

# Calculus of Residues

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Residue theory is a powerful tool to kill some integrals. I have a proof and a little analysis in here, but I left most of them as exercises for you to do on your own. To properly develop the theory we need some other stuff first ...

## 1 Complex Integrals

Since in the complex plane, we're now on a plane, not a line, there's more than one way to get from one point to another. Therefore, most the integrals you see during this lecture are contour integrals. We define an integral along some path in  $\mathbb{C}$  the same way we do in multivar for  $\mathbb{R}^2$ . One really trivial thing you should realize is that:

**Theorem 1.1** (Cauchy's Integral Theorem).  $\oint_C f(z)dz = 0$

if  $f$  is conservative inside  $C$  and continuous on  $C$ .

**Exercise 1.1.** Prove this using the Cauchy-Riemann equations.

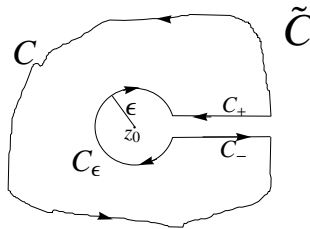
## 2 Cauchy's Integral Formula

**Theorem 2.1** (Cauchy's Integral Formula).  $\oint_C \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0)$

if  $f$  is conservative inside  $C$ .

*Proof.* Start with this contour, split as so:

$$\tilde{C} = C - C_\epsilon + C_+ + C_-$$



$$\oint_{\tilde{C}} \frac{f(z)dz}{z - z_0} = 0 = \int_C \frac{f(z)dz}{z - z_0} - \int_{C_\epsilon} \frac{f(z)dz}{z - z_0}$$

$$\int_C \frac{f(z)dz}{z - z_0} = \int_{C_\epsilon} \frac{f(z)dz}{z - z_0} = \int_{C_\epsilon} \frac{f(z_0 + \epsilon e^{i\theta})}{\epsilon e^{i\theta}} \epsilon i e^{i\theta} d\theta = i \int_0^{2\pi} f(z_0 + \epsilon e^{i\theta}) d\theta \rightarrow 2\pi i f(z_0)$$

□

Note that for this to hold,  $f$  has to be conservative, but not necessarily at  $z_0$

### 3 Laurent Series

Now we can find something interesting if we try to take series expansions using this fact, so let's do it! Let  $t$  be on some arbitrary  $C$ . We're looking for an expansion in powers of  $(z - a)$ , with  $a$  inside  $C$ .

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(t)dt}{t - z}$$

$$\frac{1}{t - z} = \frac{1}{(t - a) - (z - a)} = \frac{1}{t - a} \frac{1}{1 - \frac{z-a}{t-a}} = \frac{1}{t - a} \sum_{n=0}^{\infty} \left( \frac{z - a}{t - a} \right)^n$$

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(t)dt}{t - a} \sum_{n=0}^{\infty} \left( \frac{z - a}{t - a} \right)^n = \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z - a)^n \oint_C \frac{f(t)dt}{(t - a)^{n+1}}$$

Now, if we write  $f(z) = \sum_{n=-\infty}^{\infty} A_n(z - a)^n$ , we arrive at the nice results:

$$f^{(n)}(a) = n!A_n$$

$$A_n = \frac{1}{n!2\pi i} \oint_C \frac{f(t)dt}{(t - a)^{n+1}}$$

We've proved this for positive  $n$ , but not for negative  $n$ . That proof is left as an exercise.

**Exercise 3.1.** Prove that, for  $n < 0$ ,

$$A_n = \frac{1}{2\pi i} \oint_C \frac{f(t)dt}{(t - a)^{n+1}}$$

Since we also have negative powers of  $n$ , this expansion also works for functions that are not analytic, specifically, functions whose discontinuity is classified as a pole. The order of this pole is the lowest  $n$  for which  $A_n \neq 0$  in the expansion. A pole of order 1 is also called a simple pole. Furthermore, the residue is defined,

$$\text{Res}_{z=a} f(z) = A_{-1}$$

### 4 Residues

Here are three theorems that are useful and easy to prove:

**Theorem 4.1** (Residues for simple poles). If  $f(z) = \frac{g(z)}{h(z)}$ ,  $h(z_0) = 0$ ,  $h'(z_0) \neq 0$ , then

$$\text{Res}_{z=z_0} f(z) = \frac{g(z_0)}{h'(z_0)}$$

**Theorem 4.2** (Residues for poles of higher order). If  $z_0$  is an  $m^{\text{th}}$  order pole,

$$\text{Res}_{z=z_0} f(z) = \lim_{z \rightarrow z_0} \frac{1}{(m-1)!} D_z^{m-1} [(z - z_0)^m f(z)]$$

**Theorem 4.3** (Cauchy's Residue Theorem). If  $f$  has isolated poles at  $z_0, z_1, z_2, \dots, z_k$  inside  $C$ , and is analytic elsewhere inside  $C$ ,

$$\oint_C f(z)dz = 2\pi i \sum_{i=0}^k \text{Res}_{z=z_i} f(z)$$

**Exercise 4.1.** *Prove these*

Now that we can do integrals on contours in the complex plane, we can use them to evaluate real definite integrals. Finally, some more applied stuff.

## 5 Evaluation of Real Definite Integrals

Let's use this to find

$$\int_{-\infty}^{\infty} \frac{\cos(x)dx}{x^2 + 1}$$

First note:

$$\int_{-\infty}^{\infty} \frac{\cos(x)dx}{x^2 + 1} = \Re \left[ \int_{-\infty}^{\infty} \frac{e^{ix} dx}{(x-i)(x+i)} \right]$$

So we use this contour:

$$C = \mathcal{C}_{\infty}^0 + C_0$$

where  $C_0$  is along the real axis and  $C_{\infty}$  is the infinite upper half-circle.

$$\int_{-\infty}^{\infty} \frac{e^{ix} dx}{(x-i)(x+i)} = \int_{C_0} \frac{e^{iz} dz}{(z-i)(z+i)} = \oint_C \frac{e^{iz} dz}{(z-i)(z+i)} = 2\pi i \operatorname{Res}_{z=i} \frac{e^{iz}}{(z-i)(z+i)} = \frac{2\pi i e^{-1}}{2i} = \pi e^{-1}$$

$$\int_{-\infty}^{\infty} \frac{\cos(x)dx}{x^2 + 1} = \Re \left[ \oint_C \frac{e^{iz} dz}{(z-i)(z+i)} \right] = \Re \left[ \frac{\pi}{e} \right] = \frac{\pi}{e}$$

## 6 Lecture Problems

1.  $\int_{-\infty}^{\infty} \frac{\sin(x)dx}{x^2 + 1}$
2.  $\int_{-\infty}^{\infty} \frac{dx}{3x^2 + 2x + 10}$
3.  $\int_{-\infty}^{\infty} \frac{dx}{(x^2 + 1)^2}$
4.  $\oint_C \frac{\operatorname{Log}(z)}{(z^2 + 1)^2}$  where  $C$  is an annular, positively oriented contour around  $i$  and not around  $-i$ .
5.  $\int_0^{\infty} \frac{dx}{x^m + 1}$
6.  $\int_0^{\pi/2} \frac{d\theta}{(2 + 3 \cos \theta)^2}$