$$E_0(\mathbf{k}) = 2t_2 \cos\phi \sum_{\alpha} \cos(\mathbf{k} \cdot \mathbf{a}_{\alpha}) \tag{12}$$

and

$$\mathbf{B}(\mathbf{k}) \cdot \sigma = \left[M + 2t_2 \sin\phi \sum_{\alpha} \sin(\mathbf{k} \cdot \mathbf{a}_{\alpha}) \right] \sigma_z - \left[t_1 \sum_{\alpha} \cos(\mathbf{k} \cdot \mathbf{d}_{\alpha}) \right] \sigma_x - \left[t_1 \sum_{\alpha} \sin(\mathbf{k} \cdot \mathbf{d}_{\alpha}) \right] \sigma_y. \tag{13}$$

One can see that $B_z(\mathbf{k})$ term contains two terms. One breaks spatial inversion symmetry, and the other breaks time reversal symmetry. The spatial inversion symmetry is defined as

$$H(\mathbf{k}) \to \sigma_x H(-\mathbf{k}) \sigma_x = H(\mathbf{k}). \tag{14}$$

Because under spatial inversion transformation, $\sigma_z \to -\sigma_z$, one can see that the M-term breaks inversion symmetry while t_2-term does not. The time reversal symmetry is defined as:

$$H(\mathbf{k}) \to H^*(-\mathbf{k}) = H(\mathbf{k}), \tag{15}$$

one can see that t_2-term breaks time-reversal symmetry while M-term does not.

Now at each momentum $\mathbf{k}$ we have introduced a $\mathbf{B}(\mathbf{k})$ vector, and the eigen-state of each given band can described by a pseudo-spin which is always in the same direction of $\mathbf{B}(\mathbf{k})$ field. Thus, it defines a mapping from the momentum space of the first Brillouin zone to S^2 Bloch sphere. Such a mapping can also be classified by the homotopy group and is characterized by the Chern number. In this case, the Chern number of the lower band is defined as

$$\mathcal{C} = \frac{1}{4\pi} \int_{\mathrm{BZ}} d^2k \left(\frac{\partial \hat{\mathbf{B}}}{\partial k_x} \times \frac{\partial \hat{\mathbf{B}}}{\partial k_y} \right) \cdot \hat{\mathbf{B}}, \tag{16}$$

where $\hat{\mathbf{B}}(\mathbf{k}) = \mathbf{B}(\mathbf{k}) / |\mathbf{B}(\mathbf{k})|$. If $\mathcal{C}$ is a non-zero integer, this state is a topological nontrivial state. In this case, one can show that when $-3\sqrt{3}|t_2 \sin\phi| < M < 3\sqrt{3}|t_2 \sin\phi|$, $\mathcal{C}$ equals $+1$ or -1. That is to say, if the band gap opening

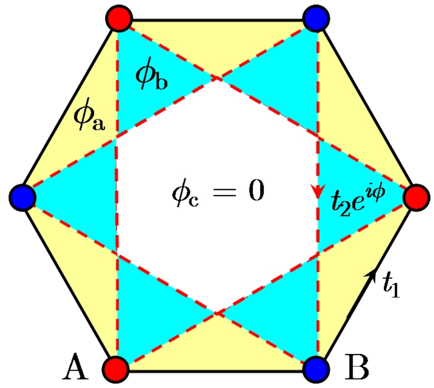

FIG. 2: The Haldane model

is primarily due to tim-reversal symmetry breaking t_2 term instead of inversion symmetry breaking M-term, it favors a topological nontrivial phase. In this case of Haldane model, it exhibits "quantum anomalous Hall effect " when chemical potential lies inside the band gap, and the system exhibits chiral edge state in a finite system.

Floquet Method for a Time-Periodically Driven System. In optical lattices, the next-nearest-neighbor hopping is usually quite small. It is the major challenging for implementing the Haldane model. To overcome this problem, one comes to the idea of using periodically driven system, for which we shall first introduce the Floquet theory for a time-periodic system.

The Schrödinger equation of a time-periodic system is given by $i\partial_t \psi = H(t)\psi(t)$ with $H(t+T) = H(t)$, and $H(t)$ can be expanded as $\sum_n e^{-in\omega t} H_n$ with $\omega = 2\pi/T$. The time evolution operator is given by:

$$U(t) = \hat{\mathrm{T}} \exp\left[-i \int_0^t dt' H(t')\right] \tag{17}$$

where $\hat{\mathrm{T}}$ is the time-ordering operator. After completing one period, the evolution operator

$$U(T) = \hat{\mathrm{T}} \exp\left[-i \int_0^T dt H(t)\right], \tag{18}$$

and $U(T)$ only depends on T. Thus, we can introduce an effective Hamiltonian H_{eff} as:

$$U(T) = \exp\left(-iH_{\text{eff}} T\right). \tag{19}$$

That is to say, this effective Hamiltonian H_{eff} can fully recover the evolution of original time-periodic system at integer period. If ω is much larger than typical energy scales in H_0, one can perform $1/\omega$ expansion to deduce the effective Hamiltonian H_{eff} as

$$H_{\text{eff}} = H_0 + \sum_{n=1}^{\infty} \frac{[H_n, H_{-n}]}{n\omega} \tag{20}$$

Topological Phases from Shaking Honeycomb Lattice. In the experiment, a honeycomb lattice potential $V(x,y)$ is realized by interference of three laser beams. By time-periodically modulating the relative phase between lasers, one can realize a shaken optical lattice whose Hamiltonian is given by

$$H = -\frac{\hbar^2 \nabla^2}{2m} + V\left[x + b\cos(\omega t), y + b\sin(\omega t)\right] \tag{21}$$

where b is the shaking amplitude, and ω is the shaking frequency. Note that here we choose a $\pi/2$ phase difference between the shaking in x and that in y direction. By making a coordinate transformation $x' = x + b\cos(\omega t)$ and $y' = y + b\sin(\omega t)$, the new Hamiltonian in the comoving frame reads

$$H(x,y,t) = \frac{\hbar^2}{2m}\left[-i\partial_x - A_x(t)\right]^2 + \frac{1}{2m}\left[-i\partial_y - A_y(t)\right]^2 + V(x,y), \tag{22}$$

where

$$A_x(t) = m\omega b \sin(\omega t)/\hbar, \quad A_y(t) = -m\omega b \cos(\omega t)/\hbar, \tag{23}$$

With this transformation, the lattice potential becomes static, but it appears a time-dependent vector potential. This model can also be used for the system of a circular polarized light applied to graphene.

Considering a tight-binding model with nearest neighbor hopping t_1 and on-site energy difference M, by employing the Peiels substitution, it leads to

$$H(\mathbf{k}, t) = \begin{pmatrix} M & -t_1 \sum_\alpha e^{-i[\mathbf{k}-\mathbf{A}(t)]\cdot\mathbf{d}_\alpha} \\ \text{h.c.} & -M \end{pmatrix}. \tag{24}$$

The Hamiltonian can be approximately written as

$$H(\mathbf{k},t) \approx H_0(\mathbf{k}) + H_1(\mathbf{k}) e^{i\omega t} + H_{-1}(\mathbf{k}) e^{-i\omega t}, \tag{25}$$

where

$$H_0(\mathbf{k}) = M\sigma_z - \left\{ t_1 J_0(f) \sum_\alpha \cos(\mathbf{k}\cdot\mathbf{d}_\alpha) \right\} \sigma_x - \left\{ t_1 J_0(f) \sum_\alpha \sin(\mathbf{k}\cdot\mathbf{d}_\alpha) \right\} \sigma_y \tag{26}$$

$$H_1(\mathbf{k}) = -it_1 J_1(f) \sum_\alpha e^{-i\theta_\alpha} \left[\sin(\mathbf{k}\cdot\mathbf{d}_\alpha)\sigma_x - \cos(\mathbf{k}\cdot\mathbf{d}_\alpha)\sigma_y \right] \tag{27}$$

$$H_{-1}(\mathbf{k}) = it_1 J_1(f) \sum_\alpha e^{i\theta_\alpha} \left[\sin(\mathbf{k}\cdot\mathbf{d}_\alpha)\sigma_x - \cos(\mathbf{k}\cdot\mathbf{d}_\alpha)\sigma_y \right] \tag{28}$$

where $f = m\omega b a/\hbar$, and θ_α is the angle of $\mathbf{d}_\alpha$. H_0 is basically the static part of the honeycomb lattice Hamiltonian, with tunneling modified by shaking. $H_{\pm 1}$ describes shaking induced tunneling between neighboring sites. With the help of Eq. (20), the effective Hamiltonian is given by:

$$\begin{aligned} H_{\text{eff}}(\mathbf{k}) &\approx H_0(\mathbf{k}) + \frac{[H_1(\mathbf{k}), H_{-1}(\mathbf{k})]}{\omega} \\ &= \left\{ M - \frac{4t_1^2 J_1^2(f)}{\omega} \sum_{\alpha\beta} \sin(\theta_\alpha - \theta_\beta)\cos(\mathbf{k}\cdot\mathbf{d}_\alpha)\sin(\mathbf{k}\cdot\mathbf{d}_\beta) \right\} \sigma_z \\ &- \left\{ t_1 J_0(f) \sum_\alpha \cos(\mathbf{k}\cdot\mathbf{d}_\alpha) \right\} \sigma_x - \left\{ t_1 J_0(f) \sum_\alpha \sin(\mathbf{k}\cdot\mathbf{d}_\alpha) \right\} \sigma_y \end{aligned} \tag{29}$$

One can see that a time-reversal symmetry breaking term emerges in $B_z(\mathbf{k})$ of the effective Hamiltonian, which is proportional to t_1^2 as a result of the second-order perturbation effect of shaking induced hopping. This term plays the same role as t_2-term in the original Haldane model Eq. 13. When this term dominates over the M-term, the system will enter a topological nontrivial insulting phase. Phase transition between topological trivial and nontrivial phase can be realized.

[1] F. D. M. Haldane, Phys. Rev. Lett. **61**, 2015 (1988)