Combinatorics 3
poker hands
and Some general probability

Play cards

13 ranks

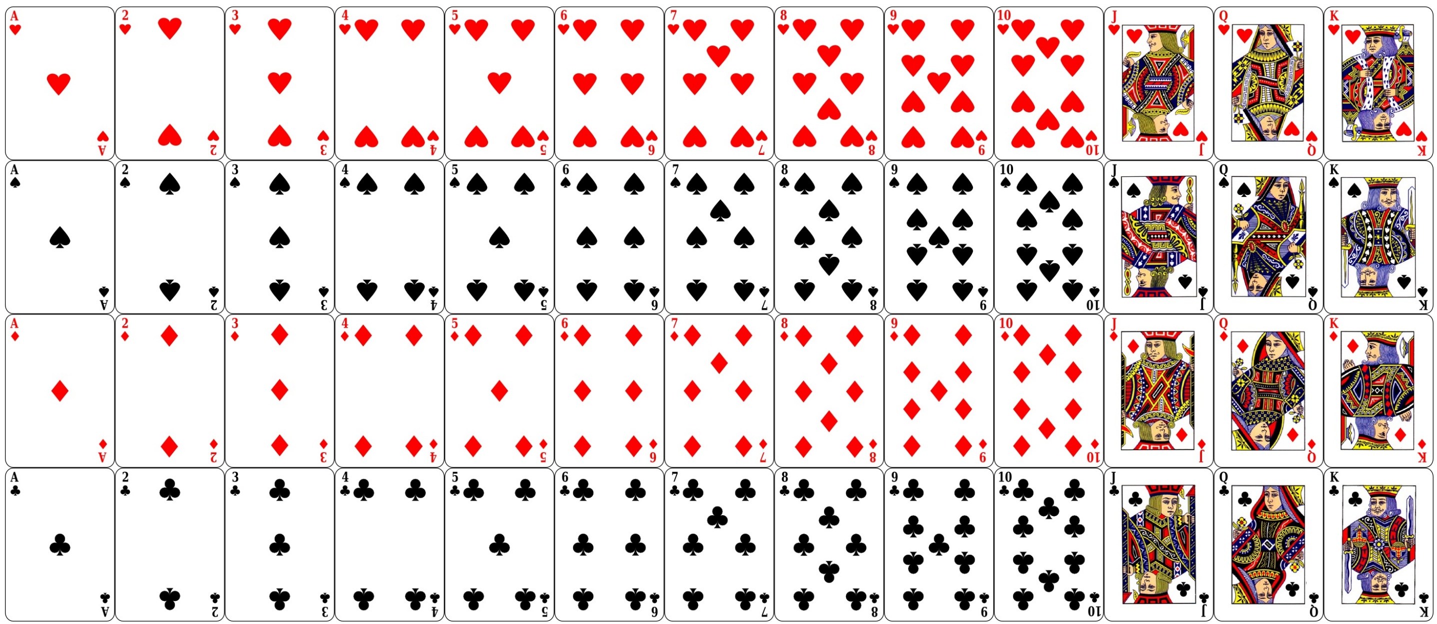
4 Suits

Heart

Spade

Diamond

Club



Total: 4X13=52 cards

You pick one card from a shuffled deck.
What is the probability that it is the Ace of Spades?
1/52

You pick one card from a shuffled deck.
What is the probability that it is a spade or a diamond?
2/4 = 1/2

You pick one card from a shuffled deck.
What is the probability that it's rank is higher than 5?

Assuming that Ace is the highest we get 9/13

Basic Poker Rules

1. ***Each player has two private cards***
2. ***There are 5 shared cards***
3. ***A hand is 5 cards***
4. ***Hand with highest rank wins***

High Rank = Low Probability

The rank of hands in poker

The basic rules of texas hold'm poker

1. *Each player is dealt 2 cards (the hole)*
2. *A round of betting*
3. *Three cards are revealed (the flop)*
4. *A round of betting*
5. *Fourth card is revealed (the turn)*
6. *A round of betting*
7. *Fifth card is revealed (the river)*
8. *Final round of betting*

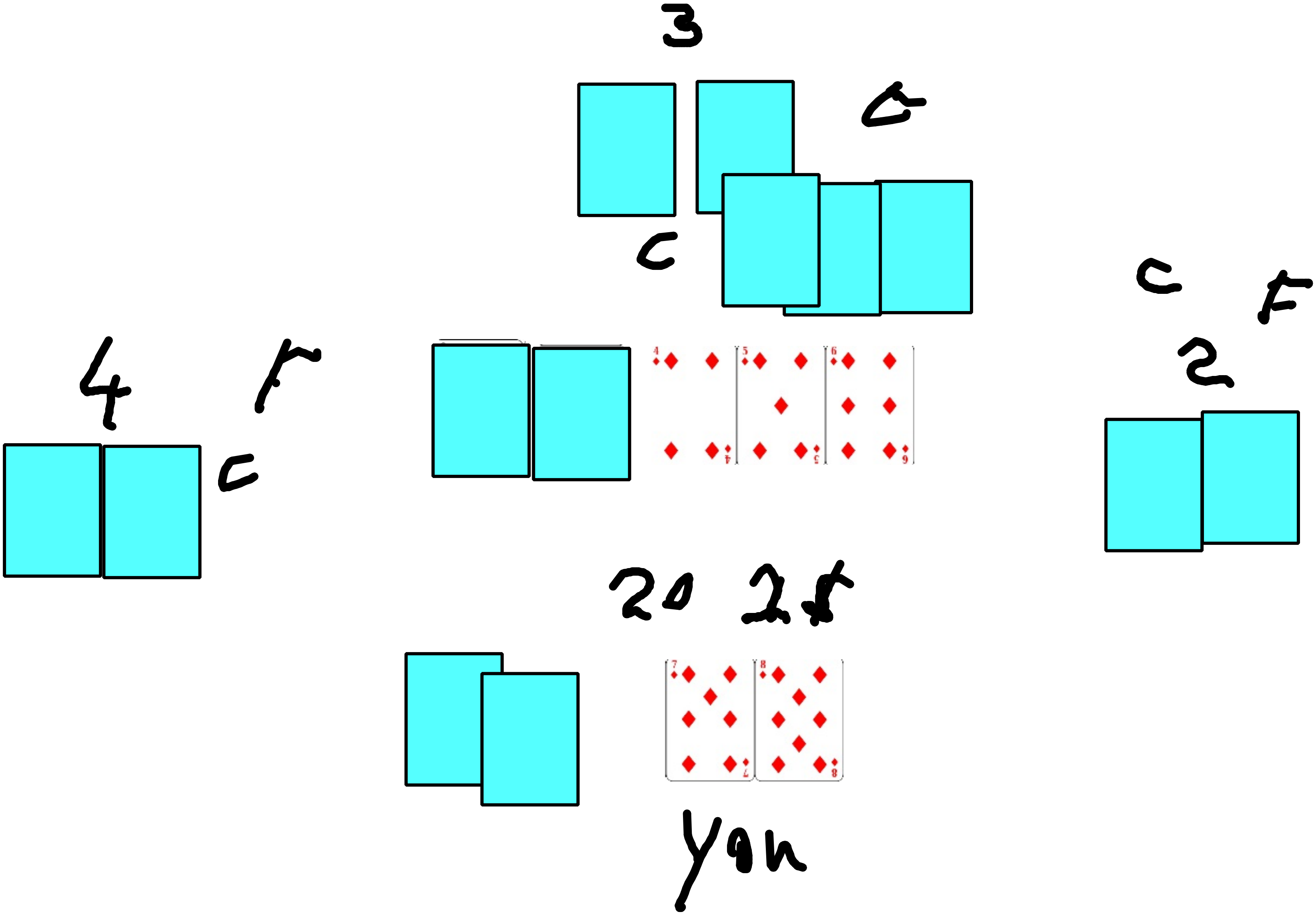
Betting rounds

1. Proceed clockwise.
2. Bets start with a minimal amount and can only increase
3. Each person has to either:

- *- Check: Match the current bet.*
- *- Raise: bet a larger amount*
- *- fold: quit the game (losing the money already put in)*

4. A round of betting repeats circling until a round where all players either checked or folded = a round in which the bet has not increased.

5. In the final round:
- if only one player remains, they win all of the bets (the pot).
- if more than one player is checked, there is a "showdown", the checked players show their cards and the one with the stronger hand wins.


3
4
2
20
You

Poker is a game of talent, not of chance

Each player tries to estimate the chances that theirs is a winning hand from the revealed cards and from the betting actions of the others.

At the high levels of the game, familiarity with the betting styles of other players is critical.

Winning or losing a single game is of little importance, it is the long term average that matters.

Curious?
Check out program about Annie Duke: Radiolab/Dealing with doubt
http://www.radiolab.org/story/278173-dealing-doubt/

At a minimum, a player has to have an intuitive knowledge of the probabilities of different hands.

Which is what we will now do.

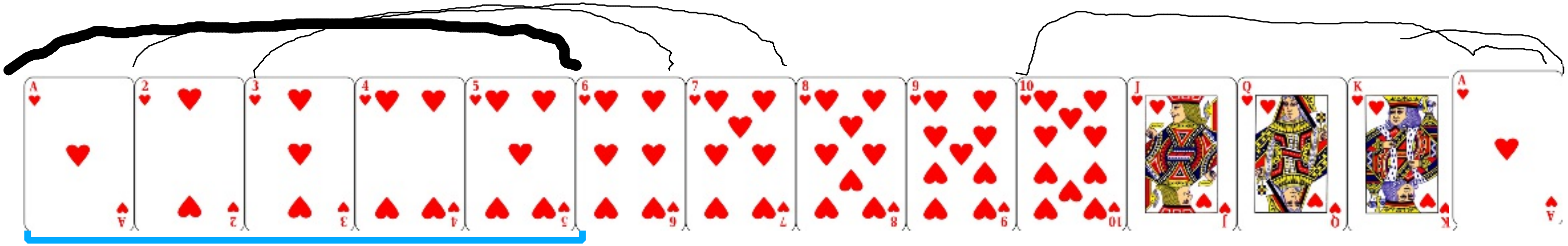
Calculating the probabilities of different hands

What is the sample space?

The sets of 5 cards out of 52.
Order does not matter
C(52,5) = 2,598,960

1 choice for the card ranks
4 choices for the suit
Prob = 4/C(52,5)

How many choices for the ranks?
the Ace can be added on either side



9 choices for the card ranks (can't be royal)
4 choices for the suit
Prob = 4*9/C(52,5)=36/C(52,5)

Number of choices for the rank of the 4 cards?
13

choices for the rank of the single?
12

choices for the suit of the single?
4

Prob = (13*12*4)/C(52,5)=624/C(52,5)

Number of choices for the rank of the triple:
13

Number of choices for the rank of the pair:
12

Number of choices for the suits of the triple:
C(4,3)=4

Number of choices for the suits of the pair :
C(4,2)

Prob = (13*12*C(4,3)*C(4,2))/C(52,5) = 3744/C(52,5)

Number of choices for the ranks of the cards?
C(13,5) = 10

choices for the suit of the cards?
4

Prob =
(C(13,5)*4)/C(52,5)=5148/C(52,5)

Number of choices for the ranks of the cards?
(excluding straight flush and royal flush)
C(13,5)-10

choices for the suit of the cards?
4

Prob =
(C(13,5)*4)/C(52,5)=5148/C(52,5)

How many choices for the card ranks?
10

How many choices for the card suits (cannot be royal flush or straight flush) ?
4^5-4

Prob =
10*(4^5-4)/C(52,5)=10,200/C(52,5)

$C(49,2)$

~10

Number of choices for the rank of the triple:
13

Number of choices for the suits of the triple:
C(4,3)=4

Number of choices for the ranks of the other 2 cards:
C(12,2)

Number of choices for the suits of the other 2 cards:
4*4

Prob = (13*4*C(12,2)*4*4)/C(52,5) = 54,912/C(52,5)

Unlike full house (2,3) the two pairs are indistinguishable
Number of choices for the ranks of the pairs?
C(13,2)

Number of choices for the rank of the single:
11

Number of choices for the suits of the pairs:
C(4,2)^2

Number of choices for the suit of the single:
4

Prob = (C(13,2)*11*(C(4,2)^2)*4)/C(52,5) = 123,552/C(52,5)

The lowest ranked hand = The hand with highest probability

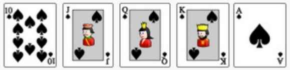
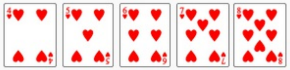
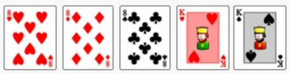
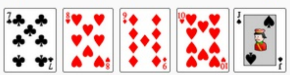
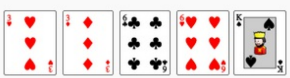
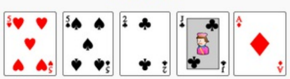
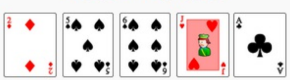
Number of choices for the pair:
13*C(4,2)

The other 3 cards must not form a pair, else the hand will be two pairs or full house.

Number of possible ranks for the 3 cards:
C(12,3)

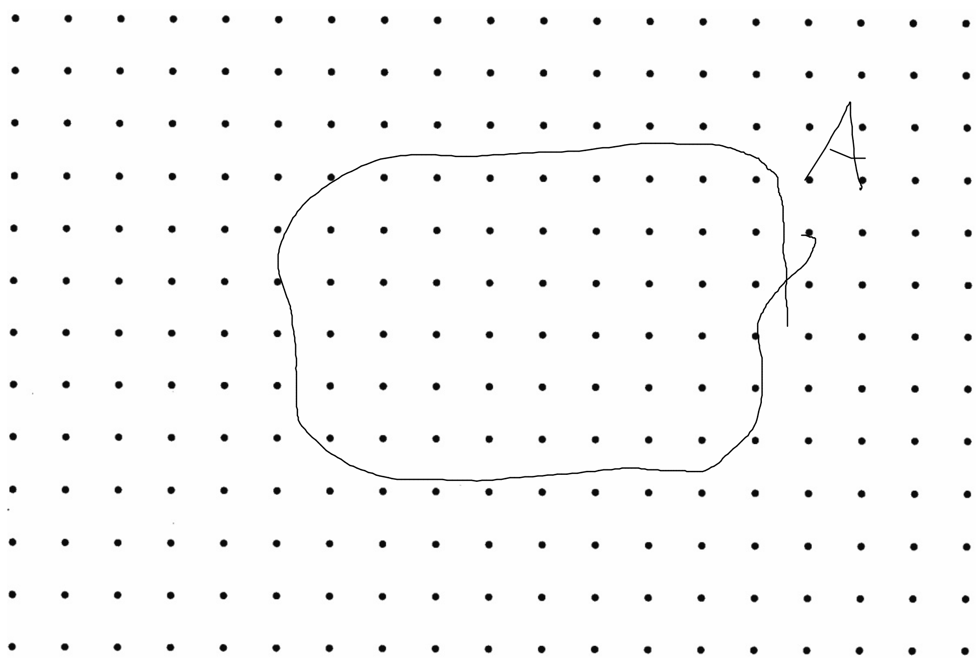
Number of possible suites for the 3 cards:
4**3

Prob=
(13*C(4,2)*C(12,3)*4**3)/C(52,5) = 1,098,240/C(52,5)

Hand	Distinct Hands	Frequency	Probability	Cumulative probability	Odds	Mathematical expression of absolute frequency
Royal flush 	1	4	0.000154%	0.000154%	649,739 : 1	$\binom{4}{1}$
Straight flush (excluding royal flush) 	9	36	0.00139%	0.00154%	72,192 : 1	$\binom{10}{1}\binom{4}{1}-\binom{4}{1}$
Four of a kind 	156	624	0.0240%	0.0256%	4,164 : 1	$\binom{13}{1}\binom{12}{1}\binom{4}{1}$
Full house 	156	3,744	0.144%	0.17%	693 : 1	$\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}$
Flush (excluding royal flush and straight flush) 	1,287	5,148	0.198%	0.367%	508 : 1	$\binom{13}{5}\binom{4}{1}-\binom{10}{1}\binom{4}{1}$
Straight (excluding royal flush and straight flush) 	10	10,200	0.392%	0.76%	254 : 1	$\binom{10}{1}\binom{4}{1}^5-\binom{10}{1}\binom{4}{1}$
Three of a kind 	858	54,912	2.11%	2.87%	46.3 : 1	$\binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}^2$
Two pair 	858	123,552	4.75%	7.62%	20.0 : 1	$\binom{13}{2}\binom{4}{2}^2\binom{11}{1}\binom{4}{1}$
One pair 	2,860	1,098,240	42.3%	49.9%	1.36 : 1	$\binom{13}{1}\binom{4}{2}\binom{12}{3}\binom{4}{1}^3$
No pair / High card 	1,277	1,302,540	50.1%	100%	0.995 : 1	$\left[\binom{13}{5}-10\right]\left[\binom{4}{1}^5-4\right]$
Total	**7,462**	**2,598,960**	**100%**	**---**	**1 : 1**	$\binom{52}{5}$

Counting probability distributions

Until now, we considered finite outcome spaces
where all outcomes the same probability.



$$P(A) = \frac{|A|}{|\Omega|}$$

All events have rational probabilities: n/m
In general, probabilities can be irrational.

Properties of general probability distributions

every event has probability between 0 and 1. $\forall \ A \subseteq \Omega, \ 0 \leq P(A) \leq 1$

The outcome space has probability 1. $P(\Omega) = 1$

The probability of a union is at most the sum of the probabilities

$$\forall \ A, B \subseteq \Omega, \ P(A \cup B) \leq P(A) + P(B)$$

The probability of a union of disjoint sets is equal to the sum of the probabilities

$$\forall \ A, B \subseteq \Omega, \ A \cap B = \emptyset$$

Disjoint sets

$$\Rightarrow P(A \cup B) = P(A) + P(B)$$

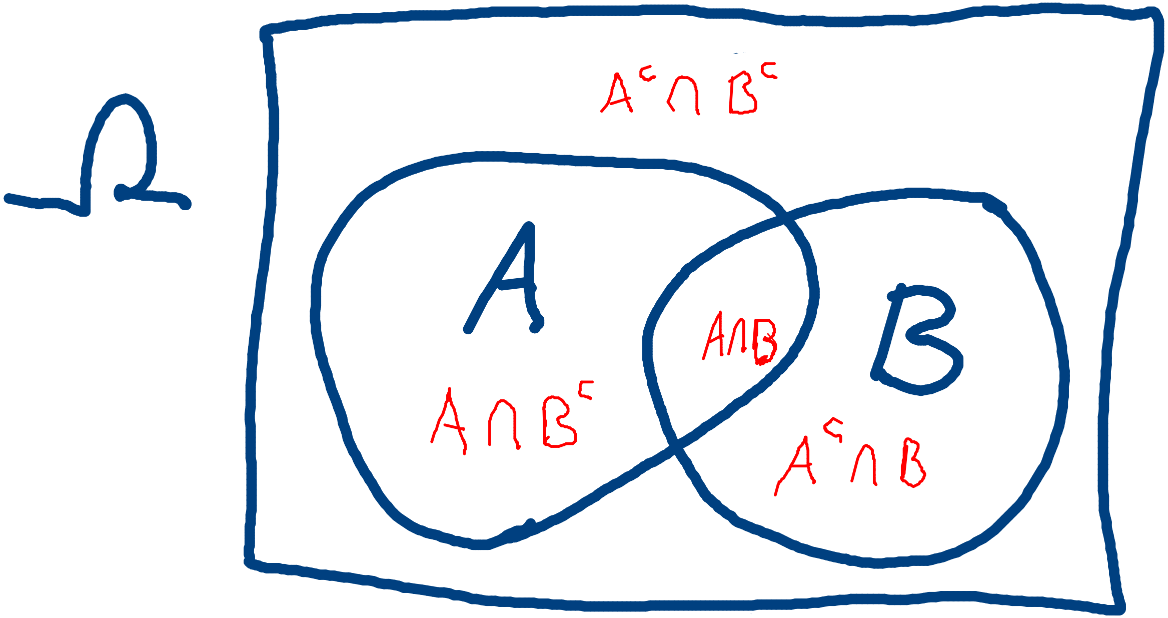
Implies that: $P(A^c) = 1 - P(A)$

$$A \cup A^c = \Omega, P(\Omega) = 1, A \cap A^c = \emptyset$$

$$\Rightarrow P(A) + P(A^c) = 1$$

The total probability equation

Partitioning a union



$$A \cup B = (A \cap B^c) \cup (A^c \cap B) \cup (A \cap B)$$

$$P(A \cup B) = P(A \cap B^c) + P(A^c \cap B) + P(A \cap B)$$

$$= P(A) + P(B) - P(A \cap B)$$

If $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$, $P(A \cap B) = \frac{1}{6}$, What is $P(A \cup B) =$?

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{1}{2} + \frac{1}{3} - \frac{1}{6} = \frac{2}{3}$$

A few simple questions:

If $P(A)=\dfrac{1}{2}$, $P(B)=\dfrac{2}{3}$, What can be said about $P(A\cap B)$?

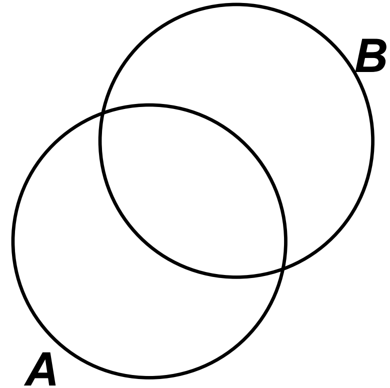
at most $\quad A\cap B \subseteq A \qquad P(A\cap B) \leq P(A) = 1/2$

at least $\quad 1 \geq P(A\cup B) = P(A) + P(B) - P(A\cap B)$

$$1/2 + 2/3$$

$$P(A\cap B) \geq \frac{1}{6}$$

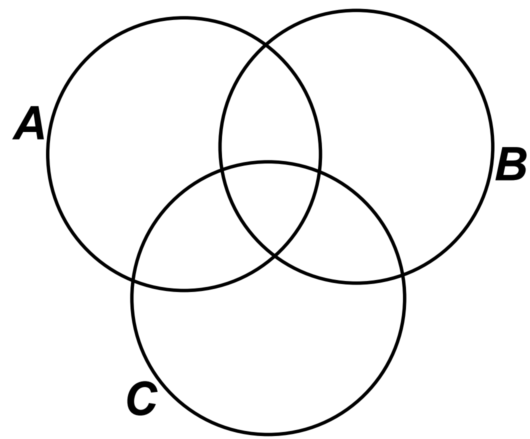
General Formula:



$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

How about:

$$P(A \cup B \cup C) = ?$$



$P(A) + P(B) + P(C)$
$- P($

For Tue.

1. Finish Week2 homework.
2. Read class notes:
- ***Section 4.5 (Poker)***
- ***Chapter 5***

The total probability equation for (countably) infinite sets

$$A_1, A_2, A_3, \ldots \subseteq \Omega$$
$$\forall i \neq j,\ A_i \cap A_j = \varnothing$$

$$A_1 \cup A_2 \cup A_3 \cup \cdots \doteq \bigcup_{i=1}^{\infty} A_i = \Omega$$

$$\text{Then } \sum_{i=1}^{\infty} P(A_i) = 1$$

Consider the natural numbers: 1,2,3,...
Is it possible to define a uniform distribution over them?

1st possibility: 0=P(1)=P(2)=... $P(\Omega) =$

2nd possibility: 0<P(1)=P(2)=.... $P(\Omega) =$