(1 pt) **setAssignment10/probs70.pg**

Suppose you throw m =13 balls into n =9 bins.

Let X_i be the number of balls that fall into bin i.

Let $T_{i,j}$ Be a random variable that is 1 if the j'th ball falls in the i'th bin.

Clearly $T_{i,j}, T_{i,k}$ are independent for $1 \leq j < k \leq m$, and $X_i = \sum_{j=1}^{m} T_{i,j}$.

Using these facts, answer the following questions:

1. What is $\mathbb{E}(T_{i,j})$?
2. What is $var(T_{i,j})$?

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$T_{i,j}$ is 1 with probability 1/9 and 0 with probability 8/9

$$E(T_{i,j}) = 0P(T_{i,j}=0) + 1P(T_{i,j}=1)$$
$$= P(T_{i,j}=1) = 1/9$$

$$\mathrm{var}(T_{i,j}) =$$
$$= (0-1/9)^2 P(T_{i,j}=0) + (1-1/9)^2 P(T_{i,j}=1)$$
$$= (1/9)^2(8/9) + (8/9)^2(1/9)$$
$$= (1/9)(8/9)(1/9+8/9) = (1/9)(8/9)$$

In General, for a binary random variable:

$$P(X=1) = p, P(X=0) = 1-p$$
$$E(X) = p, \quad \mathrm{var}(X) = p(1-p)$$

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3. What is $Pr(X_i = 0)$ (i.e. there are no balls in bin i) ?

4. What is $Pr(X_i = 1)$ (i.e. there is exactly one ball in bin i) ?

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$$\Pr(X_i = 0) = \Pr\left(\sum_{j=1}^{m} T_{i,j} = 0\right)$$

Holds if and only if $T_{i,j} = 0$ for all j

$$\Pr\left(T_{i,1} = 0, T_{i,2} = 0, \ldots, T_{i,m} = 0\right) = \left(\frac{n-1}{n}\right)^m = \left(\frac{8}{9}\right)^{13}$$

$$\Pr(X_i = 1) = \Pr\left(\sum_{j=1}^{m} T_{i,j} = 1\right)$$

Holds if and only if

$T_{i,j} = 1$ for a single j and 0 for the rest

$$\Pr\left(\exists 1 \le k \le m \text{ s.t. } T_{i,k} = 1 \text{ and } T_{i,j} = 0 \text{ for } j \ne k\right)$$

$$= \binom{m}{1}\left(\frac{n-1}{n}\right)^{m-1}\left(\frac{1}{n}\right) = \frac{m(n-1)^{m-1}}{n^m} = \frac{13 \times 8^{12}}{9^{13}}$$

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5. What is $\mathbb{E}(X_i)$?

   Hint: Recall linearity of expectations: $E(\sum_{i=1}^{n} T_{i,j}) = \sum_{i=1}^{n} E(T_{i,j})$

6. What is $var(X_i)$?

   Hint: The variance of the sum of *independent* random variables is equal to the sum of the variances.

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$$E(X_i) = E\left(\sum_{j=1}^{m} T_{i,j}\right) = \sum_{j=1}^{m} E\left(T_{i,j}\right) = mE\left(T_{i,1}\right) = \frac{m}{n} = \frac{13}{9}$$

$$\mathrm{var}(X_i) = \mathrm{var}\left(\sum_{j=1}^{m} T_{i,j}\right) = \sum_{j=1}^{m} \mathrm{var}\left(T_{i,j}\right) = m\,\mathrm{var}\left(T_{i,1}\right) = \frac{m(n-1)}{n^2} = \frac{13*8}{9^2}$$

recall that: $E(T_{i,j}) = p = 1/n = 1/9,$

$$\mathrm{var}(T_{i,j}) = p(1-p) = \frac{n-1}{n^2} = \frac{8}{81}$$

(1 pt) **setAssignment9/probs77.pg**

Expected Number of Right Positions in a Random Permutation

Pick a random permutation of $(1, 2, \ldots, n)$. Let X_i be the number that ends up in the ith position. For instance, if the permutation is $(3, 2, 4, 1)$ then $X_1 = 3$, $X_2 = 2$, $X_3 = 4$, and $X_4 = 1$.

(a) What is the expected number of positions at which $X_i \neq i$ (i.e. the number of *wrong* positions)?

Let random variable D represents the number of wrong positions, we aim to find $\mathbb{E}(D)$.

If we devise a new r.v. $Y_i = \{0, 1\}$ to represent whether or not $X_i \neq i$, then it is easy to see that, $D = Y_1 + Y_2 + \cdots + Y_n$. The linearity of expectation gives: $\mathbb{E}(D) = \mathbb{E}(Y_1 + Y_2 + \cdots + Y_n) = \mathbb{E}(Y_1) + \mathbb{E}(Y_2) + \cdots + \mathbb{E}(Y_n)$. Notice that all positions are equivalent, so all Y_i have the same distribution.

We can easily compute $\mathbb{E}(Y_i) = 0 \cdot \mathbf{Pr}(X_i = i) + 1 \cdot \mathbf{Pr}(X_i \neq i) =$ ______ .

It follows that, $\mathbb{E}(D) =$ ______ .

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It follows that, $\mathbb{E}(D) =$ ________ .

$$X_i = i \leftrightarrow Y_i = 1, \quad X_i \neq i \leftrightarrow Y_i = 0$$

$$P(Y_i = 1) = 1/n, \quad P(Y_i = 0) = (n-1)/n$$

$$E(Y_i) = 0 \times P(Y_i = 0) + 1 \times P(Y_i = 0) = 1/n$$

$$E\left(\sum_{i=1}^{n} Y_i\right) = \sum_{i=1}^{n} E(Y_i) = n \times (1/n) = 1$$

(b) What is the expected number of positions at which $X_i = i + 1$?

Similar to the previous question, we let D represent the number of positions at which $X_i = i + 1$, and let $Y_i = 0, 1$ represent whether or not $X_i = i + 1$ at a specific position. Again we use the linearity of expectations. Notice that this time, not all Y_i are the same, because Y_n is always 0, while the other Y_i's have some chances of taking both values.

In other words $\mathbb{E}(Y_n) =$ _____

and for all $1 \leq i < n$; $\mathbb{E}(Y_i) =$ _____

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$$E(Y_n) = 0; \quad \forall i < n, \; E(Y_i) = 1/n$$

$$E(D) = E\left(\sum_{i=1}^{n} Y_i\right) = \sum_{i=1}^{n} E(Y_i) = \frac{n-1}{n}$$

(c) What is the expected number of positions at which $X_i \geq i$?

In this part, the different Y_i's have different distributions, but you should be able to compute each of $\mathbb{E}(Y_i)$ easily.

$\mathbb{E}(Y_i) =$ ____ .

$\mathbb{E}(D) =$ ____ .

Hint: use the equality: $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$

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$\mathbb{E}(D) =$ ______ .

Hint: use the equality: $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$

$$Y_i = 1 \text{ if } X_i \geq i, \quad \text{otherwise } Y_i = 0$$

$$E(Y_i) = P(Y_i = 1) = \frac{n-i+1}{n}$$

$$E(D) = E\left(\sum_{i=1}^{n} Y_i\right) = \sum_{i=1}^{n} E(Y_i)$$

$$= \frac{n+(n-1)+\cdots+1}{n} = \frac{n(n+1)/2}{n} = \frac{n+1}{2}$$

(d) What is the expected number of positions at which $X_i > \mathbf{max}(X_1, \ldots, X_{i-1})$?

In this part, $\mathbb{E}(Y_i)$ is not so obvious. We know that $\mathbb{E}(Y_i) = \mathbf{Pr}(X_i > \mathbf{max}(X_1, \ldots, X_{i-1}))$, but how do we compute this probability ?

We are going to use the combinatorial method. Fix the value of i. Let A_i be the set of all permutations which obey the condition $X_i > \mathbf{max}(X_1, \ldots, X_{i-1})$. We will calculate $|A_i|$

Let us design a method for constructing the elements of A_i. We first choose a **set** S_i of i different numbers from 1 to n to put in the bins X_1 through X_i. The largest of these i numbers will be X_i, and the remaining $n - i$ numbers can be assigned arbitrarily to $X_{i+1}, \cdots, X_n$.

1. How large is the sample space, i.e. how many possibilities are there for choosing the **set** S_i when there is no restriction on the values for X_i other than that they are a subset of $\{1, \ldots, n\}$?

   (Note that S_i is a set, i.e. order does not matter).

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Si is a subset of size i of the elements {1,2,...,n}

Let Bi be the set of such sets.

$$|B_i| = \binom{n}{i}$$

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The number of ways to order (i-1) elements is (i-1)!

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3. $(n-i)!$

4. $$|A_i| = \binom{n}{i}(i-1)!(n-i)!$$

$$= \frac{n!(i-1)!(n-i)!}{i!(n-i)!} = \frac{n!}{i}$$

4. taking the product of the numbers we calculated in 1,2,3 we find that $|A_i| =$ ______

5. Finally we know the size of the entire outcome space is $n!$, dividing by $n!$ we get that

$\mathbb{E}(Y_i) = P(A_i) = \dfrac{|A_i|}{n!} =$ ______ which simplifies to $\mathbb{E}(Y_i) =$ ______

Now you should be able to compute $\mathbb{E}(D) = \sum_{i=1}^{n} \mathbb{E}(Y_i)$

For large n, $\mathbb{E}(D) \approx$ ______ .

Hint: use the approximation $\sum_{i=1}^{n} 1/i \approx \ln n$

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$$E(Y_i = 1) = P(A_i) = \frac{|A_i|}{n!} = \frac{1}{i}$$

$$\sum_{i=1}^{n} E(Y_i) = \sum_{i=1}^{n} \frac{1}{i} \approx \ln n$$