

description

These pages are a summary of the formulas used in CSE103. You can use this as a basis for your *hand-written* “cheat-sheet”. However, you cannot bring a printout of these pages to the final exam.

Combinatorics

1. The number of different tuples of length k over an alphabet of size n : n^k
2. The number of ways to order n different objects is $n!$
3. The number of sequences (i.e. order is important) of length k that can be created from a set of n different elements $\frac{n!}{(n-k)!}$
4. Binomial coefficients: The number of subsets (i.e. order is not important) of size k in a set of size n : $\binom{n}{k} = \frac{n!}{k!(n-k)!}$
5. For any two real numbers $(a+b)^n = \sum_{i=0}^n \binom{n}{i} a^i b^{n-i}$. To get the binomial distribution let $a = p$, $b = 1 - p$.
6. Suppose we have n_1 objects of type 1, n_2 objects of type 2, etc up to n_k so the total number of objects is $n = n_1 + \dots + n_k$. The number of different ways to order these n objects is $\binom{n}{n_1, n_2, \dots, n_k} = \frac{n!}{n_1! n_2! \dots n_k!}$
7. Bounds on the binomial coefficient: $\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{ne}{k}\right)^k$

Discrete probability

1. The union bound: for any events A_1, A_2, \dots, A_k ; $P(A_1 \cup \dots \cup A_k) \leq \sum_{i=1}^k P(A_i)$
2. Summation rule: If A_1, A_2, \dots, A_k are a partition of the sample space Ω , i.e. $A_1 \cup \dots \cup A_k = \Omega$ and $A_i \cap A_j = \emptyset$ if $i \neq j$. Then $\sum_{i=1}^k P(A_i) = 1$.
3. If A and B are arbitrary events $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
4. Conditional probability: $P(A|B) = P(A \cap B)/P(B)$.
5. Independence: A, B are independent events if $P(A \cap B) = P(A)P(B)$, equivalently, if $P(A|B) = P(A)$.

6. Bayes Rule: $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$

7. Conditional summation rule: If A is an event and B_i is a partition of the sample space then $P(A) = \sum_i P(A \cap B_i) = \sum_i P(A|B_i)P(B_i)$

Series

- Arithmetic sum: $1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$
- Geometric Series ($0 < r < 1$):

$$\sum_{i=0}^{\infty} r^i = \frac{1}{1-r}; \quad \sum_{i=1}^{\infty} r^i = \frac{r}{1-r}, \quad \sum_{i=1}^{\infty} i r^i = \frac{r}{(1-r)^2}$$

- If we repeatedly flip a coin whose probability of landing heads is p the expected number of flips until the first heads is:

$$\sum_{i=1}^{\infty} i(1-p)^{i-1}p = \frac{p}{1-p} \sum_{i=1}^{\infty} i(1-p)^i = \frac{p}{1-p} \frac{1-p}{p^2} = \frac{1}{p}$$

- Harmonic Sums:

$$\sum_{i=1}^n 1/i \approx \ln n, \quad \sum_{i=1}^{\infty} 1/i = \infty$$

$$\sum_{i \in \{\dots, -2, -1, 1, 2, \dots\}} \frac{1}{i} \text{ is undefined}$$
$$\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6}$$

Random variables, expectation and Variance

1. A random variable is a function from the instance space Ω to the real line.
2. Two random variables X, Y whose ranges are finite sets A, B are independent if and only if $P(X = a \text{ and } Y = b) = P(X = a)P(Y = b)$ for all $a \in A$ and $b \in B$. More on independent rv's in the next section.
3. The expected value (or mean μ) of a random variable X is defined to be

$$\mu \doteq E(X) \doteq \sum_z z P(X = z)$$

properties of the expected value (X, Y are random variables).

- (a) If a, b are constants:
 $E(aX + b) = aE(X) + b$.
- (b) Linearity of expectations:
 $E(X + Y) = E(X) + E(Y)$
- (c) Expectation of a product:
 $E(XY) = E(X)E(Y)$ if X and Y are independent.
- (d) Linearity of expectations:
 $E(X_1 + \dots + X_n) = E(X_1) + \dots + E(X_n)$
- (e) If X_1, \dots, X_n are independent then
 $\text{var}(X_1 + \dots + X_n) = \text{var}(X_1) + \dots + \text{var}(X_n)$

4. The variance and standard deviation

$$\sigma^2 \doteq \text{var}(X) \doteq E((X - E(X))^2) = E(X^2) - E(X)^2$$

$$\sigma = \text{stddev}(X) = \sqrt{\text{var}(X)}$$

- Variance of a sum: $\text{var}(X + Y) = \text{var}(X) + \text{var}(Y)$ if X and Y are independent.
- For any constants a, b : $\text{var}(aX + b) = a^2 \text{var}(X)$ (adding a constant to a RV has no effect on its variance).
- The standard deviation of n independent identically distributed (IID) random variables:

$$\text{var}\left(\frac{1}{n} \sum_{i=1}^n X_i\right) = \frac{\text{var}(X_i)}{n}$$

and therefor

$$\text{stddev}\left(\frac{1}{n} \sum_{i=1}^n X_i\right) = \frac{\text{stddev}(X_i)}{\sqrt{n}}$$

- 5. Markov Inequality: if X is a non-negative random variable and $a > 0$ is some constant value, then $P(X \geq a) \leq \frac{E(X)}{a}$
- 6. Chebyshev's inequality: If μ is the mean and σ is the standard deviation of the rv X

$$P(|X - \mu| > k\sigma) \leq \frac{1}{k^2}$$

- 7. Suppose you toss a coin with bias p and code the outcome as either $X = 1$ (with prob. p) or $X = 0$ (with prob. $1-p$). We say that X has a Bernoulli distribution. $E(X) = p$ and $\text{var}(X) = p(1-p)$.

Independence of RVs, Covariance, correlation and anti-correlation

- Two random variables X, Y are independent if and only if for any constants a, b :

$$P(X \leq a \wedge Y \leq b) = P(X \leq a)P(Y \leq b)$$

If the random variables are integer valued then they are independent if and only if for any integers i, j

$$P(X = i \wedge Y = j) = P(X = i)P(Y = j)$$

- The Covariance of two random variables X, Y is

$$\begin{aligned} \text{Cov}(X, Y) &= E((X - E(X))(Y - E(Y))) \\ &= E(XY) - E(X)E(Y) \end{aligned}$$

If X, Y are independent then $E(XY) = E(X)E(Y)$ and therefor the covariance is zero. However the other direction is not true: zero covariance does not imply independence.

- If X, Y are *integer valued* random variables

$$\begin{aligned} \text{Cov}(X, Y) &= E(XY) - E(X)E(Y) = \\ &= \sum_{i,j} ijP(X = i \wedge Y = j) - [\sum_i iP(X = i)][\sum_j jP(Y = j)] \end{aligned}$$

- The correlation coefficient is a normalized version of the covariance:

$$\text{cor}(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{var}(X) \text{var}(Y)}}$$

The correlation coefficient varies between -1 and 1. $\text{cor}(X, Y) = 1$ if and only if there are constants $a > 1, b$ such that $P(aX + b = Y) = 1$. Similarly $\text{cor}(X, Y) = -1$ if and only if $P(-aX + b = Y) = 1$. If the correlation (and the covariance) are positive, we say that X and Y are correlated, if it is negative, we say that X and Y are anti-correlated and if it is zero we say that X and Y are uncorrelated (which does *not* imply that they are independent.)

Distributions over the real line

- Suppose X is a random variable defined over the whole real line from $-\infty$ to ∞ . The probability distribution for such a random variable can *always* be represented by the Cumulative Distribution Function (CDF) $F(a) \doteq P(x \leq a)$. The CDF is monotone non-decreasing.

- The probability of a segment can be computed as $P(a < X \leq b) = F(b) - F(a)$.
- The uniform distribution on the segment $[a, b]$ corresponds to the CDF that is 0 for $x \leq a$, $\frac{x-a}{b-a}$ for $a \leq x \leq b$ and 1 for $x > b$. We denote such a distribution by $U(a, b)$
- If the derivative of the CDF is defined everywhere then we call the derivative the Probability Density Function (PDF) $f(x)$. The integral of the PDF is the CDF: $F(a) = \int_{-\infty}^a f(x)dx$.
- If the density function is defined at a the probability of the event $\{a\}$ is zero. Conversely, if the probability of a single point is non-zero then the PDF is not defined at that point and we say that the distribution contains a *point mass* at a , denoted $PM(a)$
- If P_1, P_2 are two distributions, and p a constant between 0 and 1, then $pP_1 + (1 - p)P_2$ is also a well-defined distribution P_3 . We say that P_3 is a mixture of P_1 and P_2 .

Poisson Distribution

The poisson distribution describes the number of events in a unit time when the events are distributed uniformly in (continuous, non-discretized) time.

- The number of events per unit time is

$$P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}$$

- The expected number of events per unit time is $E(X) = \lambda$.
- The Poisson distribution is the limit of the binomial distribution where $n \rightarrow \infty$ and $p = \lambda/n$:

$$\lim_{n \rightarrow \infty} \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} = e^{-\lambda} \frac{\lambda^k}{k!}$$

- When events are distributed uniformly in time and the expected number of events per unit time is λ , the time between consecutive events $d = t_{i+1} - t_i$ is distributed according to the density $f(d) = \lambda \exp(-\lambda d)$ and the CDF is:

$$P(d \leq s) = 1 - e^{-\lambda s}$$

Statistical tests

- Statistical testing is a methodology for quantifying the significance of conclusions made based on observations. The *null hypothesis* corresponds to the skeptical opinion stating that the observation is explained well by the null distribution H_0 . The alternative hypothesis H_1 represents the new explanation that the experiment is intended to confirm. For example, when testing a new drug, the Null hypothesis states that the drug has no effect and the alternative hypothesis states that it does have a beneficial effect.
- The *statistical test* is a function that takes as input the observations and a *significance values* α and outputs either “Reject Null Hypothesis” or “Fail”.
- The *significance level* or α -value of a statistical test is (an upper bound on) the probability that the test rejects the null hypothesis when the data is generated according to the null hypothesis, α is *not a random variable*. It is set to a constant value before the observation data is given..
- The p -value of a test is a *random variable*, it is the minimal value of α that would result in rejecting the null hypothesis. In other words, the test rejects the null hypothesis if $p \leq \alpha$.
- A type I error is rejecting the null hypothesis when it is correct. The probability of a type I error is bounded by the chosen value of α . A type II error is failing to reject the null hypothesis when the alternative hypothesis is correct. Usually, we have no control over type II errors. Increasing α increases the probability of type I errors and decreases the probability of type II errors.

Tests based on Normality

- **Central Limit Theorem** If X_1, X_2, \dots are independent, identically distributed random variables with mean μ and variance σ^2 and

$$Y_n = \frac{\sum_{i=1}^n X_i - n\mu}{\sigma\sqrt{n}}$$

Then, as $n \rightarrow \infty$ the CDF of Y_n converges to the CDF of the standard normal distribution $\mathcal{N}(0, 1)$, which has the density distribution:

$$p(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

- **the Normal Approximation:** Given a random variable Y whose distribution is normal $\mathcal{N}(\mu, \sigma^2)$ we can calculate the probability $P(Y > a) = P(Y \geq a)$ (or $P(Y < a) = P(Y \leq a)$) for any value of a . The common way of calculating $P(Y > a)$ (or without a computer is transforming the threshold a to a z -score and then using a table for $\mathcal{N}(0, 1)$. The formula for calculating z is $z = \frac{a - \mu}{\sigma}$. And the $P(Y > a) = P(X > z)$ where the distribution of X is $\mathcal{N}(0, 1)$.
- A few standard definitions: $Q(z) = P(X > z)$, $\Phi(z) = P(X < z)$, $Q^{-1}(p)$ is the inverse function to Q . In other words: $Q(Q^{-1}(p)) = p$.
- few useful values:

$$Q(1) \approx 15\%, \quad Q(2) \approx 2.5\%,$$

$$Q(3) \approx 0.15\%, \quad Q(4) \approx 0.003\%$$
- **Hashing n elements into a table of size n .** The location of each element is distributed uniformly over the n bins, independent of the location of the other elements. The *occupancy* of a bin is the number of elements that it holds. The probability that “the maximal occupancy (over all bins) is larger than $\log(n)$ ” goes to zero as n goes to infinity.
- **The power of two:** One very effective method for reducing the maximal occupancy is to use two hash functions instead of one. To add a new element to the table both locations are checked and the new element is added to the location with smaller occupancy. The probability that “the maximal occupancy is larger than $\log \log(n)$ ” goes to zero as n goes to infinity.
- **Min-Hash:** Here we are concerned with comparing document. We view each document as a set of words. The Jaccard similarity between two documents A, B , is defined as

$$S(A, B) = \frac{|A \cap B|}{|A \cup B|}$$

Randomized Algorithms

- **A Las Vegas Algorithm:** always produces the correct output but the time it takes to produce this output can vary. Let μ be the expected running time.
 - **A Monte Carlo Algorithm :** always completes within the same amount of time. However, the output is incorrect with probability $0 < q = 1 - p < 1$.
 - **Transforming Las Vegas to Monte Carlo:** We run a timer for time T in parallel with the algorithm. If the algorithm completes before T , we output the output of the algorithm. If the timer’s timer reaches time T , we abort the algorithm and output an incorrect output. Using Markov’s Inequality we get that the probability of failure is $q \leq \mu/T$
 - **Transforming Monte Carlo to Las Vegas:** We assume that we have an efficient way to check whether the output of the algorithm is correct. We repeatedly tune the Monte-Carlo algorithm followed by the checker algorithm until we find a correct output. Suppose the time for one iteration is T , and the probability of success in each iteration is p , then the expected running time until completion is T/p .
 - **Bloom filters:** A method for determining whether a given item has been observed in the
- Where A denotes the set words in the document A (without repetitions) and $|A|$ is the number of elements in that set.
- The min-hash method associates with each document a short “signature” so that the similarity between any two documents can be approximated efficiently from their signatures alone.
- The min-hash signature consists of k integers from a very large range (much larger than the set of possible words). Each of the k numbers is computed by using an independent hash function h_i . Each word w is mapped to its hash value $h_i(w)$. A document is then mapped to the minimum of the hash values of its words. This gives the i th min-hash value for the document.
- The *probability* that the i th min-hash values of two documents A, B match is equal to the similarity $S(A, B)$. Thus the random variables X_i which are 1 for match, 0 for no-match, are IID binary RV with expected value $S(A, B)$. Using this fact we can compute the minimal value of k required to reach a specified level of accuracy in estimating $S(A, B)$.

past. The method consists of a binary vector T of length m which is initialized to all zeros, and k hash functions that map items into the range $1, \dots, m$. An item x is mapped to the k values $h_1(x), \dots, h_k(x)$. The corresponding entries in T are checked, if all of these entries are 1 the item is declared to have been observed in the past, then all of the entries are set to 1.

- Bloom filters do not make false negative mistakes
- declaring an item to be new when it is not.
- Suppose we have already entered n elements into the filter, The probability of a false positive (declaring the $n + 1$ th item as not new when in fact it is new) is $p \approx (1 - e^{-kn/m})^k$.
- The optimal choice of k for given values of m, n is $k = \frac{m}{n} \ln 2$
- If we want to have probability at most p of a false positive, we are given the value of n and use the optimal choice for k then the minimal table size we need is: $m \geq (n \ln(1/p))/(\ln 2)^2$