COMPUTATION OF TOPOLOGICAL JACOBI FORMS

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Abstract. We compute, at the prime 2, the entire descent spectral sequence converging to the homotopy groups of the spectra of topological Jacobi forms TJF_m for every index $m \geq 1$. An explicit TMF–cellular decomposition $\mathrm{TJF}_m \simeq \mathrm{TMF} \otimes P_m$ reduces the problem to analyzing a finite complex P_m with one even cell in each dimension $\leq 2m$. We identify all differentials using the cell structure.

1. Introduction

The spectrum topological modular forms (TMF) was constructed by Hopkins and Mahowald as a height 2 spectrum with the structure map $\pi_{2*}\mathrm{TMF} \to \mathrm{MF}_*$. Through the work of Ando–Hopkins–Strickland [AHS01] and Ando-Hopkins-Rezk [AHR10], TMF arises as the target of the Witten genus

$$M\mathrm{String} \to \mathrm{TMF};$$

and consequently, it has become a widely studied object in homotopy theory. After the extensive work of Lurie on spectral algebraic geometry and elliptic cohomology ([Lur18c], [Lur09], [Lur18a], [Lur18b], [Lur19]), the G-equivariant refinement of TMF for any compact Lie group G was constructed by Gepner–Meier in [GM20]. In their work, they built a spectral Deligne-Mumford stack $\mathcal{M}_G$ and a colimit-preserving functor

$$\mathcal{E}ll_G \colon \mathrm{Sp}_G^{\mathrm{op}} \to \mathrm{QCoh}(\mathcal{M}_G)$$

satisfying

(1) $\mathcal{M}_e = \mathcal{M}_{\mathrm{ell}}^{\mathrm{or}}$, where e is the trivial group and $\mathcal{M}_{\mathrm{ell}}^{\mathrm{or}}$ is the spectral DM stack of oriented elliptic curves constructed in [Lur18b],
(2) $\mathcal{M}_{\mathbb{T}} \simeq \mathcal{E}$, where $\mathbb{T}$ is the circle and $\mathcal{E}$ is the universal oriented elliptic curves over $\mathcal{M}_{\mathrm{ell}}^{\mathrm{or}}$ (see also [Mei18, Chapter 4]),
(3) each $\mathcal{M}_G$ is a relative scheme over $\mathcal{M}_{\mathrm{ell}}^{\mathrm{or}}$,

and $\mathcal{E}ll_G$ is natural with respect to G so that G-equivariant TMF can be seen as a globally equivariant cohomology [GLP24].

Based on their work, we define a new family of spectra—the *topological Jacobi forms*

$$\mathrm{TJF}_m \coloneqq \Gamma\big(\mathcal{E}, \mathcal{O}_{\mathcal{E}}(me)\big) \qquad (m \geq 1)$$

emerged as the global sections of line bundles on the universal oriented elliptic curve $\mathcal{E} \to \mathcal{M}_{\mathrm{ell}}^{\mathrm{or}}$. These spectra package the $RO(\mathbb{T})$-graded TMF, and they are the spectral refinement of the classical ring of weight k, index $\frac{m}{2}$ integral Jacobi forms

$$\mathrm{JF}_{k,\frac{m}{2}} = \Gamma\big(E, p^*\omega^{\otimes k} \otimes \mathcal{O}_E(me)\big)$$

defined in, for example, [EZ85] and [Kra95].

The present paper gives a complete 2-local analysis of TJF_m for every $m \geq 1$. Our main contribution is an explicit computation of the homotopy groups $\pi_*\mathrm{TJF}_m$ together with exotic extensions. While the classical Jacobi forms are already subtle, the topological enrichment reveals a surprising relationship with the complex projective space via the transfer map in the equivariant homotopy theory.

1.1. Main results.

Theorem 1.1. *For each integer $m \geq 1$, there exists an equivalence of* TMF*-modules*

$$\mathrm{TJF}_m \simeq \mathrm{TMF} \otimes P_m,$$

where P_m is obtained from the stunted complex projective space $\Sigma^2\mathbb{C}P_{-1}^{m-1}$ by deleting its 2-cell (Definition 6.1). In particular, TJF_m has exactly one TMF*-cell in each even dimension $2d \leq 2m$, $d \neq 1$, and the stable attaching maps are detected by the Hopf fibrations η and ν.*

Theorem 1.2. *Let $E_r^{s,t}(m)$ denote the descent spectral sequence converging to $\pi_{t-s}\mathrm{TJF}_m$.*

(1) The E_2-term is

$$E_2^{s,2t}(m) \cong H^s\big(E; p^*\omega^{\otimes t} \otimes \mathcal{O}_E(me)\big)$$

and is explicitly described by the cohomology of a Hopf algebroid (B_m, Σ_m) obtained from the $GL_2(\mathbb{Z}/3)$-Galois cover $E' \to E$ of the universal elliptic curve E (Section 4).

(2) The descent spectral sequence enjoys a horizontal vanishing line at $s = 24$ on the E_{24}-page (Corollary 6.5).

(3) The differentials are completely determined for every m (Section 7 to 12), from the vanishing line, the Leibniz rule, and the fact that $\pi_n\mathrm{tmf}$ does not have any c_4-torsion elements in degrees $169 \leq n \leq 191$ with Adams filtration ≥ 3.

1.2. **Outline of the argument.** Our computation proceeds in three stages.

(1) **Setting up spectral sequences.** We work over the étale cover $\mathcal{M}_1(3) \to \mathcal{M}_{\mathrm{ell}}$, whose affine coordinate ring is $A = \mathbb{Z}_{(2)}[a_1, a_3, \Delta^{\pm}]$. Pulling back the universal elliptic curve E to $\mathcal{M}_1(3)$ yields a Weierstraß curve, from which we form the Hopf algebroid (B_m, Σ_m) controlling TJF_m.

(2) TJF_∞**-case.** Under the change-of-rings theorem [HS08] [Hov01], we can work out the E_2-page of the DSS for TJF_∞. It turns out that there is essentially only one d_3-differential given by the relation $\eta^4 = 0$, yielding the simple structure of $\pi_*\mathrm{TJF}_\infty$.

(3) **Transfer and cellular structure.** Exploiting the transfer sequence $\Sigma\mathrm{TMF} \to \mathrm{TMF}_{\mathbb{T}} \to \mathrm{TMF}$ studied in [GM20], we identify TJF_m as the cofiber of a map $\mathrm{TMF} \otimes \Sigma\mathbb{C}P_+^{m-1} \to \mathrm{TMF} \oplus \Sigma\mathrm{TMF}$. A comparison with the stunted projective spaces [Mil82] [Mos68] reveals the cell structure of the cofiber and proves Theorem 1.1.

(4) **Higher differentials.** The cell decomposition sharply restricts the possible d_r in the descent spectral sequence. For $m = 2$, we compute all differentials from the fact that $\mathrm{TJF}_2 \simeq \mathrm{TMF} \otimes C\nu$. The method extends, with increasing bookkeeping, to $m = 3, \ldots, 7$.

1.3. **Organization of the paper.** Section 3 recalls circle-equivariant elliptic cohomology and sets notation. Section 3 constructs the Hopf algebroid (B_m, Σ_m) and Section 4 identifies its cohomology. Section 6 proves Theorem 1.1 via the comparison with stunted projective spaces. Sections 7–12 compute the descent spectral sequences for $m = 2, \ldots, 7$ and record some exotic extensions.

1.4. **Conventions.**

(1) Sp denotes the category of spectra and Sp_G the category of genuine G-spectra.

(2) $\mathcal{S}$ denotes the category of ∞-groupoids (spaces). $\mathcal{S}^{\mathrm{fin}}$ denotes the full subcategory spanned by spaces with the homotopy type of a finite complex.

(3) $\Sigma^\infty_+ \colon \mathcal{S} \to \mathrm{Sp}$ and $\Sigma^\infty \colon \mathcal{S}_* \to \mathrm{Sp}$ denote the suspension spectrum functor. We sometimes abbreviate Σ^∞ even when we treat a pointed space as a spectrum.

(4) The letter $\mathbb{T}$ denotes the circle group $U(1)$ and $\rho\colon \mathbb{T} \hookrightarrow \mathbb{C}^\times$ its tautological representation.

(5) The symbol tmf stands for the connective spectrum of topological modular forms, and TMF denotes its localization $\mathrm{tmf}[(\Delta^8)^{-1}]$.

(6) $\mathcal{M}_{\mathrm{ell}}^{\mathrm{or}}$ denotes the spectral Deligne-Mumford stack of oriented elliptic curves constructed in [Lur18b]. In particular, TMF is equivalent to the global section of the structure sheaf $\mathrm{TMF} \simeq \Gamma(\mathcal{M}_{\mathrm{ell}}^{\mathrm{or}}, \mathcal{O}_{\mathcal{M}_{\mathrm{ell}}^{\mathrm{or}}})$. Its underlying stack $\mathcal{M}_{\mathrm{ell}}$ is the 1-stack of (classical) smooth elliptic curves.

(7) Similarly, $\mathcal{M}_1^{\mathrm{or}}(n)$ denotes the spectral moduli of oriented elliptic curves with the level structure. The global section of its structure sheaf is $\mathrm{TMF}_1(n)$.

(8) We use the same notationf for the elements in $\pi_*\mathrm{tmf}$ as in [Bau08].

(9) η, ν denotes elements in $\pi_1\mathbb{S}$ and $\pi_3\mathbb{S}$ represented by the Hopf fibrations $S^3 \to S^2$ and $S^7 \to S^4$.

(10) Whenever possible, we use the mathcal letters $\mathcal{M}$ for the spectral Deligne-Mumford stacks, and usual letters M for its underlying stacks.

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2. Preliminary

Take a spectral scheme $S = \operatorname{Spec} R$ for an even periodic $\mathbb{E}_\infty$-ring $R \in \mathrm{CAlg}$.

Definition 2.1. A spectral elliptic curve $p\colon C \to S$ is a strict abelian group object in SpSch_S such that

- p is flat,
- $\tau_{\geq 0}p$ is proper and locally almost of finite presentation, and

- for any point $i\colon \operatorname{Spec} k \to \tau_{\geq 0} S$ with k being an algebraically closed field, the pullback $i^*\mathcal{C} \to \operatorname{Spec} k$ is a (classical) elliptic curve over k.

Definition 2.2. A preorientation of a strict abelian group object C in SpSch_S is a map

$$i\colon S^2 \to \mathrm{Map}_{\mathrm{SpSch}_S}(S, C).$$

In other words, a preorientation is a map of strict abelian group objects of spaces $i\colon BU(1) \to \mathrm{Map}_{\mathrm{SpSch}_S}(S, C)$.

The base ring R defines the Quillen formal groups $\widehat{\mathbb{G}}^Q_R$ when R is even periodic in the sense of [Lur18b]. Moreover, a preorientation induces a map of formal groups $\widehat{\mathbb{G}}^Q_R \to \widehat{C}$ [Lur18b, Proposition 4.3.21].

Definition 2.3. A preorientation is called orientation if the corresponding map

$$\widehat{\mathbb{G}}^Q_R \to \widehat{C}$$

of formal groups is an equivalence.

For an oriented abelian schemes C, we can associate global equivariant cohomology associated to C as follows.

Definition 2.4. A smooth 1-stack is a sheaf valued in $\mathcal{S}$ on the category of smooth manifolds Mfd.

Let G be a compact Lie group. When a smooth manifold X admits an G-action, we can construct a groupoid object

$$\cdots \rightrightarrows G \times G \times X \rightrightarrows G \times X \longrightarrow X$$

in the (1-)category of smooth manifolds. Sheafifying the functor $M \mapsto \mathrm{Map}_{\mathrm{Mfd}}(M, X_*)$ corepresented by this groupoid object, one gets an associated smooth stack $[X/G]$. Especially, when X is a point, one gets the stack $\mathbb{B}G$ classifying principal G-bundles.

Definition 2.5. The orbit category Orb is defined as follows:

- objects are smooth stacks of the form $\mathbb{B}G$, and
- mapping space is defined by $\mathrm{Map}_{\mathrm{Orb}}(\mathbb{B}G, \mathbb{B}H) = \mathrm{Map}_{\mathrm{Lie}}(H, G)_{hG}$ where the G-action on the latter is given by conjugation.

The presheaf category $\mathcal{P}(\mathrm{Orb})$ is called the category of orbispaces. The category of orbispaces contains the category of G-equivariant for each compact Lie group G, thereby being a framework of global homotopy theory.

Proposition 2.6. *Let G be a compact Lie group. Then the functor*

$$\mathcal{S}_G \to \mathcal{P}(\mathrm{Orb})_{/\mathbb{B}G},\ X \mapsto [X/G]$$

is colimit-preserving and fully faithful.

Take the full subcategory $\mathrm{Orb}^{\mathrm{ab}} \subset \mathrm{Orb}$ spanned by objects equivalent to $\mathbb{B}G$ for a abelian compact Lie group G. Given an oriented elliptic curve $C \to S$, we can construct a functor from $\mathrm{Orb}^{\mathrm{ab}}$ to the category of locally ringed spaces $\mathrm{Top}^{\mathrm{CAlg}}_{\mathrm{loc}}$ (cite Ell 3 1.1.2) by the formula

$$\mathbb{B}G \mapsto \mathcal{X}[\widehat{G}] \coloneqq \mathrm{Map}_{\mathrm{Ab}^s(\mathrm{SpSch}_S)}(\widehat{G}, \mathcal{E})$$

where $\widehat{G}$ is the Pontryagin dual of G. As $\mathrm{Top}^{\mathrm{CAlg}}_{\mathrm{loc}}$ is cocomplete (cite something), we obtain a functor

$$\begin{tikzcd} \mathrm{Orb}^{\mathrm{ab}} \arrow[r] \arrow[d] & \mathrm{Top}^{\mathrm{CAlg}}_{\mathrm{loc}} \\ \mathcal{P}(\mathrm{Orb}^{\mathrm{ab}}) \arrow[ur, dotted] & \end{tikzcd}$$

by the left Kan extension. Finally, by precomposing with the restriction $\mathcal{P}(\mathrm{Orb}) \to \mathcal{P}(\mathrm{Orb}^{\mathrm{ab}})$, one gets the functor

$$\mathrm{Ell}_{C/S}\colon \mathcal{P}(\mathrm{Orb}) \to \mathrm{Top}^{\mathrm{CAlg}}_{\mathrm{loc}}.$$

From the construction, we can identify images

$$\mathrm{Ell}_{C/S}(\mathbb{B}S^1) \simeq C,\ \mathrm{Ell}_{C/S}(\mathbb{B}C_n) \simeq C[n].$$

Now we can define the G-equivariant elliptic cohomology functor for each compact Lie group G as follows. Denote the spectral scheme $\mathrm{Ell}_{C/S}(\mathbb{B}G)$. For a finite G-space X, embed this space to $\mathcal{P}(\mathrm{Orb})_{/\mathbb{B}G}$ by the functor $X \mapsto [X/G]$. Applying $\mathrm{Ell}_{C/S}$, we obtain a spectral scheme over $\mathrm{Ell}_{C/S}([X/G])$ over $\mathrm{Ell}_{C/S}(\mathbb{B}G)$.

Theorem 2.7. *Assume that a compact Lie group G satisfies either*

- *G is abelian, or*
- *$\pi_1 G$ is torsion-free.*

Then a locally ringed space $\mathrm{Ell}_{C/S}([X/G])$ becomes a spectral scheme for a finite G-space X.

Pushing forward the structure sheaf of $\mathrm{Ell}_{C/S}([X/G])$ to $\mathrm{Ell}_{C/S}(\mathbb{B}G)$, we obtain a quasi-coherent sheaf on $\mathrm{Ell}_{C/S}(\mathbb{B}G)$. We summarize this construction as follows:

Theorem 2.8. *Let G be a compact Lie group with $\pi_1 G$ torsion-free or G is abelian. Let $p\colon C \to S$ be an oriented elliptic curve. There is a G-equivariant elliptic cohomology functor*

$$\mathcal{E}ll_G^{C/S}\colon \mathcal{S}^{G,\mathrm{fin}} \to \mathrm{QCoh}(\mathrm{Ell}_{C/S}(\mathbb{B}G))^{\mathrm{op}}$$

satisfying the following properties:

- *$\mathcal{E}ll_G$ preserves colimits,*
- *for each orbit G/H for a closed subgroup $H \triangleleft G$, we have*

$$\mathcal{E}ll_G^{C/S}(G/H) \simeq (i_H)_* \mathcal{O}_{\mathrm{Ell}_{C/S}(\mathbb{B}H)}$$

  where $i_H\colon \mathrm{Ell}_{C/S}(\mathbb{B}H) \to \mathrm{Ell}_{C/S}(\mathbb{B}G)$ is the image of the map $\mathbb{B}H \to \mathbb{B}G$.

Since the target category $\mathrm{QCoh}(\mathrm{Ell}_{C/S}(\mathbb{B}G))$ is pointed, we can extend the domain of $\mathcal{E}ll_G$ along $(-)_+\colon \mathcal{S}^{G,\mathrm{fin}} \to \mathcal{S}_*^{G,\mathrm{fin}}$ and obtain the reduced cohomology functor

$$\widetilde{\mathcal{E}ll}_G^{C/S}\colon \mathcal{S}_*^{G,\mathrm{fin}} \to \mathrm{QCoh}(\mathrm{Ell}_{C/S}(\mathbb{B}G))^{\mathrm{op}}$$

as well. Moreover, when G is torus,[GM20, Theorem 8.1, Proposition 9.2] showed that $\mathcal{E}ll_G^{C/S}$ symmetric monoidal:

Theorem 2.9. *The functors $\mathcal{E}ll_G^{C/S}$ and $\widetilde{\mathcal{E}ll}_G^{C/S}$ are symmetric monoidal for when $G = \mathbb{T}^n$ for some $n \in \mathbb{N}$. Moreover, the domain of $\widetilde{\mathcal{E}ll}_{\mathbb{T}^n}$ can be extended along $\Sigma^\infty\colon \mathcal{S}_*^{\mathbb{T}^n,\mathrm{fin}} \to \mathrm{Sp}_{\mathbb{T}^n}^{\mathrm{fin}}$, and we obtain a functor*

$$\mathcal{E}ll_G\colon \mathrm{Sp}_{\mathbb{T}^n} \to \mathrm{QCoh}(\mathrm{Ell}_{C/S}(\mathbb{B}\mathbb{T}^n))^{\mathrm{op}} \simeq \mathrm{QCoh}(C^{\times n})^{\mathrm{op}}.$$

The main interest in this paper is the circle-equivariant $G = S^1$ case. Instead of thinking individual oriented ellipitc curves and associated elliptic cohomology, we want to the universal object. Recall that Lurie constructed the moduli of oriented elliptic curves $\mathcal{M}_{\mathrm{ell}}^{\mathrm{or}}$ [Lur18b] and showed an equivalence

$$\mathrm{TMF} \simeq \Gamma(\mathcal{M}_{\mathrm{ell}}^{\mathrm{or}}, \mathcal{O}_{\mathcal{M}_{\mathrm{ell}}^{\mathrm{or}}}).$$

Note that we can express the moduli as the colimit of affines

$$\mathcal{M}_{\mathrm{ell}}^{\mathrm{or}} \simeq \operatorname*{colim}_{\substack{\mathrm{Spec}\, R \to \mathcal{M}_{\mathrm{ell}}^{\mathrm{or}} \\ R\colon \text{ even periodic}}} \mathrm{Spec}\, R.$$

We can construct the universal oriented elliptic curve by pasting oriented elliptic curves by the same diagram

$$\mathcal{E} \coloneqq \operatorname*{colim}_{\substack{\mathrm{Spec}\, R \to \mathcal{M}_{\mathrm{ell}}^{\mathrm{or}} \\ C \to \mathrm{Spec}\, R\colon \text{ oriented}}} C$$

As the equivariant elliptic cohomology is functorial with respect to the oriented elliptic curves, we can construct the universal circle-equivariant elliptic cohomology as follows.

Theorem 2.10. *Let $p\colon \mathcal{E} \to \mathcal{M}_{\mathrm{ell}}^{\mathrm{or}}$ be the universal oriented elliptic curve. Then there is an associated circle-equivariant* TMF*-cohomology functor*

$$\mathcal{E}ll_{\mathbb{T}}\colon \mathrm{Sp}_{\mathbb{T}}^{\mathrm{fin}} \to \mathrm{QCoh}(\mathcal{E})^{\mathrm{op}}$$

satisfying the following properties:

1. *$\mathcal{E}ll_{\mathbb{T}}$ is a colimit-preserving symmetric monoidal functor,*
2. *$\mathcal{E}ll_{\mathbb{T}}(S^0) \simeq \mathcal{O}_C$,*
3. *$\mathcal{E}ll_{\mathbb{T}}(\Sigma_+^\infty(\mathbb{T}/\mathbb{T}[n])) \simeq (e_n)_* \mathcal{O}_{C[n]}$ where $e_n\colon C[n] \to C$ is the inclusion of n-torison points of C.*

3. The Descent Spectral Sequence: Setup

To define the topological Jacobi forms, we set up a space with S^1-action. Let $\rho\colon \mathbb{T} = U(1) \hookrightarrow \mathbb{C}^\times$ be the fundamental represenation of the circle $\mathbb{T}$. Then there is a cofiber sequence

$$\mathbb{T}_+ \to S^0 \to S^\rho.$$

where S^ρ is the representation sphere associated to ρ. Applying $\mathcal{E}ll_\mathbb{T}$ to this sequence, we obtain a fiber sequence of sheaves

$$\mathcal{E}ll_\mathbb{T}(\mathbb{T}_+) \simeq e_*\mathcal{O}_S \leftarrow \mathcal{O}_C \leftarrow \mathcal{E}ll_\mathbb{T}(S^\rho),$$

and see that $\mathcal{E}ll_\mathbb{T}(S^\rho)$ is the invertible sheaf $\mathcal{O}_\mathcal{E}(-e)$. As $\mathcal{E}ll_\mathbb{T}$ is symmetric monoidal, it sends the Spanier-Whitehead dual $S^{-\rho}$ in $\mathrm{Sp}_\mathbb{T}$ to $\mathcal{O}_C(e)$ and therefore $S^{-m\rho}$ to $\mathcal{O}_C(me)$.

Definition 3.1. Let $p\colon \mathcal{E} \to \mathcal{M}_{\mathrm{ell}}^{\mathrm{or}}$ be the universal oriented elliptic curve. The topological Jacobi form TJF_m of index m is the global section

$$\mathrm{TJF}_m \coloneqq \Gamma(\mathcal{E}, \mathcal{O}_\mathcal{E}(me)).$$

In other words, TJF_m is the representation spheres' $\mathbb{T}$-equivariant elliptic homology. The following two results justify this naming.

Lemma 3.2. *Over the (classical) Deligne-Mumford stack of elliptic curves $\mathcal{M}_{\mathrm{ell}}$, we have*

$$\pi_n\mathcal{O}_\mathcal{E}(me) \simeq \begin{cases} 0 & (n = 2k+1) \\ p^*\omega^k \otimes \mathcal{O}_E(me) & (n = 2k) \end{cases}$$

where E is the universal elliptic curve over $\mathcal{M}_{\mathrm{ell}}$.

Proof. Using the flatness of $p\colon \mathcal{E} \to \mathcal{M}_{\mathrm{ell}}^{\mathrm{or}}$, we compute

$$\begin{aligned} \pi_n\mathcal{O}_\mathcal{E}(me) &\simeq \pi_n p^*\mathcal{O}_{\mathcal{M}_{\mathrm{ell}}^{\mathrm{or}}} \otimes_{\pi_0\mathcal{O}_{\mathcal{M}_{\mathrm{ell}}^{\mathrm{or}}}} \pi_0\mathcal{O}_\mathcal{E}(me) \\ &\simeq \begin{cases} 0 & (n = 2k+1) \\ p^*\omega^k \otimes \mathcal{O}_E(me) & (n = 2k). \end{cases} \end{aligned}$$

∎

Definition 3.3 ([Kra95])**.** For all weights $k \in \mathbb{Z}$ and indices $m \in \frac{\mathbb{Z}}{2}$, the set of integral Jacobi forms is defined to be the global section $\Gamma(E, p^*\omega^k \otimes \mathcal{O}_E(2me))$.

Remark 3.4. A weak Jacobi form of weight $k \in \mathbb{Z}$ and index $m \in \frac{\mathbb{Z}}{2}$ is a two-variable function

$$\phi\colon \mathbb{H} \times \mathbb{C} \to \mathbb{C}$$

such that

- for $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $SL_2(\mathbb{Z})$, ϕ satisfy

$$\phi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = (c\tau+d)^k e^{\frac{2\pi imcz^2}{c\tau+d}}\phi(\tau,z),$$

- for all integers $\lambda, \mu \in \mathbb{Z}$, ϕ satisfy

$$\phi(\tau, z+\lambda\tau+\mu) = e^{-2\pi im(\lambda^2\tau+2\lambda z)}\phi(\tau,z).$$

A weak Jacobi form is called integral if its Fourier coefficients are all integral.

The evaluation at the Tate curve gives rise a map from the global section $\Gamma(E, p^*\omega^k \otimes \mathcal{O}_E(2me))$ to the group of weak integral Jacobi form with integral coefficients. Kramer [Kra95] showed that this map is injective.

The goal of this paper is to fully compute the descent spectral sequence associated with TJF_m:

$$H^p(E; p^*\omega^q \otimes \mathcal{O}_E(me)) \to \pi_{2q-p}\mathrm{TJF}_m.$$

Remark 3.5. In [CDvN24], it is shown that the descent spectral sequence for TMF is isomorphic to the signature spectral sequence of $\nu_{\mathrm{MU}}(\mathrm{TMF})$, where ν_{MU} is the synthetic analogue functor, and they computed the signature spectral sequence in [CDN24]. Moreover, their result is generalized to the quasi-coherent sheaves on $\mathcal{M}_{\mathrm{ell}}^{\mathrm{or}}$ in, and therefore the descent spectral sequence for TJF_n is isomorphic to the signature spectral sequence for $\nu_{\mathrm{MU}}(\mathrm{TJF}_n)$. We will use this identification in the later chapter 7 to 12 to apply the synthetic Leibniz rule in the computation of differentials.

4. E_2-term of DSS

From now on, we implicitly 2-localize all spectra and spectral stacks we handle. Let $\mathcal{X}$ be a spectral Deligne-Mumford stack and $\mathcal{F} \in \mathrm{QCoh}(X)$ be a quasi-coherent sheaf on $\mathcal{X}$. Take an étale cover $\mathcal{U} \to \mathcal{X}$. Then the sheaf condition says that

$$\Gamma(\mathcal{X}; \mathcal{F}) \simeq \mathrm{Tot}(\Gamma(\mathcal{U} \times_{\mathcal{X}} \mathcal{U} \times \cdots \times_{\mathcal{X}} \mathcal{U}; \pi^* \mathcal{F}))$$

where $\pi\colon \mathcal{U} \times_{\mathcal{X}} \mathcal{U} \times \cdots \times_{\mathcal{X}} \mathcal{U} \to \mathcal{X}$ is the projection map. The descent spectral sequence is the Bousfield-Kan spectral sequence associated with this totalization. In particular, its E_2-term is the Čech cohomology

$$\check{H}^p(X; \pi_q \mathcal{F})$$

associated to the étale cover $U \to X$ where U, X are the underlying Deligne-Mumford 1-stack of $\mathcal{U}, \mathcal{X}$, respectively.

In this paper, we take $X = \mathcal{E}$ to be the universal oriented elliptic curve over $\mathcal{M}^{\mathrm{or}}_{\mathrm{ell}}$. Let $q\colon \mathcal{E}' \to \mathcal{M}^{\mathrm{or}}_1(3)$ to be the pullback of $\mathcal{E}$ along the étale cover $\mathcal{M}^{\mathrm{or}}_1(3) \to \mathcal{M}^{\mathrm{or}}_{\mathrm{ell}}$ from the spectral enhancement of the moduli stack of elliptic curves with a $\Gamma_1(3)$-structure. Since $\mathcal{M}^{\mathrm{or}}_1(3) \to \mathcal{M}^{\mathrm{or}}_{\mathrm{ell}}$ is an étale cover, the induced map $\mathcal{E}' \to \mathcal{E}$ is so too.

Recall the identification on the underlying stack

$$\mathcal{M}_1(3) \simeq [\mathrm{Spec}\, A/\mathbb{G}_m]$$

where $A = \mathbb{Z}_2[a_1, a_3, \Delta^{\pm 1}]$, and the $\mathbb{G}_m$ action is given by the degrees $|a_1| = 2, |a_3| = 6$ with $\Delta = a_3^3(a_1^3 - 27a_3)$. Then the underlying elliptic curve E' on $\mathcal{M}_1(3)$ is given by the closed subscheme of weighted projective stack $\mathbb{P}(4,6)$ cut out by the Weierstraß equation

$$Y^2 Z + a_1 XYZ + a_3 YZ^2 = X^3.$$

Lemma 4.1. *The higher cohomology on E'*

$$H^i(E', q^*\omega^k \otimes \mathcal{O}_{E'}(me)) = 0 \ (i > 0)$$

vanishes. Moreover,

$$\bigoplus_{k \in \mathbb{Z}} H^0(E', q^*\omega^k \otimes \mathcal{O}_{E'}(me))$$

is a free A-module of rank m.

Proof. Note that $\mathcal{M}_1(3) \simeq [\mathrm{Spec}\, A/\mathbb{G}_m]$ has cohomolgical dimension zero. By using the Leray spectral sequence [Sta25, Tag 0782], we see that the i-th cohomology of $\mathcal{O}_{E'}(me) \otimes q^*\omega^k$ is isomorphic to the global section of i-th derived functor $R^i q_*(\mathcal{O}_{E'}(me) \otimes q^*\omega^k)$. However, by the projection formula, we have

$$R^i q_*(\mathcal{O}_{E'}(me) \otimes q^*\omega^k) \simeq R^i(q_*\mathcal{O}_{E'}(me) \otimes \omega^k),$$

and therefore the first claim follows from the semicontinuity theorem. Moreover, the global section of $q_*\mathcal{O}_{E'}(me) \otimes \omega^k$ is the degree k part of the global section of the pullback of $q_*\mathcal{O}_{E'}(me)$ to $\mathrm{Spec}\, A$. Note that the sheaf $q_*\mathcal{O}_{E'}(me)$ can be extended to $\mathrm{Spec}\, \mathbb{Z}_{(2)}[a_1, a_3]$, and because it is a polynomial ring over a PID, the global section of $q_*\mathcal{O}_{E'}(me)$ on $\mathrm{Spec}\, A$ turns out to be free, and the second claim follows (or apply the global Horrock's extension theorem). $\square$

Denote the A-module generator of $\bigoplus_k H^0(E', q^*\omega^k \otimes \mathcal{O}_{E'}(2e))$ as 1 and x, and the generator of $H^0(E', q^*\omega^\bullet \otimes \mathcal{O}_{E'}(3e))$ as $1, x$, and y. Degrees are $|x| = 4$ and $|y| = 6$. This x and y satisfy the equation $y^2 + a_1 xy + a_3 y = x^3$, and the monomials $x^i y^j$ of x, y generate $H^0(E', q^*\omega^\bullet \otimes \mathcal{O}_{E'}(me))$ as A-module for each m.

Similarly, let's consider the pullback of $\mathcal{E}$ to the intersection $\mathcal{M}_1(3) \times_{\mathcal{M}^{\mathrm{or}}_{\mathrm{ell}}} \mathcal{M}_1(3)$. Recall the underlying stack of $\mathcal{M}_1(3) \times_{\mathcal{M}^{\mathrm{or}}_{\mathrm{ell}}} \mathcal{M}_1(3)$ is equivalent to $[\mathrm{Spec}\, \Gamma/\mathbb{G}_m]$ where

$$\Gamma \coloneqq A[s,t]/I$$

and I is the ideal in A generated by

$$s^4 - 6st + a_1 s^3 - 3a_1 t - 3a_3 s,$$
$$-27t^2 + 18s^3 t + 18a_1 s^2 t - 27a_3 t - 2s^6 - 3a_1 s^5 + 9a_3 s^3 + a_1^3 s^3 + 9a_1 a_3 s^2.$$

Recall that the pair (A, Γ) forms a Hopf algebroid. The structure map is determined by the automorphism of the elliptic curve $y^2 + a_1 xy + a_3 y = x^3$ by

$$x \mapsto x - \frac{1}{3}(s^2 + a_1 s),$$
$$y \mapsto y - sx + \frac{1}{3}(s^3 + a_1 s^2) - t,$$

so that a_1, a_3 are mapped to

$$a_1 \mapsto a_1 + 2s$$
$$a_3 \mapsto a_3 + \frac{1}{3}(a_1 s^2 + a_1^2 s) + 2t.$$

Combining the argument above, we describe the E_1-term of the descent spectral sequence. Let B_m be the collection of polynomials in $A[x,y]/(y^2 + a_1xy + a_3y - x^3)$ of degree less than or equal to $2m$, where degree of x is 4 and degree of y is 6. Note that B_m is a free A-module of rank m. Then the E_1-term of our descent spectral sequence is the cochain complex given by the cobar complex

$$B_m \rightrightarrows \Gamma \otimes_A B_m \rightrightarrows \Gamma \otimes_A \Gamma \otimes_A B_m \rightrightarrows \cdots$$

where the coaction on x, y, a_1, a_3 are determined by the formula above.

To compute its cohomology, we introduce a filtration in the cobar complex (B_m, Σ_m) by letting $|x| = 4$, $|y| = 6$, and $|a| = 0$ for any elements $a \in A$. We call this a *celluar filtraion.* The cohomology of (B_m, Σ_m) can be calculated from the associated spectral sequence (algebraic Atiyah-Hirzebruch spectral sequence). Its E_1-term is the cohomology of the associated graded object. Because B_m is free of rank n generated by monomials x^iy^j with degree less than m, we see that the resulting E_1-term of algebraic AHSS is the free rank m module over E_2-term of the descent spectral sequence of TMF. We denote E_2-generator of filtration degree n as x_n. For example, x_0 is represented by $1 \in A$, x_4 is represented by $x \in B_2$, and x_6 is represented by $y \in B_3$.

The formula on the coaction of x and the fact that $-(s^2 + a_1s)/3$ represents the element ν in the E_2-term of DSS for TMF (see 7.1 for notations), we see the differential $d_2(x_4) = \nu x_0$ in algebraic AHSS. Similarly, the coaction on y shows the d_1-differential on x_6 is equal to the element represented by

$$-sx + \frac{1}{3}(s^3 + a_1 s^2) - t,$$

and because $s \in \Gamma$ represents $\eta \in \pi_1\mathrm{TMF}$, $d_1(y)$ hits ηx_4. As we can take a generator of the associated graded

$$B_{m+1}/B_m \simeq \begin{cases} A\{x^i\} & m = 2i-1, \\ A\{x^{i-1}y\} & m = 2i, \end{cases}$$

we can similarly calculate the differentials of generators x_n from the Leibniz rule. Noting that $d_2(x_{16}) = 4\nu x^3 = 0$, we obtain the following table of 16-periodic differentials.

	x_4	x_6	x_8	x_{10}	x_{12}	x_{14}	x_{16}	x_{18}
d_1	0	ηx_4	0	ηx_8	0	ηx_{14}	0	ηx_{16}
d_2	νx_0	0	$2\nu x_4$	νx_6	$3\nu x_8$	$2\nu x_{10}$	0	$3\nu x_{14}$

The computation of each E_2-term is completed in the later chapter 7 to 12.

5. DSS for TJF_∞

First, we compute the descent spectral sequence for $\mathrm{TJF}_\infty := \mathrm{colim}_m\,\mathrm{TJF}_m$. It's E_2-term is given by the cohomology of the cobar complex

$$B \to B \otimes_A \Gamma \to B \otimes_A \Gamma \otimes_A \Gamma \cdots$$

where

$$A = \mathbb{Z}_2[a_1, a_3, \Delta^{-1}],$$
$$B = A[x,y]/(y^2 + a_1xy + a_3y - x^3),$$
$$\Gamma = A[s,t]/I$$

with coactions

$$x \mapsto x - \frac{1}{3}(s^2 + a_1 s),$$
$$y \mapsto y - sx + \frac{1}{3}(s^3 + a_1 s^2) - t,$$
$$a_1 \mapsto a_1 + 2s,$$
$$a_3 \mapsto a_3 + \frac{1}{3}(a_1 s^2 + a_1^2 s) + 2t.$$

Remark 5.1. Instead, one can use the fact that the global section of $\mathcal{O}_{\mathcal{E}}(\infty e) = \operatorname{colim}_n \mathcal{O}_{\mathcal{E}}(ne)$ is equivalent to the ring of functions of the punctured oriented elliptic curve, and therefore the descent spectral sequence for TJF_∞ is isomorphic tothe one for the structure sheaf of the punctured oriented elliptic curve.

Denote $\Sigma \coloneqq \Gamma \otimes_A B$. Then (B, Σ) forms a Hopf algebroid, and the cobar complex above comes from (B, Σ). We recall the change-of-rings theorem to compute its cohomology.

Theorem 5.2 ([HS08], [Hov01])**.** *Let (A, Γ) be a Hopf algebroid and $f\colon A \to A'$ a ring map. If there exists a ring R and a ring map $A' \otimes_A \Gamma \to R$ such that the composite*

$$A \xrightarrow{f\otimes\eta_R} A' \otimes_A \Gamma \longrightarrow R$$

is faithfully flat, then the Hopf algebroid map $(A, \Gamma) \to (A', A' \otimes_A \Gamma \otimes_A A')$ induces an equivalence of comodule categories.

Lemma 5.3 ([Bau08], Remark 4.2)**.** *Let (A', Γ') be the Hopf algebroid given by*

$$\begin{aligned} A' &= \mathbb{Z}_{(2)}[a_1, a_3, a_4, a_6, \Delta^{\pm}], \\ \Gamma' &= A'[s, t] \end{aligned}$$

with coactions

$$\begin{aligned} a_1 &\mapsto a_1 + 2s, \\ a_3 &\mapsto a_3 + a_1 r + 2t, \\ a_4 &\mapsto a_4 - a_3 s - a_1 t - a_1 r s - 2st + 3r^2, \\ a_6 &\mapsto a_6 + a_4 r - a_3 t - a_1 r t - t^2 + r^3, \end{aligned}$$

where $r = \frac{1}{3}(s^2 + a_1 s)$. Then the quotient map $A' \to A$ satisfies the condition of theorem 5.2.

Denote a new Hopf algebroid

$$\begin{aligned} B' &= A'[x, y]/(y^2 + a_1 xy + a_3 y - x^3 - a_4 x - a_6) \\ &\simeq \mathbb{Z}_{(2)}[a_1, a_3, a_4, x, y, \Delta^{\pm}] \\ \Sigma' &= B'[s, t] \simeq B' \otimes_{A'} \Gamma'. \end{aligned}$$

with coactions

$$\begin{aligned} x &\mapsto x - \frac{1}{3}(s^2 + a_1 s), \\ y &\mapsto y - sx + \frac{1}{3}(s^3 + a_1 s^2) - t. \end{aligned}$$

Then the theorem 5.2 tells us that the cohomology $H^*(B, \Sigma)$ is isomorphic to $H^*(B', \Sigma')$. We compute $H^*(B', \Sigma')$ with further applications of theorem 5.2.

Lemma 5.4. *Consider the quotient map $q\colon B' \to B'/(y)$. Then the composition*

$$B' \xrightarrow{q\otimes\eta_R} B'/(y) \otimes_{B'} \Sigma'$$

is faithfully flat.

Proof. By definition, the ring map

$$\mathbb{Z}_2[a_1, a_3, a_4, x, y, \Delta^{\pm}] \to \mathbb{Z}_2[a_1, a_3, a_4, x, s, t, \Delta^{\pm}]$$

is determined by sending a_1, a_3, a_4, x to its coactions and

$$y \mapsto -sx + \frac{1}{3}(s^3 + a_1 s^2) - t.$$

Denote

$$\begin{aligned} a_1' &\coloneqq a_1 + 2s \\ a_3' &\coloneqq a_3 + \frac{1}{3}(a_1 s^2 + a_1^2 s) + 2t \\ x' &\coloneqq x - \frac{1}{3}(s^2 + a_1 s). \end{aligned}$$

Then $\mathbb{Z}_{(2)}[a_1, a_3, x, s, t]$ is isomorphic to $\mathbb{Z}_{(2)}[a'_1, a'_3, x', s, t]$ and the ring map we consider is determined by

$$\begin{aligned} a_1 &\mapsto a'_1, \\ a_3 &\mapsto a'_3, \\ x &\mapsto x', \\ y &\mapsto -sx' - t. \end{aligned}$$

The claim follows because $-sx' - t$ is monic in t. □

Remark 5.5. Note that the ring B classifies an elliptic curve and a non-unital point on it. With a suitable choice of t, we can move any point to a point whose y-coordinate is 0 via an automorphism of the elliptic curve. Therefore the stack represented by (B, Σ) must be equivalent to $(B/(y), (y)\backslash\Sigma/(y))$.

Therefore, the cohomology we want comes from the Hopf algebroid

$$\begin{aligned} B^{(2)} &:= B'/(y) = \mathbb{Z}_{(2)}[a_1, a_3, a_4, x, \Delta^{\pm}], \\ \Sigma^{(2)} &:= B^{(2)} \otimes_B \Sigma' \otimes_B B^{(2)} = B^{(2)}[s,t]/(-sx + sr - t) \simeq B^{(2)}[s] \end{aligned}$$

with coactions

$$\begin{aligned} a_1 &\mapsto a_1 + 2s, \\ a_3 &\mapsto a_3 - 2sx + \frac{1}{3}(a_1 + 2s)(s^2 + a_1 s), \\ a_4 &\mapsto a_4 - a_3 s - a_1(sr - sx) - a_1 rs - 2s(sr - sx) + 3r^2, \\ x &\mapsto x - \frac{1}{3}(s^2 + a_1 s). \end{aligned}$$

We can simplify the Hopf algebroid with further application of the change-of-rings theorem.

Lemma 5.6. *Consider the quotient map $q\colon B^{(2)} \to B^{(2)}/(x)$. Then the composition*

$$B^{(2)} \xrightarrow{q \otimes \eta_R} B^{(2)}/(x) \otimes_{B'} \Sigma^{(2)}$$

is faithfully flat.

Proof. We can apply a similar argument above, noting that $-(s^2 + a_1 s)/3$ is a monic polynomial in s. □

Therefore the cohomology of (B, Σ) is isomorphic to $(B^{(3)}, \Sigma^{(3)})$, where

$$\begin{aligned} B^{(3)} &:= B^{(2)}/(x) = \mathbb{Z}_{(2)}[a_1, a_3, a_4, \Delta^{\pm}], \\ \Sigma^{(3)} &:= B^{(3)} \otimes_{B^{(2)}} \Sigma^{(2)} \otimes_{B^{(2)}} B^{(3)} = B^{(3)}[s]/(s^2 + a_1 s), \end{aligned}$$

with coactions

$$\begin{aligned} a_1 &\mapsto a_1 + 2s, \\ a_3 &\mapsto a_3, \\ a_4 &\mapsto a_4 - a_3 s. \end{aligned}$$

We use the algebraic Novikov spectral sequence associated with an invariant ideal to compute its cohomology. Note that $I_0 := (2), I_1 := (2, a_1), I_2 := (2, a_1, a_3)$ are invariant ideals in $(B^{(3)}, \Sigma^{(3)})$. Denote the quotient Hopf algebra $(B^{(3)}/I_i, I_i\backslash\Sigma^{(3)}/I_i)$ as (B_i, Σ_i) for $i = 0, 1, 2$.

Note that $(B_2, \Sigma_2) = (\mathbb{F}_2[a_4], B_2[s]/(s^2))$ with the trivial coaction. Therefore we see that $H^*(B_2, \Sigma_2) \simeq \mathbb{F}_2[\eta, a_4]$ where $\eta \in H^1$ is the element represented by s. The coactions says that a_3, a_4^2 are the permanent cycles, and a_4 hits $a_3\eta$. From the algebraic Novikov spectral sequence, we obtain

$$H^*(B_1, \Sigma_1) \simeq \mathbb{F}_2[\eta, a_3, a_4^2]/(a_3\eta).$$

Furthermore, because the coaction of a_1 is trivial mod 2 and $a_4^2 + a_1 a_3 a_4$ is a cocycle, we have

$$H^*(B_0, \Sigma_0) \simeq \mathbb{F}_2[\eta, a_1, a_3, a_4^2 + a_1 a_3 a_4]/(a_3\eta).$$

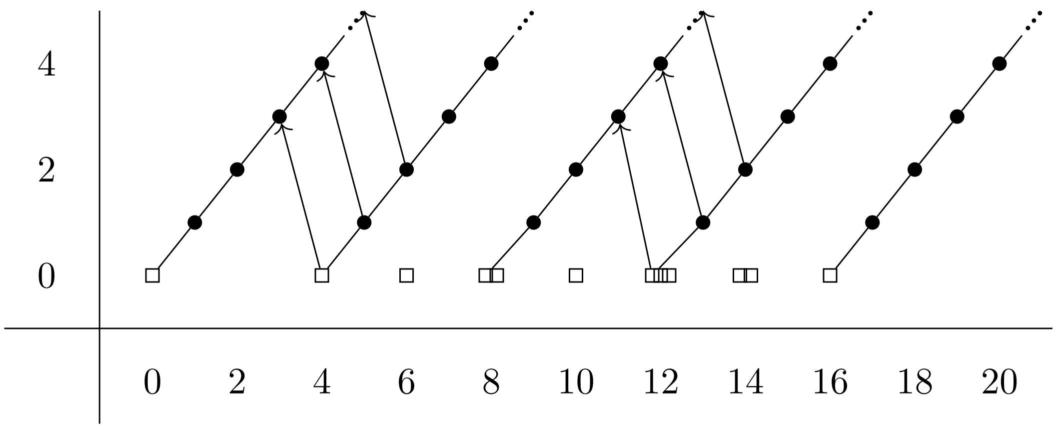

Figure 1. The E_3-page of DSS for TJF_∞ and differentials

Now we run the 2-Bockstein spectral sequence. Differentials are determined by $a_1 \mapsto 2s$. Note that a_1^2 and $a_1a_3 + 2a_4$ are cocycles. Because η times $a_4^2 + a_1a_3a_4$ is nonzero and η is 2-torsion, a_1 times $a_4^2 + a_1a_3a_4$ must hit $2\eta \cdot (a_4^2 + a_1a_3a_4)$. Therefore we conclude

$$H^*(B, \Sigma) = \mathbb{Z}_{(2)}[\eta, a_1^2, a_3, a_1a_3 + 2a_4, a_4^2 + a_1a_3a_4]/(2\eta, a_3\eta).$$

Recall that the ring of meromorphic integral modular forms MF_* is isomorphic to $\mathbb{Z}[c_4, c_6, \Delta^{\pm}]/(c_4^3 - c_6^2 - 1728\Delta)$, where c_4, c_6, and Δ is given by the equation

$$\begin{aligned} c_4 &:= a_1^4 - 24a_1a_3 - 48a_4, \\ c_6 &:= a_1^6 - 36a_1^3a_3 - 72a_1^2a_4 + 216a_3^2, \\ \Delta &:= a_1^5a_3a_4 + a_1^3a_3^3 + a_1^4a_4^2 - 30a_1^2a_3^2a_4 - 27a_3^4 - 96a_1a_3a_4^2 - 64a_4^3. \end{aligned}$$

Therefore, as a module over MF_*, the E_2-term of DSS for TJF_∞ can be written as

$$\mathrm{MF}_*[\eta, b, c, d, e]/(2\eta, c\eta, d\eta, 4e - d^2 + bc^2, c_4^3 - c_6^2 - 1728\Delta, b^2 - 24d - c_4, b^3 - 36bd + 216c^2 - c_6).$$

The remaining task is to work out the higher differentials of the descent spectral sequence for TJF_∞. Because $\eta \in H^1$ is in the image of the unit map $\mathrm{TMF} \to \mathrm{TJF}_\infty$, it must satisfy the relation $\eta^4 = 0 \in \pi_*\mathrm{TJF}_\infty$. The only class that can hit η^4 is $b\eta$, so we obtain

$$d_3(b) = \eta^3.$$

Due to degree reasons, there are no other possible differentials, and we finish the computation of $\pi_*\mathrm{TJF}_\infty$. A part of the spectral sequence is drawn in figure 1.

Remark 5.7. Although the homotopy groups of TJF_∞ only have KO-like elements and free parts, we do *not* have a spectrum decomposition of TJF_∞ into the sum of a shifted copy of KO and KU. In fact, TJF_∞ is not even $K(2)$-acyclic. One way to see this is the following: consider the inclusion $\mathcal{E}[3] \to \mathcal{E}$. Removing the basepoint of those schemes and taking the global section, we obtain a ring morphism $\mathrm{TJF}_\infty \to \mathrm{TMF}_1(3)$ by to Meier's decomposition of TMF^{C_3} [Mei18], and the latter is not $K(2)$-acyclic.

6. Transfers and Cell Structures of TJF

We make the following observation to compute the higher differentials of TJF_m. This section's main goal is to prove the theorem 6.3.

Definition 6.1. Let $m \geq 1$ be an integer, $\mathrm{tr}\colon \Sigma\Sigma_+^\infty \mathbb{C}P^\infty \to S^0$ be the circle-equivariant transfer map, and $q\colon \mathbb{C}P_+^\infty \to S^0$ be the collapsing map. Define a finite spectrum P_m to be the cofiber of the map

$$\mathrm{tr} \oplus \Sigma q\colon \Sigma\mathbb{C}P_+^{m-1} \to S^0 \oplus S^1$$

Remark 6.2. Recall that [Mil82, Lemma 2.7] the cofiber of the circle-equivariant transfer map is equivalent to the double suspension of the stunted projective space $\Sigma^2\mathbb{C}P_{-1}^\infty$. Therefore we obtain the following commutative diagram of cofiber sequences.

$$\begin{tikzcd}
\Sigma\mathbb{C}P^{m-1}_+ \arrow[r] \arrow[d, equal] & S^0 \oplus S^1 \arrow[r] \arrow[d] & P_m \arrow[d] \\
\Sigma\mathbb{C}P^{m-1}_+ \arrow[r] \arrow[d] & S^0 \arrow[r] \arrow[d] & \Sigma^2\mathbb{C}P^{m-1}_{-1} \arrow[d] \\
0 \arrow[r] & S^2 \arrow[r, equal] & S^2
\end{tikzcd}$$

The right vertical sequence exibits P_m as "$\Sigma^2\mathbb{C}P^{m-1}_{-1}$ with the 2-cell removed". In particular, P_m has one cell in every even dimension except 2, and P_2 is equivalent to the cofiber of the Hopf fibration $C\nu = \mathbb{H}P^2$.

Theorem 6.3. *For each integer $m \geq 1$, we have an equivalence of* TMF*-module*

$$\mathrm{TJF}_m \simeq \mathrm{TMF} \otimes P_m.$$

This theorem is heavily used in the computation of the descent spectral sequence because the spectral sequence for $\mathrm{tmf} \otimes P_m$ has much less possibility of differentials, and we obtain the one for TJF_m by inverting the polynomial generator $\Delta^8 \in \pi_{192}\mathrm{tmf}$.

To prove the theorem, we recall one observation in spectral algebraic geometry.

Lemma 6.4. *The dualizing sheaf $\omega_{C/S}$ for an oriented elliptic curve $p\colon C \to S$ is isomorphic to $S^{-1} \otimes \mathcal{O}_C$. For a quasi-coherent sheaf $\mathcal{F} \in \mathrm{QCoh}(C)$, the global section of the dual sheaf $\Gamma(C, \mathcal{F}^\vee)$ is isomorphic to the degree-shifted* TMF*-dual $\Sigma D(\Gamma(C, \mathcal{F}))$.*

Proof. The first assertion follows from Example 6.4.2.9 of [Lur18c] and Remark 5.2.5 of [Lur18a].The second statement follows from the observation

$$\begin{aligned}
\mathrm{Hom}_S(p_*\mathcal{F}, \mathcal{O}_S) &\simeq \mathrm{Hom}_C(\mathcal{F}, \omega_{C/S}) \\
&\simeq \mathrm{Hom}_C(\mathcal{F}, S^{-1} \otimes \mathcal{O}_C) \\
&\simeq S^{-1} \otimes \mathrm{Hom}_C(\mathcal{F}, \mathcal{O}_C) \\
&\simeq S^{-1} \otimes \Gamma(C, \mathcal{F}^\vee).
\end{aligned}$$

□

Consider the sequence of pointed spaces

$$S(m\rho)_+ \to S^0 \to S^{m\rho}.$$

where $S(m\rho)$ is the unit sphere in the representation $m\rho$. Applying $\widetilde{\mathcal{E}ll}_{\mathbb{T}}$, we obtain a fiber sequence

$$\widetilde{\mathcal{E}ll}_{\mathbb{T}}(S(m\rho)_+) \leftarrow \mathcal{O}_C \leftarrow \mathcal{O}_C(-me)$$

in $\mathrm{QCoh}(C)$. Since the group $\mathbb{T}$ acts freely on $S(m\rho)$, we have

$$\widetilde{\mathcal{E}ll}_{\mathbb{T}}(S(m\rho)_+) \simeq e_*(\mathcal{O}_S^{\mathbb{C}P^{m-1}})$$

by (3) of theorem 2.10. Taking the dual sheaf of this sequence and global section, one obtains the cofiber sequence

$$\mathrm{TMF} \otimes \Sigma\mathbb{C}P^{m-1} \to \mathrm{TMF}_{\mathbb{T}} \to \mathrm{TJF}_m.$$

of TMF-modules. Note that $\Sigma\mathrm{TMF} \hookrightarrow \mathrm{TMF} \otimes \Sigma\mathbb{C}P^{m-1} \to \mathrm{TMF}_{\mathbb{T}}$ is the transfer map defined in [GM20]. They showed that the cofiber of the transfer map $\Sigma\mathrm{TMF} \to \mathrm{TMF}_{\mathbb{T}}$ is equivalent to TMF. In particular, TJF_m is the cofiber of the map

$$\mathrm{TMF} \otimes \Sigma\mathbb{C}P^{m-1}_1 \to \mathrm{TMF} \to \mathrm{TJF}_m.$$

We induct on m using the cofiber sequence above to prove the theorem. First, we know that $\mathrm{TJF}_1 = \Gamma(C, \mathcal{O}_C(e)) \simeq \mathrm{TMF}$. (Maybe cite GM here). Notice the following commutative diagram:

$$\begin{tikzcd}
\mathrm{TMF} \otimes \Sigma S^{2m-1} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \mathrm{TMF} \otimes \Sigma^2 S^{2m-1} \arrow[d] \\
\mathrm{TMF} \otimes \Sigma \mathbb{C}P_1^{m-1} \arrow[r] \arrow[d] & \mathrm{TMF} \arrow[r] \arrow[d, "\mathrm{id}"'] & \mathrm{TJF}_m \arrow[d] \\
\mathrm{TMF} \otimes \Sigma \mathbb{C}P_1^{m} \arrow[r] & \mathrm{TMF} \arrow[r] & \mathrm{TJF}_{m+1}
\end{tikzcd}$$

where both rows and columns are cofiber sequences. The right vertical sequence shows that TJF_{m+1} is obtained by attaching the S^{2m+2}-cell to $\mathrm{TJF}_m \simeq \mathrm{TMF} \otimes P_m$. We analyze this attaching from the cobar complex described in section 4.

Let's strat the base case $m = 1$. The E_2-term of the descent spectral sequence for TJF_2 is the cohomology of the Hopf algebroid $(B_2, B_2 \otimes_A \Gamma)$, and B_2 is the free A-module of rank 2 generated by 1 and x. The coaction on x tells us that the d_1-differential on x is given by

$$d_1(x) = -\frac{1}{3}(s^2 + a_1 s),$$

and $-\frac{1}{3}(s^2 + a_1 s)$ represents ν in the E_2-page of DSS for TMF. Therefore $\nu \in \mathrm{TMF}$ becomes zero in TJF_2, and this fact forces the map $\mathrm{TMF} \otimes S^3 \to \mathrm{TJF}_1 = \mathrm{TMF}$ to be equal to $\nu \in [\Sigma^3 \mathrm{TMF}, \mathrm{TMF}]_{\mathrm{TMF}} \simeq \pi_3 \mathrm{TMF}$. In particular, TJF_2 is equivalent to the cofiber $\mathrm{TMF} \otimes C\nu$.

We proceed the induction as follows. Assume the equivalence $\mathrm{TJF}_n \simeq \mathrm{TMF} \otimes P_m$, and consider the following diagram:

$$\begin{tikzcd}
\mathrm{TMF} \otimes \Sigma^2 S^{2m-1} \arrow[r] \arrow[d, equal] & \mathrm{TMF} \otimes P_m \arrow[r] \arrow[d] & \mathrm{TMF} \otimes P_{m+1} \\
\mathrm{TMF} \otimes \Sigma^2 S^{2m-1} \arrow[r] \arrow[d, equal] & \mathrm{TJF}_m \arrow[r] \arrow[d] & \mathrm{TJF}_{m+1} \arrow[d] \\
\mathrm{TMF} \otimes \Sigma^2 S^{2m-1} \arrow[r] & \mathrm{TMF} \otimes \Sigma^2 \mathbb{C}P_1^{m-1} \arrow[r] & \mathrm{TMF} \otimes \Sigma^2 \mathbb{C}P_1^{m}.
\end{tikzcd}$$

The bottom squares are commutative, and the composition of the middle vertical map is given by the collapse of the bottom 0-cell. It suffices to show that the top left square commutes to finish the induction. Because the two maps coincide after the composition $\mathrm{TJF}_m \to \mathrm{TMF} \otimes \Sigma^2 \mathbb{C}P_1^{m-1}$, we can check the commutativity by the diagram chase from the below.

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \mathrm{TMF} \arrow[r, equal] \arrow[d] & \mathrm{TMF} \arrow[d] \\
\mathrm{TMF} \otimes \Sigma^2 S^{2m-1} \arrow[r] \arrow[d, equal] & \mathrm{TMF} \otimes P_m \arrow[r] \arrow[d] & \mathrm{TMF} \otimes P_{m+1} \arrow[d] \\
\mathrm{TMF} \otimes \Sigma^2 S^{2m-1} \arrow[r] & \mathrm{TMF} \otimes \Sigma^2 \mathbb{C}P_1^{m-1} \arrow[r] & \mathrm{TMF} \otimes \Sigma^2 \mathbb{C}P_1^{m}
\end{tikzcd}$$

As a consequence of the cell decomposition, we can deduce a horizontal vanishing line of DSS for TJF_n.

Lemma 6.5. *(see also* [BBPX22] *and* [Chu22]*) The descent spectral sequence for* TJF_n *has the horizontal vanishing line at* $s = 24$ *in* E_{24}*-page.*

Proof. Denote I be the kernel of the map $\mathrm{TMF} \to \mathrm{TMF}_1(3)$. The descent spectral sequence for TMF has a horizontal vanishing line at $s = 24$ in E_{24}-page [Bau08], and therefore the map $I^{\otimes 24} \to \mathrm{TMF}$ is phantom. As TMF is the limit of even periodic $\mathbb{E}_\infty$-rings, any phantom map to TMF is zero. Therefore the map $I^{\otimes 24} \to \mathrm{TMF}$ is zero too.

The argument in this section identifies the DSS for TJF_n with the Adams-Novikov spectral sequence of $\mathrm{TMF} \otimes P_n$ by the map $\mathrm{TMF} \to \mathrm{TMF}_1(3)$. As we know the map $I^{\otimes 24} \otimes P_n \to \mathrm{TMF} \otimes P_n$ is zero, we have the stated vanishing line from [Mat17, Proposition 2.29]. $\square$

7. TJF_2

7.1. **E_2-term.** The rest of our task is to determine the higher differentials and the E_∞-page of the descent spectral sequence for TJF_m. Let's draw its E_2-page. The E_2-term of the descent spectral sequence for TMF is computed in [Bau08] (see the figure 2).

Theorem 7.1 ([Beh19])**.** *As a ring, the E_2-page of the descent spectral sequence for* TMF *is isomorphic to*

$$E_2^{*,*} \simeq \mathbb{Z}_{(2)}[c_4, c_6, \Delta^{\pm 1}, \eta, \nu, \delta, \epsilon, \kappa, \overline{\kappa}]/I$$

where I is the ideal generated by

$$\begin{gathered} 2\eta, \eta\nu, 4\nu, 2\nu^2, \nu^3 - \eta\epsilon, \\ 2\epsilon, \nu\epsilon, \epsilon^2, 2\delta, \nu\delta, \epsilon\delta, \delta^2 = c_4\eta^2, \\ 2\kappa, \eta^2\kappa, \nu^2\kappa - 4\overline{\kappa}, \epsilon\kappa, \kappa^2, \kappa\delta, \\ \nu c_4, \nu c_6, \epsilon c_4, \epsilon c_6, \delta c_4 - \eta c_6, \delta c_6 - \eta c_4^2, \\ \kappa c_4, \kappa c_6, \overline{\kappa} c_4 - \eta^4\Delta, \overline{\kappa} c_6 - \eta^3\delta\Delta, c_6^2 - c_4^3 + 1728\Delta \end{gathered}$$

with bidegrees

$$\begin{array}{lll} |c_4| = (0,8), & |c_6| = (0,12), & |\Delta| = (0,24), \\ |\eta| = (1,2), & |\nu| = (1,4), & |\delta| = (1,6), \\ |\epsilon| = (2,10), & |\kappa| = (2,16), & |\overline{\kappa}| = (4,24). \end{array}$$

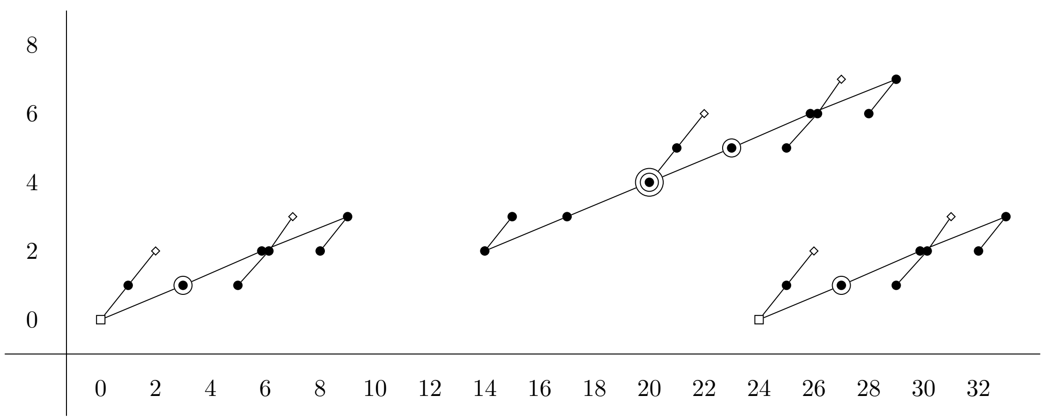

FIGURE 2. The E_2-page of DSS for the 2-local TMF

The argument in the section 6 says that the E_1-term of algebraic AHSS is the two-copy of the E_2-term of TMF generated by elements x_0 in bidegree $(0,0)$ and x_4 in bidegree $(4,0)$. The d_1-differentials are determined by $x_4 \mapsto \nu \cdot x_0$.

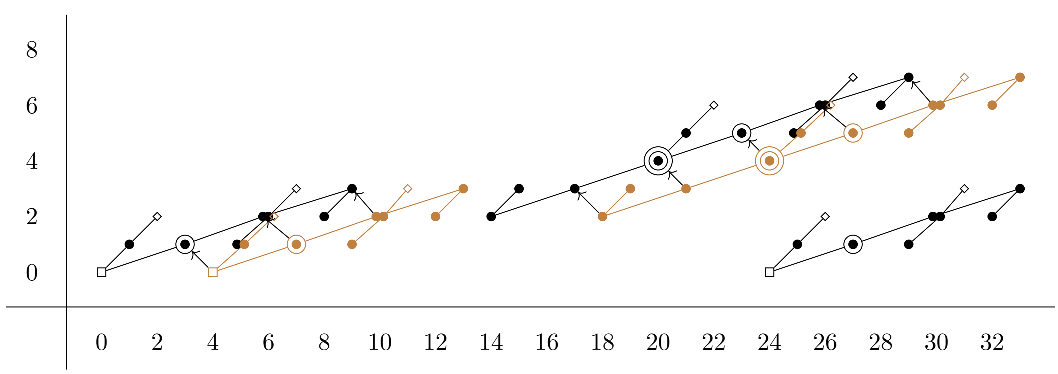

FIGURE 3. E_1-term of algebraic AHSS for TJF_2 and differentials

This yields the following E_2-page of DSS (figure 4). Note that E_2-page is Δ and $\overline{\kappa}$-linear.
We note the following multiplicative extensions.

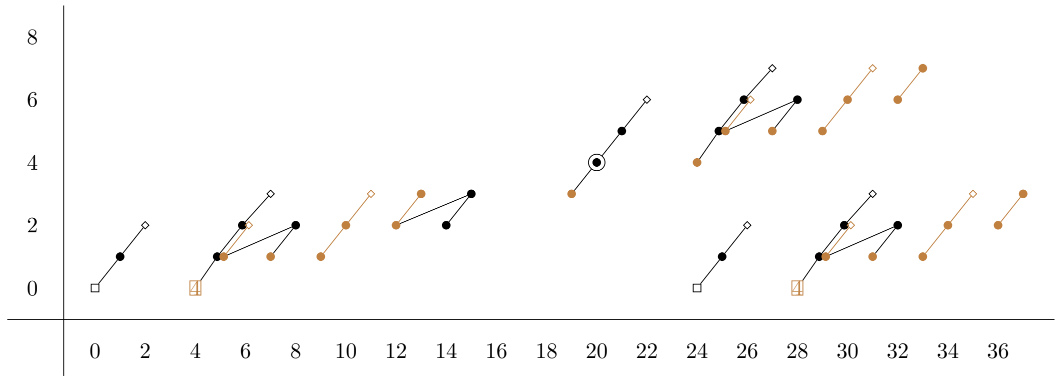

FIGURE 4. The E_2-page of DSS for TJF_2

Lemma 7.2. *There are the following multiplicative extensions in the E_2-term.*

(1) $\eta \cdot [4x] = [\delta x_0]$,
(2) $\nu \cdot [\eta x_4] = [\epsilon x_0]$,
(3) $\eta \cdot [2\nu x_4] = [\epsilon x_0]$,
(4) $\nu \cdot [\epsilon x_4] = [\eta\kappa x_0]$,
(5) $\eta\kappa \cdot [\eta x_4] = [2\overline{\kappa} x_0]$.

Proof. We use Massey brackets in the E_2-term and shuffling lemma:

(1) $\eta \cdot [4x_4] = \eta\langle 4, \nu, x_0\rangle = \langle \eta, 4, \nu\rangle x_0 = [\delta x_0]$,
(2) $\nu \cdot [\eta x_4] = \nu\langle \eta, \nu, x_0\rangle = \langle \nu, \eta, \nu\rangle x_0 = [\epsilon x_0]$,
(3) $\eta \cdot [2\nu x_4] = \eta\langle 2\nu, \nu, x_0\rangle = \langle \eta, 2\nu, \nu\rangle x_0 = [\epsilon x_0]$,
(4) $\nu \cdot [\epsilon x_4] = \nu\langle \epsilon, \nu, x_0\rangle = \langle \nu, \epsilon, \nu\rangle x_0 = [\eta\kappa x_0]$,
(5) $\eta\kappa \cdot [\eta x_4] = \eta\kappa\langle \eta, \nu, x_0\rangle = \langle \eta\kappa, \eta, \nu\rangle x_0 = [2\overline{\kappa} x_0]$.

□

7.2. **Higher Differentials.** The map $\mathrm{TMF} \to \mathrm{TJF}_2$ induces the d_3-differential $d_3([2\nu x_0]) = \eta^4 x_0$, and therefore η-linearity yields that $d_3([4x_4]) = \eta^3$. One can check that there are no other possibilities of d_3, so we obtain the E_4-page as in Figure 7.

We determine all higher differentials.

Remark 7.3. Since any element above the 3-line is divisible by $\overline{\kappa}$, it suffices to look at elements with cohomological degrees equal or less than 3 when computing the higher differential d_r. We use the similar observation for computaiton of differentials in TJF_n.

Also, for bidegree reasons, differentials d_r happen only when r is odd. Therefore, it suffices to determine $d_5, d_7, \cdots, d_{23}$, and we complete this task below.

Lemma 7.4. *The d_5-differentials are linear for Δ^2. Differentials are determined by*

(1) $d_5([\Delta\eta x]) = [\epsilon\overline{\kappa} x_0]$,
(2) $d_5([\Delta\epsilon x]) = [\eta\kappa\overline{\kappa} x_0]$,

and linearity.

Proof. Recall that $d_5(\Delta) = \nu\overline{\kappa}$ holds in the DSS for TMF. Take an element $w \in E_4^{*,*}$. The Leibnitz rule

$$d_5(\Delta w) = \nu\overline{\kappa} w + \Delta d_5(w)$$

proves the claimed differentials. Moreover, we can verify the Δ^2-linearity by

$$d_5(\Delta^2 w) = 2\Delta\nu\overline{\kappa} w + \Delta^2 d_5(w) = \Delta^2 d_5(w)$$

because ν-multiples are only 2-torsions in the E_2-page. We can see that there are no other possible differentials by sparsity. □

Lemma 7.5. *The d_7-differential is Δ^4-linear, and are determined by*

(1) $d_7([\Delta^{2i+1}\eta\kappa x_4]) = [\Delta^{2i}\eta^2\overline{\kappa}^2 x_0]$,
(2) $d_7([\Delta^{2i}\eta\kappa x_4]) = 0$,

Proof. Since $d_7(\Delta^4) = \Delta^3\eta^3\overline{\kappa} = 4\Delta^3\nu\overline{\kappa}$ in TMF, the d_7-differential for TJF_2 is Δ^4-linear by the same reason for d_5-case. Also, $\eta^2\overline{\kappa}^2 \in \pi_*\mathrm{TMF}$ is divisible by ν by an exotic extension, so $d_7([\Delta\eta\kappa x_4])$ must be nontrivial to kill $[\eta^2\overline{\kappa}^2 x_0]$.

However, the second differential can be calculated as

$$d_7([\Delta^2\eta\kappa x_4]) = \kappa d_7([\Delta^2\eta x_4]) = \kappa \cdot 0 = 0.$$

To show the third differential, we show $d_7(\Delta^7\eta\kappa x_4)$ must be nonzero and apply the Δ^4-linearity. If $d_7(\Delta^7\eta\kappa x_4) = 0$, then the element $\eta^2\overline{\kappa}^2\Delta^6 x_0$ must support some nontrivial differential because $\pi_{186}\mathrm{TMF}\otimes C\nu$ does not have any c_4-torsion elements. The only possibility is $d_{17}(\eta^2\overline{\kappa}^2\Delta^6 x_0) = \overline{\kappa}^5\Delta^3\eta\epsilon x$, but we will independently show that $d_9(\overline{\kappa}^5\Delta^3\eta\epsilon x_4) = \overline{\kappa}^5 d_9(\Delta^3\eta\epsilon x_4) \neq 0$ below.

The first claim follows from the d_5-differentials and synthetic Leibniz rule.

$$\begin{aligned}\delta_4^8([\Delta^{2i+1}\eta\kappa x_4]) &= \kappa\delta_4^8([\Delta^{2i+1}\eta x_4]) \\ &= \kappa[\Delta^{2i}\epsilon\overline{\kappa}x_0] \\ &= [\Delta^{2i}\eta^2\overline{\kappa}^2 x_0].\end{aligned}$$

□

Remark 7.6. Recall that $w^2 = [\Delta^2\eta\overline{\kappa}^2]$ is a permanent cycle in the DSS for TMF. Its image in E_∞-page of DSS for TJF_2 is zero, but image in $\pi_*\mathrm{TJF}_2$ is nonzero. It maps to the element with higher Adams filtration.

Lemma 7.7. *Suppose that $p_*(a)$ persists to $E_{r'}$-term for some $r' \geq r$ and there is a nontrivial differential $d_{r'}(p_*(a)) \neq 0$. Then there is a nontrivial differential $d_{r''}(a) \neq 0$ for some $r'' \leq r'$.*

Lemma 7.8. *The d_9-differentials are determined by the following differentials with $i = 0, 1$:*

(1) $d_9(\Delta^{2+4i}x_0) = \Delta^{4i}\overline{\kappa}^2 \cdot 2\nu x_4$,
(2) $d_9(\Delta^{3+4i}x_0) = \Delta^{4i+1}\overline{\kappa}^2 \cdot 2\nu x_4$,
(3) $d_9(\eta\Delta^2 x_4) = \overline{\kappa}^2\epsilon x_4$.

Proof. We prove the claim for $i = 0$. The same argument applies to the $i = 1$ case. The first differential follows from the fact that

$$\pi_{47}\mathrm{TMF}\otimes C\nu = 0.$$

The second one is the image of d_9-differential of TMF and η-linearity. In $\nu_{\mathrm{MU}}(\mathrm{TJF}_2)/\tau^{12}$, we compute

$$\begin{aligned}\tau^8\overline{\kappa}^2[2\nu x_4] &= \tau^8\overline{\kappa}^2\langle 2\nu, \nu, x_0\rangle \\ &= \tau^4\overline{\kappa}\langle \tau^4\overline{\kappa}, 2\nu, \nu\rangle x_0 \\ &= \tau^4\overline{\kappa}[2\nu\Delta]x_0 \\ &= 0.\end{aligned}$$

where the last equality is proven in [CDN24, Corollary 6.36]. Therefore $[\overline{\kappa}^2 2\nu x_4]$ must be a target of a d_9-differential, and $[\Delta^2 x_0]$ is the only possible domain. $d_9(\eta\Delta^2 x)$ is the consequence of lemma 7.7. □

Lemma 7.9. *The d_{11}-differentials are determined by the following with $i = 0, 1$:*

(1) $d_{11}([2\Delta^{2+4i}x_0]) = [\Delta^{4i}\overline{\kappa}^2\eta^3 x]$
(2) $d_{11}(\kappa\Delta^{2+4i}x_0) = \Delta^{4i}\eta\overline{\kappa}^3 x_0$
(3) $d_{11}(2\Delta^{3+4i}x_0) = \Delta^{1+4i}\overline{\kappa}^2\eta^3 x$
(4) $d_{11}(\kappa\Delta^{3+4i}x_0) = \Delta^{1+4i}\eta\overline{\kappa}^3 x_0$
(5) $d_{11}(\Delta^4 x_0) = \Delta^2\overline{\kappa}^2\eta^3 x_4$
(6) $d_{11}(\Delta^5 x_0) = \Delta^3\overline{\kappa}^2\eta^3 x_4$

Proof. The second and fourth differentials are the image of the differential $\mathrm{TMF} \to \mathrm{TJF}_2$. Every other differential follows from the following observation that

$$\pi_{24i-1}\mathrm{TMF}\otimes C\nu = 0, \ i = 2, 3, 4, 5, 6, 7.$$

The first and third differentials are the consequence of the synthetic Leibniz rule. (4) and (5) can be deduced similarly. □

Lemma 7.10. *There are no d_{13}, d_{15}, d_{17}-differentials.*

Proof. This follows from the degree reason. □

Lemma 7.11. *The d_{19}-differential is determined by*

(1) $d_{19}(\eta\Delta^4 x_4) = \overline{\kappa}^5 x_0$,
(2) $d_{19}(\Delta^7\eta^3 x) = \Delta^3\overline{\kappa}^5\eta^2 x_0$.

Proof. One way to see this is that $\overline{\kappa}^5$ is divisible by ν by an exotic extension in TMF. The second one comes from the observation that

$$\pi_{175}\mathrm{TMF} \otimes C\nu = 0.$$

Both differentials can be deduced from the lemma 6.5. Both targets must be hit by some differential because of the vanishing line, and there is only one possibility to accomplish them. □

Lemma 7.12. *The d_{23}-differential is determined by the following:*

(1) $d_{23}(\Delta^5\eta^2 x) = \overline{\kappa}^6\eta x$,
(2) $d_{23}(\Delta^6\eta x_0) = \overline{\kappa}^6\Delta x_0$,
(3) $d_{23}(\Delta^7\eta^2 x_0) = \overline{\kappa}^6\eta\Delta x_0$.

Proof. The first differential is the consequence of Lemma 7.7. The second and third follow from Lemma 6.5. □

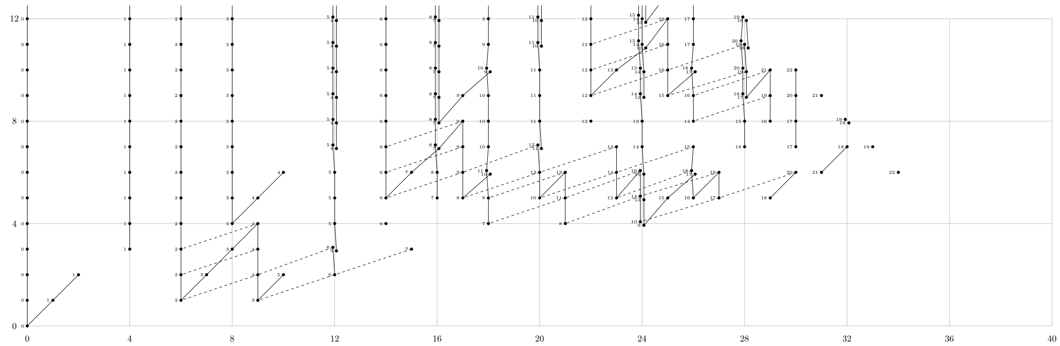

FIGURE 5. E_2-term of ASS for TJF_3

8. TJF_3

8.1. E_2**-term.** The E_1-term of algebraic AHSS is generated by x_0, x_4, x_6, and d_1-differential is determined by $d_1(x_6) = \eta x_4$. The resulting E_2-page is shown in figure 11. We record multiplicative extensions in the E_2-page below.

Proposition 8.1. *There are the following multiplicative extensions in the E_2-term:*
(1) $\eta \cdot [2x_6] = [2\nu x_4]$,
(2) $2 \cdot [2\nu x_6] = [\delta x_4]$,
(3) $\nu \cdot [2\nu x_6] = 2 \cdot [\nu^2 x_6] = [\epsilon x_4]$,
(4) $\eta \cdot [\nu\kappa x_6] = \overline{\kappa} \cdot [4x_4] = [\nu^2 \kappa x_4]$.

Proof. The first two extensions can be verified from the comparison with the Adams spectral sequence for $\mathrm{tmf} \otimes P_2$. Follows from the product formula.

- For the third one, it suffices to show
$$\nu \cdot (2[\nu^2 x_6]) = \nu \cdot [\epsilon x_4] = [\eta\kappa x_0].$$
Note the differential $d_2([\nu x_6]) = [\epsilon x_0]$ in the algebraic AHSS. This leads to the Massey product shuffling
$$\begin{aligned} 2\nu \cdot [\nu^2 x_6] &= 2\nu \langle \nu, \epsilon, x_0 \rangle \\ &= \langle 2\nu, \nu, \epsilon \rangle x_0 \\ &= [\eta\kappa x_0]. \end{aligned}$$
- One way to show this is again a comparison with ASS: there is a d_3-differential that hits the $\mathbb{Z}/2$ class in bidegree $(22, 8)$, hence we see that $\pi_{22}\mathrm{TJF}_3$ is torsion free. Therefore $[\nu^2\kappa x_4]$ must be hit by a differential, and the only possibility is $d_3([\nu\kappa x_6]) = [\nu^2\kappa x_4]$. Then the multiplicative extension follows from the η-linearity of d_3-differentials.

□

8.2. **Higher Differentials.** We determine all differentials d_r, $r \geq 5$. Differentials are depicted in Figure 12, 13, 14, and 15.

Lemma 8.2. *d_5-differentials are determined by the following:*
(1) $d_5([2\Delta^{1+2i} x_6]) = \nu[2\Delta^{2i}\overline{\kappa} x_6]$,
(2) $d_5([\Delta^{1+2i}\nu^2 x_6]) = [\Delta^{2i}\overline{\kappa}\nu^3 x_6]$,
(3) $d_5([2\Delta^{2+4i} x_6]) = 2\nu[\Delta^{1+4i}\overline{\kappa} x_6] = [\Delta^{1+4i}\overline{\kappa}\delta x_4]$,
(4) $d_5([\nu^2\Delta^{2+4i} x_6]) = [\Delta^{1+4i}\eta\kappa x_0]$.

Proof. Follows from $d_5(\Delta) = \nu\overline{\kappa}$ and the Leibniz rule. □

Lemma 8.3. *There are no d_7-differentials.*

Proof. This is a consequence of the degree reason. □

Lemma 8.4. *d_9-differentials are determined by the following:*
(1) $d_9([\Delta^2 x_0]) = [\overline{\kappa}^2 \nu x_4]$,

(2) $d_9([\Delta^2\nu x_4]) = [\overline{\kappa}^2\kappa x_0]$
(3) $d_9([\Delta^3 x_0]) = [\Delta\overline{\kappa}^2\nu x_4]$,
(4) $d_9([\Delta^3\nu x_4]) = [\Delta\overline{\kappa}^2\kappa x_0]$.

Proof. These are the image of $\mathrm{TJF}_2 \to \mathrm{TJF}_3$. □

Lemma 8.5. *d_{13}-differentials are determined by the following:*

(1) $d_{13}([\Delta^{2+4i}\eta x_0]) = [\Delta^{4i}\overline{\kappa}^3\epsilon x_4]$,
(2) $d_{13}([\Delta^5\nu^3 x_6]) = [4\Delta^2\overline{\kappa}^4 x_6]$.

Proof. The first one is the image of $\mathrm{TJF}_2 \to \mathrm{TJF}_3$. The second differential can be deduced from the vanishing line 6.5. □

Applying the Lemma 6.5, we can verify all differentials stated below.

Lemma 8.6. *d_{15}-differentials are determined by the following:*

(1) $d_{15}([\Delta^{3+4i}\delta x_4]) = [\Delta^{4i}\overline{\kappa}^4 x_0]$.

Lemma 8.7. *There are no d_{17}-differentials. The d_{19}-differentials are determined by the following:*

(1) $d_{19}([\Delta^5 x_0]) = [\Delta\overline{\kappa}^4\nu^3 x_6]$

Lemma 8.8. *The d_{21}-differentials are determined by the following:*

(1) $d_{21}([4\Delta^6 x_6]) = [\Delta^2\overline{\kappa}^5\eta x_0]$

Lemma 8.9. *The d_{23}-differentials are determined by the following:*

(1) $d_{23}([\Delta^6\eta x_0]) = [\overline{\kappa}^6 x_0]$.

9. TJF_4

9.1. **E_2-term.** The E_1-term of algebraic AHSS computing the E_2-term of DSS is generated by elements x_0, x_4, x_6, and x_8 with $d_1(x_8) = 2\nu x_4$. We note the differential $d_3(\epsilon x_8) = \eta\kappa x_0$ in AHSS from the Massey product $\eta\kappa = \langle\epsilon, \nu, 2\nu\rangle$. The resulting E_2-page of DSS is drwan in Figure 16. We verify multiplicative extensions in the E_2-term.

Proposition 9.1. *There are the following multiplicative extensions in the E_2-page:*

(1) $2[\eta x_8] = [2\nu x_6]$,
(2) $2[\nu^2 x_8] = [\kappa x_0]$,
(3) $\eta[\eta\kappa x_8] = [4\overline{\kappa} x_4]$.

Proof. (1) The differential $d_1(x_8) = 2\nu x_4$ gives an identity

$$[\delta x_4] = \langle\eta, \nu, 4x_4\rangle = \eta\cdot[2x_8] = 2\cdot[\eta x_8].$$

(2) We have

$$[\kappa x_0] = \langle\nu, 2\nu, \nu, 2\nu\rangle x_0 = \nu\langle 2\nu, \nu, 2\nu, x_0\rangle.$$

However, the class $\langle 2\nu, \nu, x_0\rangle = [2\nu x_4]$ is killed by the differential $d_1(x_8) = 2\nu x_4$, so we compute

$$\langle 2\nu, \nu, 2\nu, x_0\rangle = [2\nu x_8]$$

and obtain the desired result.

(3) Consider the element $[\eta\overline{\kappa}x_0] \in \pi_{21}\mathrm{TJF}_3$. The Toda bracket $\eta\overline{\kappa} = \langle\kappa, 2, \nu^2\rangle$ and the Moss' theorem implies the relation

$$\begin{aligned}[\eta\overline{\kappa}x_0] &= \langle\kappa, 2, \nu^2\rangle x_0\\ &= \kappa\langle 2, \nu^2, x_0\rangle\\ &= \kappa[2\nu x_4]\end{aligned}$$

in $\pi_{21}\mathrm{TJF}_3$. Therefore $[\eta\overline{\kappa}x_0]$ must be killed by a differntial in the DSS of TJF_4, and this extension is forced to kill this element by d_3.

□

9.2. **Higher Differentials.**

Lemma 9.2. *d_5-differentials are determined by the following:*

(1) $d_5([2\Delta^{1+2i}x_6]) = \nu[2\Delta^{2i}\overline{\kappa}x_6]$,
(2) $d_5([\Delta^{1+2i}\nu^2 x_6]) = [\Delta^{2i}\overline{\kappa}\nu^3 x_6]$,
(3) $d_5([2\Delta^{2+4i}x_6]) = 2\nu[\Delta^{1+4i}\overline{\kappa}x_6] = [\Delta^{1+4i}\overline{\kappa}\delta x_4]$,
(4) $d_5([\nu^2\Delta^{2+4i}x_6]) = [\Delta^{1+4i}\eta\kappa x_0]$,
(5) $d_5([\Delta^{2+4i}\nu x_6]) = [\Delta^{1+4i}\overline{\kappa}\epsilon x_4]$,
(6) $d_5([2\Delta^{1+4i}x_8]) = [2\Delta^{4i}\nu\overline{\kappa}x_8]$,
(7) $d_5([\Delta^{2+4i}\nu x_8]) = [\Delta^{1+4i}\overline{\kappa}\kappa x_0]$.

Proof. Follows from $d(\Delta)$ and the Leibniz rule. □

Lemma 9.3. *d_7-differentials are determined by the following:*

(1) $d_7([4\Delta^{1+4i}x_8]) = [\Delta^{4i}\eta^3\overline{\kappa}x_8]$,
(2) $d_7([2\Delta^{2+4i}x_8]) = [\Delta^{1+4i}\eta^2\overline{\kappa}x_8])$.

Proof. This can be checked from 7.7. □

Lemma 9.4. *d_9-differentials are determined by the following:*

(1) $d_9([\Delta^{3+4i}\eta^2 x_8]) = [\Delta^{1+4i}\overline{\kappa}^2\nu^3 x_8]$.

Proof. This is a consequence of 7.7. □

Lemma 9.5. *There are the following d_{11}-differentials:*

(1) $d_{11}([\Delta^{2+4i}\nu\kappa x_6]) = [\Delta^{4i}\eta^2\overline{\kappa}^3 x_8]$.

Proof. Note that $[\Delta^{4i}\eta^2\overline{\kappa}^3 x_8] = \eta\overline{\kappa}^3[\Delta^{4i}\eta x_8]$, but $\eta\overline{\kappa}^3 = 0$ in $\pi_{61}\mathrm{TMF}$. Therefore, $[\Delta^{4i}\eta^2\overline{\kappa}^3 x_8]$ must be the target of a differential, and this is the only possibility. □

Lemma 9.6. *There are the following d_{13}-differentials:*

(1) $d_{13}([\Delta^{3+4i}\nu^3 x_6]) = [\Delta^{4i}2\overline{\kappa}^4 x_6]$,
(2) $d_{13}([\Delta^{3+4i}x_0]) = [2\Delta^{4i}\overline{\kappa}^3\nu x_8]$,
(3) $d_{13}([\Delta^{3+4i}\nu^3 x_8]) = [\Delta^{4i}2\overline{\kappa}^4 x_8]$.

Proof. (1) is shown in TJF_3. (2) can be deduced from the relation $[2\Delta^{4i}\overline{\kappa}^3\nu x_8] = \overline{\kappa}^3\nu[2\Delta^{4i}x_8]$ and the fact $\overline{\kappa}^3\nu = 0$. (3) is deduced from 7.7. □

Lemma 9.7. *There are the following d_{15}-differentials:*

(1) $d_{15}([\Delta^{3+4i}\delta x_4]) = [\Delta^{4i}\overline{\kappa}^4 x_0]$,
(2) $d_{15}([\Delta^4\eta x_8]) = [\Delta\overline{\kappa}^4 x_0]$,
(3) $d_{15}([\Delta^6 x_0]) = [\Delta^3\overline{\kappa}^3\eta^3 x_8]$.

Proof. (1) is proven in TJF_3. (2) and (3) can be deduced from the vanishing line. □

Similarly, you can deduce the following differentials from the lemma 6.5.

Lemma 9.8. *There are the following d_{17}-differentials:*

(1) $d_{17}([\Delta^6\eta^2 x_8]) = [\Delta^2\overline{\kappa}^4\nu\kappa x_8]$.

Lemma 9.9. *There are the following d_{19}-differentials:*

(1) $d_{19}([\Delta^5 x_0]) = [\Delta\overline{\kappa}^4\nu^3 x_6]$,
(2) $d_{19}([\Delta^5\nu^3 x_6]) = [\Delta\overline{\kappa}^5\eta^2 x_6]$.

Lemma 9.10. *There are the following d_{21}-differentials:*

(1) $d_{21}([\Delta^5\kappa\nu x_8]) = [\Delta^2\overline{\kappa}^6 x_0]$.

Lemma 9.11. *There are the following d_{23}-differentials:*

(1) $d_{23}([\Delta^5\eta^2 x_8]) = [\overline{\kappa}^6\eta x_8]$,
(2) $d_{23}([\Delta^7\eta^3 x_8]) = [\Delta^2\overline{\kappa}^6\eta^2 x_8]$.

10. TJF_5

10.1. **E_2-term.** The differential in algebraic AHSS is given by $d_1(x_{10}) = \eta x_8 + \nu x_6$. The E_2-term of DSS for TJF_6 is shown in Figure 21.

Proposition 10.1. *There are the following multiplicative extensions in the E_2-page:*

(1) $\eta[\nu^3 x_{10}] = [\overline{\kappa} x_0]$,
(2) $2[\eta\kappa x_{10}] = [\nu\kappa x_8]$,
(3) $\eta[\eta\kappa x_{10}] = [2\overline{\kappa} x_6]$.

Proof. (1) This extension will be proven in section 12 by comparison with the Adams spectral sequence for $\mathrm{tmf} \otimes P_7$.

(2) Note that the cofiber of the map $\mathrm{TJF}_3 \to \mathrm{TJF}_5$ is equivalent to $\Sigma^8\mathrm{TMF} \otimes C\eta$. The E_2-term of the descent spectral sequence for $\mathrm{tmf} \otimes C\eta$ is computed in [Bau04, Figure 4.3], and this extension can be verified from the multiplicative extension happening in $E_2^{3,28}$ of DSS for $\mathrm{tmf} \otimes \Sigma^8 C\eta$.

(3) Similary, we consider the cofiber of the map $\mathrm{TJF}_3 \to \mathrm{TJF}_5$. This cofiber is $\mathrm{TMF} \otimes \mathbb{C}P_3^5$. Then we can verify the extension in $\mathrm{TMF} \otimes \mathbb{C}P_3^5$ in the same way as 7.

□

10.2. **Higher Differentials.** The d_3-differentials follow from $d_3(\delta) = \eta^4$, and the resulting E_4-page is drawn in 22. We determine differentials $d_r, r \geq 5$ below.

Lemma 10.2. *There are the following d_5-differentials:*

(1) $d_5([2\Delta^{1+4i} x_8]) = [2\Delta^{4i}\nu\overline{\kappa} x_8]$,
(2) $d_5([\Delta^{2+4i}\nu x_8]) = [\Delta^{1+4i}\overline{\kappa}\kappa x_0]$.

Proof. Can be verified by the Leibniz rule. □

Lemma 10.3. *There are no d_7-differentials, and there are the following d_9-differentials:*

(1) $d_9([\Delta^{2+4i}\nu^3 x_{10}]) = [2\Delta^{4i}\overline{\kappa}^3 x_6]$,
(2) $d_9([\Delta^{3+4i}\nu^3 x_{10}]) = [2\Delta^{1+4i}\overline{\kappa}^3 x_6]$,
(3) $d_9([\Delta^{2+4i} x_6]) = [\Delta^{4i}\overline{\kappa}^2\delta x_8]$,
(4) $d_9([\Delta^{3+4i} x_6]) = [\Delta^{1+4i}\overline{\kappa}^2\delta x_8]$.

Proof. Note the relation $[2\overline{\kappa}^4 x_6] = \overline{\kappa}^3[2\overline{\kappa} x_6] = \overline{\kappa}^3\eta[\eta\kappa x_6]$ in the E_2-term. Because $\overline{\kappa}^3\eta = 0$ holds in $\pi_*\mathrm{tmf}$, there must be a differential hitting $[2\overline{\kappa}^4 x_6]$. (1) and (2) are forced by this observation and linearity.

(3) and (4) can be verified from lemma 6.5. □

Lemma 10.4. *There are the following d_{11}-differentials:*

(1) $d_{11}([\Delta^{2+4i}\delta x_8]) = [\Delta^{4i}\overline{\kappa}^3 x_0]$,
(2) $d_{11}([\Delta^{3+4i}\delta x_8]) = [\Delta^{1+4i}\overline{\kappa}^3 x_0]$.

Proof. Because $[\overline{\kappa}^4 x_0] = 0 \in \pi_{80}\mathrm{TJF}_4$, there must be a differential hitting $[\overline{\kappa}^4 x_0]$ in the DSS for TJF_5 as well, and the claimed differential is the only possibility. □

Lemma 10.5. *There are the following d_{13}-differentials:*

(1) $d_{13}([\Delta^{3+4i} x_0]) = [\Delta^{4i}\overline{\kappa}^3\nu x_8]$,
(2) $d_{13}([\Delta^{3+4i}\nu^2 x_8]) = [\overline{\kappa}^{3+4i}\eta\kappa x_{10}]$,
(3) $d_{13}([\Delta^5\nu^2 x_8]) = [\Delta^2\overline{\kappa}^3\nu\kappa x_8]$

Proof. These are the image of d_{13}-differentials in TJF_4. □

The vanishing line lemma 6.5 proves the following long differentials.

Lemma 10.6. *There are the following d_{15}-differentials:*

(1) $d_{15}([\Delta^3 x_6]) = [\overline{\kappa}^3\nu^3 x_{10}]$

Lemma 10.7. *There are no d_{17}-differentials, and there are the following d_{19}-differentials:*

(1) $d_{19}([\Delta^5\kappa\eta x_{10}]) = [\Delta\overline{\kappa}^5\nu^2 x_8]$.

Lemma 10.8. *There are the following d_{21}-differentials:*

(1) $d_{21}([\Delta^6\eta\kappa x_{10}]) = [\Delta^2\overline{\kappa}^6 x_0]$

Lemma 10.9. *There are the following d_{23}-differentials:*

(1) $d_{23}([\Delta^6 x_0]) = [\overline{\kappa}^5\nu^3 x_{10}]$.

11. TJF_6

11.1. E_2-term.

The E_2-term of DSS for TJF_6 is shown in Figure 26.

Proposition 11.1. *There are the following multiplicative extensions in the E_2-page:*

(1) $2[\eta x_{12}] = \eta[4x_{12}] = [\delta x_8]$,
(2) $\eta[\eta\kappa x_{12}] = [2\overline{\kappa}x_8]$.

Proof. (1) First, we can see$[\delta x_8] = \langle \eta, 2\nu, 2x_8\rangle = \eta[4x_{12}]$.
(2)

□

Remark 11.2. Note that the element in coordinate $(18, 2)$ is represented by $[\nu^2 x_{12} + \epsilon x_{10}]$, so we have an extension $\eta[\nu^2 x_{12} + \epsilon x_{10}] = [\eta\epsilon x_{10}]$.

11.2. Higher Differentials.

Lemma 11.3. *There are the following d_5-differentials:*

(1) $d_5([\Delta^{2i+1}\eta\kappa x_{10}]) = [\Delta^{2i}\overline{\kappa}^2 x_8]$,
(2) $d_5([\Delta^{2i+1}\nu^2 x_{12}]) = [\Delta^{2i}\overline{\kappa}\nu^3 x_{12}]$.

Proof. It can be verified from the Leibniz rule. □

Lemma 11.4. *There are the following d_7-differentials:*

(1) $d_7([4\Delta^{2i+1}x_{12}]) = [\Delta^{2i}\eta^3\overline{\kappa}x_{12}]$.

Proof. This is a consequence of 7.7. □

Lemma 11.5. *There are the following d_9-differentials:*

(1) $d_9([\Delta^{2+4i}x_6]) = [\Delta^{4i}\overline{\kappa}^2\delta x_8]$,
(2) $d_9([\Delta^{3+4i}x_6]) = [\Delta^{1+4i}\overline{\kappa}^2\delta x_8]$,
(3) $d_9([\Delta^{2+4i}\eta x_{12}]) = [\Delta^{4i}\overline{\kappa}^3 x_{12}]$,
(4) $d_9([\Delta^{3+4i}\eta x_{12}]) = [\Delta^{1+4i}\overline{\kappa}^3 x_{12}]$.

Proof. (1) and (2) are shown in TJF_6. (3) and (4) can be deduced from 7.7. □

Lemma 11.6. *There are the following d_{11}-differentials:*

(1) $d_{11}([\Delta^{2+4i}\delta x_8]) = [\Delta^{4i}\overline{\kappa}^3 x_0]$,
(2) $d_{11}([\Delta^{3+4i}\delta x_8]) = [\Delta^{1+4i}\overline{\kappa}^3 x_0]$.
(3) $d_{11}([2\Delta^{3+4i}x_8]) = [\Delta^{1+4i}\overline{\kappa}^2\eta^3 x_{12}]$,
(4) $d_{11}([\Delta^{2+4i}\kappa\eta x_{12}]) = [\overline{\kappa}^2\eta^2 x_{12}]$,
(5) $d_{11}([\Delta^5\eta x_{12}]) = [\Delta^3\overline{\kappa}^3 x_0]$.

Proof. (1) and (2) are shown in TJF_6. (3) and (5) can be verified by 6.5. (4) is a consequence of 7.7. □

Lemma 11.7. *There are no d_{13} or d_{15}-differentials.*

Proof. This can be checked from the degree reason. □

Using the lemma 6.5, you can verify the differentials below.

Lemma 11.8. *There are the following d_{19}-differentials:*

(1) $d_{19}([\Delta^5\nu^2 x_{12}]) = [\overline{\kappa}^5\eta x_{12}]$,
(2) $d_{19}([\Delta^7 x_0]) = [\Delta^3\overline{\kappa}^4\eta^3 x_{12}]$.

Lemma 11.9. *There are no d_{21}-differentials, and there are the following d_{23}-differentials:*

(1) $d_{23}([\Delta^5\nu^3 x_{10}]) = [\overline{\kappa}^6\nu^2 x_{12}]$,
(2) $d_{23}([\Delta^6 x_0]) = [\overline{\kappa}^5\nu^3 x_{10}]$,
(3) $d_{23}([\Delta^6\eta^2 x_{12}]) = [\Delta\overline{\kappa}^6\eta x_{12}]$,
(4) $d_{23}([\Delta^7\eta^3 x_{12}]) = [\Delta^2\overline{\kappa}^6\eta^2 x_{12}]$.

12. TJF_7

12.1. E_2**-term.** The E_2-term is shown in Figure 31.

Proposition 12.1. *There are the following multiplicative extensions in the E_2-page:*

(1) $4[\nu x_{14}] = [\delta x_{12}]$,
(2) $\eta[\nu x_{14}] = [\nu^2 x_{12} + \epsilon x_{10}]$,
(3) $2[\nu^2 x_{14}] = [\epsilon x_{12}]$,
(4) $\nu[\nu^3 x_{14}] = [2\overline{\kappa} x_6]$,
(5) $\eta[\kappa\nu x_{14}] = [4\overline{\kappa} x_{12}]$.

Proof. Note that the cofiber of the inculsion $\mathrm{TJF}_5 \to \mathrm{TJF}_7$ is equivalent to $\Sigma^{12}\mathrm{TMF} \otimes C\eta$. Comparing the E_2-term of DSS for $\mathrm{tmf} \otimes C\eta$ in [Bau04, Figure 4.3], extensions (1), (2), (3), and (5) can be deduced. $\square$

12.2. **Higher Differentials.** d_3-differentials and d_5-differntials are determined by the Leibniz rule and the relation $d_3(\delta) = \eta^4$ and $d_5(\Delta) = \nu\overline{\kappa}$.

Lemma 12.2. *There are the following d_5-differentials:*

(1) $d_5([4\Delta x_{14}]) = [\overline{\kappa}\delta x_{12}]$,
(2) $d_5([\Delta\nu x_{14}]) = [\overline{\kappa}\nu^2 x_{14}]$,
(3) $d_5([\Delta^2\nu x_{14}]) = [\Delta\overline{\kappa}\epsilon x_{12}]$.

Lemma 12.3. *There are the following d_7-differentials:*

(1) $d_7([\Delta^{2i+1}\delta x_{12}]) = [\Delta^{2i}\overline{\kappa}^2 x_0]$,
(2) $d_7([\Delta^{4i+2}2\nu x_{14}]) = [\Delta^{4i+1}\overline{\kappa}^2 x_0]$,
(3) $d_7([\Delta^4\nu x_{14}]) = [\Delta^3\overline{\kappa}^2 x_0]$.

Lemma 12.4. *(1) is shown in* TJF_6*. (2) can be shown by the synthetic Leibniz rule and the relation $4\nu = \tau^2\eta^3$ in* $\mathrm{tmf}^{\mathrm{syn}}$*. (3) is the consequence of $d_7(\Delta^4) = \Delta^3\eta^3\overline{\kappa}$ in the DSS of* tmf*.*

Lemma 12.5. *There are the following d_9-differentials:*

(1) $d_9([\Delta^{2+4i}\delta\eta x_{12}]) = [\Delta^{4i}\overline{\kappa}^2\kappa\eta x_{10}]$,
(2) $d_9([\Delta^{2+4i}\kappa\eta x_{10}]) = [4\Delta^{4i}\overline{\kappa}^3 x_{12}]$,
(3) $d_9([\Delta^{3+4i}\delta\eta x_{12}]) = [\Delta^{1+4i}\overline{\kappa}^2\kappa\eta x_{10}]$,
(4) $d_9([\Delta^{3+4i}\kappa\eta x_{10}]) = [4\Delta^{1+4i}\overline{\kappa}^3 x_{12}]$.

Proof. All these differentials are images of d_9 under the map $\mathrm{TJF}_6 \to \mathrm{TJF}_7$. $\square$

Lemma 12.6. *There are the following d_{11}-differentials:*

(1) $d_{11}([4\Delta^{2+4i}x_{12}]) = [\Delta^{4i}\overline{\kappa}^2\nu^3 x_{10}]$,
(2) $d_{11}([4\Delta^{3+4i}x_{12}]) = [\Delta^{1+4i}\overline{\kappa}^2\nu^3 x_{10}]$.

Proof. Again, those differentials are images of d_{11} under the map $\mathrm{TJF}_6 \to \mathrm{TJF}_7$. $\square$

Lemma 12.7. *There are the following d_{13}-differentials:*

(1) $d_{13}([2\Delta^{3+4i}x_8]) = [2\Delta^{4i}\overline{\kappa}^3\nu x_{14}]$,
(2) $d_{13}([\Delta^{3+4i}\nu^2 x_{14}]) = [\Delta^{4i}\overline{\kappa}^3\kappa\nu x_{12}]$.

Lemma 12.8. *There are no $d_{15}, d_{17}, d_{19}, d_{21}$-differentials.*

Lemma 12.9. *There are the following d_{23}-differentials:*

(1) $d_{23}([\Delta^5\delta\eta x_{12}]) = [\overline{\kappa}^6\nu x_{14}]$,
(2) $d_{23}([\Delta^6\delta\eta x_{12}]) = [\Delta\overline{\kappa}^6\delta\eta x_{12}]$,
(3) $d_{23}([\Delta^7 x_0]) = [\Delta^2\overline{\kappa}^5\nu^3 x_{10}]$.

Appendix A. A comparison between algebraic and analytic Jacobi forms

Definition A.1. The Tate curve is the projective plane curve over $\mathbb{Z}[[q]]$ given by the equation

$$y^2 + xy = x^3 + a_4(q)x + a_6(q)$$

where $a_4(q)$ and $a_6(q)$ are defined by

$$a_4(q) = -5\sum_{n\geq 1}\frac{n^3q^n}{1-q^n}$$

and

$$a_6(q) = -\sum_{n\geq 1} \frac{q^n(7n^5+5n^3)}{12(1-q^n)}.$$

The Tate curve becomes a smooth elliptic curve on $\mathbb{Z}((q))$, and it is denoted by E_q. Note that the Tate curve has the canonical invariant differential $\omega_{\mathrm{can}} = dx/(2x+y)$. (of the differential ω.)

Definition A.2. Let R be a ring. Denote $E_R \to \mathcal{M}_{ell}(R)$ to be the universal elliptic curve over the moduli of elliptic curves over R. For all weights $k \in \mathbb{Z}$ and indices $m \in \frac{\mathbb{Z}}{2}$, the set of weak Jacobi forms with coefficients in R is defined to be the global section $\Gamma(E_R, p^*\omega^k \otimes \mathcal{O}_{E_R}(2me))$.

In other words, a Jacobi form with weight k and index m in coefficient R is a rule f that associates to each pair $(C/\operatorname{Spec} R', \widetilde{\omega})$, consisting an elliptic curve C over a R-algebra R' and a trivialization $\widetilde{\omega}$ of the differential ω, an element

$$f(C/R', \widetilde{\omega}) \in \Gamma(C, \mathcal{O}_C(2me))$$

subject to the following:

- f commutes with the base change, and
- if $\lambda \in R'^{\times}$, then
$$f(C/R', \lambda\widetilde{\omega}) = \lambda^{-k} f(C/R', \widetilde{\omega}).$$

In particular, when we evaluate an arithmetic Jacobi form f on the Tate curve $C_{\mathrm{Tate}}/\mathbb{Z}((q))$ with the canonical generator ω_{can}, we obtain an element $f(C_{\mathrm{Tate}}/\mathbb{Z}((q)), \omega_{\mathrm{can}}) \in \Gamma(C, \mathcal{O}_C(2me))$. Because C_{Tate} can be written as $\mathbb{G}_m/q^{\mathbb{Z}}$ [DR73], the global section $\Gamma(C_{\mathrm{Tate}}, \mathcal{O}_C(2me))$ is an element $s \in \mathbb{Z}((q))[u^{\pm 1}]$ that satisfy the equation

$$s(qu) = q^{-m}u^{-2m}s(u).$$

For a weak Jacobi form f, the resulting two-variable power series $f(C_{\mathrm{Tate}}, \omega_{\mathrm{can}})$ is called the q-expansion of f.

Proposition A.3 (q-expansion principle)**.** *Consider a weak Jacobi form f over a ring R. If the coefficients of q-expansion of f lands in a subring $S \subset R$, then there exists a unique weak Jacobi form $\tilde{f}$ over S that gives rise to f by the extension of scalars. Moreover, a weak Jacobi form over R is uniquely determined by its q-expansion.*

Proof. This is proven in [Kra95, Lemma 2.6]. □

We consider the q-expansion of weak Jacobi forms over $\mathbb{C}$. Recall that the moduli of complex elliptic curves can be written as the quotient $[\mathbb{H}/\operatorname{SL}_2(\mathbb{Z})]$. As an elliptic curve over $\mathbb{C}$ can be written as $\mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})$ for some $\tau \in \mathbb{H}$, we can write the universal complex elliptic curve over $\mathcal{M}_{ell}(\mathbb{C})$ as

$$E_{\mathbb{C}} \cong [\mathbb{H} \times \mathbb{C}/\operatorname{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2]$$

where the action of $\operatorname{SL}_2(\mathbb{Z})$ to $\mathbb{Z}^2$ is given by the matrix multiplication. The action of $(\gamma, (m,n)) \in \operatorname{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is given by

$$\begin{aligned}(\gamma, (\lambda, \mu)) \cdot (\tau, z) &= \left(\gamma \cdot \tau, \frac{z + \lambda\tau + \mu}{c\tau + d}\right) \\ &= \left(\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d}\right).\end{aligned}$$

Proposition A.4. *Let f be a weak Jacobi form over $\mathbb{C}$. Then, for any element $\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu)\right)$ in $\operatorname{SL}_2(\mathbb{Z}) \ltimes Z^2$, its q-expansion $\phi(\tau, z) \coloneqq f(E_q/\mathbb{C}((q)), \omega_{\mathrm{can}})$ satisfy the following equation:*

$$\phi\left(\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d}\right) = (c\tau + d)^{-k} e^{2\pi i m(\lambda^2\tau + 2\lambda z - \frac{c(z+\lambda\tau+\mu)^2}{c\tau+d})}\phi(\tau, z).$$

In other words, $\phi(\tau, z)$ becomes an analytic Jacobi forms of weight k and index m.

Proof. Recall an isomorphism of groups ([MRM08, Appendix I.2])

$$H^1(\operatorname{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2, \mathbb{C}^{\times}) \cong H^1(E, \mathcal{O}_E^{\times}) \cong \operatorname{Pic}(E).$$

We can check that the map

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu)\right) \mapsto e^{2\pi i m(\lambda^2\tau + 2\lambda z - \frac{c(z+\lambda\tau+\mu)^2}{c\tau+d})}$$

defines a 1-cocycle in $H^1(\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2, \mathbb{C}^\times)$. One can check that the corresponding line bundle is isomorphic to $\mathcal{O}_E(2me)$ by Appel-Humbert theorem (see the discussion in [Kra91, section 2]) Therefore the global sections $\Gamma(E_\mathbb{C}, \mathcal{O}_{E_\mathbb{C}}(2me))$ are functions $\phi\colon \mathbb{H}\times\mathbb{C} \to \mathbb{C}$. □

Note that the action of $(\gamma, (\lambda, \mu))$ on the invariant differential Therefore, evaluating at the Tate curve with the canonical invariant differential, we obtain an injective map

$$\Gamma(E_\mathbb{C}, p^*\omega^k \otimes \mathcal{O}_{E_\mathbb{C}}(2me)) \to \mathrm{JF}_{k,m}.$$

Theorem A.5 ([GW20], Corollary 3.6)**.** *The graded ring of (analytic) Jacobi form is generated by the following generators*

$$\bigoplus_{n,k} \mathrm{JF}_{n,k} \cong \mathbb{Z}[c_4, c_6, \Delta^{\pm 1}, E_{4,1}, E_{4,2}, E_{4,3}, E_{6,1}, E_{6,2}, F_{6,3}, \phi_{0,1}, \phi_{0,2}, \phi_{0,\frac{3}{2}}, \phi_{0,4}, \phi_{-1,\frac{1}{2}}]$$

with relations

$$\begin{aligned}
&c_4^3 - c_6^2 = 1728\Delta, && 24\phi_{0,2} = \phi_{0,1} - c_4\phi_{-1,\frac{1}{2}}^4,\\
&432\phi_{0,\frac{3}{2}}^2 = \phi_{0,1}^3 - 3c_4\phi_{0,1}\phi_{-1,\frac{1}{2}}^4 + 2c_6\phi_{-1,\frac{1}{2}}^6, && 4\phi_{0,4} = \phi_{0,1}\phi_{0,\frac{3}{2}}^2 - \phi_{0,2}^2,\\
&12E_{4,1} = c_4\phi_{0,1} - c_6\phi_{-1,\frac{1}{2}}^2, && 12E_{6,1} = c_6\phi_{0,1} - c_4^2\phi_{-1,\frac{1}{2}}^2,\\
&6E_{4,2} = E_{4,1}\phi_{0,1} - c_4\phi_{0,2}, && 6E_{6,2} = E_{6,1}\phi_{0,1} - c_6\phi_{0,2},\\
&2E_{4,3} = E_{4,1}\phi_{0,2} - c_4\phi_{0,3}, && 2F_{6,3} = E_{6,1}\phi_{0,2} - c_6\phi_{0,3}.
\end{aligned}$$

Appendix B. Descent spectral sequence charts

Here we display the descent spectral sequences for TJF_n. We use the following conventions. Throughout this section, we use Adams indexing.

- A square indicates the $\mathbb{Z}_{(2)}$-summand.
- A number n in a square indicates the $n\mathbb{Z}_{(2)}$-summand.
- A dot indicates the $\mathbb{Z}/2$-summand.
- A n concentric circle indicates $\mathbb{Z}/2^n$-summand. For example, ⊚ indicates the $\mathbb{Z}/4$-summand.
- Colors in charts indicate the cellular filtration appeared in section 4.

	x_0	x_4	x_6	x_8	x_{10}	x_{12}	x_{14}
colors	black	brown	red	orange	purple	green	blue

- non-vertical arrows with negative slope indicate differentials.
- lines with positive slope indicate the multiplication with an element in $\pi_*\mathrm{TMF}$: a vertical line indicates the multiplication by 2, a line with slope 1 indicates the multiplication by η, and a line with slope 1/3 indicates the multiplication by ν.
- A diamond ⋄ in the E_2-term is an abbreviation of repeated η-multiplication, meaning that η-tower from there survives.

B.1. TJF_2**.** This section collects diagrams of the descenet spectral sequence for TJF_2.

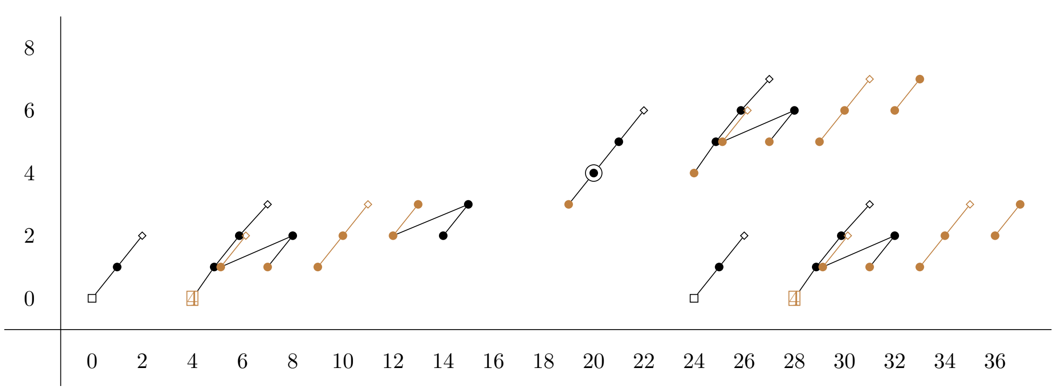

Figure 6. The E_2-page of DSS for TJF_2

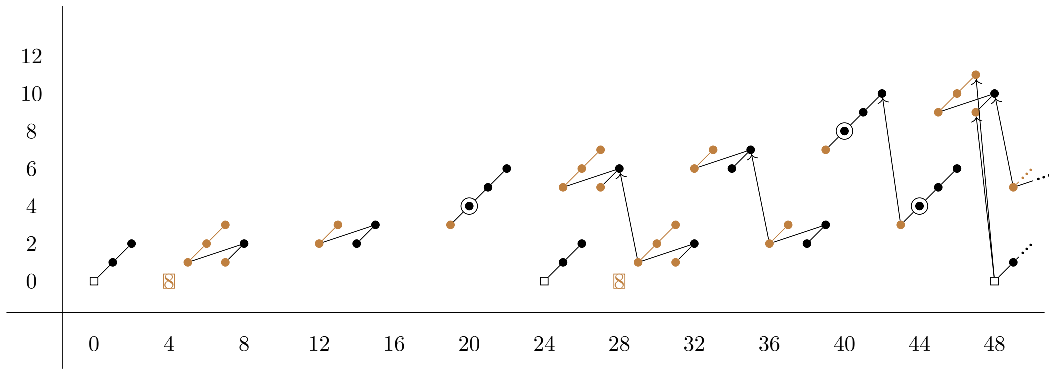

FIGURE 7. The higher differentials of DSS for TJF_2 in $0-49$ range

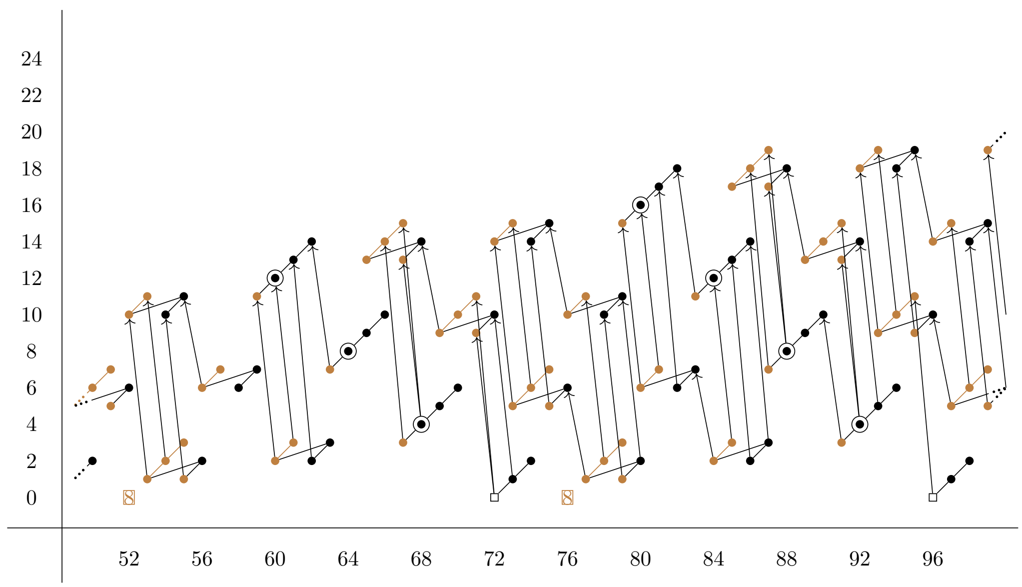

FIGURE 8. The higher pages of DSS for TJF_2 and differentials

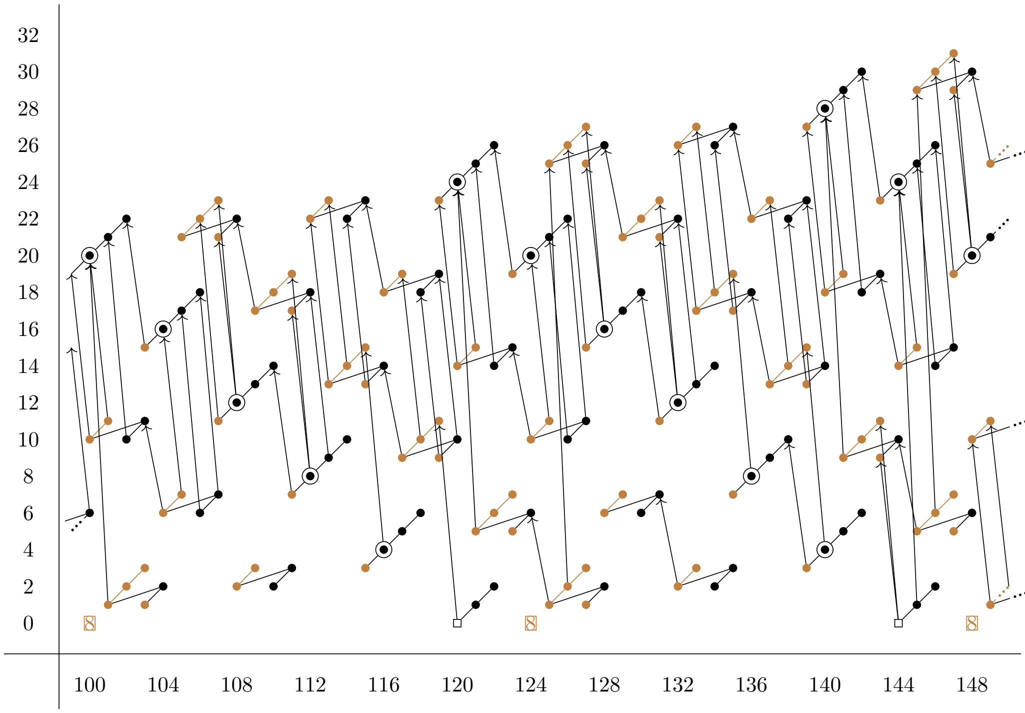

FIGURE 9. The higher pages of DSS for TJF_2 and differentials

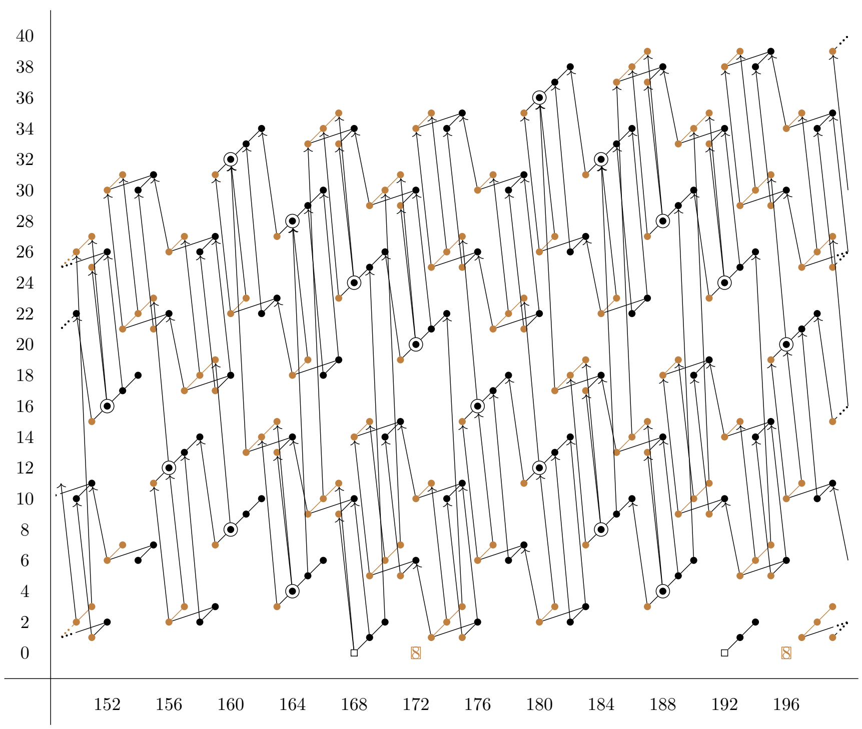

FIGURE 10. The higher pages of DSS for TJF_2 and differentials

B.2. TJF_3. TJF_3 diagrams are listed below.

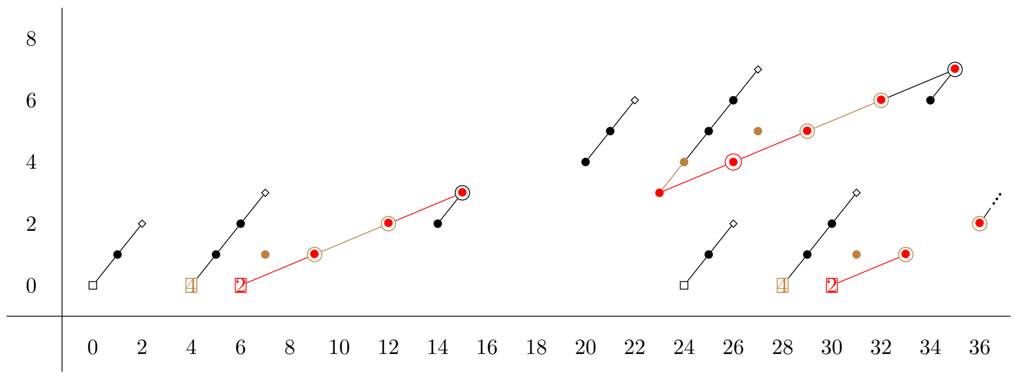

Figure 11. E_2-term of DSS for TJF_3

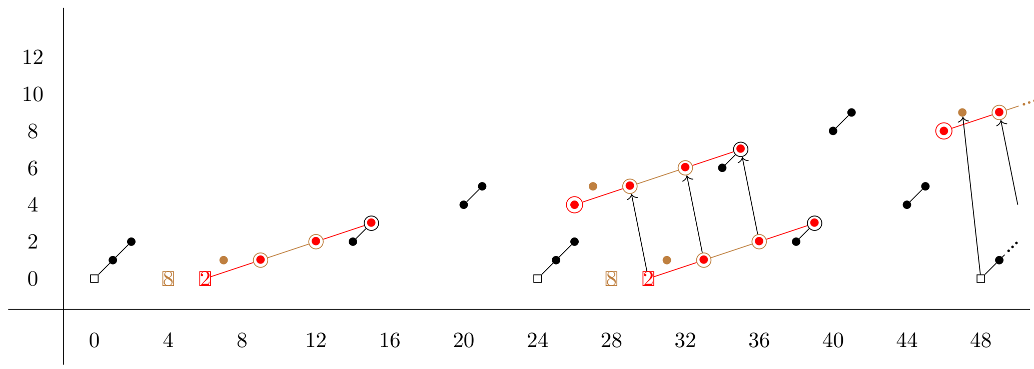

FIGURE 12. E_4-page and d_r, $r \geq 5$ of DSS for TJF_3

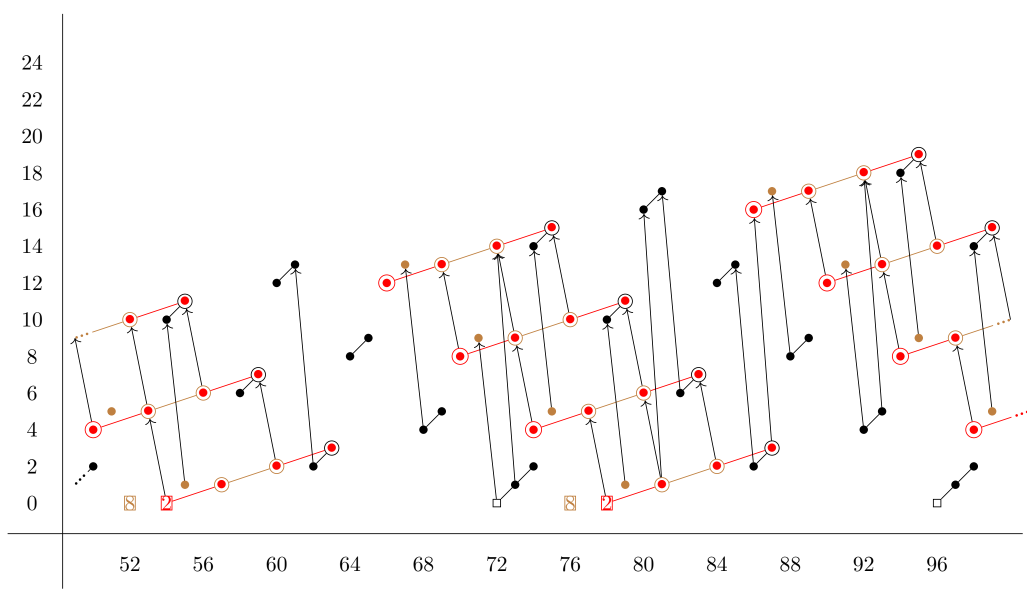

FIGURE 13. E_4-page and d_r, $r \geq 5$ of DSS for TJF_3

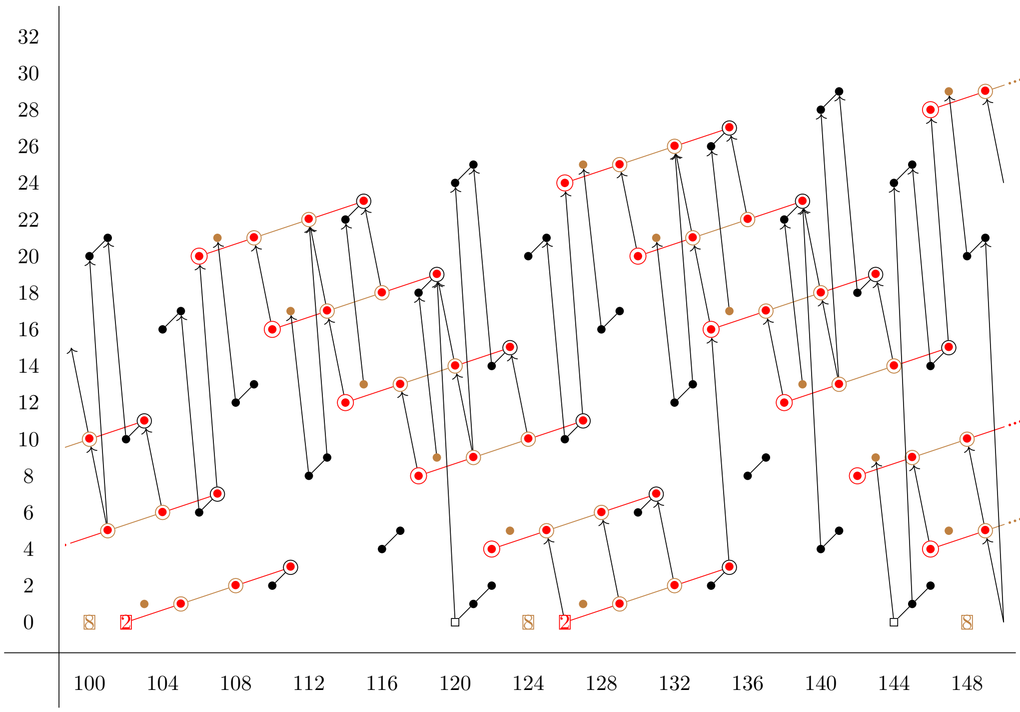

FIGURE 14. E_4-page and d_r, $r \geq 5$ of DSS for TJF_3

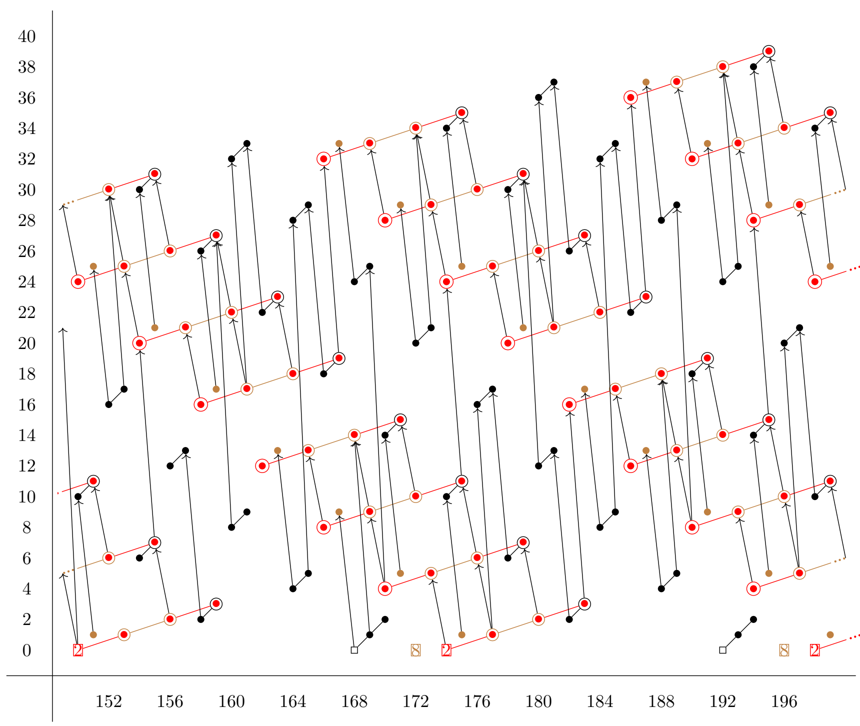

FIGURE 15. E_4-page and d_r, $r \geq 5$ of DSS for TJF_3

B.3. TJF_4. This section collects diagrams of the descenet spectral sequence for TJF_4.

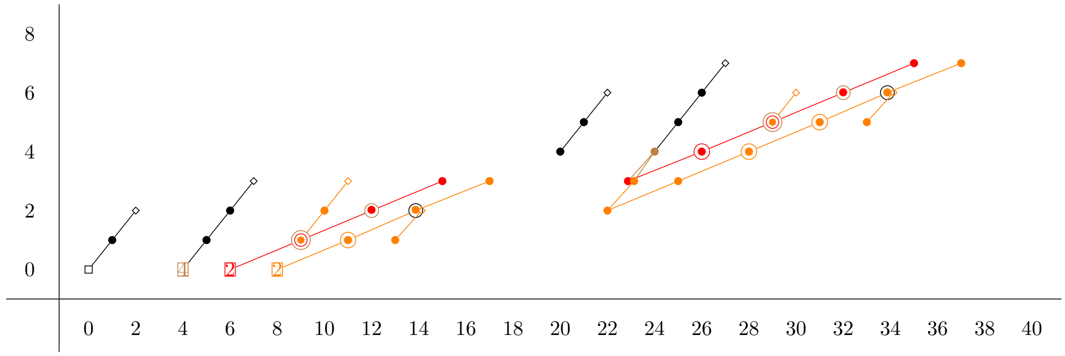

FIGURE 16. E_2-page of DSS for TJF_4

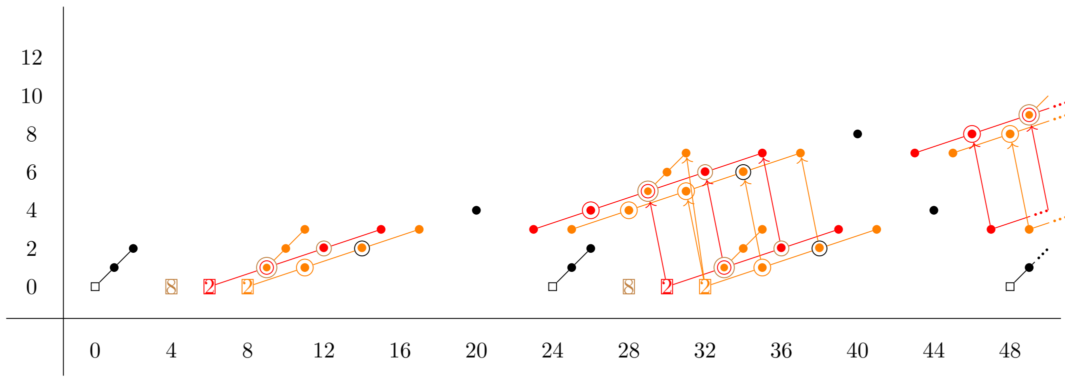

FIGURE 17. E_4-page and d_r, $r \geq 5$ of DSS for TJF_4

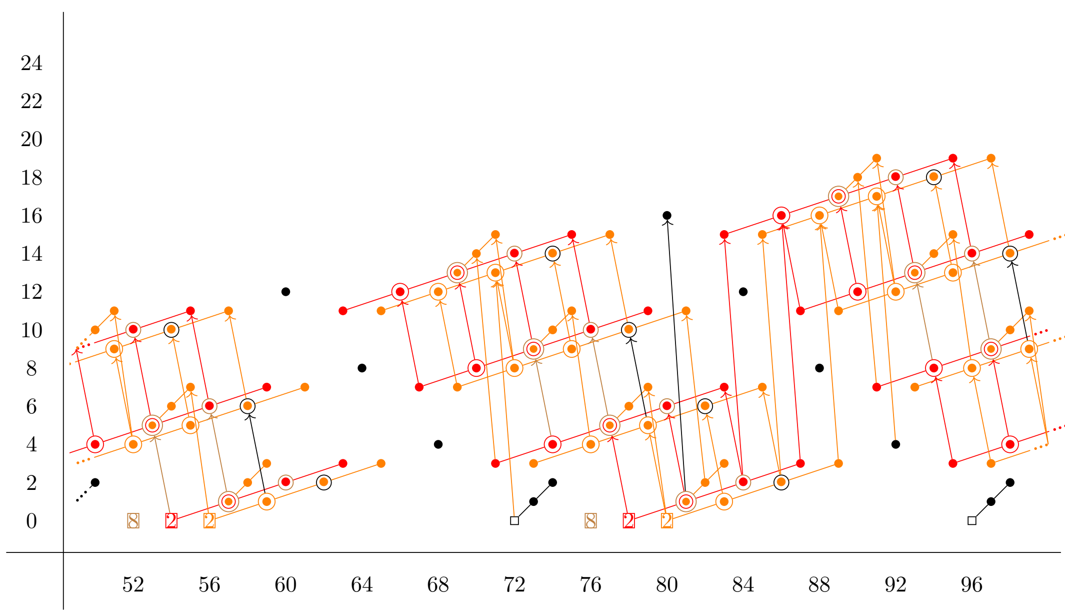

FIGURE 18. E_4-page and d_r, $r \geq 5$ of DSS for TJF_4

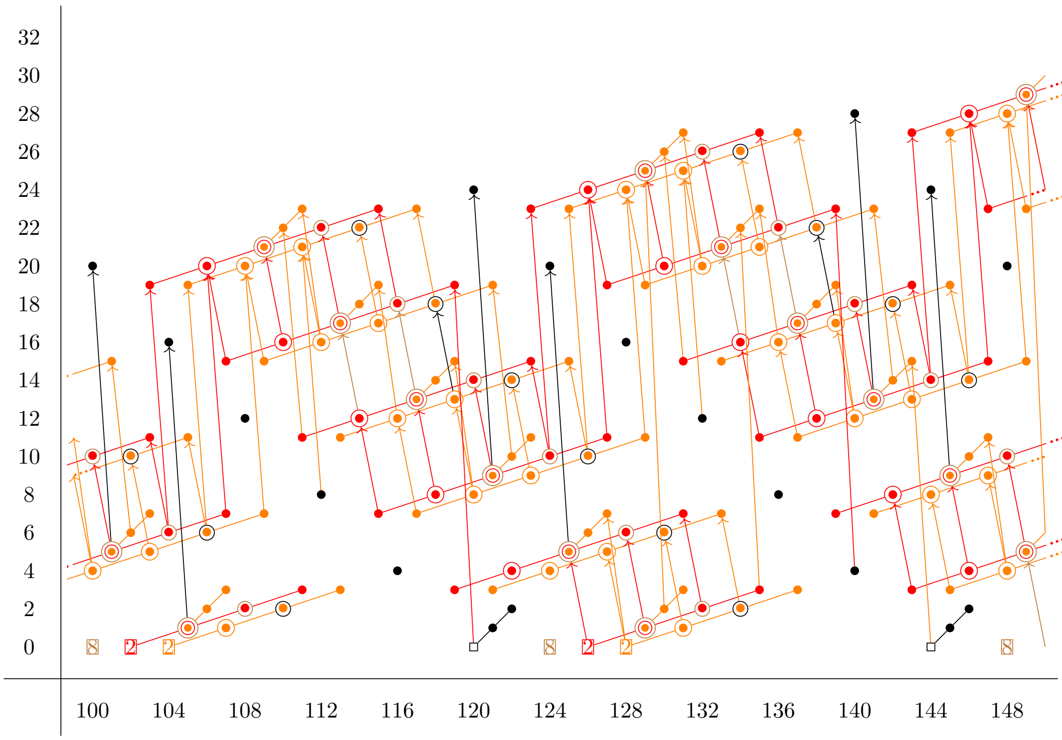

FIGURE 19. E_4-page and d_r, $r \geq 5$ of DSS for TJF_4

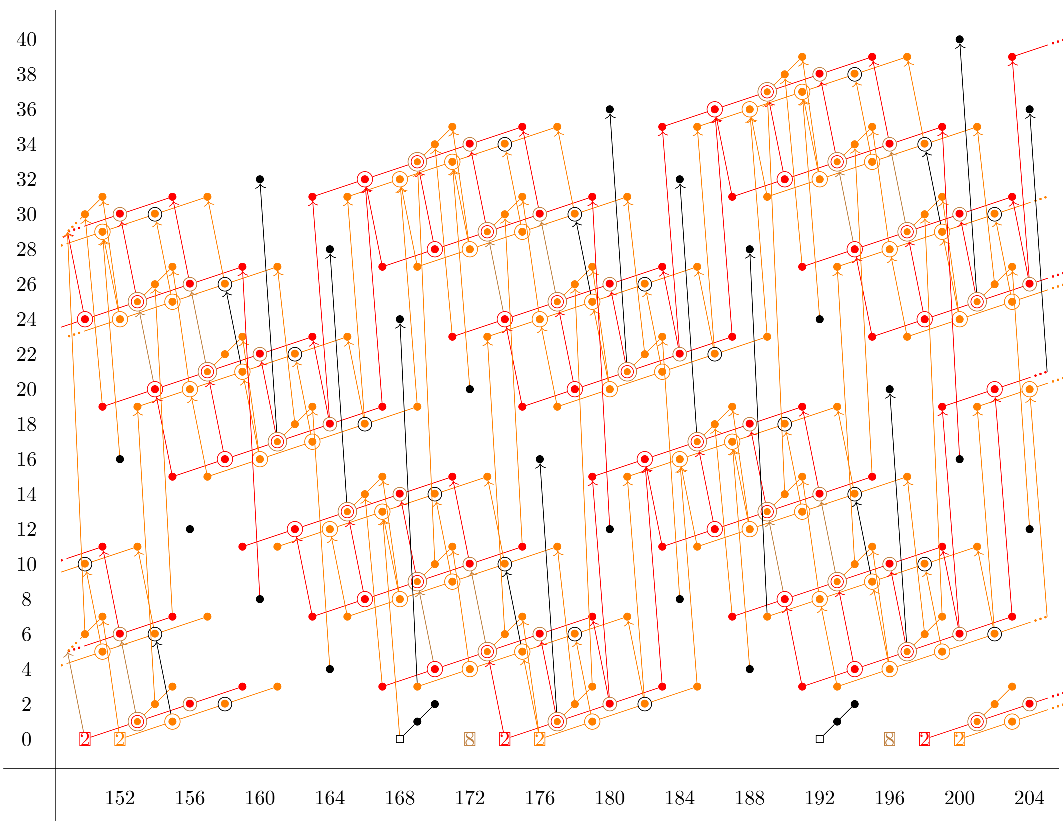

FIGURE 20. E_4-page and d_r, $r \geq 5$ of DSS for TJF_4

B.4. TJF_5. This section collects diagrams of the descenet spectral sequence for TJF_5.

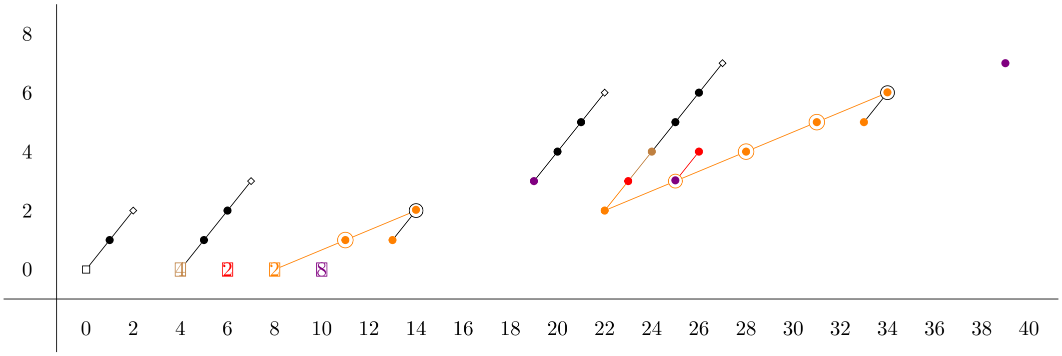

FIGURE 21. E_2-page of DSS for TJF_5

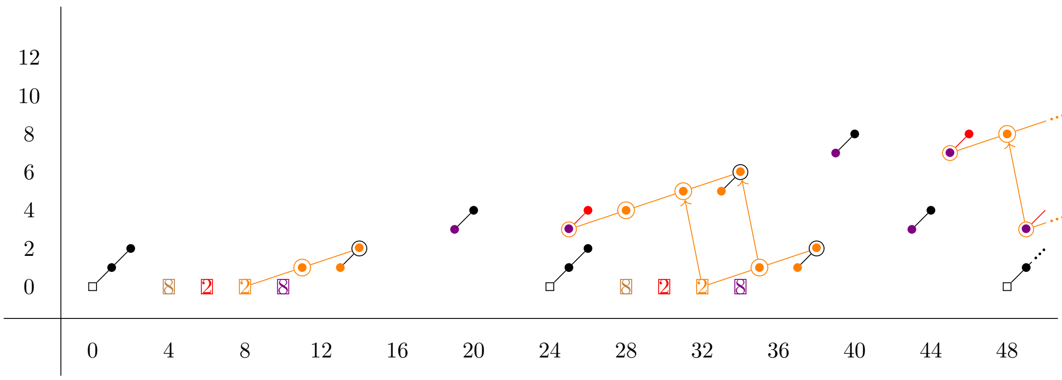

FIGURE 22. E_4-page and d_r, $r \geq 5$ of DSS for TJF_5

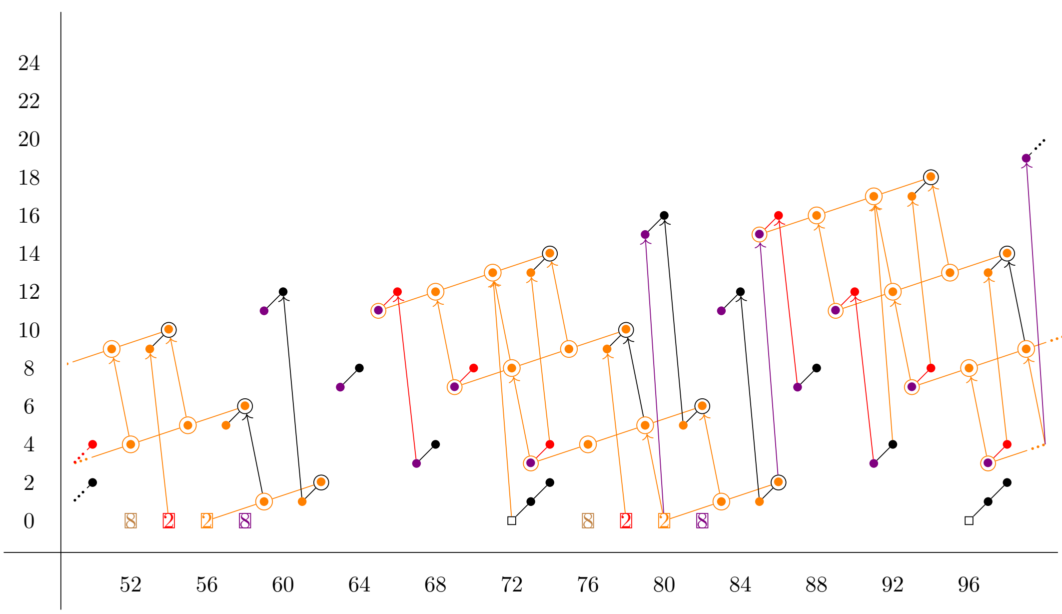

FIGURE 23. E_4-page and d_r, $r \geq 5$ of DSS for TJF_5

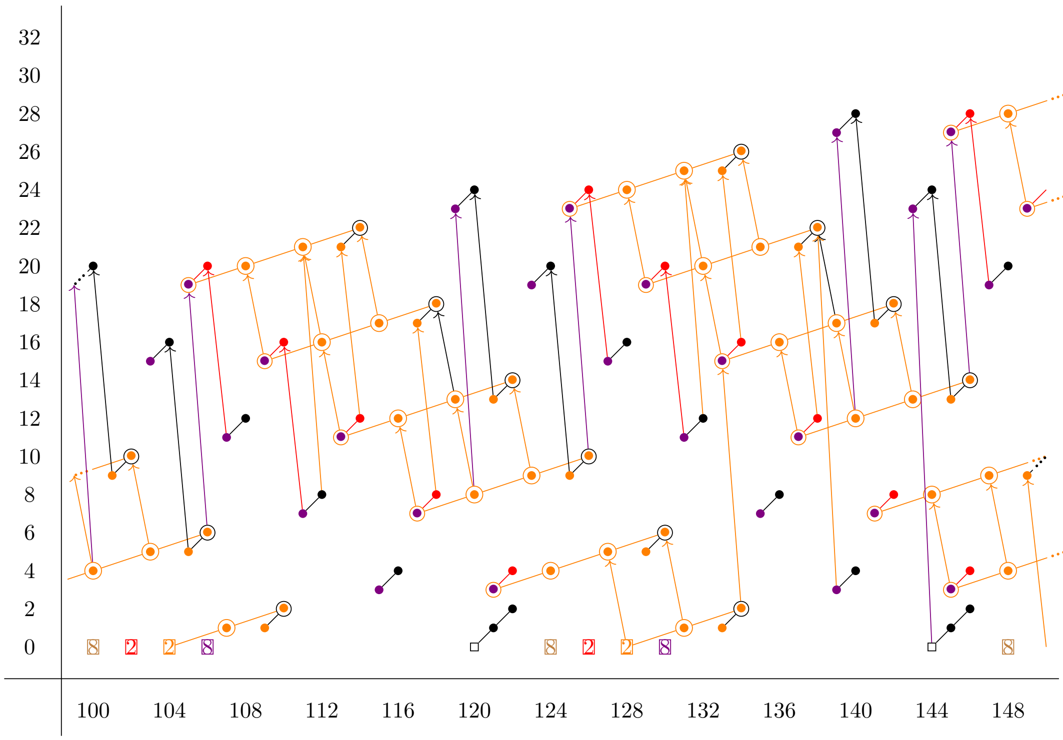

FIGURE 24. E_4-page and d_r, $r \geq 5$ of DSS for TJF_5

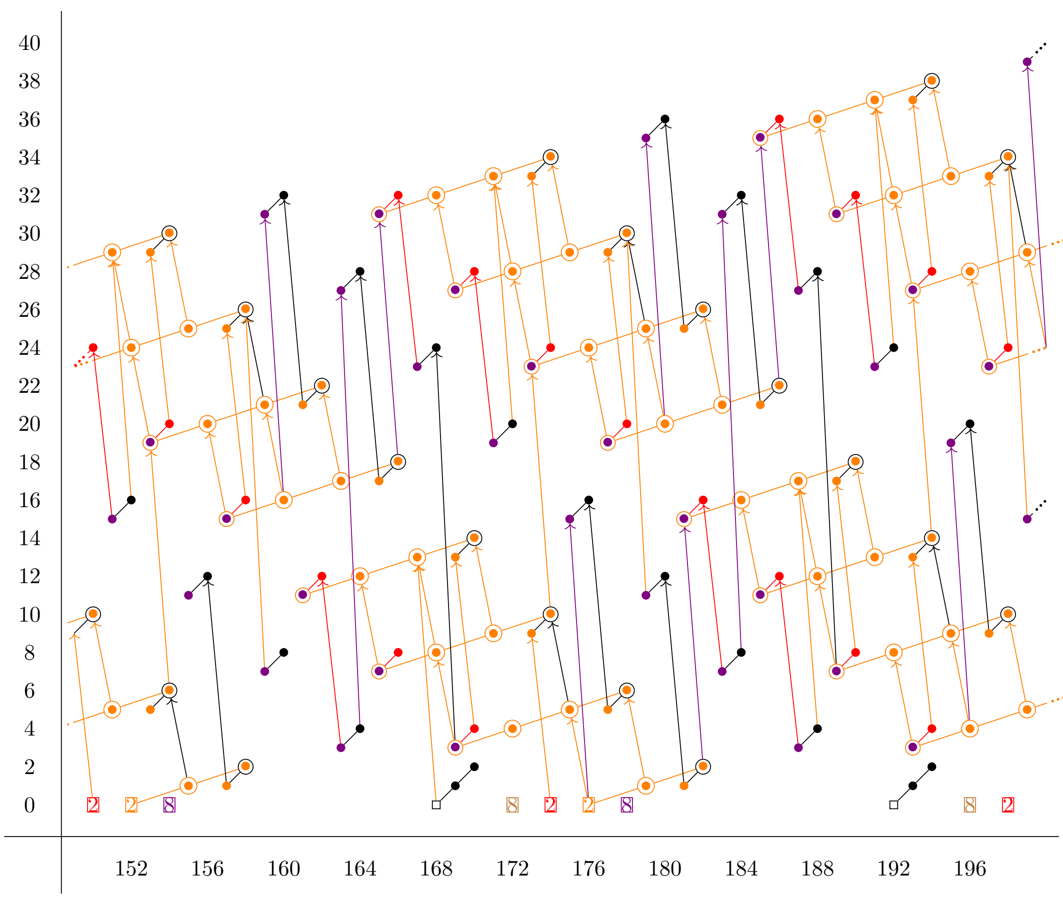

FIGURE 25. E_4-page and d_r, $r \geq 5$ of DSS for TJF_5

B.5. TJF_6. This section collects diagrams of the descenet spectral sequence for TJF_6.

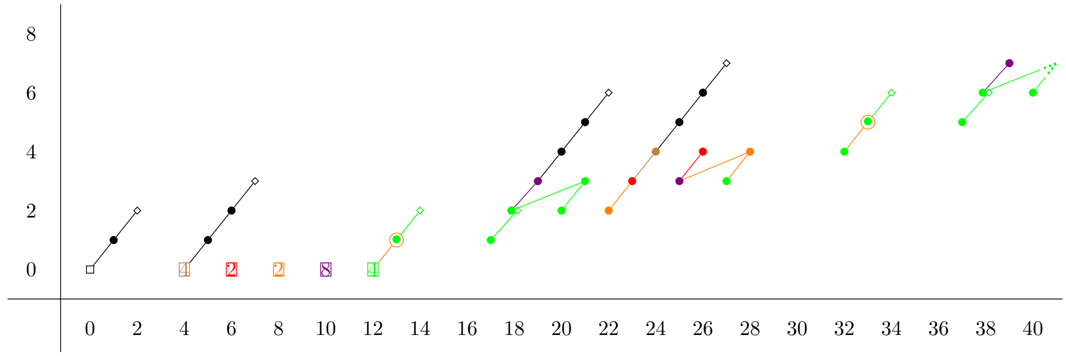

FIGURE 26. E_2-page of DSS for TJF_6

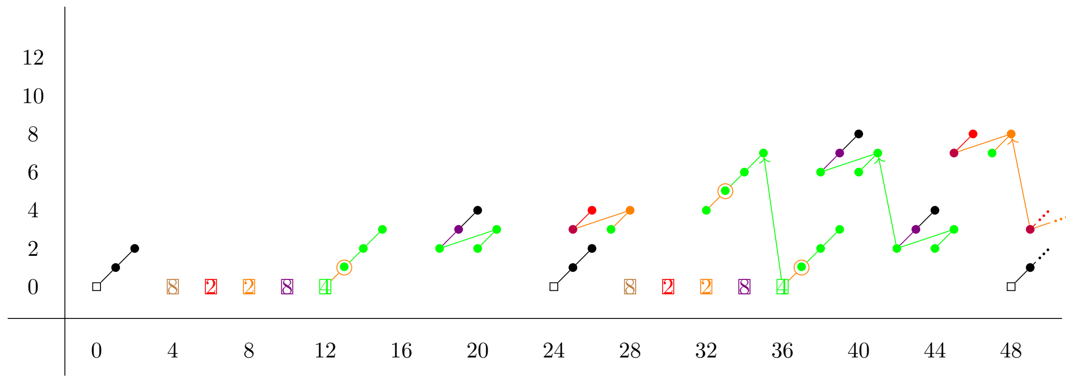

FIGURE 27. E_4-page and d_r, $r \geq 5$ of DSS for TJF_6

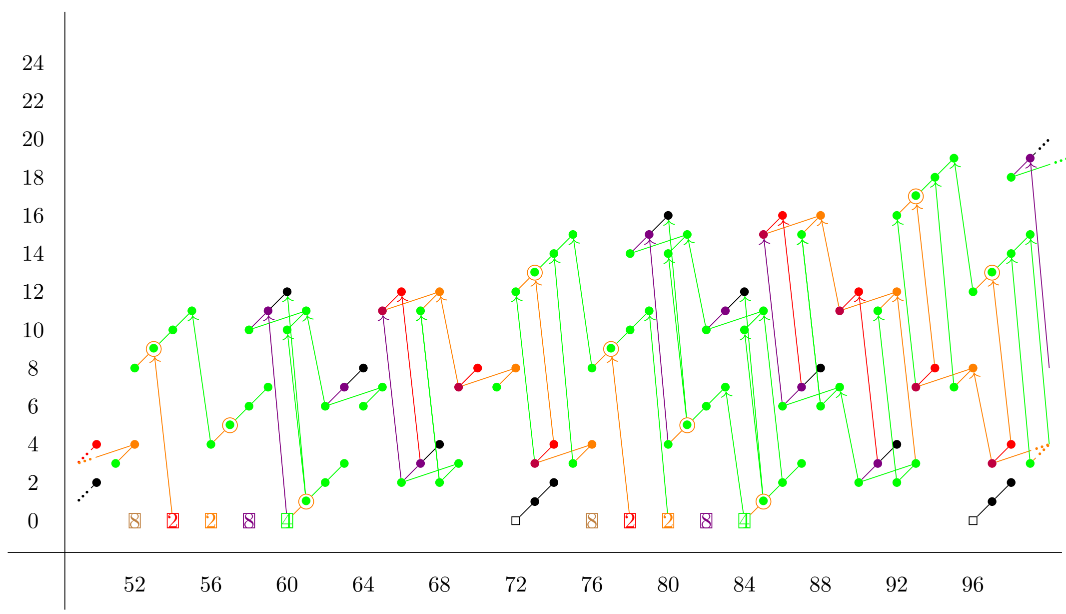

FIGURE 28. E_4-page and d_r, $r \geq 5$ of DSS for TJF_6

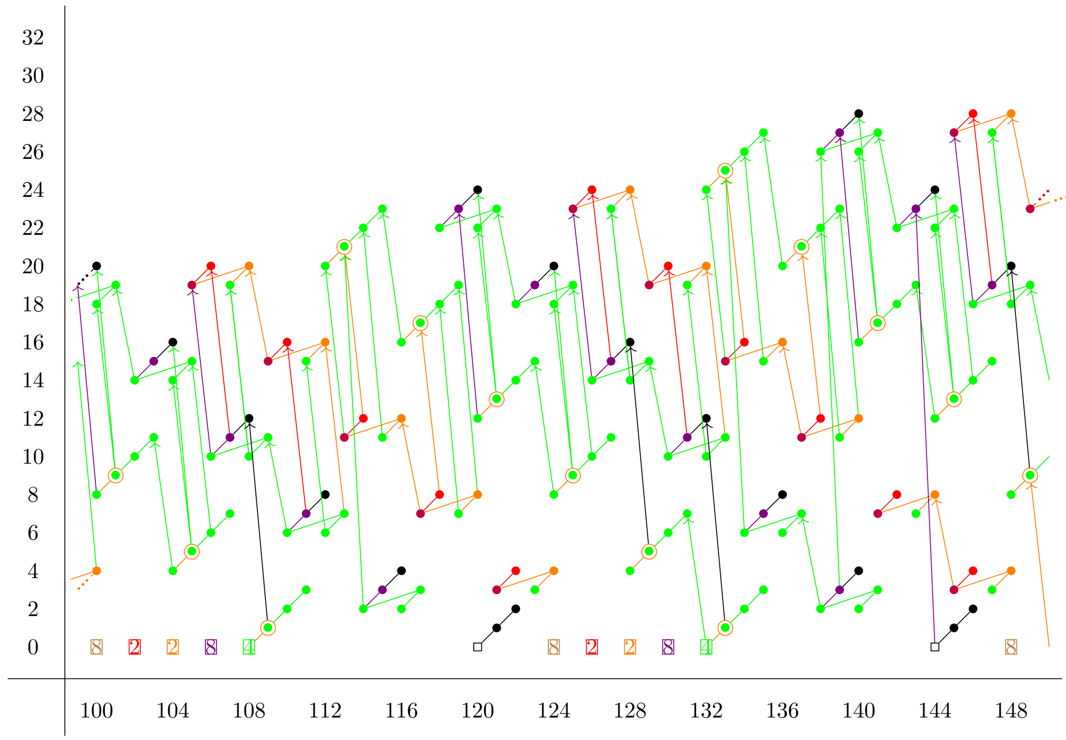

FIGURE 29. E_4-page and d_r, $r \geq 5$ of DSS for TJF_6

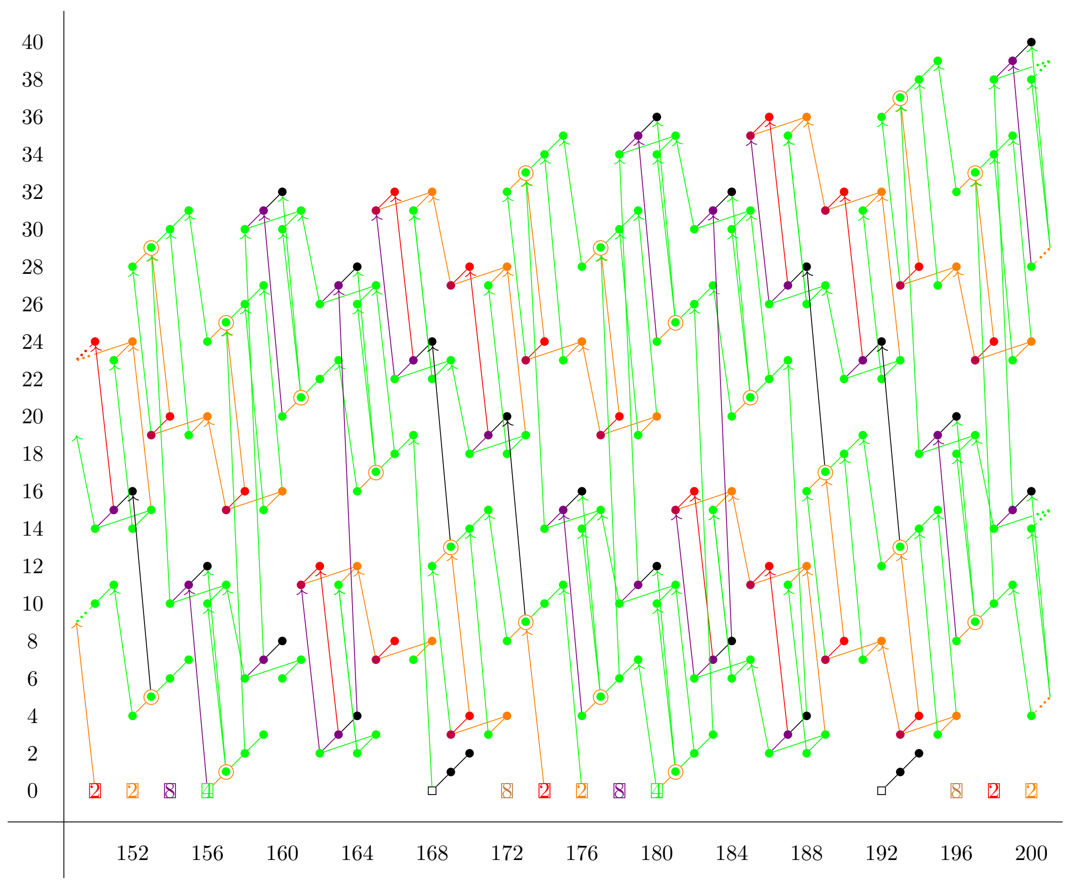

FIGURE 30. E_4-page and d_r, $r \geq 5$ of DSS for TJF_6

B.6. TJF_7. This section collects diagrams of the descenet spectral sequence for TJF_7.

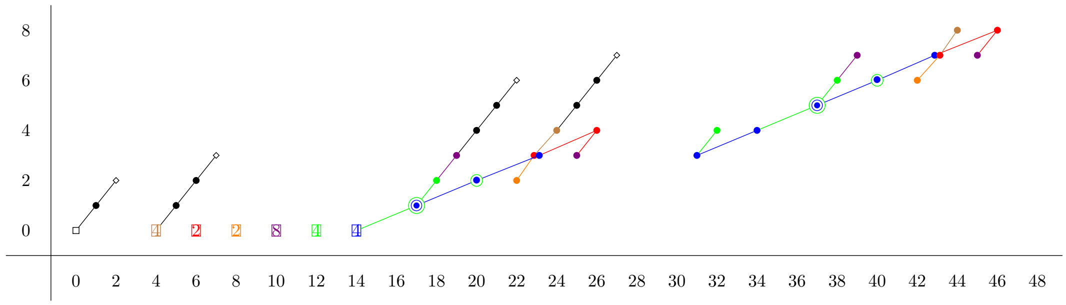

FIGURE 31. E_2-page of DSS for TJF_7

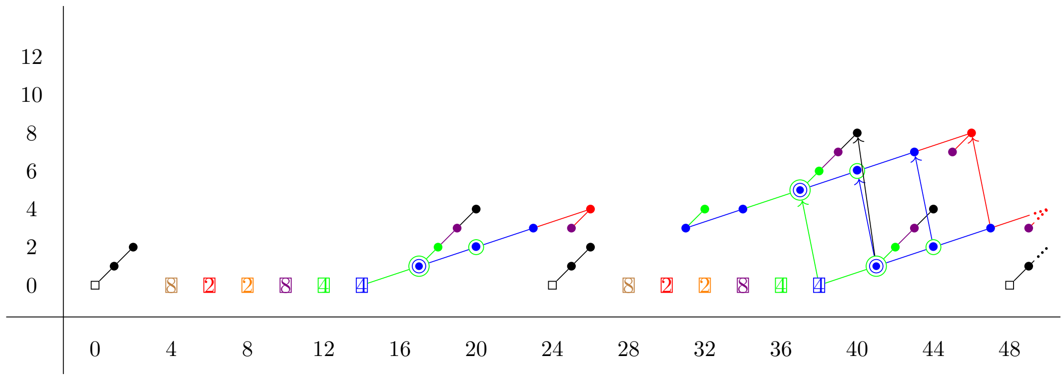

FIGURE 32. E_4-page and d_r, $r \geq 5$ of DSS for TJF_7

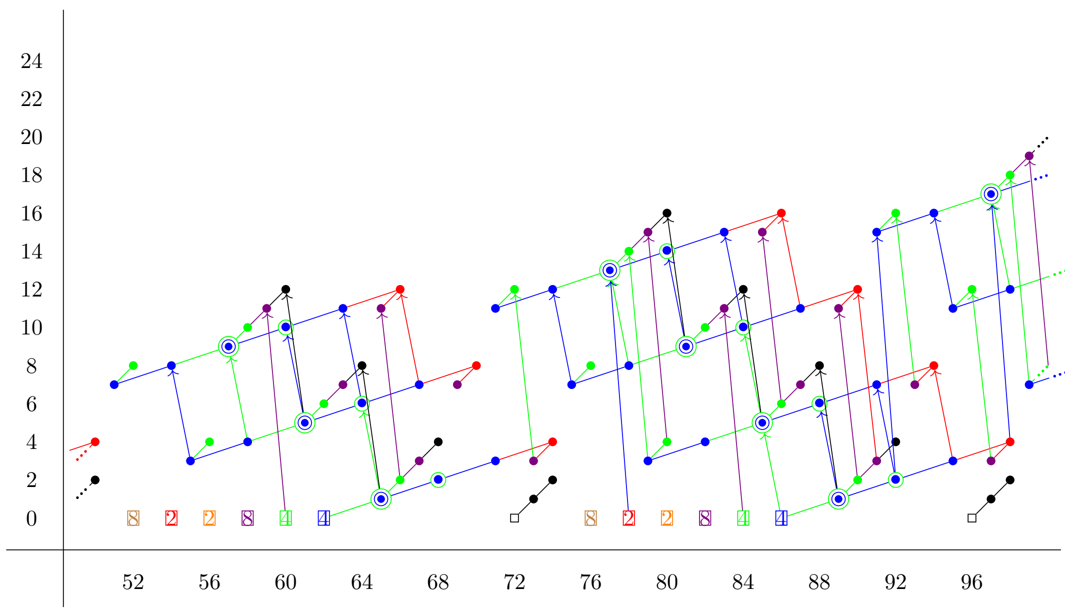

FIGURE 33. E_4-page and d_r, $r \geq 5$ of DSS for TJF_7

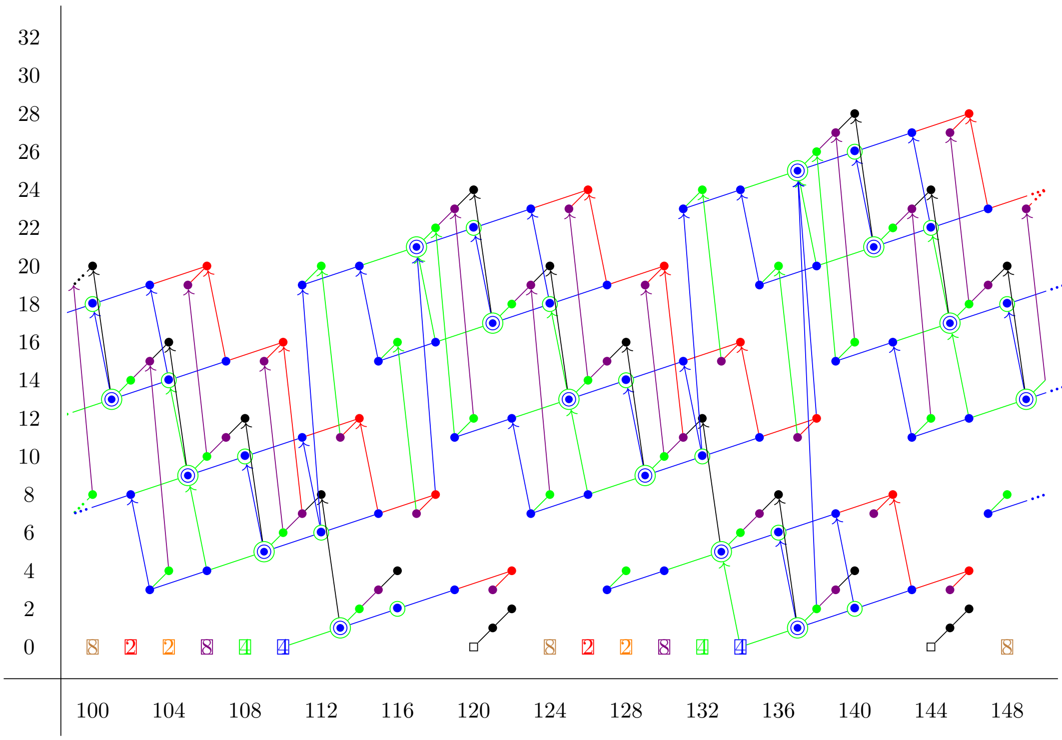

FIGURE 34. E_4-page and d_r, $r \geq 5$ of DSS for TJF_7

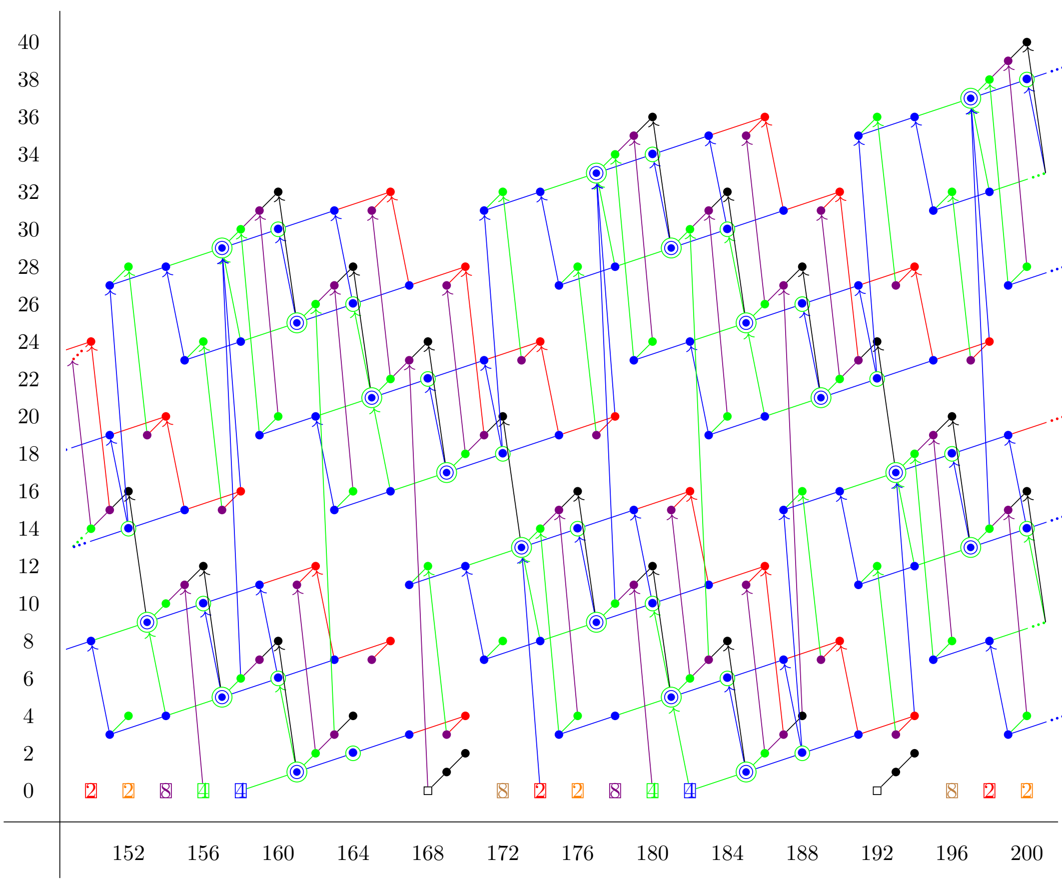

FIGURE 35. E_4-page and d_r, $r \geq 5$ of DSS for TJF_7

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