Matrix Representation of Operators and the Role of Completeness

1. Operator Representation via Completeness

Let A be a linear operator acting on a Hilbert space, and let $\{|e_n\rangle\}$ be a complete orthonormal basis of that space. The completeness relation is:

$$\sum_n |e_n\rangle\langle e_n| = \mathbb{I} \tag{1}$$

Using this, any operator A can be written as:

$$\begin{aligned}
A &= \mathbb{I}A\mathbb{I} &(2)\\
&= \left(\sum_m |e_m\rangle\langle e_m|\right) A \left(\sum_n |e_n\rangle\langle e_n|\right) &(3)\\
&= \sum_{m,n} |e_m\rangle\langle e_m|A|e_n\rangle\langle e_n| &(4)\\
&= \sum_{m,n} A_{mn}|e_m\rangle\langle e_n| &(5)
\end{aligned}$$

where the matrix elements are defined as:

$$A_{mn} = \langle e_m|A|e_n\rangle \tag{6}$$

This shows that A can be represented as a matrix once a basis is chosen.

2. Benefits of Using Eigenstates as Basis

Let A be a Hermitian operator with eigenstates $\{|a_n\rangle\}$ and eigenvalues $\{a_n\}$, satisfying:

$$A|a_n\rangle = a_n|a_n\rangle \tag{7}$$

Then the matrix elements in this basis are:

$$A_{mn} = \langle a_m|A|a_n\rangle = a_n\delta_{mn} \tag{8}$$

So the matrix representation becomes:

$$A = \sum_n a_n |a_n\rangle\langle a_n| \tag{9}$$

which is a diagonal matrix:

$$A = \begin{pmatrix} a_1 & 0 & 0 & \cdots \\ 0 & a_2 & 0 & \cdots \\ 0 & 0 & a_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Advantages of Diagonal Representation

- **Simplifies Calculations:** Functions of A, like e^{-iAt}, become easy to compute.
- **Measurement Interpretation:** The probability of measuring eigenvalue a_n in state $|\psi\rangle$ is:
$$P(a_n) = |\langle a_n|\psi\rangle|^2$$
- **Projection Operators:** Each $|a_n\rangle\langle a_n|$ projects onto the n-th eigenstate:
$$\Lambda_n = |a_n\rangle\langle a_n|, \quad \Lambda_n^2 = \Lambda_n$$
- **Spectral Decomposition:** The operator can be reconstructed from its eigenvalues and eigenstates:
$$A = \sum_n a_n |a_n\rangle\langle a_n|$$