Derivation of the Schwarz Inequality (Cauchy-Schwarz Inequality)

Let $|\psi\rangle$ and $|\phi\rangle$ be two vectors in a Hilbert space. The Schwarz inequality states:

$$|\langle\psi|\phi\rangle|^2 \leq \langle\psi|\psi\rangle \cdot \langle\phi|\phi\rangle$$

Step 1: Define a Vector

Define a new vector $|\chi\rangle$ as:

$$|\chi\rangle = |\phi\rangle - \lambda|\psi\rangle$$

for any complex number λ. Then:

$$\langle\chi|\chi\rangle = \langle\phi - \lambda\psi|\phi - \lambda\psi\rangle \geq 0$$

Step 2: Expand the Inner Product

$$\langle\chi|\chi\rangle = \langle\phi|\phi\rangle - \lambda^*\langle\phi|\psi\rangle - \lambda\langle\psi|\phi\rangle + |\lambda|^2\langle\psi|\psi\rangle$$

Step 3: Choose Optimal λ

To minimize $\langle\chi|\chi\rangle$, choose:

$$\lambda = \frac{\langle\psi|\phi\rangle}{\langle\psi|\psi\rangle}$$

Step 4: Substitute λ Back In

Substituting into the expression for $\langle\chi|\chi\rangle$:

$$\langle\chi|\chi\rangle = \langle\phi|\phi\rangle - \frac{|\langle\psi|\phi\rangle|^2}{\langle\psi|\psi\rangle}$$

Since $\langle\chi|\chi\rangle \geq 0$, we get:

$$\langle\phi|\phi\rangle - \frac{|\langle\psi|\phi\rangle|^2}{\langle\psi|\psi\rangle} \geq 0$$

Multiplying both sides by $\langle\psi|\psi\rangle$, we obtain:

$$\langle\psi|\psi\rangle\langle\phi|\phi\rangle \geq |\langle\psi|\phi\rangle|^2$$

Final Result

$$\boxed{|\langle\psi|\phi\rangle|^2 \leq \langle\psi|\psi\rangle \cdot \langle\phi|\phi\rangle}$$

Equality Condition

Equality holds if and only if:

$$|\phi\rangle = \lambda|\psi\rangle$$

i.e., the vectors $|\phi\rangle$ and $|\psi\rangle$ are linearly dependent.

April 16, 2025

Objective

We aim to derive the generalized uncertainty relation for two Hermitian operators A and B, acting on a quantum state $|\psi\rangle$:

$$\sigma_A \sigma_B \geq \frac{1}{2} \left| \langle [A, B] \rangle \right| . \tag{1}$$

This relation sets a lower bound on the product of standard deviations (uncertainties) of measurements of the observables A and B.

Step 1: Define Deviation Operators

Let the expectation values be:

$$\langle A \rangle = \langle \psi | A | \psi \rangle, \qquad \langle B \rangle = \langle \psi | B | \psi \rangle.$$

We define the deviation operators (also known as fluctuation operators):

$$\Delta A = A - \langle A \rangle, \qquad \Delta B = B - \langle B \rangle.$$

These represent how the operators differ from their mean values when acting on the state $|\psi\rangle$.

Step 2: Define Vectors in Hilbert Space

Define the following state vectors:

$$|\phi\rangle = \Delta A |\psi\rangle, \qquad |\chi\rangle = \Delta B |\psi\rangle.$$

These vectors represent how the state $|\psi\rangle$ is modified by the action of the deviation operators.

Step 3: Apply the Schwarz Inequality

The Schwarz inequality states:

$$|\langle\phi|\chi\rangle|^2 \leq \langle\phi|\phi\rangle \cdot \langle\chi|\chi\rangle.$$

Substitute the vectors:

$$|\langle\psi|\Delta A \Delta B|\psi\rangle|^2 \leq \langle\psi|(\Delta A)^2|\psi\rangle \cdot \langle\psi|(\Delta B)^2|\psi\rangle.$$

So we get:

$$|\langle\Delta A \Delta B\rangle|^2 \leq \sigma_A^2 \sigma_B^2,$$

where $\sigma_A^2 = \langle(\Delta A)^2\rangle$, and similarly for σ_B.

Step 4: Expand the Expectation Value

We now expand the complex quantity $\langle\Delta A \Delta B\rangle$:

$$\Delta A \Delta B = \frac{1}{2}[\Delta A, \Delta B] + \frac{1}{2}\{\Delta A, \Delta B\}.$$

This identity splits the operator product into a commutator (antisymmetric) and an anticommutator (symmetric) part.

Take the expectation value:

$$\langle\Delta A \Delta B\rangle = \frac{1}{2}\langle[\Delta A, \Delta B]\rangle + \frac{1}{2}\langle\{\Delta A, \Delta B\}\rangle.$$

Now, note that:

$$[\Delta A, \Delta B] = [A, B],$$

because constants commute and have zero expectation value.

So:

$$\langle\Delta A \Delta B\rangle = \frac{1}{2}\langle[A, B]\rangle + \frac{1}{2}\langle\{\Delta A, \Delta B\}\rangle.$$

Step 5: Separate Real and Imaginary Parts

We now write:

$$|\langle\Delta A \Delta B\rangle|^2 = \left(\mathrm{Re}\langle\Delta A \Delta B\rangle\right)^2 + \left(\mathrm{Im}\langle\Delta A \Delta B\rangle\right)^2.$$

Using the earlier expansion:

$$\mathrm{Re}\langle\Delta A \Delta B\rangle = \frac{1}{2}\langle\{\Delta A, \Delta B\}\rangle,$$
$$\mathrm{Im}\langle\Delta A \Delta B\rangle = \frac{1}{2i}\langle[A, B]\rangle.$$

So:

$$|\langle\Delta A \Delta B\rangle|^2 = \left(\frac{1}{2}\langle\{\Delta A, \Delta B\}\rangle\right)^2 + \left(\frac{1}{2i}\langle[A, B]\rangle\right)^2.$$

Step 6: Final Inequality

Combining with the Schwarz inequality:

$$\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2} \langle \{ \Delta A, \Delta B \} \rangle \right)^2 + \left(\frac{1}{2i} \langle [A, B] \rangle \right)^2 .$$

This is the **Robertson–Schrödinger uncertainty relation**.

Special Case: Heisenberg Uncertainty Principle

If the anticommutator term is zero or ignored, we obtain:

$$\sigma_A \sigma_B \geq \frac{1}{2} \left| \langle [A, B] \rangle \right| .$$

This is the standard uncertainty principle.