# Tsing Hua AMO SummerSchool Few Body 1 Homework and Solution

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## 1 Problem

Two particles interact with a “spherical well” potential:

$$V(r) = \begin{cases} -V_0 & , \quad r < r_0 \\ 0 & , \quad r > r_0 \end{cases}$$

1. Calculate their scattering length as a function of the strength of the interaction, $V_0$, and discuss the resonances at which the scattering length diverges.
2. Verify that at each resonance point, the scattering length has a simple pole.
3. Also verify that a new s-wave bound state appears at each resonance.

## 2 Solution

### 2.1 Scattering length as a function of the strength of the interaction, $V_0$

Using partial wave method, the wave function $\psi$ can be written as

$$\psi = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} R_l(kr) Y_l^m(\hat{r})$$

where $R_l(kr)$ is the radial function, and $\hat{r}$ indicates that $Y_l^m$ is related only to direction. Actually, we assume the incoming wave is $\sim e^{ikz}$ thus it's already the eigenstate of $l_z$. And because $[l_z, H] = 0$, $l_z$ is conserved. Thus $m \equiv 0$. Thus we have

$$\psi = \sum_{l=0}^{\infty} R_l(kr) Y_l^0(\theta)$$

$k$ indicates the incoming energy $E = \hbar^2 k^2/2\mu$. Here we consider a two-particle interaction, so $\mu = m_1 m_2/(m_1+m_2)$ is the reduced mass.

According to Schrödinger equation

$$[-\frac{\hbar^2}{2\mu}\Delta^2 + V(r)]\psi = E\psi$$

we obtain

$$\sum_{l=0}^{\infty}\left[-\frac{\hbar^2}{2\mu}\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr} + \frac{l(l+1)\hbar^2}{2\mu r^2} + V(r) - \frac{\hbar^2 k^2}{2\mu}\right]R_l(kr)Y_l = 0$$

Thus for arbitrary l, we have

$$\left[\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr} + k^2 - \frac{l(l+1)}{r^2} - \frac{2\mu V(r)}{\hbar^2}\right]R_l = 0$$

Define $R_l = u_l/r$ then it tends to be

$$\left[\frac{1}{r}\frac{d^2}{dr^2} + k^2 - \frac{l(l+1)}{r^2} - \frac{2\mu V(r)}{\hbar^2}\right]u_l = 0$$

When $l = 0$, i.e. as for s-wave, then

$$\left[\frac{d^2}{dr^2} + k^2 - \frac{2\mu V(r)}{\hbar^2}\right]u(r) = 0$$

As for this problem, the potential function is a spherical well, that

$$V(r) = \begin{cases} -V_0 & , \quad r < r_0 \\ 0 & , \quad r > r_0 \end{cases}$$

Thus

$r < r_0$ **:** $V = -V_0$. Define $V_0 = \hbar^2 k_0^2/2\mu$ , then it tends to be

$$\frac{d^2 u}{dr^2} + (k^2 + k_0^2)u = 0$$

And define $k_1^2 = k^2 + k_0^2 = \sqrt{2\mu(E^2+V_0^2)}/\hbar$, then

$$\frac{d^2 u}{dr^2} + k_1^2 u = 0$$

$r > r_0$ **:** $V = 0$. Thus

$$\frac{d^2 u}{dr^2} + k^2 u = 0$$

$r \to 0$ **:** $u \to 0$ , unless $R = u/r$ is divergent.

$r = r_0$ **:** $u|_{r=r_0^-} = u|_{r=r_0^+}$ and $u'|_{r=r_0^-} = u'|_{r=r_0^+}$

$$\Longrightarrow u(r) = \begin{cases} \sin(k_1 r) & , \quad r < r_0 \\ A\sin(kr + \delta_0) & , \quad r > r_0 \end{cases}$$

and

$$\frac{\tan(kr_0 + \delta_0)}{k} = \frac{\tan(k_1 r_0)}{k_1}$$

$$\frac{1}{k}\frac{\tan(kr_0) + \tan(k\delta_0)}{1 - \tan(kr_0)\tan(k\delta_0)} = \frac{1}{k_1}\tan(k_1 r_0)$$

So

$$\begin{aligned} \tan\delta_0 &= \frac{k\tan(k_1 r_0) - k_1\tan(kr_0)}{k_1 + k\tan(kr_0)\tan(k_1 r_0)} \\ &= kr_0\left(\frac{\tan(k_0 r_0)}{k_0 r_0} - 1\right) \\ &\uparrow \text{ on the condition of low energy approximation} \\ &\quad\ \text{where we consider only s wave} \end{aligned}$$

According to the definition http://en.wikipedia.org/wiki/Scattering_length on wikipedia of Scattering Length,

$$\lim_{k\to 0} k\cot\delta(k) = -\frac{1}{a_0}$$

where $a_0$ is scattering length and $\delta_0$ is scattering phase shift, we obtain that

$$\begin{aligned} a_0 &= r_0\left(1 - \frac{\tan(k_0 r_0)}{k_0 r_0}\right) \\ &= r_0\left(1 - \frac{\tan(\sqrt{2\mu V_0} r_0/\hbar)}{\sqrt{2\mu V_0}/r_0\hbar}\right) \end{aligned}$$

## 2.2 Resonances at which the scattering length diverges

Scattering cross section

$$\sigma = \int |f(\theta)|^2 d\Omega = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l$$

Here we consider only s wave (low energy), then

$$\begin{aligned} \sigma &= \frac{4\pi}{k^2}\sin^2\delta_0 \\ &= 4\pi r_0^2\left(1 - \frac{\tan(\sqrt{2\mu V_0} r_0/\hbar)}{\sqrt{2\mu V_0}/r_0\hbar}\right)^2 \\ &= 4\pi a_0^2 \end{aligned}$$

So when $a_0 \to \infty$, $\sigma \to \infty$. There is resonance. Resonance appears at $k_0 r_0 = \sqrt{2\mu V_0}/\hbar = (2n+1)\pi/2$, Referring to Fig.1

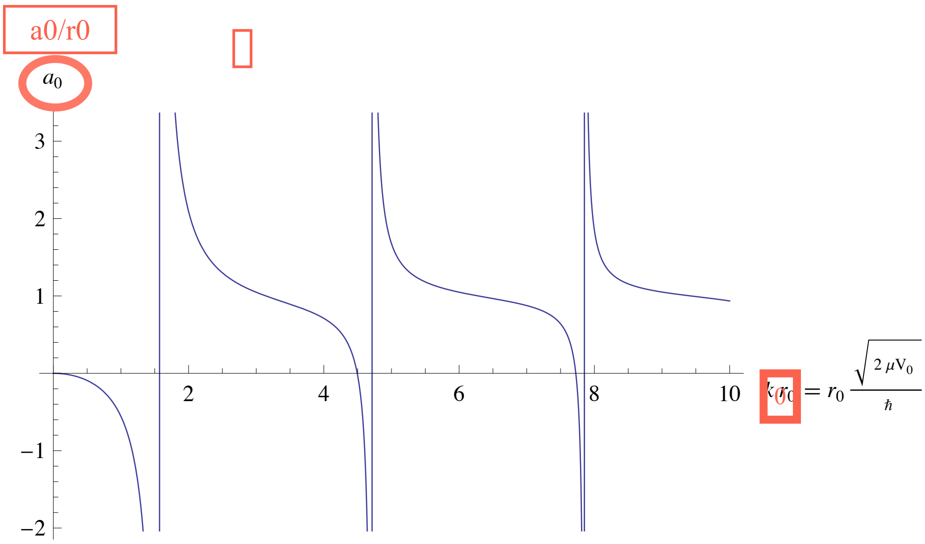


Figure 1: $a_0 \sim k_0 r_0$

## 2.3 Verify that at each resonance point, the scattering length has a simple pole

We obtain in Sec.2.1 that

$$a_0 = r_0 \left(1 - \frac{\tan(k_0 r_0)}{k_0 r_0}\right)$$

We assume $k > 0$, so all the divergence comes from $\tan k_0 r_0$ at the resonance point where $a_0 \to \infty$ (i.e. $k_0 r_0 = (2n+1)\pi/2$). Expand $tan(x)$ at $x = (2n+1)\pi/2$ as Laurent Serise that

$$\tan(x) = -\frac{1}{x - (n+1/2)\pi} + \frac{1}{3}\left(x - (n+1/2)\pi\right) + \ldots + a_n \left(x - (n+1/2)\pi\right)^{2n-1}$$

So when $k_0 r_0 \to (2n+1)\pi/2$

$$\lim_{a_0 \to \infty} \left(k_0 r_0 - \frac{(2n+1)\pi}{2}\right) \cdot a_0 = \lim_{a_0 \to \infty} \left(k_0 r_0 - \frac{(2n+1)\pi}{2}\right) \cdot r_0 \left(1 - \frac{\tan(k_0 r_0)}{k_0 r_0}\right)$$

tends to be a limited value.

Definitely, it is simple pole at resonance point (because the divergence point of tan-function is simple pole).

## 2.4 Also verify that a new s-wave bound state appears at each resonance

All above talk about scattering state, and

$$k = \frac{\sqrt{2\mu E}}{\hbar}$$

$$k_0 = \frac{\sqrt{2\mu V_0}}{\hbar}$$

$$k_1 = \frac{\sqrt{2\mu(E + V_0)}}{\hbar}$$

Now we consider bound states. We know that for a $V_0$-depth spherical well, the solution of the bound state equals to half-infinite half-finite well, which equals, moreover, the odd parity solution of one-dimension finite square well. Referring to Fig.2 we find that a new bound state appears when $k_0 r_0 = (2n+1)\pi/2$, which is consistent with the condition of resonance.

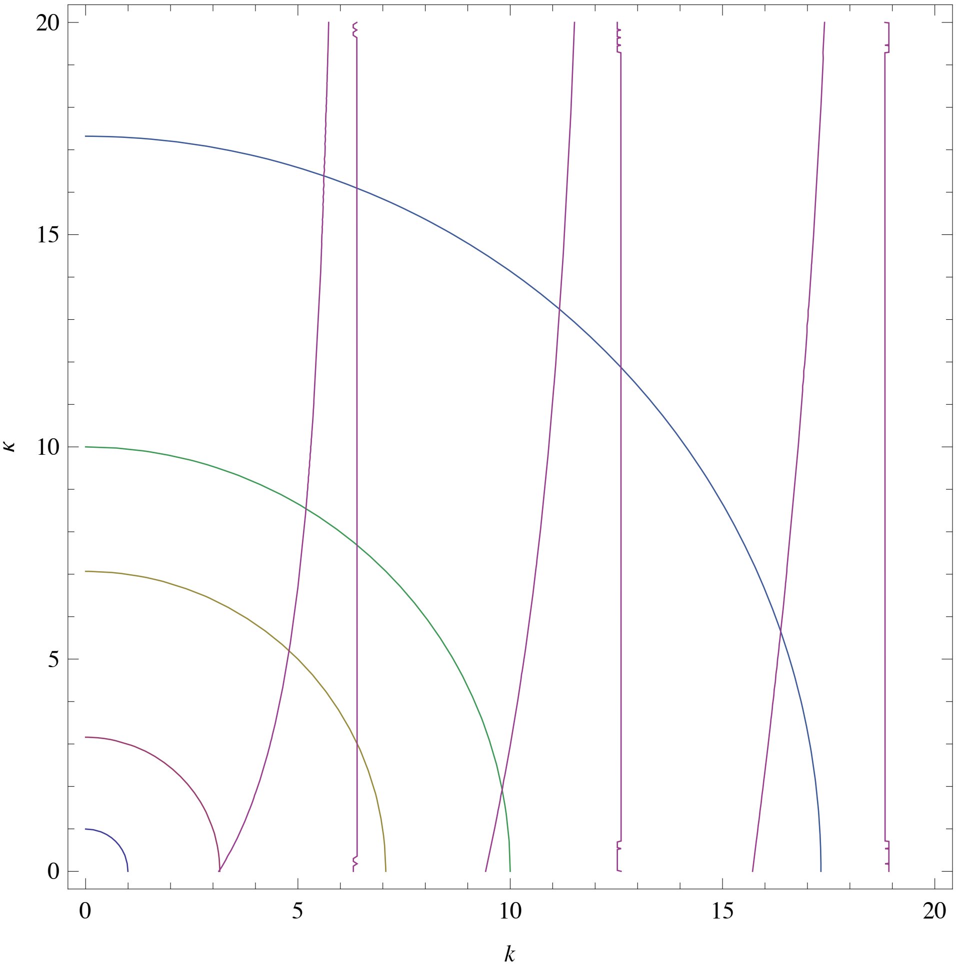


Figure 2: Sketch of odd parity solution for one-dimension finite square well