Note on One-Dimensional Optical Lattice

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We start from a time-dependent 1D optical superlattice of the form

$$V_{\mathrm{OL}} = V_1 \cos^2(\frac{2\pi x}{d}) + V_2 \cos^2(\frac{\pi x}{d} - \varphi(t))$$

where the relative phase between these two part is varied linearly with time

$$\varphi(t) = \frac{\pi t}{T}$$

Consider the V_2 (time-dependent) part. It could be easily expanded as

$$\begin{aligned} &V_2 \cos^2(\frac{\pi x}{d} - \varphi(t)) \\ &= V_2 \cos^2(\frac{\pi x}{d}) \cos(\frac{2\pi t}{T}) + V_2 \cos^2(\frac{\pi x}{d} - \frac{\pi}{4}) \sin(\frac{2\pi t}{T}) \\ &+ \frac{1}{2} V_2 (1 - \cos(\frac{2\pi t}{T}) - \sin(\frac{2\pi t}{T})) \end{aligned}$$

Or more conveniently,

$$V_2 \cos^2(\frac{\pi x}{d} - \varphi(t)) = \frac{1}{2} V_2 \cos(\frac{2\pi x}{d}) \cos(\frac{2\pi t}{T}) + \frac{1}{2} V_2 \sin(\frac{2\pi x}{d}) \sin(\frac{2\pi t}{T})$$

Drop the space independent term (the last line above), we get our time-dependent part of lattice potential as

$$\mathrm{V}_2(x,t) = V_2 \cos(\frac{2\pi t}{T}) \cos^2(\frac{\pi x}{d}) + V_2 \sin(\frac{2\pi t}{T}) \cos^2(\frac{\pi x}{d} - \frac{\pi}{4})$$

Now the dependence on time is periodic, with angular frequency $\omega = 2\pi/T$.

This section refers to [1].

1 Floquet Formalism

Floquet states

Note that for a periodically driven system, the *Floquet theory* provides a powerful framework for analysis. A class of static states we called *Floquet states*, makes good description of the evolution of system. Floquet states are defined as[1]

$$\Psi_\alpha(t) = e^{-\mathrm{i}\varepsilon_\alpha t/\hbar}\Phi_\alpha(t) \tag{1a}$$

$$\Phi_\alpha(t) = \Phi_\alpha(t+T) \tag{1b}$$

quite analogous to the *Bloch theory* where we have *Bloch wavefunction* to characterize a state[2]

$$\psi_{\vec{k}}(\vec{r}) = e^{\mathrm{i}\vec{k}\cdot\vec{r}}\phi_{\vec{k}}(\vec{r})$$
$$\phi_{\vec{k}}(\vec{r}) = \phi_{\vec{k}}(\vec{r}+\vec{a})$$

Just like the quasimomentum $\vec{k}$ is a good quantum number (conserved modulo a reciprocal lattice vector) to label a Bloch state, the quasienergy ε_α is also a good quantum number used to label a Floquet mode, and it is conserved modulo $\hbar\omega$.

Easily to verify that Floqued states satisfy the periodc condition offered by the periodical Hamiltonian. If we have a Floquet mode as a solution to the Shrödinger equation at time t, i.e. $\mathrm{i}\hbar\partial_t\Psi(t) = H(t)\Psi(t)$, this must also fit into that at time $t+T$, i.e. $\mathrm{i}\hbar\partial_t\Psi(t+T) = H(t+T)\Psi(t+T)$, since $\Psi(t) = \Psi(t+T)$ has the same periodicity as $H(t)$. Thus the Hamiltonian allows solutions of the specific form of these Floquet states.

In terms of the periodicity of $\Phi(t)$ (Eq.1b), we could expand $\Phi(t)$ under plane wave bases satisfying the same boundary conditions, i.e. $\{e^{\mathrm{i}m\omega t}\}$ with $\omega = 2\pi/T$, such that

$$\Phi_\alpha(\vec{r},t) = \sum_m e^{\mathrm{i}m\omega t}\phi_{\alpha,m}(\vec{r})$$

Here $\phi_{\alpha,m}(\vec{r})$ is the m^{th} component coefficient of Φ expanding on t, also labeled α standing for its quasienergy ε_α. Given $\Phi(t)$, We obtain ϕ_m by inversive Fourier transformation

$$\phi_m = \frac{1}{T}\int_0^T H(t)e^{-\mathrm{i}m\omega t}\mathrm{d}t$$

Extended Floquet modes

From

$$(H(t) - \mathrm{i}\hbar\partial_t)\Psi(t) = 0$$

we have

$$(H(t) - \mathrm{i}\hbar\partial_t)\Phi(t) = \varepsilon_\alpha\Phi(t)$$

[1]Here we omit the space label without confusion.

[2]with $\vec{a}$ is the primitive vector of crystal.

Thus if we define a new operator as

$$\mathcal{H}(t) = H(t) - \mathrm{i}\hbar\partial_t$$

we have $\Phi_\alpha(t)$, also called *Floquet modes*, as its eigenstates, with eigenvalue ε_α. i.e.

$$\mathcal{H}(t)\Phi_\alpha(t) = \varepsilon_\alpha\Phi_\alpha(t)$$

We called this new operator as *Floquet Hamiltonian*.

Floquets modes are complete orthogonal bases spanning the physical Hilbert space describe by $\mathcal{H}$, with inner product defined as

$$\langle\langle\Phi_\alpha(t)|\Phi_\beta(t)\rangle\rangle = \frac{1}{T}\int_0^T \langle\Phi_\alpha(t)|\Phi_\beta(t)\rangle \mathrm{d}t = \delta_{\alpha\beta}$$

where $\langle\Phi_\alpha(t)|\Phi_\beta(t)\rangle$ denotes usual inner product of other degrees of freedom. At each time Floquet modes form a set of complete bases of the system, i.e.

$$\sum_\alpha |\Phi_\alpha(t)\rangle\langle\Phi_\alpha(t)| = \mathbb{1}$$

The eigenspectrum defining above lies upto $-\hbar\omega/2 < \varepsilon_\alpha < \hbar\omega/2$, for it is easily verified that $\varepsilon_\alpha + m\hbar\omega$ corresponds to the same state as ε_α due to the periodicity of Floquet state. Actually, we have

$$\begin{aligned}
e^{-\mathrm{i}(\varepsilon_\alpha + m\hbar\omega)t/\hbar}\Phi_\alpha(t) &= e^{-\mathrm{i}\varepsilon_\alpha t/\hbar}e^{-\mathrm{i}m\omega t}\Phi_\alpha(t) \\
&= e^{-\mathrm{i}\varepsilon_\alpha t/\hbar}e^{-\mathrm{i}m\omega t}\sum_{m'} e^{\mathrm{i}m'\omega t}\phi_{\alpha,m'} \\
&= e^{-\mathrm{i}\varepsilon_\alpha t/\hbar}\sum_{m'} e^{\mathrm{i}m\omega t}\phi_{\alpha,m'+m} = e^{-\mathrm{i}\varepsilon_\alpha t/\hbar}\sum_{m'} e^{\mathrm{i}m\omega t}\phi^m_{\alpha,m'} \\
&= e^{-\mathrm{i}\varepsilon_\alpha t/\hbar}\Phi^m_\alpha(t)
\end{aligned}$$

Here we define $\Phi^m_\alpha(t) = e^{\mathrm{i}m\omega t}\Phi_\alpha$. Thus the corresponding Floquet states are identical, while not so as the Floquet modes. Within this extended definition we have the extended Hilbert space spanned by $\{\Phi^m_\alpha(t)\}$ with orthogonal relation

$$\langle\langle\Phi^m_\alpha(t)|\Phi^n_\beta(t)\rangle\rangle = \delta_{\alpha,\beta}\delta_{m,n}$$

Completeness relation could now be written as

$$\sum_{\alpha,m} |\Phi^m_\alpha(t)\rangle\langle\Phi^m_\alpha(t)| = \hat{\mathbb{1}}$$

This section refers to [2, 3].

2 One-dimensional Super lattice

In this section, we solve the single particle physics of a 1D super lattice. Specifically, we work on the quasienergy spectrum.

The Hamiltonian here is

$$
\begin{aligned}
H(x,t) = \frac{p^2}{2m} &+ V_1 \cos(2k_r x) \\
&+ V_2 \cos(k_r x)\cos(\omega t) + V_2 \sin(k_r x)\sin(\omega t)
\end{aligned}
$$

where $k_r = 2\pi/d$, $\omega = 2\pi/T$, $(1/2)V_2 \to V_2$, corresponding to the notation in previous section. Here we omit the constant term which is of no interest to us. It contributes nothing to the change of band structure but just shifting whole spectrum of a constant $V_1 + V_2$.

A typical shape of the super lattice potential at some certain moment looks like Fig.1

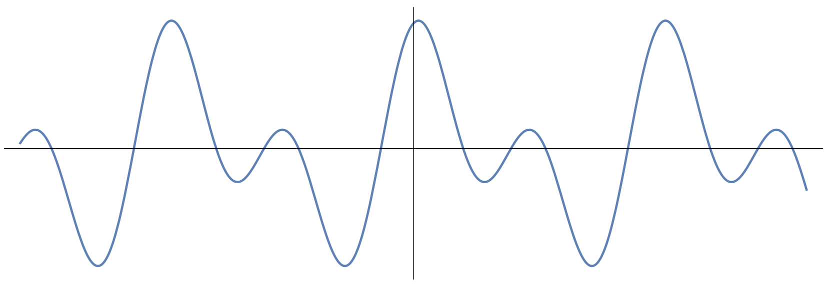


Figure 1: A typical shape of the super lattice potential

Notice that we are now dealing with a time-periodically-driving Hamiltonian, the Floquet theory fits here. From above we learned that

$$
\begin{aligned}
\mathcal{H}\Phi(t) &= \varepsilon\Phi(t) \\
\text{with} \qquad \mathcal{H} &= H - \mathrm{i}\partial_t
\end{aligned}
$$

$\Phi(t)$ is the Floquet modes satisfying the eigenvalue equation of $\mathcal{H}$. And the eigenvalues of Floquet Hamiltonian $\mathcal{H}$ are the quasienergies we wanted. Thus, to get the spectrum of quasienergy, we simply diagonalize $\mathcal{H}$ under some bases. Quite conveniently, we choose to write down $\mathcal{H}$ under the most natural bases, that is $\{e^{\mathrm{i}m\omega t}\}$ and $\{e^{\mathrm{i}lk_r x}\}$ where $m, l = 0, \pm1, \pm2, \ldots$, which satisfy the same boundary conditions as the system.

Floquet Hamiltonian Matrix

The Floquet Hamiltonian Matrix looks like

$$
\begin{bmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & H_0+2\omega & H_1 & 0 & 0 & 0 & \cdot \\
\cdot & H_{-1} & H_0+\omega & H_1 & 0 & 0 & \cdot \\
\cdot & 0 & H_{-1} & H_0 & H_1 & 0 & \cdot \\
\cdot & 0 & 0 & H_{-1} & H_0-\omega & H_1 & \cdot \\
\cdot & 0 & 0 & 0 & H_{-1} & H_0-2\omega & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{bmatrix}
$$

Where H_m are the Fourier components of the original time-dependent Hamiltonian. These are actually matrices under plane-wave bases $\{e^{\mathrm{i}lk_r x}\}$. In practice, these are always matrices truncated to orders of how many bands are we concerned. A typical matrix of some specified quasimomentum is of the form as Fig.2

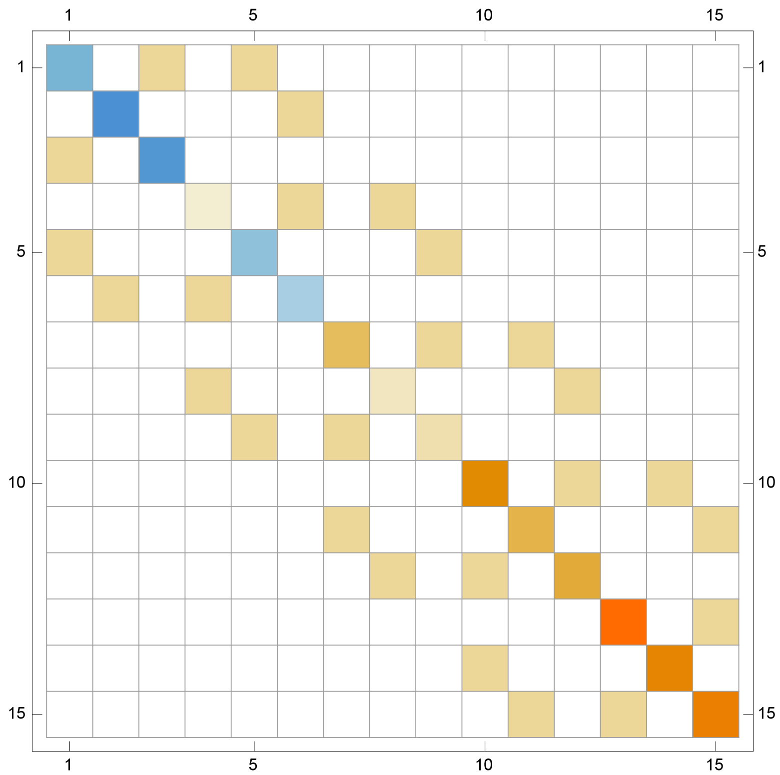


Figure 2: Mathematica plotting Floquet Hamiltonian matrix

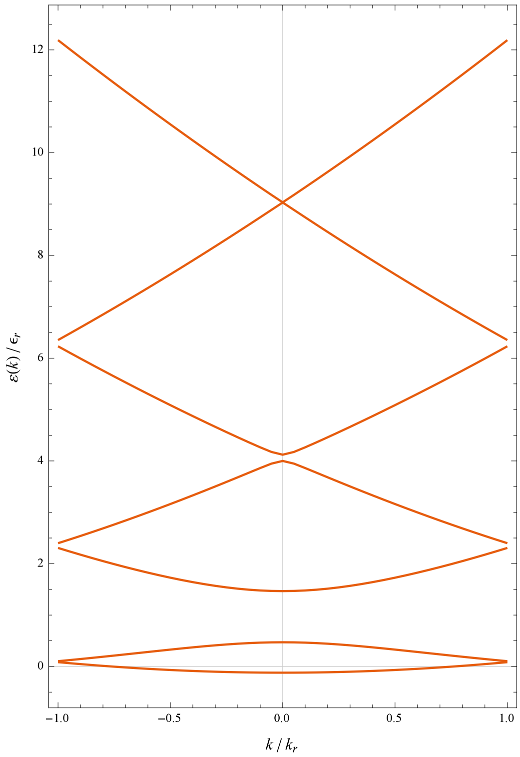
Quasienergy band

We obtain quasienergy bands by numerically diagonalizing the Floquet Hamiltonian. Here we have three parameters to modulate, V_1(potential depth of static part), V_2(potential depth of time-dependent part), ω(driving frequency). If we set $V_2 = 0$ we get a static lattice Hamiltonian. And if $\omega \approx 0$, it seems that the adiabatic condition fits.

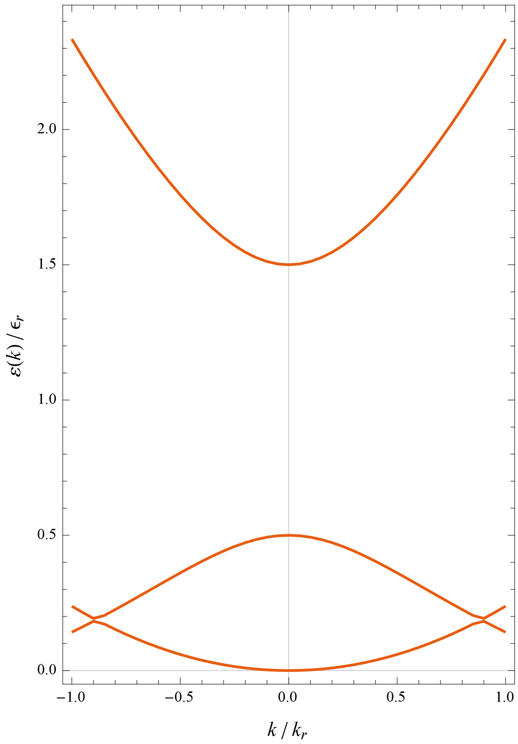
Firstly, we solve the static Hamiltonian. Namely, $V_2 = 0$, in which case the Hamiltonian reduces to

$$H(x,t) = \frac{P^2}{2m} + V_1 \cos(2k_r x)$$

The energy bands are plotted in Fig.3



(a) The lowest seven bands

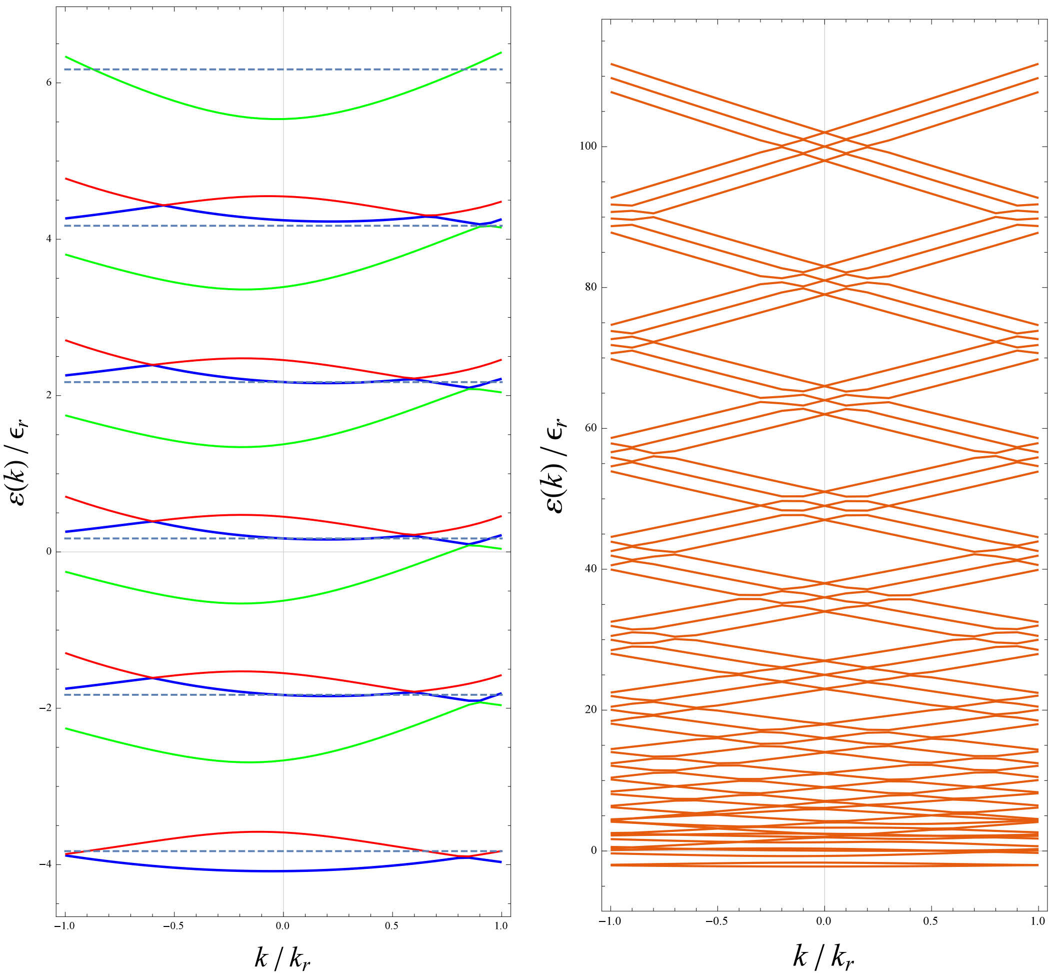


(b) The lowest three bands

Figure 3: Energy bands solved from static Hamiltonian, $V_1 = 1.0\epsilon_r$

Then we turn on driving, expecting to obtain a shifted spectrum of the original bands. Yet there's another thing to pay attention to, that is of how many bands are we concerning in all, and which are them. By this I mean, the magnitude of ω is only meaningful compared with the energy scale of the bands we are concerned with. To be specific, suppose we consider a time-dependent system with driving frequency ω fixed at $2\epsilon_r$.[3] If merely of the lowest three bands are we concerning, in which case the largest energy difference in the whole energy spectrum Δ_0 is less than $\hbar\omega$, then we will see a clear "shifting" quasienergy spectrum structure when we turn on driving (see Fig.4(a)). However, if our concern includes, to say, 20 bands in all, where $\hbar\omega$ is very small compared with the energy scale of the original bands of which we are concerned, then the quasienergy spectrum solved is a "shifted effect" mixed with the original energy bands of static potential, which would be a totally mess (see Fig.4(b)). Both cases are plotted in Fig.4

[3] Hereinafter we take the recoil energy ϵ_r as our energy unit, and set constants like $\hbar$ to be 1.



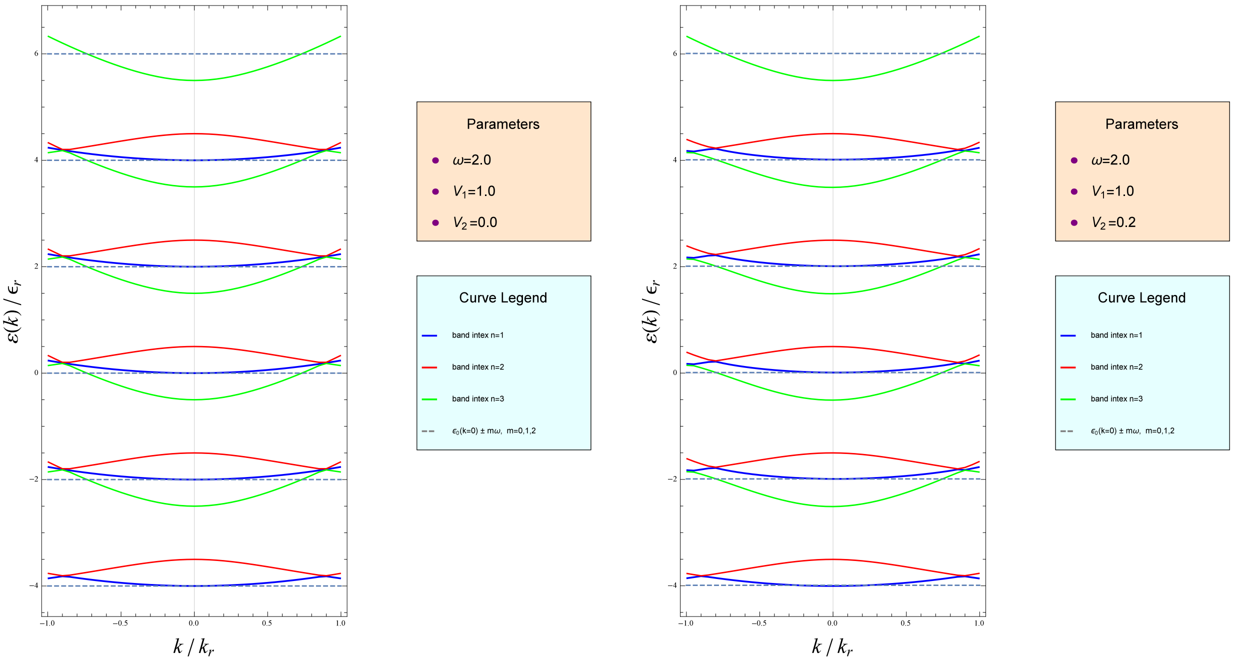
(a) Three bands in all

(b) Mand bands in all

Figure 4: Three bands vs. many bands considered in all. As we see above, the relative magnitude of ω vs. Δ_0 is crucial to the coming out of shifted structure of Floquet spectrum.

In terms of the analysis above, we concerned cases only $\hbar\omega \gg \Delta_0$. Specifically, we set $\omega = 2$, $V_1 = 1$, and increase V_2 from 0 to 1. We consider three bands in all, and truncate the Fourier expansion of $H(t)$ to order 2 ($m = 0, \pm1, \pm2$). Now our matrix is of dimension 15×15. And the quasienergy spectrum we get exhibit 15 bands in all.

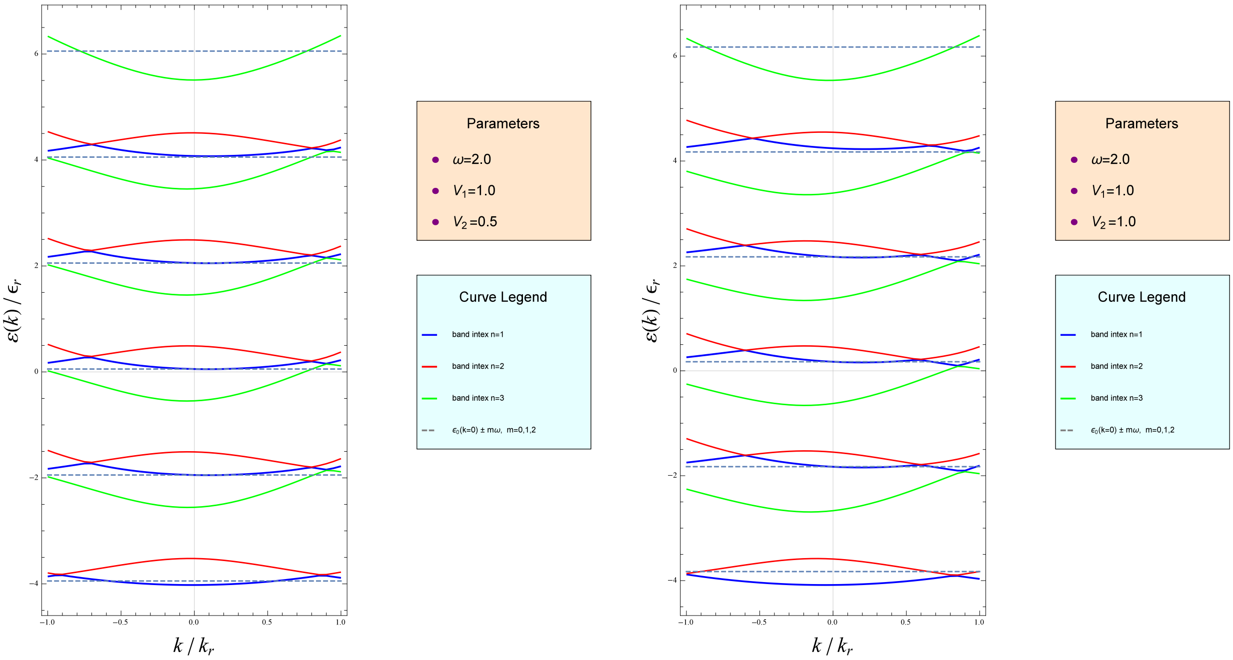
As we increase V_2 from 0 to 1. A strange thing happens. The bands structures tends from a symmetrical one to be not so symmetrical. The spectrum are shown in Fig.5 .



(a) $V_2 = 0$ (b) $V_2 = 0.2$

Figure 5: Floquet spectrum with $V_1 = 1.0, \omega = 2.0$ fixed, V_2 increased from zero to nonzero. The first one is quite symmetrical, while the second a little bit unsymmetrical.

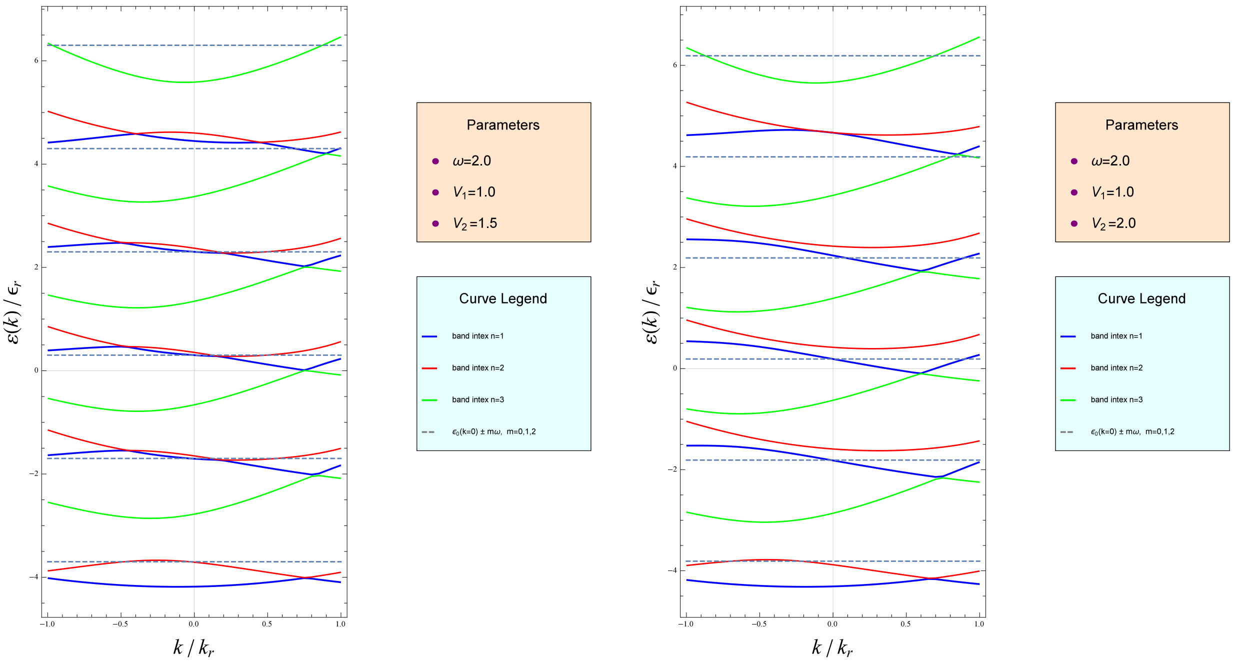
This is quite odd. I have not yet figured out why.



(a) $V_2 = 0.5$

(b) $V_2 = 1.0$

Figure 6: Floquet spectrum with $V_1 = 1.0, \omega = 2.0$ fixed, V_2 =0.5, 1.0



(a) $V_2 = 1.5$

(b) $V_2 = 2.0$

Figure 7: Floquet spectrum with $V_1 = 1.0, \omega = 2.0$ fixed, V_2=1.5, 2. Unsymmetry becomes obvious.

A subtle thing in this model is that the V_2 parameter modulate not only the time-dependency strength, but also the space lattice form. So when we ”turn on” driving, not only do we add a time-dependent part, but also have this part changed the space structure of the lattice potential. To be contrast, we will consider another model, inspired by Jon H. Shirley's earliest work[2], where the time-dependent effect is only to have the original potential moving along $\hat{x}$-axis.

References

[1] Lei Wang, Matthias Troyer and Xi Dai, Phys. Rev. Lett. **111** 026802 (2013).

[2] Jon H. Shirley, Phys. Rev. **138**, B979 (1965).

[3] Thomas Bilitewski, Nigel R. Cooper, Phys. Rev. A **91** 033601 (2015).