#3

$$xy' + (\ln x)y = 0$$

$$y' + \frac{\ln x}{x}y = 0$$

$$p(x) = \frac{\ln x}{x}$$

$$y = Ce^{-P(x)}$$

where

$$P'(x) = p(x)$$

$$P(x) = \int \frac{\ln x}{x}dx$$

$$u = \ln x$$

$$du = \frac{1}{x}dx$$

$$P(x) = \int udu$$

$$= \frac{1}{2}u^2 + k$$

(k=0)

$$= \frac{1}{2}(\ln x)^2$$

$$y = Ce^{-P(x)}$$

$$= Ce^{-\frac{1}{2}(\ln x)^2}$$

$$= \frac{C}{\sqrt{x^{\ln x}}}$$

$$y = \frac{C}{\sqrt{x^{\ln x}}}$$

and the trivial solution

$$y \equiv 0.$$

#6

$$y' + \left(\frac{x+1}{x}\right)y = 0$$
$$p(x) = \frac{x+1}{x}$$

$$P(x) = \int \frac{x+1}{x}dx$$
$$= \int \left(1 + \frac{1}{x}\right)dx$$
$= x + \ln|x|$; (k=0)

$$y = Ce^{-x}e^{-\ln|x|}$$
$$= \frac{Ce^{-x}}{|x|}$$

$$1 = C \cdot \frac{e^{-e}}{e}$$
$$C = e^{e+1}$$

$$y = e^{e+1} \cdot \frac{e^{-x}}{|x|}$$

note: i think
$$y = e^{e+1} \cdot \frac{e^{-x}}{x}$$
is also a solution, but i don't know how to prove that. it also might be that the solution *with* |x| is incorrect, (not a valid solution) but i dont know how i would prove that either.

#14
$$y' + 2xy = xe^{-x^2}$$

$$y_1' + 2xy_1 = 0$$
$$y_1 = e^{-x^2}$$

$$y = uy_1 = ue^{-x^2}$$

$$y' = u'y_1 + uy_1' = u'e^{-x^2} - 2x \cdot ue^{-x^2}$$

$$y' + 2xy = u'e^{-x^2} - 2x \cdot ue^{-x^2} + 2x \cdot ue^{-x^2}$$

$$= u'e^{-x^2}$$

$$u'e^{-x^2} = xe^{-x^2}$$
$$u' = x$$
$$u = \frac{x^2}{2} + C$$

$$y = \frac{x^2 e^{-x^2}}{2} + Ce^{-x^2}$$

#23

$$y' + (2\sin x \cos x)y = e^{-\sin^2 x}$$

$$y_1' + (2\sin x \cos x)y_1 = 0$$

$$P(x) = 2\int (\sin x \cos x)dx$$

$$\alpha = \sin x \quad \beta = \sin x$$
$$\alpha' = \cos x \; \beta' = \cos x$$

$$\int (\sin x \cos x)dx = \sin^2 x - \int (\sin x \cos x)dx$$

$$\frac{1}{2}P(x) = \sin^2 x - \frac{1}{2}P(x)$$
$P(x) = \sin^2 x$ (k=0)

$$y_1 = e^{-\sin^2 x}$$
$$y = uy_1 = ue^{-\sin^2 x}$$

$$y' = u'y_1 + uy_1' = u'e^{-\sin^2 x} - 2\sin x \cos x \cdot ue^{-\sin^2 x}$$

$$y' + (2\sin x \cos x)y = u'e^{-\sin^2 x} - 2\sin x \cos x \cdot ue^{-\sin^2 x} + 2\sin x \cos x \cdot ue^{-\sin^2 x}$$
$$= u'e^{-\sin^2 x}$$

$$u'e^{-\sin^2 x} = e^{-\sin^2 x}$$

$$u' = 1$$
$$u = x + C$$

$$y = xe^{-\sin^2 x} + Ce^{-\sin^2 x}$$

#26

$$\left(x^2+1\right)y' + 4xy = \frac{2}{x^2+1}$$

$$y' + \frac{4x}{x^2+1}y = \frac{2}{\left(x^2+1\right)^2}$$

$$y_1' + \frac{4x}{x^2+1}y_1 = 0$$

$$P(x) = 2\int \frac{2x}{x^2+1}dx$$

$$v = x^2 + 1$$
$$dv = 2xdx$$

$$P(x) = 2\int \frac{1}{u}du$$

$= 2\ln|x^2+1|$ (k=0)

$$\left(x^2 + 1 \geq 0\right)$$
$$= 2\ln\left(x^2+1\right)$$

$$y_1 = e^{-P(x)} = \frac{1}{\left(x^2+1\right)^2}$$

$$y = uy_1 = \frac{u}{\left(x^2+1\right)^2}$$

$$y' = u'y_1 + uy_1' = \frac{u'}{\left(x^2+1\right)^2} + \frac{-2u \cdot 2x}{\left(x^2+1\right)^3}$$

$$y' + \frac{4x}{x^2+1}y = \frac{u'}{\left(x^2+1\right)^2} + \frac{-4ux}{\left(x^2+1\right)^3} + \frac{4ux}{\left(x^2+1\right)^3}$$

$$= \frac{u'}{\left(x^2+1\right)^2}$$

$$\frac{u'}{\left(x^2+1\right)^2} = \frac{2}{\left(x^2+1\right)^2}$$

$$u' = 2$$

$$u = 2x + C$$

$$y = \frac{2x}{\left(x^2+1\right)^2} + \frac{C}{\left(x^2+1\right)^2}$$

$$1 = (0) + \frac{C}{(1)^2}$$

$$C = 1$$

$$y = \frac{2x+1}{\left(x^2+1\right)^2}$$

#32

$$xy' + 2y = -x^2$$

$$y' + \frac{2}{x}y = -x$$

$$y_1' + \frac{2}{x}y_1 = 0$$

$$P(x) = 2\int \frac{1}{x}dx$$

$$P(x) = 2\ln|x|$$

$$y_1 = e^{-P(x)} = \frac{1}{|x|^2}$$

$$|x|^2 = x^2$$

$$y_1 = \frac{1}{x^2}$$

$$y = uy_1 = \frac{u}{x^2}$$

$$y' = u'y_1 + uy_1' = \frac{u'}{x^2} + \frac{-2u}{x^3}$$

$$y' + \frac{2}{x}y = \frac{u'}{x^2} + \frac{-2u}{x^3} + \frac{2u}{x^3}$$

$$= \frac{u'}{x^2}$$

$$\frac{u'}{x^2} = -x^2$$

$$u' = -x^4$$

$$u = -\frac{x^5}{5} + C$$

$$y = -\frac{x^3}{5} + \frac{C}{x^2}$$

$$1 = -\frac{1}{5} + C$$

$$C = \frac{4}{5}$$

$$y = \frac{4}{5x^2} - \frac{x^3}{5}$$