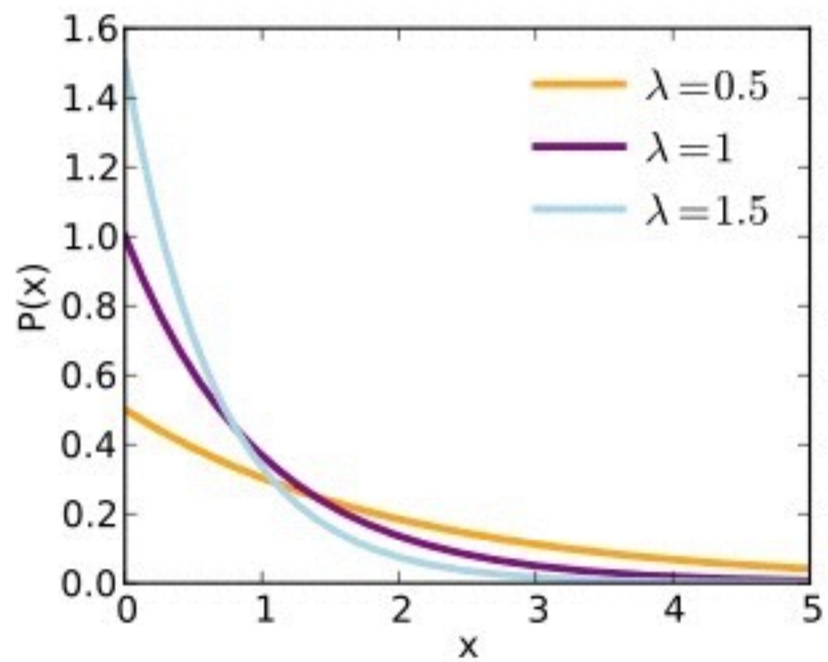
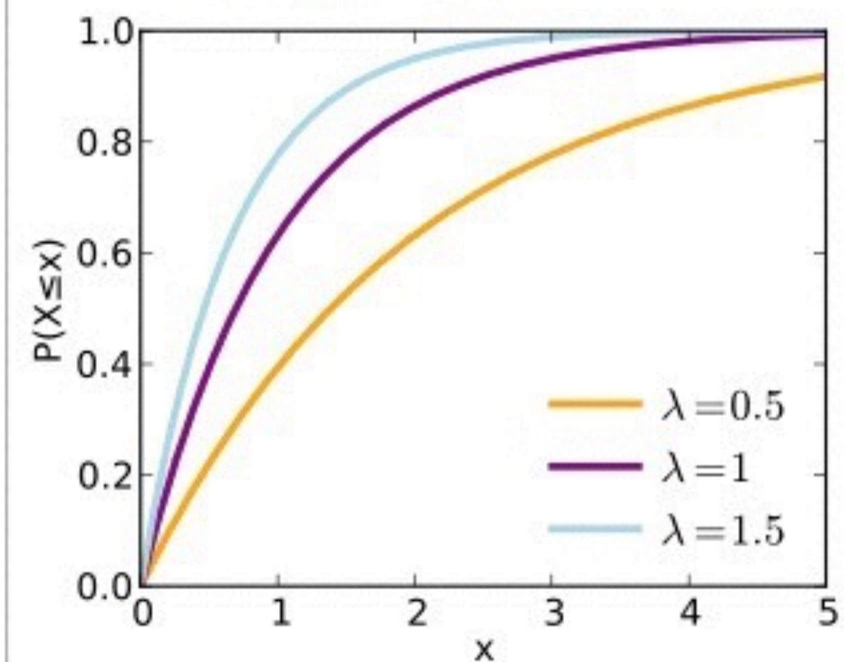
Exponential distribution

Exponential

Probability density function



Cumulative distribution function

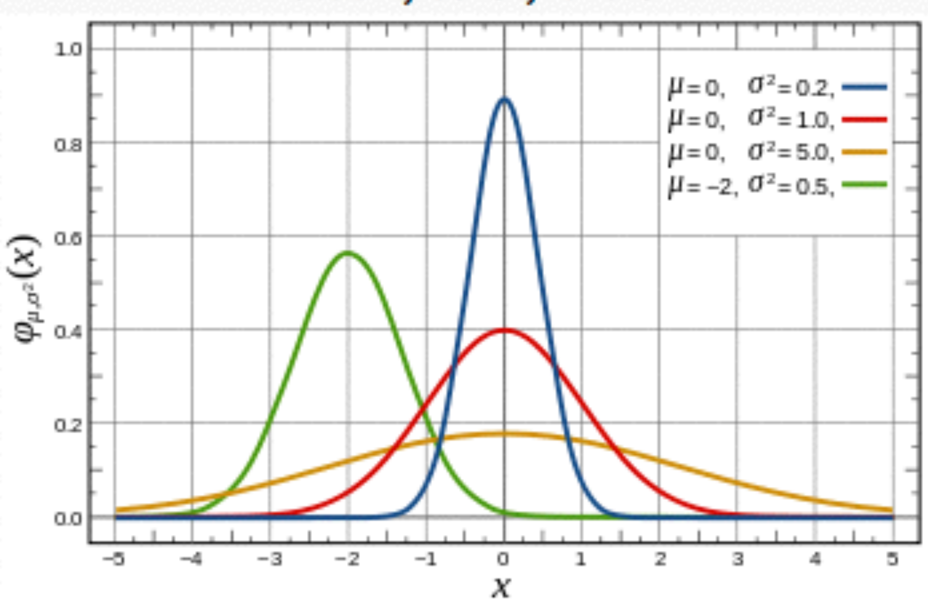


Parameters	$\lambda > 0$ rate, or inverse scale
Support	$x \in [0, \infty)$
pdf	$\lambda e^{-\lambda x}$
CDF	$1 - e^{-\lambda x}$
Quantile	$-\ln(1-F)/\lambda$
Mean	λ^{-1}

Normal Distribution

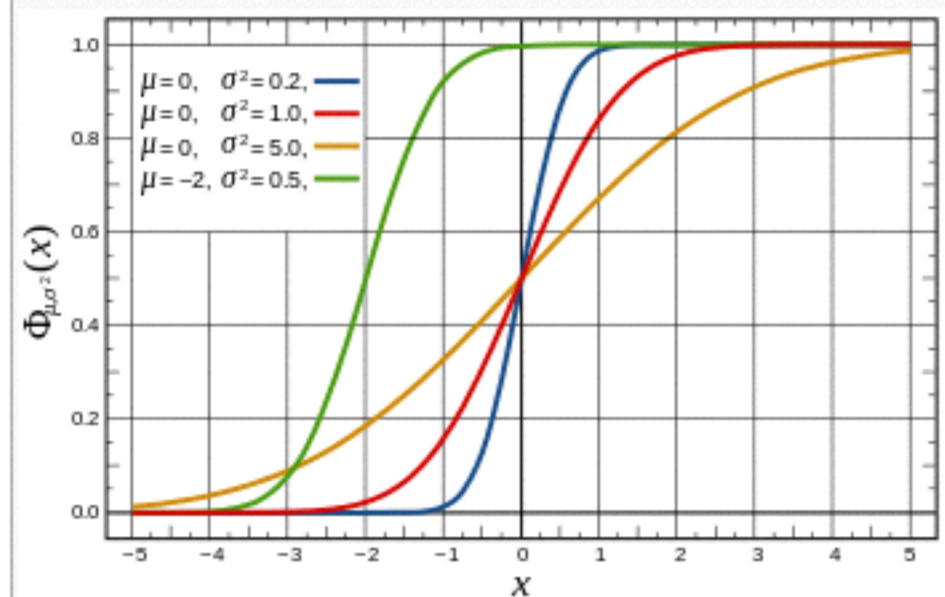
Normal

Probability density function



The red curve is the *standard normal distribution*

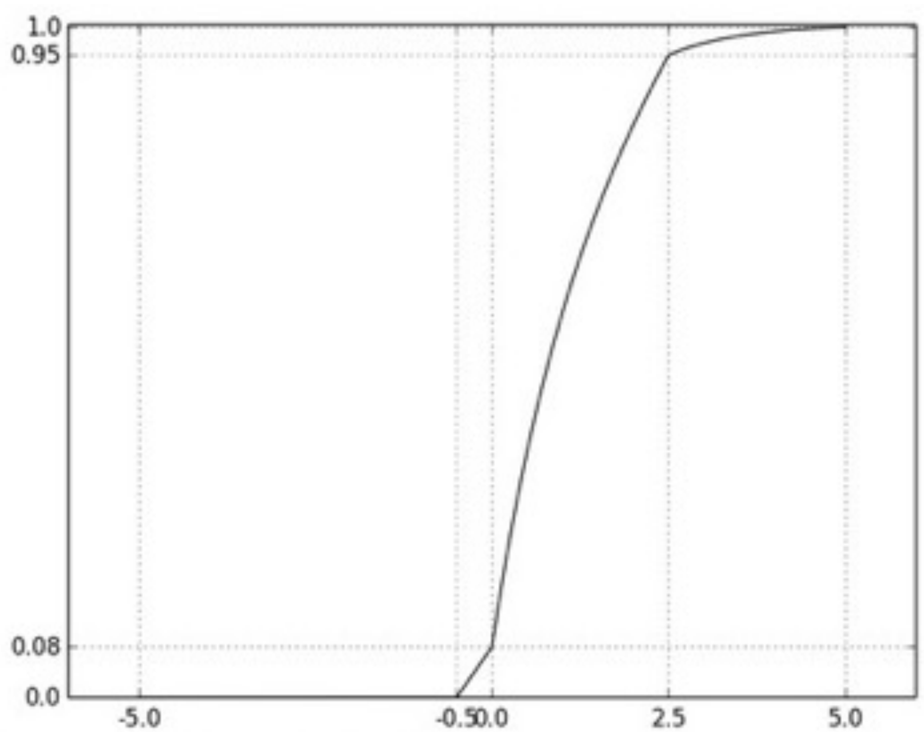
Cumulative distribution function



Notation	$\mathcal{N}(\mu,\,\sigma^2)$
Parameters	$\mu \in \mathbf{R}$ — mean (location) $\sigma^2 > 0$ — variance (squared scale)
Support	$x \in \mathbf{R}$
pdf	$\frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}$
CDF	$\frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right]$
Quantile	$\mu + \sigma\sqrt{2}\,\operatorname{erf}^{-1}(2F-1)$
Mean	μ
Median	μ
Mode	μ
Variance	σ^2

General way of solving problems involving mixture of CDF

(1 pt) **Reorganized/CumulativeDistributionFunctions/cdf_random.pg**
Below is the CDF of a mixture distribution with **two** components.
One of the components is either a normal or an exponential distribution; the other is either a point mass or a uniform distribution.
All parameters of component distributions are small multiples of 0.5.
λ of exponential components and std of normal components take on value 0.5, 1 or 1.5.
Component weights take on multiples of 0.05 and they need to sum to one.



Identify the component distributions:

- The exponential component has λ of 1.0. Its component weight is
- Uniform component on the interval (,). Its component weight is

- A mixture model is:
  - w1*P1+w2*P2+…
  - P1,P2,… are distributions
  - w1,w2,… are the weights (or probabilities) of the components. non-negative and sum to 1.
- Find the CDF at transition points.
- Use CDF formula for component to figure out its mixture coefficients.
- Use X values at transition points to figure out the parameters of each component.

Convergence to the Mean
Take 1

The average also called the empirical mean

Suppose $X_1, X_2, \ldots, X_n$ are

independent identically distributed (IID) random variables

$$\Pr\left[X_i = 1\right] = p, \quad \Pr\left[X_i = 0\right] = 1 - p, \qquad 0 \le p \le 1$$

$$E\left[X_i\right] = 1 \times p + 0 \times (1 - p) = p$$

We define the average to be another random variable

$$S_n \doteq \frac{1}{n} \sum_{i=1}^{n} X_i$$

We already know that

$$E\left[S_n\right] = E\left[\frac{1}{n} \sum_{i=1}^{n} X_i\right] = \frac{1}{n} \sum_{i=1}^{n} E\left[X_i\right] = \frac{1}{n} \sum_{i=1}^{n} p = p$$

We want to show that S_n tends to be close to p

We will use two approaches to show that.

Approach 1: using the variance

$$S_n \doteq \frac{1}{n}\sum_{i=1}^{n} X_i$$

$$E\left[S_n\right] = E\left[X_i\right] = p$$

$$Var[X_i] = p \times (1-p)^2 + (1-p) \times (0-p)^2$$

$$= (1-p+p) \times (1-p) \times p = p(1-p)$$

As X_i are IID:

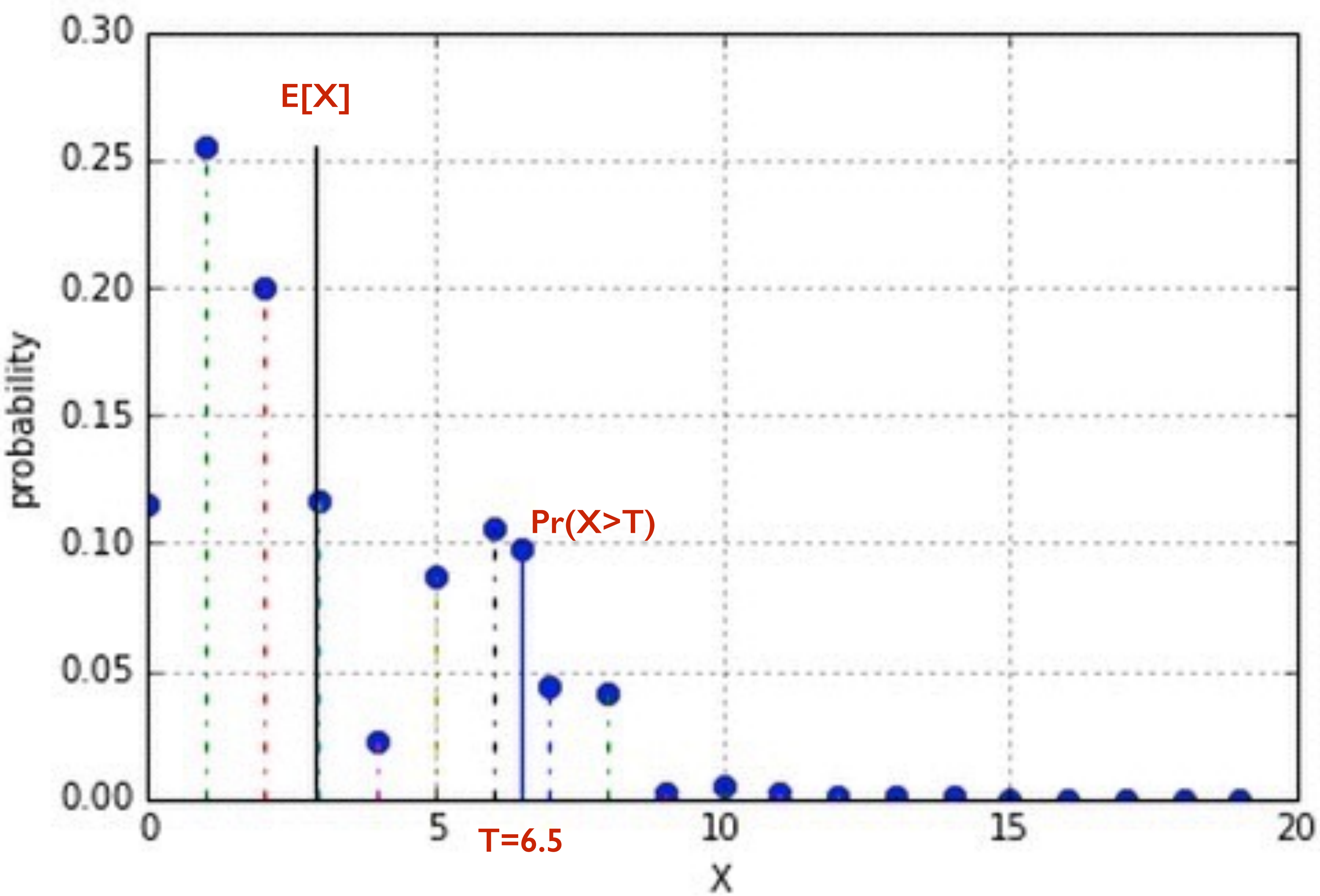
$$Var\left[S_n\right] = Var\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n^2}\sum_{i=1}^{n} Var\left[X_i\right] =$$

$$\frac{1}{n^2} np(1-p) = \frac{p(1-p)}{n}$$

$$\sigma\left(S_n\right) = \sqrt{\frac{p(1-p)}{n}}$$

Detour 1: Markov Bound

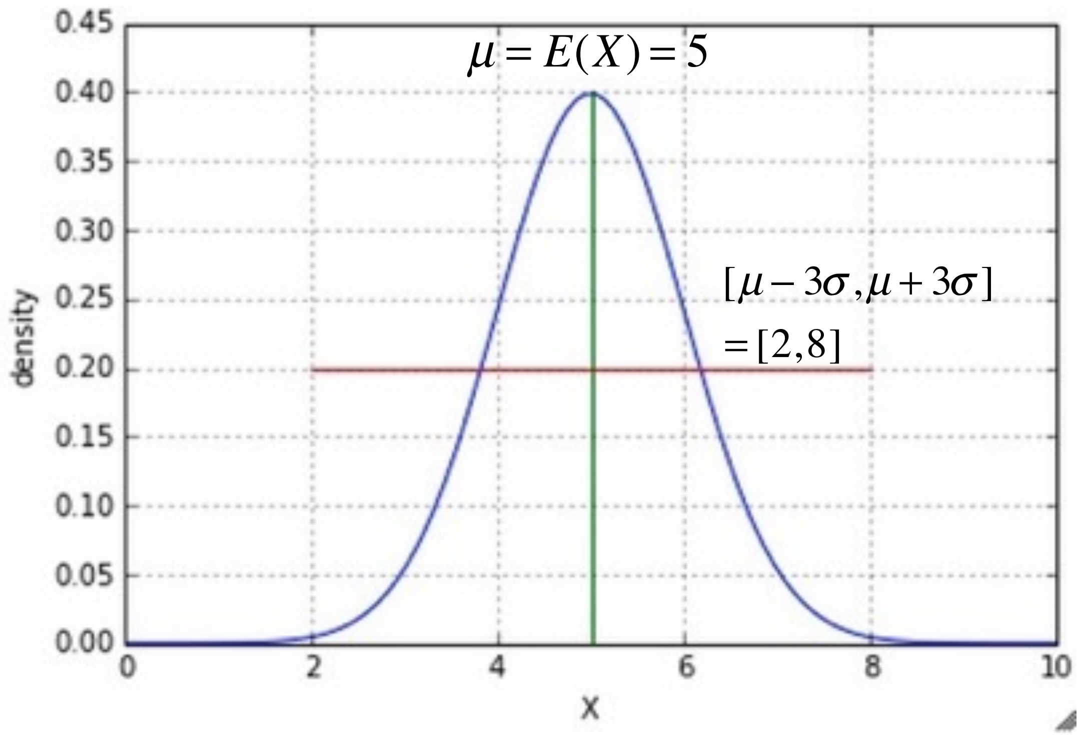
- Suppose the RV X is distributed over the **non-negative** integers 0,…,20
- Suppose we know the mean E[X]. Can we bound the probability that X>T ?



$$E[X] \geq 0 \times \Pr(X < T) + T \times \Pr(X \geq T)$$

$$\Pr(X \geq T) \leq \frac{E(X)}{T}$$

Detour 2: Chebyshev's bound

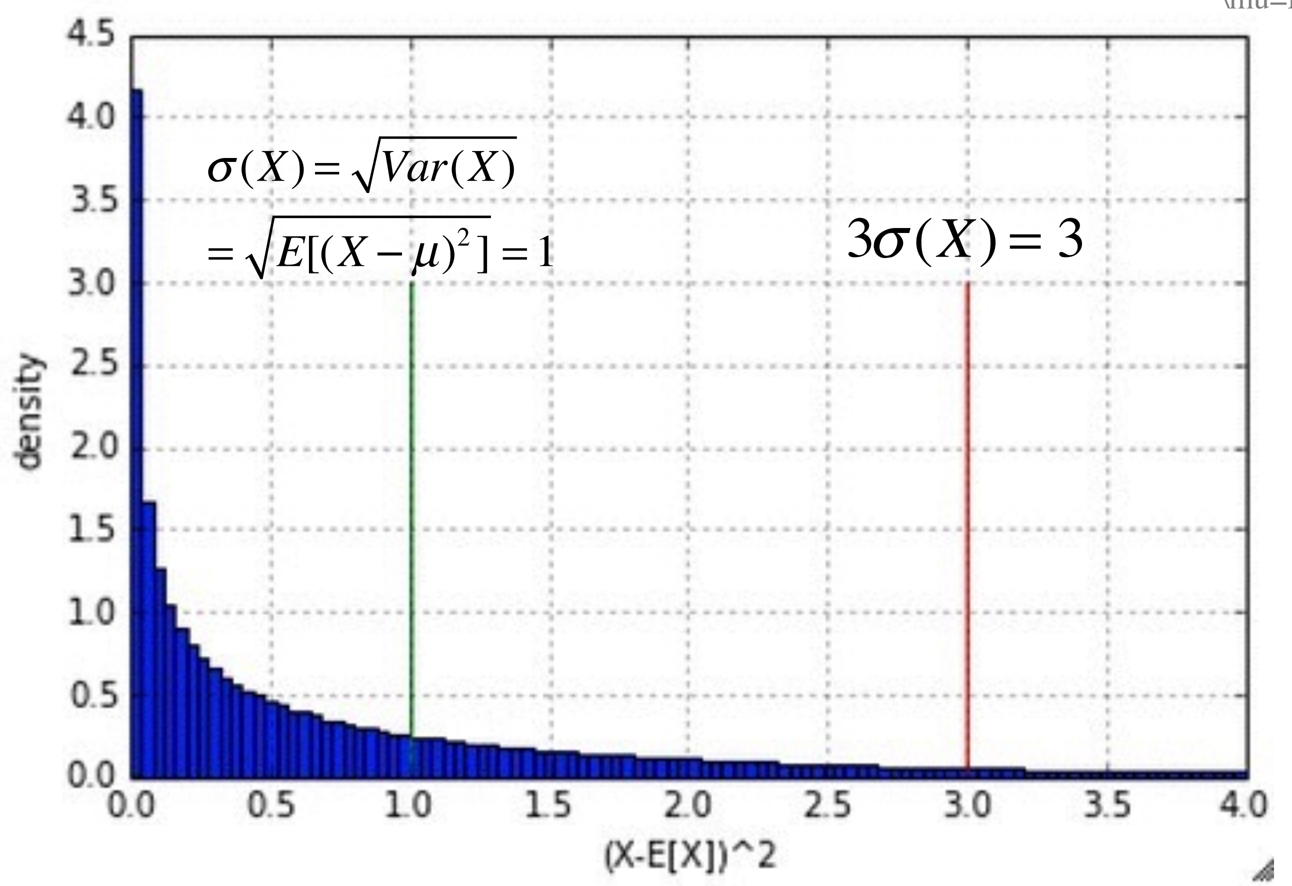


$$\Pr\left((X-\mu)^2 \geq \lambda^2\right) \leq \frac{E\left[(X-\mu)^2\right]}{\lambda^2} = \frac{Var(X)}{\lambda^2}$$

Plugging in $\lambda = k\sigma(X)$

$$\Pr\left[|X-\mu| \geq k\sigma(X)\right] \leq \frac{\sigma(X)^2}{k^2\sigma(X)^2} = \frac{1}{k^2}$$

In this example:
[\mu-
\mu=E



In the example shown

$\mu = E(X) = 5$

$\sigma = \sqrt{Var(X)} = 1$

We choose $k = 3$ to get that

$$\Pr\left(|X-5| \geq 3\right) \leq \frac{1}{k^2} = \frac{1}{9}$$

Applying Chebishev's bound

$$\Pr\left[|X-\mu| \geq k\sigma(X)\right] \leq \frac{\sigma(X)^2}{k^2\sigma(X)^2} = \frac{1}{k^2}$$

A few slides ago, we found that

$$\mu(S_n) = p; \quad \sigma(S_n) = \sqrt{\frac{p(1-p)}{n}}$$

$$\Pr\left[|S_n - p| \geq k\sqrt{\frac{p(1-p)}{n}}\right] \leq \frac{1}{k^2}$$

fixing k and letting n increase