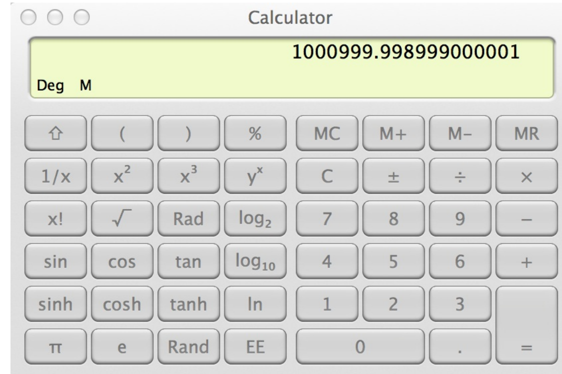
General Probability Spaces

$$\frac{1}{\frac{1}{1000}-\frac{1}{1001}}=\frac{1}{\frac{1001-1000}{1000\times 1001}}=1001000$$

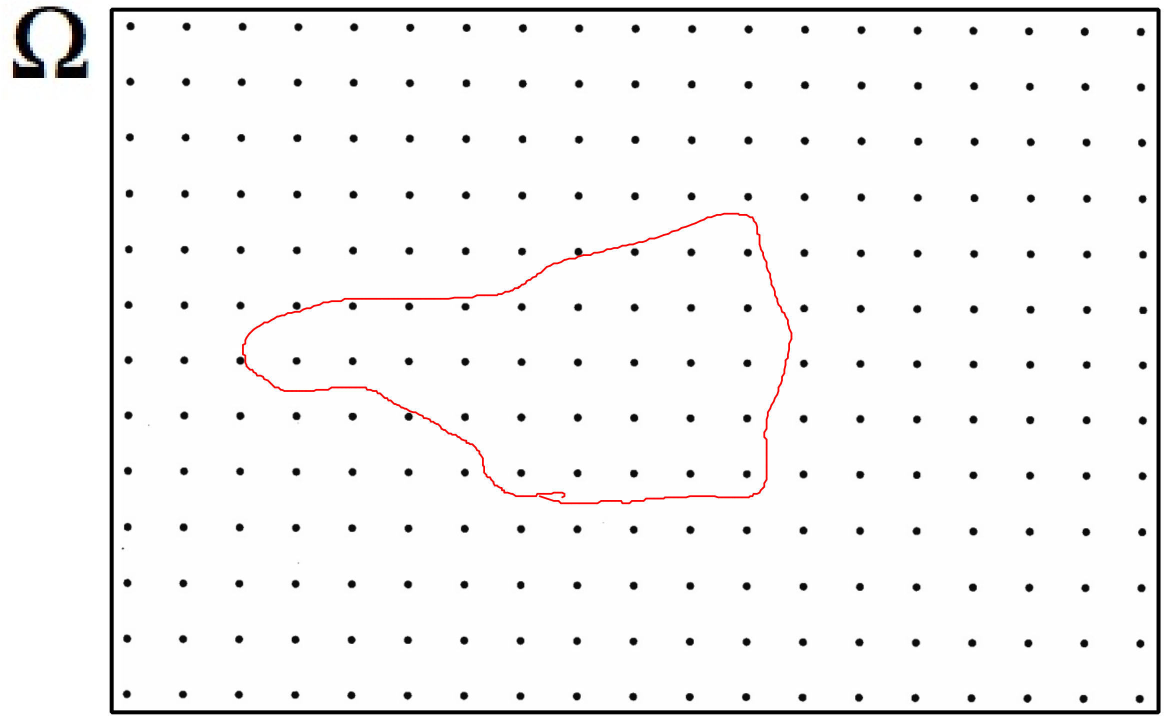


1/10000 = 1e-4 not 9.9E-5

WebWork checks your answers against the correct answers within some tolerance.
If you use a calculator your mistake might be masked and reappear at a later point in the problem.

Write complete expressions, don't use a calculator!

Discrete, finite, uniform probability spaces



So Far, we considered finite sample spaces and uniform distributions.

a	***b***	***c***	***d***	***e***
0.2	*0.2*	*0.2*	*0.2*	*0.2*

$$P(\{a,c,d\}) = \frac{|\{a,c,d\}|}{|\{a,b,c,d,e\}|} = \frac{3}{5} = 0.6$$

We now consider
finite sample spaces and
non-uniform distributions.

a	b	c	d	e
0.1	0.2	0.5	0.1	0.1

$$P(\{a,c,d\}) = p(a) + p(c) + p(d)$$
$$= 0.1 + 0.5 + 0.1 = 0.7$$

Properties of general probability distributions

$$\forall\ A \subseteq \Omega,\ 0 \le P(A) \le 1 \quad P(\Omega) = 1$$

$$\forall\ A, B \subseteq \Omega,\ P(A \cup B) \le P(A) + P(B)$$

$$\forall\ A, B \subseteq \Omega,\ A \cap B = \varnothing$$ ***Disjoint sets***

$$\Rightarrow P(A \cup B) = P(A) + P(B)$$

Implies that: $P(A^c) = 1 - P(A)$ ***The total probability equation***

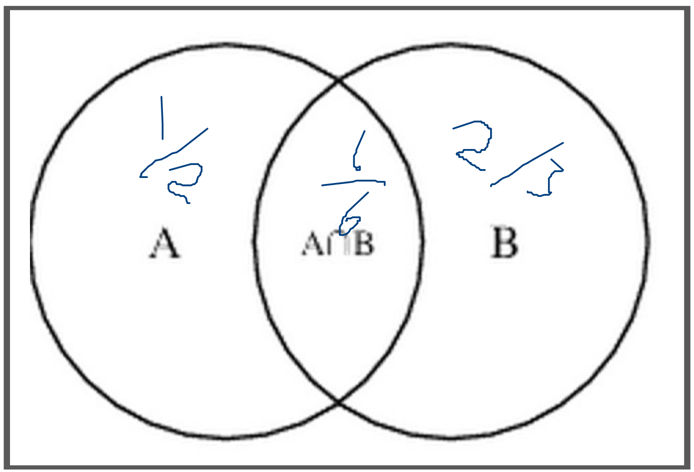
Proof: $A \cup A^c = \Omega, P(\Omega) = 1, A \cap A^c = \varnothing$

$$\Rightarrow P(A) + P(A^c) = 1$$

A few simple questions:

If $P(A) = \frac{1}{2}$, $P(B) = \frac{2}{3}$, What can be said about $P(A \cap B)$?

Ω



$p(A) + p(B) = \frac{7}{6}$

$1/6$

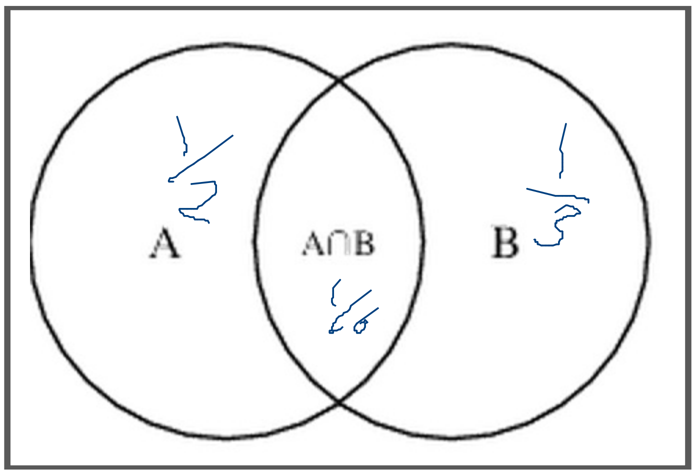
$1/2$

$\frac{2}{3} \le p(A \cap B) \le 1$

$2/3 + \frac{1}{2} - \frac{1}{6} = 1$

If $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$, $P(A \cap B) = \frac{1}{6}$, What is $P(A \cup B) =$?

Ω

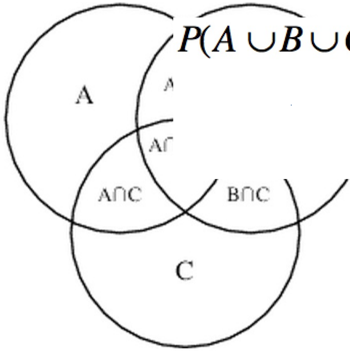


$$\frac{1}{2} + \frac{1}{3} - \frac{1}{6}$$

$$\frac{4}{6} = \frac{2}{3}$$

General Formula: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

How about: $P(A \cup B \cup C) = ?$



$$P(A \cup B \cup C) = P(A) + P(B) + P(C)$$
$$- P(A \cap B) - P(A \cap C) - P(B \cap C)$$
$$+ P(A \cap B \cap C)$$

inclusion/exclusion principle

Countably infinite sets

GET
GETTY

The natural numbers: 1,2,3,4,5....
-- an infinite set
-- represents counting
-- A set is infinitely countable if each element can be given an integer index.
-- Equivalently, if the elements can be put in a list

Other countable sets:

all integers (positive and negative): 0,-1,1,-2,2,-3,3,...
all words (not lexicographic order)(words of length 1, words of length 2,)

set index
element index
(i,j)

The union of n countable sets
(1,1),(2,1),...,(n,1); (1,2),(2,2),...,(n,2); (1,3),(2,3),...,(n,3);

The union of a countably many countable sets
(1,1); (1,2),(2,1); (1,3),(2,2),(3,1),...

The total probability equation for (countably) infinite sets

$$A_1, A_2, A_3, \ldots \subseteq \Omega$$

$$\forall i \neq j,\ A_i \cap A_j = \varnothing$$

$$A_1 \cup A_2 \cup A_3 \cup \cdots \doteq \bigcup_{i=1}^{\infty} A_i = \Omega$$

$$\text{Then } \sum_{i=1}^{\infty} P(A_i) = 1$$

Consider the natural numbers: 1,2,3,...
Is it possible to define a uniform distribution over them?

1st possibility: 0=P(1)=P(2)=... $P(\Omega) =$

2nd possibility: 0<P(1)=P(2)=.... $P(\Omega) =$

What is the meaning of $\sum_{i=1}^{\infty} p_i$?

$$p_i \geq 0, \quad \sum_{i=1}^{\infty} p_i \doteq \lim_{n\to\infty} \sum_{i=1}^{n} p_i$$

This is a non-decreasing sequence

it can either converge to

some real number or to infinity (∞)

p_i = c (constant) $\sum_{i=1}^{\infty} c = \lim_{n\to\infty} nc,$

$$If \;\; c = 0: \;\; 0,0,0,0,0,0 \to 0$$

$$If \;\; c > 0: \;\; c, 2c, 3c, 4c \to \infty$$

Is it enough if $p_i \xrightarrow{i\to\infty} 0$?

$p_i =$

Can we define a distribution of the form $P[X(\omega) = i] = \dfrac{c}{i}$?

No, because $\sum_{i=1}^{\infty} (1/i) = \infty$

$$\left(\frac{1}{1}\right), \left(\frac{1}{2}\right), \left(\frac{1}{3}, \frac{1}{4}\right), \left(\frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}\right), \dots$$

$$\sum_{i=1}^{n} \frac{1}{i} \approx \ln n$$

Can we define a distribution of the form $P[X(\omega)=i]=\frac{c}{i^2}$?

Yes, because $\displaystyle\sum_{i=1}^{\infty}\frac{1}{i^2}=\frac{\pi^2}{6}\approx 1.6449\ldots$

If we define the distribution to be $P(X=i)=\frac{6}{\pi^2 i^2}$

Then the sum of the probabilities over all natural numbers is 1

If the series is finite then we can define a distribution by dividing each term by the sum of the series = the normalization factor

$$\text{if } \alpha > 1 \qquad \sum_{i=1}^{\infty}\frac{1}{i^{\alpha}}<\infty$$

- **Geometric Series**: Let r be a number in the range $[0, 1]$, i.e. $0 \leq r \leq 1$. Then:

$$\sum_{i=0}^{\infty} r^i = \frac{1}{1-r}$$

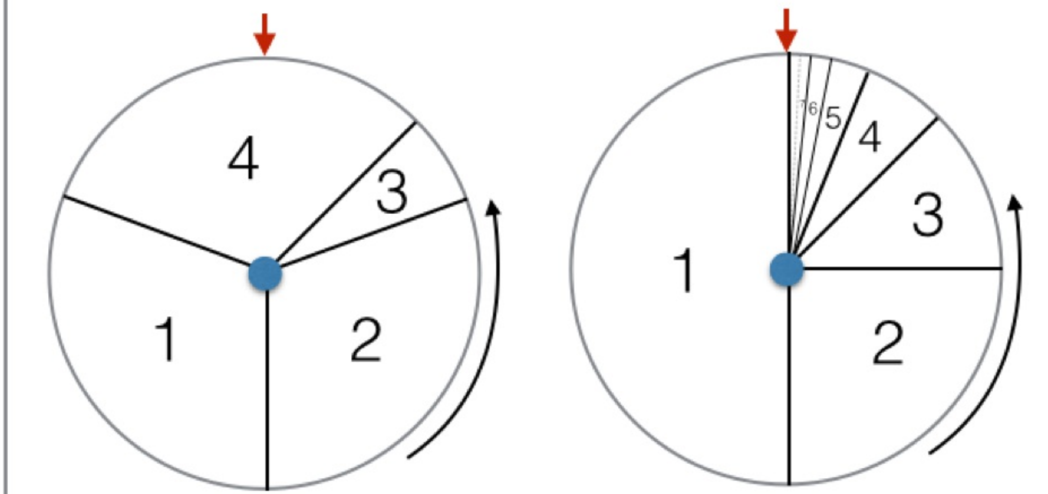
$$\sum_{i=1}^{\infty} r^i = \frac{r}{1-r}$$

and

$$\sum_{i=1}^{\infty} i r^i = \frac{r}{(1-r)^2}$$

Note that if r=1 the sums are infinite.

Probabilities over uncountable sets



(a) A wheel with four outcomes

(b) A wheel with Infinitely many outcomes

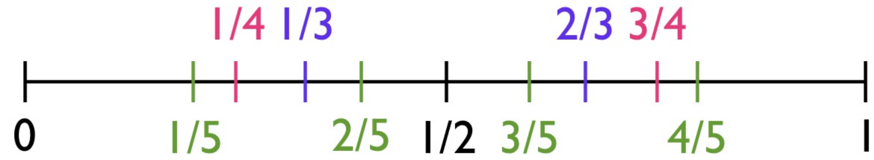
How can we define the uniform distribution over angles?
Each angle has probability 0
Summing over all angle still gives 0

It seems like we can represent the points on the line using a countable set

Numbers that can be written as i/j, where i,j are natural numbers

Each element corresponds to a pair of natural numbers. Therefor the

The rational numbers in [0,1]:



The distance between i/n and (i+1)/n is 1/n
As n increases the distance decreases to zero
---> the rationals are dense on the line
= there is a rational number arbitrarily close to any positive real number

Does that mean that the all real number are rational? NO! (sqrt(2))
Does that mean that the reals are countable? NO!

The real number 0 <=x <= 1 are uncountable

Proof by contradiction:
1. suppose they are countable.
2. write the list of all of the numbers in binary expansion

0.000001101001100011100010001000...
0.000101101001100011100010001000...
0.000000101001100011100010001000 ...
0.000001001001100011100010001000 ...
0.0000001100001100011100010001000 ...
0.0000011010000000011100010001000 ...
0.000001101001111011100010001000 ...
0.000001101001100111100010001000 ...
0.000001101001100011000010001000 ..

Construct a number that differes from the 1st element in the 1st position, from the 2nd in the 2nd position ...

0.11111001011...
This number is not in the list: contradiction

The uniform distribution over [0,1]

$$0 \leq x_1, x_2, \ldots \leq 1$$

$$P(\{x_1\}) = 0$$

$$P(\{x_1, x_2\}) = 0$$

$$P(\{x_1, x_2, x_3, \ldots\}) = 0$$

But, if $0 \leq a < b \leq 1$

$$P([a, b]) = b - a$$

This is called a density distribution.
General density distributions - on monday.

For Friday

1. ***Read Chapter 5 (it is updated)***
2. ***Start working on the homework (Get going, it is harder than previous)***
3. ***Akshay will replace me on Friday lecture.***