# Hypothesis Testing

# Central Limit Theorem

Let $X_1, X_2, \ldots, X_n$ be IID Random variables with common mean $\mu$ and variance $\sigma^2$

Define: $$Z_n = \frac{S_n - n\mu}{\sigma\sqrt{n}}$$

Then the CDF of $Z_n$ converges to the standard normal CDF:

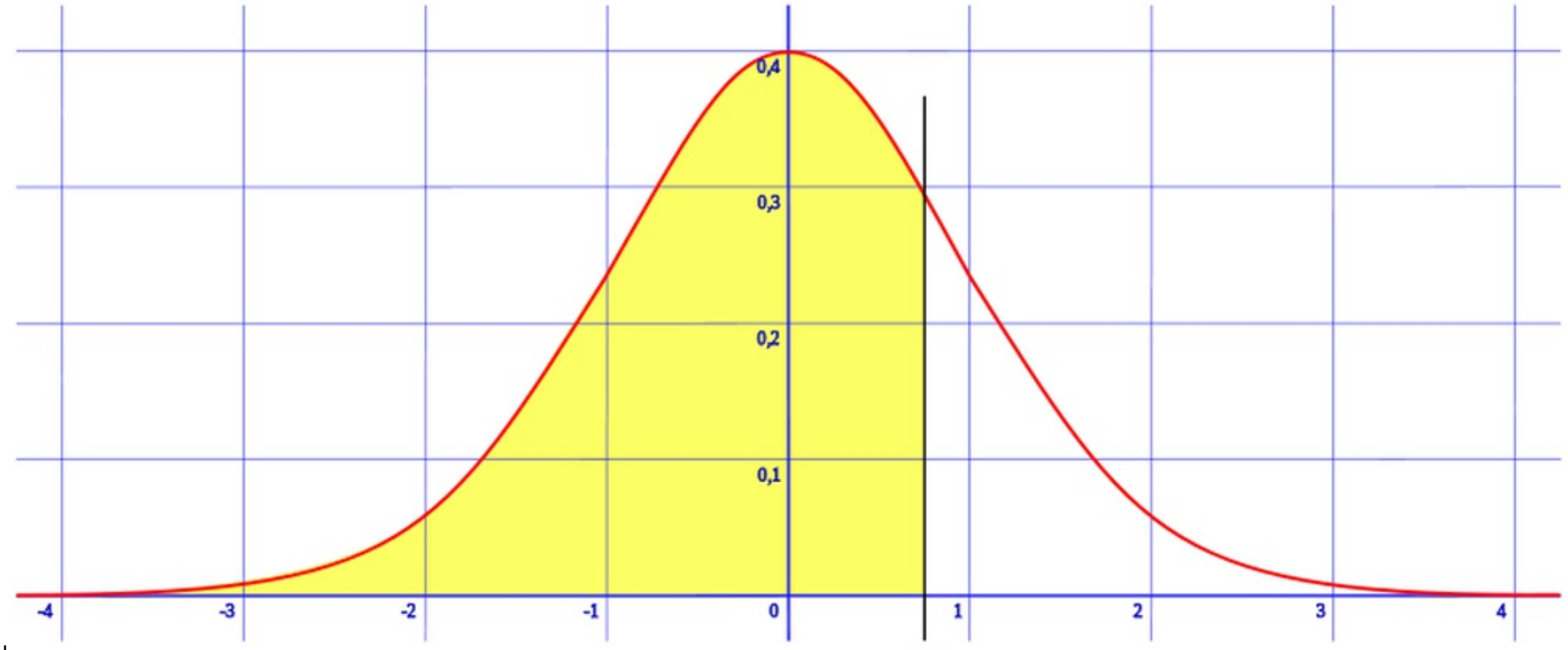
$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2} dx,$$

In the sense that

$$\forall z, \quad \lim_{n \to \infty} P\left(Z_n \leq z\right) = \Phi(z)$$

The central limit theorem is a strong justification for assuming that a distribution is normal.
Assuming normality is very common in practice.
Gives rise to the common use of Z-scores and Z-tables.

$$Z = \frac{X - \mathrm{E}[X]}{\sigma(X)}$$



- A few standard definitions: $Q(z) = P(X > z)$, $\Phi(z) = P(X < z)$, $Q^{-1}(p)$ is the inverse function to $Q$. In other words: $Q(Q^{-1}(p)) = p$.

- few useful values:

$$Q(1) \approx 15\%, \quad Q(2) \approx 2.5\%,$$

$$Q(3) \approx 0.15\%, \quad Q(4) \approx 0.003\%$$

# Example application of Hypothesis testing

We want to prove that seat-belts save a significant number of lives.

Suppose first that we know that, in general, the probability of a fatality in a car accident is $q=1\%$. (one out of 100 car accidents is fatal, ignoring whether or not seat-belts were used)

We examine $n=1000$ randomy selected records of accidents where the driver wore a car seat. We find that $k=5$ of these accidents were fatal.

With what confidence can we conclude that claim is correct?

X_i = i'th accident was fatal.

The null hypothesis is that seat belts make no difference.

P(X_i=1)=q=0.01

Alternative hypothesis, P(X_i=1)<q

$$S_n = \sum_{i=1}^{n} X_i;$$

$$\mu = E(S_n) = nq = 1000 \times 0.010 = 10$$

$$\mathrm{var}(S_n) = nq(1-q) \approx nq = 10$$

$$\sigma(S_n) = \sqrt{10} \approx 3.16$$

Therefore z-value = (10-5)/3.16= 1.58,

and the probability of getting 5 or higher is Q(1.58) = 5.94% ~ 6%

The p-value is 6%

The p value is a *random variable*

# Burden of Proof

- **Legal:** The burden of proof (Latin: onus probandi) is the imperative on a party in a trial to **produce the evidence** that will shift the conclusion **away from the default position** to **one's own position**.
  - Example: Innocent until proven guilty.

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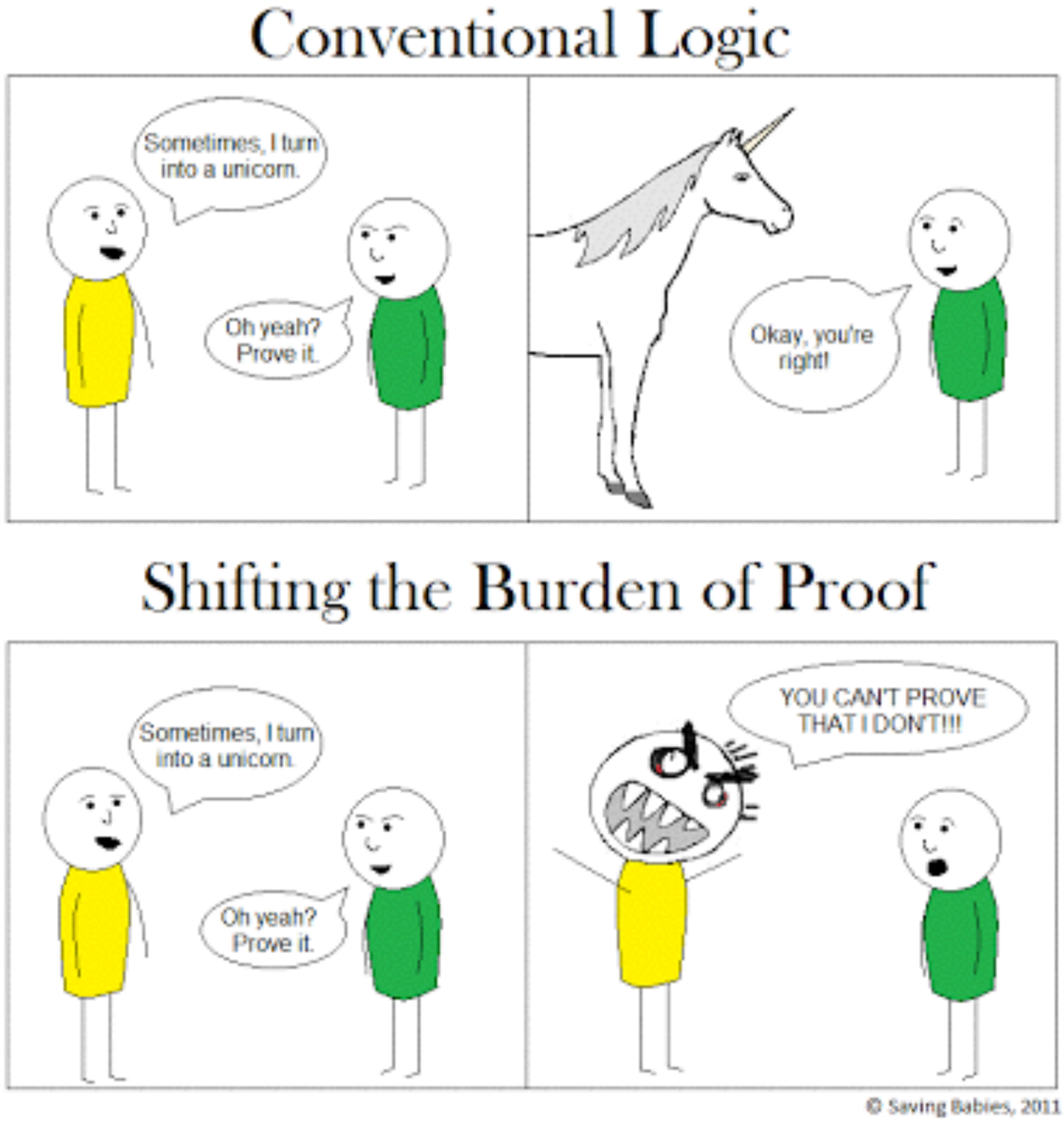
  - Example: Innocent until proven guilty.
- **Statistical:** The imperative on a scientist arguing for a new theory to **provide sufficient evident** to **reject the Null Hypothesis = the prevailing theory** and establish **his own theory: the Alternative Hypothesis**.
  - Example: proving that the police is guilty of racial profiling.

# Burden of Proof



Is 6% small enough? maybe yes maybe no, how is this decided?

Suppose that an important decision needs to be made, for example, to increase the fine for not wearing a seat-belt. We need to choose a significance level ( $\alpha$ ). And this choice has to be made before looking at the data. It should NOT be a random variable.

If $p \leq \alpha$ we say that the test rejected the null hypothesis

If $p > \alpha$ we say that the test failed to reject the null hypothesis.

In other words, if we chose $\alpha$ =0.05, then the test failed. We did not show that seatbelt save lives.

One might hope that in this case we gained some evidence that seat-belts do not save lives. Unfortunately, that is not the case, we did NOT gain evidence that seat-belts are useless. We are in a situation in which no significant conclusions can be drawn.

* A test cannot provide evidence towards the null hypothesis.
* We did not disprove that belts save lives.

# Are these details important?

Yes!
Statistical tests are at the foundation of the scientific method, medicine, and public policy.

Scientific method = repeatability of experiments. We need to decide how many successful repetitions are needed to be convinced.

Medicine = The most expensive part of drug development are the human trials.

What does it mean that the standard value of alpha used in medical journals is 5% ?

Public policy = Seat-belts? What level of chemicals in public water deems it unsafe?

# Gullability


**Fact:** most articles published in medical journals use an alpha value of 0.05=%5

# The hypothesis testing protocol

1. Define null hypothesis and alternative hypothesis. The null hypothesis represents the default or prevailing opinion which you want to overturn. the alternative hypothesis represents your opinion/belief.
   Both hypotheses are distributions over experimental outcomes.

2. Define an experiment and a statistical test:
   test: experimental outcome --> score.

3. Compute the p-value of each score.
   p(S) = prob. that a random score > S under the null hypothesis distribution.
4. Decide on a value of alpha - smaller - more convincing, larger - higher chance of rejecting the null.
5. Run experiment
6. Reject null hypothesis if p<alpha, else experiment failed.

# Hypothesis Testing Protocol for the effect of Seat-Belts

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1. Null hypothesis: probability of a fatality in an accident is q=1%, whether or not you wear a seat-belt.
   alternative hypothesis: seat belts reduce chance of fatality.
2. Experiment: Collect 1,000 records of experiments in which seat-belts were used.
3. $p(S) = Q\left(\frac{nq - S}{\sqrt{nq(1-q)}}\right)$
4. alpha=5%
5. outcome was S=7
6. p=6%, null hypothesis not rejected, hypothesis failed

Example question:
Suppose that the probability that a computer chip is defective is 0.1% and that we are manufacturing 1,000,000 chips. What is the probabillity that the number of defective chips is larger than 1100?

mean of single defect p=1/1000
n=1000000
mean number of defects=1000
var of single defect 999/1,000,000 approx 1/1000
var of number of defects= 1000. std is approximately 31

Z-score  is 100/31 more than 3, less than 4.

Probability is smaller than 0.13% (corresponding to 3X std)

# What can statistics can prove?

## Can

* Driving under the influence increases the chance of an accident
* Driving under the influence does not increase the chance of an accident by more than 2%.
* Members of the Kalenjin tribe run faster than the average.
* $E(X)>2$
* $E(X)<7$

## Cannot

* Driving under the influence does not change the chance of an accident.
* The probability of an accident  when DUI is 1.2%
* $E(X)=7$
* $P(X=3)=0.23$

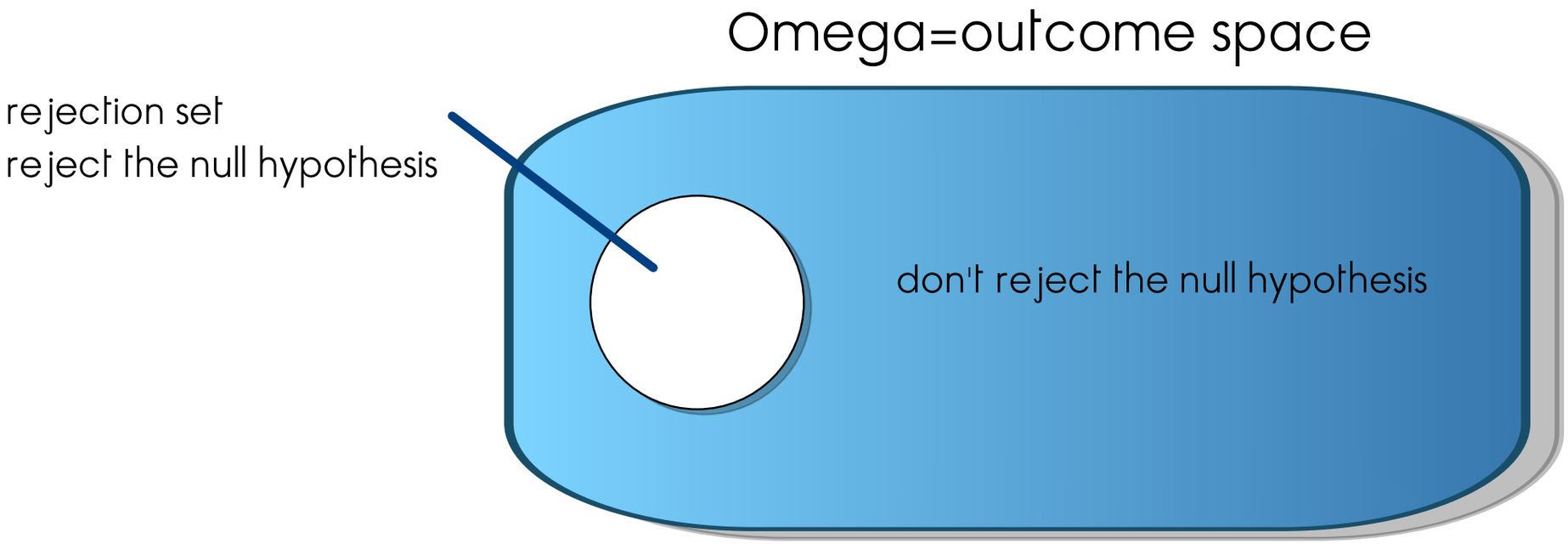
Choosing alpha is a compromise between two types of errors:
Type I error: Rejecting the null hypothesis when it is correct
Type II error: Failing to reject the null hypothesis when it is incorrect.

| | H_0 true<br>Seatbelts don't help | H_1 true<br>Seatbelts help |
|---|---|---|
| Fail to reject | + | type II |
| Reject Null | Type I | + |

Question: Increasing alpha:

A Increases Type II error, Decreases type I
B  Increases Type I Error,  Decreases type II
C Decreases both.
D Increases both.

# The probabilty theory of statistical tests




1. The each point in the outcome space corresponds to outcomes of a complete experiment - we observe only one!
2. The white circle represents the set of outcomes that will cause us to reject the null hypothesis.
3. alpha = The probability of the rejection set under the distribution defined by the null hypothesis.

# Examples of common statistical tests

## From the matlab Statistics module

| | |
|---|---|
| ranksum | Wilcoxon rank sum test. Tests if two independent samples come from identical continuous distributions with equal medians, against the alternative that they do not have equal medians. |
| runstest | Runs test. Tests if a sequence of values comes in random order, against the alternative that the ordering is not random. |
| signrank | One-sample or paired-sample Wilcoxon signed rank test. Tests if a sample comes from a continuous distribution symmetric about a specified median, against the alternative that it does not have that median. |
| signtest | One-sample or paired-sample sign test. Tests if a sample comes from an arbitrary continuous distribution with a specified median, against the alternative that it does not have that median. |
| ttest | One-sample or paired-sample *t*-test. Tests if a sample comes from a normal distribution with unknown variance and a specified mean, against the alternative that it does not have that mean. |
| ttest2 | Two-sample *t*-test. Tests if two independent samples come from normal distributions with unknown but equal (or, optionally, unequal) variances and the same mean, against the alternative that the means are unequal. |

# One-sample t-test

`h = ttest(x)` performs a *t*-test of the null hypothesis that data in the vector `x` are a random sample from a normal distribution with mean 0 and unknown variance, against the alternative that the mean is not 0. The result of the test is returned in `h`. `h = 1` indicates a rejection of the null hypothesis at the 5% significance level. `h = 0` indicates a failure to reject the null hypothesis at the 5% significance level.

tests the mean
Assumes normality

High power test - can identify a small deviation from the mean using few samples.

# two-sample t-test

`h = ttest2(x,y)` performs a *t*-test of the null hypothesis that data in the vectors `x` and `y` are independent random samples from normal distributions with equal means and equal but unknown variances, against the alternative that the means are not equal. The result of the test is returned in `h`. `h = 1` indicates a rejection of the null hypothesis at the 5% significance level. `h = 0` indicates a failure to reject the null hypothesis at the 5% significance level. `x` and `y` need not be vectors of the same length.

tests difference between mean
Assumes normality

# Lilliefors test

## Description

`h = lillietest(x)` performs a Lilliefors test of the default null hypothesis that the sample in vector `x` comes from a distribution in the normal family, against the alternative that it does not come from a normal distribution. The test returns the logical value `h = 1` if it rejects the null hypothesis at the 5% significance level, and `h = 0` if it cannot. The test treats `NaN` values in `x` as missing values, and ignores them.

alternative hypothesis is the complement of the null hypothesis.

# Ansari bradley test

`h = ansaribradley(x,y)` performs an Ansari-Bradley test of the hypothesis that two independent samples, in the vectors `x` and `y`, come from the same distribution, against the alternative that they come from distributions that have the same median and shape but different dispersions (e.g. variances). The result is `h = 0` if the null hypothesis of identical distributions cannot be rejected at the 5% significance level, or `h = 1` if the null hypothesis can be rejected at the 5% level. `x` and `y` can have different lengths.

alternative hypothesis is not the complement of the null hypothesis.

# Multiple Hypothesis testing

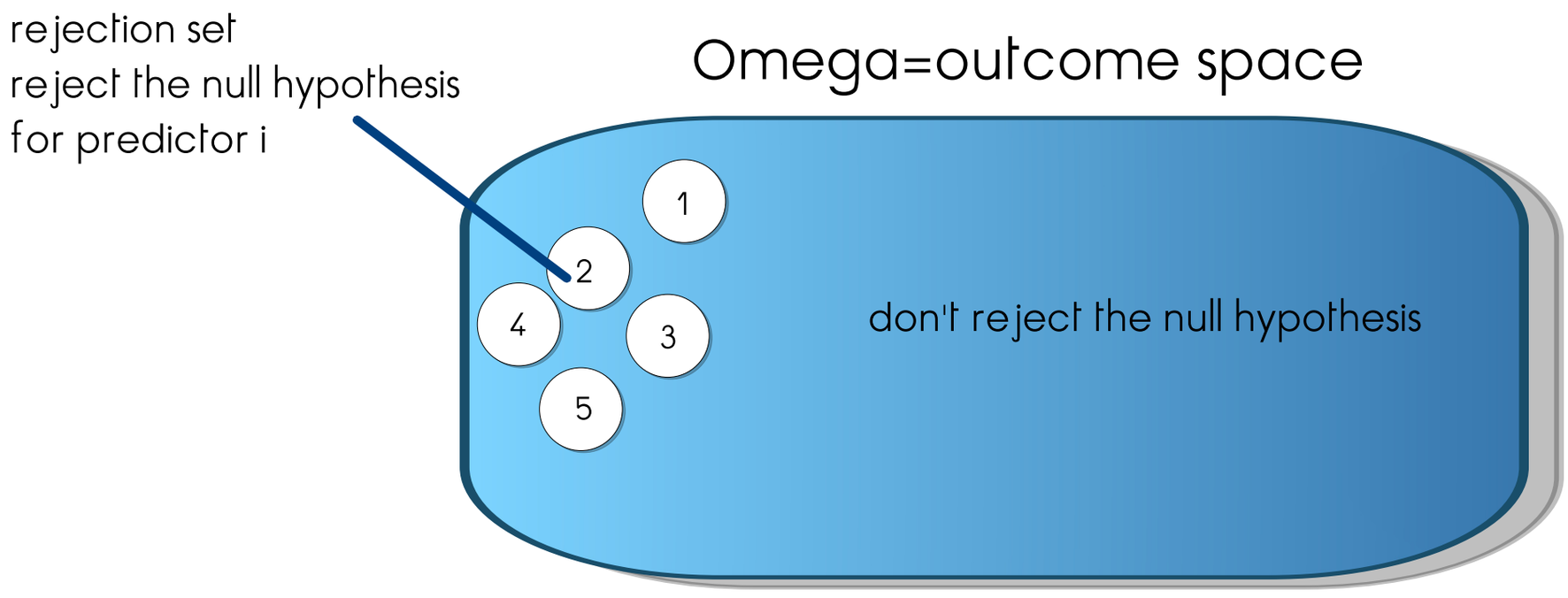
Consider the online ad problem, our goal is to maximize click-through rate. Our null hypothesis is that nothing performs better than picking one of the ads uniformly at random each time.

We have a large number of click-prediction algorithms. Each such algorithm takes as input information about the person, the web page and the ad and predicts the probability that the person will click on the ad.

We can go back in time and compute the expected number of errors each method would have made. We can use a statistical test to quantify the statistical significance of the performance of the method.

Suppose we have 100 methods and use an alpha value of 1%
Suppose for our data we found that one of the 100 methods rejects the null hypothesis at the 1% significance level. How sure can we be that the predictor that we found is better than random?

# The probabilty theory of statistical tests




We don't know what would happen of different samples than the one we observe.
In the worst case the rejection sets are disjoint.

The Bonferroni correction for multiple-hypothesis testing:

If $n$ statistical tests are performed using the same data
and the significance threshold used for all tests is $\alpha$
Then the probability that at least one of the tests will
reject the null hypothesis can be as high as $n\alpha$

# Be a skeptic:

When you read that something has been proven statistically:
1. Ask what was the null hypothesis
2. Ask what was the statistical significance
3. Ask whether similar tests were performed that did not succeed.