DMS

Graph

- **Graph $G = (V, E)$**

A graph written as $G = (V, E)$ consist of two components:

(1) The set of vertices V, also called points or nodes.

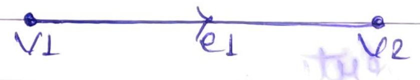
(2) The set of edges E, also called lines or arcs, connecting pair of vertices.

Each pair of vertices, connected by an edge, is called the end points or end vertices.

- **Directed Graph $G = (V, E)$**

A directed graph (V, E) consist of a set of vertices V and a set of edges E that are ordered pair of elements of V.

ex: $G_1 : (\{v_1, v_2\}, \{e_1\})$.



- **Undirected Graph.**

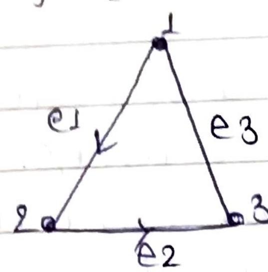
A graph $G = (V, E)$, in which every edge is associated with an unordered pair of vertices.

ex: $G_1 = (\{v_1, v_2\}, \{e\})$.

- **Mixed Graph:**

In a graph $G = (V, E)$, if som
edges are directed & some are undir
then it is called Mixed graph.

ex: $G = (\{1, 2, 3\}, \{e_1, e_2, e_3\})$



- **Isolated vertex:**

A vertex v which is not conn
with any vertex of a graph G, by an
edge is called Isolated vertex.

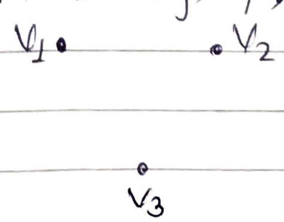
ex:

V_3 is the isolated vertex.

- **Null Graph:**

If all the vertices of a graph are
isolated [i.e set $E = \phi$], the graph is
called null graph.
(or a totally disconnected graph).

ex: $G = (\{V_1, V_2, V_3\}, \phi)$



- **Self loop:**

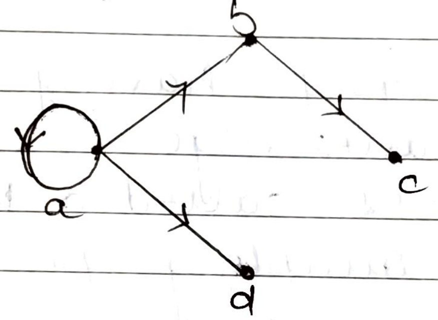
An edge having the same vertex as both its end vertices is called self-loop.

∴ a loop is an edge of the form (a, a).

ex: Let, V = (a, b, c, d)

E = {(a, a), (a, b), (a, d), (b, c)} set of edges.

then, G = (V, E).



The edge (a, a) is represented by a closed curve drawn at a and is called self-loop.

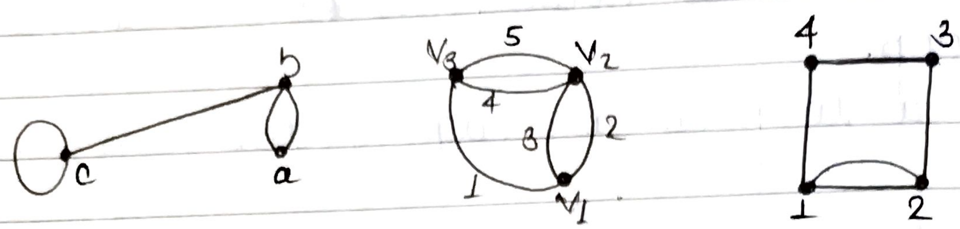
- **Initial & terminal vertices:**

Let G = (V, E) be a graph and let e ∈ E be a directed edge associated with the ordered pair of vertices (u, v). Then the edge e is said to be initially or originating in the vertex u & terminating or ending in the vertex v. The vertices u & v are called the initial & terminal vertices respectively of the edge e. An edge e ∈ E which joins the vertex u & v is said to be incident on the vertices u & v.

- **Parallel edges or Multiple edges:**

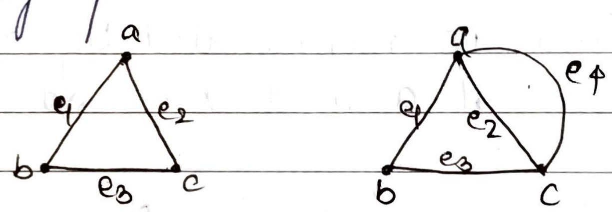
If a pair of vertices is joined by

more than one edge then these edges are called parallel edges.



Here parallel edges are a, b; 2, 3. & 4, 5; 1, 2 Etc.

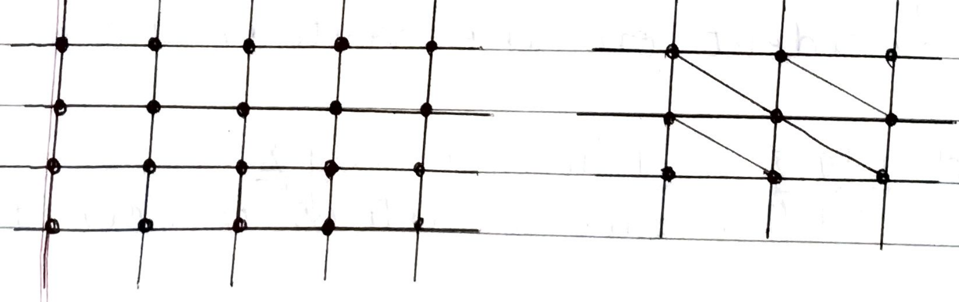
- **Simple Graph:** A graph having neither self-loop nor parallel edge is called a simple graph otherwise called a Multigraph.
Here G_1 is simple graph & G_2 is Multigraph.



G_1 G_2

- **Finite & infinite graph:** In a graph $G=(V,E)$, if the set V & E are both finite then it is called a finite graph otherwise a infinite graph.

ex: $V=\{v_1, v_2, v_3 \ldots\}$ $E=\{e_1, e_2, \ldots\}$, then the graph is infinite graph.



- **Order of a graph:**
The no. of vertices in a graph, is called its order.

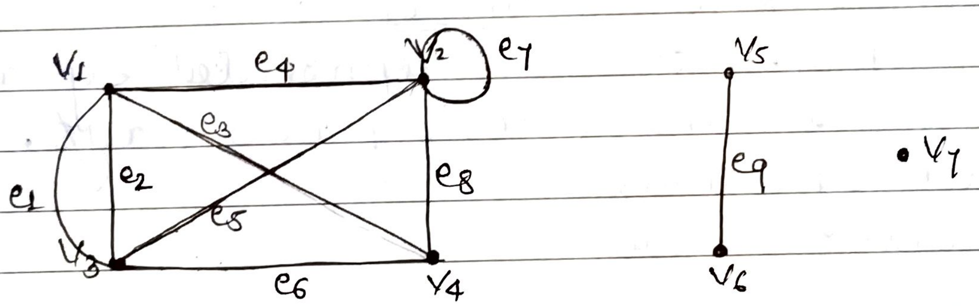
- **Size of a graph:**
The no. of edges in a graph is called its size.

NOTE: for calculating size, a loop is treated to be of size 2.

ex: Find no. of vertices, edges, loops, isolated vertex, parallel edges in the following graph.

G1: • v_1 • v_2 • v_3 • v_4.

G2:



Sol^n: G1 has only 4 vertices & denoted as disconnected graph v_1, v_2, v_3, v_4.
$G_2 = (\{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}, \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9\})$
has i) 7 vertices ii) 9 edges
iii) one self loop, e_7 iv) one isolated vertex v_7.
v) a pair of parallel edges e_1 & e_2 with end vertices v_1 & v_3.

- **Adjacent edges & Adjacent vertices:**
Two non-parallel edges are said to be adjacent if both are incident on a common vertex.

In fig. e_1 & e_2; e_2, e_3 & e_3, e_4, e_5 are adjacent edges.

Two vertices connected by an edge are called adjacent vertices.

In fig. v_1, v_2; v_2, v_3; v_3, v_4; v_4, v_5; v_4, v_6 are adjacent vertices.

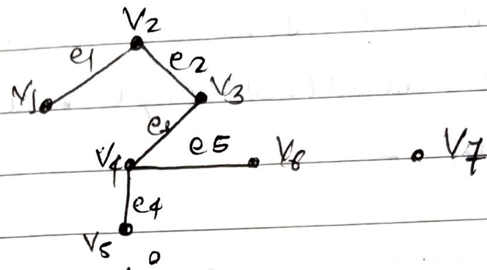


fig.

- <u>Matrix representation of graphs:</u>

A graph can be represented by a matrix in the following two ways,

(1) Adjacency matrix.
(2) Incidence matrix.

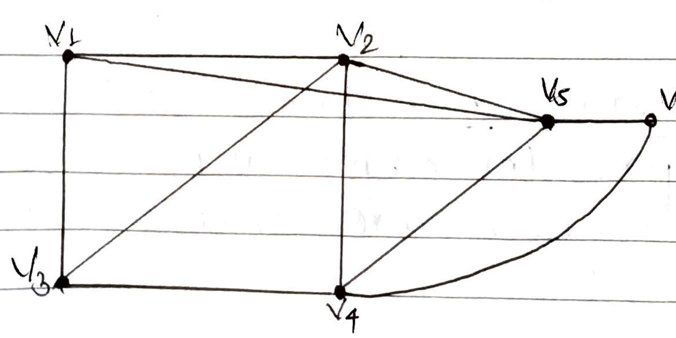
1. <u>Adjacency Matrix:</u>

Let $G = (V, E)$ be a graph with n vertices then the adjacency matrix of G, denoted by $A(G)$ is a symmetric matrix $A(G) = [a_{ij}]_{n \times n}$

where, $a_{ij} = K$.

if there are K edges b/w v_i & v_j.

ex:



then adjacency matrix $A(G)$ is displayed as:

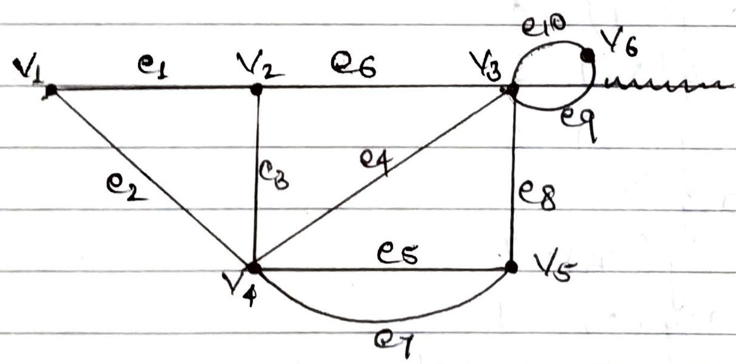
$$A(G) = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \end{array} \begin{array}{c} \begin{array}{cccccc} V_1 & V_2 & V_3 & V_4 & V_5 & V_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

- **Incidence Matrix:**

Let G be a graph with n vertices, e edges and no self loops. Then the incidence matrix of G, denoted by $I(G) = (a_{ij})_{n \times e}$, is an $n \times e$ matrix, where

$$a_{ij} = \begin{cases} 1, & \text{if } j\text{th edge } e_j \text{ is incident on } i\text{th vertex } v_i \\ 0, & \text{otherwise.} \end{cases}$$

ex:

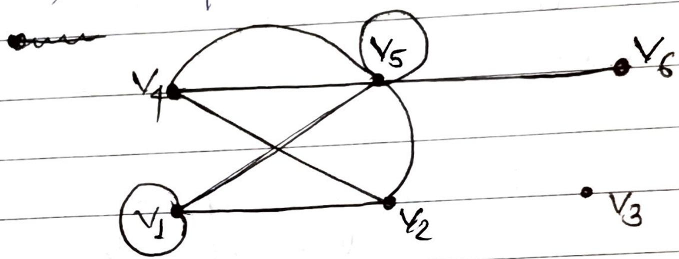


$$I(G) = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \end{array} \begin{array}{c} \begin{array}{cccccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 & e_{10} \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

- **Degree of vertices:**
  The no. of edges incident on a vertex v of a graph is called the degree of the vertex and is denoted as deg(v).

- **Pendant vertex:**
  A vertex of degree one is known as the pendant vertex.

ex: Find the degree of each vertex in the following graph. Also find if there is any isolated vertex or pendant vertex.



$deg(v_1) = 4$
$deg(v_2) = 3$
$deg(v_3) = 0$
$deg(v_4) = 3$
$deg(v_5) = 7$
$deg(v_6) = 1$

Here, v_3 is an isolated vertex since $deg(v_3) = 0$ and v_6 is a pendant vertex since $deg(v_6) = 1$.

- **Even & odd vertices:**
  In a graph, if the degree of a vertex is an even integer then the vertex is called even vertex, and if the degree of a vertex is an odd integer the vertex is known as the odd vertex.

Theorem 1: The Handshaking theorem:
The sum of the degrees of all the vertices in a graph is equal to twice the no. of edges in the graph.

Proof
Each edge contributes two to the sum of degrees of vertices because an edge is incident with exactly two vertices. Therefore if $v_1, v_2 -- v_k$ are the vertices then $d(v_1) + d(v_2) + d(v_3) + -- d(v_k) = 2 \times n$ or

$\sum_{i=1}^{k} d(V_i) = 2n$; where n is no. of edges

Theorem 2 :- The number of vertices of odd degree in graph is always even.

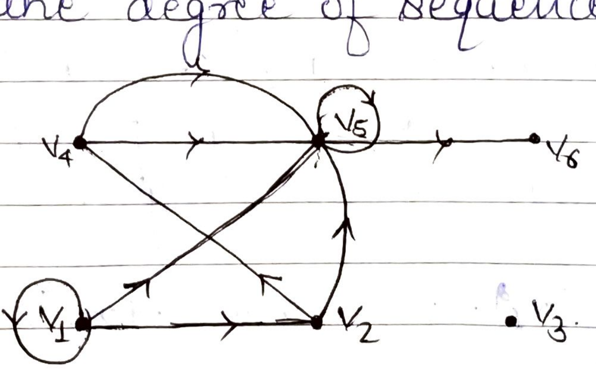
Proof :- Let $G(V, E)$ be a graph. V_e and V_o be the set of vertices of even degree & odd degree i.e $V = V_e \cup V_o$

Now $\sum d(V_i) = 2n \Rightarrow \sum d(V_e) + \sum d(V_o) = 2n$.

Now $\sum d(V_e) =$ even number $= 2k \Rightarrow \sum d(V_o) = 2n - 2k = 2(n-k)$

Degree sequence of a graph:
If we find the degree of each vertex of a graph & write them in an ascending order, the sequence so obtained is known as the degree sequence of the graph.

ex: Find the degree of sequence of graph G.



In graph G: $deg(v_1) = 4$, $deg(v_2) = 3$, $deg(v_3) = 0$
$deg(v_4) = 3$, $deg(v_5) = 7$, $deg(v_6) = 1$

The deg. of sequence: $\langle 0, 1, 3, 4, 7 \rangle$

• **Isomorphic graph:**

Two graph $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are said to be isomorphic to each other if there exist a bijection mapping f from V_1 to V_2.

i.e $f : V_1 \rightarrow V_2$ such that for each of the vertices v_i, v_j of V_1.

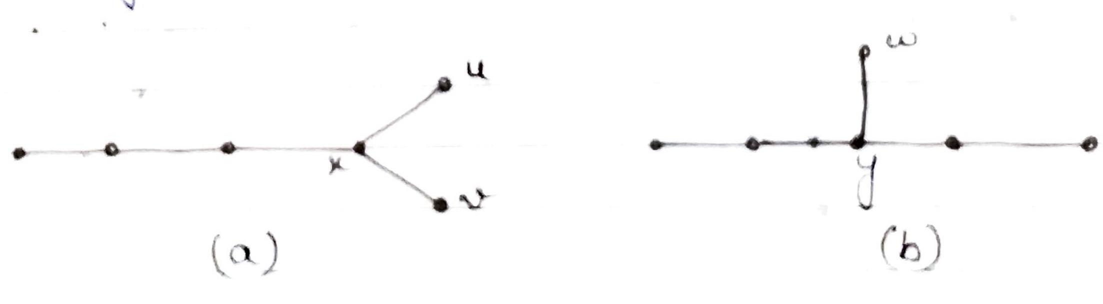
$\{v_i, v_j\} \in E_1 \Rightarrow \{f(v_i), f(v_j)\} \in E_2$.

The function f is called an isomorphism from G_1 to G_2.

NOTE: It is immediately apparent by the def^n of isomorphism that two isomorphic graphs must have:

(1) Same no. of vertices.
(2) Same no. of edges.
(3) An equal no. of vertices with a given degree (i.e same degree sequence).

However, these cond^n are by no means sufficient. For instance, the two graphs given below satisfy all 3 cond^n, yet they are not isomorphic.

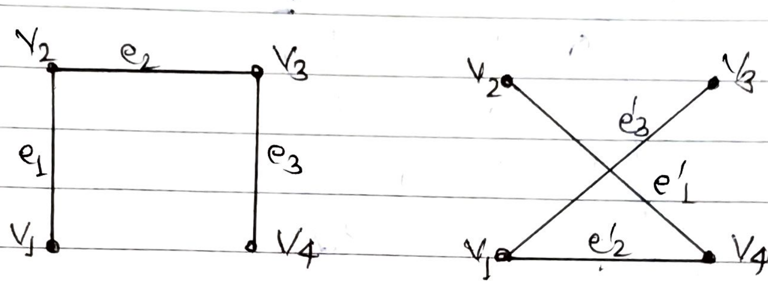


Two graphs that are not isomorphic.

- ## Self-complementary graph :

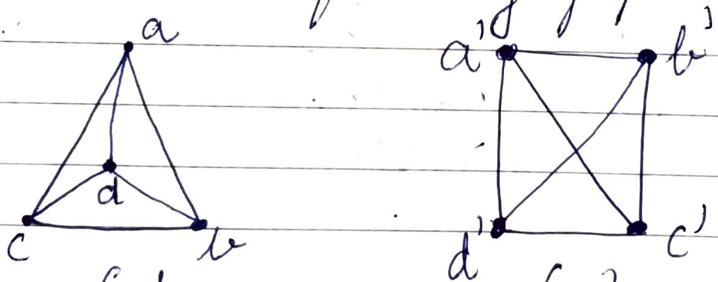
A simple graph $G = (V, E)$ is a self complementary graph if the graph G and its complementary graph $\bar{G}$ are isomorphic.

ex: The following graph G is a self-complementary graph & $\bar{G}$ is the complementary graph of G.



G $\bar{G}$

Q Show that following graphs are isomorphic



G1 G2

Sol^n 1) G1 and G2 have same no. of vertices i.e 4

2) G1 and G2 have same no of edges i.e 6

3) ~~All the vertice~~ Degree sequence of G1 and G2 are same i.e $\{3, 3, 3, 3\}$

Let us define a mapping

$f: V1 \to V2$

s.t $f(a) = a'$

$f(b) = b'$

$f(c) = c'$,

$f(d) = d'$

Now adjacency matrix of G1 and G2 is given by

$$A_{G1} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

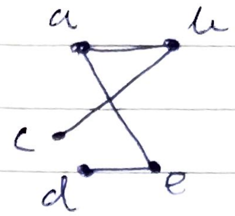
$$\cancel{A_G}.\ A_{G2} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

~~AG~~ Here $A_{G1} = A_{G2}$

$\Rightarrow G1 \cong G2$

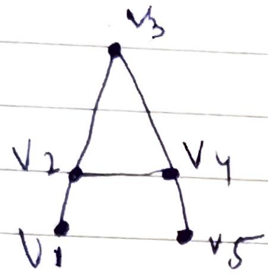
Determine whether the following graphs are isomorphic or not

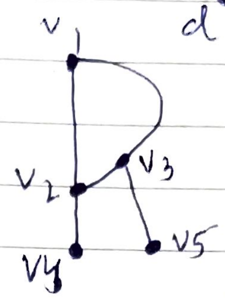
a)



(iso)

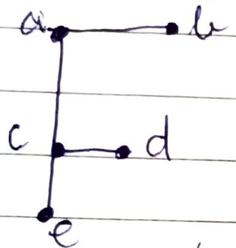
b)

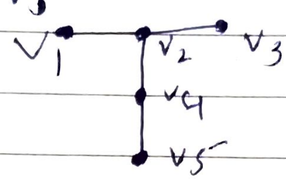




(iso)

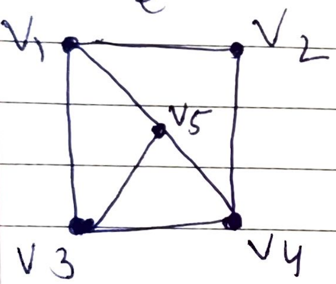
c)

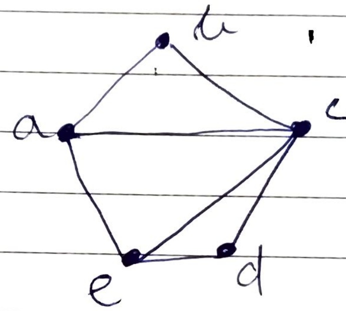




(iso)

d





(non iso)

18/05/16

Graph Theory - II

* Path, Circuits and Cycles :-

Let u and v be two vertices in a graph G. A path from u to v in G is an alternating sequence of vertices & edges of G having the form

$u = v_1, e_1, v_2, e_2, \ldots\ldots, v_{n-1}, e_{n-1}, v_n, e_n, v_{n+1} = v,$

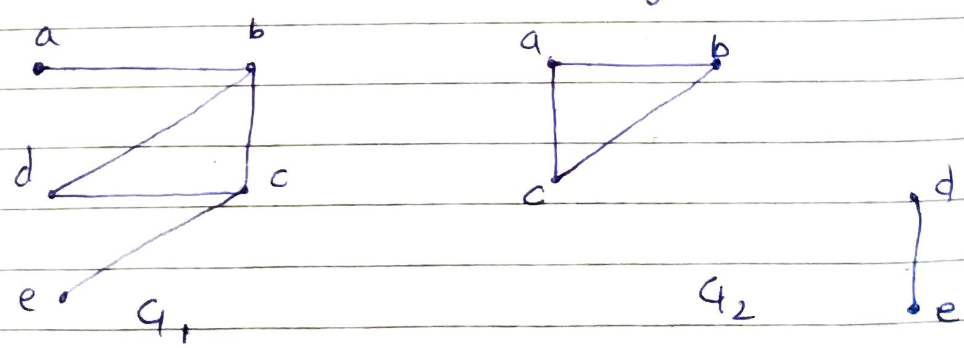
beginning with vertex u called initial vertex and ending with vertex v called the terminal vertex.

If the graph G is directed, the path is called a directed path.

* Connected graph :-

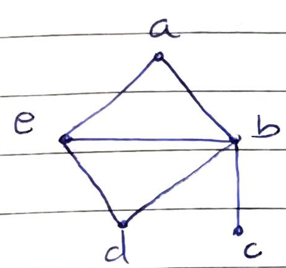
Let G be a graph. A vertex u is said to be connected to a vertex v if there is a u-v path in G.

A graph G is called a connected graph if for any two vertices u, v of G, there is a u-v path in G, otherwise it is called disconnected graph.

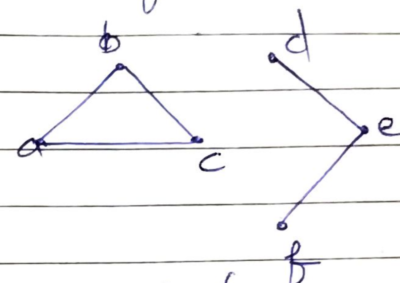


(connected graph) (Disconnected graph)

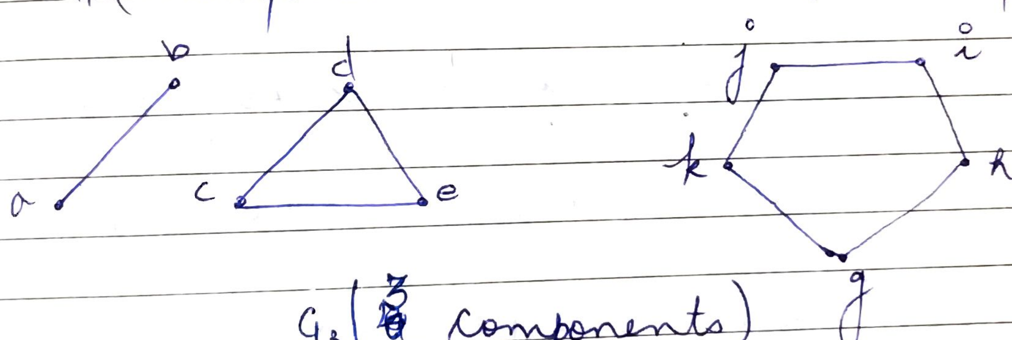
* **Component of a graph:-**
(Maximal connected sub graph of G):-
A subgraph H of graph G is called a component of G if
i) any two vertices of H are connected in H.
ii) H is not properly contained in any connected subgraph of G.



G_1 (1 component)



G_2 (2 components)



G_3 (3 components)

Theorem:- A simple graph with n vertices & k components cannot have more than $\frac{(n-k)(n-k+1)}{2}$ edges.

Proof:- Let n_i be the no.'s of vertices in i^{th} component, $1 \leq i \leq k$. Then $\sum_{i=1}^{k} n_i = n$ —(1)

A component with n_i vertices will have maximum no. of edges when it is complete. The no. of edges in a complete graph K_{n_i} is

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$$\frac{1}{2} n_i (n_i - 1) \quad \text{——— (2)}$$

Hence the maximum no. of edges is

$$\frac{1}{2} \sum_{i=1}^{k} n_i(n_i-1) = \frac{1}{2}\left(\sum_{i=1}^{k} n_i^2 - \sum_{i=1}^{k} n_i\right)$$

$$= \frac{1}{2}\left[\sum_{i=1}^{k} n_i^2 - n\right] \quad [\text{using (i)}] \quad \text{——— (3)}$$

Now for $\sum_{i=1}^{k} n_i^2$, consider

$$\sum_{i=1}^{k} (n_i - 1) = \sum_{i=1}^{k} n_i - \sum_{i=1}^{k} 1 = n - k$$

$$\Rightarrow \left[\sum_{i=1}^{k} (n_i - 1)\right]^2 = (n-k)^2 = n^2 + k^2 - 2nk$$

$$\Rightarrow \sum_{i=1}^{k} (n_i - 1)^2 + \text{non -ve cross terms} = n^2 + k^2 - 2nk$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 - 2\sum_{i=1}^{k} n_i + \sum_{i=1}^{k} 1 \le n^2 + k^2 - 2nk$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 - 2n + k \le n^2 + k^2 - 2nk$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 \le n^2 + k^2 - 2nk + 2n - k \quad \text{——— (4)}$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 \le n^2 - (k-1)(2n-k)$$

Substituting (4) in (3), we get,

$$\frac{1}{2} \sum_{i=1}^{k} n_i(n_i - 1) \le \frac{1}{2}\left[n^2 - (k-1)(2n-k) - n\right]$$

$$= \frac{1}{2}\left(n^2 - 2nk + k^2 + n - k\right)$$

$$= \frac{1}{2}(n-k)(n-k+1)$$

Hence Proved.

Theorem :- Show that a simple graph G with n vertices is connected if it has more than $\frac{1}{2}(n-1)(n-2)$ edges.

Proof :- A simple graph is connected, if it has only one component.
Let the graph is not connected and has two components. By theorem, the maximum no. of edges is
$\frac{1}{2}(n-2)(n-2+1) = \frac{1}{2}(n-1)(n-2)$

So if the no. of edges is more than $\frac{1}{2}(n-1)(n-2)$ the graph will get connected. Hence the result.

* **Cut Set : Cut Vertex and Cut edge :-**
A cut vertex (cut-point) of a connected graph is the vertex whose removal ↑ses the no. of components than in the original graph.
If v is a cut vertex of a connected graph G, $G - \{v\}$ is disconnected.

Eg :- Consider the following graph

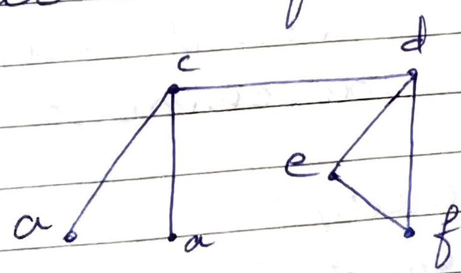


fig 5.3

Here c & d are cut-vertices

Cut edge :- An edge whose removal

produces a graph with more components than the original graph is called a cut-edge or bridge. eg- in the above graph, the edges {b, c}, {a, c}, {c, d} are cut edges or bridges.

Cut Set :- The set of all minimum no. of edges whose removal produces a subgraph with more connected components than in the original graph but no proper subset of this set of edges has this property is called a cut-set.

For example in this graph of fig 5.3, each of the set {{c, d}}, {{d, e}}, {{d, f}}, {{b, c}}, {{a, c}} is a cut set.
The edges {b, c}, {a, c}, {c, d} are cut edges or bridges. Singleton set consisting of a bridge is always a cut set of G.

* Connectivity :-
It measures the connectedness of a graph G in terms of minimum no. of vertices & edges to be removed from the graph in order to disconnect it.

* Edge Connectivity :-
The edge connectivity of a connected graph is the minimum no. of edges whose removal results in a disconnected graph and is denoted

by $\lambda(G)$. Thus,

i) If G is a disconnected graph, then $\lambda(G) = 0$

ii) If G consists of cut edge (bridge) then $\lambda(G) = 1$.

* Vertex connectivity :-

The vertex connectivity of a connected graph is the minimum no. of vertices whose removal results in a disconnected graph. It is denoted by $k(G)$.

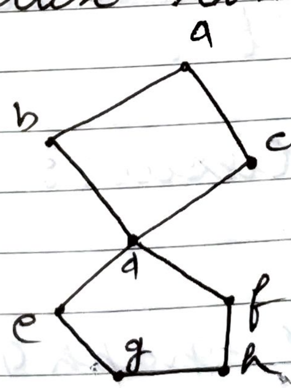
i) If G is a disconnected graph, then $k(G) = 0$.

ii) Vertex connectivity of a graph having a bridge is always one.

iii) For a path $k(G) = 1$ and for a cycle $C_n (n \geq 4)$, $k(C_n) = 2$.

Q. Determine the edge connectivity and vertex connectivity of following graph

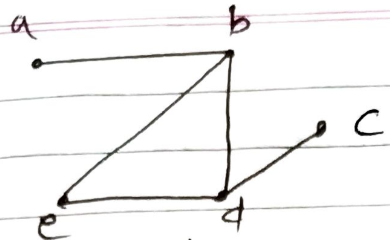
i)



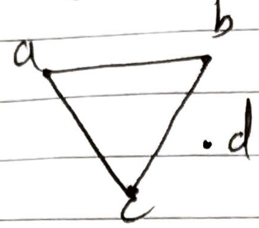
Edge connectivity of G_1 is 2 & vertex connectivity is 1. That is
$\lambda(G_1) = 2$, $k(G_1) = 1$

ii)

For G_2, $\lambda(G_2) = 2$, $k(G_2) = 2$

iii) 

For G_3, $\lambda(G_3)=1$, $k(G_3)=1$
$\because$ edges $\{a,b\}$ & $\{c,d\}$ are bridges

iv) 

For G_4, $\lambda(G_4)=0=k(G_4)$
$\because$ G_4 is disconnected.

* Eulerian Path & Circuits :-

Euler Path :- A path is a connected graph G is called Euler path if it includes every edge exactly once. Since the path contains every edge exactly once, it is also called Euler trail.

Euler Circuit :- An euler path that is a circuit is called Euler (or Eulerian) ~~graph.~~ circuit i.e a closed Euler path is Euler circuit.

* Hamiltonian Path & Circuit :-

Hamiltonian Path :-
A path is a connected graph G is called Hamiltonian ~~cycle~~ path if it ~~contains~~ includes every vertex exactly once.

Hamiltonian cycle :-
A cycle in a connected graph G

is called Hamiltonian cycle if it contains each vertex of G exactly once except the starting & ending vertex, which are same (edges of cycle are distinct).

A graph G which has a Hamiltonian cycle is called a Hamiltonian graph.

Q. Give an example of connected graph that has:-

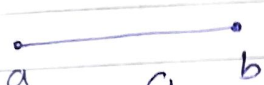
i) Neither an Euler circuit Nor a Hamiltonian cycle.

ii) A Euler circuit but no Harmiltonian Cycle

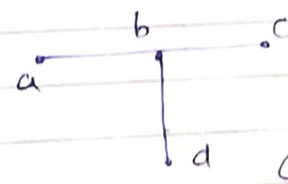
iii) A Hamiltonian cycle but no Euler circuit

iv) Both " " and " "

Ans i)

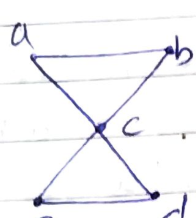


G_1

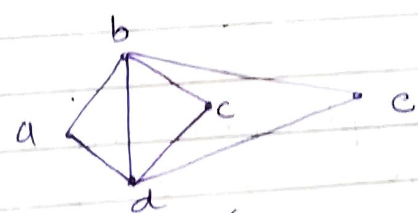


G_2

ii)



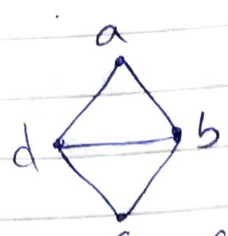
G_1



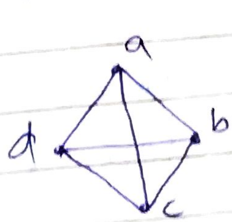
G_2

Euler circuit : c, a, b, c, d, e, c in G_1
and a, b, d, c, b, e, d, a in G_2

iii)

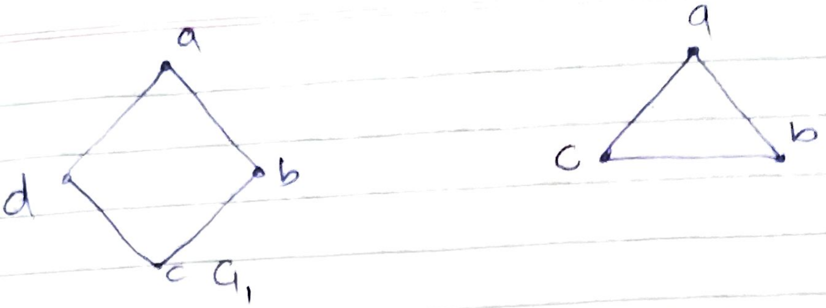


G_1



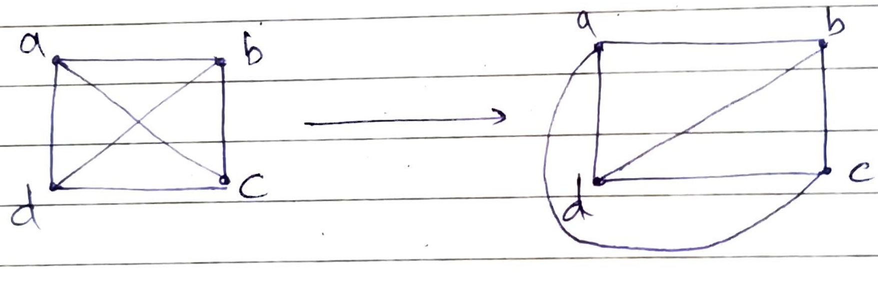
G_2

Hamiltonian Cycle : a, b, c, d

iv) 

In G_1 Euler circuit : a, b, c, d, a
Hamiltonian cycle : a, b, c, d, a
In G_2 Euler circuit and Hamiltonian cycle are a, b, c, a.

* Planar graphs:-
A graph is called planar if it can be drawn in a plane such that no two edges intersect except at their common end vertices, if any.

a) 

Theorem :- (Euler Theorem) Let G be a connected planar graph with n_v vertices n_e edges and n_f faces. Then $n_v - n_e + n_f = 2$. This formula is known as Euler's Formula.

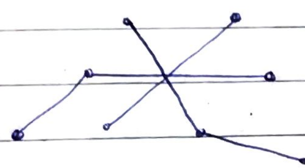
Proof:- We prove the theorem by induction on n_e (no. of edges)

Basis Step: Let $n_e = 0$. Since the graph is connected, the graph has only one vertex i.e $n_v = 1$ & hence only one region i.e $n_f = 1$.

Then $n_v - n_e + n_f = 1 - 0 + 1 = 2$
Hence, the result is true.

Inductive Hypothesis :- let $n_e = k$, k be a +ve integer. Assume that $n_v - n_e + n_f = 2$ for any connected planar graph with $n_e = k$

Inductive Step :- let G be a connected planar graph with $n_e = k+1$ edges & $n_v = t+1$ vertices



Suppose G has no cycles, than G has no interior region & has only exterior region $\therefore n_f = 1$, we now show that G contains a vertex of degree 1.
choose vertex v in G
if $\deg(v) = 1$, we are done
if $\deg(v) \neq 1 \Rightarrow \deg(v) > 1$. let v_1 be the vertex adjacent to v. Because G has no cycles, v_1 is diff. from v. If $\deg(v_2) \neq 1$, we find a vertex v_3 adjacent to v_1 diff from v, v_1 & v_2. Since G has a finite no. of vertices, it follows there is a vertex u of degree 1. we now delete this vertex u from the graph G & form a new connected planar graph H with k edges & t vertices.
By the inductive hypothesis, for a graph H with $n_e = k$ edges

$$n_v - n_e + n_f = 2 \text{ in } H$$
$$\Rightarrow t - k + n_f = 2$$
$$\Rightarrow (t+1) - (k+1) + n_f = 2$$
$$\Rightarrow n_v - n_e + n_f = 2 \text{ in } G$$

Now, suppose that G has a cycle C. let e be an edge in C. Construct a new graph H from G by deleting the edge e in C. i.e $H = G - \{e\}$. H is still connected planar graph with $n_v = t+1$, $n_e = k$. let $n_f = m$ in G. After deleting edge e, $C - \{e\}$ is not a cycle in H & thus will not form a boundary in H. $\therefore n_f = m-1$ in H

By inductive hypothesis for H
$$n_v - n_e + n_f = 2 \text{ in } H$$
$$\Rightarrow (t+1) - k + (m-1) = 2$$
$$\Rightarrow (t+1) - (k+1) + m = 2$$
$$\Rightarrow n_v - n_e + n_f = 2 \text{ in } G$$

Hence the result follows from induction

Theorem :- If a connected graph G is Eulerian then every vertex of G has even degree

Proof :- let G be an Eulerian graph. Then G has an Euler circuit which begins & ends at u (say) and is of the form
$$u = v_1, e_1, v_2, e_2, \ldots, v_{i-1}, e_{i-1}, v_i, \ldots, v_n, e_n, v_{n+1} = u$$
when we travel along the circuit,

then each time, we visit a vertex v_i, we use two edges e_{i-1} & e_i (one in & out). This is also true for starting vertex u because we also ends at u. Since an Euler circuit crosses every edge once, each occurrence of v in circuit represents a contribution of 2 to its degree.

Thus, degree of all vertices is even.

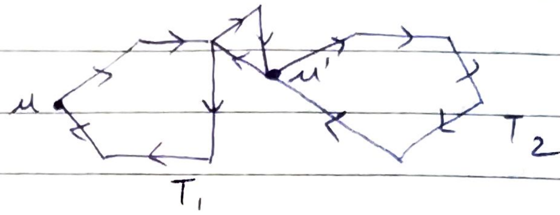
Conversely, suppose that G is connected and every vertex is even. We construct a circuit T_1 at any edge e beginning with u. T_1 by adding an edge after the other.

If T_1 is not closed at any step i.e the end vertex $v \neq u$, then only an odd no. of the edges incident on v appear in T_1. But v is of even degree hence we can extend T_1 by another edge incident on v. Thus, we can continue to extend T_1 until T_1 returns to its initial vertex u. If T_1 includes all the edges of G then T_1 is an Euler circuit.

Suppose T_1 does not include all edges of G. Consider the graph H obtained by deleting all edges of T_1 from G. H may not be connected but each vertex of H has even degree since each vertex in T_1 contains even no. of edges incident on it.

Since G is connected, there is

an edge e' of H which has an endpoint u' in T_1. We construct a trail T_2 in H beginning at u' & using e'. Since all vertices of H are of even degree, we continue to extent T_2 until T_2 returns to u'.



Clearly T_2 can be put in T_1 to form a larger closed trail (circuit) in G. We continue this process until all the edges of G are used. We finally obtain an Euler circuit in G. So G is Eulerian graph.

Non-Planar Graph:-
A graph which cannot be drawn without intersecting its edges is called non-planar graph.

Regions OF A Graph:-
Every planar representation of a graph divides the plane into diff. region called phaces of the graph.

Property OF a region:-
- A region is known by set of edges & vertices constructing its boundary.
- Region is not define in non-planar graph.

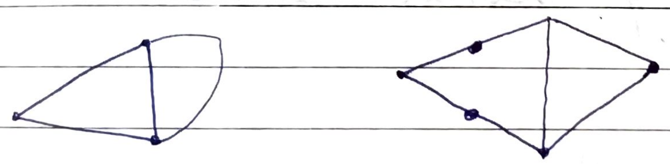
Region is a property of specific plane representation of a graph.

Degree Of Region :-

Let R be a region in a planar representation of graph, the degree of region r is represented by deg (R) is the no. of edges traversed in a shortest closed path about the boundary of R.

Homeomorphic Graph :-

2 graphs are said to be homeomorphic graph if one can be obtained from the other by creation of edges in series (i.e insertion of vertices of degree 2) the following is eg. of homeomorphic graph :-

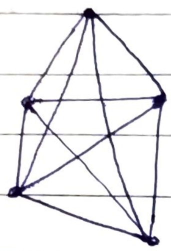


Kurtowski's Graph :-

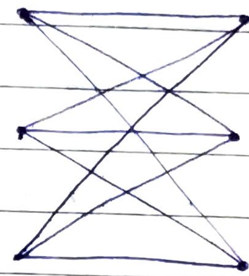
The complete graph K_5 & $K_{3,3}$ (complete Bipartite graph) are non-planar graph & these are known as kurtowski's graph 1st & 2nd graph resp. These 2 graphs are very imp. because these are used to find whether given graph is planar or not by using the property that if 2 graphs are homeomorphic than they are simultaneously planar or non-planar.

Kuratowski's Theorem :-
A graph is non-planar if & only if it contains a sub graph i.e homeomorphic to K_5 or $K_{3,3}$

K_5



$K_{3,3}$



Q. Suppose that a connected planar graph has 30 vertices each of degree 3. In how many regions the plane is divided by planar representation of this graph.

$$n_v - n_e + n_f = 2$$

$30 - 3$

$$\Sigma \deg v_i = 2e$$

$$30 \times 3 = 2e$$

$$e = 45$$

$$30 - 45 + n_f = 2$$

$$\boxed{n_f = 17}$$

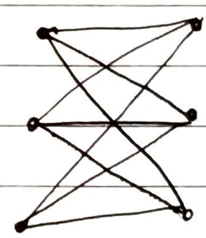
Q. Suppose that connected planar graph has 30 edges & its planner rep divides the plane into 20 region how many vertices this graph has.

$$n_v - 30 + 20 = 2$$

$$\boxed{n_v = 12}$$

Q. Show that completcompeetite graph $k_{3,3}$ is not planar graph.

Ans. We shall find the sol^n to this problem by method of contradiction.
Let $k_{3,3}$ be a planar graph.
By Eulers theorem:-



$$n_v - n_e + n_f = 2$$

here $n_v = 6$

$n_e = 9$.

$$6 - 9 + n_f = 2$$

$$\boxed{n_f = 5}$$

here $k_{3,3}$ consists cycle of length 4 ∴ the total no. of appearances of edges in boundary of 5 ~~phases~~ faces is $>=$ ~~equal~~ to $5 \cdot 4$ but in planar rep. 9 edge may appear in atmost 2 diff faces.
Thus the total no. of appearances of 9 edges ≤ 18. Hence, a contradiction
∴ $k_{3,3}$ is Non planar graph.

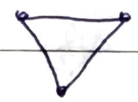
Q. Let G be a connected simple planar graph with $N_v \geq 3$ & n_e edges than prove that $n_e \leq 3n_v - 6$.

Ans. Consider a planar representation of a graph:-

Case I:- When $n_v = 3$

$n_e \leq 3 \cdot 3 - 6$

$\boxed{9 - 6 = 3}$



When $n_v = 3 \to G$ is a simple graph $\therefore$ max^n no. of edges $= 3$, $n_e \leq 3$.
Hence $3n_v - 6 = 3 \cdot 3 - 6 = 3$
Hence Theorem is true.

Case II:- $n_v > 3$.

If G does not contain any cycle than

$n_e = n_v - 1$

$3n_v - 6 = (n_v - 1) + (n_v - 2) + (n_v - 3)$

$3n_v - 6 = n_e + (n_v - 2) + (n_v - 3)$

$$\boxed{3n_v - 6 \geq n_e}$$

Further we suppose that $n_v > 3$ & G consists of a cycle with min no. of 3edges. $\therefore$ the no. of edges in the boundary of a face is $>= 3$ & hence we obtain

$\boxed{3n_f < 2n_e}$

than by Eulers theorem :-

$n_v - n_e + n_f = 2$

$n_f = 2 - n_v + n_e$

$3(2 - n_v + n_e) \leq 2n_e$

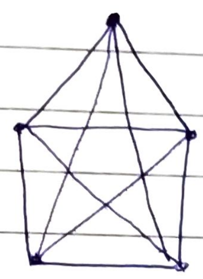
$6 - 3n_v + 3n_e \leq 2n_e$

$-n_e \geq 6 - 3n_v$

$\boxed{3n_v - 6 \geq n_e}$

Q. Show that graph k_5 is not a planar graph.

Ans. Here k_5 is a simple graph



Here $n_v = 5$ $\therefore$ max no. of edges =

$$\frac{n_v(n_v-1)}{2}$$

$$= \frac{5(5-1)}{2}$$

$$= 10$$

We suppose that k_5 is planar

$\Rightarrow n_e \leq 3n_v - 6$

$10 \leq 3 \cdot 5 - 6$

$10 \leq 15 - 6$

$10 \leq 9$

Not possible

$\Rightarrow k_5$ is Non planar