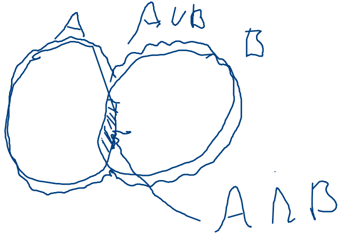
Basic Combinatorics

Sets

$4 \in A$

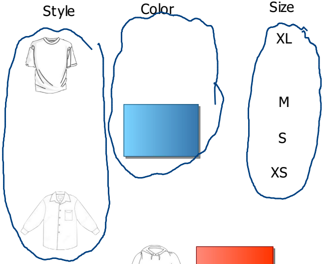
$5 \notin A$

- Explicit : A={1,4,2}
- Implicit: A={i|i is an odd number}
- Intersection: $x \in A \cap B$ *if* $x \in A$ *and* $x \in B$
- Union: $x \in A \cup B$ *if* $x \in A$ *or* $x \in B$



Products of sets

- Taking all possible combinations.



|style| = 3
|color| = 2
|size| = 5

Product Set={ (,  , L),(, ,),...,}

Size of Product Set= $3 \times 2 \times 5$

Raising a set to a power

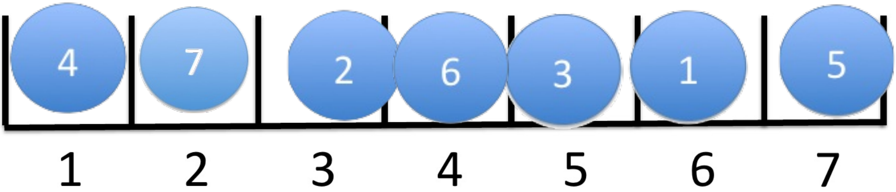
- The set of all binary sequences of length 7:
  - 0000000, 0000001, 0000010,…
  - 1111101, 1111110, 1111111
- Using product notation:
  - $\{0,1\} \times \{0,1\} \times \{0,1\} \times \{0,1\} \times \{0,1\} \times \{0,1\} \times \{0,1\} = \{0,1\}^7$
- Size:
  - $2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^7 = 128$

The Factorial Function

How many ways are there to order n different objects?

How many ways are there to order 7 different objects?

$$7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 7!$$



The Factorial function

- The number of possible ways to put n different objects into n different slots is

$$n * (n-1) * (n-2) * \cdots * 2 * 1 \doteq n!$$

- We say “n factorial”

Permutations

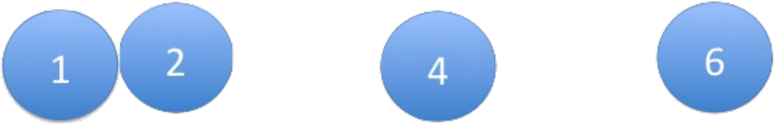
How many ways are there

To pick k out of n elements

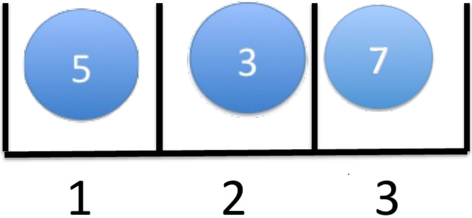
When the order matters

How many ways are there to pick 3 out of 7 elements

When the order matters



$$7 \times 6 \times 5 = P(7,3)$$



The Permutation Function

- ~~The number of possible ways to put k<n different objects into n different slots is~~
- For n=7, k=3:

$$7*6*5=\frac{7*6*5*4*3*2*1}{4*3*2*1}=\frac{7!}{(7-3)!}$$

- In general:

$$P(n,k)\doteq\frac{n!}{(n-k)!}$$

combinations

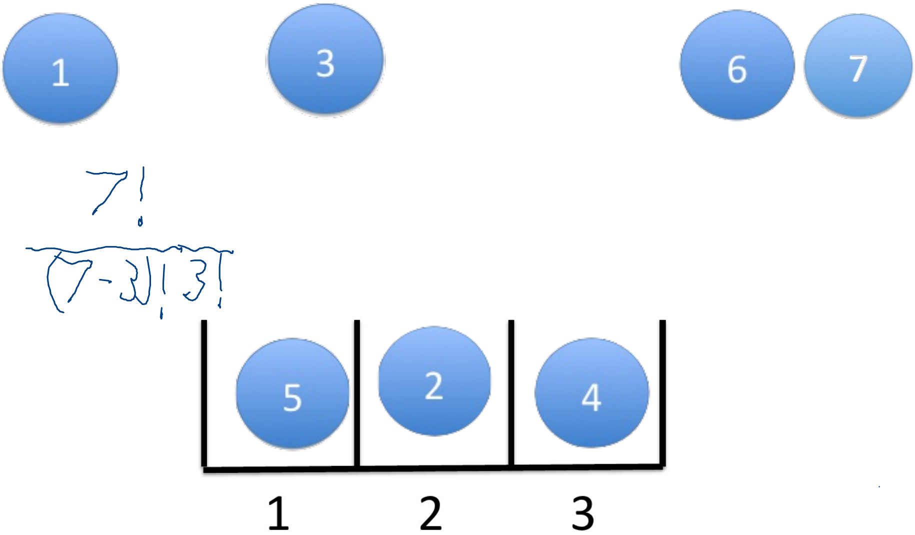
How many ways are there

To pick k out of n elements

<u>When the order does not matter</u>

How many ways are there to pick 3 out of 7 elements

When the order does not matter

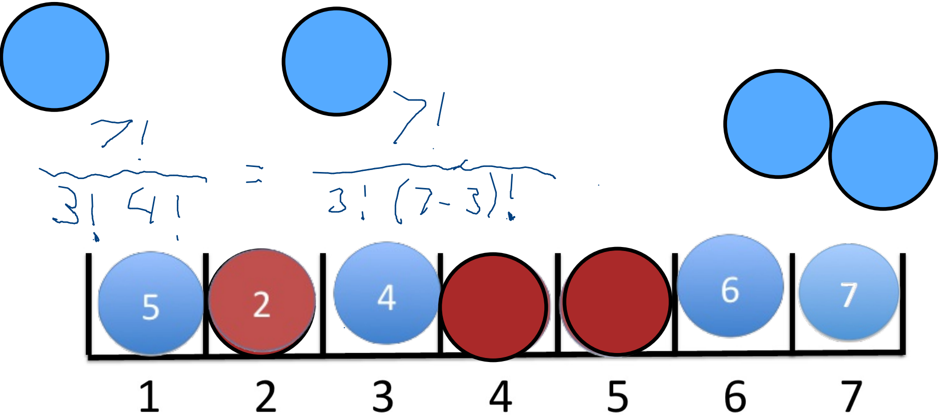


$$\frac{7!}{(7-3)!\,3!}$$

The Combinatorial function

- The number of possible ways to place k identical objects into n different slots is
- $C(n,k) \doteq \binom{n}{k} \doteq \frac{P(n,k)}{k!} = \frac{n!}{k!(n-k)!}$
- We Say “n choose k”

How many ways are there to order 3 red and 4 blue balls?



$$\frac{7!}{3!\,4!} = \frac{7!}{3!\,(7-3)!}$$

this problem is identical to the previous one

How many different 3 digit numbers from the digits 1-9 ?

2 2 1   1 2 3

$9 \times 9 \times 9 = \boxed{9^3}$

10^3

How many different 4 digit numbers from the digits 1-9 where all of the digits are different?

$$P(9,4) = \frac{9!}{(9-4)!} = 9 \times 8 \times 7 \times 6$$

How many different 4 digit numbers from the digits 1-9 where all of the digits are different and the digits are in increasing order?

2 5 6 8 ✓

5 2 6 8 ✗

$$C(9,4) = \binom{9}{4} = \frac{9!}{4!\,5!}$$

Suppose you choose 4 different digits from the set 1-9 and you place them in increasing order. What is the probability that the first digit is 3 ?

$$P(A) = \frac{|A|}{|\Omega|}$$

What is the size of the sample space?

$$|\Omega| = C(9,4)$$

What is the size of the event?

$$\boxed{3}\,\overbrace{\boxed{\ }\,\boxed{\ }\,\boxed{\ }}^{\{4-9\}}$$

$$C(6,3)$$

$$|A| = C(6,3)$$

$$P(A) = \frac{C(6,3)}{C(9,4)}$$

How many different words can be created by rearranging (all) the letters in the word MISSISSIPPI ?

P S I S I S M P I S I

$$\frac{11!}{4!\,4!\,2!\,1!} \qquad \binom{11}{4,4,2,1}$$

Suppose the letters of the word MISSISSIPPI are put in a random order. What is the probability that the result is again MISSISSIPPI?

M

I I I I

S S S S

P P

What is the size of the sample space?

$11!$

What is the size of the event?

$4!\,4!\,2!$

$$P(A) = \frac{4!\,4!\,2!}{11!}$$

A fair coin is flipped 11 times, what is the probability of 4 heads and 7 tails ?

What is the size of the sample space?

$$|\{0,1\}^{11}| = 2^{11}$$

What is the size of the event?

$$|A| = C(11,4)$$

$$P(A) = \frac{C(11,4)}{2^{11}}$$