Theorem: Diagonality of Operators in Nondegenerate Eigenbasis

Theorem

Statement: Suppose A and B are compatible observables (i.e., they commute), and the eigenvalues of A are nondegenerate. Then the matrix elements $\langle a'|B|a\rangle$ are all diagonal:

$$\langle a'|B|a\rangle = 0 \quad \text{for } a' \neq a$$

Proof

Step 1: Eigenstates of A

Let $A|a\rangle = a|a\rangle$, with each eigenvalue a nondegenerate. This implies that each eigenvalue corresponds to a unique (up to a phase) eigenstate $|a\rangle$.

Step 2: Commutation Implies Shared Eigenbasis

Since A and B are compatible, they commute:

$$[A, B] = AB - BA = 0$$

Applying both sides to $|a\rangle$, we get:

$$AB|a\rangle = BA|a\rangle = aB|a\rangle \Rightarrow A(B|a\rangle) = a(B|a\rangle)$$

This shows that $B|a\rangle$ is also an eigenvector of A with eigenvalue a.

Step 3: Use Nondegeneracy of A

Since the eigenvalue a is nondegenerate, the corresponding eigenspace is one-dimensional. Therefore, $B|a\rangle$ must be proportional to $|a\rangle$:

$$B|a\rangle = b_a|a\rangle$$

This means $|a\rangle$ is also an eigenstate of B.

Step 4: Compute Matrix Elements

Now consider the matrix element $\langle a'|B|a\rangle$ for $a' \neq a$. Using the result from above:

$$B|a\rangle = b_a|a\rangle \Rightarrow \langle a'|B|a\rangle = b_a\langle a'|a\rangle = b_a \cdot 0 = 0$$

since $\langle a'|a\rangle = 0$ when $a' \neq a$ (orthonormality of eigenstates of Hermitian operators with different eigenvalues).

Conclusion

$$\langle a'|B|a\rangle = 0 \quad \text{for } a' \neq a$$

Thus, the operator B is diagonal in the basis of eigenstates of A.