A Gentle introduction to probability

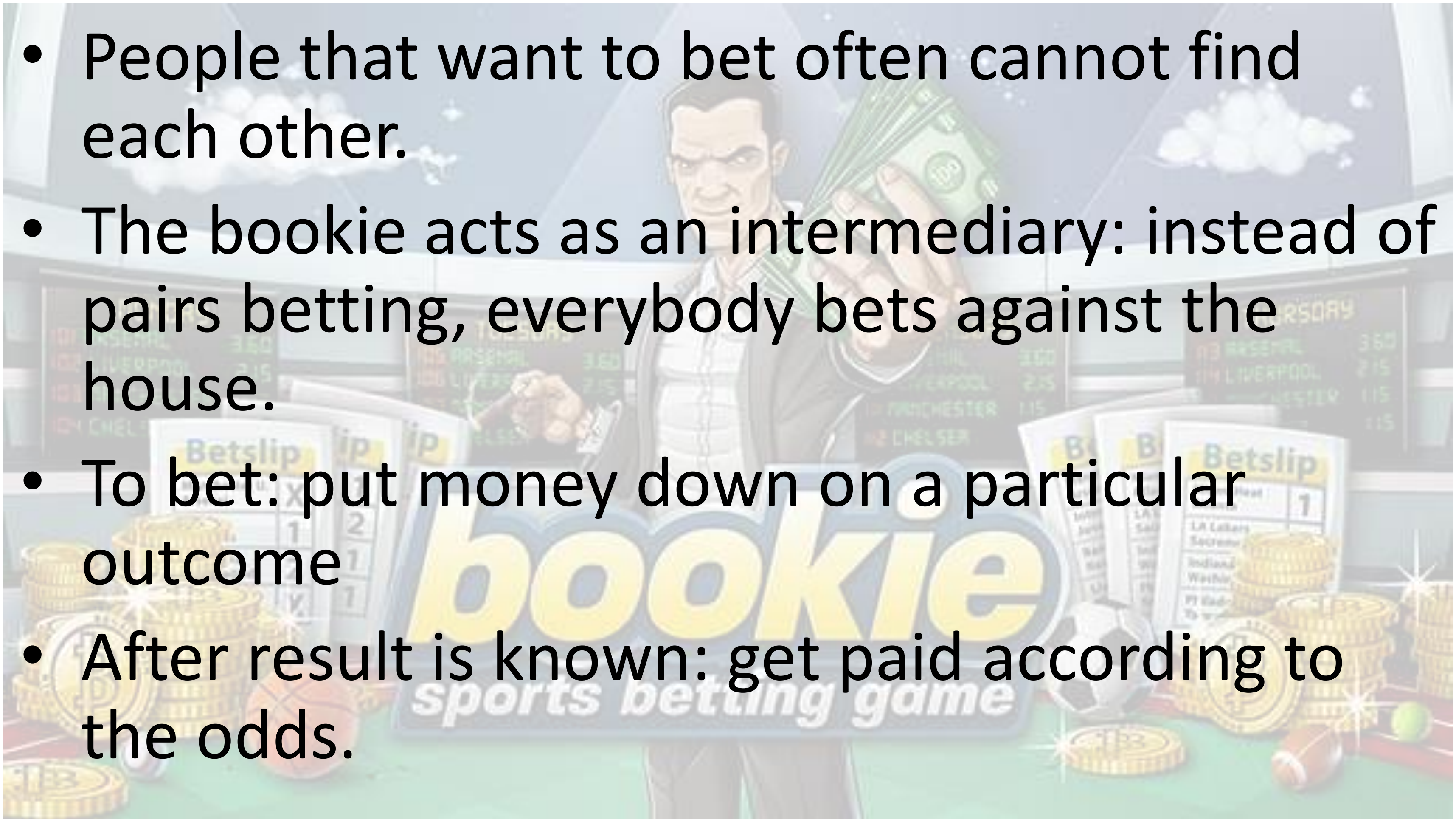
The goal of this class

- Probability is a branch of math.
- Solving complex problems requires mathematical tools and mathematical definitions.
- It might not be obvious how the math relates to the intuition.
  - Some times the intuition is wrong!
- Today I will introduce some central concepts in an intuitive way.
- In later classes we will give more formal definitions.
- The concepts are:
  - Outcomes
  - Expected value / fair price.
  - Events
  - Event trees.
  - Probabilities / probability distribution.
  - Conditional Probability.

Bets between two people

- John: I bet the chargers will win their next game.
- Kathy: I bet they will lose. Do you want to put money on it?
- John, sure. In fact, <u>**I am so sure they will win**</u> that if they lose I'll pay you 90$, if they win, you pay me just 10$.
- Kathy: You are on!
- The odds are: 9 to 1
- Equivalent to john thinking that the probability the chargers will win is at least 90%
- Why? Because 0.9*10-0.1*90=0
- If John – kathy would have many bets, the long terms average will have john at least break even.

Bets against the house.



- People that want to bet often cannot find each other.
- The bookie acts as an intermediary: instead of pairs betting, everybody bets against the house.
- To bet: put money down on a particular outcome
- After result is known: get paid according to the odds.

Fair odds: in words

- In the betting games we will talk about, the probability of each <u>outcome</u> is known.
- The bet is **<u>fair if:</u>**
  - The long term average of gains/losses is zero.
  - The **expected value** is zero.

Fair odds: in symbols

probabilities of outputs: $p_1, p_2, \ldots, p_n$

money gained for each outcome: $g_1, g_2, \ldots, g_n$

price of ticket: T

At each iteration, player pays T and gains one of $g_1, g_2, \ldots, g_n$

The expected gain of the player is $\sum_{i=1}^{n} p_i g_i - T$

The game is fair if $\sum_{i=1}^{n} p_i g_i - T = 0$

Equivalently: the price is fair if $T = \sum_{i=1}^{n} p_i g_i$

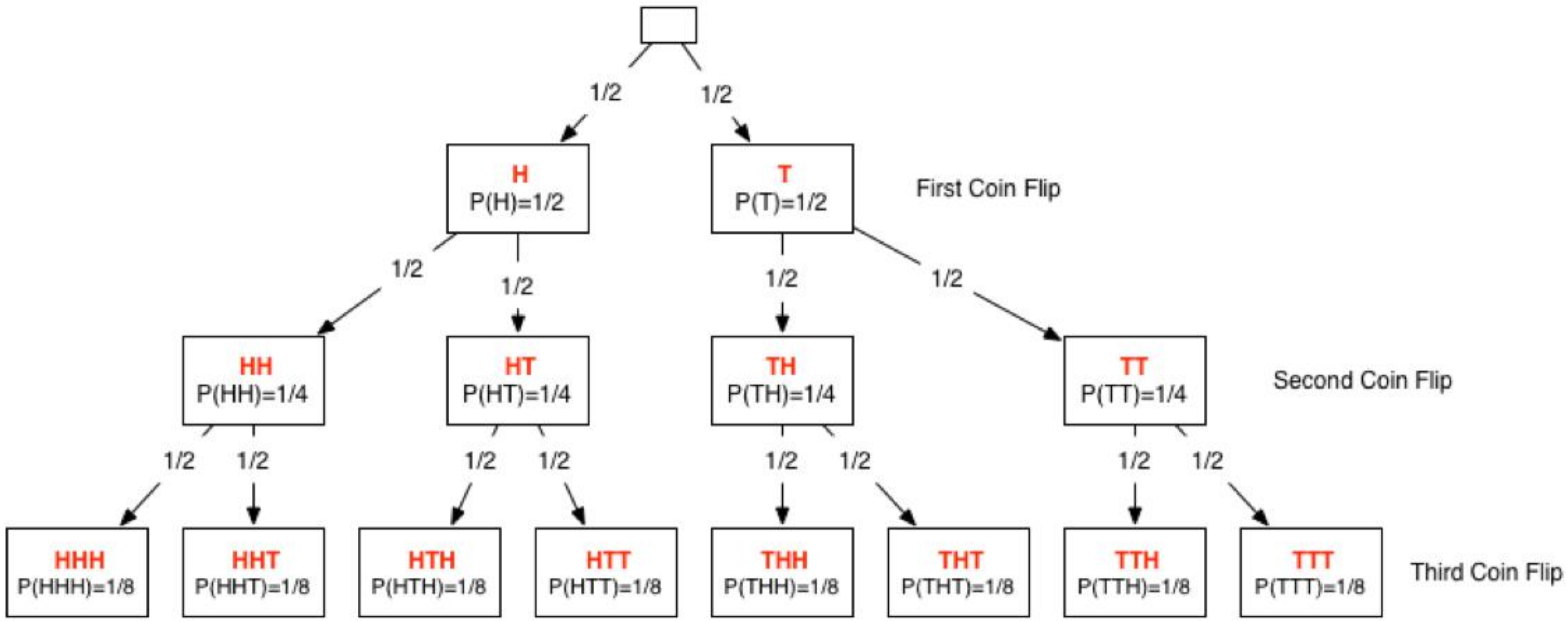
First Example

- House flips an unbiased coin.
  - “heads” : house pays player $1
  - “tails”: house pays player $2
- Outcomes: “heads”,”tails”
- What is the fair ticket price?
  - $1.5
  - Why? Because 0.5*1 + 0.5*2 = 1.5

Second example

- The house flips the coin three times in a row.
- Eight outcomes:
  HHH,HHT,HTH,HTT, THH,THT,TTH,TTT
- Each outcome has probability 1/8
- Each outcome consists of three coin flips.
- It sometimes helps our understanding to consider each coin flip separately, one by one.

The 3 coin flips event tree



What is an “Event”

- An event is **a set** of outcomes.
  - The event “the first coin flip is H”. Corresponds to the set: {HHH,HHT,HTH,HTT}
  - The event “the first coin flip is T”. Corresponds to the set: {THH,THT,TTH,TTT}
  - The event “the first 2 coin flips are HH”. Corresponds to the set: {HHH,HHT}
  - The event “the first 2 coin flips are HT”. Corresponds to the set: {HTH,HTT}
  - …
  - The event “the three coin flips are HHH” corresponds to the set: {HHH}
- The probability of an event is the number of outcomes in the set, divided by 8.
- The event that contains all possible outcomes is called the “outcome space” and is denoted by Ω
- The probability of the whole outcome space is always 1:

$$\mathrm{Prob}(\Omega)=8/8=1$$

What is a set?

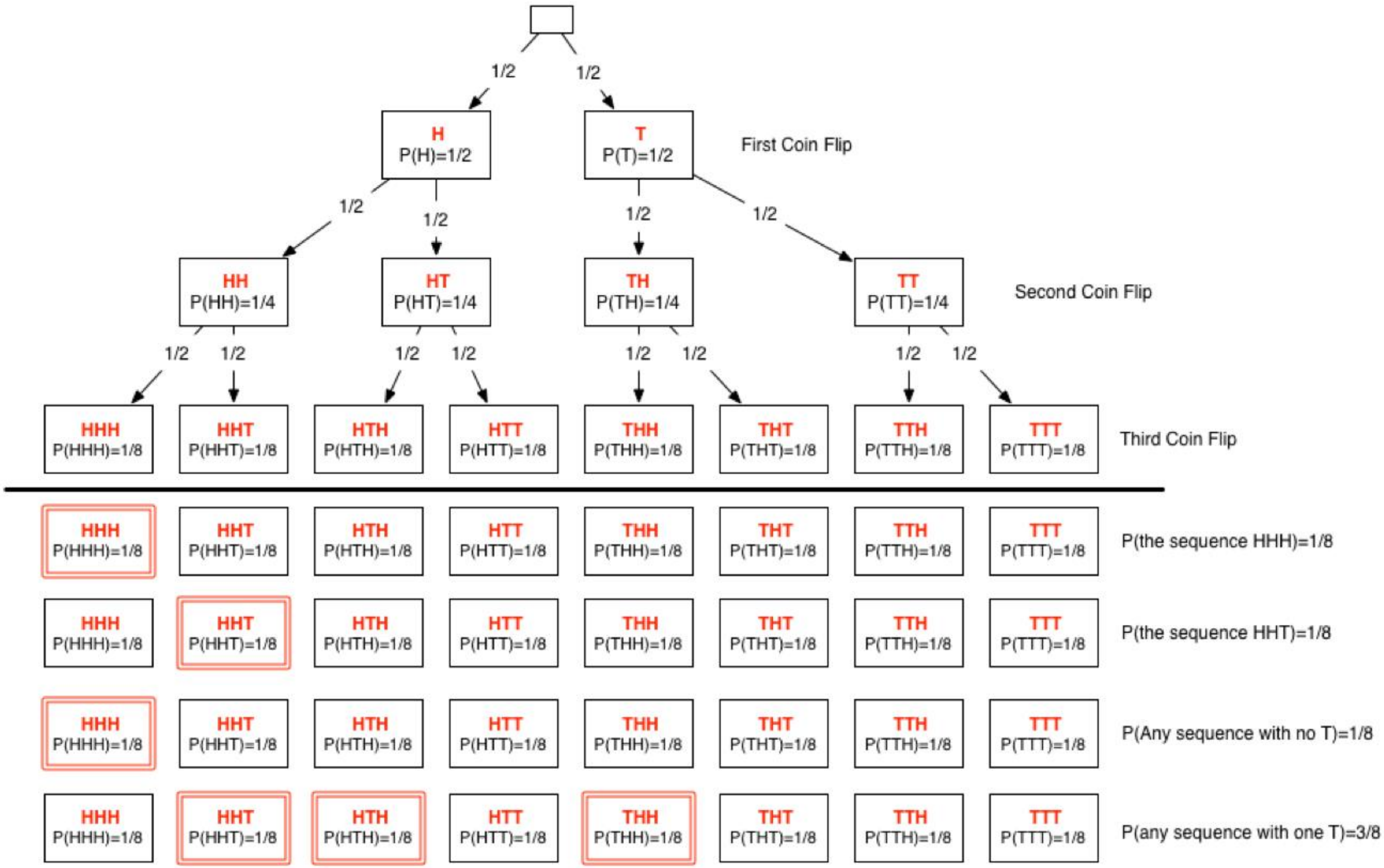
- A set is a collection of items: $A = \{b, d, e\}$
- A set cannot have the same item twice.
- Some Notation:
  - Item *b* is a member of the set *A*: $b \in A$
  - A is a subset of B: $A \subset B$
  - All events are subsets of Ω
  - The empty set $\emptyset = \{\}$ contains no element
- Explicit set definition: $A = \{b, d, e\}$
- Implicit set definition: $B = \{i | i \text{ is prime}\}$
- *A* is finite, *B* is infinite

Calculating probabilities of events

- The probability of an event is the number of outcomes in the set, divided by 8.
- The probability of an event is the number of outcomes in the event divided by the total number of outcomes (the number of outcomes in Ω which is 8 in out case.
- Prob({HHH})=Prob({HHT})=1/8

Slightly more complex events

- P({The sequence contains no T})=P({HHH})=1/8
- P({The sequence contains one T})=
  P({HHT,HTH,THH}) = 3/8
- While HHH,HHT,HTH,…. All have the same probability, the event defined by “one T” has three times the probability of “no T”.
- The main task is to count the number of outcomes in the event. This is done using “combinatorics”


1/2
1/2
H
P(H)=1/2
T
P(T)=1/2
First Coin Flip
1/2
1/2
1/2
1/2
HH
P(HH)=1/4
HT
P(HT)=1/4
TH
P(TH)=1/4
TT
P(TT)=1/4
Second Coin Flip
1/2
1/2
1/2
1/2
1/2
1/2
1/2
1/2
HHH
P(HHH)=1/8
HHT
P(HHT)=1/8
HTH
P(HTH)=1/8
HTT
P(HTT)=1/8
THH
P(THH)=1/8
THT
P(THT)=1/8
TTH
P(TTH)=1/8
TTT
P(TTT)=1/8
Third Coin Flip
P(the sequence HHH)=1/8
P(the sequence HHT)=1/8
P(Any sequence with no T)=1/8
P(any sequence with one T)=3/8

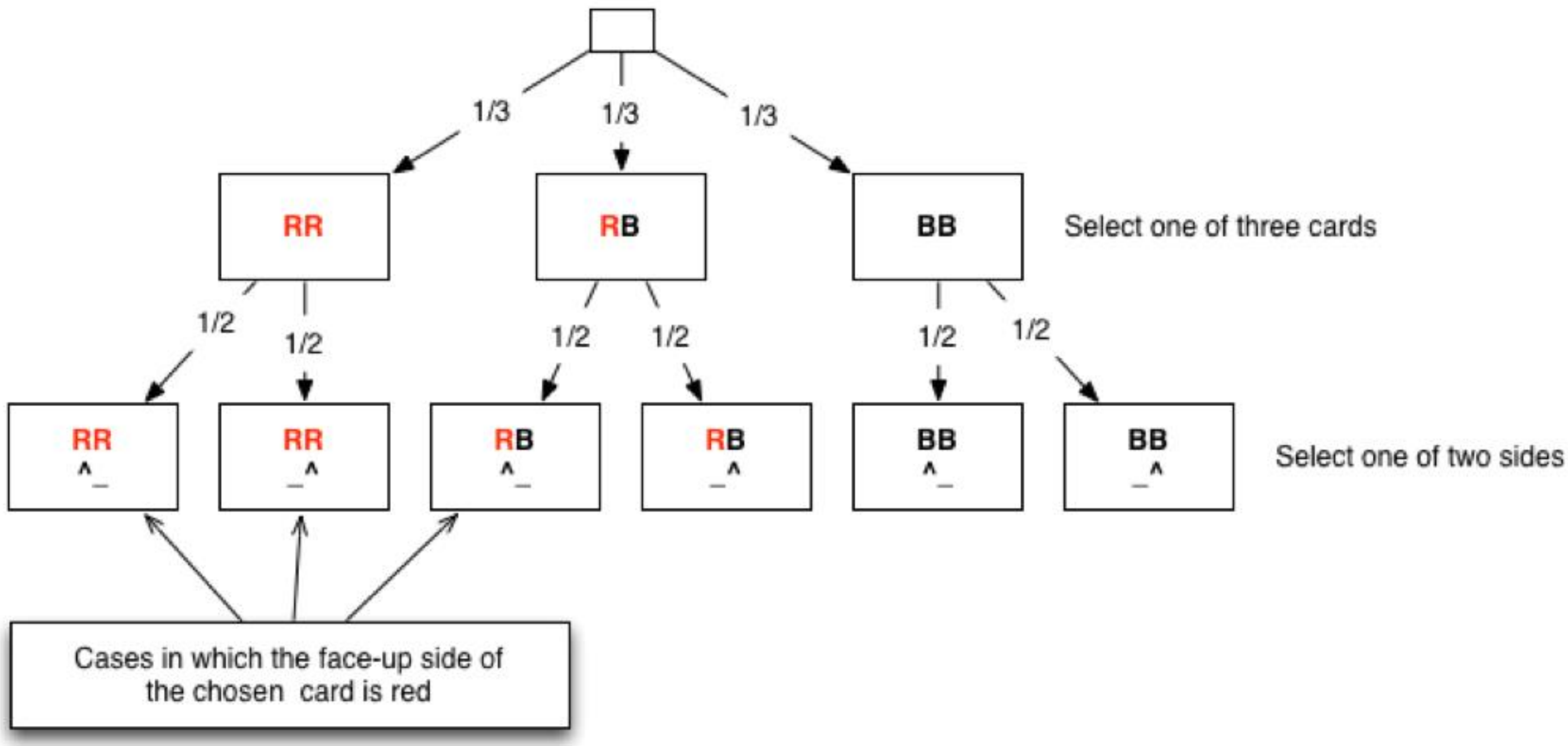
Ticket prices

- Suppose that the house pays you \$1 if the specified even happens, zero otherwise. What is the fair price?
- $T(E) = 1 * P(E) = P(E)$
- $T(\{HHH\})=T(\{HHT\})=T(\{\text{no } T\})=1/8=12.5$ ¢
- $T(\{\text{one } T\})=3/8=37.5$¢

The three card problem

- There are three cards in a hat. Each side of each card is colored red **R** or black **B**.
- The colors of the cards are **RR**, **RB**, **BB**
- I pick one of the cards at random and put it on a random side.
- I say: if the color of the other side is the same, you give me one dollar, if it is different, I give you one dollar.
- Is this fair?

Event tree for three cards



Conditional probability

- The probability that the seen color is R (B) is ½.
- The probability that the other side is R (B) given that the seen color is R(B) is 2/3.

Review

- Fair bets: bets whose expected value is zero.
- Expected value: $\sum_{i=0}^{n} p_i g\_i$
- Outcome: the output of a single experiment
- Ω : The outcome space: the set of all possible outcomes. $P(\Omega) = 1$
- Events: subsets of Ω
- In finite domains with uniform distribution for any event *A:* $P(A) = \frac{|A|}{|\Omega|}$

For wed.

- Read Chapter 2.
- Finish Week1 homework on webwork.

- Wed: Basic combinatorics.