Probability
a gentle introduction

The goal of this class

- We all have intuitions about probabilities.
  - But sometime these intuitions are wrong!
- In order to arrive at the correct answer we need more than intuition – we need a well defined system of definitions and axioms.
- The common system for reasoning under uncertainty is probability theory, which is a branch of mathematics.
  - Other systems exist, such as Fuzzy Logic, but we will not cover them in this course.
- Moving from intuitive thinking to formal thinking is hard!!
- Today we will introduce some central concepts from probability theory in an intuitive way.
  - We'll also show how intuition alone can lead us astray.
  - In later classes we will give more formal definitions.
- The concepts are:
  - Outcomes
  - Expected value / fair price.
  - Events
  - Event trees.
  - Probabilities / probability distribution.
  - Conditional Probability.

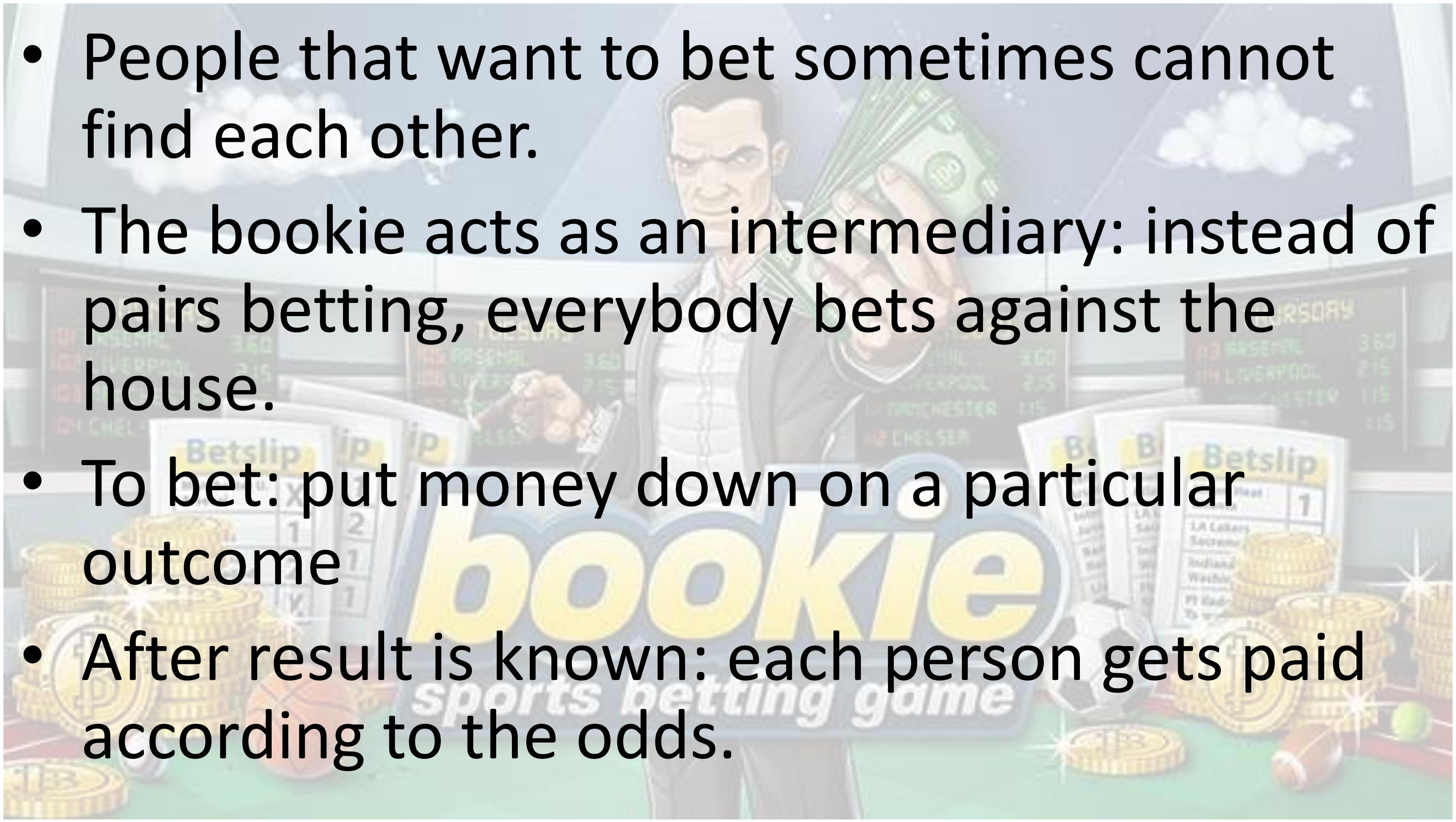
Bets between two people

- John: I bet the chargers will win their next game.
- Kathy: I bet they will lose. Do you want to put money on it?
- John, sure. In fact, **I am so sure they will win** that if they lose I'll pay you 90$, if they win, you pay me just 10$.
- Kathy: You are on!
- The odds are: 9 to 1
- Equivalent to john thinking that the probability the chargers will win is at least 90%
- Why? Because 0.9*10-0.1*90=0

What does it mean that john is right?

- Suppose John is right = the probability that the chargers win is at least 90%
- What does this mean?
- If we focus on just one bet, john either wins or lose. The probability has very little meaning.
- If John and Kathy bet before each game of the chargers, and john always gives the same odds, then **on the long run**, John will not lose money to Kathy.

Bets against the house.



- People that want to bet sometimes cannot find each other.
- The bookie acts as an intermediary: instead of pairs betting, everybody bets against the house.
- To bet: put money down on a particular outcome
- After result is known: each person gets paid according to the odds.

Fair odds: in words

- In the betting games we will talk about, the probability of each <u>outcome</u> is known.
- The bet is <u>**fair if:**</u>
  - The long term average of gains/losses is zero.
  - In other words: the **expected value** is zero.

Fair odds: in symbols

n: number of outcomes

Probabilities of outcomes $p_1, p_2, \ldots, p_n$

money gained for each outcome: $g_1, g_2, \ldots, g_n$

price of ticket: T

At each iteration, player pays T and gains one of $g_1, g_2, \ldots, g_n$

The expected gain of the player is $\sum_{i=1}^{n} p_i g_i - T$

The game is fair if $\sum_{i=1}^{n} p_i g_i - T = 0$

Equivalently: the price is fair if $T = \sum_{i=1}^{n} p_i g_i$

Fair Price=Expected value ≈long term average

- From the previous slide : $T = \sum_{i=1}^{n} p_i g_i$
- This quantity is also called the **expected value** and denoted as $E[g] \doteq \sum_{i=1}^{n} p_i g_i$
- The **long-term-average** converges to the expected value.

First Example

- House flips an unbiased coin.
  - “heads” : house pays player \$1
  - “tails”: house pays player \$2
- Outcomes: “heads”,”tails”
- What is the fair ticket price?
  - \$1.5
  - Why? Because 0.5*1 + 0.5*2 = 1.5

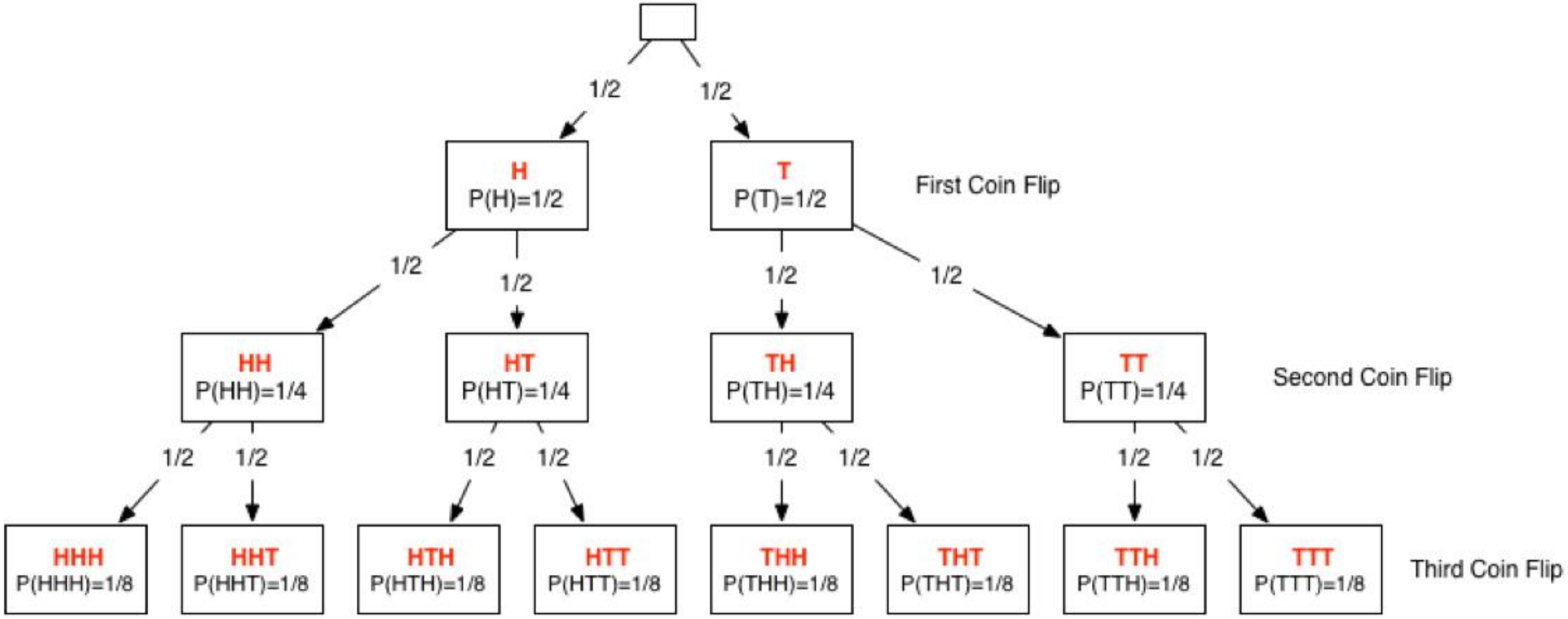
Second example

- The house flips the coin three times in a row.
- Eight outcomes:
  HHH,HHT,HTH,HTT, THH,THT,TTH,TTT
- Each outcome has probability 1/8
- Each outcome consists of three coin flips.
- You win \$1 if there is exactly one T, 0\$ otherwise, what is the fair price of the ticket?

Event Trees

- It sometimes helps our understanding to consider each coin flip separately, one by one.
- The result of considering all the possibilities is called the event tree.

The 3 coin flips event tree



What is an “Event” ?

- An event is a **set** of outcomes.
- Each node of an event tree defines an event
- The event of each node is a subset of the event of the parent.
  - The event “the first coin flip is H”. Corresponds to the set: {HHH,HHT,HTH,HTT}
  - The event “the first coin flip is T”. Corresponds to the set: {THH,THT,TTH,TTT}
  - The event “the first 2 coin flips are HH”. Corresponds to the set: {HHH,HHT}
  - The event “the first 2 coin flips are HT”. Corresponds to the set: {HTH,HTT}
  - …
  - The event “the three coin flips are HHH” corresponds to the set: {HHH}
- The probability of an event is the number of outcomes in the set, divided by 8.
- The event that contains all possible outcomes is called the “outcome space” and is denoted by Ω
- The probability of the whole outcome space is always 1:

  $$\mathrm{Prob}(\Omega)=8/8=1$$

Calculating probabilities of events

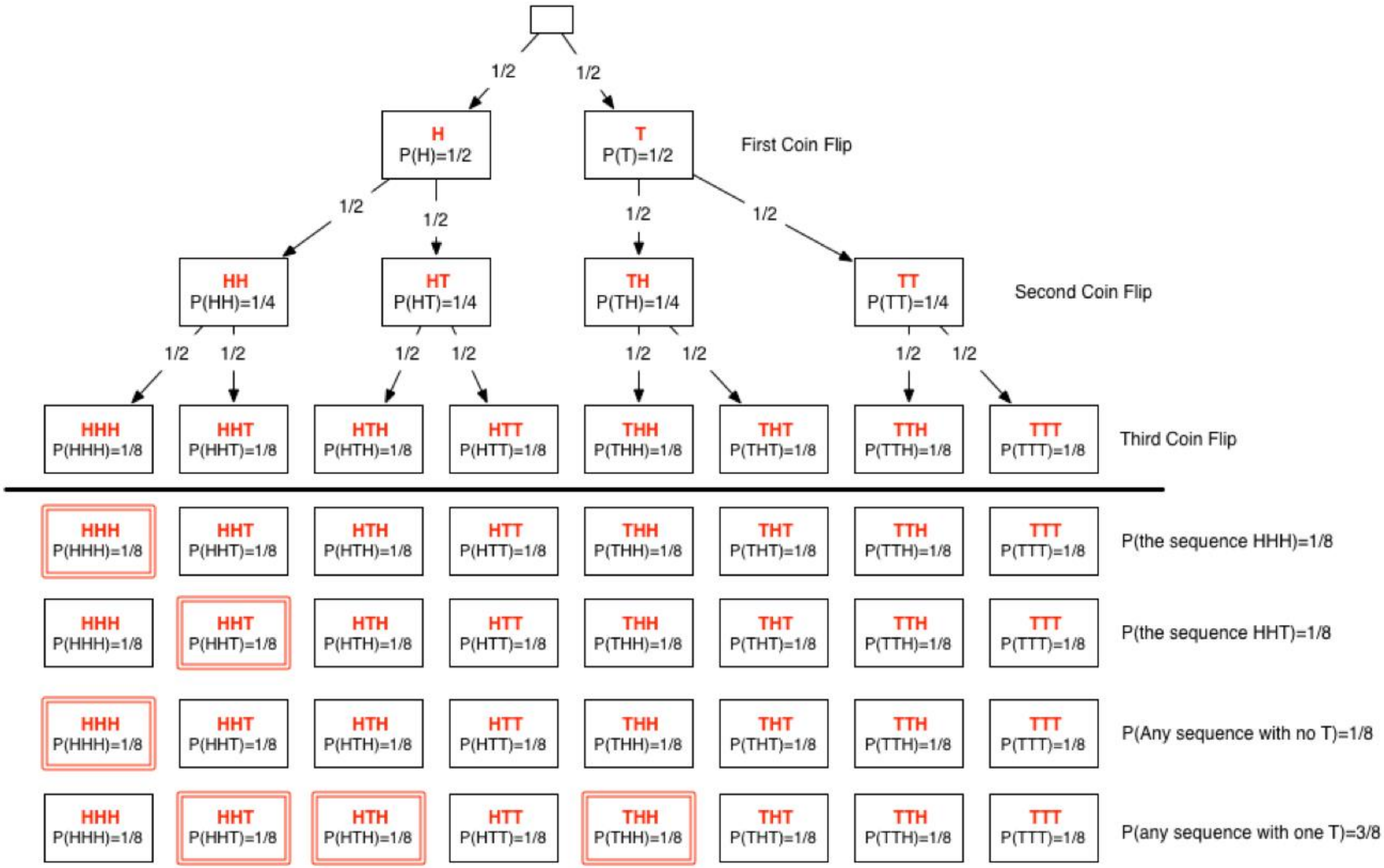
- The probability of an event (in the second example) is the number of outcomes in the set, divided by 8.
- The probability of an event is the number of outcomes in the event divided by the total number of outcomes in Ω (which is 8 in our case).
- Prob({HHH})=Prob({HHT})=1/8

The size of sets

- The number of elements in a set is also called the **size**, or **cardinality**, of the set.
- The size of the event A is denoted $|A|$
- The probability of the event A is $\mathrm{P}(A) = \frac{|A|}{|\Omega|}$
- *The fine print:* This is true for the special case of "uniform distribution over a finite sample space". Which will occupy us for the first 2 weeks or so.

Slightly more complex events

- P({The sequence contains no T})=P({HHH})=1/8
- P({The sequence contains one T})=
  P({HHT,HTH,THH}) = 3/8
- While HHH,HHT,HTH,…. All have the same probability, the event defined by “one T” has three times the probability of “no T”.
- The main task is to count the number of outcomes in the event. This is done using “combinatorics”


1/2
1/2
H
P(H)=1/2
T
P(T)=1/2
First Coin Flip
1/2
1/2
1/2
1/2
HH
P(HH)=1/4
HT
P(HT)=1/4
TH
P(TH)=1/4
TT
P(TT)=1/4
Second Coin Flip
1/2
1/2
1/2
1/2
1/2
1/2
1/2
1/2
HHH
P(HHH)=1/8
HHT
P(HHT)=1/8
HTH
P(HTH)=1/8
HTT
P(HTT)=1/8
THH
P(THH)=1/8
THT
P(THT)=1/8
TTH
P(TTH)=1/8
TTT
P(TTT)=1/8
Third Coin Flip
P(the sequence HHH)=1/8
P(the sequence HHT)=1/8
P(Any sequence with no T)=1/8
P(any sequence with one T)=3/8

Ticket prices

- Suppose that the house pays you $1 if a specified event happens, zero otherwise. What is the fair price?
- $T(E) = 1* P(E) = P(E)$
  - In this special case, probability and expectation are the same.
- T({HHH})=T({HHT})=T({no T})=1/8=¢12.5
- T({one T})=$3/8= ¢37.5

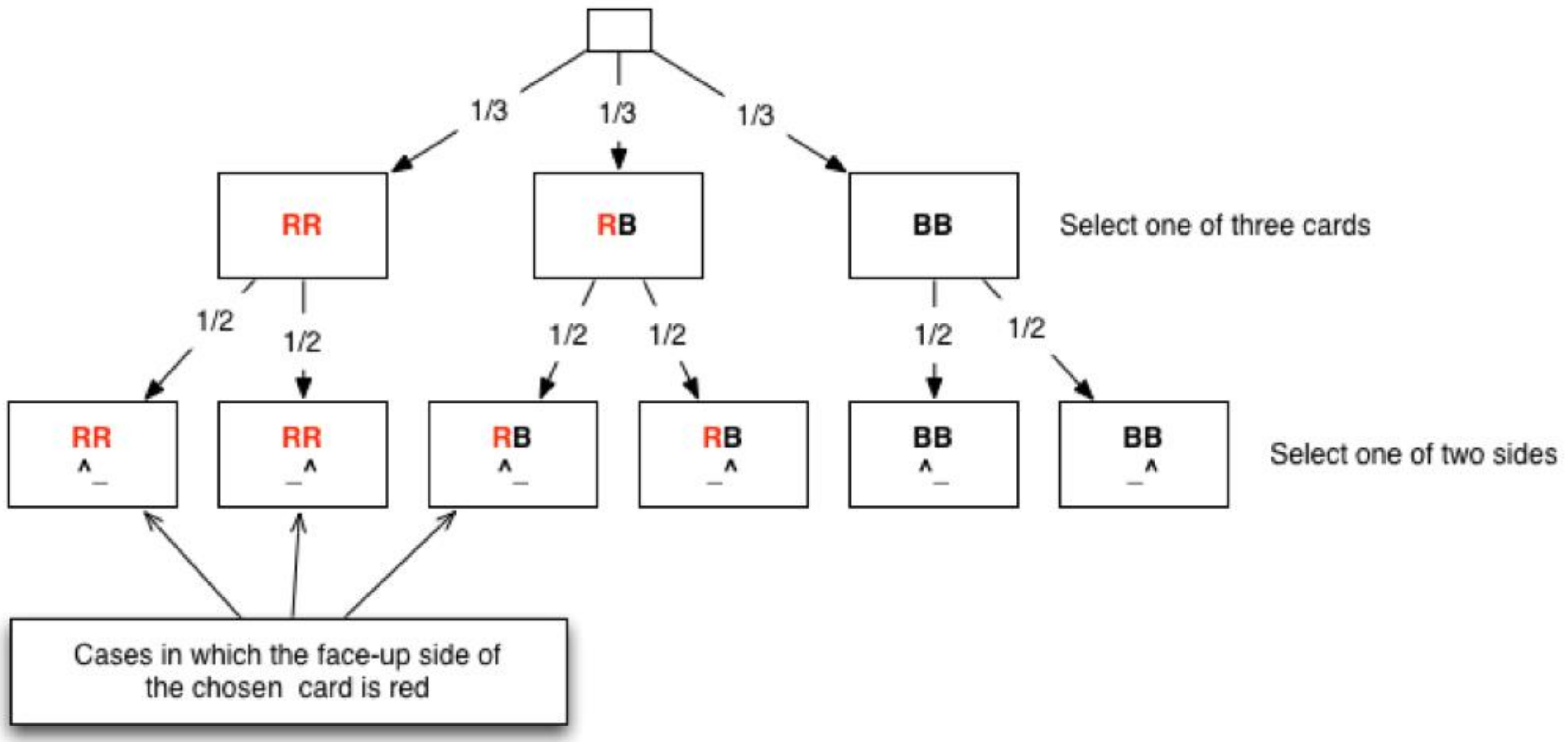
Do we really need all this?

- Maybe you think:
  “Calculating the probabilities and expected values for what we have seen so far can be done intuitively, do we really need ‘events’ and ‘event trees’?”
- Here is an example which challenges the intuition.

The three card problem

- There are three cards in a hat. Each side of each card is colored red **R** or black **B**.
- The colors of the cards are **RR, RB, BB**
- I pick one of the cards at random and put it on a random side.
- I say: if the color of the other side is the same, you give me $1, if it is different, I give you $1.
- My reasoning why this is fair:
  - Suppose that the side we see is **R,**
  - then two cards are possible: **RR** or **RB.**
  - Therefor there are equal probabilities that the opposite side would be **R** or **B.**
  - Similar argument can be made if the side we see is **B.**
- Is this argument correct? Is the expected gain equal zero?

Event tree for three cards



Conditional probability

- The probability that the seen color is R (B) is ½.
- The probability that the other side is R (B) given that the seen color is R(B) is 2/3.

For thu.

- Read Chapter 2.
- Finish Week1 homework on webwork.

- Thu: Basic combinatorics.
- If there is time, we’ll do some of next class now, and redo it on Thursday.