Exam 2 Solutions

Problem 1

(a): The matrix is orthogonal, thus

$$\det\begin{pmatrix} q_1 & q_2 & q_3 \end{pmatrix} = \pm 1$$

(b): By row reduction,

$$\det\begin{pmatrix} q_1+q_2 & q_2+q_3 & q_3+q_1 \end{pmatrix} = \det\begin{pmatrix} q_1+q_2 & -q_1+q_3 & q_3+q_1 \end{pmatrix}$$

$$= \det\begin{pmatrix} q_1+q_2 & -q_1+q_3 & 2q_3 \end{pmatrix}$$

$$= 2\det\begin{pmatrix} q_1+q_2 & -q_1 & q_3 \end{pmatrix}$$

$$= -2\det\begin{pmatrix} q_2 & q_1 & q_3 \end{pmatrix}$$

$$= \pm 2$$

(c): $\pm 1 \cdot \pm 1 = +1$ since they are paired.

Problem 2

For

$$Ax = B \Rightarrow \begin{bmatrix} 1 & -10 \\ \vdots & \\ 1 & 0 \\ \vdots & \\ 1 & 10 \end{bmatrix} \begin{bmatrix} c \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

solve instead

$$A^T A \hat{x} = A^T B$$

$$\begin{bmatrix} 21 & 0 \\ 0 & 770 \end{bmatrix} \begin{bmatrix} \hat{c} \\ \hat{D} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and thus $\hat{c} = 1/21$, $\hat{D} = 0$

Problem 3

(a): We have $P_a = A(A^T A)^{-1} A^T$, and $P_q = QQ^T$.

(b): Yes, they project onto the same subspace

(c): 3.

Problem 4

(a): The maximum degree is 2, since the largest possible set of expansions is

$$x\left(x(\text{stuff}_1) \cancel{+} x(\text{stuff}_2) + x(\text{stuff}_3)\right)$$

(b): Factor out x;

$$\det A = x \det \begin{vmatrix} 1 & x & x & x \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{vmatrix} = x\left(\left(\det \begin{matrix} x & x & x \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{matrix}\right) - \left(\det \begin{matrix} 1 & x & x \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{matrix}\right)\right)$$

$$= x(1-3x)$$