Combinatorics
&
Uniform,finite Distributions

The Combinatorial function C(n,r):
The number of ways to choose a subset of size r from a set of size n.

Alternative notation: $\binom{n}{r}$,

Expressed verbally as "n choose r"

$$n \geq 0,\ 0 \leq r \leq n, \quad \binom{n}{0} = \binom{n}{n} = 1$$

Binomial Expansion

$$(a+b)^2 = (a+b)(a+b) = a^2 + 2ab + b^2$$

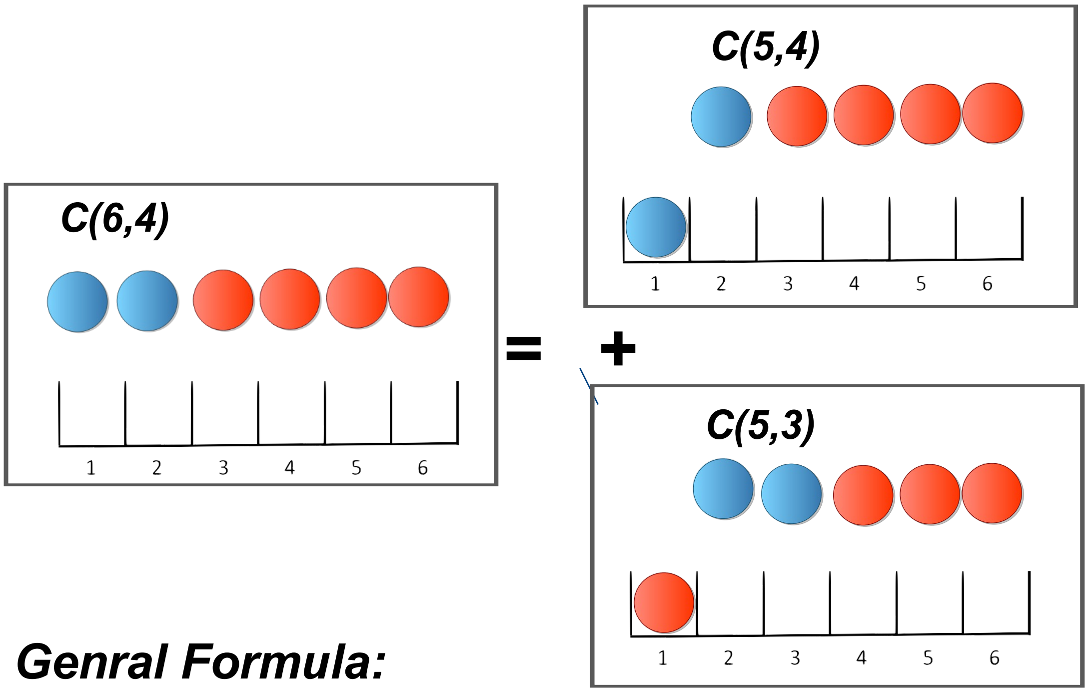
$$(a+b)^3 = (a+b)(a+b)(a+b) = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a+b)^3 = \binom{3}{0}a^3 + \binom{3}{1}a^2b + \binom{3}{2}ab^2 + \binom{3}{3}b^3$$

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$$

Incremental computation of C(n,r)

number of different patterns of placing
r red balls and n-r blue balls in n bins



Genral Formula:
$C(n,r)=C(n-1,r)+C(n-1,r-1)$

Pascal Triangle

Calculating the binomial coefficients using the recursion

$$c(n,r)=c(n-1,r)+c(n-1,r-1)$$

and the boundary conditions

$$c(n,0)=c(n,n)=1$$

C(0,0) 1

C(1,0) 1 C(1,1) 1

C(2,0) 1 C(2,1) 2 C(2,2) 1

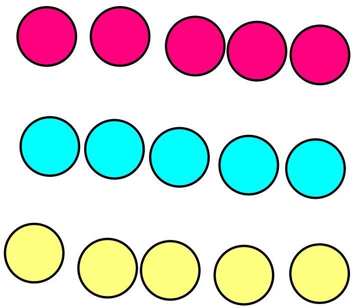
C(3,0) 1 C(3,1) 3 C(3,2) 3 C(3,3) 1

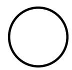
C(4,0) 1 C(4,1) 4 C(4,2) 6 C(4,3) 4 C(4,4) 1

The Pigeon-Hole Principle

There are 3 colors of marbles. How many marbles do we need in order to guarantee that there is at least one color for which we have at least 6 marbles?

Find out what is the maximal number of marbles you can have without having 6 marbles of the same color





now add one more

$3*(6-1)+1$

The Birthday Paradox

How many people do you need in the room so that at least two of them have the same birthday?

For sure? 366

With probability at least half?

Assume all days have the same probability (1/365)

K = the number of people in the room.

We want to calculate P(A) for the event
A={K birthdays such that at least two are the same}

$$P(A) = \frac{|A|}{|\Omega|}$$

$$\Omega = \{1, \ldots, 365\}^K$$

$$|\Omega| = 365^K$$

How many people do you need in the room so that at least two of them have the same birthday?

$$A = \left\{ (i_1, i_2, \ldots, i_K), 1 \le i_j \le 365 \middle| \exists \ 1 \le j_1 < j_2 \le K, i_{j_1} = i_{j_2} \right\}$$

Consider the complement,
No two people have the same birthday

$$A^c = \left\{ (i_1, i_2, \ldots, i_K), 1 \le i_j \le 365 \middle| \forall \ 1 \le j_1 < j_2 \le K, i_{j_1} \ne i_{j_2} \right\}$$

$$A^c \doteq \{x \in \Omega, \ x \notin A\} \qquad A^c = \Omega - A$$

A sequence of K birthdates and no 2 have the same birthday
-> K days out of 365

$$\left| A^c \right| = P(365, K) = \frac{365!}{(365 - K)!}$$

Putting it all together

$$|\Omega| = 365^K$$

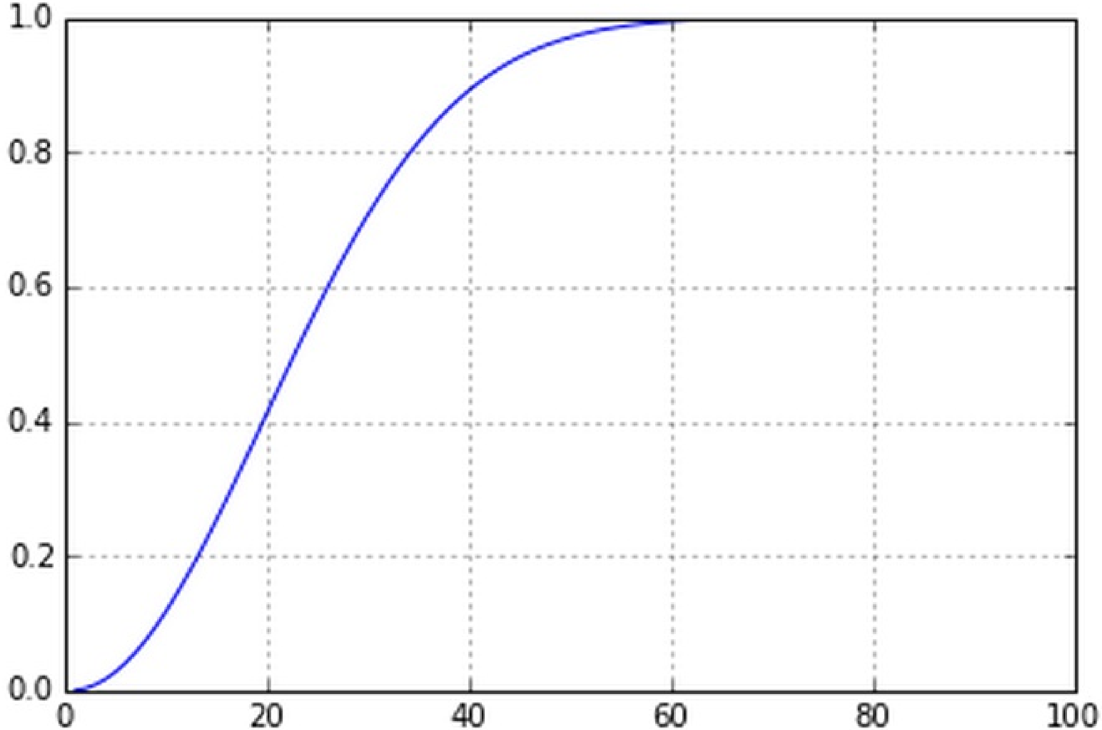
$$|A^c| = \binom{365}{K} = C(365,K) = \frac{365!}{K!(365-K)!}$$

$$|A| = |\Omega| - |A^c|$$

$$P(A) = \frac{|A|}{|\Omega|} = \frac{|\Omega| - |A^c|}{|\Omega|} = 1 - \frac{|A^c|}{|\Omega|} = 1 - P(A^c)$$

$$P(A) = 1 - \frac{P(365,K)}{365^K} == 1 - \frac{365!}{(365-K)!} = 1 - \left(\frac{365}{365}\right) \times \left(\frac{364}{365}\right) \times \left(\frac{363}{365}\right) \times \cdots \times \left(\frac{365-K+1}{365}\right)$$

1	:	0.00000000
2	:	0.00273973
3	:	0.00820417
4	:	0.01635591
5	:	0.02713557
6	:	0.04046248
7	:	0.05623570
8	:	0.07433529
9	:	0.09462383
10	:	0.11694818
11	:	0.14114138
12	:	0.16702479
13	:	0.19441028
14	:	0.22310251
15	:	0.25290132
16	:	0.28360401
17	:	0.31500767
18	:	0.34691142
19	:	0.37911853
20	:	0.41143838
21	:	0.44368834
22	:	0.47569531
23	:	0.50729723
24	:	0.53834426
50	:	0.97037358
100	:	0.99999969



How many strings contain 3 letters and two digits?

(digits and letters can repeat and there is no restrictions on their order)

Number of ways to combine 3 letters and 2 digits: C(5,2)

Set of possible 3 letter tuples = {A,...,Z}^3
*The size of this set is 26*26*26 = 26^3*

*Set of 2 digits, size of this set is 10*10=100*

$C(5,2) \times 26^3 \times 10^2$

What is the number of strings that start with a digit followed by 4 letters, followed by 2 digits?

Answer: this is a product set:

$10*26*26*26*26*10*10 = 26^4*10^3$

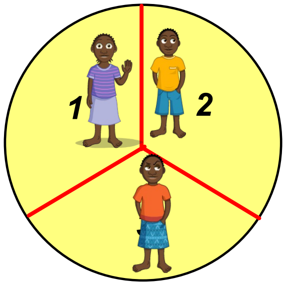
What is the probability that a random word of length 4 with distinct letters has the letters in increasing alphabetical order?

Outcome space Omega: all words with 4 distinct letters: 26*25*24*23=P(26,4)

Event A: the number of words of length 4 that have 4 distinct letters in increasing order:
P(26,4)/4! = C(26,4)

P(A)=|A| / |Omega| = C(26,4)/P(26,4) = 1/4!=1/24

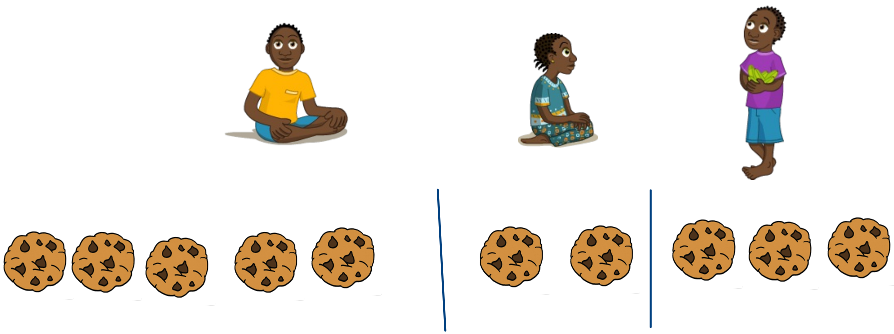
How many ways to sit 3 out of 7 kids on a marry-go-round with three identical seats?



Number of ways of choosing 3 out of 7 kids when the order matters P(7,3)

The marry-go-round can be rotated to 3 indistinguishable positions: P(7,3)/3

How many ways to divide 10 cookies among three children? (the cookies are identical and cannot be broken)



We have one fewer vertical line than children

$$C(10+3-1,3-1) = C(10+3-1,10)$$

How many ways to split 10 cookies among 3 kids if each kid has to get at least 2 cookies?

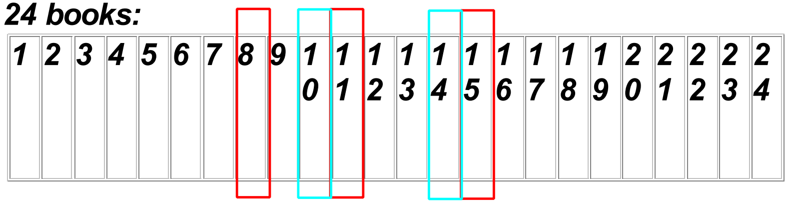
First, give each kid 2 cookies, 4 cookies are left.

Second, divide the remaining cookies among the 3 kids.

$C(4+3-1,3-1) = C(4+3-1,4)$

You have a shelf with 24 books on it. You can pick any 3 books but no two picked books can be right next to each other. How many choices do you have?

24 books:



Red rectangle: chosen book
Cyan rectangle: "buffer" book

Equivalent to choosing 3 out of 24-2=22 books:
If we care about order of chosen books: P(24-2,3)
If we don't care about order of chosen books: C(24-2,3)

If you choose 3 out of 24 books at random, what is the probability that at least 2 would be next to each other?

Size of sample space (Omega):
number of way to choose 3 out of 24 books:
If we care about order of chosen books: P(24,3)
If we don't care about order of chosen books: C(24,3)

$$P(A) = 1 - \frac{|A^c|}{|\Omega|} = 1 - \frac{P(22,3)}{P(24,3)} = 1 - \frac{C(22,3)}{C(24,3)}$$

For monday:

1. Review class (slides are on web site)
2. Read chapter 4 up to (and not including) 4.5
3. You should now be able to finish the HW (Due on wed).

Next time: Poker and non-uniform distributions.