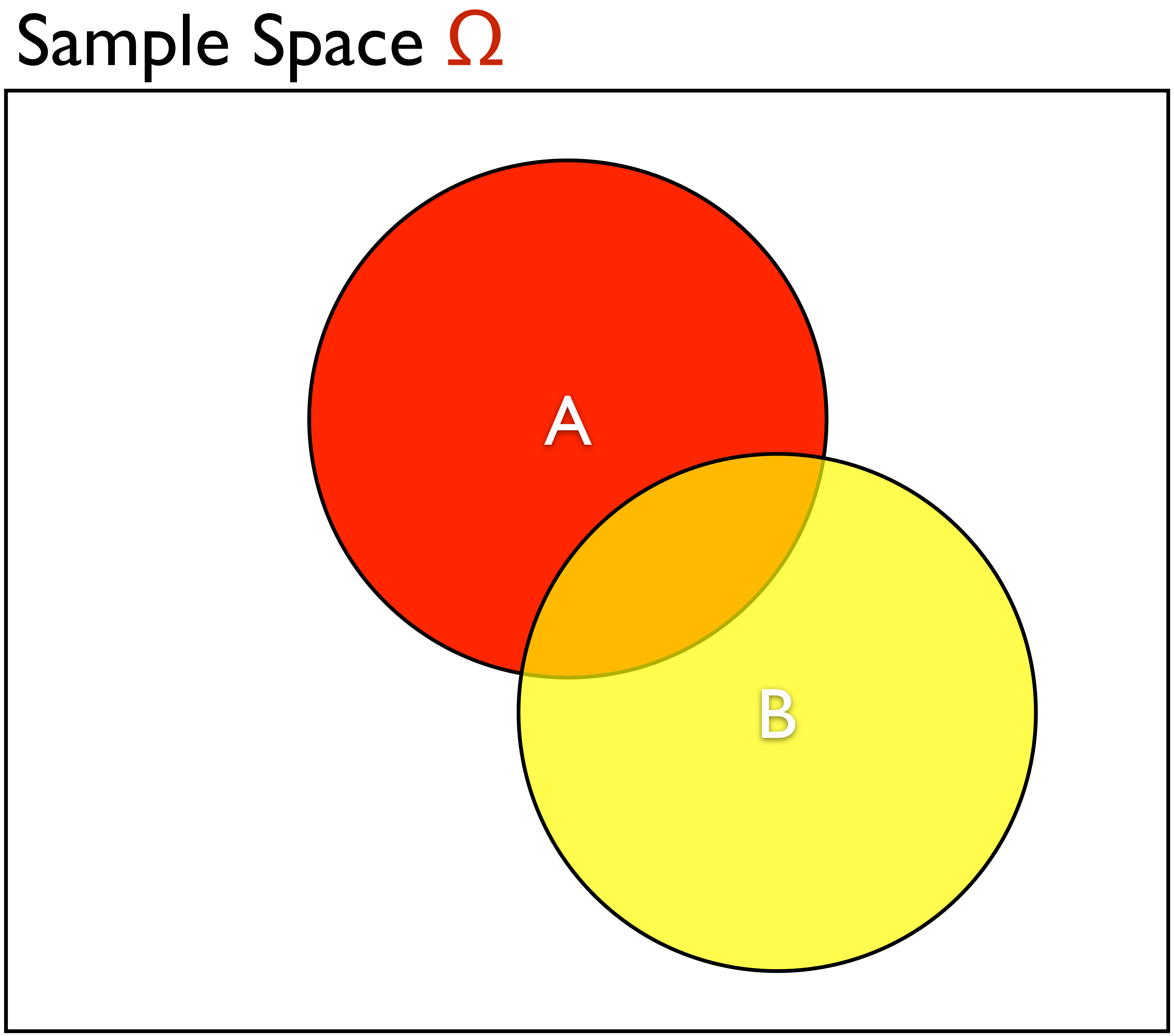
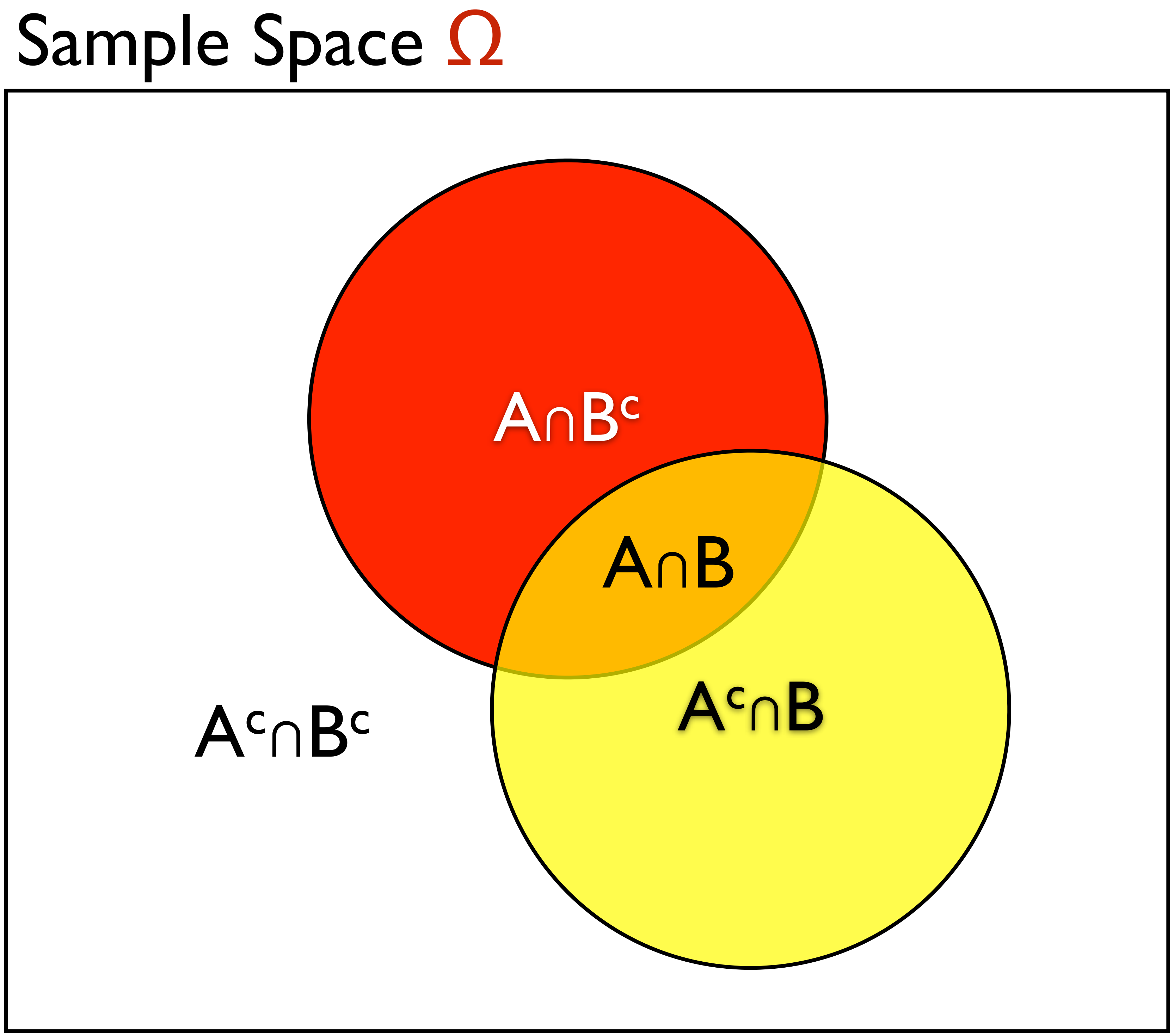
Conditional Probabilities
and
Independence

Events



Partitioning into disjoint events



Independence: Definition

$$P(A \cap B) = P(A) \times P(B) = P(A)P(B)$$

Example of Independence

Suppose we have a biased coin $P(H) = 0.9, P(T) = 0.1$

We flip the coin 10 times. What is the probability of

$H, T, H, H, H, T, H, H, H, H$?

The coin flips define independent events, therefor

$P(H, T, H, H, H, T, H, H, H, H) =$

$= 0.9 \times 0.1 \times 0.9 \times 0.9 \times 0.9 \times 0.1 \times 0.9 \times 0.9 \times 0.9 \times 0.9$

$0.9^8 \times 0.1^2$

Dependence: Example

- Sample Space: all 22 year old men.
- Set A: men playing in the NBA
- Set B: men taller than 6’ 5”
- $P(A \cap B) = P(\text{NBA players that are at least 6'5"}) \gg P(A)P(B)$

Conditional probability definition

The probability of A given B is:

$$P(A \mid B) \doteq \frac{P(A \cap B)}{P(B)}$$

Expressing independence using conditional prob.

Definition of independence: $P(A \cap B) = P(A)P(B)$

Implication 1: $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{P(A)P(B)}{P(B)} = P(A)$

Implication 2: $P(B \mid A) = \dfrac{P(A \cap B)}{P(A)} = \dfrac{P(A)P(B)}{P(A)} = P(B)$

- Interpretation: If A and B are independent, knowing that A happened does not change the probability of B and vice versa

- Note: we are speaking of states of knowledge, NOT of causality.

Dependence works both ways.

- If A depends on B then B depends on A
- Dependence/independence is **not** a directional relationship.
- Conditioning A on B rather than B on A is a decision of the observer - it is not inherent to the observed system.

independence in tabular form

	marginal on b	A	A^c
marginal on A		4/12 =1/3	8/12 =2/3
B	3/12 =1/4	1/12	2/12
B^c	9/12 =3/4	3/12	6/12

$$P(A \cap B) = 1/12$$

$$P(A) = 1/3$$

$$P(B) = 1/4$$

$$P(A \cap B) = P(A)P(B)$$

- A and B are independent
- Are A and B^c Independent?

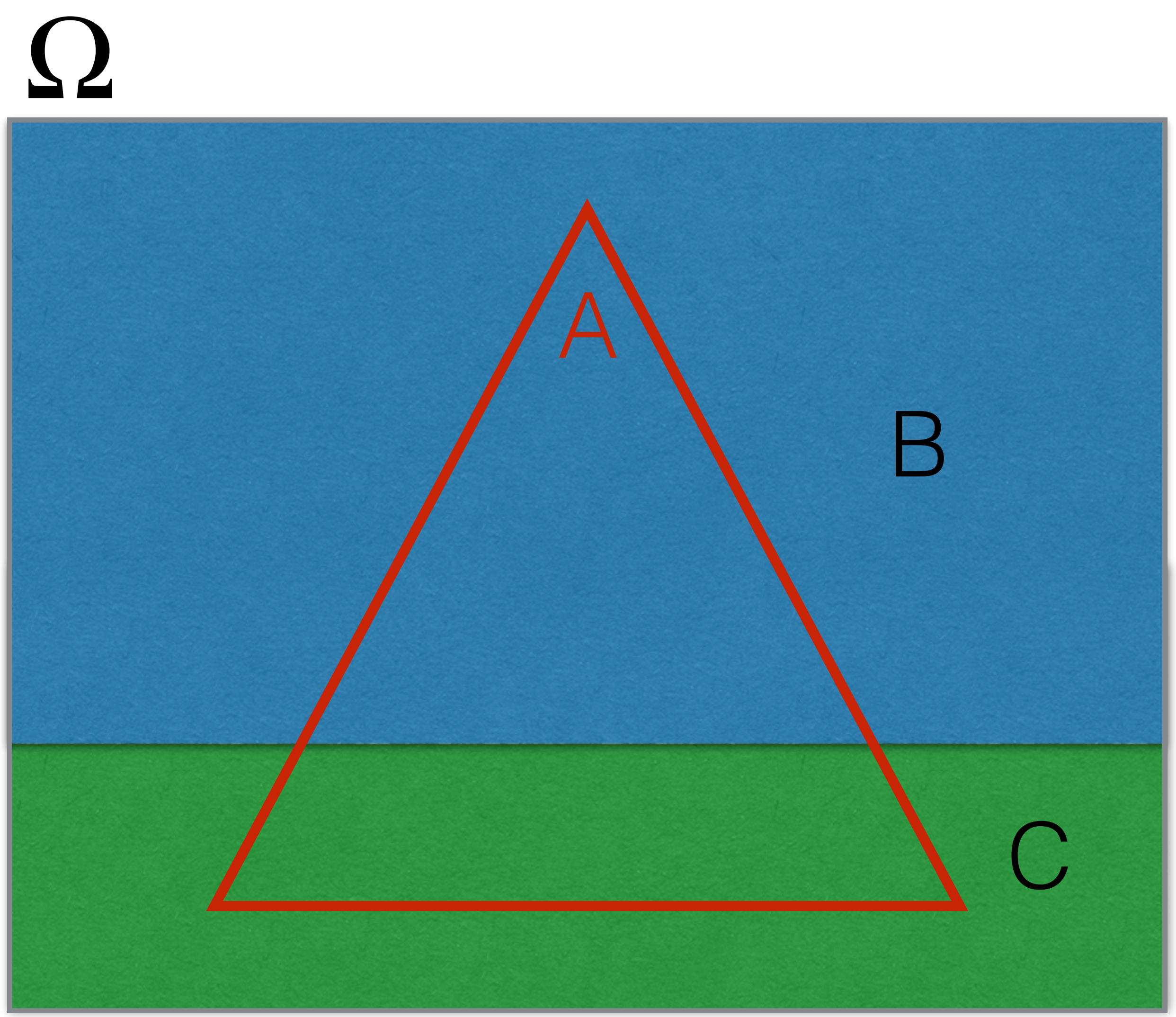
Are A and B^c Independent?

- Yes, because

$$P(A \cap B^c) = P(A) - P(A \cap B) =$$
$$= P(A) - P(A)P(B) = P(A)\left(1 - P(B)\right) = P(A)P(B^c)$$

- Same for: A^c and B , A^c and B^c

Conditional summation rule



Standard summation rule

$$P(B)+P(C)=1$$

Conditional summation rule

$$P(B \mid A)+P(C \mid A)=1$$

Because:

$$P(B,A)+P(C,A)=P(A)$$

Dividing all by $P(A)$ we get:

$$\frac{P(B,A)}{P(A)}+\frac{P(C,A)}{P(A)}=1$$

Example using Poker

- Suppose your private cards (hole) are Ace ♡,2♤
- Suppose the community cards are 3♤,4♤,5♤,K♧,J♡
- You have one opponent. What is the probability that you opponent has a Straight Flush? (All cards of the same suit and sequential)
- Pr(opponent's cards complete a straight flush
  | hole=(Ace ♡,2♤) and community=(3♤,4♤,5♤,K♧,J♡))
- There are three combination the 2 cards of the opponent that would give her a straight flush: (1♤,2♤),(2♤,6♤),(6♤,7♤). However the first two pairs are impossible because you hold the 2♤.
- The conditional probability of a straight flush is therefor:
  Pr(opponent's cards complete a straight flush
  | hole=(Ace ♡,2♤) and community=(3♤,4♤,5♤,K♧,J♡))=2/C(52-7,2)

pair-wise vs complete independence

- Suppose we have three coins.
- We flip two of the coins to get HH,HT,TH,TT
- We chose the side of the third coin so that the number of Heads is even. (i.e. H if HT or TH. T otherwise)
- We have therefor a uniform distribution over 4 possibilities: HHT,HTH,THH,TTT.
- Focusing on any pair of coins, the outcomes of the coins are independent (Check!)
- However, consider the combination HHH
  - $P(HHH)=0$, while $P(H)P(H)P(H)=1/8$
- Even though each pair of coins are independent, the set of 3 coins are not independent!

Bayesian Inference

Bayesian Inference

- Who crashed the car?
- Suppose the only possible drivers are Rob or Sarah.
- We know the following probabilities.
  - P(R rob drove)=10%, P(C crash | R rob drove)=50%
  - P(S Sarah drove)=90%,   P(C crash | S sarah drove)=1%

$$P(C)=P(C,R)+P(C,S)=P(C\mid R)P(R)+P(C\mid S)P(S)$$

$$P(R\mid C)=\frac{P(R,C)}{P(C)}=\frac{P(C\mid R)P(R)}{P(C)}=\frac{P(C\mid R)P(R)}{P(C\mid R)P(R)+P(C\mid S)P(S)}$$

$$=\frac{0.1\times 0.5}{0.1\times 0.5+0.9\times 0.01}=0.85 \qquad P(S\mid C)=0.15$$

- Prior probabilities: Sarah drives the car 90% of the time, Rob 10% of the time.
- Posterior probabilities: given that there was an accident, the probability that the driver was Rob jumps to 85% because he is much more accident prone.

The following example is taken from *Probabilistic Reasoning in Intelligent Systems* by Judea Pearl:

> You wake up in the middle of the night to the shrill sound of your burglar alarm. What is the chance that a burglary attempt has taken place?

The relevant facts are:

- There is a 95% chance that an attempted burglary attempt will trigger the alarm. That is,

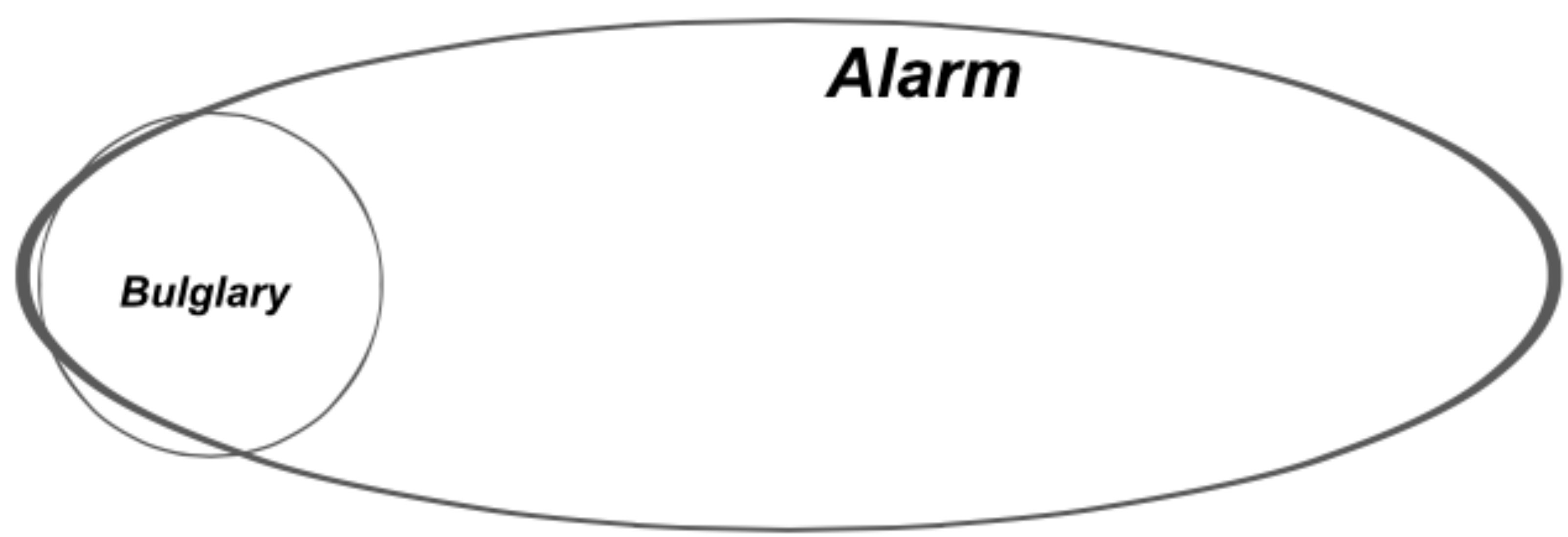
$$\Pr(\text{alarm}|\text{burglary}) = 0.95.$$

- There is a 1% chance of a false alarm.

$$\Pr(\text{alarm}|\text{no burglary}) = 0.01.$$

- Based on local crime statistics, there is a one-in-10,000 chance that a house will be burglarized on a given night.

$$\Pr(\text{burglary}) = 10^{-4}.$$



We are interested in the chance of a burglary given that the alarm has sounded. We can use the conditional probability formula for this:

$$\Pr(\text{burglary}|\text{alarm}) = \frac{\Pr(\text{burglary, alarm})}{\Pr(\text{alarm})} = \frac{\Pr(\text{alarm}|\text{burglary})\Pr(\text{burglary})}{\Pr(\text{alarm})}.$$

The one term we don't immediately know is Pr(alarm). By the summation rule,

$$\Pr(\text{alarm}) = \Pr(\text{alarm}|\text{burglary})\Pr(\text{burglary}) + \Pr(\text{alarm}|\text{no burglary})\Pr(\text{no burglary}).$$

Putting it all together,

$$\Pr(\text{burglary}|\text{alarm}) = \frac{0.95 \times 10^{-4}}{0.95 \times 10^{-4} + 0.01 \times (1 - 10^{-4})} = 0.00941,$$

about 0.94%. Thus our belief in a burglary has risen approximately a hundredfold from its default value of 10^{-4}, on account of the alarm.

It is frequently the case, as in this example, that we wish to update the chances of an event H based on new evidence E. In other words, we wish to know $\Pr(H|E)$. The derivation above implicitly uses the following formula, called **Bayes' rule**:

$$\Pr(H|E) = \frac{\Pr(E|H)\Pr(H)}{\Pr(E)}.$$

To calculate Pr(E) we use the summation rule

$$\Pr(H \mid E) = \frac{\Pr(E \mid H)\Pr(H)}{\Pr(E)} = \frac{\Pr(E \mid H)\Pr(H)}{\Pr(E \mid H)\Pr(H) + \Pr(E \mid H^c)\Pr(H^c)}$$

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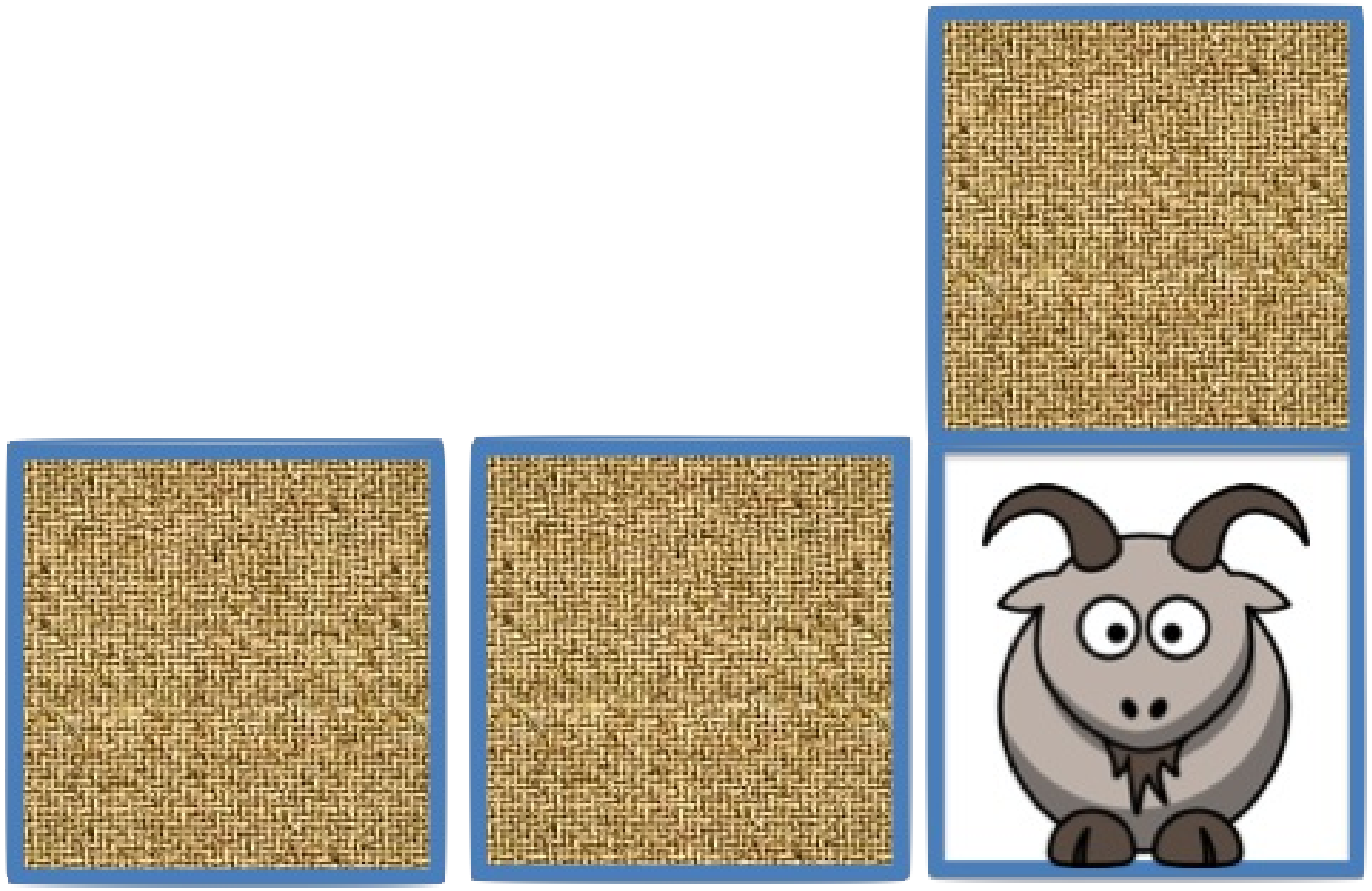
The Monty Hall Puzzle

- Monty Hall was a variety show on TV.
- In one of the games there are three doors, one hiding a treasure, two hiding goats.
- Your goal is to select the door with the treasure.


I am betting
on this door



Monty opens this door

I am betting on this door

I am allowed to switch, should I?

Argument that it does not matter:

The chance that the treasure is behind each of the doors 50%.

As the probabilities are equal, it does not matter whether we switch or not.

Argument for choosing one of the two unopen doors at random.

Before I had to choose between 3 doors - my probability of sucsses was 1/3

Now I am choosing between between two doors, my probability of success is 1/2
So random is better than staying on the same door.

Argument for Switching.

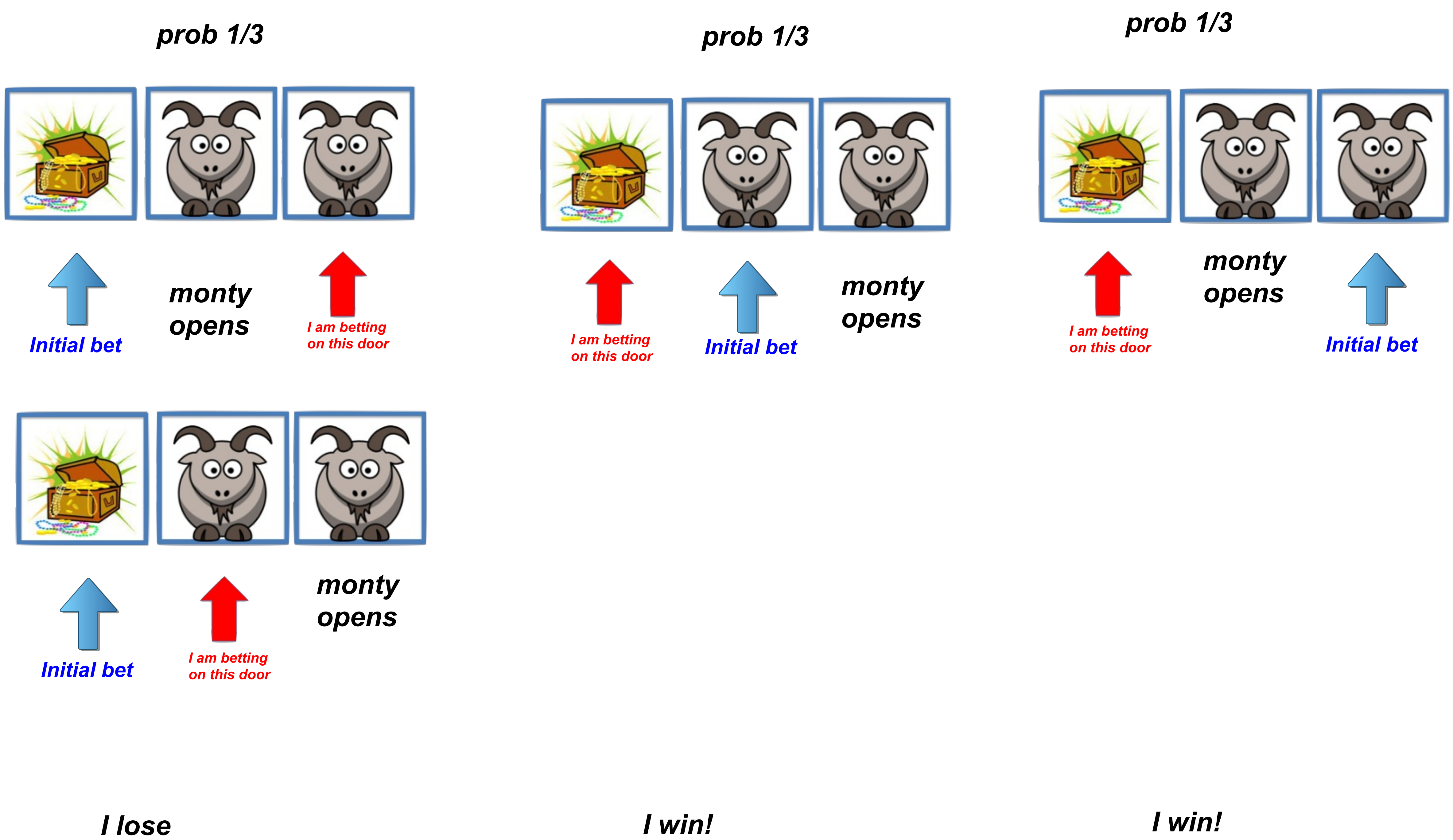
The probability that the treasure is behind the door I chose did not change.
Therefor the probability that switching will put me on the treasure must be 2/3:

$1/2*1/3+1/2*2/3 = 1/2$

Arguments against switching:

I know already that one of the other doors has a goat behind it. So getting the information does not tell me anything new.

Analysis for always switching



Hidden Assumption: *monty always opens a door to reveal a goat.*

In fact, he might have his own goals:

If Monty wants us to lose: *open door only when we choose the treasure door.*

If Monty wants us to win: *open door only when we choose a goat door.*

For us the only SAFE thing to do is not to switch.

This is called the "Min-Max" strategy.

Min-Max is the strategy the guarantees us the best outcome in the worst case.

More on that - game theory.