Topological Floquet States in Two-Dimensional Shaking Triangle Optical Lattice

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I. TWO-DIMENSIONAL SHAKING OPTICAL LATTICE

The potential given by a general optical lattice performs like this

$$\mathcal{V}(r) = V[\cos(\vec{k}\cdot\vec{r})]^2$$

is equivalent to $V\cos(2\vec{k}\cdot\vec{r})$ by a phase shift about the reference frame.

In the following, I choose the later form to clarify my work, that potential $\mathcal{V}(\vec{r})$ is in the form of $V\cos(2\vec{k}\cdot\vec{r})$.

I A . Setup

Consider a two-dimensional triangle optical lattice:

$$\mathcal{H} = T + \mathcal{V} = \frac{p^2}{2m} + V_1\cos(2\vec{k_1}\cdot\vec{r}) + V_2\cos(2\vec{k_2}\cdot\vec{r})$$

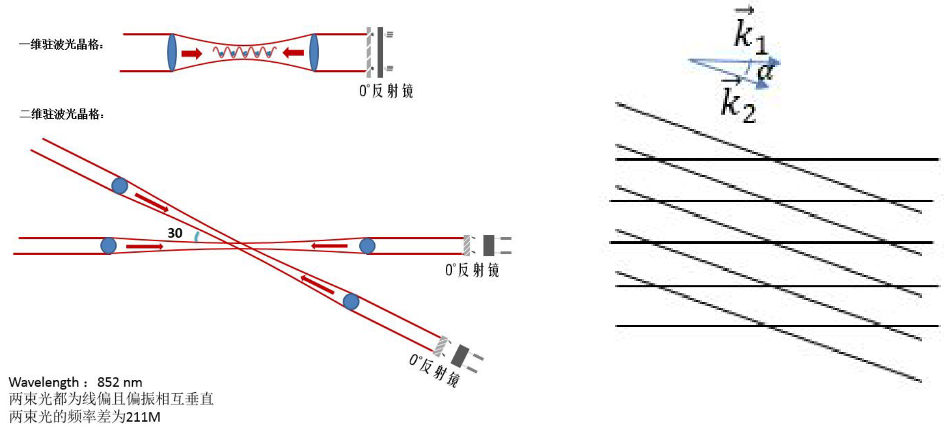
Here $\hat{r}_1 = \hat{e}_x$, $\hat{r}_2 = \cos(30^\circ)\hat{e}_x - \sin(30^\circ)\hat{e}_y$ represents the directions of two laser beams generating this two-dimensional triangle optical lattics.

Shaking was added along the directions of these two beams, giving a time-dependent Hamiltonian that

$$\begin{aligned} h(t) &= T + V(t) \\ &= \frac{p^2}{2m} - V_1\cos(2\vec{k_1}\cdot\vec{r} + f_1\cos(\omega t)) - V_2\cos(2\vec{k_2}\cdot\vec{r} + f_2\cos(\omega t + \alpha)) \end{aligned} \tag{1}$$

The general setup of this model is shown in Fig. 1 (Fig. 1a and Fig. 1b). A two-dimensional triangle lattice is formed with two lattices in two directions with an intersection angle of 30°. Each lattice is produced by two counterpropagating lasers. Periodically modulating the relative phase between the two counterpropagating laser beams in each lattice, one can produce the shaking lattice describe by a time-dependent Hamiltonian in Eq. (1).

To simplify, we consider only a simple case $f_1 = f_2$, $\omega_1 = \omega_2$ in the following, while generally α does not equal to zero. To fulfill this, one can modulate the time-dependent term in the same way[1] along the two directions of the lattices.



(a) Setup of the two-dimensional triangle optical lattice with 30° intersection angle

(b) Sketch of two laser beams creating the 2d optical lattice.

Figure 1: Experimental Setup

I B . Shaking Lattice in Co-moving Frame of Reference

Noted that, in the time-dependent Hamiltonian above, shaking effect is mixed with the space-periodic lattice effect, it is not a easy thing to solve such a time-dependent Hamiltonian directly. Meanwhile the intrinsic physics could not be obviously foreseen through it. However, by transferring to the co-moving frame of reference, $\vec{r} \to \vec{r} + \vec{\xi}(t)$, the Hamiltonian acquires a time-dependent vector potential term as $\dot{\vec{\xi}}(t) \cdot \vec{p}$. Another advantage is that the static lattice term is separated with shaking term, providing a way that we could make use of the clear results of static optical lattice straightforward. Thus we transfer the time-dependent Hamiltonian into co-moving frame of reference

[1]By the same way, it means the same shaking amplitude and frequency.

first.

Schrödinger equation reads

$$h(t) - i\hbar\partial_t)\Psi \;=\; 0 \tag{2}$$

Do translational transformation to it, we have

$$T_{\vec{\xi}}(t)(h(t) - i\partial_t)T^{\dagger}_{\vec{\xi}}(t)T_{\vec{\xi}} = \Big[T_{\vec{\xi}}(t)(h(t) - i\partial_t)T^{\dagger}_{\vec{\xi}}(t) - i\hbar\partial_t\Big]T_{\vec{\xi}}\Psi = 0 \tag{3}$$

where

$$T_{\vec{\xi}} = e^{-\frac{i}{\hbar}\vec{\xi}\cdot\vec{p}},\ \vec{\xi}(t) = \frac{f_1}{2k_1}\cos(\omega t)\hat{r}_1 + \frac{f_2}{2k_2}\cos(\omega t + \alpha)\hat{r}_2$$

Thus in co-moving frame of reference, Hamiltonian reads

$$\begin{aligned}
H(t) &= T_{\vec{\xi}}(t)(\mathscr{H}(t) - \mathrm{i}\partial_t)T^{\dagger}_{\vec{\xi}}(t) \\
&= \frac{\vec{p}^2}{2m} + V_0\Big(\cos^2(\vec{k}_1\cdot\vec{r}) + \cos^2(\vec{k}_2\cdot\vec{r})\Big) + \dot{\vec{\xi}}(t)\cdot\vec{p} \\
&= \frac{\vec{p}^2}{2m} + V_0\Big(\cos^2(\vec{k}_1\cdot\vec{r}) + \cos^2(\vec{k}_2\cdot\vec{r})\Big) - f_1\omega\sin(\omega t)\vec{k}_1 - f_2\omega\sin(\omega t+\alpha)\vec{k}_2
\end{aligned}$$

This seperates the static part and the moving part. And it could be written as

$$H(t) = \sum_{n=0,\pm1} H_n e^{\mathrm{i}\omega t}$$

$$\begin{aligned}
H(t) &= T_{\vec{\xi}}(t)(h(t) - i\partial_t)T^{\dagger}_{\vec{\xi}}(t) \\
&= \frac{p^2}{2m} - V_1\cos(2\vec{k}_1\cdot\vec{r}) - V_2\cos(2\vec{k}_2\cdot\vec{r}) + \dot{\vec{\xi}}(t)\cdot\vec{p} \\
&= \frac{p^2}{2m} - V_1\cos(2\vec{k}_1\cdot\vec{r}) - V_2\cos(2\vec{k}_2\cdot\vec{r}) \\
&\quad -(\frac{\omega f_1}{2k_1}\sin(\omega t) + \frac{\omega f_2}{2k_2}\sin(\omega t+\alpha)\cos(30^\circ))p_x \\
&\quad +\frac{\omega f_2}{2k_2}\sin(\omega t+\alpha)\sin(30^\circ)p_y \\
&= \frac{p^2}{2m} - V_1\cos(2\vec{k}_1\cdot\vec{r}) - V_2\cos(2\vec{k}_2\cdot\vec{r}) \\
&\quad +i\frac{\hbar\omega f}{2k_0}\left[\Big(\sin(\omega t) + \sin(\omega t+\alpha)\cos(30^\circ)\Big)\partial_x - \sin(\omega t+\alpha)\sin(30^\circ)\partial_y\right]
\end{aligned}$$

Writing Hamiltonian in the form of

$$H(t) \;=\; \sum_n H_n e^{in\omega t} \tag{4}$$

we obtain

$$\begin{aligned} H(t) =& \frac{p^2}{2m} - V_1\cos(2\vec{k}_1\cdot\vec{r}_1) - V_2\cos(2\vec{k}_2\cdot\vec{r}_2) \\ & + \mathrm{i}\frac{\hbar\omega f}{2k_0}\Big[\sin(\omega t)\partial_x + \sin(\omega t+\alpha)\big(\cos 30^\circ\partial_x - \sin 30^\circ\partial_y\big)\Big] \\ =& \frac{p^2}{2m} - V_1\cos(2\vec{k}_1\cdot\vec{r}_1) - V_2\cos(2\vec{k}_2\cdot\vec{r}_2) \\ & + \frac{\hbar\omega f}{4k_0}\Big[(1+\cos(30^\circ)e^{i\alpha})\partial_x - \sin(30^\circ)e^{i\alpha}\partial_y\Big]\cdot e^{i\omega t} \\ & - \frac{\hbar\omega f}{4k_0}\Big[(1+\cos(30^\circ)e^{-i\alpha})\partial_x - \sin(30^\circ)e^{-i\alpha}\partial_y\Big]\cdot e^{-i\omega t} \\ =& H_0 + H_1 e^{i\omega t} + H_{-1}e^{-i\omega t} \end{aligned} \tag{5}$$

with

$$H_0 \;=\; \frac{p^2}{2m} - V_1\cos(2\vec{k}_1\cdot\vec{r}_1) - V_2\cos(2\vec{k}_2\cdot\vec{r}_2) \tag{6a}$$

$$H_1 \;=\; \frac{\hbar\omega f}{4k_0}\Big[(1+\cos(30^\circ)e^{i\alpha})\partial_x - \sin(30^\circ)e^{i\alpha}\partial_y\Big] \tag{6b}$$

$$H_{-1} \;=\; H_1^\dagger = -\frac{\hbar\omega f}{4k_0}\Big[(1+\cos(30^\circ)e^{-i\alpha})\partial_x - \sin(30^\circ)e^{-i\alpha}\partial_y\Big] \tag{6c}$$

I C . Floquet Theory and Effective Hamiltonian

Generally, it might be quite complicated to study the evolution of a system governed by a time-dependent Hamiltonian $H(t)$. However, for a system where the Hamiltonian depends periodically on time, i.e. $H(t+T) = H(t)$ for some driving period T, Floquet theory provides a powerful frame work for analysis [21–23]. Floquet operator for a periodic time-dependent Hamiltonian is defined as

$$\hat{F} = \hat{U}(T_i+T, T_i) = \hat{T} = \hat{\mathcal{T}}\exp\left(-\mathrm{i}\int_{T_i}^{T_i+T}\hat{H}(t)\mathrm{d}t\right)$$

where $\hat{\mathcal{T}}$ denotes time order, and T_i is the initial time.

The eigen equation of Floquet operator $\hat{F}$ gives

$$\hat{F}|\varphi_n\rangle = e^{-\mathrm{i}\varepsilon_n T}|\varphi_n\rangle$$

with eigenvalue $-\pi/T < \varepsilon_n < \pi/T$ and $|\varphi_n\rangle$ the corresponding eigenstate for the n$^{\text{th}}$ quasi-energy band. With the quasi-energy bands spectrum of the Floquet operator, topological properties could be characterized by some topological invariants like winding numbers.

Beside numerically evaluating the Floquet operator directly, one efficient method is to introduce an time-independent Effective Hamiltonian H_{eff} via

$$\hat{F} = e^{-\mathrm{i}H_{\text{eff}}T}$$

Expanding $\hat{H}(t)$ as

$$\hat{H}(t) = \sum_{n=-\infty}^{\infty} \hat{H}_n(t)e^{i\omega t}$$

H_{eff} can be deduced as[2]

$$\hat{H}_{\text{eff}} = \hat{H}_0 + \sum_{n=1}^{\infty}\left(\frac{[\hat{H}_n, \hat{H}_{-n}]}{n\omega} - \frac{[\hat{H}_n, \hat{H}_0]}{e^{-2\pi n i\alpha}n\omega} + \frac{[\hat{H}_{-n}, \hat{H}_0]}{e^{2\pi n i\alpha}n\omega}\right)$$

[2]For details see Appendix A.

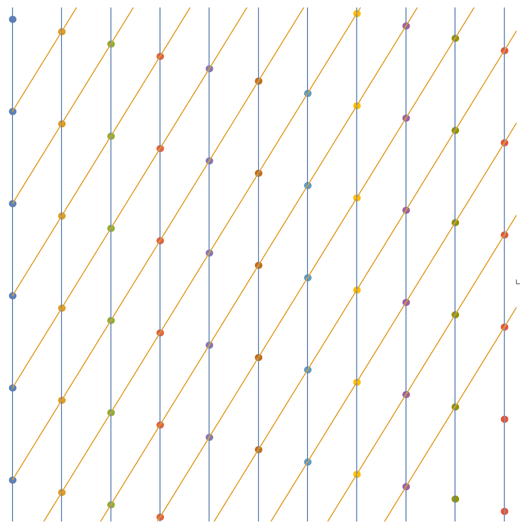
II. STATIC LATTICE SOLVING

The Hamiltonian of static two-dimensional triangle optical lattice with intersection angle of 30° is

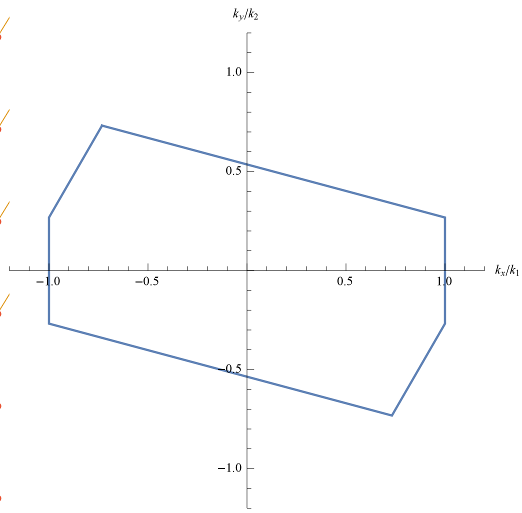
$$H = \frac{p^2}{2m} + V(\vec{r})$$

$$V(\vec{r}) = V_1 \cos(2\vec{k}_1 \cdot \vec{r}) + V_2 \cos(2\vec{k}_2 \cdot \vec{r})$$

Structure of this lattice in real space is shown in Fig. 2a. To help studying the physics in quasi-momentum space, we plot the First Brillioun-Zone of the reciprocal lattice also, which is shown in Fig. 2b. In Section III A we will construct tight bounding approximation model of this optical lattice, where we evaluate hopping terms between the Nearest Neighbour sites in the lattice. To see it straightforward, hopping sketch around one specific site has been plotted in Fig. 3.



(a) Lattice structure of two-dimensional triangle optical lattice with 30° intersection angle in real space



(b) First Brillouin Zone of the Lattice

Figure 2: Lattice Structure

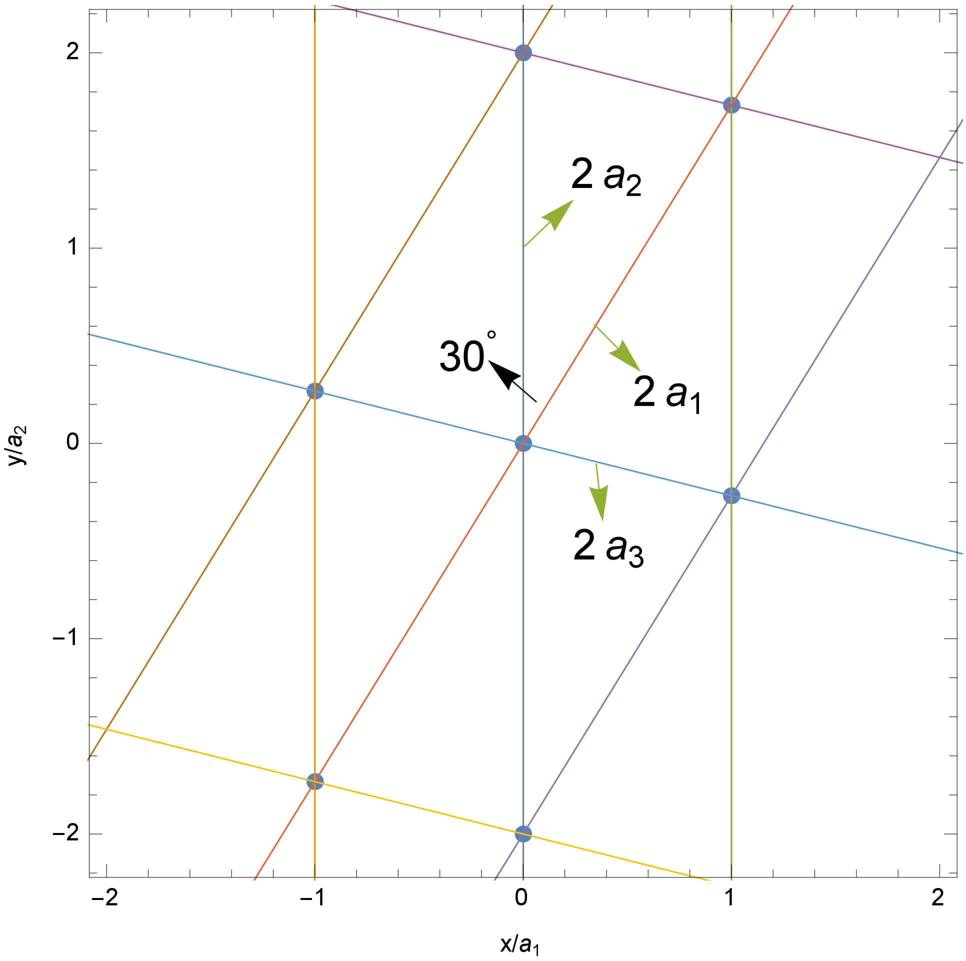


Figure 3: Tight bounding model hopping scheme

Here $2a_1$ and $2a_2$ are the lattice constants we consider to make lattice cell of in real space, with $a_1 = \pi/k_1$, $a_2 = \pi/k_2$. While a_3 is the nearest sites distance in real space in our frame, the value of which could be calculated from a_1, a_2 and the intersection angle, which is, in our model, 30°. For the simple case that $k_1 = k_2 = k_0$, the lattice constants are $a_1 = a_2 = \pi/k_0$ and $a_3 = (\sqrt{6} - \sqrt{2})\pi/k_0$.

The general way to solving such a lattice for energy bands and wave functions is to solving the time-independent Schrödinger equation $H|\psi\rangle = \varepsilon|\psi\rangle$ in a Bloch bases. This is an eigen-problem which could be numerically solved by cutting the Hamiltonian to the subspace expanded by a set of finite bases, i.e., numerically diagonalizing the cut-off Hamiltonian matrix. Energy bands are then acquired, as well as Bloch wave functions. Thus the Wannier functions, localized wave functions in real space, could be obtained by a Fourier transform of the Bloch Functions with respect to the quasi-momentum $\vec{k}$ into real space. After that, Tight-Bounding approximation model could be establish using the results above, which would be studied in Section III A .

II A . Eigen Problem Solving under Bloch Bases

Eigen equation reads

$$H|\psi_q^{(n)}(\vec{r})\rangle = \varepsilon^{(n)}(\vec{k})|\psi_q^{(n)}(\vec{r})\rangle$$

with

$$H = \frac{p^2}{2m} + V_1 \cos(2\vec{k}_1 \cdot \vec{r}_1) + V_2 \cos(2\vec{k}_2 \cdot \vec{r}_2)$$

Here superscript n denotes the n$^{\text{th}}$ energy band, and $\vec{k}$ refers to the quasi-momentum.

Expand the Bloch wave function with quasi-momentum $\vec{q}$ on the bases of plane wave functions (Bloch bases), we get

$$\psi_q(\vec{r}) = \sum_{l_1,l_2} C_{l_1,l_2}(\vec{q}) e^{i(2l_1\vec{k}_1 + 2l_2\vec{k}_2 + \vec{q})\cdot\vec{r}}$$

The matrix elements of Hamiltonian of quasi-momentum $\vec{q}$ under this plane wave bases $\{e^{i(2l_1\vec{k}_1+2l_2\vec{k}_2+\vec{q})\cdot\vec{r}}\}$ could be easily figured out exactly, that

$$
\begin{aligned}
&H_{l_1'l_2',l_1l_2}(\vec{q})\\
&=\langle l_1'l_2'|H|l_1l_2\rangle\\
&=(\frac{1}{2\pi})^2\int e^{-i(2l_1\vec{k}_1+2l_2\vec{k}_2+\vec{q})\cdot\vec{r}}\Big(-\frac{\hbar^2}{2m}(\partial_x^2+\partial_y^2)\Big)e^{i(2l_1\vec{k}_1+2l_2\vec{k}_2+\vec{q})\cdot\vec{r}}\mathrm{d}^2r\\
&\quad+(\frac{1}{2\pi})^2\int e^{-i(2l_1\vec{k}_1+2l_2\vec{k}_2+\vec{q})\cdot\vec{r}}V_1\cos(2\vec{k}_1\cdot\vec{r}_1)e^{i(2l_1\vec{k}_1+2l_2\vec{k}_2+\vec{q})\cdot\vec{r}}\mathrm{d}^2r\\
&\quad+(\frac{1}{2\pi})^2\int e^{-i(2l_1\vec{k}_1+2l_2\vec{k}_2+\vec{q})\cdot\vec{r}}V_2\cos(2\vec{k}_2\cdot\vec{r}_2)e^{i(2l_1\vec{k}_1+2l_2\vec{k}_2+\vec{q})\cdot\vec{r}}\mathrm{d}^2r\\
&=\frac{\hbar^2}{2m}(2l_1\vec{k}_1+2l_2\vec{k}_2+\vec{q})^2(\frac{1}{2\pi})^2\int e^{i[(l_1-l_1')2\vec{k}_1+(l_2-l_2')2\vec{k}_2]\cdot\vec{r}}\mathrm{d}^2r\\
&\quad+\frac{1}{2}V_1\Big[(\frac{1}{2\pi})^2\int e^{i[(l_1-l_1'-1)2\vec{k}_1+(l_2-l_2')2\vec{k}_2]\cdot\vec{r}}\mathrm{d}^2r\\
&\qquad+(\frac{1}{2\pi})^2\int e^{i[(l_1-l_1'+1)2\vec{k}_1+(l_2-l_2')2\vec{k}_2]\cdot\vec{r}}\mathrm{d}^2r\Big]\\
&\quad+\frac{1}{2}V_2\Big[(\frac{1}{2\pi})^2\int e^{i[(l_1-l_1')2\vec{k}_1+(l_2-l_2'-1)2\vec{k}_2]\cdot\vec{r}}\mathrm{d}^2r\\
&\qquad+(\frac{1}{2\pi})^2\int e^{i[(l_1-l_1')2\vec{k}_1+(l_2-l_2'+1)2\vec{k}_2]\cdot\vec{r}}\mathrm{d}^2r\Big]\\
&=\frac{\hbar^2}{2m}(2l_1\vec{k}_1+2l_2\vec{k}_2+\vec{q})^2\delta_{l_1'l_1}\delta_{l_2'l_2}\\
&\quad+\frac{1}{2}V_1(\delta_{l_1,l_1'-1}\delta_{l_2,l_2'}+\delta_{l_1,l_1'+1}\delta_{l_2,l_2'})+\frac{1}{2}V_2(\delta_{l_1,l_1'}\delta_{l_2,l_2'-1}+\delta_{l_1,l_1'}\delta_{l_2,l_2'+1})
\end{aligned}
$$

With the help of this exact expression of each elements in the (cut-off) Hamiltonian matrix, a numerical method of diagonalization is approached to this cut-off matrix and we obtain the energy bands of this two-dimensional triangle lattice with the corresponding Bloch wave functions.

Some energy bands with different potential depth are plotted in Fig. 4 - Fig. 7. Here we take $\epsilon_0=\hbar^2k_0^2/2m$ as energy unit. To examine whether the results are reasonable, we calculate the energy bands for a very small potential depth, that $V_1=V_2=0.05\epsilon_0$, approximate to zero. Theoretically, the asymptotic behavior when potential depth close to zero tends to be like a free particle, which means the energy gap tends to disappear that next energy bands getting touched, and the whole energy spectrum is a little bit varied from a quadratic form. Our numerical results show this character as expected, which are plotted in Fig. 8 . It shows from Fig. 8a that four lowest energy bands of potential depth $V_1 = V_2 = 0.05\epsilon_0$ almost touch together, which is quite distinguished from the large potential depth cases where energy bands are separated by large gaps, referring to Fig. 4a ($V_1 = V_2 = 1.0\epsilon_0$), Fig. 5a ($V_1 = V_2 = 2.0\epsilon_0$), Fig. 6a ($V_1 = V_2 = 8.0\epsilon_0$), and Fig. 7a ($V_1 = V_2 = 16.0\epsilon_0$). Besides, Fig. 8d is a top view to the lowest energy band, from which we see clearly what the first Brillioun Zone looks like. It shows the the same contour as we have showed in Fig. 2b.

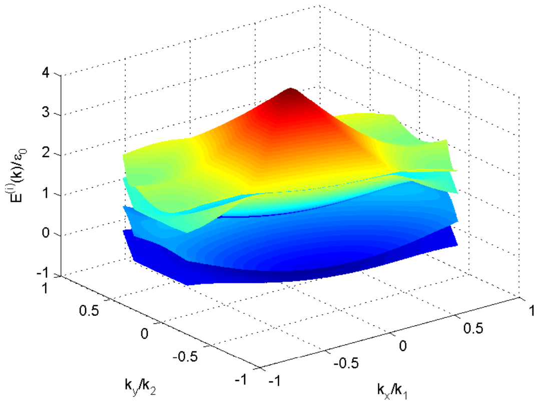
II B . Wannier Functions

Analytically, Wannier functions can be obtained from the Bloch functions by a Fourier transform of the Bloch functions into real space with respect to the quasi-momentum $\vec{k}$.

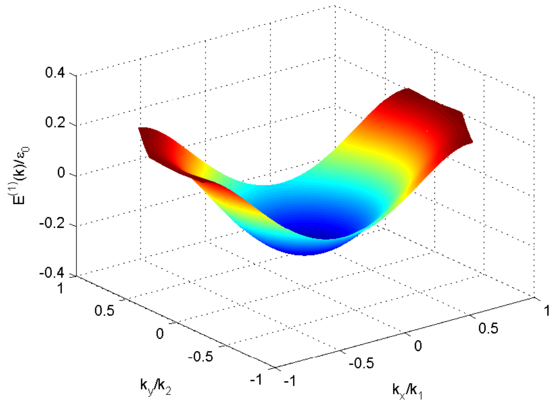
$$w_n(\vec{R}_m, \vec{r}) = w_n(\vec{r} - \vec{R}_m) = \frac{1}{\sqrt{N}} \sum_{\vec{k} \in BZ} e^{-i\vec{k}\cdot\vec{R}_m} \psi_{n\vec{k}}(\vec{r})$$

Here $\psi_{n\vec{k}}$ is the bloch function with quasi-momentum of $\vec{k}$, and $\vec{R}_m$ refers to the lattice sites.

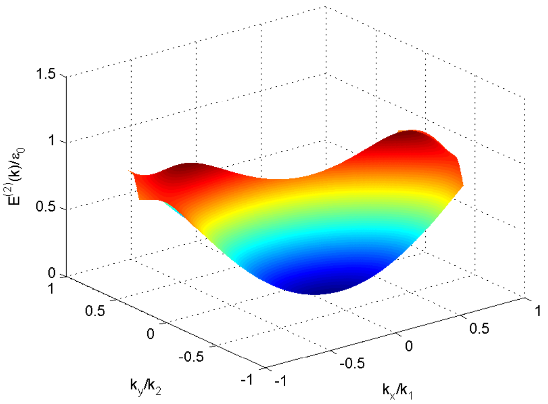
We numerically approach this method to get Wannier functions localized around the origin of the real space by quantizing the following Section II B



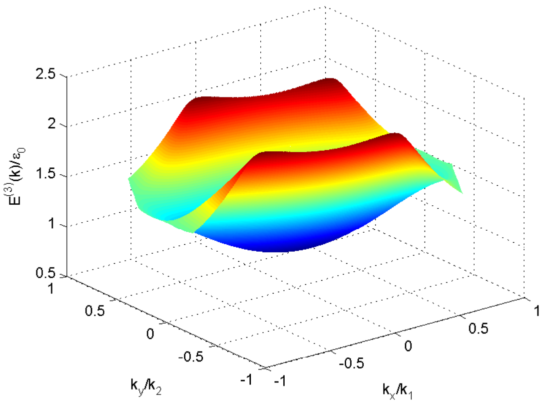
(a) The lowest four bands



(b) Band 1



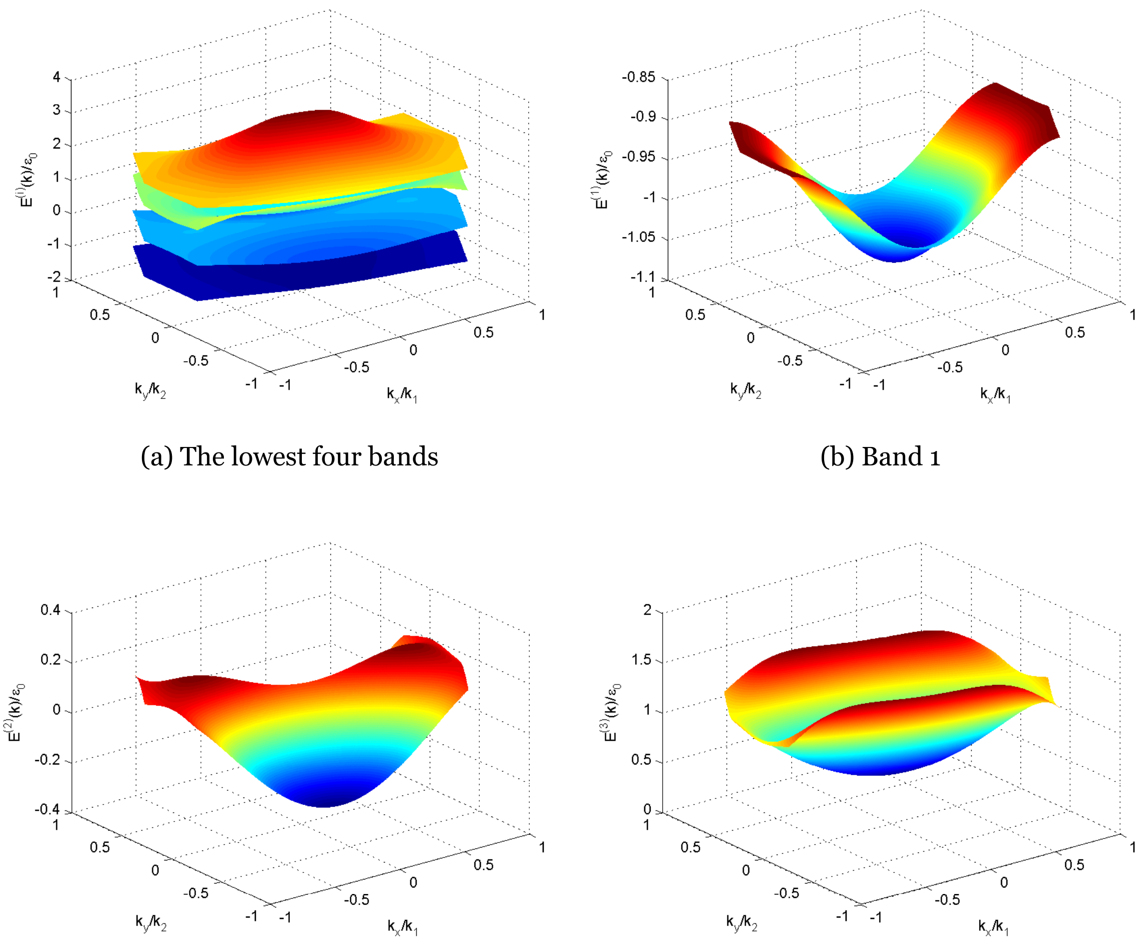
(c) Band 2



(d) Band 3

Figure 4: Energy bands plotting within the First Brilllioun Zone. Potential depth $V_1 = V_2 = 1.0$

Taking $\epsilon_0 = \hbar^2 k_0^2/2m$ as energy unit, similarly hereinafter.



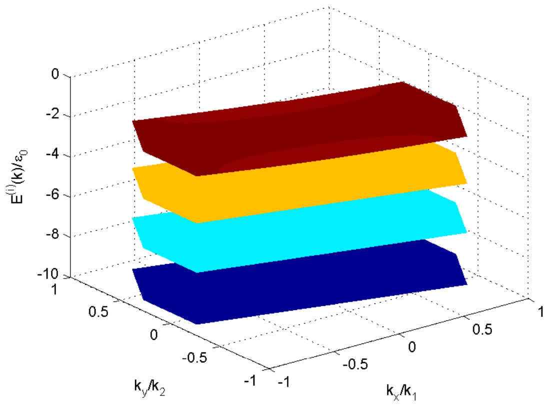
(a) The lowest four bands

(b) Band 1

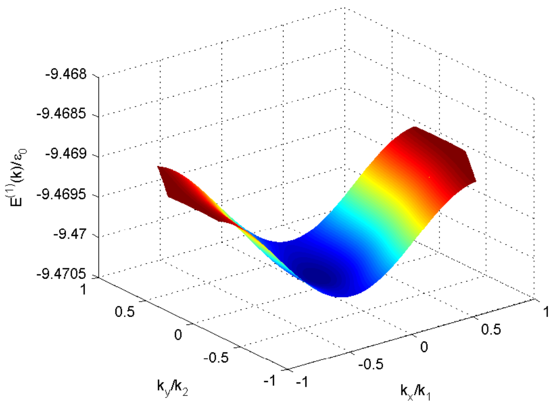
(c) Band 2

(d) Band 3

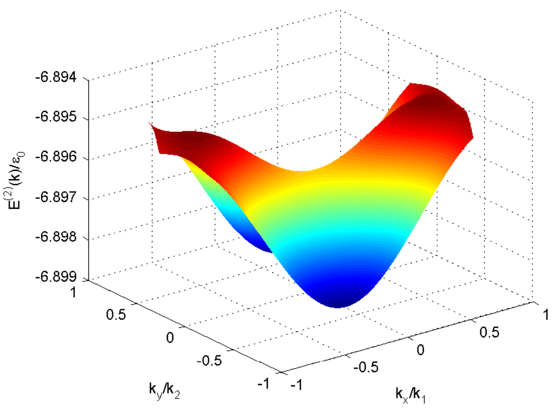
Figure 5: Energy bands with potential depth $V_1 = V_2 = 2.0$



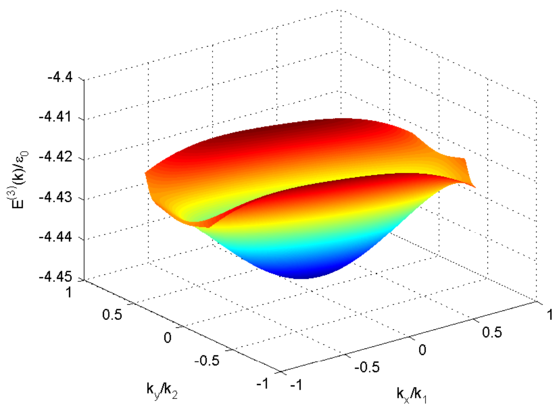
(a) The lowest four bands



(b) Band 1

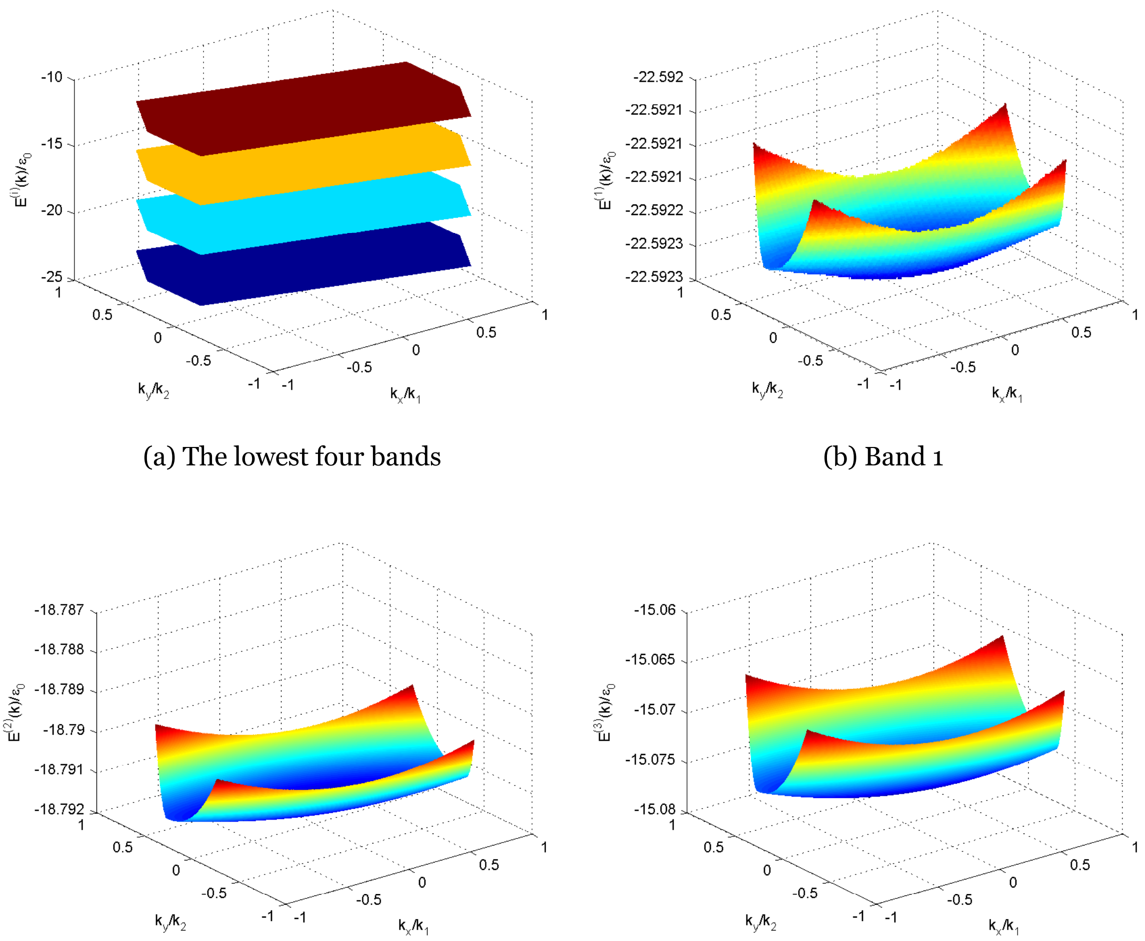


(c) Band 2



(d) Band 3

Figure 6: Energy bands with potential depth $V_1 = V_2 = 8.0$



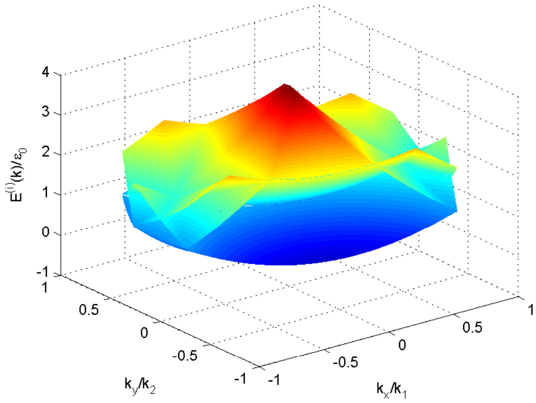
(a) The lowest four bands

(b) Band 1

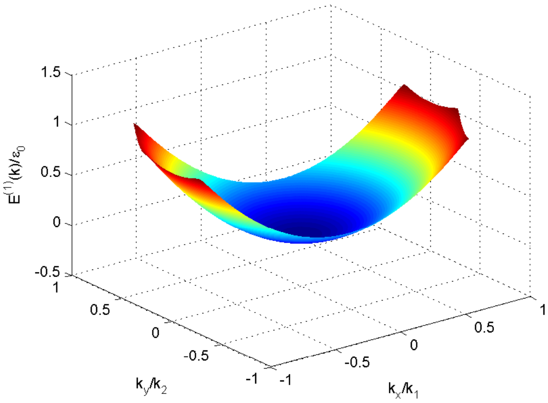
(c) Band 2

(d) Band 3

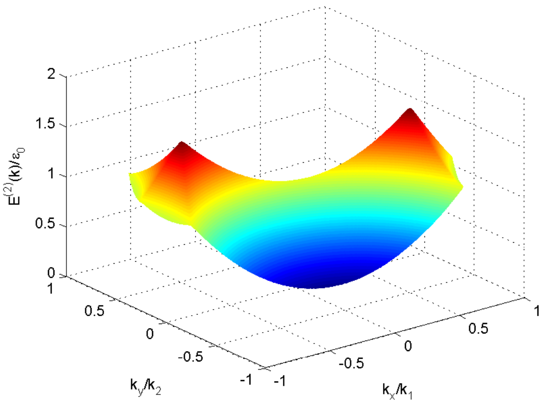
Figure 7: Energy bands with potential depth $V_1 = V_2 = 16.0$



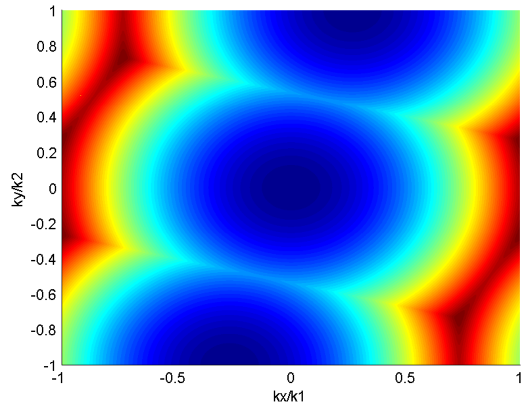
(a) The lowest four bands



(b) Band 1



(c) Band 2



(d) Top View for the lowest band

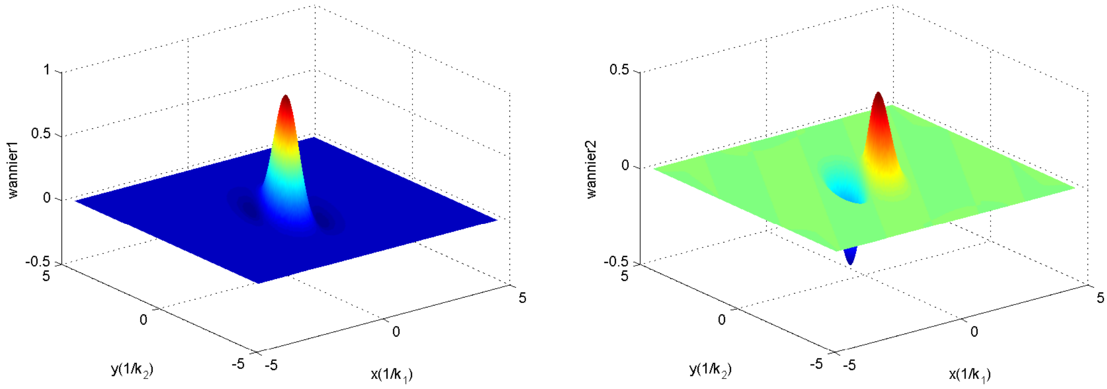
Figure 8: Energy bands with potential depth $V_1 = V_2 = 0.05$

and numerically evaluate it.

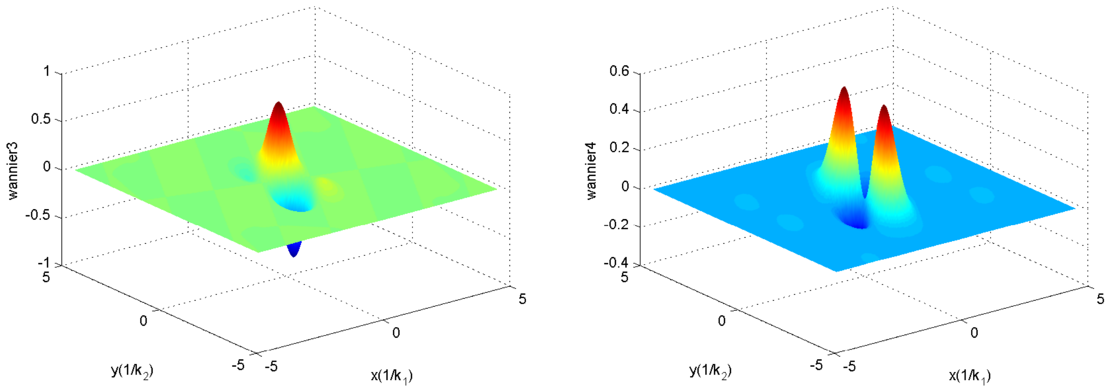
$$
\begin{aligned}
w_n(\vec{r}) = w_n(0,\vec{r}) &= \frac{1}{\sqrt{N}} \sum_{\vec{q}\in BZ} \psi_{n\vec{q}}(\vec{r}) \\
&= \frac{1}{\sqrt{N}} \sum_{\vec{q}\in BZ} \sum_{l_1,l_2} C_{l_1,l_2}(\vec{q}) e^{i(2l_1\vec{k}_1+2l_2\vec{k}_2+\vec{q})\cdot\vec{r}}
\end{aligned} \tag{7}
$$

Here plot some Wannier Functions in Fig. 9 . In these figures, we choose the potential depth to be $V_1 = V_2 = 8.0\epsilon_0$, and plotting the Wannier functions of the lowest four bands. The square of modulus of the Wannier functions for the lowest two bands are also plotted in Fig. 10.

Figure 9 shows that the Wannier functions are well localized in real space, and tends to disappear with a rapid damped oscillation around zero when going away from the origin.

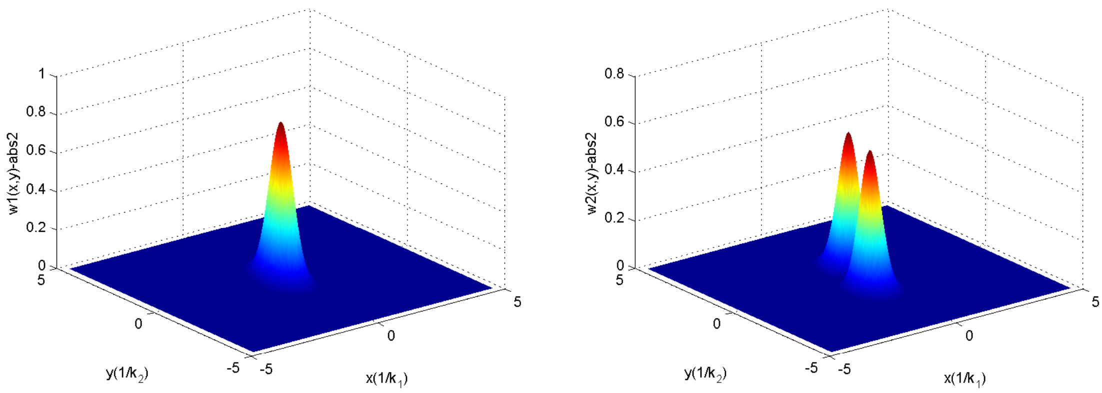


(a) Wannier function for the 1^{st} band $w_1(x, y)$ (b) Wannier function for the 2^{nd} band $w_2(x, y)$



(c) Wannier function for the 3^{th} band $w_3(x, y)$ (d) Wannier function for the 4^{th} band $w_4(x, y)$

Figure 9: Wannier functions with potential detph $V_1 = V_2 = 8.0$



(a) Square Modulus of $w_1(x, y)$ (b) Square Modulus of $w_2(x, y)$

Figure 10: Square modulus of the Wannier functions in Fig. 9

III. EFFECTIVE HAMILTONIAN FOR TIGHT BOUNDING APPROXIMATION MODEL

III A . Tight Bounding Approximation

Second quantized Hamiltonian under Wannier bases reads

$$\hat{H}(t) = \int \mathrm{d}^2 r \hat{\Psi}^\dagger(\vec{r}) H(t) \hat{\Psi}(\vec{r}) = \sum_{\substack{\lambda'\lambda \\ \vec{m}\vec{n}}} H_{\lambda'\lambda}^{(\vec{m,n})}(t) \hat{a}^\dagger_{\lambda'\vec{m}} \hat{a}_{\lambda\vec{n}}$$

with

$$\hat{\Psi}(\vec{r}) = \sum_{\lambda\vec{n}} \hat{a}_{\lambda\vec{n}} w_\lambda(\vec{r} - \vec{R}_n)$$

$$H_{\lambda'\lambda}^{(\vec{m,n})}(t) = \int \mathrm{d}^2 r w^*_{\lambda'}(\vec{r} - \vec{R}_m) H(t) w_\lambda(\vec{r} - \vec{R}_n)$$

and $\hat{a}^\dagger$ and $\hat{a}$ are the creative and annihilative operators. For translational symmetry possessed by the lattice, $H_{\lambda'\lambda}^{(\vec{m,n})}(t) = H_{\lambda'\lambda}^{(\vec{m}-\vec{n})}(t)$ depends only on $\vec{m} - \vec{n}$.

Take Fourier transform with respect to $\vec{r}$,

$$\hat{a}^\dagger_{\lambda'\vec{m}} = \frac{1}{\sqrt{L}} \sum_{\vec{k}} e^{-i\vec{k}\cdot\vec{R}_m} \hat{a}^\dagger_{\lambda'\vec{k}} \ , \quad \hat{a}_{\lambda\vec{n}} = \frac{1}{\sqrt{L}} \sum_{\vec{k}} e^{i\vec{k}\cdot\vec{R}_n} \hat{a}_{\lambda\vec{k}}$$

Thus Hamiltonian in the quasi-momentum space, with $\vec{\delta} = \vec{R}_m - \vec{R}_n$, writes

$$\hat{H}(t) = \sum_{\vec{k}} \hat{H}_{\lambda'\lambda}(\vec{k}) \ , \quad \hat{H}_{\lambda'\lambda}(\vec{k}) = \sum_{\lambda'\lambda,\vec{\delta}} e^{i\vec{k}\cdot\vec{\delta}} H^{\vec{\delta}}_{\lambda'\lambda}(t) \hat{a}^\dagger_{\lambda'\vec{k}} \hat{a}_{\lambda\vec{k}}$$

Cutting off the Hamiltonian within the subspace expanded by the lowest two bands (denoted as s band and p band), it becomes

$$\hat{H}(t) = \sum_{\vec{k}} (\hat{a}^\dagger_{s\vec{k}} \ \hat{a}^\dagger_{p\vec{k}}) H(\vec{k}, t) \begin{pmatrix} \hat{a}_{s\vec{k}} \\ \hat{a}_{p\vec{k}} \end{pmatrix}$$

and

$$H(\vec{k},t) = H_0(\vec{k}) + H_1(\vec{k})e^{i\omega t} + H_{-1}(\vec{k})e^{-i\omega t}$$

Tight bounding approximation considers merely the nearest sites hopping. So $\vec{\delta} = \vec{R}_m - \vec{R}_n = 0, or\vec{\gamma}_i$, where$\vec{\gamma}_i$ is the crystal lattice vectors or nearest sites vectors. Thus we could write $H(\vec{k})$ as

$$H_0(\vec{k}) = \begin{bmatrix} H_{11}^{(0)} & H_{12}^{(0)} \\ H_{21}^{(0)} & H_{22}^{(0)} \end{bmatrix} \quad H_1(\vec{k}) = \begin{bmatrix} H_{11}^{(1)} & H_{12}^{(1)} \\ H_{21}^{(1)} & H_{22}^{(1)} \end{bmatrix} \quad H_{-1}(\vec{k}) = \begin{bmatrix} H_{11}^{(-1)} & H_{12}^{(-1)} \\ H_{21}^{(-1)} & H_{22}^{(-1)} \end{bmatrix}$$

where

$$H_{\lambda'\lambda}^{(i)} = \langle w_{\lambda'}(\vec{r})|H_i|w_\lambda(\vec{r})\rangle + \sum_{\text{n.n}} e^{i\vec{k}\cdot\vec{R}_m}\langle w_{\lambda'}(\vec{r}-\vec{R}_m)|H_i|w_\lambda(\vec{r})\rangle$$

Here n.n means that summation includes only the nearest sites.

Specifically,

$$\begin{aligned}
H_{11}^{(0)} &= \epsilon_s + 2\Big(t_s^{(1)}\cos(2\vec{k}\cdot\vec{a}_1) + t_s^{(2)}\cos(2\vec{k}\cdot\vec{a}_2) + t_s^{(3)}\cos(2\vec{k}\cdot\vec{a}_3)\Big) \\
H_{22}^{(0)} &= \epsilon_p + 2\Big(t_p^{(1)}\cos(2\vec{k}\cdot\vec{a}_1) + t_p^{(2)}\cos(2\vec{k}\cdot\vec{a}_2) + t_p^{(3)}\cos(2\vec{k}\cdot\vec{a}_3)\Big) \\
H_{12}^{(0)} &= \mathrm{i}\Big[t_{sp}^{(0)} + 2\Big(t_{sp}^{(1)}\sin(2\vec{k}\cdot\vec{a}_1) + t_{sp}^{(2)}\sin(2\vec{k}\cdot\vec{a}_2) + t_{sp}^{(3)}\sin(2\vec{k}\cdot\vec{a}_3)\Big)\Big] \\
H_{21}^{(0)} &= H_{12}^{(0)\dagger} \\
H_{11}^{(1)} &= \mathrm{i}2\Big(h_s^{(1)}\sin(2\vec{k}\cdot\vec{a}_1) + h_s^{(2)}\sin(2\vec{k}\cdot\vec{a}_2) + h_s^{(3)}\sin(2\vec{k}\cdot\vec{a}_3)\Big) \\
H_{22}^{(1)} &= \mathrm{i}2\Big(h_p^{(1)}\sin(2\vec{k}\cdot\vec{a}_1) + h_p^{(2)}\sin(2\vec{k}\cdot\vec{a}_2) + h_p^{(3)}\sin(2\vec{k}\cdot\vec{a}_3)\Big) \\
H_{12}^{(1)} &= h_{sp}^{(0)} + 2\Big[h_{sp}^{(1)}\cos(2\vec{k}\cdot\vec{a}_1) + h_{sp}^{(2)}\cos(2\vec{k}\cdot\vec{a}_2) + h_{sp}^{(3)}\cos(2\vec{k}\cdot\vec{a}_3)\Big] \\
H_{21}^{(1)} &= H_{12}^{(1)\dagger}
\end{aligned}$$

where the hopping integral are, exactly, as follow:

$$\begin{aligned}
\epsilon_s &= \langle s|H_0|s\rangle = -\frac{\hbar^2}{2m}\langle s|\partial_x^2+\partial_y^2|s\rangle + \langle s|V(\vec{r})|s\rangle \\
\epsilon_p &= \langle p|H_0|p\rangle = -\frac{\hbar^2}{2m}\langle p|\partial_x^2+\partial_y^2|p\rangle + \langle p|V(\vec{r})|p\rangle \\
t_s^{(j)} &= \langle s(j)|H_0|s\rangle = -\frac{\hbar^2}{2m}\langle s(i)|\partial_x^2+\partial_y^2|s\rangle + \langle s|V(\vec{r})|s\rangle \\
t_p^{j)} &= \langle p(j)|H_0|p\rangle = -\frac{\hbar^2}{2m}\langle p(i)|\partial_x^2+\partial_y^2|p\rangle + \langle p(i)|V(\vec{r})|p\rangle \\
t_{sp}^{(j)} &= \langle s(j)|H_0|p\rangle = -\frac{\hbar^2}{2m}\langle s(i)|\partial_x^2+\partial_y^2|p\rangle + \langle s(i)|V(\vec{r})|p\rangle \\
h_s^{(j)} &= \langle s(j)|H_1|s\rangle = \frac{\hbar\omega f}{4k_0}\Big[(1+\cos(30^\circ)e^{i\alpha})\langle s(i)|\partial_x|s\rangle + \sin(-30^\circ)e^{i\alpha}\langle s(i)|\partial_y|s\rangle\Big] \\
h_p^{(j)} &= \langle p(j)|H_1|p\rangle = \frac{\hbar\omega f}{4k_0}\Big[(1+\cos(30^\circ)e^{i\alpha})\langle p(i)|\partial_x|p\rangle + \sin(-30^\circ)e^{i\alpha}\langle p(i)|\partial_y|p\rangle\Big] \\
h_{sp}^{(j)} &= \langle s(j)|H_1|p\rangle = \frac{\hbar\omega f}{4k_0}\Big[(1+\cos(30^\circ)e^{i\alpha})\langle s(i)|\partial_x|p\rangle + \sin(-30^\circ)e^{i\alpha}\langle s(i)|\partial_y|p\rangle\Big]
\end{aligned}$$

Actually, due to the symmetry of the lattice, there is

$$\langle\lambda'(j)|\partial_x|\lambda\rangle = \cos 30^\circ\langle\lambda'(j)|\partial_x|\lambda\rangle + \sin(-30^\circ)\langle\lambda'(j)|\partial_y|\lambda\rangle$$

So

$$\begin{aligned}
h_{\lambda\lambda'}^{(j)} &= \frac{\hbar f\omega}{4k_0}(1+e^{\mathrm{i}\alpha})u_{\lambda\lambda'}^{(j)} \\
&= \bar{f}\omega(1+e^{\mathrm{i}\alpha})u_{\lambda\lambda'}^{(j)}
\end{aligned} \tag{10}$$

with

$$u_{\lambda\lambda'}^{(j)} = \langle\lambda'(j)|\partial_x|\lambda\rangle$$

and

$$\bar{f} = \frac{\hbar f}{4k_0}$$

Thus we obtain

$$H_0(\vec{k}) = \begin{pmatrix} E_s & iK \\ -iK & E_p \end{pmatrix} \quad H_1(\vec{k}) = \begin{pmatrix} \Lambda_s & \Omega \\ -\Omega & \Lambda_p \end{pmatrix} \quad H_{-1}(\vec{k}) = H_1(\vec{k})^\dagger = \begin{pmatrix} \Lambda_s^* & -\Omega^* \\ \Omega^* & \Lambda_p^* \end{pmatrix}$$

The elements of Hamiltonian matrix therefore could be written in a more compact form,

$$E_s = \epsilon_s + 2\sum_{j=1}^{3} t_s^{(j)} \cos(2\vec{k}\cdot\vec{a}_j) \tag{11a}$$

$$E_p = \epsilon_p + 2\sum_{j=1}^{3} t_p^{(j)} \cos(2\vec{k}\cdot\vec{a}_j) \tag{11b}$$

$$K = t_{sp}^{(0)} + 2\sum_{j=1}^{3} t_{sp}^{(j)} \sin(2\vec{k}\cdot\vec{a}_j) \tag{11c}$$

$$\Lambda_s = \mathrm{i}\bar{f}\omega(1+e^{\mathrm{i}\alpha})2\sum_{j=1}^{3} u_s^{(j)} \sin(2\vec{k}\cdot\vec{a}_j) \tag{11d}$$

$$\Lambda_p = \mathrm{i}\bar{f}\omega(1+e^{\mathrm{i}\alpha})2\sum_{j=1}^{3} u_p^{(j)} \sin(2\vec{k}\cdot\vec{a}_j) \tag{11e}$$

$$\Omega = \bar{f}\omega(1+e^{\mathrm{i}\alpha})\Big(u_{sp}^{(0)} + 2\sum_{j=1}^{3} u_{sp}^{(j)} \cos(2\vec{k}\cdot\vec{a}_j)\Big) \tag{11f}$$

where

$$\begin{aligned}
t_{\lambda'\lambda}^{(j)} &= \langle\lambda'(j)|H_0|\lambda\rangle \\
&= -\frac{\hbar^2}{2m}\langle\lambda'(j)|\partial_x^2+\partial_y^2|\lambda\rangle + \langle\lambda'(j)|V(\vec{r})|\lambda\rangle \\
u_{\lambda'\lambda}^{(j)} &= \langle\lambda'(j)|\partial_x|\lambda\rangle \\
\epsilon_s &= t_s^{(0)}\,, \quad \epsilon_p = t_p^{(0)}
\end{aligned}$$

From above we see, t is real, while h is complex. And t is related to the potential depth V_1, V_2, while h is related to the shaking frequency ω and shaking amplitude f . Thus for a given potential, $u_{\lambda'\lambda}^{(j)}$ possesses fixed real value, while $\Lambda_s(\vec{k})$, $\Lambda_p(\vec{k})$ and $\Omega(\vec{k})$ are proportional to $\bar{f}$ and $\bar{\omega}$, and related to the shaking phase-difference α . The real parts and imaginary parts of Λ_s, Λ_p, Ω are[3]

[3]Here $\Re$ and $\Im$ means the real part and imaginary part of a complex number, respectively.

$$\begin{aligned}\Re(\Lambda_s) &= -2\bar{f}\omega\sin\alpha\sum_{j=1}^{3}u_s^{(j)}\sin(2\vec{k}\cdot\vec{a}_j) && \text{(13a)}\\ \Im(\Lambda_s) &= 2\bar{f}\omega(1+\cos\alpha)\sum_{j=1}^{3}u_s^{(j)}\sin(2\vec{k}\cdot\vec{a}_j) && \text{(13b)}\end{aligned}$$

$$\begin{aligned}\Re(\Lambda_p) &= -2\bar{f}\omega\sin\alpha\sum_{j=1}^{3}u_p^{(j)}\sin(2\vec{k}\cdot\vec{a}_j) && \text{(14a)}\\ \Im(\Lambda_p) &= 2\bar{f}\omega(1+\cos\alpha)\sum_{j=1}^{3}u_p^{(j)}\sin(2\vec{k}\cdot\vec{a}_j) && \text{(14b)}\end{aligned}$$

$$\begin{aligned}\Re(\Omega) &= \bar{f}\omega(1+\cos\alpha)\Big(u_{sp}^{(0)}+\sum_{j=1}^{3}u_{sp}^{(j)}\cos(2\vec{k}\cdot\vec{a}_j)\Big) && \text{(15a)}\\ \Im(\Omega) &= \bar{f}\omega\sin\alpha\Big(u_{sp}^{(0)}+2\sum_{j=1}^{3}u_{sp}^{(j)}\cos(2\vec{k}\cdot\vec{a}_j)\Big) && \text{(15b)}\end{aligned}$$

III B . One-photon resonance Effective Hamiltonian

For one-photon resonance condition, do a unitary transformation to the Hamiltonian, that

$$\begin{aligned}\widetilde{H}(\vec{k},t) &= U_1(t)(H(\vec{k},t)-\mathrm{i}\partial_t)U_1^\dagger(t)\\ &= \sum_n \widetilde{H}_n e^{in\omega t} = \widetilde{H}_0+\sum_{n=\pm1,\pm2}\widetilde{H}_n e^{in\omega t}\end{aligned}$$

with

$$U_1(t)=\begin{pmatrix} e^{-i\omega t} & 0\\ 0 & 1\end{pmatrix}$$

We obtain

$$\widetilde{H}_0(\vec{k}) = \begin{pmatrix} E_s + \omega & \Omega \\ \Omega^* & E_p \end{pmatrix}$$

$$\widetilde{H}_1(\vec{k}) = \begin{pmatrix} \Lambda_s & 0 \\ -iK & \Lambda_p \end{pmatrix} \qquad \widetilde{H}_{-1}(\vec{k}) = \begin{pmatrix} \Lambda_s^* & iK \\ 0 & \Lambda_p^* \end{pmatrix} = H_1^\dagger$$

$$\widetilde{H}_2(\vec{k}) = \begin{pmatrix} 0 & 0 \\ -\Omega & 0 \end{pmatrix} \qquad \widetilde{H}_{-2}(\vec{k}) = \begin{pmatrix} 0 & -\Omega^* \\ 0 & 0 \end{pmatrix} = H_2^\dagger$$

Effective Hamiltonian is[4]

$$\hat{H}_{\text{eff}} = \hat{H}_0 + \sum_{n=1}^{\infty} \left(\frac{[\hat{H}_n, \hat{H}_{-n}]}{n\omega} - \frac{[\hat{H}_n, \hat{H}_0]}{e^{-2\pi n i\alpha} n\omega} + \frac{[\hat{H}_{-n}, \hat{H}_0]}{e^{2\pi n i\alpha} n\omega} \right)$$

Using the Pauli matrices[5] σ , the Effective Hamiltonian can be written as

$$\begin{aligned} H_{\text{eff}}(\vec{k}) &= \frac{1}{2}(E_s + \omega + E_p) \\ &\quad + \left(\frac{E_s + \omega - E_p}{2} - \frac{|\Omega|^2}{2\omega} \right) \sigma_z - \frac{K^2}{\omega} \\ &\quad + \left(\Re(\Omega) - \frac{1}{\omega}[\Im(\Lambda_s) - \Im(\Lambda_p)]K \right) \sigma_x \\ &\quad + \left(-\Im(\Omega) - \frac{1}{\omega}[\Re(\Lambda_s) - \Re(\Lambda_p)]K \right) \sigma_y \end{aligned}$$

Or,

$$\begin{aligned} \widetilde{H}_{\text{eff}}(\vec{k}) &= \varepsilon_0 + \vec{B}(\vec{k}) \cdot \vec{\sigma} \\ &= \varepsilon_0 + B_x(\vec{k})\sigma_x + B_y(\vec{k})\sigma_y + B_z(\vec{k})\sigma_z \end{aligned}$$

---

[4]See Appendix Appendix A

[5] The Pauli matrices are

$$\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$\sigma_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \sigma_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$or\,, \quad \sigma_\pm = \frac{1}{2}(\sigma_x \pm i\sigma_y)$$

where

$$
\begin{aligned}
\varepsilon_0 &= \frac{1}{2}(E_s + \omega + E_p) && \text{(16a)}\\
B_z(\vec{k}) &= \frac{E_s + \omega - E_p}{2} - \frac{1}{\omega}(\frac{|\Omega|^2 + K^2}{2}) && \text{(16b)}\\
B_x(\vec{k}) &= \Re(\Omega) + \frac{1}{\omega}\cdot[\Im(\Lambda_p) - \Im(\Lambda_s)]K && \text{(16c)}\\
B_y(\vec{k}) &= -\Im(\Omega) + \frac{1}{\omega}\cdot[\Re(\Lambda_p) - \Re(\Lambda_s)]K && \text{(16d)}
\end{aligned}
$$

III C . Two-photon resonance Effective Hamiltonian

For two-phton resonance, the unitary transformation matrix is

$$
U_2(t) = \begin{bmatrix} e^{-\mathrm{i}\omega t} & 0 \\ 0 & e^{\mathrm{i}\omega t} \end{bmatrix}
$$

and

$$
\begin{aligned}
\widetilde{H}(\vec{k}, t) &= U_2(t)(H(\vec{k}, t) - \mathrm{i}\partial_t)U_2^\dagger(t)\\
&= \sum_n \widetilde{H}_n e^{\mathrm{i}n\omega t} = \sum_{n=0,\pm1,\pm2,\pm3} \widetilde{H}_n e^{\mathrm{i}n\omega t}
\end{aligned}
$$

with

$$
\widetilde{H}_0(\vec{k}) = \begin{pmatrix} E_s + \omega & 0 \\ 0 & E_p + \omega \end{pmatrix}
$$

$$
\widetilde{H}_1(\vec{k}) = \begin{pmatrix} \Lambda_s & 0 \\ \Omega^* & \Lambda_p \end{pmatrix} \quad \widetilde{H}_{-1}(\vec{k}) = \begin{pmatrix} \Lambda_s^* & \Omega \\ 0 & \Lambda_p^* \end{pmatrix} = \widetilde{H}_1^\dagger
$$

$$
\widetilde{H}_2(\vec{k}) = \begin{pmatrix} 0 & 0 \\ -\mathrm{i}K & 0 \end{pmatrix} \quad \widetilde{H}_{-2}(\vec{k}) = \begin{pmatrix} 0 & \mathrm{i}K \\ 0 & 0 \end{pmatrix} = \widetilde{H}_2^\dagger
$$

$$
\widetilde{H}_3(\vec{k}) = \begin{pmatrix} 0 & 0 \\ -\Omega & 0 \end{pmatrix} \quad \widetilde{H}_{-3}(\vec{k}) = \begin{pmatrix} 0 & -\Omega^* \\ 0 & 0 \end{pmatrix} = \widetilde{H}_3^\dagger
$$

Thus the Effective Hamiltonian is

$$
\begin{aligned}
H_{\text{eff}} = \frac{E_s + E_p}{2} \quad & + \Big[\frac{E_s - E_p + 2\omega}{2} - \frac{1}{\omega}\left(\frac{4}{3}|\Omega|^2 + \frac{K^2}{2}\right)\Big]\sigma_z \\
& + \frac{1}{\omega}\Re\Big(\Omega(\Lambda_s - \Lambda_p)\Big)\sigma_x \\
& - \frac{1}{\omega}\Im\Big(\Omega(\Lambda_s - \Lambda_p)\Big)\sigma_y
\end{aligned}
$$

which can be written as

$$
\begin{aligned}
H_{\text{eff}} &= \varepsilon_0 + \vec{B}(\vec{k}) \cdot \vec{\sigma} \\
&= \varepsilon_0 + \vec{B}_z(\vec{k}) \cdot \vec{\sigma}_z + \vec{B}_x(\vec{k}) \cdot \vec{\sigma}_x + \vec{B}_y(\vec{k}) \cdot \vec{\sigma}_y
\end{aligned}
$$

with

$$\varepsilon_0 = \frac{E_s + E_p}{2} \tag{17a}$$

$$\vec{B}_z(\vec{k}) = \frac{E_s - E_p + 2\omega}{2} - \frac{1}{\omega}\left(\frac{4}{3}|\Omega|^2 + \frac{K^2}{2}\right) \tag{17b}$$

$$\vec{B}_x(\vec{k}) = \frac{1}{\omega}\Re\Big(\Omega(\Lambda_s - \Lambda_p)\Big) \tag{17c}$$

$$\vec{B}_y(\vec{k}) = -\frac{1}{\omega}\Im\Big(\Omega(\Lambda_s - \Lambda_p)\Big) \tag{17d}$$

IV. CHERN NUMBERS IN TWO-DIMENSIONAL SYSTEMS

The first Chern number is a topological number employed to characterize two-dimensional band insulators such as the integer quantum Hall systems, and the direct relation between the Chern number and Hall conductance [28]. Generally, the Chern numbers can be defined for quantum states with two periodic parameters as integral over a two-dimensional compact surface, such as the Brillouin zone, of a fictitious magnetic fields, which is actually the field strengths of the Berry connection [29].

For example, the Chern number assigned to the nth band is defined by

$$c_n = \frac{1}{2\pi \mathrm{i}} \int_{T^2} F_{12}(\vec{k}) \mathrm{d}^2 k$$

where n denotes the nth Bloch band, and the Berry connection $A_\mu(\vec{k})(\mu = 1, 2)$ and the associated field strength $F_{12}(\vec{k})$ are given by

$$\begin{aligned} A_\mu(\vec{k}) &= \langle \psi_n(\vec{k}) | \partial_\mu | \psi_n(\vec{k}) \rangle \\ F_{12}(\vec{k}) &= \partial_1 A_2(\vec{k}) - \partial_2 A_1(\vec{k}) \end{aligned}$$

Here T^2 means the surface of the two-dimensional Brillouin zone, and $|\psi_n(\vec{k})\rangle$ are normalized Bloch wave functions.

In our model, the Effective Hamiltonian H_{eff} is a two-band Hamiltonian which can be written in this form

$$H_{\text{eff}} = \vec{B}(\vec{k}) \cdot \vec{\sigma} = \varepsilon(k) \hat{n}(k) \cdot \vec{\sigma}$$

where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is a vector of Pauli matrices acting in the sublattice space and $\hat{n}$ is a three-dimentional unit vector. When $\varepsilon(k) \neq 0$, π/T for all k

, each quasienergy band of the Effective Hamiltonian could be characterized topologically by the first Chern number, that [20]

$$C_{\pm} = \frac{\pm 1}{4\pi} \int_{\mathrm{FBZ}} \hat{n} \cdot (\partial_{k_x} \hat{n} \times \partial_{k_y} \hat{n}) d^2k$$

Calculation shows that for one-photon resonance case there is no topological nontrivial states found (no non-zero Chern-number could be obtained from the effective Hamiltonian), while for two-photon resonance case,

APPENDICES

Appendix A. EFFECTIVE HAMILTONIAN CORRESPONDING TO A FLOQUET OPERATOR

Suppose our system is subjected to a periodically varying, time-dependent Hamiltonian $H(t) = H(t+T)$. Here T is the period of the driving cycle. The behavior of the system is analyzed in terms of an Floquet operator [25] which is the evolution operator of the system over one full period of the driving, $U(t)$, defined as

$$U(t) = \mathcal{T} e^{-i\int_0^T H(t)\mathrm{d}t}$$

where the $\mathcal{T}$ is the time-ordering operator.

One method to study such a periodically driving system is to introduce a time-independent effective Hamiltonian $\hat{H}_{\text{eff}}$ via $\hat{F} = e^{-i\hat{H}_{\text{eff}}T}$ [27]. Expanding $\hat{H}(t)$ as $\hat{H}(t) = \sum_{n=-\infty}^{\infty} \hat{H}_n(t)e^{in\omega t}$ with $\omega = 2\pi/T$, and for a situation that the static component $\hat{H}_0$ contains energy bands within an energy range of Δ and $\omega \gg \Delta$, it is straightforward to show that to the leading order of Δ/ω, the effective Hamiltonian $\hat{H}_{\text{eff}}$ could be deduced as [27]

$$\hat{H}_{\text{eff}} = \hat{H}_0 + \sum_{n=1}^{\infty}\left(\frac{[\hat{H}_n, \hat{H}_{-n}]}{n\omega} - \frac{[\hat{H}_n, \hat{H}_0]}{e^{-2\pi n i\alpha}n\omega} + \frac{[\hat{H}_{-n}, \hat{H}_0]}{e^{2\pi n i\alpha}n\omega}\right)$$

Here we give an simple deduction. Expanding $U(t)$ to T order, that

$$
\begin{aligned}
U(t) &= \lim_{N\to\infty}\prod_{j=0}^{N} e^{-iH(t_j)\delta t} \\
&= \lim_{N\to\infty}\prod_{j=0}^{N}(1-iH(t_j)\delta t) \\
&= \lim_{N\to\infty}\prod_{j=0}^{N}\left[1-i\Delta t\sum_{j=0}^{N}H(t_j)+(-i\Delta t)^2\sum_{j>k}H(t_j)H(t_k)+\cdots\right] \\
&= 1-i\int_0^T \mathrm{d}tH(t)+(-i)^2\int_0^T \mathrm{d}t_1\int_0^{t_1}H(t_1)H(t_2)+\cdots
\end{aligned}
$$

Comparing coefficients, we obtain

$$
H_{\text{eff}} = H_0+\frac{1}{\hbar\omega}\sum_{n>0}\frac{1}{n}\Big(e^{i\eta n\pi}[H_0,H_n]-e^{-i\eta n\pi}[H_0,H_{-n}]+[H_n,H_{-n}]\Big)+O(\frac{1}{\omega^2})
$$

Appendix B. CALCULATIONS ON EFFECTIVE HAMILTONIAN FOR ONE-PHOTON RESONANCE

The unitary matrix for one-photon resonance case is

$$
U_1(t)=\begin{pmatrix} e^{-\mathrm{i}\omega t} & 0 \\ 0 & 1 \end{pmatrix}
$$

$$
\widetilde{H}(\vec{k},t) = U_1(t)(H(\vec{k},t)-\mathrm{i}\partial_t)U_1^\dagger(t)
$$

Thus

$$\widetilde{H}_0(\vec{k}) = \begin{pmatrix} E_s+\omega & \Omega \\ \Omega^* & E_p \end{pmatrix} = \frac{E_s+\omega+E_p}{2} + \frac{E_s+\omega-E_p}{2}\sigma_z + \Omega\sigma_+ + \Omega^*\sigma_-$$

$$\widetilde{H}_1(\vec{k}) = \begin{pmatrix} \Lambda_s & 0 \\ -iK & \Lambda_p \end{pmatrix} = \frac{\Lambda_s+\Lambda_p}{2} + \frac{\Lambda_s-\Lambda_p}{2}\sigma_z - iK\sigma_-$$

$$\widetilde{H}_{-1}(\vec{k}) = \begin{pmatrix} \Lambda_s^* & iK \\ 0 & \Lambda_p^* \end{pmatrix} = \frac{\Lambda_s^*+\Lambda_p^*}{2} + \frac{\Lambda_s^*-\Lambda_p^*}{2}\sigma_z + iK\sigma_+$$

$$\widetilde{H}_2(\vec{k}) = \begin{pmatrix} 0 & 0 \\ -\Omega & 0 \end{pmatrix} = -\Omega\sigma_-$$

$$\widetilde{H}_{-2}(\vec{k}) = \begin{pmatrix} 0 & -\Omega^* \\ 0 & 0 \end{pmatrix} = -\Omega^*\sigma_+$$

$$\begin{aligned}[\widetilde{H}_1(\vec{k}), \widetilde{H}_{-1}(\vec{k})] &= \begin{pmatrix} -K^2 & +iK(\Lambda_s-\Lambda_p) \\ -iK(\Lambda_s^*-\Lambda_p^*) & K^2 \end{pmatrix} \\ &= -K^2\sigma_z + iK(\Lambda_s-\Lambda_p)\sigma_+ - iK(\Lambda_s^*-\Lambda_p^*)\sigma_-\end{aligned}$$

$$\widetilde{H}_2(\vec{k}), \widetilde{H}_{-2}(\vec{k}) = \begin{pmatrix} -|\Omega|^2 & 0 \\ 0 & |\Omega|^2 \end{pmatrix} = -|\Omega|^2\sigma_z$$

Therefore

$$
\begin{aligned}
\widetilde{H}_{\text{eff}}(\vec{k}) &= \widetilde{H}_0(\vec{k}) + \frac{1}{\omega}[\widetilde{H}_1(\vec{k}), \widetilde{H}_{-1}(\vec{k})] + \frac{1}{2\omega}[\widetilde{H}_2(\vec{k}), \widetilde{H}_{-2}(\vec{k})] \\
&= \frac{1}{2}(E_s + \omega + E_p) + \frac{1}{2}(E_s + \omega - E_p)\sigma_z + \Omega\sigma_+ + \Omega^*\sigma_- \\
&\quad + \frac{1}{\omega}\Big(- K^2\sigma_z + iK(\Lambda_s - \Lambda_p)\sigma_+ - iK(\Lambda_s^* - \Lambda_p^*)\sigma_-\Big) - \frac{|\Omega|^2}{2\omega}\sigma_z \\
&= \frac{1}{2}(E_s + \omega + E_p) + \Big(\frac{E_s + \omega - E_p}{2} - \frac{K^2}{\omega} - \frac{|\Omega|^2}{2\omega}\Big)\sigma_z \\
&\quad + \Big(\Omega + i\frac{K}{\omega}(\Lambda_s - \Lambda_p)\Big)\sigma_+ + \Big(\Omega^* - i\frac{K}{\omega}(\Lambda_s^* - \Lambda_p^*)\Big)\sigma_-
\end{aligned}
$$

Appendix C. CALCULATIONS ON EFFECTIVE HAMILTONIAN FOR TWO-PHOTON RESONANCE

$$
U_2(t) = \begin{pmatrix} e^{-\mathrm{i}\omega t} & 0 \\ 0 & e^{\mathrm{i}\omega t} \end{pmatrix}
$$

$$
\begin{aligned}
\widetilde{H}(\vec{k}, t) &= U_2(t)(H(\vec{k}, t) - \mathrm{i}\partial_t)U_2^\dagger(t) \\
&= U_2(t)H(\vec{k}, t)U_2^\dagger(t) - \mathrm{i}U_2(t)\partial_t U_2^\dagger(t)
\end{aligned}
$$

$$
\partial_t U_2^\dagger(t) = \partial_t \begin{pmatrix} e^{\mathrm{i}\omega t} & 0 \\ 0 & e^{-\mathrm{i}\omega t} \end{pmatrix} = \begin{pmatrix} \mathrm{i}\omega e^{\mathrm{i}\omega t} & 0 \\ 0 & -\mathrm{i}\omega e^{-\mathrm{i}\omega t} \end{pmatrix} = \mathrm{i}\omega \begin{pmatrix} e^{\mathrm{i}\omega t} & 0 \\ 0 & -e^{\mathrm{i}\omega t} \end{pmatrix}
$$

So

$$
U_2(t)(-\mathrm{i}\partial_t)U_2^\dagger(t) = \omega
$$

$$
U_2(t)H(\vec{k}, t)U_2^\dagger(t) = \sum_j U_2(t)H_j(\vec{k})e^{\mathrm{i}j\omega t}U_2^\dagger(t)
$$

$$
\begin{aligned}
U_2(t)H_0(\vec{k})U_2^\dagger(t) &= \begin{pmatrix} e^{-\mathrm{i}\omega t} & 0 \\ 0 & e^{\mathrm{i}\omega t} \end{pmatrix} \begin{pmatrix} E_s & \mathrm{i}K \\ -\mathrm{i}K & E_p \end{pmatrix} \begin{pmatrix} e^{\mathrm{i}\omega t} & 0 \\ 0 & e^{-\mathrm{i}\omega t} \end{pmatrix} \\
&= \begin{pmatrix} e^{-\mathrm{i}\omega t} & 0 \\ 0 & e^{\mathrm{i}\omega t} \end{pmatrix} \begin{pmatrix} E_s e^{\mathrm{i}\omega t} & \mathrm{i}K e^{-\mathrm{i}\omega t} \\ -\mathrm{i}K e^{\mathrm{i}\omega t} & E_p e^{-\mathrm{i}\omega t} \end{pmatrix} \\
&= \begin{pmatrix} E_s & \mathrm{i}K e^{-2\mathrm{i}\omega t} \\ -\mathrm{i}K e^{2\mathrm{i}\omega t} & E_p \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
U_2(t)H_1(\vec{k})e^{\mathrm{i}\omega t}U_2^\dagger(t) &= \begin{pmatrix} e^{-\mathrm{i}\omega t} & 0 \\ 0 & e^{\mathrm{i}\omega t} \end{pmatrix} \begin{pmatrix} \Lambda_s & \Omega \\ -\Omega & \Lambda_p \end{pmatrix} e^{\mathrm{i}\omega t} \begin{pmatrix} e^{\mathrm{i}\omega t} & 0 \\ 0 & e^{-\mathrm{i}\omega t} \end{pmatrix} \\
&= \begin{pmatrix} e^{-\mathrm{i}\omega t} & 0 \\ 0 & e^{\mathrm{i}\omega t} \end{pmatrix} \begin{pmatrix} \Lambda_s & \Omega \\ -\Omega & \Lambda_p \end{pmatrix} \begin{pmatrix} e^{\mathrm{i}2\omega t} & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} e^{-\mathrm{i}\omega t} & 0 \\ 0 & e^{\mathrm{i}\omega t} \end{pmatrix} \begin{pmatrix} \Lambda_s e^{\mathrm{i}2\omega t} & \Omega \\ -\Omega e^{\mathrm{i}2\omega t} & \Lambda_p \end{pmatrix} \\
&= \begin{pmatrix} \Lambda_s e^{\mathrm{i}\omega t} & {}^{-\mathrm{i}\omega t} \\ \_^{\mathrm{i}3\omega t} & \Lambda_p e^{\mathrm{i}\omega t} \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
U_2(t)H_{-1}(\vec{k})e^{-\mathrm{i}\omega t}U_2^\dagger(t) &= \begin{pmatrix} e^{-\mathrm{i}\omega t} & 0 \\ 0 & e^{\mathrm{i}\omega t} \end{pmatrix} \begin{pmatrix} \Lambda_s^* & -\Omega^* \\ \Omega^* & \Lambda_p^* \end{pmatrix} e^{-\mathrm{i}\omega t} \begin{pmatrix} e^{\mathrm{i}\omega t} & 0 \\ 0 & e^{-\mathrm{i}\omega t} \end{pmatrix} \\
&= \begin{pmatrix} e^{-\mathrm{i}\omega t} & 0 \\ 0 & e^{\mathrm{i}\omega t} \end{pmatrix} \begin{pmatrix} \Lambda_s^* & -\Omega^* \\ \Omega^* & \Lambda_p^* \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{-\mathrm{i}2\omega t} \end{pmatrix} \\
&= \begin{pmatrix} e^{-\mathrm{i}\omega t} & 0 \\ 0 & e^{\mathrm{i}\omega t} \end{pmatrix} \begin{pmatrix} \Lambda_s^* & -\Omega^* e^{-\mathrm{i}2\omega t} \\ \Omega^* & \Lambda_p^* e^{-\mathrm{i}2\omega t} \end{pmatrix} \\
&= \begin{pmatrix} \Lambda_s^* e^{-\mathrm{i}\omega t} & -\Omega^* e^{-\mathrm{i}3\omega t} \\ \Omega^* e^{\mathrm{i}\omega t} & \Lambda_p^* e^{-\mathrm{i}\omega t} \end{pmatrix}
\end{aligned}
$$

Thus

$$
\begin{aligned}
&U_2(t)(H(\vec{k},t) - \mathrm{i}\partial_t)U_2^\dagger(t) \\
&= \omega + \begin{pmatrix} E_s & \mathrm{i}Ke^{-2\mathrm{i}\omega t} \\ -\mathrm{i}Ke^{2\mathrm{i}\omega t} & E_p \end{pmatrix} + \begin{pmatrix} \Lambda_s e^{\mathrm{i}\omega t} & {}^{-\mathrm{i}\omega t} \\ -{}^{\mathrm{i}3\omega t} & \Lambda_p e^{\mathrm{i}\omega t} \end{pmatrix} + \begin{pmatrix} \Lambda_s^* e^{-\mathrm{i}\omega t} & -\Omega^* e^{-\mathrm{i}3\omega t} \\ \Omega^* e^{\mathrm{i}\omega t} & \Lambda_p^* e^{-\mathrm{i}\omega t} \end{pmatrix} \\
&= \begin{pmatrix} E_s + \omega & 0 \\ 0 & E_p + \omega \end{pmatrix} \\
&\quad + \begin{pmatrix} \Lambda_s & 0 \\ \Omega^* & \Lambda_p \end{pmatrix} e^{\mathrm{i}\omega t} + \begin{pmatrix} \Lambda_s^* & \Omega \\ 0 & \Lambda_p^* \end{pmatrix} e^{-\mathrm{i}\omega t} \\
&\quad + \begin{pmatrix} 0 & 0 \\ -\mathrm{i}K & 0 \end{pmatrix} e^{\mathrm{i}2\omega t} + \begin{pmatrix} 0 & \mathrm{i}K \\ 0 & 0 \end{pmatrix} e^{-\mathrm{i}2\omega t} \\
&\quad + \begin{pmatrix} 0 & 0 \\ -\Omega & 0 \end{pmatrix} e^{\mathrm{i}3\omega t} + \begin{pmatrix} 0 & -\Omega^* \\ 0 & 0 \end{pmatrix} e^{-\mathrm{i}3\omega t}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\widetilde{H}_0(\vec{k}) = \begin{pmatrix} E_s + \omega & 0 \\ 0 & E_p + \omega \end{pmatrix} \\
&\widetilde{H}_1(\vec{k}) = \begin{pmatrix} \Lambda_s & 0 \\ \Omega^* & \Lambda_p \end{pmatrix} \quad \widetilde{H}_{-1}(\vec{k}) = \begin{pmatrix} \Lambda_s^* & \Omega \\ 0 & \Lambda_p^* \end{pmatrix} = \widetilde{H}_1^\dagger \\
&\widetilde{H}_2(\vec{k}) = \begin{pmatrix} 0 & 0 \\ -\mathrm{i}K & 0 \end{pmatrix} \quad \widetilde{H}_{-2}(\vec{k}) = \begin{pmatrix} 0 & \mathrm{i}K \\ 0 & 0 \end{pmatrix} = \widetilde{H}_2^\dagger \\
&\widetilde{H}_3(\vec{k}) = \begin{pmatrix} 0 & 0 \\ -\Omega & 0 \end{pmatrix} \quad \widetilde{H}_{-3}(\vec{k}) = \begin{pmatrix} 0 & -\Omega^* \\ 0 & 0 \end{pmatrix} = \widetilde{H}_3^\dagger
\end{aligned}
$$

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