FEFLOW PLUG-IN FOR FULLY COUPLED LAND SUBSIDENCE MODELS BASED ON BIOT'S CONSOLIDATION THEORY

THEORY AND IMPLEMENTATION

1. PROBLEM DEFINITION

1.1. Consolidation processes

An element of porous media contains soil grains (or solid skeletons) and pore space (Fig. 1-1). When the element is saturated, 100% pore space is filled with water. The volume of pore space is V_{void}, the volume of soil grains is V_{solid}, and the total volume is V_{total}.

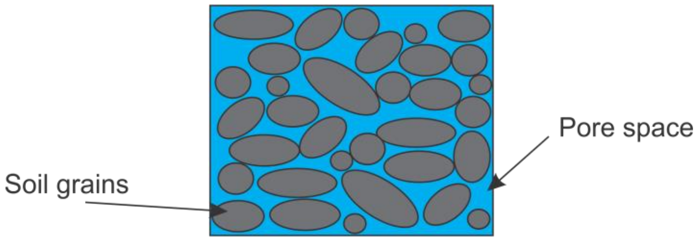


Fig. 1-1: A representative volume element of porous media

The void ratios is defined as:

$$e = \frac{V_{void}}{V_{solid}} \tag{1.1}$$

And the porosity n is:

$$n = \frac{e}{1+e} = \frac{V_{void}}{V_{total}} \tag{1.2}$$

Fig. 1-2 shows the saturated element which is subjected to a load σ on the top via a porous stone under following boundary conditions:

- Water can only flow via the porous stone on the top. The other boundaries are undrained.
- The left and the right have no horizontal movement (or roller boundary condition).
- The bottom has no vertical movement (roller boundary condition).

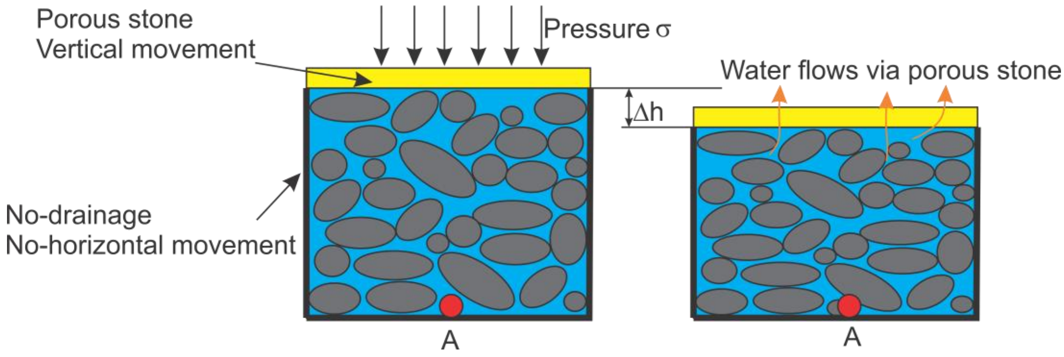


Fig. 1-2: Behavior of the saturated element under a load

Immediately after increasing the load σ, due to the low permeability of the porous media, the water cannot escape via the porous stone. Hence, almost σ is transferred to the excess pore pressure p. Therefore, the vertical deformation Δh is approximately zero, because the compressibility of water is so small.

Over time, the water is squeezed and flows out via the porous stone. The vertical deformation Δh increases, and the excess pore pressure decreases. Fig. 1-3 shows changes of Δh and the excess pore pressure of point A over time.

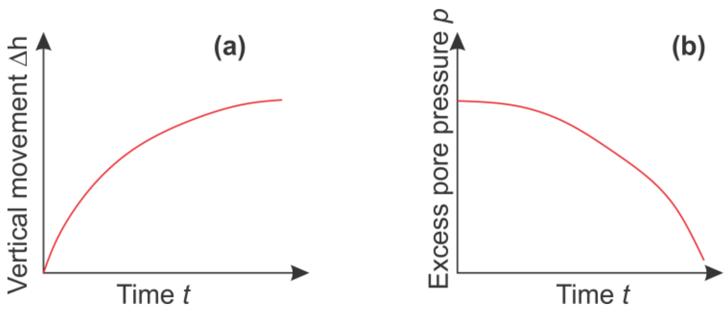


Fig. 1-3: Development of the vertical deformation and excess pore pressure over time

This process is called consolidation process. The theory for consolidation analyses was first developed by Terzaghi for one-dimensional problems. Then, it was extended by Biot for three-dimensional cases. Generally, the theories of consolidation are based on the following assumptions:

- The water flow in porous media is governed by Darcy's law.
- Soil grains, water, and pore space are compressed under a load. The total volume change of soil grains, water, and pore space is equal to the volume of escaped water.

1.2. The principle of effective stress

According to Terzaghi, the effective stress is the stress between soil grains. It is defined as:

$$\sigma^{'} = \sigma - p \tag{1.3}$$

where $\sigma^{'}$ is the effective stress, σ is the total stress, and p is the pore pressure. The pore pressure p has two components: hydrostatic pressure p_s and excess pore pressure p_e.

$$p = p_s + p_e = (H - Ele)\gamma_w \tag{1.4}$$

where H is the total head, Ele is the elevation, and γ_w is the unit weight of the water.

The equation (1.3) is rewritten for 3D problems:

$$\begin{Bmatrix} \sigma_x^{'} \\ \sigma_y^{'} \\ \sigma_z^{'} \\ \tau_{xy}^{'} \\ \tau_{xz}^{'} \\ \tau_{yz}^{'} \end{Bmatrix} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix} - \begin{Bmatrix} \alpha p \\ \alpha p \\ \alpha p \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix} - \begin{Bmatrix} \alpha(H - Ele)\gamma_w \\ \alpha(H - Ele)\gamma_w \\ \alpha(H - Ele)\gamma_w \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{1.5}$$

where α is the Biot's coefficient, σ_{ij} are stress components (Fig. 1-4).

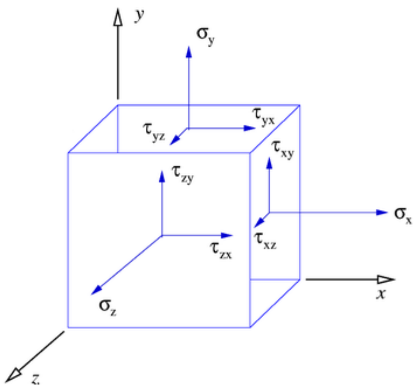


Fig. 1-4: Stress components in 3D coordinate

We denote the compressibility of water, solid grains, and porous media as C_f, C_s and C_m respectively. The Biot's coefficient is defined as:

$$\alpha = 1 - \frac{C_s}{C_m} \quad \textbf{(1.6)}$$

If $C_s \sim 0$, $\alpha \sim 1$. The equation (1.5) is similar to the Terzaghi's theory (equation 1.3).

1.3. The relation between stress and strain: Hook's law

In 3D coordinate, displacement components of an element of the x-direction, y-direction, and z-direction are denoted as u, v, w respectively. The strain components are:

$$\begin{aligned}
&\varepsilon_{xx} = \frac{\partial u}{\partial x}; \varepsilon_{yy} = \frac{\partial v}{\partial y}; \varepsilon_{zz} = \frac{\partial w}{\partial z}; \varepsilon = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} \\
&\varepsilon_{xy} = \varepsilon_{yx} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\
&\varepsilon_{yz} = \varepsilon_{zy} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\
&\varepsilon_{xz} = \varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)
\end{aligned} \quad \textbf{(1.7)}$$

According to the Hook's law, the relation between stress and strain are:

$$\begin{aligned}
&\sigma'_{xx} = -\left(K + \frac{4}{3}G\right)\varepsilon_{xx} - \left(K - \frac{2}{3}G\right)\varepsilon_{yy} - \left(K - \frac{2}{3}G\right)\varepsilon_{zz} \\
&\sigma'_{yy} = -\left(K + \frac{4}{3}G\right)\varepsilon_{yy} - \left(K - \frac{2}{3}G\right)\varepsilon_{xx} - \left(K - \frac{2}{3}G\right)\varepsilon_{zz} \\
&\sigma'_{zz} = -\left(K + \frac{4}{3}G\right)\varepsilon_{zz} - \left(K - \frac{2}{3}G\right)\varepsilon_{xx} - \left(K - \frac{2}{3}G\right)\varepsilon_{yy} \\
&\sigma'_{xy} = \sigma'_{yx} = 2G\varepsilon_{xy} \\
&\sigma'_{xz} = \sigma'_{zx} = 2G\varepsilon_{xz} \\
&\sigma'_{yz} = \sigma'_{zy} = 2G\varepsilon_{yz}
\end{aligned} \quad \textbf{(1.8)}$$

where ε is total volume strain, ε_{ij} are strain components, K is bulk modulus, G is shear modulus. In the equation (1.8), the minus sign means the compressive stresses are positive.

The bulk modulus K:

$$K = \frac{1}{C_m} \quad \textbf{(1.9)}$$

The shear modulus G:

$$G = \frac{3K(1-2\mu)}{2(1+\mu)} \quad \textbf{(1.10)}$$

where μ is the Poisson's ratio.

According to the equation (1.8), changes of effective stresses lead to deformation of porous media. If effective stresses increase, porous media are compacted and vice versa. To increase the effective stress, there are three possibilities:

- (1) Increase total stress
- (2) Decrease pore pressure
- (3) Combine both (1) and (2)

1.4. Land subsidence mechanism related to consolidation processes

The land subsidence refers to compaction of porous media. As previously mentioned, there are three cases:

- (1) Increase total stress
- (2) Decrease pore pressure
- (3) Combine both (1) and (2)

1.4.1. *Increase total stress – unconfined aquifers*

We consider an aquifer system includes an unconfined aquifer on the top, an aquitard, a confined aquifer, and an aquitard between the confined aquifer and bedrock on the bottom (Fig. 1-5).

The water table in the unconfined aquifer decreases ΔH from an initial water level. The unconfined aquifer has:

- Unit weight of the unsaturated zone: γ_{unsat_a}
- Unit weight of the saturated zone: γ_{sat_a}

The saturated unit weight of the aquitard is γ_{sat_q}.

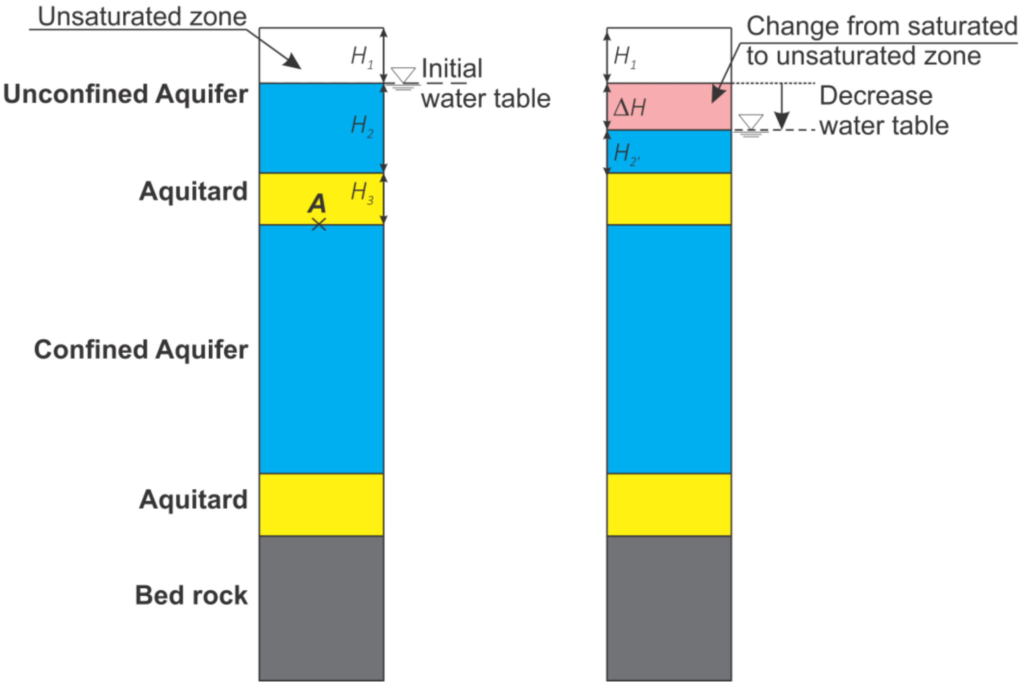


Fig. 1-5: Increase of the total stress due to lowering the water table in the unconfined aquifer

At point A, the total vertical stress of the initial state is (before lowering of the water table):

$$\sigma_{A0} = \gamma_w\left(H_2 + H_3\right) + \gamma_{unsat\_a}H_1 + \left(\gamma_{sat\_a} - \gamma_w\right)H_2 + \left(\gamma_{sat\_q} - \gamma_w\right)H_3 \quad \textbf{(1.11)}$$

Or the effective stress is:

$$\sigma'_{A0} = \gamma_{unsat\_a}H_1 + \left(\gamma_{sat\_a} - \gamma_w\right)H_2 + \left(\gamma_{sat\_q} - \gamma_w\right)H_3 \quad \textbf{(1.12)}$$

When the water table decreases ΔH, the total stress now is:

$$\sigma_{A1} = \gamma_w\left(H_2 - \Delta H + H_3\right) + \gamma_{unsat\_a}\left(H_1 + \Delta H\right) + \left(\gamma_{sat\_a} - \gamma_w\right)\left(H_2 - \Delta H\right) + \left(\gamma_{sat\_q} - \gamma_w\right)H_3 \quad \textbf{(1.13)}$$

Or the effective stress is:

$$\sigma'_{A1} = \gamma_{unsat\_a}\left(H_1 + \Delta H\right) + \left(\gamma_{sat\_a} - \gamma_w\right)\left(H_2 - \Delta H\right) + \left(\gamma_{sat\_q} - \gamma_w\right)H_3 \quad \textbf{(1.14)}$$

Compare equation (1.12) and (1.14), the change of the effective stress is:

$$\Delta\sigma'_A = \sigma'_{A1} - \sigma'_{A0} = \gamma_{unsat\_a}\Delta H - \left(\gamma_{sat\_a} - \gamma_w\right)\Delta H \quad \textbf{(1.15)}$$

If the unsaturated unit weight is equal to the saturated unit weight, the change of effective stress is:

$$\Delta\sigma_A^{'} = \sigma_{A1}^{'} - \sigma_{A0}^{'} = \gamma_w \Delta H \quad \textbf{(1.16)}$$

In summary, lowering of the groundwater table leads to the increase of the total stress which results in increasing effective stress. However, due to the low permeability of the aquitard, consolidation processes happen, and the land subsidence magnitude develops over time.

1.4.2. *Decrease pore pressure in confined aquifers*

In Fig. 1-6, initially, the water table of the unconfined aquifer is coincident with the potentiometric of the unconfined aquifer. Afterward, the potentiometric decreases ΔH (m).

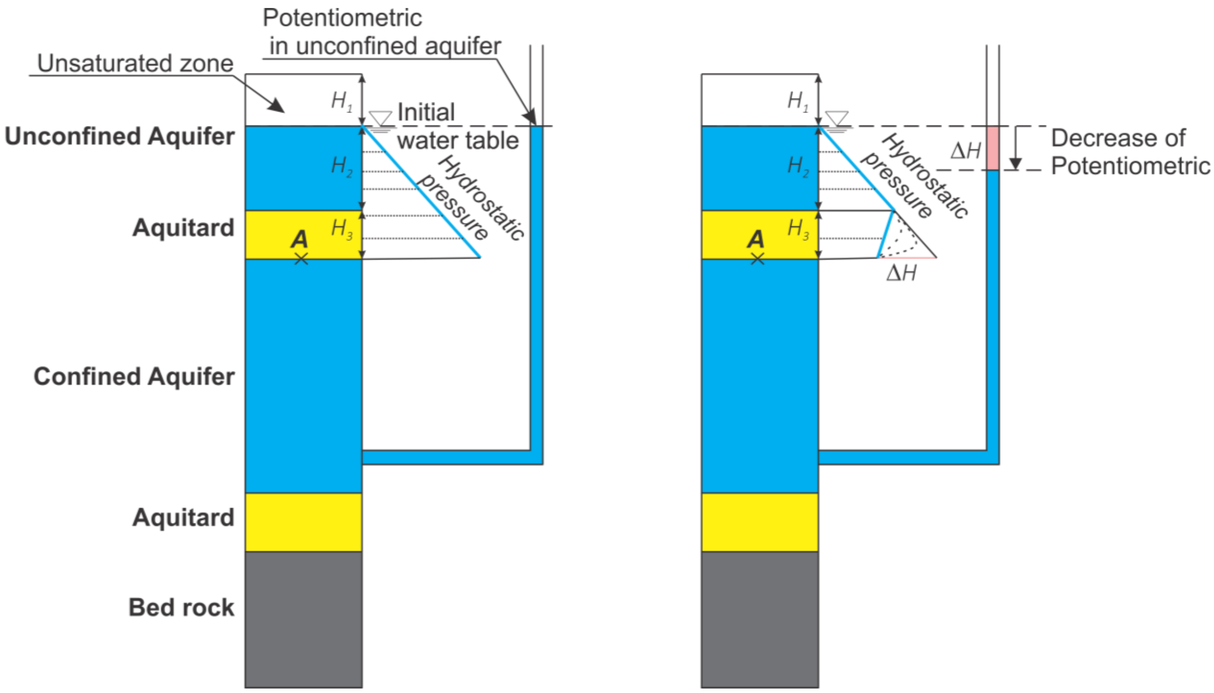


Fig. 1-6: Increase of effective stress due to a decrease of pore pressure in confined aquifers

In this case, the total vertical stresses are constant. At the point A, the total stress is σ_A. At the beginning, the pore pressure of the point A is the hydrostatic pressure:

$$p_A = p_s = \gamma_w (H_2 + H_3) \quad \textbf{(1.17)}$$

When the potentiometric drops ΔH, the pore pressure of the point A decreases:

$$\Delta p_A = -\gamma_w \Delta H \quad \textbf{(1.18)}$$

Then, the increase of the effective stress is:

$$\Delta\sigma_A^{'} = -\Delta p_A = \gamma_w \Delta H \quad \textbf{(1.19)}$$

1.4.3. *Summary*

- Decreases of groundwater tables lead to increases of effective stresses.
- Increases of effective stresses result in land subsidence. Aquitards are compacted mostly due to high compressibility. Compaction of aquifers is much smaller than compaction of aquitards.
- Consolidation processes happen in aquitards due to low permeability.

1.5. Three-dimensional consolidation theory: Biot's theory

Biot's theory includes equilibrium equations and storage equation. The storage equation is:

$$\alpha \frac{\partial \varepsilon}{\partial t} + S\gamma_w \frac{\partial H}{\partial t} - \frac{\partial}{\partial x}\left(k_x \frac{\partial H}{\partial x}\right) - \frac{\partial}{\partial y}\left(k_y \frac{\partial H}{\partial y}\right) - \frac{\partial}{\partial z}\left(k_z \frac{\partial H}{\partial z}\right) = 0 \quad \textbf{(1.20)}$$

where ε is the total volume strain, t is the time, H is the total head, k_i is the hydraulic conductivity, S is storage specific. S is defined as:

$$S = nC_f + (\alpha - n)C_s \tag{1.21}$$

where n is the porosity, C_f is the water compressibility, α is the Biot's coefficient, C_s is the soil grains compressibility, C_m is the compressibility of porous media. For shallow aquitards, $C_m \gg C_s$ and $C_m \gg C_f$. Hence, in the equation 1.20, the component $S\dfrac{\partial H}{\partial t}$ is much smaller than $\alpha\dfrac{\partial \varepsilon}{\partial t}$.

The equilibrium equations are:

$$\begin{aligned}
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} - fx = 0 \\
\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} - fy = 0 \\
\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} - fz = 0
\end{aligned} \tag{1.22}$$

where σ_{ij} are the total stress components, f_i are the body forces. The total stresses are related to effective stresses and pore pressure via equation (1.8).

1.6. Backward interpolation

We denote ε_0 and H_0 as the total volume strain and the total head at the time $t = t_0$, ε_1 and H_1 as values at the time $t = t_1$, and $\Delta t = t_1 - t_0$.

Equation 1.20 is integrated with the back-ward interpolation scheme, or it is rewritten as:

$$\alpha(\varepsilon_1 - \varepsilon_0) + S\gamma_w(H_1 - H_0) - \Delta t\left[\frac{\partial}{\partial x}\left(k_x\frac{\partial H_1}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y\frac{\partial H_1}{\partial y}\right) + \frac{\partial}{\partial z}\left(k_z\frac{\partial H_1}{\partial z}\right)\right] \tag{1.23}$$

1.7. Finite element approximation

Any 3D geometry can be approximated by using four primitive shapes: tetrahedron, hexahedron, prism, and pyramid. Fig. 1-7 shows the second-order of four element types.

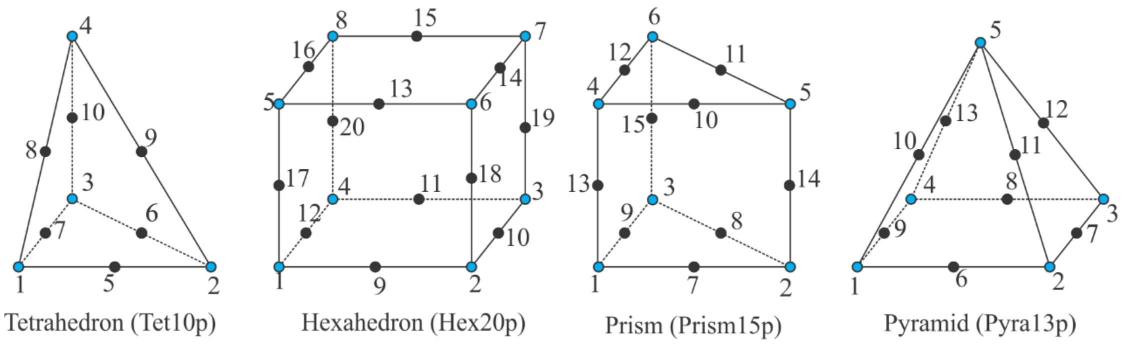


Fig. 1-7: Different type of 3D elements

We call *non* is the total number of nodes, *noe* is the total of elements. For each element with n nodes, the vectors of element displacements and the total head are u_e, v_e, w_e and H_e respectively. The displacements and the total head of a point inside an element are approximated by Galerkin method:

$$
\begin{aligned}
u &= N_d u_e \\
v &= N_d v_e \\
w &= N_d w_e \\
H &= N_p H_e, \text{ or } p_e = N_p \gamma_w \left(H_e - Ele_e \right)
\end{aligned} \quad \textbf{(1.24)}
$$

where N_d is the shape function which is used for the displacement field, and N_p is the shape function for the total head field, y_e is the vector of elevation. If the first-order element type is used for both the placement and total head field, $N_d \equiv N_p$.

1.7.1. *X-direction*

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} - fx = 0 => \sum_1^{noe} \iiint_V N_d^T \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} - fx \right) dV = 0 \quad \textbf{(1.25)}$$

1.7.2. *Y-direction*

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} - fy = 0 => \sum_1^{noe} \iiint_V N_d^T \left(\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} - fy \right) dV = 0 \quad \textbf{(1.26)}$$

1.7.3. *Z-direction*

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} - fz = 0 => \sum_1^{noe} \iiint_V N_d^T \left(\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} - fx \right) dV = 0 \quad \textbf{(1.27)}$$

1.7.4. *Storage equation*

$$
\begin{aligned}
&\alpha \left(\varepsilon_1 - \varepsilon_0 \right) + S\gamma_w \left(H_1 - H_0 \right) - \Delta t \left[\frac{\partial}{\partial x} \left(k_x \frac{\partial H_1}{\partial x} \right) + \frac{\partial}{\partial y} \left(k_y \frac{\partial H_1}{\partial y} \right) + \frac{\partial}{\partial z} \left(k_z \frac{\partial H_1}{\partial z} \right) \right] \\
&\Rightarrow \sum_1^{noe} \iiint_V \left\{ \alpha \left(\varepsilon_1 - \varepsilon_0 \right) + S\gamma_w \left(H_1 - H_0 \right) - \Delta t \left[\frac{\partial}{\partial x} \left(k_x \frac{\partial H_1}{\partial x} \right) + \frac{\partial}{\partial y} \left(k_y \frac{\partial H_1}{\partial y} \right) + \frac{\partial}{\partial z} \left(k_z \frac{\partial H_1}{\partial z} \right) \right] \right\} dV = 0
\end{aligned} \quad \textbf{(1.28).}
$$

1.7.5. *System of equations*

From equations (1.25) to (1.28), and the equation (1.8), a final system of equations is:

$$
\begin{bmatrix}
[A] & [B] & [C] & [D] \\
[E] & [F] & [G] & [H] \\
[I] & [J] & [L] & [M] \\
[N] & [O] & [P] & [Q] + \Delta t [R]
\end{bmatrix}
\begin{pmatrix} \{u_1\} \\ \{v_1\} \\ \{w_1\} \\ \{H_1\} \end{pmatrix}
=
\begin{pmatrix} \{F_{x1}\} + [D]\{H_{ini}\} \\ \{F_{y1}\} + [G]\{H_{ini}\} \\ \{F_{z1}\} + [I]\{H_{ini}\} \\ \{QQ\} \end{pmatrix} \quad \textbf{(1.29)}
$$

where:

- $\{u_1\}, \{v_1\}, \{w_1\}, \{H_1\}$ are the solution vectors at the current time step $t = t_1$
- $\{u_0\}, \{v_0\}, \{w_0\}, \{H_0\}$ are the solution vectors at the previous time step $t = t_0$
- $\{F_{x1}\}, \{F_{y1}\}, \{F_{z1}\}$ are the load vectors at the current time step $t = t_1$
- $\{H_{ini}\}$ is the initial water head
- $[A] = \sum_1^{noe} \left[\iiint_V \left(K + \frac{4}{3} G \right) \frac{\partial N_d^T}{\partial x} \frac{\partial N_d}{\partial x} dV + \iiint_V G \frac{\partial N_d^T}{\partial y} \frac{\partial N_d}{\partial y} dV + \iiint_V G \frac{\partial N_d^T}{\partial z} \frac{\partial N_d}{\partial z} dV \right]$

- $$\left[B\right]=\sum_{1}^{noe}\left[\iiint_V\left(K-\frac{2}{3}G\right)\frac{\partial N_d^T}{\partial x}\frac{\partial N_d}{\partial y}dV+\iiint_V G\frac{\partial N_d^T}{\partial y}\frac{\partial N_d}{\partial x}dV\right]$$
- $$\left[C\right]=\sum_{1}^{noe}\left[\iiint_V\left(K-\frac{2}{3}G\right)\frac{\partial N_d^T}{\partial x}\frac{\partial N_d}{\partial z}dV+\iiint_V G\frac{\partial N_d^T}{\partial z}\frac{\partial N_d}{\partial x}dV\right]$$
- $$\left[D\right]=\sum_{1}^{noe}\iiint_V -\alpha\gamma_w\frac{\partial N_d^T}{\partial x}N_p dV$$
- $$\left[E\right]=\left[B\right]^T$$
- $$\left[F\right]=\sum_{1}^{noe}\left[\iiint_V\left(K+\frac{4}{3}G\right)\frac{\partial N_d^T}{\partial y}\frac{\partial N_d}{\partial y}dV+\iiint_V G\frac{\partial N_d^T}{\partial x}\frac{\partial N_d}{\partial x}dV+\iiint_V G\frac{\partial N_d^T}{\partial z}\frac{\partial N_d}{\partial z}dV\right]$$
- $$\left[G\right]=\sum_{1}^{noe}\left[\iiint_V\left(K-\frac{2}{3}G\right)\frac{\partial N_d^T}{\partial y}\frac{\partial N_d}{\partial z}dV+\iiint_V G\frac{\partial N_d^T}{\partial z}\frac{\partial N_d}{\partial y}dV\right]$$
- $$\left[H\right]=\sum_{1}^{noe}\iiint_V -\alpha\gamma_w\frac{\partial N_d^T}{\partial y}N_p dV$$
- $$\left[I\right]=\left[C\right]^T$$
- $$\left[J\right]=\left[G\right]^T$$
- $$\left[L\right]=\sum_{1}^{noe}\left[\iiint_V\left(K+\frac{4}{3}G\right)\frac{\partial N_d^T}{\partial z}\frac{\partial N_d}{\partial z}dV+\iiint_V G\frac{\partial N_d^T}{\partial x}\frac{\partial N_d}{\partial x}dV+\iiint_V G\frac{\partial N_d^T}{\partial y}\frac{\partial N_d}{\partial y}dV\right]$$
- $$\left[M\right]=\sum_{1}^{noe}\iiint_V -\alpha\gamma_w\frac{\partial N_d^T}{\partial z}N_p dV$$
- $$\left[N\right]=-\frac{\left[D\right]^T}{\gamma_w}$$
- $$\left[O\right]=-\frac{\left[H\right]^T}{\gamma_w}$$
- $$\left[P\right]=-\frac{\left[M\right]^T}{\gamma_w}$$
- $$\left[Q\right]=\sum_{1}^{noe}\iiint_V S\gamma_w N_p^T N_p dV$$
- $$\left[R\right]=\sum_{1}^{noe}\iiint_V\left(k_x\frac{\partial N_p^T}{\partial x}\frac{\partial N_p}{\partial x}+k_y\frac{\partial N_p^T}{\partial y}\frac{\partial N_p}{\partial y}+k_z\frac{\partial N_p^T}{\partial z}\frac{\partial N_p}{\partial z}\right)dV$$
- $$\left[QQ_1\right]=\left[N\right]\left\{u_0\right\}+\left[O\right]\left\{v_0\right\}+\left[P\right]\left\{w_0\right\}+\left[Q\right]\left\{H_0\right\}$$

2. FEFLOW MESH AND GROUND WATER EQUATION

2.1. FEFLOW mesh

By default, FEFLOW uses the first-order prism element for the finite element mesh. For simplicity's sake, we also use the first order element for the displacement field.

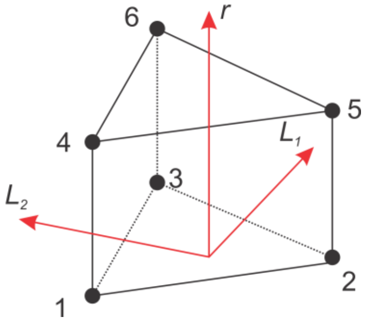


Fig. 2-1: FEFLOW element type and the local coordinate

The shape functions for the displacement field and the total head field are similar. The shape function:

$$\{N_d\} \equiv \{N_p\} = \{N_1 \quad N_2 \quad N_3 \quad N_4 \quad N_5 \quad N_6\} \tag{2.1}$$

where:

$$\begin{aligned}
L_3 &= 1 - L_1 - L_2 \\
N_1 &= \frac{1}{2} L_1 (1 - r) \\
N_2 &= \frac{1}{2} L_2 (1 - r) \\
N_3 &= \frac{1}{2} L_3 (1 - r) \\
N_4 &= \frac{1}{2} L_1 (1 + r) \\
N_5 &= \frac{1}{2} L_2 (1 + r) \\
N_6 &= \frac{1}{2} L_3 (1 + r)
\end{aligned} \tag{2.2}$$

2.2. FEFLOW groundwater equation – a simple form of Biot's theory

The storage equation of Biot's theory is:

$$\alpha \frac{\partial \varepsilon}{\partial t} + S\gamma_w \frac{\partial H}{\partial t} - \frac{\partial}{\partial x}\left(k_x \frac{\partial H}{\partial x}\right) - \frac{\partial}{\partial y}\left(k_y \frac{\partial H}{\partial y}\right) - \frac{\partial}{\partial z}\left(k_z \frac{\partial H}{\partial z}\right) = 0 \tag{2.3}$$

In the conventional groundwater theory, there is no horizontal deformation. If we consider the gravity direction is the negative z-direction (Fig. 2-2), we have:

$$\varepsilon_{xx} = \varepsilon_{yy} = 0 \tag{2.4}$$

And

$$\varepsilon = \varepsilon_{zz} \tag{2.5}$$

If porous media behave as an elastic material, then:

$$\varepsilon = \varepsilon_{zz} = -m_v \sigma'_{zz} = -m_v \left[\sigma_{zz} - \alpha\gamma_w (H - Ele)\right] \tag{2.6}$$

where m_v is the confined compressibility of porous media.

$$m_v = \frac{1}{K + \frac{4}{3}G} \tag{2.7}$$

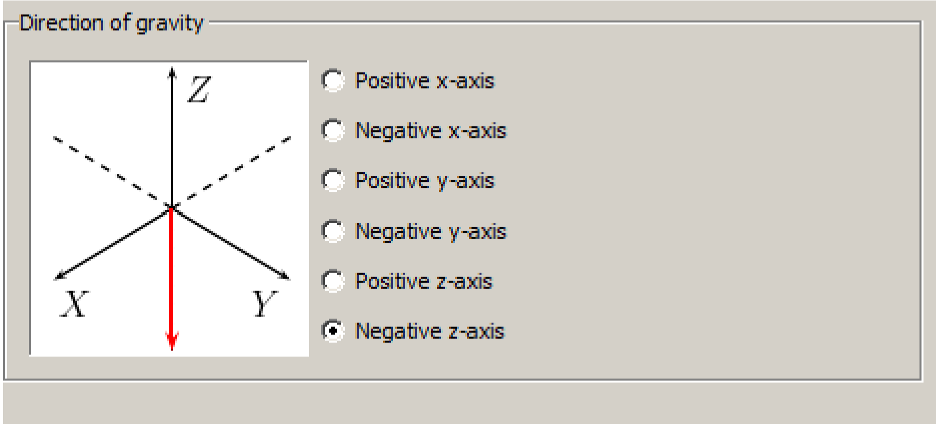


Fig. 2-2: Default gravity direction in FEFLOW

From the equation (2.6), we have:

$$\frac{\partial \varepsilon}{\partial t} = -m_v \frac{\partial \sigma_{zz}}{\partial t} + m_v \gamma_w \alpha \frac{\partial H}{\partial t} - 0 = -m_v \frac{\partial \sigma_{zz}}{\partial t} + m_v \alpha \gamma_w \frac{\partial H}{\partial t} \tag{2.8}$$

Substitution into the equation (2.3) gives:

$$\left(\alpha^2 m_v + S\right)\gamma_w \frac{\partial H}{\partial t} = m_v \frac{\partial \sigma_{zz}}{\partial t} + \nabla\left(k \nabla H\right) \tag{2.9}$$

where ∇ is the Laplace operator.

If the total stress is constant over time, $\frac{\partial \sigma_{zz}}{\partial t} = 0$, the equation (2.8) becomes:

$$\begin{aligned} &\left(\alpha^2 m_v + S\right)\gamma_w \frac{\partial H}{\partial t} = \nabla\left(k \nabla H\right) \\ &\Rightarrow S_s \frac{\partial H}{\partial t} = \nabla\left(k \nabla H\right) \end{aligned} \tag{2.10}$$

Equation (2.10) is the groundwater equation in FEFLOW. The new parameter S_s includes compressibility of water, soil grains, and porous media. In other words, the equation (2.10) is the uncoupled form of Biot's theory. Hence, land subsidence problems can be solved by two steps models:

- **Step 1**: The groundwater equation (2.9) is solved to obtain the total head field for each time step.
- **Step 2**: Calculate vertical deformation for each time step according to change of the water head.

3. SOLVING FULLY COUPLED EQUATIONS

3.1. Solving full coupled matrix system

The system of equations (1.29) can be solved directly with either a direct solver or an iterative solver.

3.2. Newton Block method

For easier expression, we reorganize the equation (1.29) as:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{pmatrix} d \\ H \end{pmatrix} = \begin{pmatrix} F \\ Q \end{pmatrix} \tag{3.1}$$

The equation (3.1) can be solved with the Newton Block method by following steps:

- **Step 1:** Calculate $[K] = [A^{-1}][B]$
- **Step 2:** Solve the equation $[A]q = F$ to obtain q
- **Step 3:** Calculate $r = Q - [C]q$
- **Step 4:** Solve $[E]H = r$ to obtain H. The matrix $[E] = [D] - [C][K] = [D] - [P]$
- **Step 5:** Calculate $d = q - [K]H$

Theoretically, **Step 4** can be implemented using FEFLOW. The other steps can be done using Plug-in. However, in Step 1, it is time-consuming and extensive memory usage to inverse and to store matrix $[A]$, especially when the dimension of the matrix $[A]$ becomes larger and larger. The inverse matrix $[A^{-1}]$ is not the sparse matrix anymore.

Hence, we do not use this method in developing the Plug-in.

4. PLUG-IN DESIGN

4.1. Plug-in description

Fig. 4-1 shows a program flow of a FEM model in FeFlow. Generally, FeFlow has three main modules:

- **Problem Editor:** To edit geometry, input parameters, boundary conditions... of the model.
- **Simulator:** To solve systems of equations.
- **Post-Processing:** To visualize results.

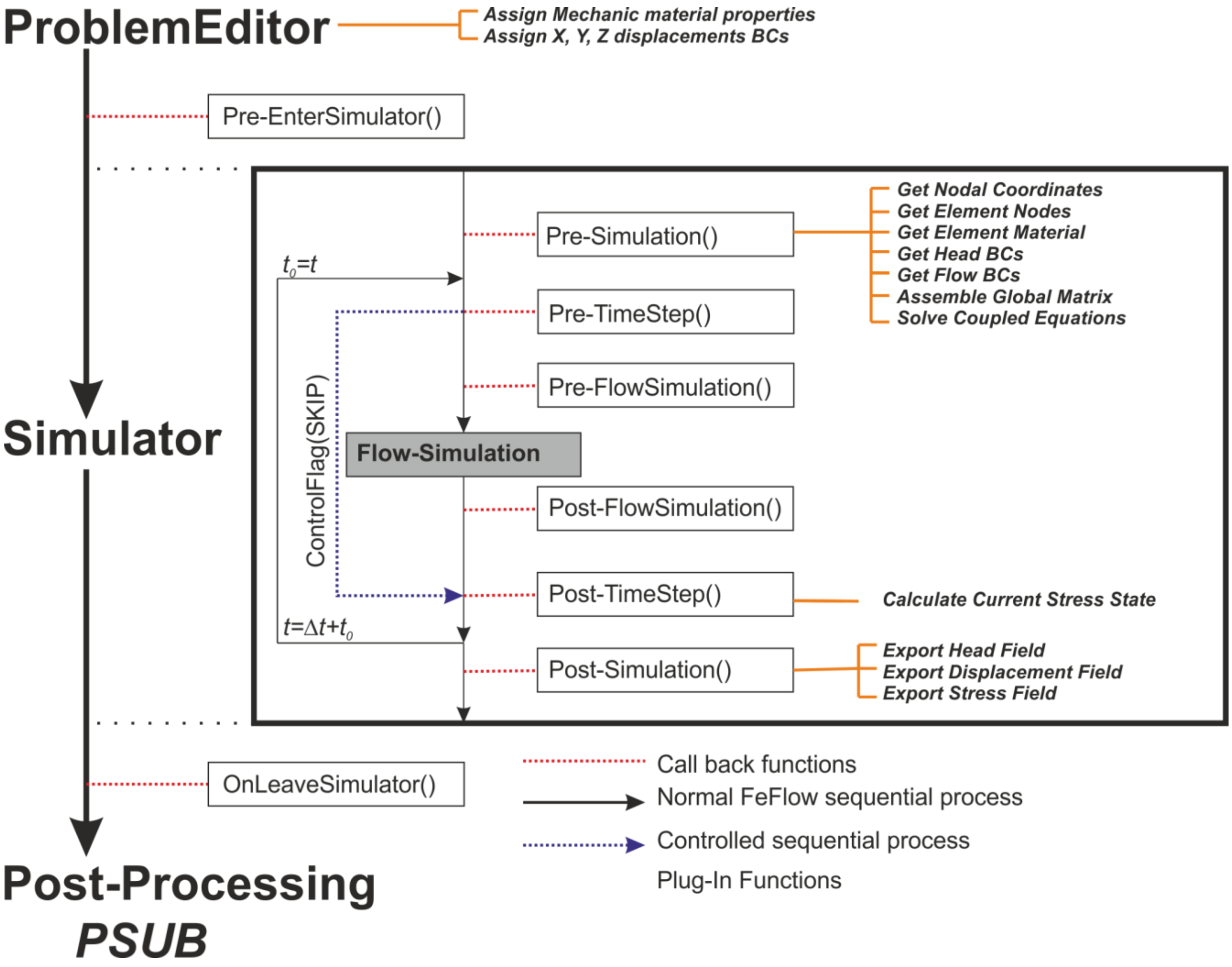


Fig. 4-1: Develop Plug-in

Within simulation processes (or in the simulator), FeFlow loops over each time step until the final simulation time is reached. FeFlow provides APIs which are a combination of call-back functions and API-Functions.

The main structure of the Plug-in is shown in Fig. 4-1:

- **Problem Editor Phase:**
  - Plug-in defines and assigns the mechanical properties.
  - Plug-in defines and assigns Dirichlet boundary conditions.
- **Simulation Phase:**
  - Plug-in ignores FeFlow solver by setting the control flag.
  - Plug-in gets and write to ASCII files: nodal coordinates (*coordinates.dat*), elements array (*elements.dat*), material properties (*properties.dat*)
  - For each time step, Plug-in assembles fully coupled matrix (equation 1.29).
  - Plug-in solves the matrix system using Pardiso direct solver.

- Plug-in calculates effective stresses, pore pressure for all elements.
- Plug-in write results to ASCII file: *X-displacement.txt, Y-displacement.txt, Z-discplacment.txt, Head.txt*

5. VERIFICATION – 1D TERZAGHI'S ANALYTICAL SOLUTION

In this section, a consolidation process of a soil column is analyzed by various approaches:

- (**1**) The Terzaghi's analytical solution.
- (**2**) A FEM model is developed in FEFLOW. Results from FEFLOW are compared to the analytical solution to verify that, the groundwater equation in FEFLOW can be derived from the uncoupled solution of Biot's theory.
- (**3**) Using Plug-in.

5.1. Problem description

We consider a column of soil which has $1.0m$ radius, and $10.0m$ height in a 3D coordinate as Fig. 5-1. The groundwater table is constant at $10.0m$ elevation. A pressure $\sigma = 98.06 kN/m^2$ acts on the top of the column under these following boundary conditions:

- The water can only escape via the top of the column.
- The bottom boundary has no vertical movement ($w = 0$)
- The outer boundary has no horizontal movement ($u = 0;\ v = 0$)

Properties of the soil are:

- Soils are isotropic
- Hydraulic conductivity $k_x = k_y = k_z = 1e-7(m/s) = 8.64e-3(m/d)$
- The compressibility of water $C_f = 1e-7(m^2/kN)$
- The compressibility of soil grains $C_s = 1e-10(m^2/kN)$
- The compressibility of porous media $C_m = 0.002(m^2/kN)$

  or the bulk modulus $K = 500(kN/m^2)$.
- The Poisson's ratio is $\mu = 0.3$.
- The shear modulus $G = \dfrac{3K(1-2\mu)}{2(1+\mu)} = 230.77(kN/m^2)$
- The confined compressibility $m_v = \dfrac{1}{K + \dfrac{4}{3}G} = 0.001238$
- The porosity $n = 0.6$
- The Biot's coefficient $\alpha \sim 1$
- The storage-specific for Biot's theory is $S = nC_f + (\alpha - n)C_s = 6e-8(m^2/kN)$
- The storage-specific for an uncoupled solution (in FEFLOW) $S_s = \gamma_w(\alpha^2 m_v + S) = 1.238e-2(m^{-1})$

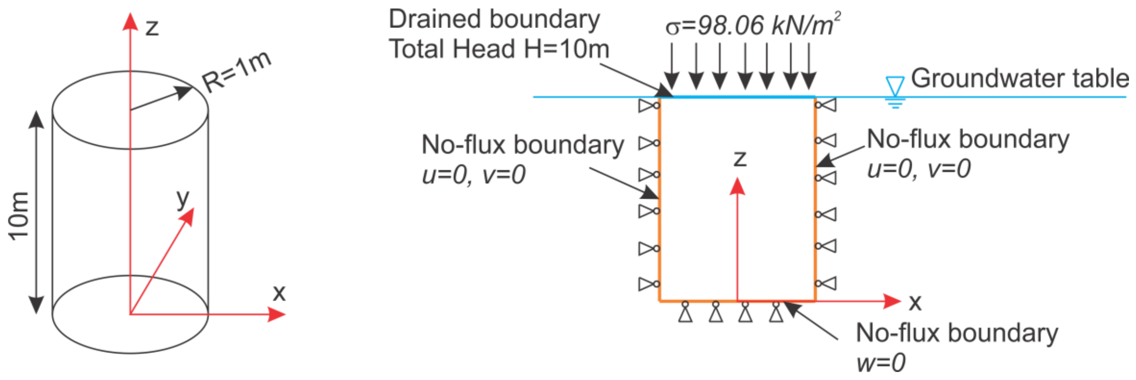


Fig. 5-1: Model for 1D Terzaghi's solution

The total consolidation time is $T = 100$ (days). The total time step is $ns = 100$, and the constant time step is $\Delta t = 1$ (day).

We will calculate the total head at the bottom boundary during the consolidation time.

5.2. Terzaghi's 1D analytical solution

At $t = 0$, the static head at the bottom is:

$$H_{0s} = 10.0(m) \tag{5.1}$$

With $\sigma = 98.06\left(kN/m^2\right)$ and at $t = 0$, the soil is compressed under an undrained condition. The approximated excess pore pressure is:

$$p_{0e} \sim \sigma = 98.06\left(kN/m^2\right) \tag{5.2}$$

Or the total head at the $t = 0$ is:

$$H_0 = 10.0 + \frac{98.06}{9.806} = 20(m) \tag{5.3}$$

The excess pore pressure at the time t and position z is:

$$p_{et} = p_{0e}\frac{4}{\pi}\sum_{j=1}^{\infty}\frac{-1^{j-1}}{2j-1}\cos\left[(2j-1)\frac{\pi}{2}\frac{z}{H}\right]\exp\left[-(2j-1)^2\frac{\pi^2}{4}\frac{c_v t}{H^2}\right] \tag{5.4}$$

where c_v is the consolidation coefficient.

$$c_v = \frac{k_z}{\gamma_f}\frac{K + \frac{4}{3}G}{\alpha^2 + \left(K + \frac{4}{3}G\right)S} = 8.236e^{-7}\left(m^2/s\right) = 0.0711\left(m^2/d\right) \tag{5.5}$$

The total head at the time t and position z is:

$$H_t = H_{0s} + \gamma_w p_{et} \tag{5.6}$$

5.3. FEFLOW uncoupled solution

The model and boundary conditions in FEFLOW is shown in Fig. 5-2.

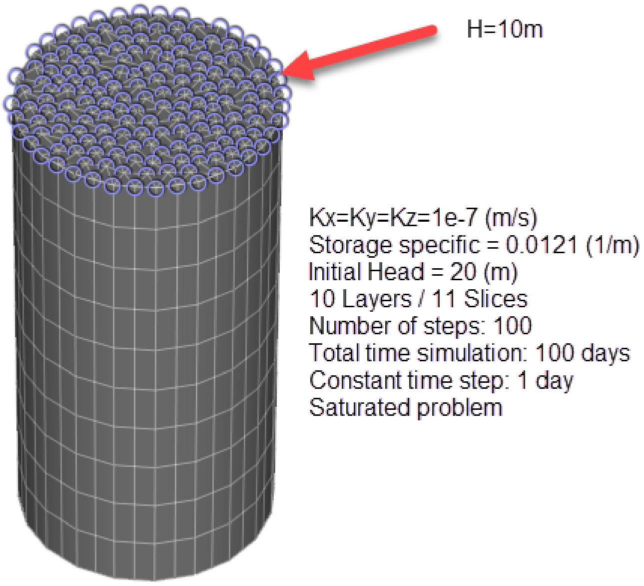


Fig. 5-2: FEFLOW model

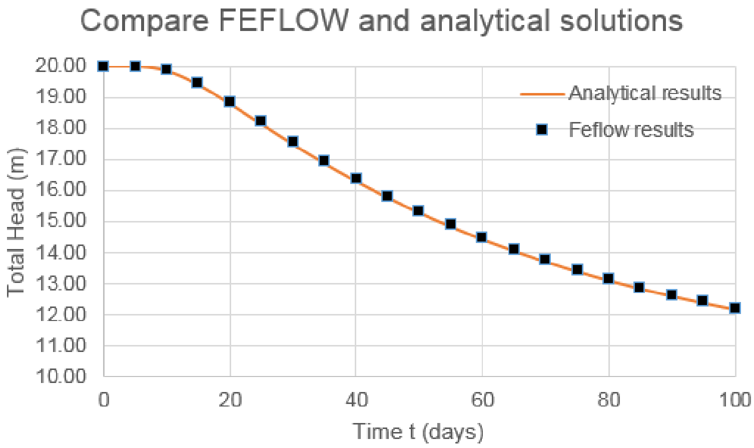


Fig. 5-3: Comparison of results between FEFLOW and analytical solutions

Fig. 5-3 compares the results between FEFLOW and analytical method. They are identical. It proves that the groundwater equation in FEFLOW is an uncoupled form of the Biot's theory.

5.4. Plug-in

The input parameters for Plug-in are:

- Bulk modulus *K*, shear modulus *G*.
- Biot's coefficient α
- The storage coefficient S
- Total of calculation step, time step.

The other information is obtained via FeFlow.

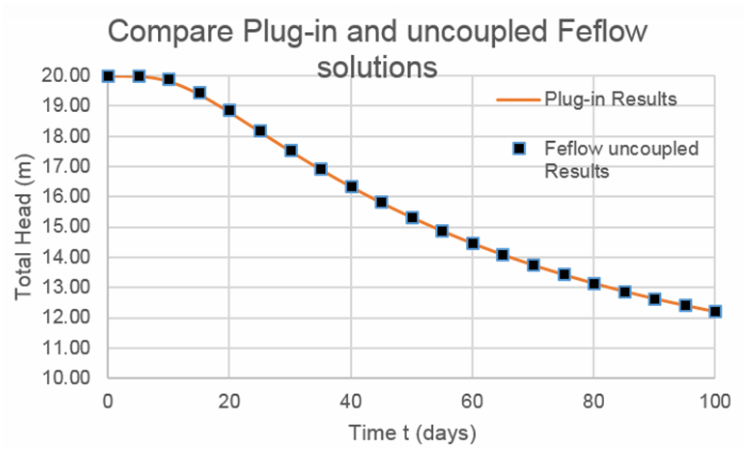


Fig. 5-4: Comparison between Plug-in and uncoupled FeFlow

6. BENCHMARK – CRYER'S EFFECT – DE LEEUW'S PROBLEM

6.1. Problem description

A cylindrical soil sample, which has the diameter $2a = 2.0(m)$ and the height $H = 1.0(m)$, is constrained by two plates on the top and the bottom (Fig. 6-1). The sample is loaded by a uniform pressure $q = 98.604(kN/m^2)$ at the outer boundary, which is also the drained boundary. The water table is at $z = 1.0(m)$, or the total head before applying load q is $H_{s0} = 1.0(m)$.

Immediately after applying load q, the initial excess pore pressure is $p_0 \sim q = 98.604(kN/m^2)$, or the initial total is: $H_0 = H_{s0} + \dfrac{p_0}{\gamma_w} = 1 + 10 = 11.0(m)$

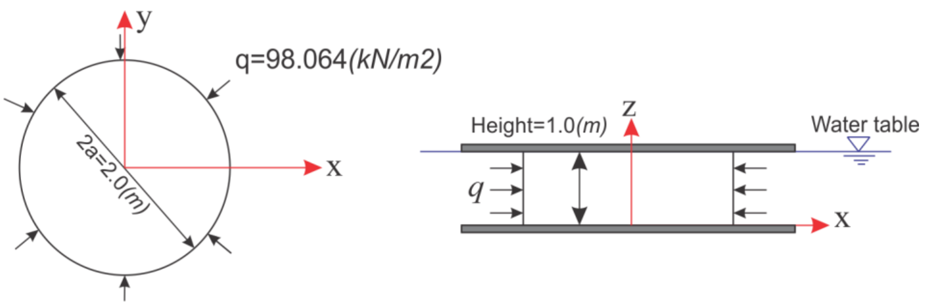


Fig. 6-1: De Leeuw's problem

The total head at position which has radius $r(m)$ and at the time $t(day)$ is calculated as:

$$H = H_{s0} + \frac{p_0}{\gamma_w}\sum_{j=1}^{\infty}\frac{J_0(\xi_j) - J_0(\xi_j r/a)}{\left(1 - m_c\xi_j^2 - 1/4m_c\right)J_0(\xi)}\exp\left(-\xi_j^2 c_v t/a^2\right) \qquad \textbf{(6.1)}$$

where J_0, J_1 is the Bessel function of the first kind zero order and first order; The parameter $\eta = \dfrac{1-\mu}{1-2\mu}$; μ is the Poisson's ratio; The parameter $m_c = \dfrac{1}{2}\eta\dfrac{\alpha^2 + S\left(K + \frac{1}{3}G\right)}{\alpha^2}$;S is storage specific (different to S_s in Feflow); α is the Biot's coefficient, K and G are the bulk modulus and the shear modulus respectively; ξ_j for $j=1,2,3\ldots$ are the roots of function: $J_1(\xi_j) = 2m_c\xi J_0(\xi_j)$; c_v is the consolidation cofficient.

For this benchmark, we use the following soil parameters:

- The compressibility of water $C_f = 10^{-7}(m^2 / kN)$
- The compressibility of soil grain $C_s = 0$
- The bulk modulus: $K = 500\left(kN / m^2\right)$ or $C_m = 1/K = 0.002\left(m^2 / kN\right)$
- The Poisson's ratio $\mu = 0.1$, the shear modulus: $G = 545.45\left(kN / m^2\right)$
- The hydraulic conductivity $k_x = k_y = k_z = 10^{-9}\left(m/s\right) = 8.64e^{-5}\left(m/d\right)$
- The porositiy $n = 0.64$
- The Biot's coefficient: $\alpha = 1 - C_s / C_m = 1$
- The specicific storage $S = nC_f + (\alpha - n)C_s = 6.4e^{-8}\left(m^2 / kN\right)$
- The compressiblity $m_v = \dfrac{1}{K + 4G/3} = 8.148e^{-4}(m^2 / kN)$
- The consolidation cofficient: $c_v = \dfrac{k}{\gamma_w}\dfrac{1}{\left(\alpha^2 m_v + S\right)} = 0.0108\left(m^2 / d\right)$
- The specific storage in Feflow: $S_s = \gamma_w\left(\alpha^2 m_v + S\right) = 0.008(1/m)$
- Total simulation time $t = 20\left(days\right)$
- The constant time step $\Delta t = 0.1\left(day\right)$, or the number of step is 200.

6.2. FeFlow model

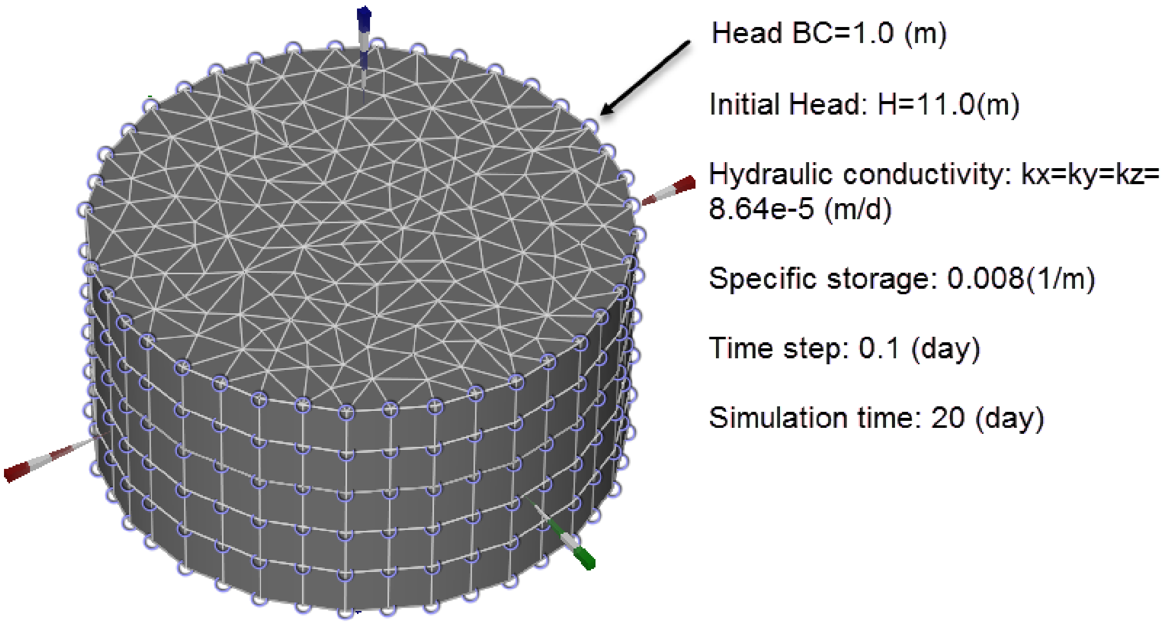


Fig. 6-2: 3D model in Feflow

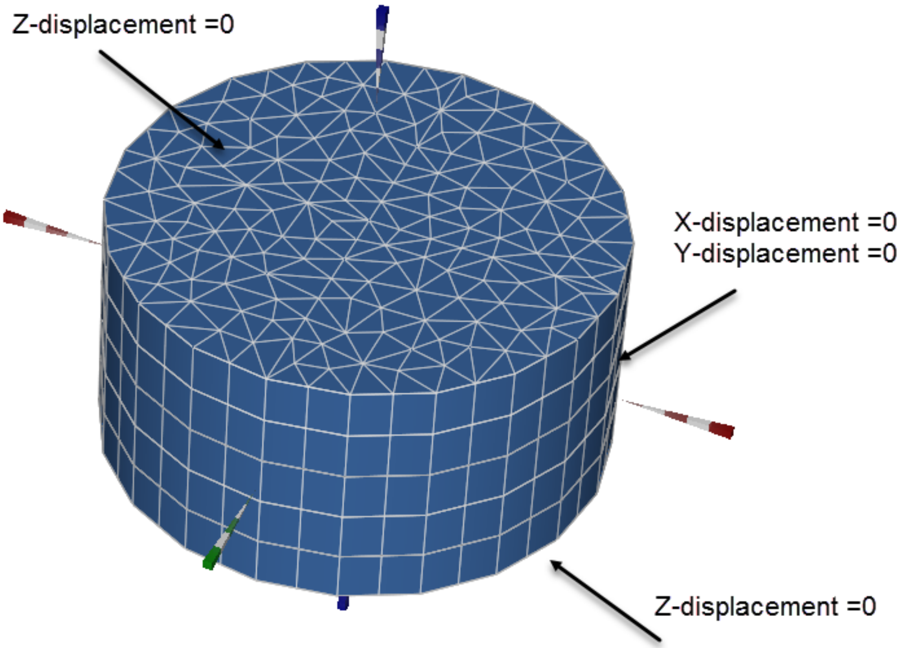
The FeFlow's model is shown in Fig. 6-2.

6.3. Plug-in model

For coupled model, the boundary conditions are:

- At the top surface ($z = 1.0m$) and the bottom surface ($z = 0.0m$) , there is no movement along z-direction.
- For the nodes which has radius $r = 1.0m$ (or the outer boundary), there is no movement along x and y direction.

The head field boundary condition is taken from FeFlow.



6.4. Compare results between uncoupled FeFlow model and coupled Plug-in model

We compare three different results of the center point. The center point has radius $r = 0.0$, or the coordinate in FEM model are $x = 0; y = 0$.

- 1st: Analytical results (from Matlab file).
- 2nd: FeFlow uncoupled results.
- 3rd: Plug-in results.

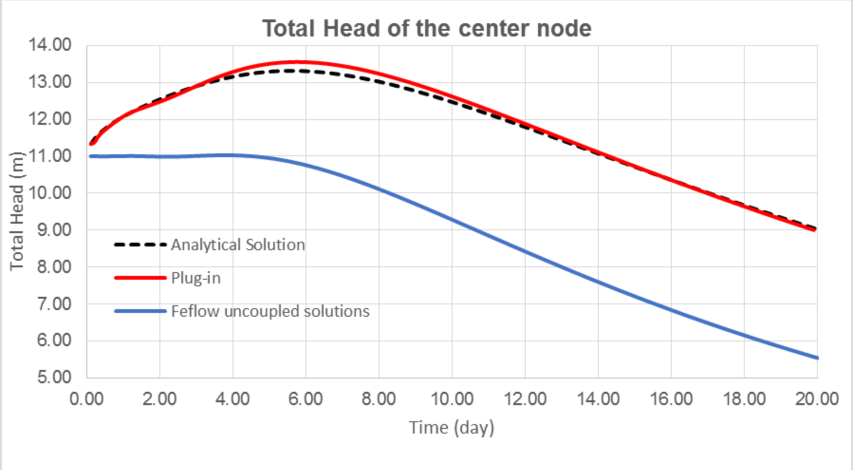


Fig. 6-3: Compare results of the center point between three different approaches

In this benchmark, results of Plug-in are similar to analytical solutions. However, Feflow uncoupled model fails to model this problem. We can see from Fig. 6-3, according to analytical solutions and the Plug-in results, the maximum total head ($H_{\max} \sim 13.5m$) is greater than the initial head ($H_0 = 11.0m$). It is called the Mandrel-Cryer's effect.