Completeness Relation and Projection Operators

Expansion in an Orthonormal Basis

Let $\{|a\rangle\}$ be a complete orthonormal set of eigenkets of a Hermitian operator A. Then any state $|\alpha\rangle$ can be expressed as:

$$|\alpha\rangle = \sum_a |a\rangle \langle a| \, \alpha \tag{1.61}$$

The operator

$$\sum_a |a\rangle \langle a| = \mathbb{I} \tag{1.65}$$

is known as the **completeness relation** or the **closure relation**. It represents the identity operator in the Hilbert space.

Normalization Condition

Consider the inner product $\langle\alpha|\alpha\rangle$. Inserting the identity operator:

$$\begin{aligned}
\langle\alpha|\alpha\rangle &= \langle\alpha| \, \mathbb{I} \, |\alpha\rangle && (1)\\
&= \langle\alpha| \left(\sum_a |a\rangle \langle a| \right) |\alpha\rangle && (2)\\
&= \sum_a | \langle a|\alpha\rangle |^2 && (1.66)
\end{aligned}$$

This shows that if $|\alpha\rangle$ is normalized, the coefficients in the expansion (1.61) must satisfy:

$$\sum_a |c_a|^2 = \sum_a | \langle a|\alpha\rangle |^2 = 1 \tag{1.67}$$

Projection Operators

The term $|a\rangle \langle a|$ is an **outer product** and acts as an operator on any ket $|\alpha\rangle$ as follows:

$$(|a\rangle \langle a|) |\alpha\rangle = |a\rangle \langle a|\alpha\rangle = c_a |a\rangle \tag{1.68}$$

This operation extracts the component of $|\alpha\rangle$ that lies along $|a\rangle$. Therefore, $|a\rangle \langle a|$ is called the **projection operator** onto $|a\rangle$, and is denoted as:

$$\Lambda_a \equiv |a\rangle \langle a| \tag{1.69}$$

Thus, the completeness relation can also be written as:

$$\sum_a \Lambda_a = \mathbb{I} \tag{3}$$