CSE 103: Notes on Union Bounds, Conditional Probability and Bayes' Rule

October 30, 2014

1 Basic Probability Formulas

In lecture, we "proved" some basic probability formulas using Venn diagrams. Let Ω be any outcome space, and A and B be any two events in it.

One such formula is the definition of conditional probability:

$$Pr(A \mid B) = \frac{Pr(A \cap B)}{Pr(B)}$$

Another is the *summation rule* given in the lecture notes, also called the "law of total probability":

$$Pr(A) = Pr(A \cap B^c) + Pr(A \cap B) = Pr(A \mid B^c)Pr(B^c) + Pr(A \mid B)Pr(B)$$

Finally, we can state the *union bound*: for any events $A, B \subseteq \Omega$,

$$Pr(A \cup B) \leq Pr(A) + Pr(B)$$

All these formulas are **always** valid for any non-empty events A, B.

A quick note: we can extend the summation rule to any sets B_1, B_2, B_3 forming a *partition* of Ω (such all the sets are mutually disjoint, but $B_1 \cup B_2 \cup B_3 = \Omega$). This is because the set $B_1 \cup B_2$ is the complement of A, so

$$Pr(A) = Pr(A \cap (B_2 \cup B_3)) + Pr(A \cap B_1) = Pr(A \cap B_1) + Pr(A \cap B_2) + Pr(A \cap B_3)$$

This extends to any number of sets in the partition. Similarly, the union bound can also be extended: it holds for any (countable) number of sets.

2 Coupon Collector Problem

A child is buying cereal. Each cereal box contains a random action figure. Suppose there are n different types of action figure. How many boxes must the child buy to be fairly sure of (with high probability) collecting at least one of each type of action figure?

Clearly, buying more boxes cannot hurt the child's chances, but after all the figures are collected it will not help either. We wish to know how many boxes guarantee that all figures are probably collected.

This is a well-known problem called the coupon collector problem, solved in Chapter 6 (Section 6.2) of the lecture notes.

In lecture, we solved an equivalent problem, stated like this: Suppose we choose m people at random. What's the minimum m we can choose so that with high probability (let's say ≥ 0.95), each day of the year is the birthday of at least one of the m people we've chosen? (Assume there are 365 days in every year, and that each person's birthday is independently and uniformly chosen as one of the days $\{1, 2, \ldots, 365\}$.)

This is an identical problem to the cereal box coupon collector, where instead of getting all n action figures we are getting all 365 days of the year. In both cases, each new cereal box / person gives us exactly one action figure / day, chosen at random independently of the others.

The derivation given in lecture is basically the same as in Chapter 6, which also contains several other examples of similar problems and their solutions.

3 Bayes' Rule

Take any non-empty events A and B. From the formula for conditional probability,

$$Pr(A \cap B) = Pr(B)Pr(A \mid B)$$

Using this fact and the conditional probability formula again,

$$Pr(B \mid A) = \frac{Pr(A \cap B)}{Pr(A)} = \frac{Pr(A \mid B)}{Pr(A)} Pr(B)$$

This is an extremely important equation, called *Bayes' Rule*.

To interpret it, we can think of B as being some event whose probability we're interested in, and A as representing some intermediate knowledge about the experiment (e.g., a partially drawn poker hand restricts the original outcome space to a smaller event). Initially, our knowledge is represented by $Pr(B)$. Bayes' Rule tells us how to update this knowledge once we're given A: just multiply $Pr(B)$ by $\frac{Pr(A|B)}{Pr(A)}$.

A final question: what does Bayes' Rule tell us if A and B are independent?