Algebriac Topology III (MAT484)

Lecture Notes

Contents

1 Singular Homology Theory

§1.1 Singular Homology Groups

Let $\mathbb{R}^\infty$ denote the generalized Euclidean space $\mathbb{E}^J$, with J being the set of positive integers. An element of the vector space $\mathbb{R}^\infty$ is an infinite sequence of real numbers (functions from $\mathbb{N}$ to $\mathbb{R}$) with finitely many nonzero entries. Let Δ_p denote the p-simplex in $\mathbb{R}^\infty$ having vertices

$$\begin{aligned}
\varepsilon_0 &= (1, 0, 0, \ldots, 0, \ldots)\ , \\
\varepsilon_1 &= (0, 1, 0, \ldots, 0, \ldots)\ , \\
&\cdots \\
\varepsilon_p &= (0, 0, 0, \ldots, \underbrace{1}_{(p+1)\text{-th entry}}, \ldots)\,.
\end{aligned}$$

We call Δ_p the **standard p-simplex**. In this notation, Δ_{p-1} is a face of Δ_p.

> **Definition 1.1** (Singular p-simplex). Let X be a topological space. We define a **singular p-simplex** of X to be a continuous map $T : \Delta_p \to X$. The free abelian group generated by singular p-simplices of X is denoted by $S_p(X)$, and is called the **singular chain group** of X in dimension p. We shall denote an element of $S_p(X)$ by a $\mathbb{Z}$-linear combination of singular p-simplices of X.

Singular means that T could be a "bad" map, i.e. it may not be an imbedding. All we want that T is just continuous. Now, recall that

$$\Delta_p = \left\{ (x_0, x_1, \ldots, x_p, 0, \ldots) \in \mathbb{R}^\infty | 0 \le x_i \le 1 \text{ and } \sum_{i=0}^{p} x_i = 1 \right\}. \tag{1.1}$$

Given $a_0, a_1, \ldots, a_p \in \mathbb{R}^\infty$, there is a unique affine map $l_{(a_0,\ldots,a_p)} : \Delta_p \to \mathbb{R}^\infty$ that maps ε_i to a_i. It is defined by

$$\begin{aligned}
l_{(a_0,\ldots,a_p)}(x_0, x_1, \ldots, x_p, 0, \ldots) &= \sum_{i=0}^{p} x_i a_i = \sum_{i=0}^{p} x_i a_i + a_0 - \sum_{i=0}^{p} x_i a_0 \\
&= a_0 + \sum_{i=0}^{p} x_i (a_i - a_0)\,.
\end{aligned} \tag{1.2}$$

We call this map the **linear singular simplex** determined by $a_0, a_1, \ldots, a_p \in \mathbb{R}^\infty$. Now, what is $l_{(\varepsilon_0,\ldots,\varepsilon_p)}$? Observe that

$$l_{(\varepsilon_0,\ldots,\varepsilon_p)}\varepsilon_i = l_{(\varepsilon_0,\ldots,\varepsilon_p)}(0, \ldots, 0, \underbrace{1}_{(i+1)\text{-th entry}}, 0, \ldots) = \varepsilon_i. \tag{1.3}$$

Therefore, $l_{(\varepsilon_0,\ldots,\varepsilon_p)}$ maps ε_i to itself, for every $i = 0, 1, \ldots, p$. Also,

$$l_{(\varepsilon_0,\ldots,\varepsilon_p)}(x_0, x_1, \ldots, x_p, 0, \ldots) = \sum_{i=0}^{p} x_i \varepsilon_i = (x_0, x_1, \ldots, x_p, 0, \ldots)\,. \tag{1.4}$$

Therefore, $l_{(\varepsilon_0,\ldots,\varepsilon_p)}$ is just the inclusion map of Δ_p into $\mathbb{R}^\infty$. Now, suppose $(x_0, x_1, \ldots, x_{p-1}, 0, \ldots) \in \Delta_{p-1}$, so that $\sum_{i=0}^{p-1} x_i = 1$. Then

$$\begin{aligned}
l_{(\varepsilon_0,\ldots,\widehat{\varepsilon_i},\ldots,\varepsilon_p)}(x_0, x_1, \ldots, x_{p-1}, 0, \ldots) &= x_0\varepsilon_0 + \cdots + x_{i-1}\varepsilon_{i-1} + 0 \cdot \varepsilon_i + x_{i+1}\varepsilon_{i+1} + \cdots + x_{p-1}\varepsilon_p \\
&= (x_0, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_{p-1}, 0, \ldots)\,,
\end{aligned} \tag{1.5}$$

which is a point on the face of Δ_p opposite to the vertex ε_i. In fact, $l_{(\varepsilon_0,\ldots,\widehat{\varepsilon_i},\ldots,\varepsilon_p)}$ is a linear homomorphism of Δ_{p-1} into the face of Δ_p that is opposite to the vertex ε_i. In other words,

$$l_{(\varepsilon_0,\ldots,\widehat{\varepsilon_i},\ldots,\varepsilon_p)} : \Delta_{p-1} \to \Delta_p$$

maps Δ_{p-1} to the face of Δ_p opposite to the vertex ε_i. Therefore, given a singular p-simplex $T : \Delta_p \to X$, one can form the composite

$$T \circ l_{(\varepsilon_0,\ldots,\widehat{\varepsilon_i},\ldots,\varepsilon_p)} : \Delta_{p-1} \to X,$$

which is a singular $(p-1)$-simplex. We think of it as the i-th face of the singular p-simplex T.

Definition 1.2 (Boundary homomorphism)**.** We define $\partial : S_p(X) \to S_{p-1}(X)$ as follows. If $T : \Delta_p \to X$ is a singular p-simplex, we define ∂T to be

$$\partial T = \sum_{i=0}^{p} (-1)^i T \circ l_{(\varepsilon_0,\ldots,\widehat{\varepsilon_i},\ldots,\varepsilon_p)}. \tag{1.6}$$

In other words, ∂T is a formal sum of singular simplices of dimension $p-1$, which are the faces of T.

Remark 1.1. Note that only the singular p-simplices are maps, not the singular p-chains. The p-chains are just formal sum of continuous maps from Δ_p to X. If T_1 and T_2 are two singular p-simplices, i.e. continuous maps $\Delta_p \to X$, then $T_1 + T_2$ is **NOT** a map. The sum present here is nothing but a formal notation. For the same reason, ∂T_1 is not a map. It is merely a formal linear combination of the continuous maps $T \circ l_{(\varepsilon_0,\ldots,\widehat{\varepsilon_i},\ldots,\varepsilon_p)}$.

If $f : X \to Y$ is a continuous map, we define a group homomorphism $f_\# : S_p(X) \to S_p(Y)$ by defining it on singular p-simplices by the equation

$$f_\#(T) = f \circ T \tag{1.7}$$

for a singular p-simplex T.

$$\begin{tikzcd} \Delta_p \arrow[r, "T"] \arrow[rr, bend right, "f\circ T"'] & X \arrow[r, "f"] & Y \end{tikzcd}$$

Theorem 1.1

The homomorphism $f_\#$ commutes with ∂. Furthermore, $\partial^2 = 0$.

Proof. Given a singular p-simplex T,

$$\partial f_\#(T) = \partial(f \circ T) = \sum_{i=0}^{p} (-1)^i (f \circ T) \circ l_{(\varepsilon_0,\ldots,\widehat{\varepsilon_i},\ldots,\varepsilon_p)}. \tag{1.8}$$

$$f_\#(\partial T) = f_\#\left(\sum_{i=0}^{p} (-1)^i T \circ l_{(\varepsilon_0,\ldots,\widehat{\varepsilon_i},\ldots,\varepsilon_p)}\right) = \sum_{i=0}^{p} (-1)^i f \circ T \circ l_{(\varepsilon_0,\ldots,\widehat{\varepsilon_i},\ldots,\varepsilon_p)}. \tag{1.9}$$

Therefore, $\partial f_\#(T) = f_\#(\partial T)$. Now, to prove $\partial^2 = 0$, we first compute ∂ for linear singular simplices $l_{(a_0,\ldots,a_p)}$.

$$\partial l_{(a_0,\ldots,a_p)} = \sum_{i=0}^{p} (-1)^i l_{(a_0,\ldots,a_p)} \circ l_{(\varepsilon_0,\ldots,\widehat{\varepsilon_i},\ldots,\varepsilon_p)}. \tag{1.10}$$

Observe that

$$\begin{aligned} l_{(a_0,\ldots,a_p)} \circ l_{(\varepsilon_0,\ldots,\widehat{\varepsilon_i},\ldots,\varepsilon_p)}(x_0,\ldots,x_{p-1},0,\ldots) &= l_{(a_0,\ldots,a_p)}(x_0,\ldots,x_{i-1},0,x_i x_{p-1},0,) \\ &= x_0a_0 + \cdots + x_{i-1}a_{i-1} + 0\cdot a_i + x_ia_{i+1} + \cdots x_{p-1}a_p \\ &= l_{(a_0,\ldots,\widehat{a_i},\ldots a_p)}(x_0,\ldots,x_{p-1},0,\ldots). \end{aligned} \tag{1.11}$$

Hence,

$$l_{(a_0,\dots,a_p)} \circ l_{(\varepsilon_0,\dots,\widehat{\varepsilon_i},\dots,\varepsilon_p)} = l_{(a_0,\dots,\widehat{a_i},\dots a_p)}. \tag{1.12}$$

Therefore, from 1.10, it follows that

$$\partial l_{(a_0,\dots,a_p)} = \sum_{i=0}^{p} (-1)^i \, l_{(a_0,\dots,\widehat{a_i},\dots a_p)}. \tag{1.13}$$

Let's now evaluate $\partial \partial l_{(a_0,\dots,a_p)}$.

$$\begin{aligned}
\partial \partial l_{(a_0,\dots,a_p)} &= \sum_{i=0}^{p} (-1)^i \, \partial l_{(a_0,\dots,\widehat{a_i},\dots a_p)} \\
&= \sum_{i=0}^{p} (-1)^i \sum_{j<i} (-1)^j \, l_{(a_0,\dots,\widehat{a_j},\dots \widehat{a_i},\dots a_p)} + \sum_{i=0}^{p} (-1)^i \sum_{j>i} (-1)^{j-1} \, l_{(a_0,\dots,\widehat{a_i},\dots \widehat{a_j},\dots a_p)} \\
&= \sum_{i=0}^{p} \sum_{j<i} (-1)^{i+j} \, l_{(a_0,\dots,\widehat{a_j},\dots \widehat{a_i},\dots a_p)} - \sum_{i=0}^{p} \sum_{j>i} (-1)^{i+j} \, l_{(a_0,\dots,\widehat{a_i},\dots \widehat{a_j},\dots a_p)}.
\end{aligned} \tag{1.14}$$

Now fix $0 \leq j_0 < i_0 \leq p$. In the first summand of 1.14, the contribution of $i = i_0, j = j_0$ is

$$(-1)^{i_0+j_0} \, l_{\left(a_0,\dots,\widehat{a_{j_0}},\dots \widehat{a_{i_0}},\dots a_p\right)}. \tag{1.15}$$

On the other hand, in the second summand of 1.14, the contribution of $i = j_0, j = i_0$ is also

$$(-1)^{i_0+j_0} \, l_{\left(a_0,\dots,\widehat{a_{j_0}},\dots \widehat{a_{i_0}},\dots a_p\right)}. \tag{1.16}$$

These two contributions cancel each other. This way, one finds that the RHS of 1.14 vanishes. Hence,

$$\partial \partial l_{(a_0,\dots,a_p)} = 0. \tag{1.17}$$

In particular,

$$\partial \partial l_{(\varepsilon_0,\dots,\varepsilon_p)} = 0. \tag{1.18}$$

Now, $l_{(\varepsilon_0,\dots,\varepsilon_p)} : \Delta_p \to \Delta_p$ is continuous, so $l_{(\varepsilon_0,\dots,\varepsilon_p)} \in S_p\left(\Delta_p\right)$. Furthermore, it is the identity map as we have seen in 1.4. Since $T : \Delta_p \to X$ is continuous, we can form $T_\# : S_p\left(\Delta_p\right) \to S_p\left(X\right)$.

$$T_\# \left(l_{(\varepsilon_0,\dots,\varepsilon_p)}\right) = T \circ l_{(\varepsilon_0,\dots,\varepsilon_p)} = T \circ \mathrm{id}_{\Delta_p} = T. \tag{1.19}$$

Therefore, using the fact that $T_\#$ commutes with ∂, we obtain

$$\partial \partial T = \partial \partial T_\# \left(l_{(\varepsilon_0,\dots,\varepsilon_p)}\right) = T_\# \left(\partial \partial l_{(\varepsilon_0,\dots,\varepsilon_p)}\right) = 0. \tag{1.20}$$

Hence, $\partial^2 T = 0$. ■

Definition 1.3 (Singular homology groups)**.** Th family of groups $S_p\left(X\right)$ and homomorphisms $\partial_p : S_p\left(X\right) \to S_{p-1}\left(X\right)$ is called **singular chain complex** of X, and is denoted by $\mathcal{S}\left(X\right)$. We will be attaching the index p with the homomorphism while dealing with singular chain complex:

$$\cdots \longrightarrow S_{p+1}(X) \xrightarrow{\partial_{p+1}} S_p(X) \xrightarrow{\partial_p} S_{p-1}(X) \longrightarrow \cdots$$

The homology groups of this chain complex are called the **singular homology groups** of X, and are denoted by $H_p\left(X\right)$.

Definition 1.4 (Augmentation map)**.** The chain complex $\mathcal{S}(X)$ is augmented by the homomorphism $\epsilon : S_0(X) \to \mathbb{Z}$ defined by setting $\epsilon(T) = 1$ for each singular 0-simplex $T : \Delta_0 \to X$. (A generic singular 0-chain is a $\mathbb{Z}$-linear combination of singular 0-simplices.)

It's immediate that if T is a singular 1-simplex, then $\epsilon(\partial T) = 0$. Indeed,

$$\epsilon(\partial T) = \epsilon\left(T \circ l_{(\widehat{\varepsilon_0},\varepsilon_1)}\right) - \epsilon\left(T \circ l_{(\varepsilon_0,\widehat{\varepsilon_1})}\right) = 0. \tag{1.21}$$

Definition 1.5 (Reduced homology groups)**.** The homology groups of $\{\mathcal{S}(X), \epsilon\}$ are called the **reduced singular homology groups** of X, and are denoted by $\widetilde{H}_p(X)$.

Now, given continuous map $f : X \to Y$ and $T : \Delta_0 \to X$ a singular 0-simplex on X, then $f_\#(T) = f \circ T : \Delta_0 \to Y$.

$$\begin{tikzcd} \Delta_0 \arrow[r, "T"] \arrow[rr, bend right, "f\circ T"'] & X \arrow[r, "f"] & Y \end{tikzcd}$$

Now, consider the augmented singular chain complexes $\left\{\mathcal{S}(X), \epsilon^X\right\}$ and $\left\{\mathcal{S}(Y), \epsilon^Y\right\}$. Noting continuous $T : \Delta_0 \to X$ and $f_\#(T) : \Delta_0 \to Y$, one obtains $\epsilon^X(T) = 1$ and $\epsilon^Y(f_\#(T)) = 1$. In other words, the following diagram commutes

$$\begin{tikzcd} S_0(X) \arrow[r, "\epsilon^X"] \arrow[d, "(f_\#)_0"'] & \mathbb{Z} \arrow[d, "\mathrm{id}"] \\ S_0(Y) \arrow[r, "\epsilon^Y"] & \mathbb{Z} \end{tikzcd}$$

Therefore, $f_\# : S_p(X) \to S_p(Y)$ is an **augmentation preserving chain map** between $\left\{\mathcal{S}(X), \epsilon^X\right\}$ and $\left\{\mathcal{S}(Y), \epsilon^Y\right\}$. Thus, $f_\#$ induces a homomorphism f_* in both ordinary and reduced singular homology.

In Theorem 1.1, we saw that the chain map $f_\#$ commutes with the boundary operator ∂. In other words, $(f_\#)_p : S_p(X) \to S_p(Y)$ takes cycles to cycles and boundaries to boundaries. Suppose $c_p \in Z_p(X) = \operatorname{Ker} \partial_p^X$, so that $\partial_p^X c_p = 0$. Now,

$$\partial_p^Y\left((f_\#)_p c_p\right) = (f_\#)_{p-1}\left(\partial_p^X c_p\right) = 0. \tag{1.22}$$

Hence, $(f_\#)_p c_p \in Z_p(Y)$. On the other hand, let $b_p \in B_p(X) = \operatorname{Im} \partial_{p+1}^X$. Then $b_p = \partial_{p+1}^X d_{p+1}$ for some $d_{p+1} \in S_{p+1}(X)$. Then

$$(f_\#)_p b_p = (f_\#)_p\left(\partial_{p+1}^X d_{p+1}\right) = \partial_{p+1}^Y\left((f_\#)_{p+1} d_{p+1}\right). \tag{1.23}$$

In other words, $(f_\#)_p b_p \in B_p(Y)$. This reflects the fact that $(f_\#)_p : S_p(X) \to S_p(Y)$ induces a homomorphism between the singular homology groups $(f_*)_p : H_p(X) \to H_p(Y)$. $(f_*)_p$ is given by

$$(f_*)_p(c_p + B_p(X)) = (f_\#)_p c_p + B_p(Y). \tag{1.24}$$

If the reduced homology groups of X vanishes in all dimensions, we say that X is **acyclic** (in singular homology).

Theorem 1.2

If $i : X \to X$ is the identity, then so is $(i_*)_p : H_p(X) \to H_p(X)$. If $f : X \to Y$ and $g : Y \to Z$ are continuous, then $((g \circ f)_*)_p = (g_*)_p \circ (f_*)_p$.

Proof. It is sufficient to show that the equations hold at the chain level. We know from the definition of $(f_\#)_p : S_p(X) \to S_p(Y)$ that it maps $T \in S_p(X)$ to $f \circ T \in S_p(Y)$. Since $i : X \to X$ is the identity map,

$$(i_\#)_p(T) = i \circ T = T. \tag{1.25}$$

So $(i_\#)_p : S_p(X) \to S_p(X)$ is the identity homomorphism. As a result,

$$(i_*)_p(c_p + B_p(X)) = (i_\#)_p c_p + B_p(X) = c_p + B_p(X). \tag{1.26}$$

Therefore, $(i_*)_p = \mathrm{id}_{H_p(X)}$.

Given continuous $f : X \to Y$ and $g : Y \to Z$, $\left((g \circ f)_\#\right)_p : S_p(X) \to S_p(Z)$ is defined by

$$\left((g \circ f)_\#\right)_p T = (g \circ f) \circ T = g \circ (f \circ T) = (g_\#)_p \left((f_\#)_p T\right). \tag{1.27}$$

Therefore, $\left((g \circ f)_\#\right)_p = (g_\#)_p \circ (f_\#)_p$. Now, at the homology level, for $c_p + B_p(X) \in H_p(X) = Z_p(X)/B_p(X)$

$$((g \circ f)_*)_p(c_p + B_p(X)) = \left((g \circ f)_\#\right)_p c_p + B_p(Z) = (g_\#)_p \left((f_\#)_p c_p\right) + B_p(Z). \tag{1.28}$$

Also,

$$(g_*)_p \circ (f_*)_p(c_p + B_p(X)) = (g_*)_p \left((f_\#)_p c_p + B_p(Y)\right) = (g_\#)_p \left((f_\#)_p c_p\right) + B_p(Z). \tag{1.29}$$

From 1.28 and 1.29, we can deduce that $((g \circ f)_*)_p = (g_*)_p \circ (f_*)_p$. ■

Corollary 1.3

If $h : X \to Y$ is a homeomorphims, then $(h_*)_p : H_p(X) \to H_p(Y)$ is an isomorphism.

Proof. Both $h : X \to Y$ and $h^{-1} : Y \to X$ are continuous, and $h \circ h^{-1} = \mathrm{id}_Y$. Therefore,

$$(h_*)_p \circ \left((h^{-1})_*\right)_p = \left((h \circ h^{-1})_*\right)_p = ((\mathrm{id}_Y)_*)_p = \mathrm{id}_{H_p(Y)}. \tag{1.30}$$

Similarly, starting with $h^{-1} \circ h = \mathrm{id}_X$, we will get $\left((h^{-1})_*\right)_p \circ (h_*)_p = \mathrm{id}_{H_p(X)}$. Therefore, $\left((h^{-1})_*\right)_p$ is the inverse of $(h_*)_p$. In other words, $(h_*)_p$ is an invertible homomorphism, i.e. an isomorphism. ■

Theorem 1.4

Let X be a topological space. Then $H_0(X)$ is free abelian. If $\{X_\alpha\}$ is the collection of path components of X, and if T_α is a singular 0-simplex with image in X_α for each α, then the homology classes of the chains T_α form a basis for $H_0(X)$. The group $\widetilde{H}_0(X)$ is also free abelian; it vanishes if X is path connected. Otherwise, let α_0 be a fixed index, then the homology classes of the chains $T_\alpha - T_{\alpha_0}$ for $\alpha \neq \alpha_0$ form a basis for $\widetilde{H}_0(X)$.

Proof. Let $x_\alpha = T_\alpha(\Delta_0) \in X_\alpha$, with $T_\alpha : \Delta_0 \to X$ being a singular 0-simplex. Here, Δ_0 consists of the point $\varepsilon_0 = (1, 0, 0, \ldots) \in \mathbb{R}^\infty$. Also, let $T : \Delta_0 \to X$ be any singular 0-simplex such that $T(\Delta_0) \in X_\alpha$. Since X_α is path connected, there is a path connecting $T(\Delta_0)$ and $T_\alpha(\Delta_0)$. In other words, there is a singular 1-simplex $f : \Delta_1 \to X$ such that

$$f(1, 0, 0\ldots) = T(\Delta_0) \text{ and } f(0, 1, 0\ldots) = T_\alpha(\Delta_0). \tag{1.31}$$

Then we have
$$\partial_1 f = f \circ l_{(\widehat{\varepsilon_0},\varepsilon_1)} - f \circ l_{(\varepsilon_0,\widehat{\varepsilon_1})}. \tag{1.32}$$
Now,
$$f \circ l_{(\varepsilon_0,\widehat{\varepsilon_1})}(1,0,0,\dots) = f(1,0,0,\dots) = T(\Delta_0) = T(1,0,0,\dots), \tag{1.33}$$
$$f \circ l_{(\widehat{\varepsilon_0},\varepsilon_1)}(1,0,0,\dots) = f(0,1,0,\dots) = T_\alpha(\Delta_0) = T_\alpha(1,0,0,\dots). \tag{1.34}$$
Therefore, $\partial_1 f = T_\alpha - T$.

An arbitrary singular 0-chain is a $\mathbb{Z}$-linear combination of singular 0-simplices. Let's take $c \in S_0(X)$. Then $c = \sum_\beta m_\beta T'_\beta$, with $m_\beta \in \mathbb{Z}$ and T'_β being singular 0-simplices. Each $T'_\beta(\Delta_0)$ belongs to some X_α, and hence homologous to T_α. Therefore, c is homologous to some $\mathbb{Z}$-linear combination $\sum_\alpha n_\alpha T_\alpha$ of the T_α's. We will now show that no such nontrivial 0-chain $\sum_\alpha n_\alpha T_\alpha$ bounds.

Assume the contrary that $\sum_\alpha n_\alpha T_\alpha = \partial_1 d$ for some $d \in S_1(X)$. Now, the singular 1-chain d is a formal linear combination of singular 1-simplices with path connected image, i.e. the image lies in one of the path components X_α. Thus we can write $d = \sum_\alpha d_\alpha$, where d_α consists of the terms whose images are in X_α. Therefore,
$$\sum_\alpha n_\alpha T_\alpha = \partial_1 d = \sum_\alpha \partial_1 d_\alpha. \tag{1.35}$$
Hence, we get
$$n_\alpha T_\alpha = \partial_1 d_\alpha \tag{1.36}$$
for each α. Applying ϵ to both sides of 1.36, we get
$$\epsilon(n_\alpha T_\alpha) = \epsilon(\partial_1 d_\alpha) \implies n_\alpha = 0. \tag{1.37}$$
Therefore, no non-trivial 0-chain $\sum_\alpha n_\alpha T_\alpha$ bounds. Since every 0-chain is automatically a 0-cycle, an element of $H_0(X)$ is homologous to a 0-chain of the form $\sum_\alpha n_\alpha T_\alpha$. Hence, the homology classes of the singular 0-simplices $\{T_\alpha\}$ form a basis for the free abelian group $H_0(X)$.

$$S_1(X) \xrightarrow{\partial_1} S_0(X) \xrightarrow{\epsilon} \mathbb{Z}$$

$\widetilde{H}_0(X)$ is defined as $\widetilde{H}_0(X) = \operatorname{Ker}\epsilon / \operatorname{Im}\partial_1$. Given a singular 0-chain $T \in S_0(X)$, we've seen that T is homologous to a 0-chain of the form $T' = \sum_\alpha n_\alpha T_\alpha$; and T' bounds iff $T' = 0$, i.e. $n_\alpha = 0$ for every α. If further $T \in \operatorname{Ker}\epsilon$, then $\epsilon(T) = 0$. Since T and T' are homologous, $T = T' + \partial_1 d$ for some $d \in S_1(X)$. Therefore,
$$0 = \epsilon(T) = \epsilon(T') + \epsilon(\partial_1 d) = \epsilon\left(\sum_\alpha n_\alpha T_\alpha\right) = \sum_\alpha n_\alpha. \tag{1.38}$$
If X is path connected, there is only one component, and hence there is only one n_α involved. Thus $n_\alpha = 0$ from 1.38. This gives us $T' = 0$, leading to the fact that every $T \in \operatorname{Ker}\epsilon$ is homologous to 0, i.e. $T = 0 + \partial_1 d$ for some $d \in S_1(X)$. So $\operatorname{Ker}\epsilon = \operatorname{Im}\partial_1$. Therefore, $\widetilde{H}_0(X) = 0$, when X is path connected.

Now, suppose X has more than one path components. Fix α_0. Then from 1.38, we get
$$0 = \sum_\alpha n_\alpha = n_{\alpha_0} + \sum_{\alpha \neq \alpha_0} n_\alpha \implies n_{\alpha_0} = -\sum_{\alpha \neq \alpha_0} n_\alpha. \tag{1.39}$$
Then T' is
$$T' = \sum_\alpha n_\alpha T_\alpha = \sum_{\alpha \neq \alpha_0} n_\alpha T_\alpha + n_{\alpha_0} T_{\alpha_0} = \sum_{\alpha \neq \alpha_0} n_\alpha T_\alpha - \sum_{\alpha \neq \alpha_0} n_\alpha T_{\alpha_0} = \sum_{\alpha \neq \alpha_0} n_\alpha (T_\alpha - T_{\alpha_0}). \tag{1.40}$$
1.40 suggests that T' is a linear combination of the singular 0-chains $\{T_\alpha - T_{\alpha_0}\}_{\alpha \neq \alpha_0}$. And T' bounds iff it is trivial, as shown earlier. Therefore, the homology classes of 0-chains $\{T_\alpha - T_{\alpha_0}\}_{\alpha \neq \alpha_0}$ form a basis for $\widetilde{H}_0(X)$. ■

Theorem 1.4 illustrates the following result:
$$H_p(X) = \begin{cases} \widetilde{H}_p(X) & \text{if } p > 0 \\ \widetilde{H}_0(X) \oplus \mathbb{Z} & \text{if } p = 0 \end{cases}. \tag{1.41}$$

§1.2 Bracket Operation

Definition 1.6 (Star convex set). A set $X \subseteq \mathbb{E}^J$ is said to be **star convex** relative to the point $w \in X$, if for each $x \in X$, the line segment from x to w lies in X.

Definition 1.7 (Bracket operation). Suppose $X \in \mathbb{E}^J$ is star convex relative to w. We define bracket operation on singular chains of X. Let us first define it for singular p-simplices. Let $T : \Delta_p \to X$ be a singular p-simplex of X. Define a singular $(p+1)$-simplex

$$[T, w] : \Delta_{p+1} \to X$$

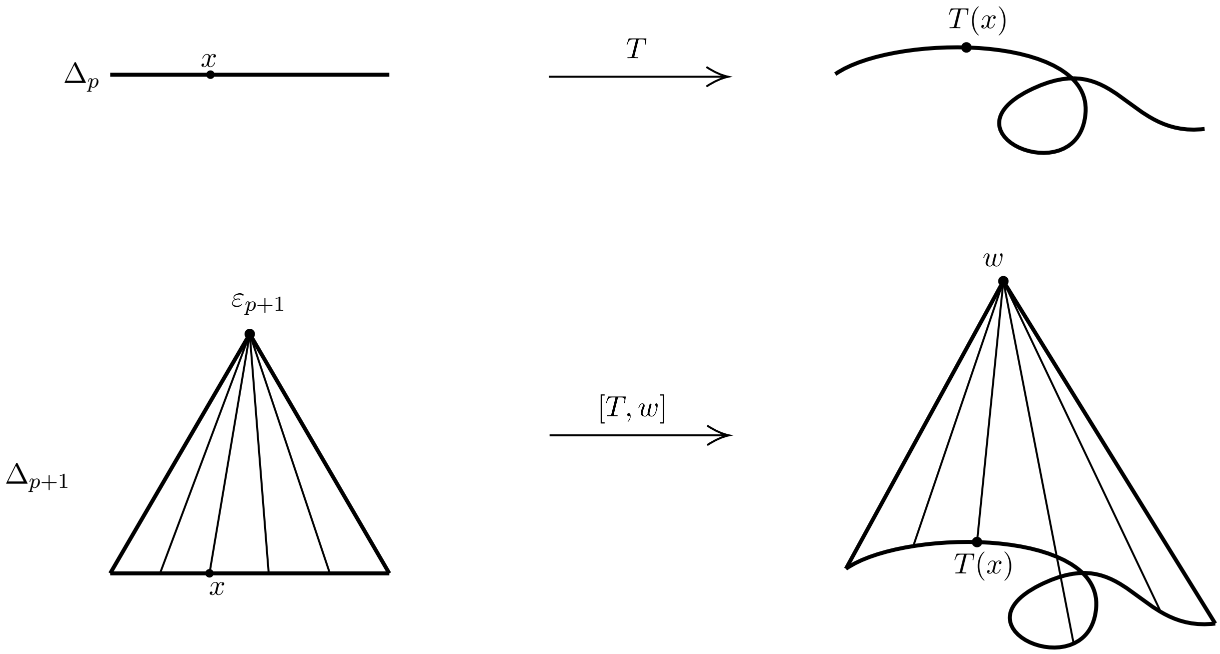
by letting $[T, w]$ carry the line segment from x to ε_{p+1}, for $x \in\in \Delta_p$ (the collection of all such line segments as x varies in Δ_p constitutes Δ_{p+1}), linearly onto the line segment $T(x)$ to w in X. In other words,

$$[T, w]\left(t\varepsilon_{p+1} + (1-t)\, x\right) = tw + (1-t)\, T(x)\,, \tag{1.42}$$

for $t \in [0, 1]$. Now, extend the definition of bracket operation to arbitrary p-chains as follows: if $c = \sum n_i T_i$ is a singular p-chain of X with each T_i being a singular p-simplex, then we define

$$[c, w] = \sum n_i \left[T_i, w\right]. \tag{1.43}$$

In other words, $[\cdot\,, w] : S_p(X) \to S_{p+1}(X)$, $c \mapsto [c, w]$ is a homomorphism.



From the diagram above, it's immediate that the restriction of $[T, w]$ to the face Δ_p of Δ_{p+1} is just the map T. Now, consider the case when T is the linear singular simplex $l_{(a_0,\dots,a_p)}$ for $a_0, \dots, a_p \in \mathbb{R}^\infty$. We want to calculate what $\left[l_{(a_0,\dots,a_p)}, w\right]$ is.

Recall that $l_{(a_0,\dots,a_p)} : \Delta_p \to \mathbb{R}^\infty$ is defined as

$$l_{(a_0,\dots,a_p)}\left(x_0, \dots, x_p\right) = \sum_{i=0}^{p} x_i a_i. \tag{1.44}$$

Consider a point $(x_0, \dots, x_p, x_{p+1}, 0, \dots) \in \Delta_{p+1}$. We want to see where $\left[l_{(a_0,\dots,a_p)}, w\right]$ takes this point to. Since $(x_0,\ldots,x_p,x_{p+1},0,\ldots)\in\Delta_{p+1}$, each x_i is nonnegative with $\sum_{i=0}^{p+1}x_i=1$. Now,

$$\sum_{i=0}^{p}\frac{x_i}{1-x_{p+1}}=1, \tag{1.45}$$

so $\left(\frac{x_0}{1-x_{p+1}},\frac{x_1}{1-x_{p+1}},\ldots,\frac{x_p}{1-x_{p+1}},0,\ldots\right)\in\Delta_p$. Therefore,

$$(x_0,\ldots,x_p,x_{p+1},0,\ldots)=(1-x_{p+1})\left(\frac{x_0}{1-x_{p+1}},\frac{x_1}{1-x_{p+1}},\ldots,\frac{x_p}{1-x_{p+1}},0,\ldots\right)+x_{p+1}\varepsilon_{p+1}. \tag{1.46}$$

By the definition of bracket operation,

$$\begin{aligned}
&\left[l_{(a_0,\ldots,a_p)},w\right](x_0,\ldots,x_p,x_{p+1},0,\ldots)\\
=\;&(1-x_{p+1})\,l_{(a_0,\ldots,a_p)}\left(\frac{x_0}{1-x_{p+1}},\frac{x_1}{1-x_{p+1}},\ldots,\frac{x_p}{1-x_{p+1}},0,\ldots\right)+x_{p+1}w\\
=\;&(1-x_{p+1})\sum_{i=0}^{p}\frac{x_i}{1-x_{p+1}}a_i+x_{p+1}w\\
=\;&\sum_{i=0}^{p}x_ia_i+x_{p+1}w.
\end{aligned} \tag{1.47}$$

Furthermore,

$$l_{(a_0,\ldots,a_p,w)}(x_0,\ldots,x_p,x_{p+1},0,\ldots)=x_0a_0+\cdots+x_pa_p+x_{p+1}w=\sum_{i=0}^{p}x_ia_i+x_{p+1}w. \tag{1.48}$$

Equating 1.47 and 1.48, we get

$$\left[l_{(a_0,\ldots,a_p)},w\right]=l_{(a_0,\ldots,a_p,w)}. \tag{1.49}$$

Now we will showo that $[T,w]:\Delta_{p+1}\to X$ is continuous. We have seen earlier that given $x\in\Delta_p$, a point in Δ_{p+1} is expressed as $t\varepsilon_{p+1}+(1-t)\,x$, with $0\le t\le 1$. Hence, we are concerened with the following quotient map $\pi:\Delta_p\times[0,1]\to\Delta_{p+1}$ defined by

$$\pi(x,t)=t\varepsilon_{p+1}+(1-t)\,x. \tag{1.50}$$

If $x=(x_0,\ldots,x_p,0,\ldots)\in\Delta_p$, then 1.50 takes the familiar form

$$\pi((x_0,\ldots,x_p,0,\ldots),t)=((1-t)\,x_0,\ldots,(1-t)\,x_p,t,0,\ldots). \tag{1.51}$$

Observe that $\pi\big|_{\Delta_p\times[0,1)}:\Delta_p\times[0,1)\to\Delta_{p+1}$ is 1-1, and $\pi(\Delta_p\times\{1\})=\{\varepsilon_{p+1}\}$, showing that π collapses $\Delta_p\times\{1\}$ to the $(p+1)$-th vertex ε_{p+1} of Δ_{p+1}. Now, the continuous map $f:\Delta_p\times[0,1]\to X$ defined by

$$f(x,t)=tw+(1-t)\,T(x) \tag{1.52}$$

is constant on $\Delta_p\times\{1\}$. In fact, $f(\Delta_p\times\{1\})=\{w\}$. Since π is $1-1$ for other points, f is seen to be constant for $\pi^{-1}(y)$ with $y\in\Delta_{p+1}\setminus\{\varepsilon_{p+1}\}$. In other words, $f:\Delta_p\times[0,1]\to X$ is constant for each $\pi^{-1}(y)$ with $y\in\Delta_{p+1}$. Therefore, f induces a unique continuous map $\widetilde{f}:\Delta_{p+1}\to X$ such that the following diagram commutes

$$\begin{tikzcd}
\Delta_p\times[0,1] \arrow[d, "\pi"'] \arrow[dr, "f"] & \\
\Delta_{p+1} \arrow[r, "\widehat{f}"] & X
\end{tikzcd}$$

This unique map $\widehat{f}$ is precisely $[T,w]$, since

$$([T,w]\circ\pi)(x,t)=[T,w](t\varepsilon_{p+1}+(1-t)\,x)=tw+(1-t)\,T(x)=f(x,t). \tag{1.53}$$

Therefore, $\widehat{f}=[T,w]$, and hence it is continuous. So $[T,w]$ is indeed a singular $(p+1)$-simplex.

Lemma 1.5

Let X be a star convex set with respect to w; let c be a singular p-chain of X. Then

$$\partial\left[c,w\right]=\begin{cases}\left[\partial c,w\right]+\left(-1\right)^{p+1}c & \text{if } p>0\\ \epsilon\left(c\right)T_w-c & \text{if } p=0\end{cases}, \tag{1.54}$$

where T_w is the singular 0-simplex mappting Δ_0 to w.

Proof. If T is a singular 0-simplex, $[T,w]$ is a singular 1-simplex. Then

$$\partial\left[T,w\right]=\left[T,w\right]\circ l_{(\widehat{\varepsilon}_0,\varepsilon_1)}-\left[T,w\right]\circ l_{(\varepsilon_0,\widehat{\varepsilon}_1)}. \tag{1.55}$$

Now, recall $[T,w]:\Delta_1\to X$ maps the line joining ε_1 to ε_0 to the line joining w to $T\left(\varepsilon_0\right)$. So

$$\left[T,w\right]\left(1-t,t,0,\ldots\right)=tw+\left(1-t\right)T\left(\varepsilon_0\right). \tag{1.56}$$

Now,

$$\left(\left[T,w\right]\circ l_{(\widehat{\varepsilon}_0,\varepsilon_1)}\right)\left(1,0,\ldots\right)=\left[T,w\right]\left(0,1,0,\ldots\right)=w=T_w\left(1,0,\ldots\right). \tag{1.57}$$

Therefore, $\left(\left[T,w\right]\circ l_{(\widehat{\varepsilon}_0,\varepsilon_1)}\right)=T_w$.

$$\left(\left[T,w\right]\circ l_{(\varepsilon_0,\widehat{\varepsilon}_1)}\right)\left(1,0,\ldots\right)=\left[T,w\right]\left(1,0,\ldots\right)=T\left(\varepsilon_0\right)=T\left(1,0,\ldots\right), \tag{1.58}$$

so $[T,w]\circ l_{(\varepsilon_0,\widehat{\varepsilon}_1)}=T$. By 1.55, we get

$$\partial\left[T,w\right]=T_w-T. \tag{1.59}$$

Now, let $c=\sum_i n_iT_i$ be a singular 0-chain with T_i being singular 0-simplices. Then

$$\partial\left[\sum_i n_iT_i,w\right]=\sum_i n_i\partial\left[T_i,w\right]=\sum_i n_i\left(T_w-T_i\right)=\left(\sum_i n_i\right)T_w-\sum_i n_iT_i. \tag{1.60}$$

Now, applying the augmentation map to c, we get

$$\epsilon\left(c\right)=\epsilon\left(\sum_i n_iT_i\right)=\sum_i n_i\epsilon\left(T_i\right)=\sum_i n_i. \tag{1.61}$$

Therefore, 1.60 gives us

$$\partial\left[c,w\right]=\epsilon\left(c\right)T_w-c. \tag{1.62}$$

Now we shall consider the case when T is a singular p-simplex, and we shall prove that $\partial\left[T,w\right]=\left[\partial T,w\right]+\left(-1\right)^{p+1}T$.

$$\begin{aligned}\partial\left[T,w\right]&=\sum_{i=0}^{p+1}\left(-1\right)^i\left[T,w\right]\circ l_{(\varepsilon_0,\ldots\widehat{\varepsilon}_i,\ldots,\varepsilon_{p+1})}\\&=\sum_{i=0}^{p}\left(-1\right)^i\left[T,w\right]\circ l_{(\varepsilon_0,\ldots\widehat{\varepsilon}_i,\ldots,\varepsilon_{p+1})}+\left(-1\right)^{p+1}\left[T,w\right]\circ l_{(\varepsilon_0,\ldots,\varepsilon_p,\widehat{\varepsilon}_{p+1})}.\end{aligned} \tag{1.63}$$

$l_{(\varepsilon_0,\ldots,\varepsilon_p,\widehat{\varepsilon}_{p+1})}$ is the inclusion map of Δ_p into Δ_{p+1}. So $[T,w]\circ l_{(\varepsilon_0,\ldots,\varepsilon_p,\widehat{\varepsilon}_{p+1})}$ is nothing but the restriction of $[T,w]$ to Δ_p, which is the same as T. Now we want to show that

$$\left[T,w\right]\circ l_{(\varepsilon_0,\ldots\widehat{\varepsilon}_i,\ldots,\varepsilon_{p+1})}=\left[T\circ l_{(\varepsilon_0,\ldots\widehat{\varepsilon}_i,\ldots,\varepsilon_p)},w\right]. \tag{1.64}$$

Both sides of 1.64 are maps from Δ_p to X. Let $(x_0,\ldots,x_p,0,\ldots)\in\Delta_p$. Then

$$\left(\left[T,w\right]\circ l_{(\varepsilon_0,\ldots\widehat{\varepsilon}_i,\ldots,\varepsilon_{p+1})}\right)\left(x_0,\ldots,x_p,0,\ldots\right)=\left[T,w\right]\left(x_0,\ldots,x_{i-1},0,x_i,\ldots,x_{p-1},x_p,0,\ldots\right). \tag{1.65}$$

Now, $(x_0, \ldots, x_{i-1}, 0, x_i, \ldots, x_{p-1}, x_p, 0, \ldots)$ is a point in Δ_{p+1}. We can write it as

$$(x_0, \ldots, x_{i-1}, 0, x_i, \ldots, x_{p-1}, x_p, 0, \ldots) = (1-x_p)\left(\frac{x_0}{1-x_p}, \ldots, \frac{x_{i-1}}{1-x_p}, 0, \frac{x_i}{1-x_p}, \ldots, \frac{x_{p-1}}{1-x_p}, 0, \ldots\right) + x_p\varepsilon_{p+1}. \tag{1.66}$$

Now, $\left(\frac{x_0}{1-x_p}, \ldots, \frac{x_{i-1}}{1-x_p}, 0, \frac{x_i}{1-x_p}, \ldots, \frac{x_{p-1}}{1-x_p}, 0, \ldots\right)$ is a point in Δ_p since its nonzero components are all non-negative and they add to 1. Therefore,

$$\begin{aligned} &[T, w](x_0, \ldots, x_{i-1}, 0, x_i, \ldots, x_{p-1}, x_p, 0, \ldots) \\ = \ &(1-x_p)\, T\left(\frac{x_0}{1-x_p}, \ldots, \frac{x_{i-1}}{1-x_p}, 0, \frac{x_i}{1-x_p}, \ldots, \frac{x_{p-1}}{1-x_p}, 0, \ldots\right) + x_p w. \end{aligned} \tag{1.67}$$

On the other hand, we can write $(x_0, \ldots, x_p, 0, \ldots)$ as

$$(x_0, \ldots, x_p, 0, \ldots) = (1-x_p)\left(\frac{x_0}{1-x_p}, \ldots, \frac{x_{p-1}}{1-x_p}, 0, \ldots\right) + x_p\varepsilon_p, \tag{1.68}$$

where $\left(\frac{x_0}{1-x_p}, \ldots, \frac{x_{p-1}}{1-x_p}, 0, \ldots\right) \in \Delta_{p-1}$. So

$$\begin{aligned} &\left[T \circ l_{(\varepsilon_0, \ldots \widehat{\varepsilon}_i, \ldots, \varepsilon_p)}, w\right](x_0, \ldots, x_p, 0, \ldots) \\ = \ &x_p w + (1-x_p)\left(T \circ l_{(\varepsilon_0, \ldots \widehat{\varepsilon}_i, \ldots, \varepsilon_p)}\right)\left(\frac{x_0}{1-x_p}, \ldots, \frac{x_{p-1}}{1-x_p}, 0, \ldots\right) \\ = \ &x_p w + (1-x_p)\, T\left(\frac{x_0}{1-x_p}, \ldots, \frac{x_{i-1}}{1-x_p}, 0, \frac{x_i}{1-x_p}, \ldots, \frac{x_{p-1}}{1-x_p}, 0, \ldots\right). \end{aligned} \tag{1.69}$$

Combining 1.65, 1.67 and 1.69, we get that 1.64 indeed holds, i.e.

$$[T, w] \circ l_{(\varepsilon_0, \ldots \widehat{\varepsilon}_i, \ldots, \varepsilon_{p+1})} = \left[T \circ l_{(\varepsilon_0, \ldots \widehat{\varepsilon}_i, \ldots, \varepsilon_p)}, w\right].$$

Now, from 1.63, we then get

$$\begin{aligned} \partial\,[T, w] &= \sum_{i=0}^{p} (-1)^i \left[T \circ l_{(\varepsilon_0, \ldots \widehat{\varepsilon}_i, \ldots, \varepsilon_p)}, w\right] + (-1)^{p+1}\, T \\ &= \left[\sum_{i=0}^{p} (-1)^i\, T \circ l_{(\varepsilon_0, \ldots \widehat{\varepsilon}_i, \ldots, \varepsilon_p)}, w\right] + (-1)^{p+1}\, T \\ &= [\partial T, w] + (-1)^{p+1}\, T. \end{aligned} \tag{1.70}$$

Now, if $c = \sum_i n_i T_i$ is a singular p-chain with T_i being singular 0-simplices, then

$$\partial\,[c, w] = \sum_i n_i \partial\,[T_i, w] = \sum_i n_i\,[\partial T_i, w] + (-1)^{p+1} \sum_i n_i T_i = [\partial c, w] + (-1)^{p+1}\, c. \tag{1.71}$$

■

Theorem 1.6

Let $X \subseteq \mathbb{E}^J$ be star convex with respect to w. Then X is acyclic in singular homology.

Proof. To show that $\widetilde{H}_0(X) = 0$, let $c \in \operatorname{Ker} \epsilon$.

$$S_1(X) \xrightarrow{\partial_1} S_0(X) \xrightarrow{\epsilon} \mathbb{Z}$$

So $\epsilon(c) = 0$. Now, by Lemma 1.5,

$$\partial_1 [c, w] = \epsilon(c) T_w - c = -c. \tag{1.72}$$

Hence, $c \in \operatorname{Im} \partial_1$ leading to $\operatorname{Ker} \epsilon \subseteq \operatorname{Im} \partial_1$. We already know Hence, $\operatorname{Im} \partial_1 \subseteq \operatorname{Ker} \epsilon$. Therefore, $\widetilde{H}_0(X) = 0$.

Now we shall show that $H_p(X) = 0$ for $p > 0$. Let $z \in \operatorname{Ker} \partial_p$. Then $\partial_p z = 0$. By Lemma 1.5 again,

$$\partial_{p+1} [z, w] = [\partial_p z, w] + (-1)^{p+1} z = (-1)^{p+1} z. \tag{1.73}$$

Hence, $z \in \operatorname{Im} \partial_{p+1}$. Therefore, $H_p(X) = 0$. In other words, $\widetilde{H}_p(X) = 0$ for all p, i.e. X is acyclic. ■

Corollary 1.7

Any simplex is acyclic in singular homology.