Theorem: Properties of Hermitian Operators

Theorem

The eigenvalues of a Hermitian operator A are real; the eigenkets of A corresponding to different eigenvalues are orthogonal.

Key Equations and Derivation

Let $A\,|a\rangle = a\,|a\rangle$. (1.55)

Since A is Hermitian, we also have:

$$\langle a'|\,A = a'^*\,\langle a'| \tag{1.56}$$

Multiply Eq. (1.55) on the left by $\langle a'|$, and Eq. (1.56) on the right by $|a\rangle$. Subtracting, we get:

$$(a - a'^*)\,\langle a'|a\rangle = 0 \tag{1.57}$$

Case 1: $a = a'$

In this case, equation (1.57) becomes:

$$(a - a^*)\,\langle a|a\rangle = 0$$

Since $|a\rangle \neq 0$, we have $\langle a|a\rangle \neq 0$, and therefore:

$$a = a^* \tag{1.58}$$

which proves that the eigenvalue is real.

Case 2: $a \neq a'$

Then equation (1.57) implies:

$$\langle a'|a\rangle = 0 \quad (a \neq a') \tag{1.59}$$

which proves that eigenkets corresponding to different eigenvalues are orthogonal.

Normalization

It is conventional to normalize the eigenkets so that:

$$\langle a'|a\rangle = \delta_{a'a} \tag{1.60}$$

Completeness

By construction of the Hilbert space, the set of eigenkets $\{|a\rangle\}$ of a Hermitian operator A forms a complete basis.