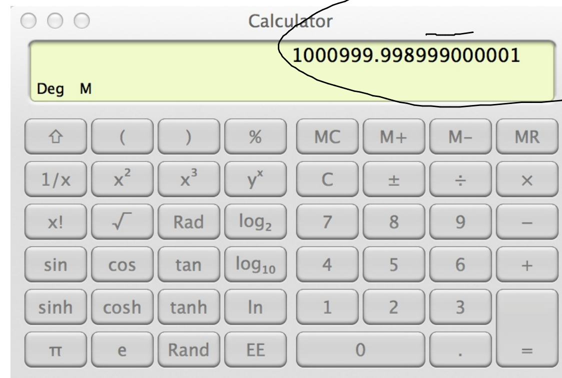
General Probability Spaces

$$\frac{1}{\frac{1}{1000}-\frac{1}{1001}} = \frac{1}{\frac{1001-1000}{1000\times1001}} = 1001000$$

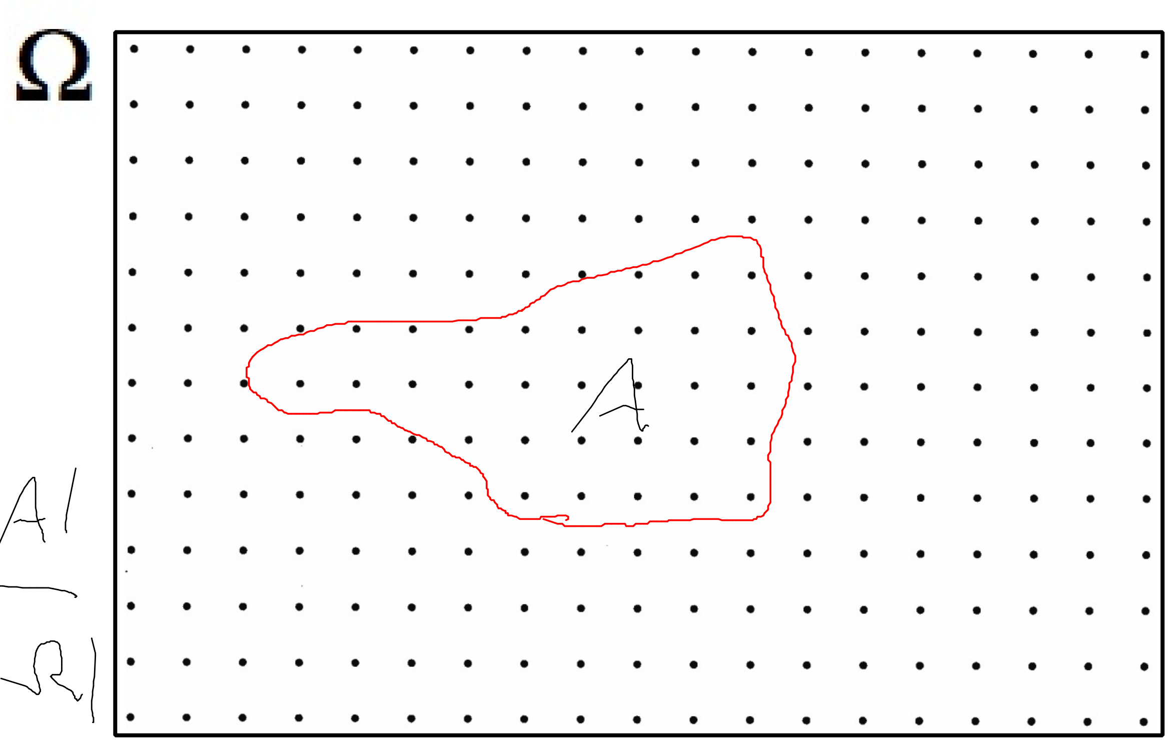


1/10000 = 1e-4 not 9.9E-5

WebWork checks your answers against the correct answers within some tolerance.
If you use a calculator your mistake might be masked and reappear at a later point in the problem.

Write complete expressions, don't use a calculator!

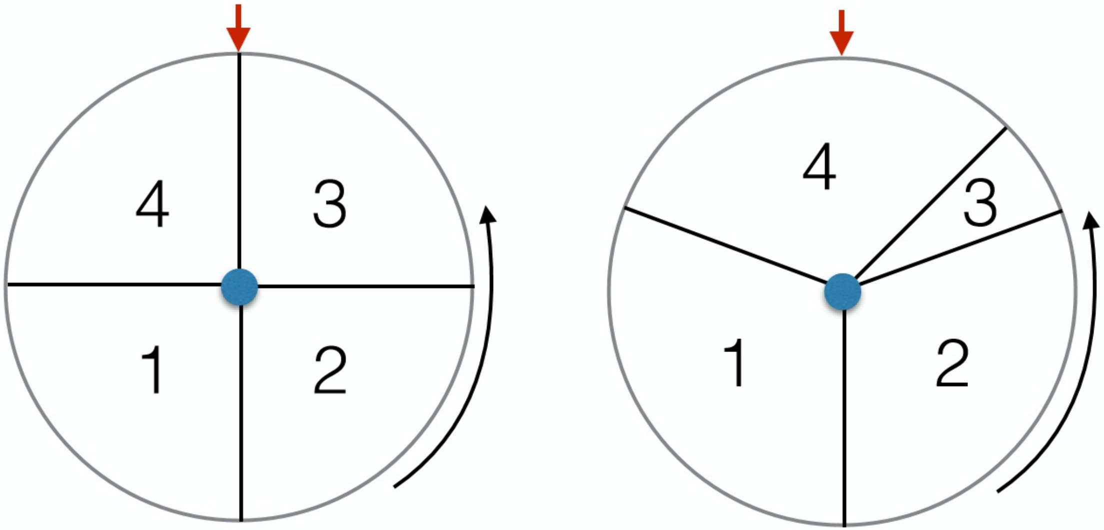
Discrete, finite, uniform probability spaces



$$P(A) = \frac{|A|}{|\Omega|}$$

Wheels of Fortune representing uniform and non-uniform distributions

show video



(a) 4 outcomes
uniform dist.

(b) 4 outcomes
non-uniform dist.

So Far, we considered finite sample spaces and uniform distributions.

a	b	c	d	e
0.2	0.2	0.2	0.2	0.2

$$P(\{a,c,d\}) = \frac{|\{a,c,d\}|}{|\{a,b,c,d,e\}|} = \frac{3}{5} = 0.6$$

We now consider
finite sample spaces and
non-uniform distributions.

a	***b***	***c***	***d***	***e***
0.1	*0.2*	*0.5*	*0.1*	*0.1*

$$P(\{a,c,d\}) = p(a) + p(c) + p(d)$$
$$= 0.1 + 0.5 + 0.1 = 0.7$$

Properties of general probability spaces

The Sample Space: Ω

Events: $A_1, A_2, A_3, \ldots$

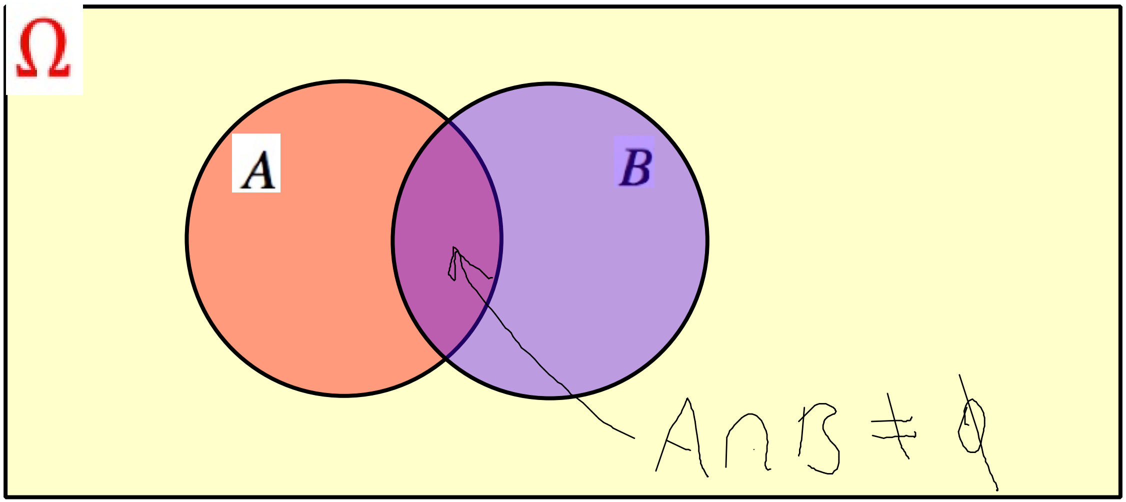
are subsets of the Sample Space: $A_i \subseteq \Omega$

$$\mathrm{P}(\Omega) = 1$$

$$\forall A \subseteq \Omega: \quad 0 \leq \mathrm{P}(\mathrm{A}) \leq 1$$

$\forall A, B \subseteq \Omega,\ A \cap B = \varnothing$ ***Disjoint sets***

$$\Rightarrow P(A \cup B) = P(A) + P(B)$$



$$\forall A, B \subseteq \Omega,\ P(A \cup B) \leq P(A) + P(B)$$

Partitioning the unit:

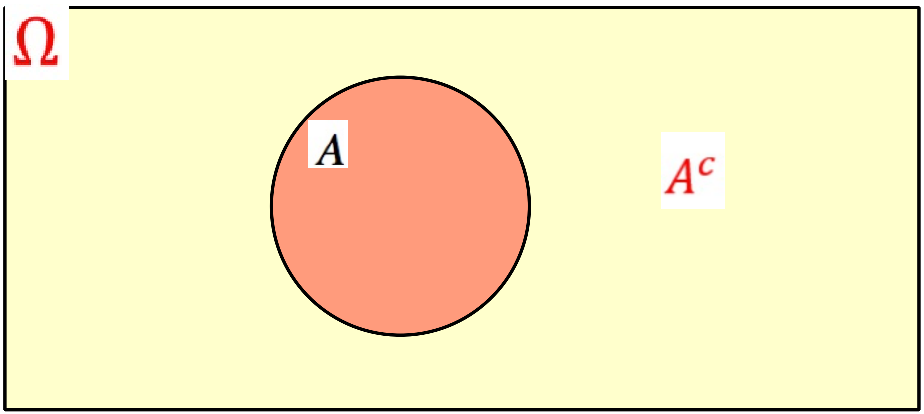
Suppose $B = A^c$

Then A and B are disjoint: $A \cap A^c = \emptyset$

and their union is the sample space: $A \cup A^c = \Omega$

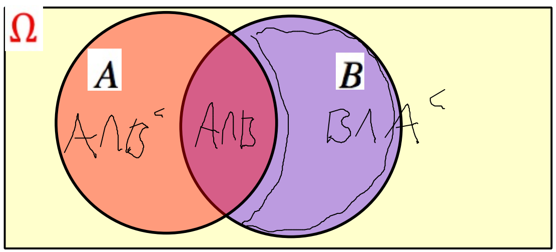
$\Rightarrow P(A) + P(A^c) = P(\Omega) = 1$

$\Rightarrow P(A^c) = 1\text{-}P(A)$



Partitioning a union of 2

If $P(A)=\frac{1}{2}$, $P(B)=\frac{1}{3}$, $P(A\cap B)=\frac{1}{6}$, What is $P(A\cup B) = \frac{1}{2}+\frac{1}{3}-\frac{1}{6}$?



$$P(A\cup B) = \underline{P(A\cap B^c) + P(A\cap B) + P(A^c\cap B)}$$

$$P(A) = P(A\cap B) + P(A\cap B^c)$$

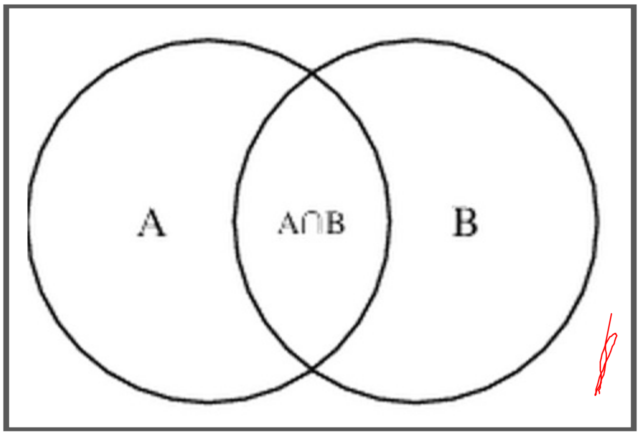
$$P(B) = P(A\cap B) + P(A^c\cap B)$$

$$P(A) + P(B) - P(A\cap B) = P(A\cap B^c) + P(A^c\cap B) + \cancel{2}P(A\cap B) \cancel{-P(A\cap B)}$$

Bounds on the probability of the intersection

If $P(A) = \frac{1}{2}$, $P(B) = \frac{2}{3}$, What can be said about $P(A \cap B)$?

Ω



$P(A \cap B) \leq P(A) = \frac{1}{2}$

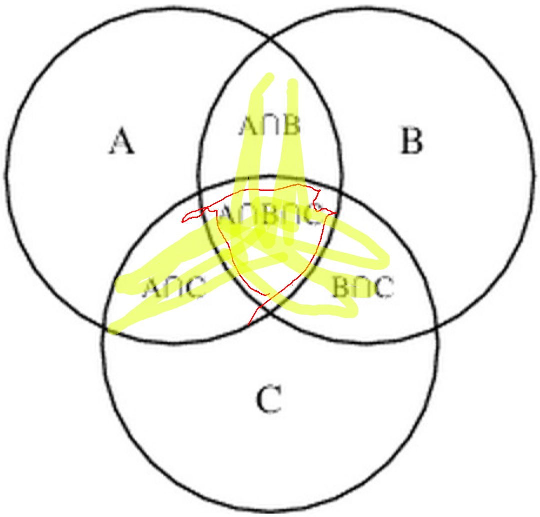
$P(B)$

$$1 \geq P(A \cup B) = \underset{1/2}{P(A)} + \underset{2/3}{P(B)} \quad 7/6$$

$$- P(A \cap B) \geq \frac{1}{6}$$

General Formula: $$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

How about: $$P(A \cup B \cup C) = ?$$



The inclusion/exclusion principle

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

Countably Infinite sample spaces:

Countably infinite sets

The natural numbers: 1,2,3,4,5....
-- an infinite set
-- represents counting
-- A set is infinitely countable if each element can be associated with a different integer index.
-- Equivalently, if the elements can be put in a list

Other countable sets:

$A_1 = (1,1), (1,2), (1,3)$

all integers (positive and negative): 0,-1,1,-2,2,-3,3,...
all words *good order:* words of length 1, words of length 2, ...
Lexicographical is a bad order: ace,ada,....,bad

$A_n = (n,1), (n,2), (n,3)$

The union of n countable sets
(1,1),(2,1),....,(n,1); (1,2),(2,2),....,(n,2); (1,3),(2,3),....,(n,3);

set index
element index
(i,j)

The union of a countably many countable sets
(1,1); (1,2),(2,1); (1,3),(2,2),(3,1),...

The total probability equation for (countably) infinite sets

$$A_1, A_2, A_3, \ldots \subseteq \Omega$$

$$\forall i \neq j,\ A_i \cap A_j = \varnothing$$

$$A_1 \cup A_2 \cup A_3 \cup \cdots \doteq \bigcup_{i=1}^{\infty} A_i = \Omega$$

$$\text{Then } \sum_{i=1}^{\infty} P(A_i) = 1$$

c, c+c, c+c+c ...

Consider the natural numbers: 1,2,3,...
Is it possible to define a uniform distribution over them?

1st possibility: 0=P(1)=P(2)=... $P(\Omega) =$ 0

2nd possibility: 0<P(1)=P(2)=.... $P(\Omega) =$

c > 0

Infinite sums

What is the meaning of $\sum_{i=1}^{\infty} p_i$?

$$p_i \geq 0, \quad \sum_{i=1}^{\infty} p_i \doteq \lim_{n\to\infty} \sum_{i=1}^{n} p_i$$

This is a non-decreasing sequence

it can either converge to

some real number or to infinity (∞)

p_i = c (constant) $\quad \sum_{i=1}^{\infty} c = \lim_{n\to\infty} nc,$

$$\mathit{If} \ \ c = 0: \ \ 0,0,0,0,0,0 \to 0$$

$$\mathit{If} \ \ c > 0: \ c, 2c, 3c, 4c \to \infty$$

Power series

$$p_i = 1/2^i$$

$$\sum_{i=1}^{\infty} p_i = \frac{1}{2}, \frac{1}{2} + \frac{1}{4}, \frac{1}{2} + \frac{1}{4} + \frac{1}{8}, \ldots$$
$$= \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \ldots \to 1$$

In General:
If $0 < a < 1$:
Then:

$$\sum_{i=1}^{\infty} a^i = \frac{a}{1-a}; \qquad \sum_{i=0}^{\infty} a^i = \frac{1}{1-a}$$

$$\sum_{i=1}^{\infty} i a^i = \frac{a}{(1-a)^2}$$

if a<1 the power series converges to a finite number

Dirichlet series

For any d>0 the dirichlet series is:

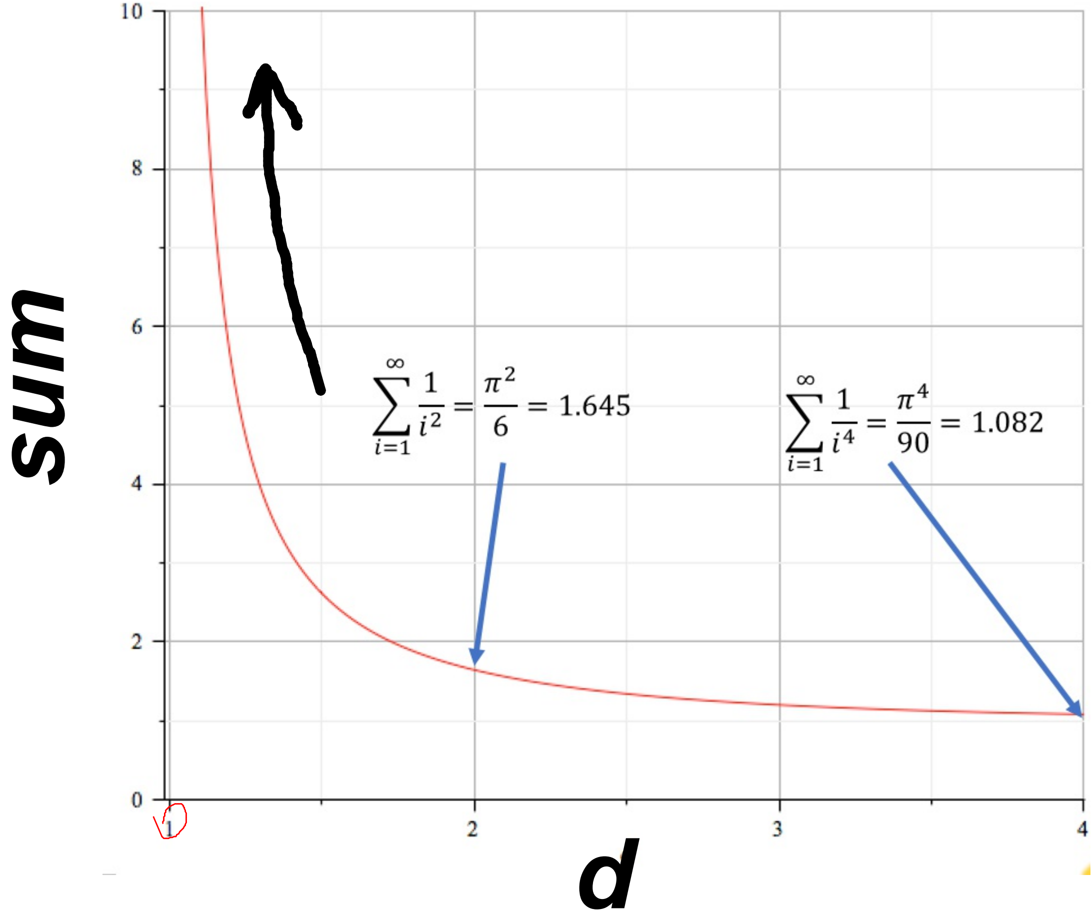
$$\sum_{i=1}^{\infty} \frac{1}{i^d}$$

for $d=3$

$$= \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots$$

When is this sum finite?

Dirichlet series as a function of d



for d=1 $\sum_{i=1}^{\infty}(1/i) = \infty$

Partition terms into groups:

1. ***smaller than 2, larger-equal to 1***
2. ***smaller than 1, larger-equal to 1/2***
3. ***smaller than 1/2, larger-equal to 1/4***
4. ***smaller than 1/4, larger-equal to 1/8***
5. ***.....***

$$\underbrace{\left(\frac{1}{1}\right)}_{\geq \frac{1}{2}}^{1}, \underbrace{\left(\frac{1}{2}\right)}_{\geq \frac{1}{2}}^{1}, \underbrace{\left(\frac{1}{3}, \frac{1}{4}\right)}_{\geq 1/2}^{2}, \underbrace{\left(\frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}\right)}_{\geq \frac{1}{2}}^{4}, \ldots \quad 8 \quad 16$$

$$\sum_{i=1}^{n} \frac{1}{i} \approx \ln n$$

Why are finite sums important?

Can we define a distribution over the natural numbers where

$$P_i = \frac{1}{Z}\frac{1}{i^2} \quad \text{Where } Z < \infty \quad ?$$

YES: we set the *normalization* factor

$$Z = \sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6}$$

we get that

$$\sum_{i=1}^{\infty} P_i = \sum_{i=1}^{\infty} \frac{1}{Z}\frac{1}{i^2} = \frac{\sum_{i=1}^{\infty} \frac{1}{i^2}}{\sum_{i=1}^{\infty} \frac{1}{i^2}} = 1 = P(\Omega)$$

the total probability equation is satisfied

Why are finite sums important?

Can we define a distribution over the natural numbers where

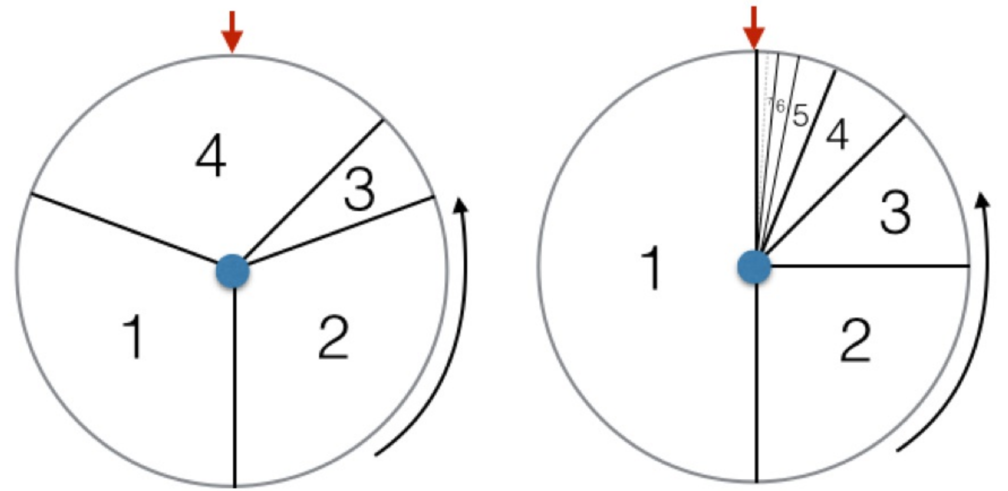
$$P_i = \frac{1}{Z}\frac{1}{i} \quad \text{Where } Z < \infty \quad ?$$

NO: If $Z < \infty$ the sum of the probabilities over the natural numbers will be infinite:

$$\sum_{i=1}^{\infty} P_i = \sum_{i=1}^{\infty} \frac{1}{z}\frac{1}{i} = \frac{\infty}{Z} = \infty > 1 = P(\Omega)$$

the total probability equation is not satisfied

Probabilities over finite and countable sets



(a) A wheel with four outcomes

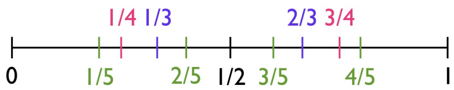
(b) A wheel with Infinitely many outcomes

Infinite sample spaces:
2. UNCountably infinite

It might seem like we can represent all points on the segment [0,1] using a countable set: the rationals

Rationals Numbers that can be written as i/j, where i,j are natural numbers

The rationals are a countable set:
1/1; 1/2; 1/3,2/2; 1/4,2/3; 1/5,2/4,3/3; 1/6,2/5,3/4; ....



We can get arbitrarily close to any point in the segment [0,1]?

----- True: There is a rational number arbitrarily close to any point in segment [0,1]

We can represent (precisely) any point on the line?

---- False: there are irrational numbers, such as sqrt(2)

Suppose that we add to the rational the roots of rational numbers.

----- There is a countable number of those, so the union with the rationals is still a countable set.

------ We are still missing numbers such as pi. Suppose we add it to the mix.

------ Continuing with this process Can we eventually have all of the points in the segment??

---NO! A union of a countable collection of countable set is still countable!

--- And the set of points in [0,1] is uncountable!!

Next time: 1. why is the set uncountable?
2. How do we define distributions over uncountable sample spaces.