Differential Geometry (MAT313)

Lecture Notes

Preface

This series of lecture notes has been prepared for aiding students who took the BRAC University course **Differential Geometry (MAT313)** in Summer 2022 semester. These notes were typeset under the supervision of mathematician **Dr. Syed Hasibul Hassan Chowdhury**. The recorded video lectures can be found here. The main goal of this typeset is to have an organized digital version of the notes, which is easier to share and handle. If you see any mistakes or typos, please send me an email at `atonuroychowdhury@gmail.com`

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References:

- *An Introduction to Manifolds*, by Loring W. Tu
- *An Introduction to Differentiable Manifolds and Riemannian Geometry*, by William Boothby
- *Introduction to Smooth Manifolds*, by John M. Lee
- *Lectures on Differential geometry*, by S.S Chern, W.H. Chen and K.S. Lam

Contents

1 Topology Review

§1.1 Euclidean Space $\mathbb{R}^n$

Before embarking on the concept of general topological space, let us look at the Euclidean space $\mathbb{R}^n$. $\mathbb{R}^n$ is equipped with the notion of distance between 2 points p and q.

Definition 1.1.1 (Distance). Let the coordinates of p and q be $(p^1, p^2, ...p^n)$ and $(q^1, q^2, ...q^n)$, respectively. The distance between p and q is given by

$$d(p, q) = \left[\sum_{i=1}^{n}(p^i - q^i)^2\right]^{\frac{1}{2}}$$

Definition 1.1.2 (Open ball). An open ball $B(p, r)$ in $\mathbb{R}^n$ with center $p \in \mathbb{R}^n$ and radius $r > 0$ is defined as the set

$$B(p, r) = \{x \in \mathbb{R}^n \ : \ d(x, p) < r\}$$

A set equipped with the notion of distance between its elements is called a metric space[1]. Thus the Euclidean space $\mathbb{R}^n$ is a metric space. And we can talk about open balls in $\mathbb{R}^n$ using this metric. We can define open sets in $\mathbb{R}^n$ using open balls $B(p, r)$ defined above.

Definition 1.1.3 (Open Set in $\mathbb{R}^n$). A set U in $\mathbb{R}^n$ is said to be open if for every p in U, there is an open ball $B(p, r)$ such that $B(p, r) \subseteq U$.

Proposition 1.1.1

The union of an arbitrary collection of $\{U_\alpha\}$ of open sets is open. The intersection of finite collection of open sets is open.

Proof. Trivial. ∎

Example 1.1.1

The intervals $\left(-\frac{1}{n}, \frac{1}{n}\right)$, $n = 1, 2, 3, ...$ are all open in $\mathbb{R}$ but their intersection

$$\bigcap_{n\in\mathbb{N}}\left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$$

is not open.

The metric d in $\mathbb{R}^n$ allows us to define open sets in $\mathbb{R}^n$. In other words, given a subset of $\mathbb{R}^n$, we can tell if it is open or not. This situation is a special case called **metric topology in** $\mathbb{R}^n$.

§1.2 Topology

[1]There are some properties that a metric (distance) function should have. We won't go into much details

Definition 1.2.1 (Topology). A topology on a set S is a collection $\mathcal{T}$ of subsets of S containing both the empty set $\varnothing$ and the S such that $\mathcal{T}$ is closed under arbitrary union and finite intersection. In other words,

- If $U_\alpha \in \mathcal{T}$ for all α in an index set A, then $\bigcup\limits_{\alpha \in A} U_\alpha \in \mathcal{T}$
- If $U_i \in \mathcal{T}$ for $i \in \{1, 2, ..., n\}$, then $\bigcap\limits_{i=1}^{n} U_i \in \mathcal{T}$

The elements of $\mathcal{T}$ are called open sets.

Definition 1.2.2 (Topological Space). The pair $(S, \mathcal{T})$ consisting of a set S together with a topology $\mathcal{T}$ on S is called a **topological space**.

Abuse of Notation. We shall often say "S is a topological space" in short. But there is always a topology $\mathcal{T}$ on S, which we recall when necessary.

Definition 1.2.3 (Neighborhood). A **neighbourhood** of a point $p \in S$ is called an open set U containing p.

Definition 1.2.4 (Closed Set). The complement of an open set is called a **closed set**.

Proposition 1.2.1

The union of a finite collection of closed sets is closed. The intersection of an arbitrary collection of closed sets is closed.

Proof. Let $\{F_i\}_{i=1}^n$ be a finite collection of closed sets. Then, $\{S \setminus F_i\}_{i=1}^n$ is a finite collection of open sets. The intersection of a finite collection of open sets is open, therefore $\bigcap\limits_{i=1}^{n} (S \setminus F_i)$ is open. By De Morgan's law,

$$\bigcap_{i=1}^{n} (S \setminus F_i) = S \setminus \left(\bigcup_{i=1}^{n} F_i \right) \quad \text{is open} \implies \bigcup_{i=1}^{n} F_i \quad \text{is closed}$$

Therefore, the union of a finite collection of closed sets is closed.
Now, let $\{F_\alpha\}_{\alpha \in A}$ be an arbitrary collection of closed sets with A being an index set. Then $\{S \setminus F_\alpha\}_{\alpha \in A}$ is an arbitrary collection of open sets. We know that the union of an arbitrary collection of open sets is open, therefore $\bigcup_{\alpha \in A} (S \setminus F_\alpha)$ is open. By De Morgan's law,

$$\bigcup_{\alpha \in A} (S \setminus F_\alpha) = S \setminus \left(\bigcap_{\alpha \in A} F_\alpha \right) \quad \text{is open} \implies \bigcap_{\alpha \in A} F_\alpha \quad \text{is closed}$$

Therefore, the intersection of an arbitrary collection of closed sets is closed. ∎

Definition 1.2.5 (Subspace Topology). Let $(S, \mathcal{T})$ be a topological space and A a subset of S. Define $\mathcal{T}_A$ to be the collection of subsets

$$\mathcal{T}_A = \{U \cap A \mid U \in \mathcal{T}\}$$

$\mathcal{T}_A$ is called the **subspace topology** of A in S.

It is not hard to see that $\mathcal{T}_A$ satisfies the conditions of a Topology. Firstly, $\mathcal{T}_A$ contains both $\varnothing$ and A. For these, taking $U = \varnothing$ and $U = S$, respecitvely, suffices. By the distributive property of union and intersection

$$\bigcup_{\alpha} (U_\alpha \cap A) = \left(\bigcup_{\alpha} U_\alpha \right) \cap A \text{ and } \bigcap_{i=1}^{n} (U_i \cap A) = \left(\bigcap_{i=1}^{n} U_i \right) \cap A$$

which shows that $\mathcal{T}_A$ is closed under arbitrary union and finite intersection. So $\mathcal{T}_A$ is a Topology indeed.

Example 1.2.1

Consider the subset $A = [0, 1]$ of $\mathbb{R}$. In the subspace topology, the half-open interval $\left[0, \frac{1}{2}\right)$ is an open subset of A, because $\left[0, \frac{1}{2}\right) = \left(-\frac{1}{2}, \frac{1}{2}\right) \cap [0, 1]$

Lemma 1.2.2

Let Y be a subspace of X (that is Y has the subspace topology inherited from X). If U is open in Y and Y is open in X, then U is open in X.

Proof. Since U is open in Y, $U = Y \cap V$ for some V open in X. Both Y and V are open in X, hence $Y \cap V = U$ is also open in X. ∎

The same conclusion holds if you replace "open" by "closed".

Lemma 1.2.3

Let Y be a subspace of X. If F is closed in Y and Y is closed in X, then F is closed in X.

Proof. Since F is open in Y, $F = Y \cap K$ for some K closed in X. Both Y and K are closed in X, hence $Y \cap K = F$ is also closed in X. ∎

Definition 1.2.6 (Closure)**.** Let S be a topological space and A a subset of S. The **closure** of A in S, denoted by $\overline{A}$ or $\mathrm{cl}_S(A)$, is defined to be the intersection of all the closed sets containing A.

As an intersection of closed sets, $\overline{A}$ is a closed set. It is the smallest closed set containing A in the sense that any closed set containing A contains $\overline{A}$.

Proposition 1.2.4

A is closed if and only if $\overline{A} = A$.

Proof. If $A = \overline{A}$, then A is closed because $\overline{A}$ is closed. Now, suppose A is closed. Then A is a closed set containing A, so $\overline{A} \subseteq A$. Clearly, $A \subseteq \overline{A}$. Therefore, $A = \overline{A}$. ∎

Proposition 1.2.5

If $A \subseteq B$ in a topological space S, then $\overline{A} \subseteq \overline{B}$.

Proof. Since $\overline{B}$ contains B, it also contains A. As a closed subset of S containing A, $\overline{B}$ also contains $\overline{A}$. ∎

Lemma 1.2.6

Let A be a subset of a topological space S. Then $x \in \overline{A}$ if and only if every open set U containing x intersects A.

Proof. We shall prove the contrapositive statements in both directions. So we need to show that

$$x \notin \overline{A} \iff \exists\, U \ni x \text{ such that } U \text{ is open, and } U \cap A = \varnothing\,.$$

Let $x \notin \overline{A}$. We take $U = X \setminus \overline{A}$. This set is open, contains x, and does not intersect A.
Now conversely, suppose U is a open set containing x, and it does not intersect A. Then $X \setminus U$ is closed and it contains A. $\overline{A}$ is the intersection of all closed sets containing A, therefore $\overline{A} \subseteq X \setminus U$. That's why $x \notin \overline{A}$. ■

Proposition 1.2.7
$\overline{A} \cup \overline{B} = \overline{A \cup B}$.

Proof. $A \subseteq A \cup B$, so by Proposition 1.2.5, $\overline{A} \subseteq \overline{A \cup B}$. Similarly, $\overline{B} \subseteq \overline{A \cup B}$. Therefore, $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$.
$A \subseteq \overline{A}$, and $B \subseteq \overline{B}$. So, $A \cup B \subseteq \overline{A} \cup \overline{B}$. Therefore, $\overline{A \cup B} \subseteq \overline{\overline{A} \cup \overline{B}}$. But $\overline{A} \cup \overline{B}$ is closed, so $\overline{\overline{A} \cup \overline{B}} = \overline{A} \cup \overline{B}$. Hence, $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$. Therefore, we have proved that $\overline{A} \cup \overline{B} = \overline{A \cup B}$. ■

§1.3 Bases and Countability

Definition 1.3.1 (Basis and Basic Open Sets). A subcollection $\mathcal{B}$ of a topology $\mathcal{T}$ is a **basis** for $\mathcal{T}$ if given an open set U and a point p in U, there is an open set $\mathrm{B} \in \mathcal{B}$ such that $p \in \mathrm{B} \subseteq U$. An element of $\mathcal{B}$ is called a **basic open set**.

Example 1.3.1
The collection of all open balls $B(p, r)$ in $\mathbb{R}^n$ with $p \in \mathbb{R}^n$ and $r > 0$ is a basis for the standard topology (metric topology) on $\mathbb{R}^n$.

Proposition 1.3.1
A collection $\mathcal{B}$ of open sets of S is a basis if and only if every open set in S is a union of sets in $\mathcal{B}$.

Proof. ($\Rightarrow$) We are given a collection of $\mathcal{B}$ of open sets of S that is a basis. U is any open set in S. Also, let $p \in U$. Therefore, there is a basic open set $B_p \in \mathcal{B}$ such that $p \in B_p \subseteq U$. Hence, one can show that $U = \bigcup\limits_{p \in U} B_p$.
($\Leftarrow$) Suppose, every open set in S is a union of open sets in $\mathcal{B}$. Now, given an open set U and a point $p \in U$, since $U = \bigcup\limits_{B_\alpha \in \mathcal{B}} B_\alpha$, there is a $B_\alpha \in \mathcal{B}$, such that $p \in B_\alpha \subseteq U$. Hence $\mathcal{B}$ is a basis. ■

Proposition 1.3.2
A collection $\mathcal{B}$ of subsets of a set S is a basis for some topology $\mathcal{T}$ on S if and only if

(i) S is the union of all the sets in $\mathcal{B}$, and

(ii) given any two sets B_1 and $B_2 \in \mathcal{B}$ and a point $p \in B_1 \cap B_2$, there is a set $B \in \mathcal{B}$ such that $p \in B \subset B_1 \cap B_2$.

Proof. **($\Rightarrow$)** (i) follows from Proposition 1.3.1.
(ii) If $\mathcal{B}$ is a basis, then B_1 and B_2 are open sets and hence so is $B_1 \cap B_2$. By the definition of a basis, there is a $B \in \mathcal{B}$ such that $p \in B \subseteq B_1 \cap B_2$.

(⇐) Define $\mathcal{T}$ to be the collection consisting of all sets that are unions of sets in $\mathcal{B}$. Then the empty set $\varnothing$ and the set S are in $\mathcal{T}$ and $\mathcal{T}$ is clearly closed under arbitrary union. To show that $\mathcal{T}$ is closed under finite intersection, let $U = \bigcup_\mu B_\mu$ and $V = \bigcup_\nu B_\nu$ be in $\mathcal{T}$, where $B_\mu, B_\nu \in \mathcal{B}$. Then

$$U \cap V = \left(\bigcup_\mu B_\mu\right) \cap \left(\bigcup_\nu B_\nu\right) = \bigcup_{\mu,\nu} (B_\mu \cap B_\nu) \, .$$

Thus, any p in $U \cap V$ is in $B_\mu \cap B_\nu$ for some μ, ν. By (ii) there is a set B_p in $\mathcal{B}$ such that $p \in B_p \subseteq B_\mu \cap B_V$. Therefore,

$$U \cap V = \bigcup_{p \in U \cap V} B_p \in \mathcal{T} \, .$$

Therefore, $\mathcal{B}$ generates a topology on S. ■

We say that a point in $\mathbb{R}^n$ is rational if all of its coordinates are rational numbers. Let $\mathbb{Q}$ be the set of rational numbers and $\mathbb{Q}^+$ the set of positive rational numbers.

Lemma 1.3.3

Every open set in $\mathbb{R}^n$ contains a rational point.

Proof. An open set U in $\mathbb{R}^n$ contains an open ball $B(p,r)$ which, in turn, contains an open cube $\prod_{i=1}^{n} I_i$ where I_i is the open interval $\left(p^i - \frac{r}{\sqrt{n}}, p^i + \frac{r}{\sqrt{n}}\right)$. Here is a visual example for $n = 2$.

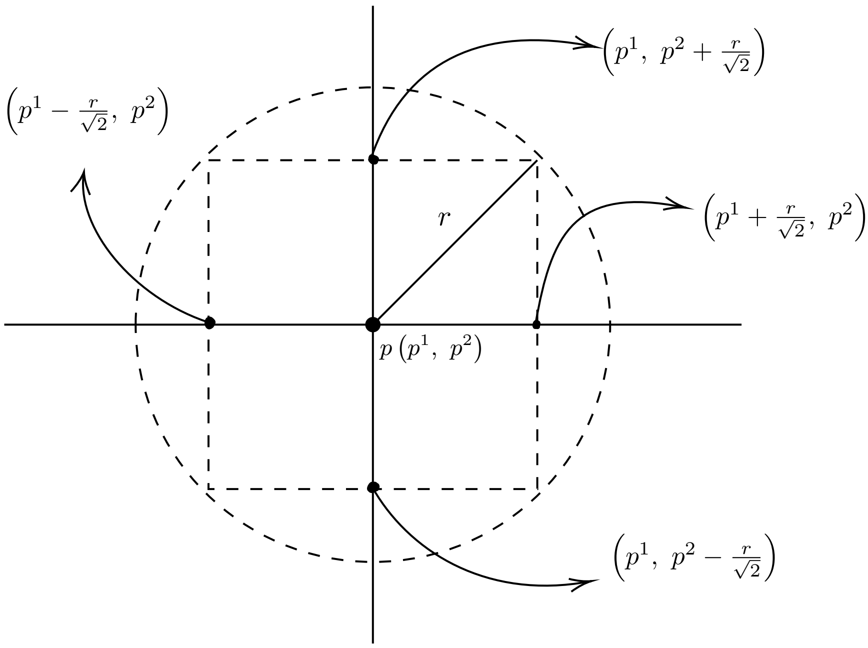


Figure 1.1: $B(p,r)$ contains $\left(p^1 - \frac{r}{\sqrt{n}}, p^1 + \frac{r}{\sqrt{n}}\right) \times \left(p^2 - \frac{r}{\sqrt{n}}, p^2 + \frac{r}{\sqrt{n}}\right)$

Now back to general n. For each i, let q^i be a rational number in I_i. Then $(q^1, q^2, ..., q^n)$ is a rational point in $\prod_{i=1}^{n} I_i \subseteq B(p,r)$. Therefore, every open set contains a rational point. ■

Proposition 1.3.4

The collection $\mathcal{B}_{\mathbb{Q}}$ of all open balls in $\mathbb{R}^n$ with rational centers and rational radii is a basis for $\mathbb{R}^n$.

Proof. Given an open set U in $\mathbb{R}^n$ and $p \in U$, there is an open ball $B(p, r')$ with positive real radius r' such that $p \in B(p, r') \subseteq U$. Take a rational number $r \in (0, r')$. Then we have

$$p \in B(p, r) \subseteq B(p, r') \subseteq U$$

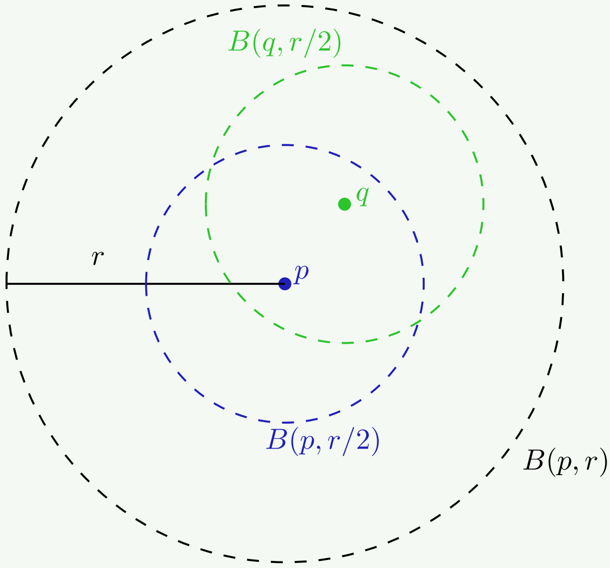
By Lemma 1.3.3 , there is a rational point in the smaller ball $B\left(p, \frac{r}{2}\right)$.

Claim — $p \in B\left(q, \frac{r}{2}\right) \subseteq B(p, r)$

Proof. Since $d(p, q) < \frac{r}{2}$, we have $p \in B\left(q, \frac{r}{2}\right)$. Next, if $x \in B\left(q, \frac{r}{2}\right)$, then by triangle inequality

$$d(x, p) \leq d(x, q) + d(q, p) < \frac{r}{2} + \frac{r}{2} = r$$

Therefore, $x \in B(p, r)$.



So, $p \in B\left(q, \frac{r}{2}\right)$ and $B\left(q, \frac{r}{2}\right) \subseteq B(p, r)$. □

As a result, $p \in B\left(q, \frac{r}{2}\right) \subseteq B(p, r) \subseteq B\left(p, r'\right) \subseteq U$. Hence we proved,

$$p \in B\left(q, \frac{r}{2}\right) \subseteq U$$

In other words, the collection $\mathcal{B}_{\mathbb{Q}}$ of open balls with rational centers and rational radii is a basis for $\mathbb{R}^n$. ■

Both the sets $\mathbb{Q}$ and $\mathbb{Q}^+$ are countable. Since the centers of the open balls in $\mathcal{B}_{\mathbb{Q}}$ are indexed by $\mathbb{Q}^n$, a countable set, and the radii are indexed by $\mathbb{Q}^+$, also a countable set, the collection $\mathcal{B}_{\mathbb{Q}}$ is countable.

Definition 1.3.2 (Second Countable). A topological space is said to be second countable if it has a countable basis.

Proposition 1.3.4 shows that $\mathbb{R}^n$ with its standard topology is second countable.

Proposition 1.3.5

Let $\mathcal{B} = \{B_\alpha\}$ be a basis for S, and A a subspace of S. Then $\{B_\alpha \cap A\}$ is a basis for A.

Proof. Let U' be any open set in A and $p \in U'$. By the definition of subspace topology, $U' = U \cap A$ for some open set U in S. Since $p \in U \cap A \subset U$, there is a basic open set B_α such that $p \in B_\alpha \subset U$. Then

$$p \in B_\alpha \cap A \subset U \cap A = U',$$

which proves that the collection $\{B_\alpha \cap A \mid B_\alpha \in \mathcal{B}\}$ is a basis for A. ■

Corollary 1.3.6

Subspace of a second countable space is also second countable.

Definition 1.3.3 (Neighborhood Basis). Let S be a topological space and p be a point in S. A **basis of neighbourhoods** or a **neighbourhood basis** at p is a collection $\mathcal{B} = \{B_\alpha\}$ of neighbourhoods of p such that for any neighbourhood U of p there is a $B_\alpha \in \mathcal{B}$ such that $p \in B_\alpha \subseteq U$.

Definition 1.3.4 (First Countable). A topological space S is first countable if it has a countable basis of neighbourhoods at every point $p \in S$.

Example 1.3.2

For $p \in \mathbb{R}^n$, let $B\left(p, \frac{1}{n}\right)$ be the open ball of center p and radius $\frac{1}{n}$ in $\mathbb{R}^n$. Then $\left\{B\left(p, \frac{1}{n}\right)\right\}_{n=1}^{\infty}$ is a neighbourhood basis at p. Thus $\mathbb{R}^n$ is first countable.

An important note: An uncountable discrete topological space is first countable but not second countable. A second countable topological space is always first countable.

§1.4 Hausdorff Space

Definition 1.4.1 (Hausdorff Space). A topological space S is Hausdorff if given any 2 distinct points x, y in S there exist disjoint open sets U, V such that $x \in U$ and $y \in V$.

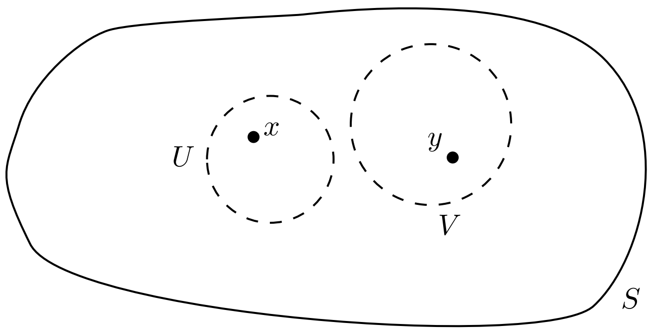


Figure 1.2: Here S is a Hausdorff space, U and V are disjoint open sets containing x and y respectively.

Proposition 1.4.1

Every singleton set (a one-point set) in a Hausdorff space S is closed.

Proof. Let $x \in S$. We want to prove that $\{x\}$ is closed, i.e. $S \setminus \{x\}$ is open.

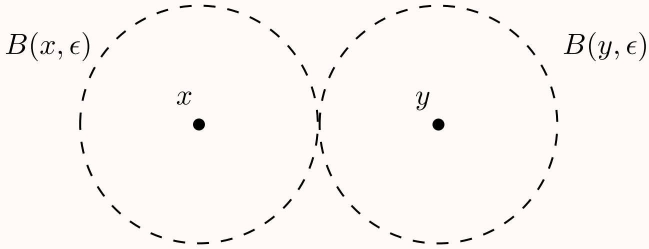
Let $y \in S \setminus \{x\}$. Since S is Hausdorff, we can find disjoint open sets U_y and V_y such that $x \in U_y$ and $y \in V_y$. No such V_y contains x. Therefore

$$S \setminus \{x\} = \bigcup_{y \in S \setminus \{x\}} V_y$$

So $S \setminus \{x\}$ is union of open sets, hence open. So $\{x\}$ is closed. ∎

Example 1.4.1

The Euclidean space $\mathbb{R}^n$ (equipped with standard/ metric topology) is Hausdorff, for given distinct points x, y in $\mathbb{R}^n$, if $\epsilon = \frac{1}{2}d(x, y)$, then the open balls $B(x, \epsilon)$ and $B(y, \epsilon)$ will be disjoint.



In a similar manner, one can show that every metric space is Hausdorff.

Lemma 1.4.2

Let A be a subspace of X. If X is a Hausdorff space, then so is A.

Proof. Take $x, y \in A \subseteq X$ with $x \neq y$. As X is Hausdorff, we can find disjoint open sets U and V in X, such that $U \ni x$ and $V \ni y$. $x \in A$ and $x \in U$, so $x \in A \cap U$. Similarly, $y \in A \cap V$.

Now, both $A \cap U$ and $A \cap V$ are open in A, with respect to the subspace topology. Furthermore, $(A \cap U) \cap (A \cap V) = A \cap (U \cap V) = \varnothing$. Therefore, for $x, y \in A$ we've found disjoint open sets $A \cap U$ and $A \cap V$, containing x and y respectively. So A is Hausdorff. ∎

§1.5 Continuity and Homeomorphism

Definition 1.5.1 (Continuous Maps)**.** Let $f : X \to Y$ be a map of topological spaces. f is said to be continuous if for each open subset V of Y, the set $f^{-1}(V)$ is an open subset of X.

Proposition 1.5.1

$f : X \to Y$ is continuous if and only if for every closed subset B of Y, the set $f^{-1}(B)$ will be closed in X.

Proof. ($\Rightarrow$) Suppose f is continuous. B is closed, so $Y \setminus B$ is open in Y. Therefore, by the continuity of f, $f^{-1}(Y \setminus B) = X \setminus f^{-1}(B)$ is open in X, so $f^{-1}(B)$ is closed.

($\Leftarrow$) Suppose $f^{-1}(B)$ is closed in X for any closed $B \subseteq Y$. Take any open set U in Y. Choose $B = Y \setminus U$. Then by the assumption $f^{-1}(Y \setminus U) = X \setminus f^{-1}(U)$ is closed in X. This gives us $f^{-1}(U)$ is open. So f is continuous. ∎

Definition 1.5.2 (Homeomorphism)**.** Let X and Y be topological spaces; let $f : X \to Y$ be a bijection. If both f and the inverse function $f^{-1} : Y \to X$ are continuous, then f is called a **homeomorphism**.

Example 1.5.1

The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 3x + 1$ is a homeomorphism. We define $g : \mathbb{R} \to \mathbb{R}$ by $g(y) = \frac{1}{3}(y - 1)$. Then we have

$$f(g(y)) = y \quad \text{and} \quad g(f(x)) = x \qquad \forall\, x, y \in \mathbb{R}$$

This proves $g = f^{-1}$. It is easy to see that both f and g are continuous functions. Therefore f is a homeomorphism.

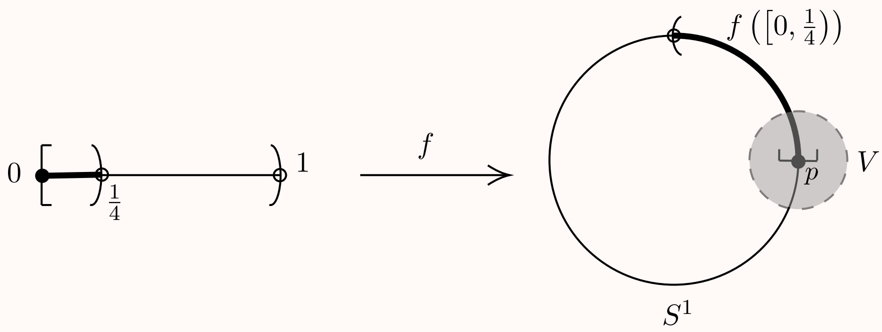
However, a bijective function can be continuous without being a homeomorphism.

Example 1.5.2

Let S^1 denote the unit circle in $\mathbb{R}^2$; that is $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$, considered as a **subspace**[a] of the space $\mathbb{R}^2$. Let $f : [0, 1) \to S^1$ be the

$$f(t) = (\cos 2\pi t, \sin 2\pi t)$$

It is left as an exercise for the reader to show that f is a continuous bijective function. But the function f^{-1} is not continuous.



$U = \left[0, \frac{1}{4}\right)$ is an open set in $[0, 1)$ according to the subspace topology. We want to show that $f(U)$ is not open in S^1. That would prove the discontinuity of f^{-1}.
Let p be the point $f(0)$. And $p \in f(U)$. We need to find an open set of S^1 in subspace topology containing $p = f(0)$ and contained in $f(U)$ to show that $f(U)$ is open in S^1, i.e we have to find an open set in V of $\mathbb{R}^2$ such that $f(0) = p \in V \cap S^1 \subseteq f(U)$. But it is impossible as is evident from the figure above. No matter what V we choose, some part of $V \cap S^1$ would lie outside $f(U)$.

[a] Subset of $\mathbb{R}^2$ equipped with subspace topology.

Lemma 1.5.2 (Pasting Lemma)

Let $X = A \cup B$, where A and B are closed in X. Let $f : A \to Y$ and $g : B \to Y$ be continuous. If $f(x) = g(x)$ for every $x \in A \cap B$, then f and g combine to give a continuous function $h : X \to Y$ defined by

$$h(x) = \begin{cases} f(x) & \text{if } \ x \in A \\ g(x) & \text{if } \ x \in B \end{cases}$$

Proof. Let C be a closed subset of Y. Now,

$$h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$$

Since f is continuous, $f^{-1}(C)$ is closed in A, hence closed in X. Similarly, $g^{-1}(C)$ is closed in X. So $h^{-1}(C)$ is the union of two closed sets in X, hence it is closed in X. Therefore, h is continuous. ∎

Lemma 1.5.3

Let X, Y be topological spaces and $f : X \to Y$. Then f is continuous if and only if for every $x \in X$ and each neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subseteq V$.

Proof. ($\Rightarrow$) Suppose f is continuous. Let $x \in X$ and $V \ni f(x)$ is open in Y. We take $U = f^{-1}(V)$. Since f is open and U is preimage of open set, so U is open. Also,

$$f(x) \in V \implies x \in f^{-1}(V) = U \text{ and } f(U) = f\left(f^{-1}(V)\right) \subseteq V$$

($\Leftarrow$) Let $V \subseteq Y$ be open. We need to show that $f^{-1}(V)$ is open. Take $x \in f^{-1}(V)$. Then $f(x) \in V$, so V is a neighborhood of $f(x)$. By assumption, there exists open $U \ni x$ such that

$$f(U) \subseteq V \implies U \subseteq f^{-1}(V)$$

So for every $x \in f^{-1}(V)$, there exists a neighborhood of x that is contained in $f^{-1}(V)$. So $f^{-1}(V)$ is open, and hence f is continuous. ■

§1.6 Product Topology

The Cartesian product of two sets A and B is the set $A \times B$ of all ordered pairs (a, b) with $a \in A$ and $b \in B$. Given two topological spaces X and Y, consider the collection $\mathcal{B}$ of subsets of $X \times Y$ of the form $U \times V$, with U open in X and V open in Y. We will call elements of $\mathcal{B}$ basic open sets in $X \times Y$. If $U_1 \times V_1$ and $U_2 \times V_2$ are in $\mathcal{B}$, then

$$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \times V_2) \ ,$$

which is also in $\mathcal{B}$. From this, it follows easily that $\mathcal{B}$ satisfies the conditions of Proposition 1.3.2 for a basis and generates a topology on $X \times Y$, called the **product topology**. Unless noted otherwise, this will always be the topology we assign to the product of two topological spaces.

Proposition 1.6.1

Let $\{U_i\}$ and $\{V_j\}$ be bases for the topological spaces X and Y, respectively. Then $\{U_i \times V_j\}$ is a basis for $X \times Y$.

Proof. Given an open set W in $X \times Y$ and point $(x, y) \in W$, we can find a basic open set $U \times V$ in $X \times Y$ such that $(x, y) \in U \times V \subset W$. Since U is open in X and $\{U_i\}$ is a basis for X, $x \in U_i \subset U$ for some U_i. Similarly, $y \in V_j \subset V$ for some V_j. Therefore,

$$(x, y) \in U_i \times V_j \subset U \times V \subset W.$$

By the definition of a basis, $\{U_i \times V_j\}$ is a basis for $X \times Y$. ■

Corollary 1.6.2

The product of two second-countable spaces is second countable.

Proposition 1.6.3

The product of two Hausdorff spaces X and Y is Hausdorff.

Proof. Given two distinct points $(x_1, y_1), (x_2, y_2)$ in $X \times Y$, without loss of generality we may assume that $x_1 \neq x_2$. Since X is Hausdorff, there exist disjoint open sets U_1, U_2 in X such that $x_1 \in U_1$ and $x_2 \in U_2$. Then $U_1 \times Y$ and $U_2 \times Y$ are disjoint neighborhoods of (x_1, y_1) and (x_2, y_2), so $X \times Y$ is Hausdorff. ■

The product topology can be generalized to the product of an arbitrary collection $\{X_\alpha\}_{\alpha \in \mathrm{A}}$ of topological spaces. Whatever the definition of the product topology, the projection maps

$$\pi_{\alpha_i} : \prod_\alpha X_\alpha \to X_{\alpha_i} \ , \pi_{\alpha_i}\left(\prod x_\alpha\right) = x_{\alpha_i}$$

should all be continuous. Thus, for each open set U_{α_i} in X_{α_i}, the inverse image $\pi_{\alpha_i}^{-1}(U_{\alpha_i})$ should be open in $\prod_\alpha X_\alpha$. By the properties of open sets, a finite intersection $\bigcap_{i=1}^r \pi_{\alpha_i}^{-1}(U_{\alpha_i})$ should also be open. Such a finite intersection is a set of the form $\prod_{\alpha \in A} U_\alpha$, where U_α is open in X_α and $U_\alpha = X_\alpha$ for all but finitely many $\alpha \in A$. We define the product topology on the Cartesian product $\prod_{\alpha \in \mathrm{A}} X_\alpha$ to be the topology with basis consisting of sets of this form.

Theorem 1.6.4
Let $f : A \to \prod_{\alpha \in J} X_\alpha$ be given by the equation

$$f(a) = (f_\alpha(a))_{\alpha \in J},$$

where $f_\alpha : A \to X_\alpha$ for each α. Let $\prod X_\alpha$ have the product topology. Then the function f is continuous if and only if each function f_α is continuous.

Proof. ($\Rightarrow$) Suppose f is continuous. Then $f_\alpha = \pi_\alpha \circ f$ is the composition of two continuous maps, hence continuous.

($\Leftarrow$) Now suppose f_α is continuous for every α. Let $U \subseteq \prod_{\alpha \in J} X_\alpha$ be a basic open set. Then U is of the form $\prod U_\alpha$ where U_α is open in X_α for every α, and $U_\alpha \neq X_\alpha$ for only fintely many α's. Then we have

$$\begin{aligned} f^{-1}(U) = \bigcap_\alpha f_\alpha^{-1}(U_\alpha) &= \left(\bigcap_{U_\alpha \neq X_\alpha} f_\alpha^{-1}(U_\alpha) \right) \cap \left(\bigcap_{U_\alpha = X_\alpha} f_\alpha^{-1}(U_\alpha) \right) \\ &= \left(\bigcap_{U_\alpha \neq X_\alpha} f_\alpha^{-1}(U_\alpha) \right) \cap \left(\bigcap_{U_\alpha = X_\alpha} A \right) \\ &= \bigcap_{U_\alpha \neq X_\alpha} f_\alpha^{-1}(U_\alpha) \end{aligned}$$

Since each f_α is continuous, $f_\alpha^{-1}(U_\alpha)$ is open in A. Therefore, as a finite intersection of open sets, $f^{-1}(U)$ is open, proving the continuity of f. ■

Proposition 1.6.5
If X and Y are topological spaces, then the projection map $\pi : X \times Y \to X$, $\pi(x, y) = x$ is an open map (it maps open sets to open sets).

Proof. Let $\{U_\alpha\}$ and $\{V_\beta\}$ be bases for the topological spaces X and Y, respectively. Then, by Proposition 1.6.1, $\mathcal{B} = \{U_\alpha \times V_\beta\}$ is a basis for $X \times Y$. Therefore, if W is an open subset of $X \times Y$, then W can be expressed as the union of some basic open sets.

$$W = \bigcup_{i,j} (U_i \times V_j).$$

Then we have,

$$\pi(W) = \pi\left(\bigcup_{i,j} (U_i \times V_j) \right) = \bigcup_{i,j} \pi(U_i \times V_j) = \bigcup_i U_i.$$

Since U_i are basic open sets of X, $\bigcup_i U_i$ is an open subset of X. In other words, for W open in $X \times Y$, $\pi(W)$ is open in X. Therefore, π is an open map. ■

§1.7 Quotient Topology

Quotient topology is defined using an equivalence relation. An equivalence relation is a binary relation on a set that has some properties.

Definition 1.7.1 (Equivalence Relation and Equivalence Class). An equivalence relation $\sim$ on a set S is a binary relation which is reflexive, symmetric and transitive. That is

(i) $a \sim a$ for every $a \in S$

(ii) $a \sim b \implies b \sim a$

(iii) $a \sim b\ ,\ b \sim c \implies a \sim c$

The equivalence class $[x]$, if $x \in S$, is the set of all elements in S equivalent to x.

An equivalence relation on S partitions S into disjoint equivalence classes. We denote the set of all equivalence classes with $S/\!\sim$ and call this the quotient of S by the equivalence relation $\sim$. There is a natural projection map $\pi : S \to S/\!\sim$ which projects $x \in S$ to its own equivalence class $[x] \in S/\!\sim$.

Abuse of Notation. Ideally $[x]$ denotes a point in $S/\!\sim$. But we will use the same notation $[x]$ to identify a set in S whose elements are all equivalent to each other under the given equivalence relation.

Definition 1.7.2 (Quotient Topology). Let S be a topological space. We define a topology called **quotient topology** on $S/\!\sim$ by declaring a set U in $S/\!\sim$ to be open if and only if $\pi^{-1}(U)$ is open in S.

It's not hard to see that quotient topology is a well defined topology. Note that $\pi^{-1}(\varnothing) = \varnothing$ and $\pi^{-1}(S/\!\sim) = S$ and hence $\varnothing$ and $S/\!\sim$ are both open sets in quotient topology. Now let $\{U_\alpha\}_{\alpha\in A}$ be an arbitrary collection of open sets in $S/\!\sim$. Then $\left\{\pi^{-1}(U_\alpha)\right\}_{\alpha\in A}$ is an an arbitrary collection of open sets in S. So,

$$\bigcup_{\alpha\in A} \pi^{-1}(U_\alpha) = \pi^{-1}\left(\bigcup_{\alpha\in A} U_\alpha\right) \text{ is open in } S \implies \bigcup_{\alpha\in A} U_\alpha \text{ open in } S/\!\sim$$

So arbitrary union of open sets is open in $S/\!\sim$. Now for a finite collection of open sets $\{U_i\}_{i=1}^n$ in $S/\!\sim$, $\left\{\pi^{-1}(U_i)\right\}_{i=1}^n$ is a finite collection of open sets in S. So,

$$\bigcap_{i=1}^{n} \pi^{-1}(U_i) = \pi^{-1}\left(\bigcap_{i=1}^{n} U_\alpha\right) \text{ is open in } S \implies \bigcap_{i=1}^{n} U_\alpha \text{ open in } S/\!\sim$$

So finite intersection of open sets is open in $S/\!\sim$. Therefore, we've verified that the open sets defined on $S/\!\sim$ indeed form a topology.

Continuity on Quotient Topology

Let $\sim$ be a equivalence relation on the topological space S and give $S/\sim$ the quotient topology. Suppose that the function $f : S \to Y$ is continuous from S to another topological space Y. Further assume that f is constant on each equivalence class. Then f induces a map

$$\bar{f} : S/\!\sim \to Y\ ;\ \bar{f}([p]) = f(p) \quad \forall\, p \in S$$

Note that this latter function $\bar{f}$ wouldn't be well-defined had we not assumed f to be constant on each equivalence class in $S/\!\sim$.

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "\pi"'] & Y \\ S/\!\sim \arrow[ur, "\overline{f}"'] & \end{tikzcd}$$

$$f = \overline{f} \circ \pi$$
$$f(p) = \overline{f}(\pi(p)) = f([p])$$

Proposition 1.7.1

The induced map $\bar{f} : S/{\sim} \to Y$ is continuous if and only if the map $f : S \to Y$ is continuous.

Proof. ($\Rightarrow$). Suppose $f : S \to Y$ is continuous. Let V be open in Y. Then $f^{-1}(V) = \pi^{-1}\left(\bar{f}^{-1}(V)\right)$ is open in S. Therefore, by the definition of quotient topology, then $\bar{f}^{-1}(V)$ is open in $S/{\sim}$. Hence, we've shown that for a given open set V in Y, $\bar{f}^{-1}(V)$ is open in $S/{\sim}$. So, $\bar{f} : S/{\sim} \to Y$ is continuous.

($\Leftarrow$). If $\bar{f} : S/{\sim} \to Y$ is continuous, then $f = \bar{f} \circ \pi$ is the composition of two continuous maps, hence continuous. ■

Identification of a subset to a point

If A is a subspace of a topological space S, we can define a relation $\sim$ on S by declaring

$$x \sim x \ , \ \forall \ x \in S \qquad \text{and} \qquad x \sim y \ , \ \forall \ x, y \in A$$

It is immediate that $\sim$ is an equivalence relation. We say that the quotient space $S/{\sim}$ is obtained from S by identifying A to a point.

§1.8 Compactness

Definition 1.8.1 (Open Cover)**.** Let S be a topological space. A collection $\{U_\alpha\}_{\alpha \in I}$ of open subsets of S is said to be an open cover of S if

$$S \subseteq \bigcup_{\alpha \in I} U_\alpha$$

Since the open sets are in the topology of S and consequently $U_\alpha \subseteq S$ for every $\alpha \in I$, one has $\bigcup_{\alpha \in I} U_\alpha \subseteq S$. Therefore, the open cover condition in this case reduces to $S = \bigcup_{\alpha \in I} U_\alpha$.

With the subspace topology, a subset A of a topological space S is a topological space by its own right. The subspace A can be covered by **open sets in** A or **by open sets in** S.

- An **open cover of** A **in** S is a collection $\{U_\alpha\}_\alpha$ of open sets in S that covers A. In other words, $A \subseteq \bigcup_\alpha U_\alpha$ (Note that in this case $A = \bigcup_\alpha U_\alpha$ might not hold in general).
- An **open cover of** A **in** A is a collection $\{U_\alpha\}_\alpha$ of open sets in A in subsapce topology that covers A. In other words, $A \subseteq \bigcup_\alpha U_\alpha$ (Here, in fact, $A = \bigcup_\alpha U_\alpha$ as each $U_\alpha \subseteq A$).

Definition 1.8.2 (Compact Set)**.** Let S be a topological space and $A \subseteq S$. A is **compact** if and only if every open cover of A in A has finite subcover.

Proposition 1.8.1

A subspace A of a topological space S is **compact** if and only if every **open cover of** A **in** S has a finite subcover.

Proof. ($\Rightarrow$) Assume A is compact and let $\{U_\alpha\}$ be an open cover of A in S. This means that $A \subseteq \bigcup\limits_\alpha U_\alpha$. Hence,

$$A \subseteq \left(\bigcup_\alpha U_\alpha\right) \bigcap A = \bigcup_\alpha \left(U_\alpha \bigcap A\right)$$

Now, $\{U_\alpha \cap A\}_\alpha$ is an open cover of A in A. Since A is compact, every open cover of A in A has a finite subcover. Let the finite sub-cover be $\{U_{\alpha_i} \cap A\}_{i=1}^n$. Thus,

$$A \subseteq \bigcup_{i=1}^n U_{\alpha_i}$$

which means that $\{U_{\alpha_i}\}_{i=1}^n$ is a finite sub-cover of the open cover $\{U_\alpha\}_\alpha$ of A in S.

($\Leftarrow$) Suppose every open cover of A in S has a finite subcover, and let $\{V_\alpha\}_\alpha$ be an open cover of A in A. Then each V_α is an open set of A in subspace topology. According to the definition of subspace topology, there is an open set U_α in S such that $V_\alpha = U_\alpha \cap A$. Now,

$$A \subseteq \bigcup_\alpha V_\alpha = \bigcup_\alpha (U_\alpha \cap A) = \left(\bigcup_\alpha U_\alpha\right) \cap A \subseteq \bigcup_\alpha U_\alpha$$

Therefore, $\{U_\alpha\}_\alpha$ is an open cover of A in S. By hypothesis, there are finitely many sets $\{U_{\alpha_i}\}_{i=1}^n$ such that $A \subseteq \bigcup\limits_{i=1}^n U_{\alpha_i}$. Hence,

$$A \subseteq \left(\bigcup_{i=1}^n U_{\alpha_i}\right) \cap A = \bigcup_{i=1}^n (U_{\alpha_i} \cap A) = \bigcup_{i=1}^n V_{\alpha_i}$$

So $\{V_{\alpha_i}\}_{i=1}^n$ is a finite subcover of $\{V_\alpha\}$ that covers A in A. Therefore, A is compact. ■

Proposition 1.8.2

Every compact subset of K of a Hausdorff space S is closed.

Proof. We shall prove that $S \setminus K$ is open. Let's take $x \in S \setminus K$. We claim that there is a neighborhood U_x of x that is disjoint from K.

Since S is Hausdorff, for each $y \in K$, we can choose disjoint open sets U_y and V_y such that $U_y \ni x$ and $V_y \ni y$. The collection $\{V_y \ : \ y \in K\}$ is an open cover of K in S. Since K is compact, there exists a finite subcover $\{V_{y_i}\}_{i=1}^n$. That is $K \subseteq \bigcup_{i=1}^n V_{y_i}$. Since $U_{y_i} \cap V_{y_i}$ for every i, we have

$$\left(\bigcap_{i=1}^n U_{y_i}\right) \cap \left(\bigcup_{i=1}^n V_{y_i}\right) = \varnothing \implies U_x \cap K = \varnothing \text{ where } U_x = \bigcap_{i=1}^n U_{y_i}$$

U_x is the finite intersection of open sets, hence open. Also, every U_{y_i} contains x, hence their intersection U_x also contains x. So U_x is the desired open set that is disjoint from K, in other words $x \in U_x \subseteq S \setminus K$. As a result,

$$S \setminus K \subseteq \bigcup_{x \in S \setminus K} U_x \subseteq S \setminus K \implies S \setminus K = \bigcup_{x \in S \setminus K}$$

$S \setminus K$ is the union of open sets, hence open. Therefore K is closed. ■

Proposition 1.8.3

The image of a compact set under a continuous map is compact.

Proof. Let $f : X \to Y$ be a continuous and K a compact subset of X. Suppose $\{U_\alpha\}$ is an open cover of $f(K)$ by open subsets of Y. Since, f is continuous, the inverse images of $f^{-1}(U_\alpha)$ are all open in X. Moreover,

$$K \subseteq f^{-1}(f(K)) \subseteq f^{-1}\left(\bigcup_\alpha U_\alpha\right) = \bigcup_\alpha f^{-1}(U_\alpha)$$

So $\{f^{-1}(U)_\alpha\}$ is an open cover of K in X. By Proposition 1.8.1, there is a finite sub-collection $\{f^{-1}(U_{\alpha_i})\}_{i=1}^n$ such that

$$K \subseteq \bigcup_{i=1}^n f^{-1}(U_{\alpha_i}) = f^{-1}\left(\bigcup_{i=1}^n U_{\alpha_i}\right) \implies f(K) \subseteq f\left(f^{-1}\left(\bigcup_{i=1}^n U_{\alpha_i}\right)\right) \subseteq \bigcup_{i=1}^n U_{\alpha_i}$$

Thus $f(K)$ is compact. ■

Lemma 1.8.4

A closed subset F of a compact topological space S is compact.

Proof. Let $\{U_\alpha\}_\alpha$ be an open cover of F in S. The collection $\{U_\alpha, S \setminus F\}$ is an open cover of S itself. By compactness of S, there is a finite sub-cover $\{U_{\alpha_i}, S \setminus F\}_{i=1}^n$ of S, that is,

$$F \subseteq S \subseteq \left(\bigcup_{i=1}^n U_{\alpha_i} \right) \cup (S \setminus F) \implies F \subseteq \bigcup_{i=1}^n U_{\alpha_i}$$

Therefore, $\{U_{\alpha_i}\}_{i=1}^n$ is a finite subcover of the open cover $\{U_\alpha\}$ of F in S. Hence, F is also compact. ■

Proposition 1.8.5

A continuous map $f : X \to Y$ form a compact space X to a Hausdorff space Y is a closed map (a map that takes closed sets to closed sets).

Proof. Let $F \subseteq X$ be closed. Then F is compact by Lemma 1.8.4. Since $f : X \to Y$ is a continuous map, by Proposition 1.8.3, $f(F)$ is compact in Y. Since Y is Hausdorff, by Proposition 1.8.2, $f(F)$ is closed in Y. Hence, f is a closed map. ■

Corollary 1.8.6

A continuous bijection $f : X \to Y$ from a compact space X to a Hausdorff space is a homeomorphism.

Proof. We want to show thaat $f^{-1} : Y \to X$ is continuous. And in order to that it suffices to show that for every closed set F in X, $\left(f^{-1}\right)^{-1}(F) = f(F)$ is closed in Y. In other words, it suffices to show that f is a closed map. The corollary then follows from Proposition 1.8.5. ■

Definition 1.8.3 (Bounded Set)**.** A subset A of $\mathbb{R}^n$ is said to be bounded if it is contained in some open ball $B(p,r)$. otherwise, it is unbounded.

Theorem 1.8.7 (Heine-Borel Theorem)

A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

Definition 1.8.4 (Diameter of Set)**.** Let $A \subseteq X$ be a bounded subset of a metric space (X,d). The diameter of A is defined by

$$\operatorname{diam}(A) := \sup \{d(a_1, a_2) \ : \ a_1, a_2 \in A\}$$

Lemma 1.8.8 (Lebesgue Number Lemma)

Let (X,d) be a compact metric space. Given an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in J}$ of X, there exists a number $\delta > 0$ — called the Lebesgue number associated with the cover — such that for a given $A \subseteq X$ with $\operatorname{diam}(A) < \delta$, one must have $A \subseteq U_\alpha$ for some $\alpha \in J$.

Proof. Take $x \in X$. As $\mathcal{U}$ covers X, we can find $U_\alpha \in \mathcal{U}$ such that $x \in U_\alpha$. Since U_α is open and $x \in U_\alpha$, there exists $r_x > 0$ such that

$$B(x, r_x) \subseteq U_\alpha$$

We do this for every $x \in X$. So we get an open cover of X

$$X = \bigcup_{x \in X} B\left(x, \frac{r_x}{2}\right)$$

Since X is compact, there exists a finite subcover of this open cover. So

$$X = \bigcup_{i=1}^{n} B\left(x_i, \frac{r_{x_i}}{2}\right)$$

We define $\delta > 0$ in the following way:

$$\delta = \min\left\{\frac{r_{x_i}}{2} \ : \ i = 1, 2, \dots, n\right\}$$

We claim that this δ is our desired Lebesgue number of the open cover $\mathcal{U}$. Let $A \subseteq X$ with $\operatorname{diam}(A) < \delta$. Fix $a \in A$. Then there exists $j \in \{1, 2, \dots, n\}$ such that

$$a \in B\left(x_j, \frac{r_{x_j}}{2}\right) \implies \boxed{d\left(x_j, a\right) < \frac{r_{x_j}}{2}}$$

By the construction of r_{x_j}, there exists $U_\beta \in \mathcal{U}$ such that $B\left(x_j, r_{x_j}\right) \subseteq U_\beta$. We claim that $A \subseteq U_\beta$. Take any $b \in A$.

$$d\left(a, b\right) \leq \operatorname{diam}(A) < \delta \leq \frac{r_{x_j}}{2} \implies \boxed{d\left(a, b\right) < \frac{r_{x_j}}{2}}$$

$$d\left(x_j, b\right) \leq d\left(x_j, a\right) + d\left(a, b\right) < \frac{r_{x_j}}{2} + \frac{r_{x_j}}{2} = r_{x_j} \implies b \in B\left(x_j, r_{x_j}\right)$$

For every $b \in A$, we have $b \in B\left(x_j, r_{x_j}\right)$. Therefore, $A \subseteq B\left(x_j, r_{x_j}\right) \subseteq U_\beta$. ■

§1.9 Quotient Topology Continued

Let I be the closed interval $[0, 1]$ in the standard topology of $\mathbb{R}^n$ and $I/\sim$ be the quotient space obtained from I by identifying the 2 points $\{0, 1\}$ to a point. Denote by S^1 the unit circle in the complex plane. Define f by $f(x) = e^{2\pi i x}$.

Now the function $f : I \to S^1$ defined above assumes the same value at 0 and 1 and based on the discussion prior to Proposition 1.7.1, f induces the map $\bar{f} : I/{\sim} \to S^1$.

Proposition 1.9.1

The function $\bar{f} : I/{\sim} \to S^1$ is a homeomorphism.

Proof. The function $f : I \to S^1$ defined by $f(x) = e^{2\pi i x}$ is continuous (check!). Therefore, by Proposition 1.7.1, $\bar{f} : I \to S^1$ is also continuous.

Note that $I = [0, 1]$ in $\mathbb{R}$ is closed and bounded and hence by Heine-Borel Theorem, I is compact. Since the projection $\pi : I \to I/\sim$ is continuous, by Proposition 1.8.3, the image of I under π, *i.e.*, $I/{\sim}$ is compact.

It should also be obvious that $\bar{f} : I/{\sim} \to S^1$ is a bijection. Since S^1 is a of the Hausdorff space $\mathbb{R}^2$, by Lemma 1.4.2, S^1 is also Hausdorff. Hence, $\bar{f}$ is a continuous bijection from the compact space $I/\sim$ to the Hausdorff topological space S^1. Therefore, by Corollary 1.8.6, $\bar{f} : I/{\sim} \to S^1$ is a homeomorphism. ■

Necessary Condition for a Hausdorff quotient

Even if S is a Hausdorff space, the quotient space $S/{\sim}$ may fail to be Hausdorff.

Proposition 1.9.2

If the quotient space $S/\sim$ is Hausdorff, then the equivalence class $[p]$ of any point p in S is closed in S.

Proof. By Proposition 1.4.1, every singleton set is closed in a Hausdorff topological space. Now, consider the canonical projection map $\pi : S \to S/\sim$. For a point $p \in S$, $\{\pi(p)\}$ is a singleton set in $S/\sim$.

Since, by hypothesis $S/\sim$ is Hausdorff, $\{\pi(p)\}$ must be closed in $S/\sim$ with respect to quotient topology. By continuity of π, $\pi^{-1}(\{\pi(p)\})$ is closed in S. But $\pi^{-1}(\{\pi(p)\}) = [p]$. Hence, $[p]$ is a closed set in S. ■

Remark 1.9.1. In order to prove that a quotient space $S/\sim$ is not Hausdorff it is sufficient to prove that the equivalence class $[p]$ of some point $p \in S$ is not closed in S. We have the following example to elucidate this remark.

Example 1.9.1

Define an equivalence relation $\sim$ on $\mathbb{R}$ by identifying the open interval $(0, \infty)$ to a point. The resulting quotient space $\mathbb{R}/\sim$ is not Hausdorff since the equivalence class $(0, \infty)$ is not a closed subset of $\mathbb{R}$.

§1.10 Open Equivalence Relations

Definition 1.10.1. An equivalence relation $\sim$ on a topological space S is said to be open if the underlying projection map $\pi : S \to S/\sim$ is open (maps open sets to open sets).

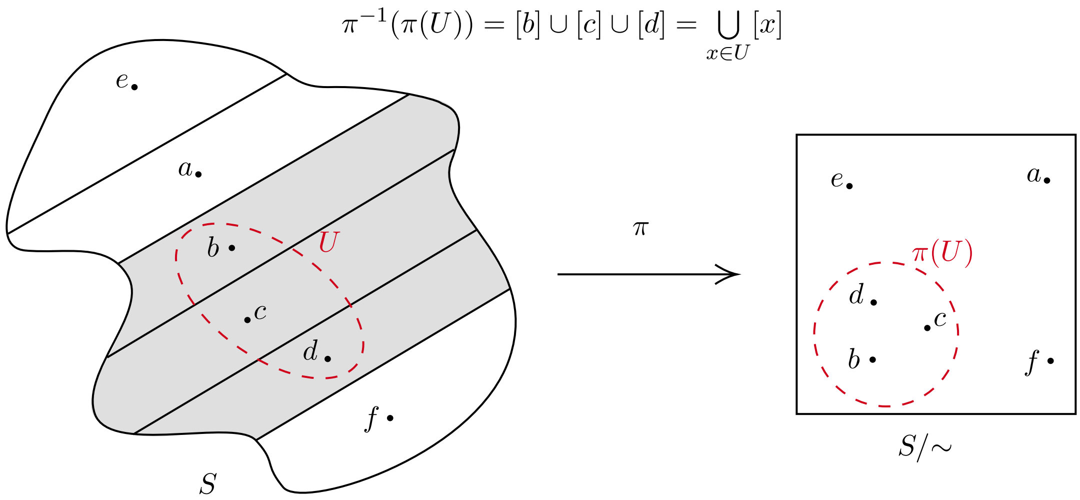


Figure 1.3: Indeed $\pi^{-1}(\pi(U)) = \bigcup_{x \in U} [x]$

In other words, the equivalence relation $\sim$ on S is open if and only if for every open set $U \in S$, the set $\pi(U) \in S/\sim$ is open. Or equivalently, by definition of quotient topology,

$$\pi^{-1}(\pi(U)) = \bigcup_{x \in U} [x] \quad \text{is open in } S$$

$\bigcup_{x \in U} [x]$ denotes all points equivalent to some point of U (shaded region in Figure 1.3).

Example 1.10.1

The projection map onto a quotient space is, in general, not open. For example, let $\sim$ be the equivalence relation on the real line $\mathbb{R}$ that identifies the two points 1 and -1, and $\pi : \mathbb{R} \to \mathbb{R}/\sim$ the projection map.

The map π is open if and only if for every open set V in $\mathbb{R}$, its image $\pi(V)$ is open in $\mathbb{R}/\sim$, or equivalently $\pi^{-1}(\pi(V))$ is open in $\mathbb{R}$. Let V be the open interval $(-2,0)$ in $\mathbb{R}$. Then,

$$\pi^{-1}(\pi(V)) = (-2,0) \cup \{1\}\,, \qquad [\text{Since } \pi(1) \in \pi(V)]$$

which is not open in $\mathbb{R}$ and hence π is not an open map. In other words, the equivalence relation $\sim$ is not open.

Definition 1.10.2 (Graph of Equivalence Relation)**.** Given an equivalence relation $\sim$ on S, let R be the subset of $S \times S$ that defines the relation $R = \{(x,y) \in S \times S \mid x \sim y\}$. We call R the **graph** of the equivalence relation $\sim$.

We have a necessary and sufficient condition for a quotient space to be Hausdorff if the underlying equivalence relation is an open equivalence relation.

Theorem 1.10.1

Suppose $\sim$ is an open equivalence relation on a topological space S. Then the quotient space $S/\sim$ is Hausdorff if and only if the graph R of $\sim$ is closed in $S \times S$.

Proof. **(⇐)** Suppose R is closed in $S \times S$. Then $R^c = (S \times S) \setminus R$ is open. Therefore, for every $(x,y) \in R^c$, there exists basic open set $U \times V$ containing (x,y) such that $U \times V \subseteq R^c$. This is equivalent to saying, no element of U is equivalent to any element of V, and vice versa.

Since $\sim$ is an open equivalence relation, $\pi(U)$ and $\pi(V)$ are open sets containing $[x]$ and $[y]$, respectively. Since no element of U is equivalent to any element of V, $\pi(U)$ and $\pi(V)$ are disjoint. Therefore, for $[x] \neq [y]$, we have found their disjoint neighborhoods. Hence, $S/\sim$ is Hausdorff.

(⇒) Now suppose $S/\sim$ is Hausdorff. Take $[x] \neq [y]$ from $S/\sim$. Then there exist disjoint neighborhoods $A \ni [x]$ and $B \ni [y]$. A and B are open, so $U = \pi^{-1}(A)$ and $V = \pi^{-1}(B)$ are open in S.

$$\pi(U) = \pi\left(\pi^{-1}(A)\right) = A \;\text{ and } \pi(V) = \pi\left(\pi^{-1}(B)\right) = B\,.$$

So $\pi(U)$ and $\pi(V)$ are disjoint. In other words, no element of U is equivalent to any element of V. Therefore, $U \times V \subseteq R^c$. $[x] \in A$, and $U = \pi^{-1}(A)$, so $x \in U$. Similarly, $y \in V$. Therefore,

$$(x,y) \in U \times V \subseteq R^c\,.$$

So R^c is open, and hence R is closed. ■

If the equivalence relation $\sim$ is equality, *i.e.*, $x \sim y$ iff $x = y$, then the quotient space $S/\sim$ is S itself and the graph R of $\sim$ is simply the diagonal $\Delta = \{(x,x) \in S \times S\}$.

Corollary 1.10.2

A topological space is Hausdorff if and only if the diagonal Δ is closed in $S \times S$.

Theorem 1.10.3

Let $\sim$ be an open equivalence relation on a topological space S with projection $\pi : S \to S/\sim$. If $\mathcal{B} = \{B_\alpha\}$ is a basis for S, then its image $\{\pi(B_\alpha)\}$ under π is a basis for $S/\sim$.

Proof. Since π is open, $\{\pi(B_\alpha)\}$ is a collection of open sets in $S/\sim$. Let W be an open set in $S/\sim$ and $[x] \in W$ with $x \in S$. So $\pi(x) \in W$, *i.e.*, $x \in \pi^{-1}(W)$. Since $\pi^{-1}(W)$ is open in S, there is a basic open set $B \in \mathcal{B}$ such that, $x \in B \subseteq \pi^{-1}(W)$. Hence

$$[x] = \pi(x) \in \pi(B) \subseteq \pi(\pi^{-1}(W)) \subseteq W$$

Now, we have seen that given W open in $S/\sim$ and $[x] \in W$, there exists an open set $\pi(B)$ in the collection $\{\pi(B_\alpha)\}$ such that $[x] \in \pi(B) \subseteq W$. This proves that $\{\pi(B_\alpha)\}$ is a basis for $S/\sim$. ∎

Corollary 1.10.4

If $\sim$ is an open equivalence relation on a second-countable topological space, then the quotient space $S/\sim$ is second countable.

2 Multivariable Calculus Review

§2.1 Differentiabiliy

Consider $f : \mathbb{R}^2 \to \mathbb{R}$ defined as follows:

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

For the piecewise defined function stated above, note that along the x-axis $y = 0$. So $f(x,0) = 0$ for every $x \in \mathbb{R}$. In other words, f is constant and identically 0 on the x-axis. Therefore,

$$\left.\frac{\partial f}{\partial x}(x,y)\right|_{y=0} = 0\,.$$

Similarly, along the y-axis $x = 0$. So $f(0,y) = 0$ for every $y \in \mathbb{R}$. In other words, f is constant and identically 0 on the y-axis. Therefore,

$$\left.\frac{\partial f}{\partial y}(x,y)\right|_{x=0} = 0\,.$$

Therefore, both the partial derivatives exist at $(0,0)$, and are equal to 0. We will now show that f is not even continuous at $(0,0)$. Consider the line $y = x$, and we shall evaluate the limit of $f(x,y)$ as $(x,y) \to (0,0)$ along this line.

$$\lim_{x\to 0} f(x,x) = \lim_{x\to 0} \frac{x \cdot x}{x^2 + x^2} = \frac{1}{2} \neq 0\,.$$

So we get,

$$\begin{aligned} \lim_{(x,y)\to(0,0)} f(x,y) &= 0\ ,\ \text{along } x\text{-axis;} \\ \lim_{(x,y)\to(0,0)} f(x,y) &= 0\ ,\ \text{along } y\text{-axis;} \\ \lim_{(x,y)\to(0,0)} f(x,y) &= \frac{1}{2}\ ,\ \text{along the line } y = x. \end{aligned}$$

Therefore, f is not even continuous at $(0,0)$, let alone being differentiable. Therefore, mere existence of partial derivatives of order doesn't guarantee differentiability at a given point.

We will, first, consider functions whose domain is $U \subseteq \mathbb{R}^n$ and codomain is $\mathbb{R}$. If $f : U \to \mathbb{R}^n$ is such a function, then $f(\vec{x}) = f\left(x^1, x^2, \ldots, x^n\right)$ denotes its value at $\vec{x} \equiv \left(x^1, x^2, \ldots, x^n\right) \in U$. We also assume that the underlying domain of f is an open set $U \subseteq \mathbb{R}^n$. At each $\vec{a} \in U$, the partial derivative $\left.\frac{\partial f}{\partial x^j}\right|_{\vec{a}}$ of f with respect to x^j is the following limit, if it exists

$$\left.\frac{\partial f}{\partial x^j}\right|_{\vec{x}=\vec{a}} = \lim_{h\to 0} \frac{f\left(a^1, \ldots, a^j + h, \ldots, a^n\right) - f\left(a^1, \ldots, a^j, \ldots, a^n\right)}{h}\,.$$

If $\frac{\partial f}{\partial x^j}$ is defined, that is, the limit above exists at each point of U for $1 \leq j \leq n$, this defines n functions on U. Should these functions be continuous on U for $1 \leq j \leq n$, f is said to be continuously differentiable on U, denoted by $f \in C^1(U)$.

We shall say that f is differentiable at $\vec{a} \in U$ if there is a homogenous linear expression $\sum_{i=1}^{n} b_i \left(x^i - a^i\right)$ such that the inhomogenous expression $f(\vec{a}) + \sum_{i=1}^{n} b_i \left(x^i - a^i\right)$ approximates $f(\vec{x})$ near $\vec{a}$ in the following sense:

$$\lim_{\vec{x} \to \vec{a}} \frac{f(\vec{x}) - f(\vec{a}) - \sum_{i=1}^{n} b_i \left(x^i - a^i\right)}{\|\vec{x} - \vec{a}\|} = 0\,.$$

In other words, if there exist constants $b_1, b_2, \ldots, b_n$ and a real valued function $r(\vec{x}, \vec{a})$ defined on a neghborhood V of $\vec{a} \in U$ such that the following two conditions hold:

$$f(\vec{x}) = f(\vec{a}) + \sum_{i=1}^{n} b_i \left(x^i - a^i\right) + \|\vec{x} - \vec{a}\|\, r(\vec{x}, \vec{a}) \quad \text{and} \quad \lim_{\vec{x} \to \vec{a}} r(\vec{x}, \vec{a}) = 0\,.$$

b_i's are uniquely determined, and they are the partial derivatives at $\vec{a}$:

$$b_i = \left.\frac{\partial f}{\partial x^i}\right|_{\vec{x}=\vec{a}}\,.$$

In fact,

$$f(\vec{x}) = f(\vec{a}) + \sum_{i=1}^{n} \left.\frac{\partial f}{\partial x^i}\right|_{\vec{x}=\vec{a}} \left(x^i - a^i\right) + \|\vec{x} - \vec{a}\|\, r(\vec{x}, \vec{a})\,.$$

Actually, existence of partial derivatives and their continuity guarantees differentiability at a given point $\vec{a} \in U \subseteq \mathbb{R}^n$.

§2.2 Chain Rule

By a differentiable curve in $\mathbb{R}^n$, we mean $f : (a, b) \to \mathbb{R}^n$, with $f(t) = \left(x^1(t), x^2(t), \ldots, x^n(t)\right)$, where the n coordinate functions $x^i(t)$ are all differentiable on (a, b). Recall that, for a function of one variable, differentiability is equivalent to existence of derivative.

Here, $\left(x^i(t)\right)$ are real valued functions of one variable. And you must be familiar with the notion of C^r-differentiability of real valued functions of one variable. For example, $h(t) = t^{\frac{1}{3}}$ is not C^1, because its derivative does not exist at $t = 0$. Similarly, $k(t) = t^{\frac{4}{3}}$ is C^1, but not C^2.

Now, let's suppose $f : (a, b) \to \mathbb{R}^n$ is a C^r differentiable curve in the sense that all the n coordinate functions $x^i(t)$ are C^r differentiable. Take t_0 with $a < t_0 < b$, and $f : (a, b) \to U \subseteq \mathbb{R}^n$. Let g be a C^r-differentiable function from U to $\mathbb{R}$. In particular, $g : U \to \mathbb{R}$ is differentiable at $f(t_0) \in U$. Then $g \circ f : (a, b) \to \mathbb{R}$ is differentiable at t_0, and the derivative is given by:

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}(g \circ f)(t)\right|_{t=t_0} = \sum_{i=1}^{n} \left.\frac{\partial g(f(t))}{\partial x^i}\right|_{f(t_0)} \cdot \left.\frac{\mathrm{d}x^i(t)}{\mathrm{d}t}\right|_{t=t_0}\,.$$

This result is known as the chain rule for real-valued functions.

Now, we can generalize this idea to functions on subsets U of $\mathbb{R}^n$, whose range is not in $\mathbb{R}$, but in $\mathbb{R}^n$. In other words, we consider $F : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^m$.

$$\vec{x} \equiv \left(x^1, x^2, \ldots, x^n\right) \in U\ ;\ F(\vec{x}) = \left(F^1(\vec{x}), F^2(\vec{x}), \ldots, F^m(\vec{x})\right)\,.$$

Now take a point $\vec{p} \in U$ with coordinate $\left(p^1, p^2, \ldots, p^n\right)$. Then $F(\vec{p})$ is a point in V with coordinate $\left(F^1(\vec{p}), F^2(\vec{p}), \ldots, F^m(\vec{p})\right)$. Now let $G : V \subseteq R^m \to R^l$. Write a point $\vec{y} \equiv \left(y^1, y^2, \ldots, y^m\right) \in V \subseteq \mathbb{R}^m$. Then

$$G(\vec{y}) = \Big(G^1(\vec{y}), G^2(\vec{y}), \ldots, G^l(\vec{y})\Big)\,.$$

In other words, $G^i : V \to \mathbb{R}$. Then we have $G^i \circ F : U \subseteq \mathbb{R}^n \to \mathbb{R}$. In this case, the chain rule is

$$\frac{\partial \left(G^i \circ F\right)}{\partial x^j}(\vec{p}) = \sum_{k=1}^{m} \frac{\partial G^i}{\partial y^k}(F(\vec{p})) \cdot \frac{\partial F^k}{\partial x^j}(\vec{p})\,.$$

§2.3 Differential of a Map

Let $F : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^m$. Let $T_p\mathbb{R}^n$ denote the tangent space on $\mathbb{R}^n$ to the point $p \in \mathbb{R}^n$. (For convenience, we'll drop arrows in $\vec{p}$) The differential of F at p is a map $DF_p : T_p\mathbb{R}^n \to T_{F(p)}\mathbb{R}^m$. $T_p\mathbb{R}^n$ is clearly isomorphic to $\mathbb{R}^n$ as vector space. Hence, $DF_p : \mathbb{R}^n \to \mathbb{R}^m$. Let's try to see that DF_p is related to the Jacobian matrix of $F : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^m$.

$$\left\{ \left.\frac{\partial}{\partial x^1}\right|_p, \left.\frac{\partial}{\partial x^2}\right|_p, \dots, \left.\frac{\partial}{\partial x^n}\right|_p \right\}$$

is a basis of $T_p\mathbb{R}^n$, which can be treated as $\mathbb{R}^n$ with origin at p. Similarly,

$$\left\{ \left.\frac{\partial}{\partial y^1}\right|_{F(p)}, \left.\frac{\partial}{\partial y^2}\right|_{F(p)}, \dots, \left.\frac{\partial}{\partial y^m}\right|_{F(p)} \right\}$$

is a basis of $T_{F(p)}\mathbb{R}^m$, which can be treated as $\mathbb{R}^m$ with origin at $F(p)$.

Geometric tangent vectors like $\left.\frac{\partial}{\partial x^i}\right|_p$ or $\left.\frac{\partial}{\partial y^j}\right|_{F(p)}$ act on smooth functions of $\mathbb{R}^n$ or $\mathbb{R}^m$, respectively, and spit out real numbers.

$$\left.\frac{\partial}{\partial x^i}\right|_p f = \frac{\partial f}{\partial x^i}(p) \in \mathbb{R}\,.$$

Since DF_p is a linear map between two vector spaces, in order to express DF_p as a matrix, we need to find where the basis vectors are getting mapped. So we want to find $DF_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right)$. This is a vector in $T_{F(p)}\mathbb{R}^m$, and hence can be written as a linear combination of $\left.\frac{\partial}{\partial y^j}\right|_{F(p)}$'s. Now we wish to find the coefficients in the linear combination.

$DF_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right)$ acts on $f \in C^\infty(\mathbb{R}^m)$ and yields a real number.

$$DF_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) f := \left.\frac{\partial}{\partial x^i}\right|_p (f \circ F)\,.$$

This makes perfect sense as $f \circ F : U \subseteq \mathbb{R}^n \to \mathbb{R}$. By chain rule,

$$\left.\frac{\partial}{\partial x^i}\right|_p (f \circ F) = \frac{\partial (f \circ F)}{\partial x^i}(p) = \sum_{j=1}^{m} \left.\frac{\partial f}{\partial y^j}\right|_{F(p)} \left.\frac{\partial F^j}{\partial x^i}\right|_p\,.$$

$$\therefore DF_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) f = \sum_{j=1}^{m} \left.\frac{\partial F^j}{\partial x^i}\right|_p \left.\frac{\partial}{\partial y^j}\right|_{F(p)} f \implies \boxed{DF_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = \sum_{j=1}^{m} \frac{\partial F^j}{\partial x^i}(p) \cdot \left.\frac{\partial}{\partial y^j}\right|_{F(p)}}$$

Therefore, DF_p can be represented by the following $m \times n$ matrix:

$$\begin{bmatrix} \frac{\partial F^1}{\partial x^1}(p) & \frac{\partial F^1}{\partial x^2}(p) & \cdots & \frac{\partial F^1}{\partial x^n}(p) \\ \frac{\partial F^2}{\partial x^1}(p) & \frac{\partial F^2}{\partial x^2}(p) & \cdots & \frac{\partial F^2}{\partial x^n}(p) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F^m}{\partial x^1}(p) & \frac{\partial F^m}{\partial x^2}(p) & \cdots & \frac{\partial F^m}{\partial x^n}(p) \end{bmatrix}$$

F is differentiable at $p \in U \subseteq \mathbb{R}^n$ if all the entries in the $m \times n$ matrix DF exist and are continuous at p. If F is differentiable at every $p \in U$, we say that F is of class C^1. DF is called the total derivative in the language of multivariable calculus.

Similarly, if all the second order partial derivatives exist and are continuous at p, then we say F is twice differentiable at p. If F is twice differentiable at every $p \in U$, we say F is of class C^2. In a similar manner, we define maps of class C^r. If a map F is of class C^r for every $r \in \mathbb{N}$, we say F is **smooth** or **infinitely differentiable**, or F belongs in the class C^∞.

§2.4 Inverse Function Theorem

Definition 2.4.1. Let U and V be open subsets of $\mathbb{R}^n$. A map $F : U \to V$ is said to be a $\boldsymbol{C^r}$**-diffeomorphism** if F is a homeomorphism, and both F and F^{-1} are of class C^r. When $r = \infty$, we just say F is a **diffeomorphism**.

Theorem 2.4.1 (Inverse Function Theorem)

Let W be an open subset of $\mathbb{R}^n$ and $F : W \to \mathbb{R}^n$ a C^∞ mapping. If $p \in W$ and DF_p is nonsingular, then there exists a neighborhood U of p in W such that $V = F(U)$ is open and $F : U \to V$ is a diffomorphism. If $x \in U$, then

$$DF^{-1}_{F(x)} = (DF_x)^{-1} .$$

We are not going to prove it here. We will see an example now.

Example 2.4.1. Let's consider the conversion of polar to rectangular coordinate. $F : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$F\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} .$$

Then the differential DF is

$$DF = \begin{bmatrix} \dfrac{\partial F^1}{\partial r} & \dfrac{\partial F^1}{\partial \theta} \\ \dfrac{\partial F^2}{\partial r} & \dfrac{\partial F^2}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

Hence, $\det DF = r$. So $DF_{(r,\theta)}$ is differentiable for $r \neq 0$. Choose $r = \sqrt{2}$ and $\theta = \frac{\pi}{4}$. Then

$$F\begin{pmatrix} \sqrt{2} \\ \frac{\pi}{4} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} .$$

$$DF_{(\sqrt{2},\frac{\pi}{4})} = \begin{bmatrix} \dfrac{1}{\sqrt{2}} & -1 \\ \dfrac{1}{\sqrt{2}} & 1 \end{bmatrix} .$$

By the Inverse Function Theorem, there is a local inverse

$$DF^{-1}_{(1,1)} = \left(DF_{(\sqrt{2},\frac{\pi}{4})}\right)^{-1} .$$

Now, F^{-1} is given by

$$F^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sqrt{x^2+y^2} \\ \tan^{-1}\left(\frac{y}{x}\right) \end{pmatrix} .$$

Therefore,

$$DF^{-1} = \begin{bmatrix} \dfrac{2x}{2\sqrt{x^2+y^2}} & \dfrac{2y}{2\sqrt{x^2+y^2}} \\ \dfrac{-y}{x^2+y^2} & \dfrac{x}{x^2+y^2} \end{bmatrix} = \begin{bmatrix} \dfrac{x}{\sqrt{x^2+y^2}} & \dfrac{y}{\sqrt{x^2+y^2}} \\ \dfrac{-y}{x^2+y^2} & \dfrac{x}{x^2+y^2} \end{bmatrix} .$$

As a result,

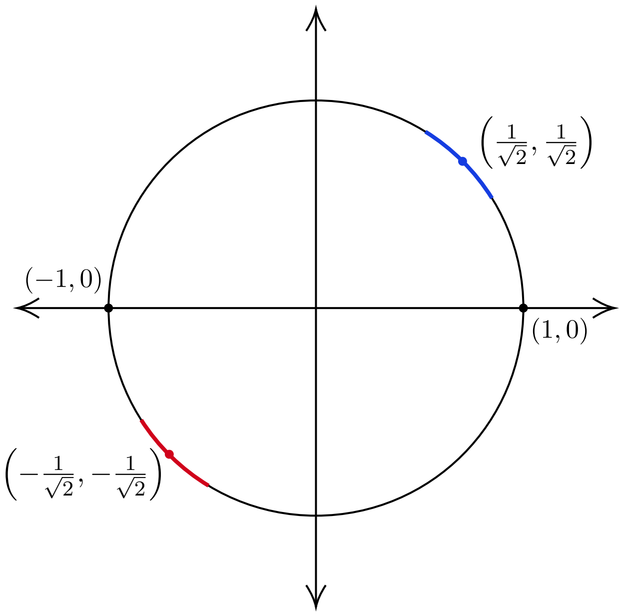
$$DF^{-1}_{(1,1)} = \begin{bmatrix} \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \\ \dfrac{-1}{2} & \dfrac{1}{2} \end{bmatrix} .$$

One can indeed check that

$$\begin{bmatrix} \dfrac{1}{\sqrt{2}} & -1 \\ \dfrac{1}{\sqrt{2}} & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \\ \dfrac{-1}{2} & \dfrac{1}{2} \end{bmatrix} .$$

§2.5 Implicit Function Theorem

Let us consider the equation of a unit circle in $\mathbb{R}^2$; $x^2 + y^2 = 1$.



The graph of the unit circle above does not represent a function. Because, for a given value of x, there are 2 values for y that satisfy the equation. Choose a point, say $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, on the unit circle. Then one can consider an arc (colored blue in the figure above) containing $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ that indeed represents a function given by $y = \sqrt{1 - x^2}$. Had we started with the point $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$, we could find an arc (colored red in the figure above) containing $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ that represents a function given by $y = -\sqrt{1 - x^2}$. The only problematic points are $(1, 0)$ and $(-1, 0)$. No matter how small an arc we choose about these points, it is not going to be represented by a function. Because, for those arcs, for a given x, there will be multiple values for y.

Now let us address the following 2-dimensional problem: Given an equation $F(x, y) = 0$, which is not globally a functional relationship (in the unit circle example, $F(x, y) = x^2 + y^2 - 1$), does there exist a point (x_0, y_0) satisfying $F(x_0, y_0) = 0$ so that there exists a neighborhood of (x_0, y_0) where y can be written as $y = f(x)$ for some real valued function f of one variable? In other words, $F(x, f(x)) = 0$ should hold for all values of x in that neighborhood. In the unit circle example, this f was given by $f(x) = \sqrt{1 - x^2}$ or $f(x) = -\sqrt{1 - x^2}$, depending on the choice of the point (x_0, y_0) in the upper or lower semicircle, respectively. The Implicit Function Theorem guarantees the **local existence** of such a function provided the initial point (x_0, y_0) was chosen *appropriately*. In the unit circle example, $(1, 0)$ and $(-1, 0)$ were two *inappropriate* points. As required by the Implicit Function Theorem, one must have

$$\frac{\partial F}{\partial y}(x_0, y_0) \neq 0 \, .$$

But in this case, for $F(x, y) = x^2 + y^2 - 1$,

$$\frac{\partial F}{\partial y} = 2y \implies \frac{\partial F}{\partial y}(1, 0) = 0 = \frac{\partial F}{\partial y}(1, 0) \; .$$

Therefore, in the light of Implicit Function Theorem, $(1, 0)$ and $(-1, 0)$ are not *appropriate* points on the unit circle around which we can construct a locally functional relationship. Now we state the most general form of Implicit Function Theorem.

Theorem 2.5.1 (Implicit Function Theorem)
Let U be an open set in $\mathbb{R}^n \times \mathbb{R}^m$ and $F : U \to \mathbb{R}^m$ a C^∞ map. Write $(x, y) = \left(x^1, \ldots, x^n, y^1, \ldots, y^m\right)$ for a point in U. Suppose the matrix

$$\left[\frac{\partial F^i}{\partial y^j}(x_0, y_0)\right]_{1 \le i,j \le m}$$

is non-singular for a point $(x_0, y_0) \in U$ satisfying $F(x_0, y_0) = 0$. Then there exists a neighborhood $X \times Y$ of (x_0, y_0) in U and a unique C^∞ map $f : X \to Y$ such that in $X \times Y \subseteq U \subseteq \mathbb{R}^n \times \mathbb{R}^m$,

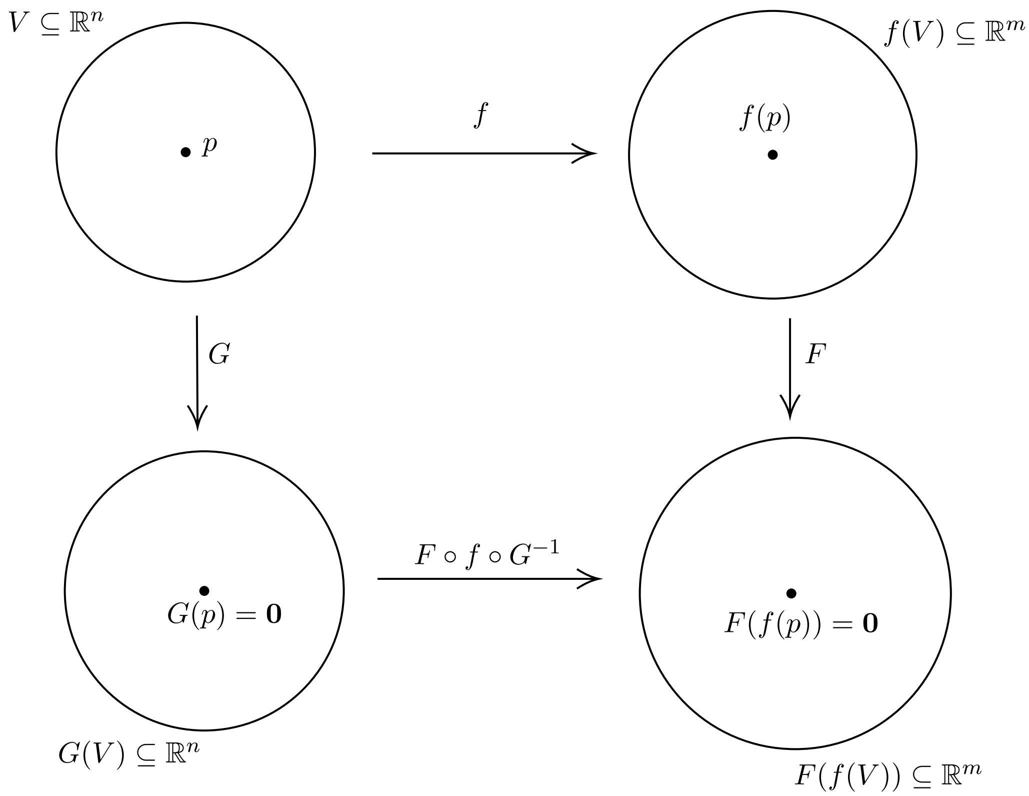
$$F(x, y) = 0 \iff y = f(x) .$$

§2.6 Constant Rank Theorem

Definition 2.6.1 (Rank of a Smooth Map at a Point)**.** Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be a C^∞ map. The **rank** of f at $p \in U$ is the rank of its Jacobian matrix $\left[\frac{\partial f^i}{\partial x^j}(p)\right]$.

Theorem 2.6.1 (Constant Rank Theorem)
Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be a C^∞ map. Suppose f has a constant rank k in a neighborhood of $p \in U$. Then there are a diffeomorphism G of a neighborhood V of $p \in U$ sending p to $\mathbf{0} \in \mathbb{R}^n$, and a diffeomorphism F of a neighborhood W of $f(p) \in \mathbb{R}^m$ sending $f(p)$ to $\mathbf{0} \in \mathbb{R}^m$ such that

$$\left(F \circ f \circ G^{-1}\right)\left(x^1, \ldots, x^n\right) = \left(x^1, \ldots, x^k, 0, \ldots, 0\right) .$$



Remark 2.6.1. Many textbooks include $f(V) \subseteq W$ in the statement of Constant Rank Theorem. Because if $f(V)$ is not a subset of W, we can always find a smaller V such that $f(V) \subseteq W$. Since G^{-1} is a map from $G(V)$ to V, we need to restrict f on V in order to form the composition $f \circ G^{-1}$. Then

$f\big|_V \circ G^{-1}$ is a map from $G(V)$ to $f(V)$. Then we need to restrict F on $f(V)$ so that the composition $F \circ f \circ G^{-1}$ makes sense. We can do this because the domain W of F contains $f(V)$. Therefore, $F \circ f \circ G^{-1}$ is actually

$$F\big|_{f(V)} \circ f\big|_V \circ G^{-1} : G(V) \to F(f(V)) \ .$$

Oftentimes we just write $F \circ f \circ G^{-1}$ when what we actually mean is $F\big|_{f(V)} \circ f\big|_V \circ G^{-1}$.

3 Manifolds

§3.1 Topological Manifolds

Definition 3.1.1 (Locally Euclidean Space). A topological space M is **locally Euclidean** of dimension n if every point in M has a neighborhood U such that there is a homeomorphism φ from U onto an open subset of $\mathbb{R}^n$. We call the pair $(U, \varphi : U \to \mathbb{R}^n)$ a **chart**, U a **coordinate neighborhood** and φ a **coordinate system** on U. We also say that a chart (U, φ) is centered at $p \in U$ if $\varphi(p) = \vec{0}$.

Definition 3.1.2 (Topological Manifold). A **topological manifold** of dimension n is a Hausdorff, second countable, locally Euclidean space of dimension n.

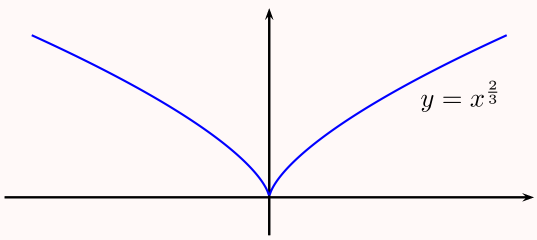
Example 3.1.1

The Euclidean space $\mathbb{R}^n$ is covered by a single chart $(\mathbb{R}^n, \mathbb{1}_{\mathbb{R}^n})$, where $\mathbb{1}_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n$ is the identity map. Every open subset U of $\mathbb{R}^n$ is also a topological manifold with the chart $(U, \mathbb{1}_U)$.

Recall that the hausdorff condition and second countability are "hereditary properties". That is, they are inherited by subspaces: a subspace of a Hausdorff space is also Hausdorff, and a subspace of a second countable space is also second countable. Hence, any subspace of $\mathbb{R}^n$ is Hausdorff and second countable.

Example 3.1.2 (The Cusp)

The graph of $y = x^{\frac{2}{3}}$ in $\mathbb{R}^2$ is a topological manifold.

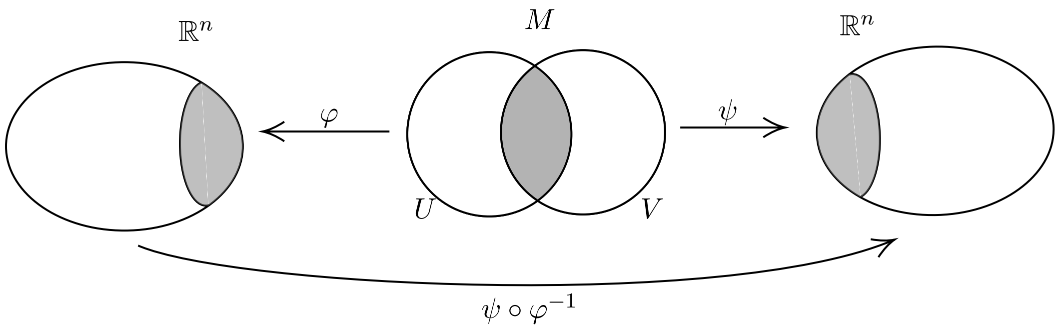


As a subspace of $\mathbb{R}^2$, it is Hausdorff and second countable. It is locally Euclidean, because it is homeomorphic to $\mathbb{R}$ via the map $\left(x, x^{2/3}\right) \mapsto x$. This map is continuous since it is just the projection onto first coordinate. The inverse map $x \mapsto \left(x, x^{2/3}\right)$ is continuous, as both $x \mapsto x$ and $x \mapsto x^{2/3}$ are continuous.

Definition 3.1.3 (Compatible Charts). Two charts $(U, \varphi : U \to \mathbb{R}^n)$ and $(V, \psi : V \to \mathbb{R}^n)$ of a topological manifold are ***C*$^\infty$-compatible** if the two maps

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V) \quad \text{and} \quad \varphi \circ \psi^{-1} : \psi(U \cap V) \to \varphi(U \cap V)$$

are both C^∞. These two maps are called **transition functions** between the charts. If $U \cap V$ is empty, then the two charts are automatically compatible.



Definition 3.1.4 (Atlas). A ***C*$^\infty$-atlas** or simply an **atlas** on a locally Euclidean space M is a collection $\mathscr{U} = \{(U_\alpha, \varphi_\alpha)\}$ of pairwise C^∞-compatible charts that cover M. In other words,

$$M = \bigcup_\alpha U_\alpha \,.$$

Example 3.1.3

The unit circle S^1 in the complex plane can be described as the set of points $\left\{e^{it} \in \mathbb{C} \mid 0 \leq t < 2\pi\right\}$. Let U_1 and U_2 be the following two open subsets of S^1:

$$U_1 = \left\{e^{it} \in \mathbb{C} \mid -\pi < t < \pi\right\} \text{ and } U_2 = \left\{e^{it} \in \mathbb{C} \mid 0 < t < 2\pi\right\} .$$

Define $\varphi_i : U_i \to \mathbb{R}$ by

$$\begin{aligned} \varphi_1\left(e^{it}\right) &= t\,, \quad -\pi < t < \pi\,; \\ \varphi_2\left(e^{it}\right) &= t\,, \quad 0 < t < 2\pi\,. \end{aligned}$$

(U_1, φ_1) and (U_2, φ_2) are charts on S^1. Their intersection $U_1 \cap U_2$ consists of two disjoint subsets of S^1 denoted by A and B.

$$A = \left\{e^{it} \in \mathbb{C} \mid -\pi < t < 0\right\} \text{ and } B = \left\{e^{it} \in \mathbb{C} \mid 0 < t < \pi\right\} .$$

$U_1 \cap U_2 = A \sqcup B$. Now,

$$\begin{aligned} \varphi_1\left(U_1 \cap U_2\right) = \varphi_1\left(A \sqcup B\right) = \varphi_1\left(A\right) \sqcup \varphi_1\left(B\right) = (-\pi, 0) \sqcup (0, \pi) \\ \varphi_2\left(U_1 \cap U_2\right) = \varphi_2\left(A \sqcup B\right) = \varphi_2\left(A\right) \sqcup \varphi_2\left(B\right) = (\pi, 2\pi) \sqcup (0, \pi) \end{aligned}$$

Now, the transition function $\varphi_2 \circ \varphi_1^{-1} : \varphi_1\left(U_1 \cap U_2\right) \to \varphi_2\left(U_1 \cap U_2\right)$ is given by:

$$\left(\varphi_2 \circ \varphi_1^{-1}\right)(t) = \begin{cases} t + 2\pi & \text{for } t \in (-\pi, 0) \\ t & \text{for } t \in (0, \pi) \end{cases}$$

Similarly, the transition function $\varphi_1 \circ \varphi_2^{-1} : \varphi_2\left(U_1 \cap U_2\right) \to \varphi_1\left(U_1 \cap U_2\right)$ is given by:

$$\left(\varphi_1 \circ \varphi_2^{-1}\right)(t) = \begin{cases} t - 2\pi & \text{for } t \in (\pi, 2\pi) \\ t & \text{for } t \in (0, \pi) \end{cases}$$

These two transition functions are C^∞. Therefore, (U_1, φ_1) and (U_2, φ_2) are C^∞-compatible charts on S^1 and form an atlas.

Remark 3.1.1. Although the C^∞-compatibility of charts is clearly reflexive and symmetric, it is not transitive. The reason is as follows. Suppose (U_1, φ_1) is C^∞-compatible with (U_2, φ_2), and (U_2, φ_2) is C^∞-compatible with (U_3, φ_3). Note that the three coordinate functions are simultaneously defined only on the triple intersection $U_1 \cap U_2 \cap U_3$. Thus, the composite

$$\varphi_3 \circ \varphi_1^{-1} = \left(\varphi_3 \circ \varphi_2^{-1}\right) \circ \left(\varphi_2 \circ \varphi_1^{-1}\right)$$

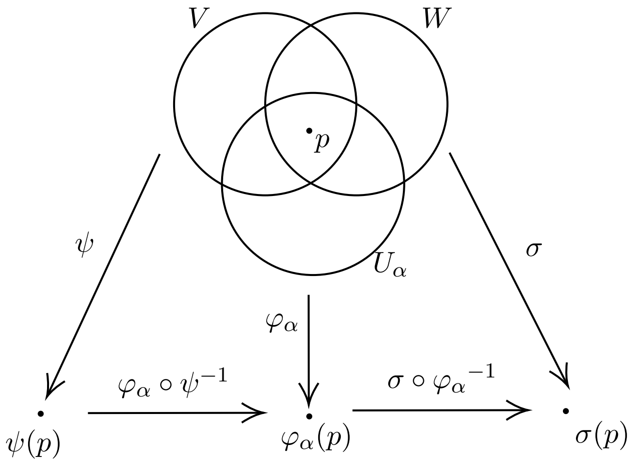
is C^∞, but only on $\varphi_1 (U_1 \cap U_2 \cap U_3)$, not necessarily on $\varphi_1 (U_1 \cap U_3)$. A priori we know nothing about $\varphi_3 \circ \varphi_1^{-1}$ on $\phi_1 ((U_1 \cap U_3) \setminus (U_1 \cap U_2 \cap U_3))$ and so we cannot conclude that (U_1, ϕ_1) and (U_3, ϕ_3) are C^∞-compatible.

We say that a chart (V, ψ) is compatible with an atlas $\{(U_\alpha, \varphi_\alpha)\}$ if it is compatible with all the charts $(U_\alpha, \varphi_\alpha)$ of the atlas.

Lemma 3.1.1

Let $\{(U_\alpha, \varphi_\alpha)\}$ be an atlas on a locally Euclidean space M. If two charts (V, ψ) and (W, σ) are both compatible with the atlas $\{(U_\alpha, \varphi_\alpha)\}$, then they are compatible with each other.

Proof. Let $p \in V \cap W$. First, we need to show that $\sigma \circ \psi^{-1}$ is C^∞ at $\psi(p)$.



Since $\{(U_\alpha, \varphi_\alpha)\}$ is an atlas for M, $p \in U_\alpha$ for some α. Hence, $p \in V \cap W \cap U_\alpha$. By the remark above,

$$\sigma \circ \psi^{-1} = \left(\sigma \circ \varphi_\alpha^{-1}\right) \circ \left(\varphi_\alpha \circ \psi^{-1}\right)$$

is C^∞ on $\psi(V \cap W \cap U_\alpha)$, and hence at $\psi(p)$. Since p was an arbitrary point of $V \cap W$, this proves that $\sigma \circ \psi^{-1}$ is C^∞ on $\psi(V \cap W)$. Similarly, $\psi \circ \sigma^{-1}$ is C^∞ on $\sigma(V \cap W)$. ■

Remark 3.1.2. In the equality $\sigma \circ \psi^{-1} = \left(\sigma \circ \varphi_\alpha^{-1}\right) \circ \left(\varphi_\alpha \circ \psi^{-1}\right)$, the maps on the two sides of the equality sign have different domains. What the equality means is that the two maps are equal on their common domain.

§3.2 Smooth Manifold

Definition 3.2.1 (Maximal Atlas)**.** An atlas $\mathscr{M}$ on a locally Euclidean space is said to be **maximal** if it is not contained in a larger atlas. In other words, if $\mathscr{U}$ is any other atlas containing $\mathscr{M}$, then $\mathscr{U} = \mathscr{M}$.

Definition 3.2.2 (Smooth Manifold)**.** A **smooth** or $\boldsymbol{C^\infty}$ **manifold** is a topological manifold M together with a maximal atlas $\mathscr{M}$. To avoid confusion, we can denote it as a pair $(M, \mathscr{M})$ of a topological manifold M and a maximal atlas $\mathscr{M}$ on M. The maximal atlas is also called a *differentiable structure* on M.

Abuse of Notation. Often instead of writing $(M, \mathscr{M})$ is a smooth manifold, we shall say M is a smooth manifold. Whenever we say that we take a chart (U, φ) on a smooth manifold M, we mean that (U, φ) is contained in the differentiable structure (*i.e.*, maximal atlas) $\mathscr{M}$ on M.

In practice, to check that a topological manifold M is a smooth manifold, it is not necessary to exhibit a maximal atlas. The existence of *any* atlas on M will do, because of the following proposition.

Proposition 3.2.1

Any atlas $\mathscr{U} = \{(U_\alpha, \varphi_\alpha)\}$ on a locally Euclidean space is contained in a unique maximal atlas.

Proof. Adjoin to the atlas $\mathscr{U}$ all the charts (V_i, ψ_i) that are compatible with $\mathscr{U}$. By Lemma 3.1.1, the charts (V_i, ψ_i) are compatible with one another. So the enlarged collection of charts is an atlas. Can we enlarge this new atlas any further? Any chart compatible with the new atlas (that we wish to adjoin to the new atlas) must be compatible with the original atlas $\mathscr{U}$ and so by construction belongs to the new atlas. This proves that the new atlas is maximal.

Now we need to prove the uniqueness. Let $\mathscr{M}$ be the maximal atlas that we constructed in the preceeding paragraph. If $\mathscr{M}'$ is another maximal atlas containing $\mathscr{U}$, then all the charts in $\mathscr{M}'$ are compatible with $\mathscr{U}$ and so by construction must belong to $\mathscr{M}$. This proves that $\mathscr{M}' \subseteq \mathscr{M}$. $\mathscr{M}'$ is a maximal atlas contained in another atlas, so $\mathscr{M}$ and $\mathscr{M}'$ must be the same. Therefore, the maximal atlas containing $\mathscr{U}$ is unique. ∎

In summary, to show that a topological space M is a smooth manifold, it suffices to check that

(i) M is Hausdorff and second countable,

(ii) M has a C^∞ atlas (not necessarily maximal).

Multiple Differentiable Structures on the Same Topological Manifold

From Proposition 3.2.1, if we have an atlas $\mathscr{U}$ on a topological manifold M, then $\mathscr{U}$ is contained in a unique maximal atlas $\mathscr{M}$. However, if we start with a different atlas $\mathscr{V}$ on M, $\mathscr{V}$ is contained in a unique maximal atlas $\mathscr{N}$. Then $\mathscr{M}$ and $\mathscr{N}$ are not, in general, the same. Therefore, $(M, \mathscr{M})$ and $(M, \mathscr{N})$ are two different smooth manifolds with the same underlying topological manifold.

For example, $\mathscr{U} = \{(\mathbb{R}, \mathbb{1}_{\mathbb{R}})\}$ is an atlas on $\mathbb{R}$ with a single chart. This atlas is contained in a maximal atlas, say $\mathscr{M}$. Then $(\mathbb{R}, \mathscr{M})$ is a smooth manifold with the usual differentiable structure. If we consider the map $\varphi : \mathbb{R} \to \mathbb{R}$ that sends x to x^3, then φ is a homeomorphism. Therefore, $\mathscr{V} = \{(\mathbb{R}, \varphi)\}$ is also an atlas on $\mathbb{R}$ with a single chart $(\mathbb{R}, \varphi)$. This $\mathscr{V}$ is contained in another maximal atlas $\mathscr{N}$. Then $(\mathbb{R}, \mathscr{N})$ is also a smooth manifold. This example is important as we will see an useful example using these two smooth manifolds in the following chapter.

Recall that we can put several topologies on a set, and then the set becomes different topological spaces under different topologies. In a similar spirit, we can have multiple maximal atlases (say $\mathscr{M}_1$ and $\mathscr{M}_2$) on a topological manifold M, and M can become different smooth manifolds $(M, \mathscr{M}_1)$ and $(M, \mathscr{M}_2)$ when equipped with different differentiable structures.

Some Notations

From now on, a "manifold" will mean a "smooth manifold". Also we shall use the terms "smooth" and "C^∞" interchangeably. Let $\vec{v} \in \mathbb{R}^n$ be a vector, or an n-tuple. The function $r^i : \mathbb{R}^n \to \mathbb{R}$ is defined as $r^i(\vec{v}) = v^i$. Let (U, φ) be a chart of the n-dimensional manifold M and let $p \in U$. Since $\varphi : U \to \mathbb{R}^n$, we write

$$\varphi(p) = \left(x^1(p), x^2(p), \ldots, x^n(p)\right) ,$$

with each component x^i of φ being a real valued function $x^i : U \to \mathbb{R}$ such that $x^i = r^i \circ \varphi$. The functions $x^1, x^2, \ldots, x^n$ are called *coordinates* or *local coordinates* on U. We sometimes write $\varphi = \left(x^1, x^2, \ldots, x^n\right)$ and the chart $(U, \varphi) = \left(U, x^1, x^2, \ldots, x^n\right)$.

Example 3.2.1 (Euclidean Space)

The Euclidean space $\mathbb{R}^n$ is a smooth manifold with a single chart $\left(\mathbb{R}^n, r^1, r^2, \ldots, r^n\right)$, where $r^1, r^2, \ldots, r^n$ are the standard coordinates on $\mathbb{R}^n$.

Example 3.2.2 (Open Subset of a Manifold)

Any open subset V of a manifold M is also a manifold. If $\{(U_\alpha, \varphi_\alpha)\}$ is an atlas for M, then

$$\mathscr{U}_V = \left\{ \left(U_\alpha \cap V, \varphi_\alpha\big|_{U_\alpha \cap V} \right) \right\}$$

is an atlas for V. Notice that V, equipped with the subspace topology inherited from M, is indeed Hausdorff and second countable. It is a topological manifold because $U_\alpha \cap V$ is open in M, and φ_α is an open map; hence, as a restriction of a homeomorphism, $\varphi_\alpha\big|_{U_\alpha \cap V}$ is a homeomorphism mapping $U_\alpha \cap V$ to an open subset of $\mathbb{R}^n$. Now we are left to show that any two charts in the collection $\mathscr{U}_V$ are compatible.

$$\varphi_\alpha\big|_{U_\alpha \cap V} \circ \varphi_\beta\big|_{U_\beta \cap V}^{-1} = \left(\varphi_\alpha \circ \varphi_\beta^{-1} \right) \big|_{\varphi_\beta(U_\alpha \cap U_\beta \cap V)} .$$

As a restriction of a C^∞ map, this is also a C^∞ map. Hence $\mathscr{U}_V$ is truly an atlas for V.

Example 3.2.3 (General Linear Groups)

For any two positive integers m and n, let $\mathbb{R}^{m\times n}$ be the vector space of all $m \times n$ matrices. Since $\mathbb{R}^{m\times n}$ is isomorphic to $\mathbb{R}^{mn}$, we give it the topology of $\mathbb{R}^{mn}$. The definition of general linear group $\mathrm{GL}\,(n, \mathbb{R})$ is as follows:

$$\mathrm{GL}\,(n, \mathbb{R}) := \left\{ A \in \mathbb{R}^{n\times n} \mid \det A \neq 0 \right\} .$$

Consider the determinant function $\det : \mathbb{R}^{n\times n} \to \mathbb{R}$. It is a polynomial of the entries, hence continuous. In terms of this continuous function, the pre-image of $\mathbb{R} \setminus \{0\}$ is precisely $\mathrm{GL}\,(n, \mathbb{R})$.

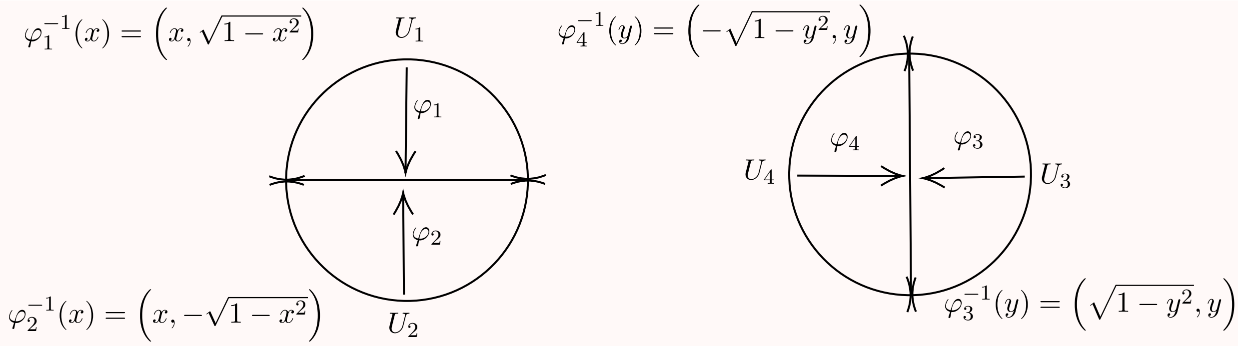
$$\mathrm{GL}\,(n, \mathbb{R}) = \det^{-1} (\mathbb{R} \setminus \{0\}) .$$

Since det is a continuous function from $\mathbb{R}^{n\times n} \cong \mathbb{R}^{n^2}$ to $\mathbb{R}$, and $\mathbb{R} \setminus \{0\}$ is open in $\mathbb{R}$, $\det^{-1} (\mathbb{R} \setminus \{0\})$ will be open in $\mathbb{R}^{n^2}$. Therefore, by Example 3.2.2, $\mathrm{GL}\,(n, \mathbb{R})$ is a manifold.

Example 3.2.4 (Unit circle in the (x, y)-plane)

In Example 3.1.3, we found a C^∞ atlas with 2 charts on the unit circle S^1 in the complex plane $\mathbb{C}$. We'll now view S^1 as the unit circle in $\mathbb{R}^2$ with defining equation $x^2 + y^2 = 1$. We can cover S^1 with 4 open sets: the upper and lower semicirles U_1 and U_2, the right and left semicircles U_3 and U_4. The homeomorphisms are:

$$\varphi_i : U_i \to (-1, 1) \;\; , \;\; \varphi_i\,(x, y) = \begin{cases} x & \text{if } i = 1, 2 \\ y & \text{if } i = 3, 4 \end{cases}$$



Let us check that on $U_1 \cap U_3$,

$$\left(\varphi_3 \circ \varphi_1^{-1}\right)\left(\varphi_1\left(x, y\right)\right) = \left(\varphi_3 \circ \varphi_1^{-1}\right)(x) = \varphi_3\left(x, \sqrt{1-x^2}\right) = \sqrt{1-x^2}\,.$$

Since $(1, 0) \notin U_1 \cap U_3$, we can conclude that $\varphi_3 \circ \varphi_1^{-1}$ is C^∞. Also, on $U_2 \cap U_4$,

$$\left(\varphi_2 \circ \varphi_4^{-1}\right)\left(\varphi_4\left(x, y\right)\right) = \left(\varphi_2 \circ \varphi_4^{-1}\right)(y) = \varphi_2\left(-\sqrt{1-y^2}, y\right) = -\sqrt{1-y^2}\,.$$

Since $(0, -1) \notin U_2 \cap U_4$, we can conclude that $\varphi_2 \circ \varphi_4^{-1}$ is C^∞. In a similar manner, one can check that $\varphi_i \circ \varphi_j^{-1}$ is C^∞ for every i, j. Therefore, $\{(U_i, \varphi_i) \mid 1 \le i \le 4\}$ is indeed a C^∞ atlas on S^1.

If M and N are manifolds, it's natural to think that $M \times N$ should also be a manifold. Now we shall demonstrate it. $M \times N$ with its product topology is Hausdorff and second countable (Proposition 1.6.3 and Corollary 1.6.2). To show that $M \times N$ is a manifold, it remains to exhibit an atlas on it. Recall that the product of two set maps $f : X \to X'$ and $g : Y \to Y'$ is

$$f \times g : X \times Y \to X' \times Y'\ ,\ \left(f \times g\right)(x, y) = \left(f\left(x\right), g\left(y\right)\right)\,.$$

Proposition 3.2.2 (Atlas for Product Manifold)

If $\{(U_\alpha, \varphi_\alpha)\}$ and $\{(V_i, \psi_i)\}$ are C^∞ atlases for the manifolds M and N of dimensions m and n, respectively, then the collection

$$\{(U_\alpha \times V_i, \varphi_\alpha \times \psi_i : U_\alpha \times V_i \to \mathbb{R}^m \times \mathbb{R}^n)\}$$

of charts is a C^∞ atlas on $M \times N$. Therefore, $M \times N$ is a C^∞ manifold of dimension $m + n$.

Proof. φ_α is a homeomorphism of U_α onto $\varphi_\alpha\left(U_\alpha\right) = \overline{U_\alpha} \subseteq \mathbb{R}^m$, and ψ_i is a homeomorphism of V_i onto $\psi_i\left(V_i\right) = \overline{V_i} \subseteq \mathbb{R}^n$. Now,

$$\left(\varphi_\alpha \times \psi_i\right)(a, b) = \left(\varphi_\alpha\left(a\right), \psi_i\left(b\right)\right) = \left(\left(\varphi_\alpha \circ \pi_1\right)(a, b), \left(\psi_i \circ \pi_2\right)(a, b)\right)\,,$$

where π_1 and π_2 are projection on first and second coordinate, respectively. Both $\varphi_\alpha \circ \pi_1$ and $\psi_i \circ \pi_2$ are composition of continuous maps, hence continuous. Therefore, by Theorem 1.6.4, $\varphi_\alpha \times \psi_i$ is continuous. One can show that

$$\left(\varphi_\alpha \times \psi_i\right)^{-1} = \varphi_\alpha^{-1} \times \psi_i^{-1}\,.$$

Using an analogous argument as above, $\varphi_\alpha^{-1} \times \psi_i^{-1}$ is continuous. Therefore, $\varphi_\alpha \times \psi_i : U_\alpha \times V_i \to \overline{U_\alpha} \times \overline{V_i} \subseteq \mathbb{R}^{m+n}$ is a homeomorphism. Furthermore,

$$\bigcup_{\alpha, i}\left(U_\alpha \times V_i\right) = \bigcup_\alpha\left(U_\alpha \times \left(\bigcup_i V_i\right)\right) = \bigcup_\alpha\left(U_\alpha \times N\right) = \left(\bigcup_\alpha U_\alpha\right) \times N = M \times N\,.$$

Now, we are only left to show that any two charts are compatible with each other. It suffices to show that $(\varphi_\alpha \times \psi_i) \circ (\varphi_\beta \times \psi_j)^{-1} : \overline{U_\beta} \times \overline{V_j} \to \overline{U_\alpha} \times \overline{V_i}$ is a C^∞ map.

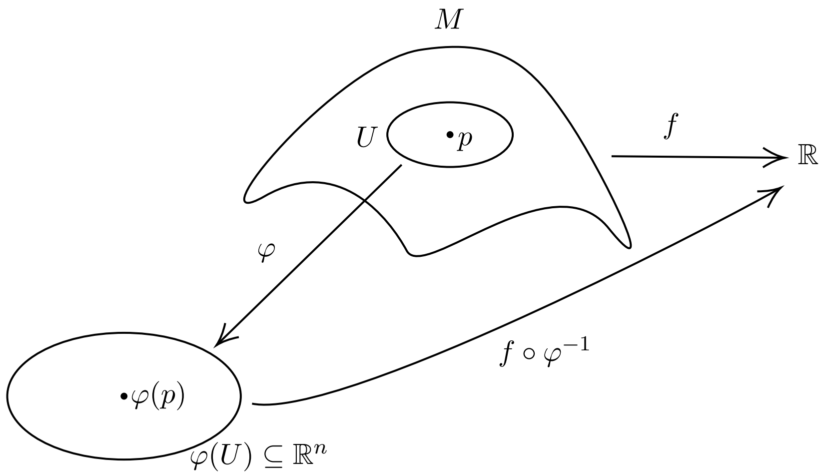
$$\begin{aligned}
(\varphi_\alpha \times \psi_i) \circ (\varphi_\beta \times \psi_j)^{-1} &= (\varphi_\alpha \times \psi_i) \circ \left(\varphi_\beta^{-1} \times \psi_j^{-1}\right) \\
\left((\varphi_\alpha \times \psi_i) \circ \left(\varphi_\beta^{-1} \times \psi_j^{-1}\right)\right)(x, y) &= (\varphi_\alpha \times \psi_i) \left(\varphi_\beta^{-1}(x), \psi_j^{-1}(y)\right) \\
&= \left(\left(\varphi_\alpha \circ \varphi_\beta^{-1}\right)(x), \left(\psi_i \circ \psi_j^{-1}\right)(y)\right) \\
\therefore (\varphi_\alpha \times \psi_i) \circ (\varphi_\beta \times \psi_j)^{-1} &= \left(\varphi_\alpha \circ \varphi_\beta^{-1}\right) \times \left(\psi_i \circ \psi_j^{-1}\right)
\end{aligned}$$

Both $\varphi_\alpha \circ \varphi_\beta^{-1}$ and $\psi_i \circ \psi_j^{-1}$ are C^∞ maps since $\{(U_\alpha, \varphi_\alpha)\}$ and $\{(V_i, \psi_i)\}$ are C^∞ atlases. Therefore, as a cartesian product of C^∞ maps, $(\varphi_\alpha \times \psi_i) \circ (\varphi_\beta \times \psi_j)^{-1}$ is also C^∞. This completes the proof. ■

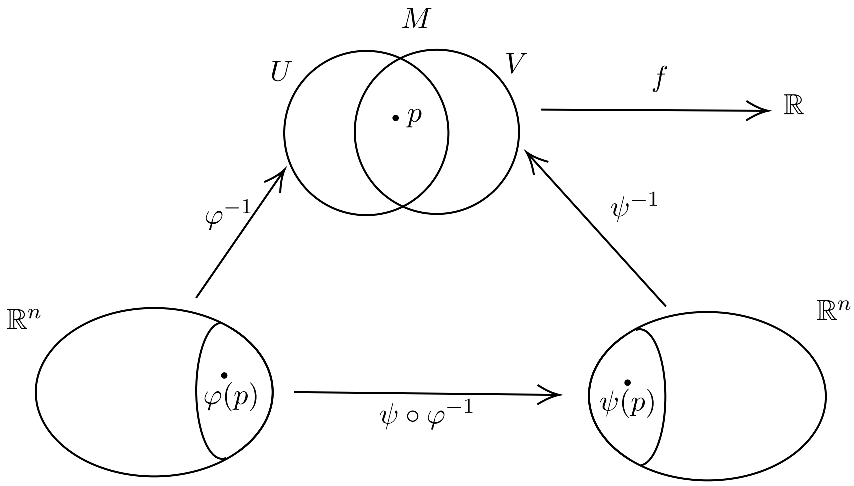
4 Smooth Maps on Manifold

§4.1 Smooth Functions on Manifold

Definition 4.1.1. Let M be a smooth manifold of dimension n. A function $f : M \to \mathbb{R}$ is said to be C^∞ or smooth at a point $p \in M$ if there is a chart (U, φ) about p in M such that $f \circ \varphi^{-1} : \varphi(U) \subseteq \mathbb{R}^n \to \mathbb{R}$ is C^∞ at $\varphi(p)$. The function f is said to be C^∞ on M if it is C^∞ at every point of M.



Remark 4.1.1. The definition of the smoothness of a function f at a given point on the manifold is independent of the chart (U, φ). Let us check this.



Suppose that $f \circ \varphi^{-1}$ is C^∞ at $\varphi(p)$ for a given chart (U, φ) about $p \in M$. Let (V, ψ) be any other chart about p. Then on $\psi(U \cap V)$,

$$f \circ \psi^{-1} = \left(f \circ \varphi^{-1}\right) \circ \left(\varphi \circ \psi^{-1}\right) .$$

$\varphi \circ \psi^{-1}$ is C^∞ by compatibility of charts. Therefore, as a composition of C^∞ maps, $f \circ \psi^{-1}$ is C^∞ at $\psi(p)$. This proves the independence of chart to determine the smoothness of a function at a given point.

Proposition 4.1.1

Let M be a manifold of dimension n, and $f : M \to \mathbb{R}$ a real-valued function on M. The following are equivalent:

(i) The function $f : M \to \mathbb{R}$ is C^∞.

(ii) The manifold M has an atlas such that for every chart (U, φ) in the atlas, $f \circ \varphi^{-1} : \varphi(U) \subseteq \mathbb{R}^n \to \mathbb{R}$ is C^∞.

(iii) For every chart (V, ψ) on M, the function $f \circ \psi^{-1} : \psi(V) \subseteq \mathbb{R}^n \to \mathbb{R}$ is C^∞.

Proof. **(ii)⇒(i):** Since (ii) holds, one can find for every $p \in M$, a coordinate neighborhood (U, φ) such that $f \circ \varphi^{-1}$ is C^∞ at $\varphi(p)$. Therefore, from the definition of C^∞ function on a manifold, $f : M \to \mathbb{R}$ is C^∞.

(i)⇒(iii): Let (V, ψ) be an arbitrary chart on M and $p \in V$. Since (i) holds, $f \circ \psi^{-1}$ is C^∞ at $\psi(p)$ (by the remark). Since p is an arbitrary point on V, $f \circ \psi^{-1}$ is C^∞ on $\psi(V)$.

(iii)⇒(ii): Obvious. ■

Definition 4.1.2 (Pullback). Let $F : N \to M$ be a map and h a function on M. The **pullback** of h by F, denoted by F^*h, is the composite function $h \circ F$.

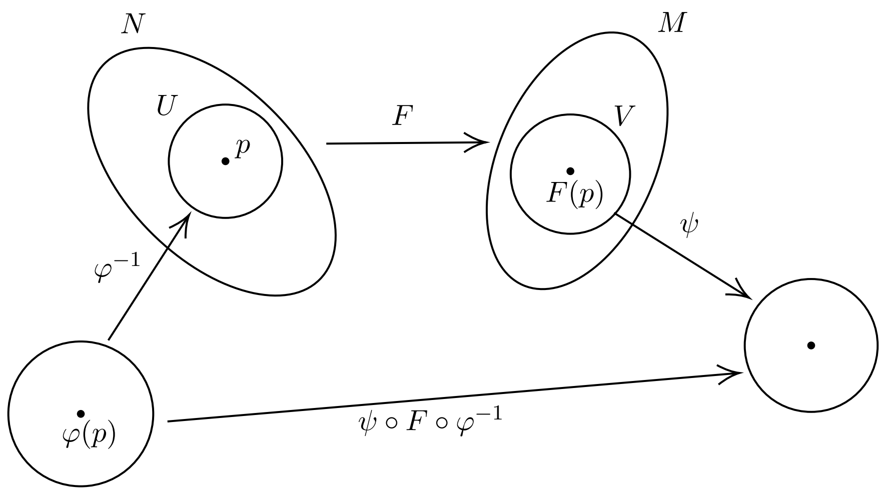
Using this terminology of pullback, a function f on M is C^∞ on a chart (U, φ) if and only if its pullback $\left(\varphi^{-1}\right)^* f$ by φ^{-1} is C^∞ on the subset $\varphi(U)$ of Euclidean space.

§4.2 Smooth Maps Between Manifolds

Definition 4.2.1. Let N and M be manifolds of dimension n and m, respectively. A continuous map $F : N \to M$ is C^∞ at a point $p \in N$ if there are charts (V, ψ) about $F(p) \in M$ and (U, φ) about $p \in N$ such that the composition

$$\psi \circ F \circ \varphi^{-1} : \varphi\left(F^{-1}(V) \cap U\right) \subseteq \mathbb{R}^n \to \mathbb{R}^m$$

is C^∞ at $\varphi(p)$. The continuous map $F : N \to M$ is said to be C^∞ if it is C^∞ at every point of N.



Remark 4.2.1. Note that in the definition of smooth map between manifolds, one must have a continuous map to start with. We require $F : N \to M$ to be continuous so that $F^{-1}(V)$ is open and $\varphi\left(F^{-1}(V) \cap U\right)$ becomes an open subset of $\mathbb{R}^n$.

Proposition 4.2.1

Suppose $F : N \to M$ is C^∞ at $p \in N$. If (U, φ) is any chart about $p \in N$ and (V, ψ) is any chart about $F(p) \in M$, then $\psi \circ F \circ \varphi^{-1}$ is C^∞ at $\varphi(p)$.

Proof. Since F is C^∞ at $p \in N$, there are charts $(U_\alpha, \varphi_\alpha)$ about $p \in N$ and (V_β, ψ_β) about $F(p) \in M$ such that $\psi_\beta \circ F \circ \varphi_\alpha^{-1}$ is C^∞ at $\varphi_\alpha(p)$. By the C^∞ compatibility of charts in a differentiable structure, both $\varphi_\alpha \circ \varphi^{-1}$ and $\psi \circ \psi_\beta^{-1}$ are C^∞ on open subsets of Euclidean spaces. Hence, the composite

$$\psi \circ F \circ \varphi^{-1} = \left(\psi \circ \psi_\beta^{-1}\right) \circ \left(\psi_\beta \circ F \circ \varphi_\alpha^{-1}\right) \circ \left(\varphi_\alpha \circ \varphi^{-1}\right)$$

is C^∞ at $\varphi(p)$. ■

Proposition 4.2.2 (Smoothness of a map in terms of charts)

Let N and M be smooth manifolds, and $F : N \to M$ a continuous map. The following are equivalent:

(i) The map $F : N \to M$ is C^∞.

(ii) There are atlases $\mathscr{U}$ for N and $\mathscr{V}$ for M such that for every chart (U, φ) in $\mathscr{U}$ and (V, ψ) in $\mathscr{V}$, the map

$$\psi \circ F \circ \varphi^{-1} : \varphi\left(U \cap F^{-1}(V)\right) \to \mathbb{R}^m$$

is C^∞.

(iii) For every chart (U, φ) on N and (V, ψ) on M, the map

$$\psi \circ F \circ \varphi^{-1} : \varphi\left(U \cap F^{-1}(V)\right) \to \mathbb{R}^m$$

is C^∞.

Proof. **(ii)⇒(i):** Let $p \in N$. Suppose (U, φ) is a chart about p in $\mathscr{U}$ and (V, ψ) is a chart about $F(p)$ in $\mathscr{V}$. Now, (ii) implies that $\psi \circ F \circ \varphi^{-1}$ is C^∞ at $\varphi(p)$. By the definition of a C^∞ map, $F : N \to M$ is C^∞ at p. Since p was an arbitrary point of N, the map $F : N \to M$ is C^∞.

(i)⇒(iii): Suppose (U, φ) and (V, ψ) are charts on N and M, respectively, such that $U \cap F^{-1}(V) \neq \varnothing$. Let $p \in U \cap F^{-1}(V)$ so that $p \in U$ and $F(p) \in V$. Then (U, φ) is a chart about p and (V, ψ) is a chart about $F(p)$. By Proposition 4.2.1, $\psi \circ F \circ \varphi^{-1}$ is C^∞ at $\varphi(p)$. Since $\varphi(p)$ was an arbitrary point of $\varphi\left(U \cap F^{-1}(V)\right)$, the map $\psi \circ F \circ \varphi^{-1} : \varphi\left(U \cap F^{-1}(V)\right) \to \mathbb{R}^m$ is C^∞.

(iii)⇒(ii): Take $\mathscr{U}$ and $\mathscr{V}$ to be the maximal atlases of N and M, respectively. ■

Smoothness of a Map Depends on the Choice of Differentiable Structure

Consider the map $\varphi : \mathbb{R} \to \mathbb{R}$ that takes x to x^3. φ is continuous, so is its inverse $x \mapsto x^{1/3}$. Therefore, φ is a homeomorphism. So $(\mathbb{R}, \varphi)$ is a chart on $\mathbb{R}$. The collection $\mathscr{V} = \{(\mathbb{R}, \varphi)\}$ is an atlas on $\mathbb{R}$ with a single chart $(\mathbb{R}, \varphi)$. This $\mathscr{V}$ is contained in another maximal atlas $\mathscr{N}$. Then $M_1 = (\mathbb{R}, \mathscr{N})$ is also a smooth manifold.

Furthermore, $\mathscr{U} = \{(\mathbb{R}, \mathbb{1}_{\mathbb{R}})\}$ is an atlas on $\mathbb{R}$ with a single chart. This atlas is contained in a maximal atlas, say $\mathscr{M}$. Then $M_2 = (\mathbb{R}, \mathscr{M})$ is a smooth manifold with the usual differentiable structure. Although the underlying topological manifolds of M_1 and M_2 are the same, they are, nevertheless, different manifolds because the differentiable structures are not the same. That's why we denote them with different symbols.

Now we want to check whether $\varphi : \mathbb{R} \to \mathbb{R}$ is smooth. In order to do that, we need to choose which differentiable structure we put on the domain and range spaces. If we equip both the domain space and range space with the usual differentiable structure $\mathscr{M}$, then it's easy to check that $\varphi : M_2 \to M_2$

is smooth. However, in this case, φ^{-1} is not smooth. Because, if we take the charts $(\mathbb{R}, \mathbb{1}_{\mathbb{R}})$ from both manifolds, then

$$\mathbb{1}_{\mathbb{R}} \circ \varphi^{-1} \circ \mathbb{1}_{\mathbb{R}}^{-1} : \mathbb{R} \to \mathbb{R}$$

is just the map $\varphi^{-1} : \mathbb{R} \to \mathbb{R}$, $\varphi^{-1}(x) = x^{1/3}$, which, as a map between two Euclidean spaces, is not even C^1, let alone being C^∞. Therefore, $\varphi^{-1} : M_2 \to M_2$ is not smooth.

Now we consider $\varphi : M_1 \to M_2$. Recall that $M_1 = (\mathbb{R}, \mathscr{N})$ and $M_2 = (\mathbb{R}, \mathscr{M})$. Then φ is indeed smooth. Because if we take the atlas $\mathscr{V} = \{(\mathbb{R}, \varphi)\}$ from $\mathscr{N}$ and the atlas $\mathscr{U} = \{(\mathbb{R}, \mathbb{1}_{\mathbb{R}})\}$ from $\mathscr{M}$, then

$$\mathbb{1}_{\mathbb{R}} \circ \varphi \circ \varphi^{-1} = \mathbb{1}_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$$

is indeed a smooth map between Euclidean spaces. Therefore, Proposition 4.2.2 (ii)⇒(i) guarantees that φ is smooth. Furthermore, $\varphi^{-1} : M_2 \to M_1$ is also smooth. Because if we take the same atlases as above,

$$\varphi \circ \varphi^{-1} \circ \mathbb{1}_{\mathbb{R}}^{-1} = \mathbb{1}_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$$

is indeed a smooth map between Euclidean spaces. Therefore, Proposition 4.2.2 (ii)⇒(i) guarantees that φ^{-1} is smooth.

This might seem disturbing at first. Because the description of φ^{-1} does not change when we impose different differentiable structures on the domain and range spaces. $\varphi^{-1}(x) = x^{1/3}$ stays the same function. We have a hardwired notion that we cannot differentiate it at $x = 0$, that's why it is not C^1, let alone being C^∞. However, when we are talking about a map between two manifolds, the notion of smoothness depends solely on the differentiable structures on the domain and range spaces.

We have seen that depending on the choice of differentiable structures, the same map can be both smooth and non-smooth. Drawing the analogy with topology, in a calculus class, we say that the identity map $\mathbb{1}_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$ is continuous. But this is only true when the domain and range sets are equipped with the same topology. If we equip the range set with the discrete topology and the domain set with the usual topology, then $\mathbb{1}_{\mathbb{R}}$ no longer stays continuous. Thus, depending on the topologies on the domain and range sets, the same map can be both continuous and discontinuous. In a similar spirit, the same map can be smooth and non-smooth depending on the choice of differentiable structures.

Proposition 4.2.3 (Composition of C^∞ maps)

If $F : N \to M$ and $G : M \to P$ are C^∞ maps of manifolds, then the composite $G \circ F : N \to P$ is C^∞.

Proof. Let $(U, \varphi), (V, \psi)$, and (W, σ) be charts on N, M, and P, respectively. Then

$$\sigma \circ (G \circ F) \circ \varphi^{-1} = \left(\sigma \circ G \circ \psi^{-1}\right) \circ \left(\psi \circ F \circ \varphi^{-1}\right).$$

Since F and G are C^∞, by Proposition 4.2.2 (i)⇒(iii), $\sigma \circ G \circ \psi^{-1}$ and $\psi \circ F \circ \varphi^{-1}$ are C^∞ maps on their respective domains. As a composite of C^∞ maps of open subsets of Euclidean spaces, $\sigma \circ (G \circ F) \circ \varphi^{-1}$ is C^∞. In particular,

$$\sigma \circ (G \circ F) \circ \varphi^{-1} : \varphi\left(U \cap F^{-1}(V)\right) \cap \psi\left(V \cap G^{-1}(W)\right) \to \mathbb{R}^p$$

is C^∞ provided N, M and P are of dimension n, m and p, respectively. By (iii)⇒(i) of Proposition 4.2.2, $G \circ F$ is C^∞. ■

Definition 4.2.2 (Diffeomorphism). A **diffeomorphism** of manifolds is a bijective C^∞ map $F : N \to M$ whose inverse F^{-1} is also C^∞.

Proposition 4.2.4

If (U, φ) is a chart on a manifold M of dimension n, then the coordinate map $\varphi : U \to \varphi(U) \subseteq \mathbb{R}^n$ is a diffeomorphism.

Proof. By definition, φ is a homeomorphism. So it suffices to check that both φ and φ^{-1} are smooth. In order to check the smoothness of $\varphi : U \to \varphi(U)$, we shall use the atlas $\{(U, \varphi)\}$ on the manifold U, and the atlas $\{(\varphi(U), \mathbb{1}_{\varphi(U)})\}$ on the manifold $\varphi(U)$. Observe that,

$$\mathbb{1}_{\varphi(U)} \circ \varphi \circ \varphi^{-1} : \varphi(U) \to \varphi(U)$$

is just the identity map on $\varphi(U)$, hence C^∞. Therefore, by (ii)⇒(i) of Proposition 4.2.2, φ is C^∞.

We shall use the same atlas as above to show the smoothness of $\varphi^{-1} : \varphi(U) \to U$. Now,

$$\varphi \circ \varphi^{-1} \circ \mathbb{1}_{\varphi(U)}^{-1} = \mathbb{1}_{\varphi(U)} : \varphi(U) \to \varphi(U) .$$

Identity map is C^∞, hence by (ii)⇒(i) of Proposition 4.2.2, φ^{-1} is C^∞. ■

Proposition 4.2.5

Let U be an open subset of a manifold M of dimension n. If $F : U \to F(U) \subseteq \mathbb{R}^n$ is a diffeomorphism onto an open subset of $\mathbb{R}^n$, then (U, F) is a chart in the maximal atlas of M.

Proof. For any chart $(U_\alpha, \varphi_\alpha)$ in the maximal atlas of M, both φ_α and φ_α^{-1} are C^∞ by Proposition 4.2.4. As compositions of C^∞ maps, $F \circ \varphi_\alpha^{-1}$ and $\varphi_\alpha \circ F^{-1}$ are C^∞ maps. Therefore, (U, F) is compatible with every chart of the maximal atlas. Hence, it is compatible with the maximal atlas. Therefore, by the maximality of the atlas, the chart (U, F) is in the maximal atlas. ■

Proposition 4.2.6 (Smoothness of a vector-valued function)

Let N be a manifold and $F : N \to \mathbb{R}^m$ a continuous map. The following are equivalent:

(i) The map $F : N \to \mathbb{R}^m$ is C^∞.

(ii) The manifold N has an atlas such that for every chart (U, φ) in the atlas, the map $F \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^m$ is C^∞.

(iii) For every chart (U, φ) on N, the map $F \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^m$ is C^∞.

Proof. **(ii)⇒(i):** In Proposition 4.2.2(ii), take the atlas $\mathscr{V}$ of $\mathbb{R}^m$ to be $\{(\mathbb{R}^m, \mathbb{1}_{\mathbb{R}^m})\}$. Now, $\varphi(U) = \varphi(U \cap N) = \varphi\left(U \cap F^{-1}(\mathbb{R}^m)\right)$. Therefore,

$$\mathbb{1}_{\mathbb{R}^m} \circ F \circ \varphi^{-1} : \varphi\left(U \cap F^{-1}(\mathbb{R}^m)\right) \to \mathbb{R}^m$$

is the same as $F \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^m$, which is C^∞. Hence by (ii)⇒(i) of Proposition 4.2.2, F is C^∞.

(i)⇒(iii): In Proposition 4.2.2(iii), let (V, ψ) be the chart $(\mathbb{R}^m, \mathbb{1}_{\mathbb{R}^m})$ on $\mathbb{R}^m$. Hence,

$$\mathbb{1}_{\mathbb{R}^m} \circ F \circ \varphi^{-1} : \varphi\left(U \cap F^{-1}(\mathbb{R}^m)\right) \to \mathbb{R}^m$$

is C^∞, which is the same as $F \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^m$.

(iii)⇒(ii): Choose the maximal atlas of N. ■

Proposition 4.2.7 (Smoothness in terms of components)

Let N be a manifold. A vector-valued function $F : N \to \mathbb{R}^m$ is C^∞ if and only if its component functions $F^1, \ldots, F^m : N \to \mathbb{R}$ are all C^∞.

Proof. $F : N \to \mathbb{R}^m$ is C^∞ if and only if for every chart (U, φ) on N, $F \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^m$ is C^∞ (Proposition 4.2.6). Now, $F \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^m$ is C^∞ if and only if $r^i \circ \left(F \circ \varphi^{-1}\right)$ is C^∞ for every $1 \leq i \leq m$.

$$r^i \circ \left(F \circ \varphi^{-1}\right) = F^i \circ \varphi^{-1} : \varphi(U) \to \mathbb{R} .$$

Therefore, F being smooth is equivalent to each $F^i \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}$ being smooth for every chart (U, φ). By Proposition 4.1.1, this is equivalent to each $F^i : N \to \mathbb{R}$ being smooth. Therefore, $F : N \to \mathbb{R}^m$ is C^∞ if and only if each $F^i : N \to \mathbb{R}$ is C^∞. ■

Proposition 4.2.8 (Smoothness of a map in terms of vector-valued functions)
Let $F : N \to M$ be a continuous map between two manifolds of dimensions n and m respectively. The following are equivalent:

(i) The map $F : N \to M$ is C^∞.

(ii) The manifold M has an atlas such that for every chart $(V, \psi) = \left(V, y^1, \ldots, y^m\right)$ in the atlas, the vector-valued function $\psi \circ F : F^{-1}(V) \to \mathbb{R}^m$ is C^∞.

(iii) For every chart $(V, \psi) = \left(V, y^1, \ldots, y^m\right)$ on M, the vector-valued function $\psi \circ F : F^{-1}(V) \to \mathbb{R}^m$ is C^∞.

Proof. **(ii)⇒(i):** Let $\mathscr{V}$ be the atlas for M in (ii), and let $\mathscr{U} = \{(U, \varphi)\}$ be an arbitrary atlas for N. For each chart (V, ψ) in the atlas $\mathscr{V}$, the collection $\left\{\left(U \cap F^{-1}(V), \varphi|_{U \cap F^{-1}(V)}\right)\right\}$ is an atlas for $F^{-1}(V)$. Since $\psi \circ F : F^{-1}(V) \to \mathbb{R}^m$ is C^∞, by (i)⇒(ii) of Proposition 4.2.6,

$$\psi \circ F \circ \varphi^{-1} : \varphi\left(U \cap F^{-1}(V)\right) \to \mathbb{R}^m$$

is C^∞. It then follows from (ii)⇒(i) of Proposition 4.2.2 that $F : N \to M$ is C^∞.

(i)⇒(iii): ψ is C^∞ by Proposition 4.2.4. As a composition of smooth maps, $\psi \circ F$ is C^∞.

(iii)⇒(ii): Trivial. Just take the maximal atlas of M and N. ■

Proposition 4.2.8 and Proposition 4.2.7 altogether gives rise to the following proposition.

Proposition 4.2.9 (Smoothness of a map in terms of components)
Let $F : N \to M$ be a continuous map between two manifolds of dimensions n and m respectively. The following are equivalent:

(i) The map $F : N \to M$ is C^∞.

(ii) The manifold M has an atlas such that for every chart $(V, \psi) = \left(V, y^1, \ldots, y^m\right)$ in the atlas, the components $y^i \circ F : F^{-1}(V) \to \mathbb{R}$ of F relative to the chart are all C^∞.

(iii) For every chart $(V, \psi) = \left(V, y^1, \ldots, y^m\right)$ on M, the components $y^i \circ F : F^{-1}(V) \to \mathbb{R}$ of F relative to the chart are all C^∞.

Example 4.2.1

Let M and N be manifolds and $\pi : M \times N \to M$, $\pi(p, q) = p$ be the projection onto the first factor. We want to show that π is a C^∞ map.

Let (p, q) be an arbitrary point of $M \times N$. Suppose $(U, \varphi) = \left(U, x^1, \ldots, x^m\right)$ and $(V, \psi) = \left(V, y^1, \ldots, y^n\right)$ are coordinate neighborhoods of p and q in M and N, respectively. By Proposition 3.2.2,

$$(U \times V, \varphi \times \psi) = \left(U \times V, x^1, \ldots, x^m, y^1, \ldots, y^n\right)$$

is a coordinate neighborhood of (p, q). Therefore, given $\left(a^1, \ldots, a^m, b^1, \ldots, b^n\right) \in (\varphi \times \psi)(U \times V) \subseteq \mathbb{R}^{m+n}$,

$$\begin{aligned}\left(\varphi \circ \pi \circ (\varphi \times \psi)^{-1}\right)\left(a^1, \ldots, a^m, b^1, \ldots, b^n\right) &= (\varphi \circ \pi)\left(\varphi^{-1}\left(a^1, \ldots, a^m\right), \psi^{-1}\left(b^1, \ldots, b^n\right)\right) \\ &= \varphi\left(\varphi^{-1}\left(a^1, \ldots, a^m\right)\right) = \left(a^1, \ldots, a^m\right)\end{aligned}$$

Therefore, $\varphi \circ \pi \circ (\varphi \times \psi)^{-1} : (\varphi \times \psi)(U \times V) \subseteq \mathbb{R}^{m+n} \to \mathbb{R}^m$ is just the projection onto the first m coordinates, which is a C^∞ map. Hence, $\pi : M \times N \to M$ is C^∞ at (p, q). Since (p, q) was chosen arbitrarily from $M \times N$, $\pi : M \times N \to M$ is C^∞ on $M \times N$.

Lemma 4.2.10

Let M_1, M_2 and N be manifolds of dimensions m_1, m_2 and n, respectively. Prove that a map $(f_1, f_2) : N \to M_1 \times M_2$ is C^∞ if and only if $f_i : N \to M_i$, $i = 1, 2$ are both C^∞.

Proof. Let $(f_1, f_2) = f$, and $\pi_i : M_1 \to M_2 \to M_i$ be projection maps for $i = 1, 2$. Both π_i are smooth, as proved in Example 4.2.1. If f is smooth, then $f_i = \pi_i \circ f : N \to M_i$ is composition of smooth maps, hence smooth.

Conversely, suppose both $f_i : N \to M_i$ are smooth. Then both f_i are continuous, hence so is f (Theorem 1.6.4). Let $p \in N$, and take coordinate neighborhoods (U, φ), (V_1, ψ_1), (V_2, ψ_2) of p, $f_1(p)$, $f_2(p)$, respectively. We can choose U sufficiently small so that $f(U) \subseteq V_1 \times V_2$. Since f_i is smooth,

$$\psi_i \circ f_i \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^{m_i}$$

is smooth at $\varphi(p)$. Now, $(V_1 \times V_2, \psi_1 \times \psi_2)$ is a coordinate neighborhood of $f(p) = (f_1(p), f_2(p))$. Given $a \in \varphi(U) \subseteq \mathbb{R}^n$,

$$\begin{aligned} \left((\psi_1 \times \psi_2) \circ f \circ \varphi^{-1}\right)(a) &= (\psi_1 \times \psi_2)\left(\left(f_1 \circ \varphi^{-1}\right)(a), \left(f_2 \circ \varphi^{-1}\right)(a)\right) \\ &= \left(\left(\psi_1 \circ f_1 \circ \varphi^{-1}\right)(a), \left(\psi_2 \circ f_2 \circ \varphi^{-1}\right)(a)\right) \\ \therefore (\psi_1 \times \psi_2) \circ f \circ \varphi^{-1} &= \left(\psi_1 \circ f_1 \circ \varphi^{-1}, \psi_2 \circ f_2 \circ \varphi^{-1}\right) \end{aligned}$$

Both $\psi_i \circ f_i \circ \varphi^{-1}$ are smooth at $\varphi(p)$. Therefore, $(\psi_1 \times \psi_2) \circ f \circ \varphi^{-1}$ is also smooth at $\varphi(p)$. In other words, f is smooth at p. Since p was chosen arbitrarily from N, f is smooth on N. ∎

§4.3 Partial Derivatives

On a manifold M of dimension n, let (U, φ) be a chart and $f : M \to \mathbb{R}$ a C^∞ function. As a function into $\mathbb{R}^n$, φ has n components: $x^1, x^2, \ldots, x^n$. Let $r^1 . r^2, \ldots, r^n$ be standard coordinates on $\mathbb{R}^n$. That is, if $\vec{v} \equiv \left(v^1, v^2, \ldots, v^n\right) \in \mathbb{R}^n$, then $r^i(\vec{v}) = v^i$ for $1 \le i \le n$.

Now, $x^i = r^i \circ \varphi$. For $p \in U$, one defines the partial derivative $\frac{\partial f}{\partial x^i}$ of f with respect to x^i at p to be

$$\left.\frac{\partial}{\partial x^i}\right|_p f := \frac{\partial f}{\partial x^i}(p) := \frac{\partial\left(f \circ \varphi^{-1}\right)}{\partial r^i}(\varphi(p)) = \left.\frac{\partial}{\partial r^i}\right|_{\varphi(p)} \left(f \circ \varphi^{-1}\right).$$

Since $p = \varphi^{-1}(\varphi(p))$, the equation can be rewritten as

$$\frac{\partial f}{\partial x^i}\left(\varphi^{-1}(\varphi(p))\right) = \frac{\partial\left(f \circ \varphi^{-1}\right)}{\partial r^i}(\varphi(p)) \implies \left(\frac{\partial f}{\partial x^i} \circ \varphi^{-1}\right)(\varphi(p)) = \frac{\partial\left(f \circ \varphi^{-1}\right)}{\partial r^i}(\varphi(p)).$$

Thus, as functions on $\varphi(U)$,

$$\frac{\partial f}{\partial x^i} \circ \varphi^{-1} = \frac{\partial\left(f \circ \varphi^{-1}\right)}{\partial r^i}.$$

The partial derivative $\frac{\partial f}{\partial x^i}$ is C^∞ on U because its pullback $\frac{\partial f}{\partial x^i} \circ \varphi^{-1}$ is C^∞ on $\varphi(U)$.

Proposition 4.3.1

Suppose $\left(U, x^1, \ldots, x^n\right)$ is a chart on a manifold. Then $\dfrac{\partial x^i}{\partial x^j} = \delta^i_j$.

Proof. At a point $p \in U$, using $x^i = r^i \circ \varphi$,

$$\frac{\partial x^i}{\partial x^j}(p) = \frac{\partial\left(x^i \circ \varphi^{-1}\right)}{\partial r^j}(\varphi(p)) = \frac{\partial\left(r^i \circ \varphi \circ \varphi^{-1}\right)}{\partial r^j}(\varphi(p)) = \frac{\partial r^i}{\partial r^j}(\varphi(p)) = \delta^i_j.$$

∎

Definition 4.3.1. Let $F : N \to M$ be a smooth map, and let $(U, \varphi) = \left(U, x^1, \ldots, x^n\right)$ and $(V, \psi) = \left(V, y^1, \ldots, y^m\right)$ be charts on N and M respectively such that $F(U) \subset V$. Denote by

$$F^i := y^i \circ F = r^i \circ \psi \circ F : U \to \mathbb{R}$$

the i-th component of F in the chart (V, ψ). Then the $m \times n$ matrix $\left[\frac{\partial F^i}{\partial x^j}\right]$ is called the **Jacobian matrix** of F relative to the charts (U, φ) and (V, ψ). In case N and M have the same dimension, the determinant of the Jacobian matrix is called the **Jacobian determinant** of F relative to the two charts. The Jacobian determinant is also written as

$$\det\left[\frac{\partial F^i}{\partial x^j}\right] = \frac{\partial\left(F^1, \ldots, F^n\right)}{\partial\left(x^1, \ldots, x^n\right)}.$$

Example 4.3.1 (Jacobian matrix of a transition map)

Let $(U, \varphi) = \left(U, x^1, \ldots, x^n\right)$ and $(V, \psi) = \left(V, y^1, \ldots, y^n\right)$ be overlapping charts on a manifold M. The transition map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is a diffeomorphism of open subsets of $\mathbb{R}^n$. Then its Jacobian matrix $J\left(\psi \circ \varphi^{-1}\right)$ at $\varphi(p)$ is the matrix $\left[\frac{\partial y^i}{\partial x^j}\right]$ of partial derivatives at p.

Since $\psi \circ \varphi^{-1}$ is a map between two open subsets of Euclidean spaces, $J\left(\psi \circ \varphi^{-1}\right) = \left[\frac{\partial\left(\psi \circ \varphi^{-1}\right)^i}{\partial r^j}\right]$.

$$\begin{aligned}
\frac{\partial\left(\psi \circ \varphi^{-1}\right)^i}{\partial r^j}(\varphi(p)) &= \frac{\partial\left(r^i \circ \psi \circ \varphi^{-1}\right)}{\partial r^j}(\varphi(p)) \\
&= \frac{\partial\left(y^i \circ \varphi^{-1}\right)}{\partial r^j}(\varphi(p)) \\
&= \frac{\partial y^i}{\partial x^j}(p)
\end{aligned}$$

Definition 4.3.2. A C^∞ map $F : N \to M$ is **locally invertible** at $p \in N$ if p has a neighborhood U on which $F\big|_U : U \to F(U)$ is a diffeomorphism.

Theorem 4.3.2 (Inverse Function Theorem for Manifolds)

Let $F : N \to M$ be a C^∞ map between two manifolds of the same dimension, and $p \in N$. Suppose for some charts $(U, \varphi) = \left(U, x^1, \ldots, x^n\right)$ about $p \in N$ and $(V, \psi) = \left(V, y^1, \ldots, y^n\right)$ about $F(p) \in M$, $F(U) \subseteq V$. Set $F^i = y^i \circ F$. Then F is locally invertible at p if and only if its Jacobian determinant $\det\left[\frac{\partial F^i}{\partial x^j}(p)\right]$ is nonzero.

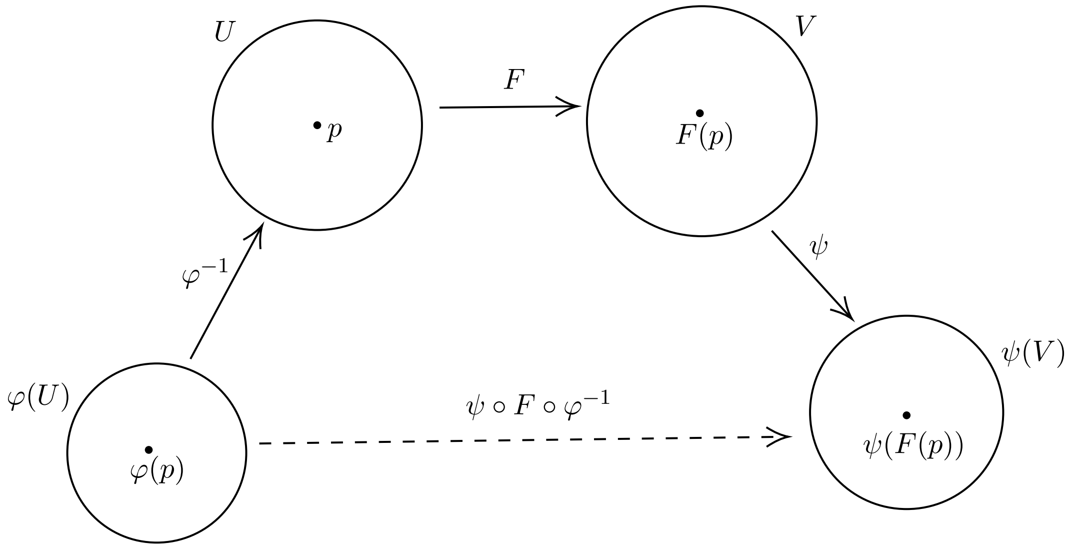
Proof. Since $F^i = y^i \circ F = r^i \circ \psi \circ F$, the Jacobian matrix of F relative to the charts (U, φ) and (V, ψ) is

$$\left[\frac{\partial F^i}{\partial x^j}(p)\right] = \left[\frac{\partial\left(r^i \circ \psi \circ F\right)}{\partial x^j}(p)\right] = \left[\frac{\partial\left(r^i \circ \psi \circ F \circ \varphi^{-1}\right)}{\partial r^j}(\varphi(p))\right] =$$

which is the Jacobian matrix of the map

$$\psi \circ F \circ \varphi^{-1} : \varphi(U) \subseteq \mathbb{R}^n \to \psi(V) \subseteq \mathbb{R}^n$$

between two open subsets of $\mathbb{R}^n$.



By Inverse Function Theorem for $\mathbb{R}^n$, $\psi \circ F \circ \varphi^{-1}$ is locally invertible at $\varphi(p)$ if and only if

$$\det\left[\frac{\partial F^i}{\partial x^j}(p)\right] = \det\left[\frac{\partial\left(\psi \circ F \circ \varphi^{-1}\right)^i}{\partial r^j}(\varphi(p))\right] \neq 0\,.$$

By Proposition 4.2.4, φ and ψ are diffeomorphisms. Therefore, local invertibility of $\psi \circ F \circ \varphi^{-1}$ at $\varphi(p)$ is equivalent to local invertibility of F at p. ■

Corollary 4.3.3

Let N be a manifold of dimension n. A set of n smooth functions $F^1, F^2, \ldots, F^n$ defined on a coordinate neighborhood $\left(U, x^1, x^2, \ldots, x^n\right)$ of a point $p \in N$ forms a coordinate system about p if and only if the Jacobian determinant $\det\left[\frac{\partial F^i}{\partial x^j}(p)\right]$ is nonzero.

Proof. **(⇒):** Let $F = \left(F^1, F^2, \ldots, F^n\right) : U \to \mathbb{R}^n$. If there exists a coordinate neighborhood $\left(W, F^1, F^2, \ldots, F^n\right)$ [1] about p in the maximal atlas of N, then F is a coordinate map, and hence $F : W \to F(W) \subseteq \mathbb{R}^n$ is a diffeomorphism by Proposition 4.2.4. In other words, F is locally invertible at p. Therefore, by Inverse Function Theorem for Manifolds, $\det\left[\frac{\partial F^i}{\partial x^j}(p)\right] \neq 0$.

(⇐): Since $\det\left[\frac{\partial F^i}{\partial x^j}(p)\right]$ is nonzero, $F : U \to \mathbb{R}^n$ is locally invertible at p (Inverse Function Theorem for Manifolds). In other words, there is a neighborhood W of $p \in N$ such that $F : W \to F(W) \subseteq \mathbb{R}^n$ is a diffeomorphism. Then by Proposition 4.2.5, there is a coordinate neighborhood $\left(W, F^1, F^2, \ldots, F^n\right)$ in the maximal atlas of N. ■

[1] Technically, it should be $F^i\big|_W$ instead of just F^i

5 Some Interesting Manifolds

§5.1 Real Projective Space

Define an equivalence relation $\sim$ on $\mathbb{R}^{n+1} \setminus \{0\}$ by

$$x \sim y \iff y = tx \text{ for some nonzero real number } t\,,$$

where $x, y \in \mathbb{R}^{n+1} \setminus \{0\}$. The **real projective space** is the quotient space of $\mathbb{R}^{n+1} \setminus \{0\}$ by this equivalence relation and is denoted by $\mathbb{R}P^n$. We denote the equivalence class of a point $\left(a^0, a^1, \ldots, a^n\right) \in \mathbb{R}^{n+1} \setminus \{0\}$ by $\left[a^0, a^1, \ldots, a^n\right]$ and let $\pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}P^n$ be the underlying projection map. We call $\left[a^0, a^1, \ldots, a^n\right]$ homogenous coordinates on $\mathbb{R}P^n$.

Geometrically, two nonzero points of $\mathbb{R}^{n+1}$ are equivalent if and only if they lie on the same line through the origin. So $\mathbb{R}P^n$ can be though of as the set of all lines through the origin in $\mathbb{R}^{n+1}$. A line through the origin in $\mathbb{R}^{n+1}$ is just a point in $\mathbb{R}P^n$.

Each line through the origin in $\mathbb{R}^{n+1}$ meets the unit sphere S^n in a pair of antipodal points. Conversely, a pair of antipodal points on S^n determines a unique line in $\mathbb{R}^{n+1}$.

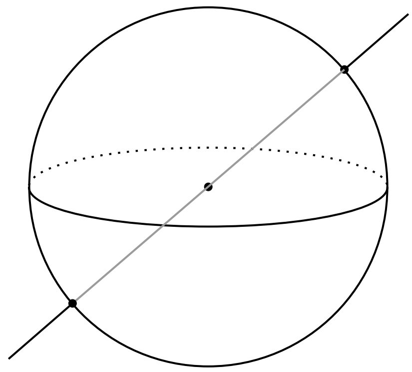


Figure 5.1: A line through 0 in $\mathbb{R}^3$ corresponds to a pair of antipodal points on S^2.

This suggests that we can define an equivalence relation $\sim$ on S^n by identifying the antipodal points:

$$x \sim y \iff x = \pm y\,, \quad x, y \in S^n\,.$$

We then have a bijection $\mathbb{R}P^n \leftrightarrow S^n/\sim$. We shall now see that this bijection is a homeomorphism.

Lemma 5.1.1

$\mathbb{R}P^n$ is homeomorphic to $S^n/\sim$.

Proof. Consider $f : \mathbb{R}^{n+1} \setminus \{0\} \to S^n$ defined by $f(x) = \frac{x}{\|x\|}$. Then f is continuous. Note that, for a nonzero real t,

$$f(tx) = \frac{tx}{\|tx\|} = \frac{t}{|t|}\frac{x}{\|x\|} = \begin{cases} f(x) & \text{if } t > 0 \\ -f(x) & \text{if } t < 0 \end{cases}$$

Now we define $\overline{f} : \mathbb{R}P^n \to S^n/\sim$ by $\overline{f}([x]) = [f(x)]$. This map is well-defined, since

$$\overline{f}([tx]) = [f(tx)] = [\pm f(x)] = [f(x)] = \overline{f}([x])\,.$$

Let $\pi_1 : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}P^n$ and $\pi_2 : S^n \to S^n/\sim$ be the respective projection maps. Now we have a commutative diagram.

$$\begin{tikzcd}
\mathbb{R}^{n+1} \setminus \{0\} \arrow[r, "f"] \arrow[d, "\pi_1"'] \arrow[dr, "\pi_2 \circ f"] & S^n \arrow[d, "\pi_2"] \\
\mathbb{R}P^n \arrow[r, "\overline{f}"'] & S^n/\sim
\end{tikzcd}$$

Now, $\pi_2 \circ f$ is the composition of two continuous maps, hence continuous. Therefore, by Proposition 1.7.1, $\overline{f}$ is continuous.

Now, let $g : S^n \to \mathbb{R}^{n+1} \setminus \{0\}$ be the inclusion map given by $g(x) = x$. We know that g is continuous. It induces another map $\overline{g} : S^n/\sim \to \mathbb{R}P^n$ defined by $\overline{g}([x]) = [x]$. $\overline{g}$ is well-defined, because

$$\overline{g}([-x]) = [-x] = [x] = \overline{g}([x]) \ .$$

As before, we have another commutative diagram.

$$\begin{tikzcd}
S^n \arrow[r, "g"] \arrow[d, "\pi_2"'] \arrow[dr, "\pi_1 \circ g"] & \mathbb{R}^{n+1} \setminus \{0\} \arrow[d, "\pi_1"] \\
S^n/\sim \arrow[r, "\overline{g}"'] & \mathbb{R}P^n
\end{tikzcd}$$

$\pi_1 \circ g$ is the composition of two continuous maps, hence continuous. Therefore, by Proposition 1.7.1, $\overline{g}$ is continuous. Now we are only left to show that $\overline{f}$ and $\overline{g}$ are inverses of one another. For $[x] \in \mathbb{R}P^n$,

$$\left(\overline{g} \circ \overline{f}\right)[x] = \overline{g}\left[\frac{x}{\|x\|}\right] = \left[\frac{x}{\|x\|}\right] = [x] \ ,$$

because $x \sim \frac{x}{\|x\|}$ in $\mathbb{R}^{n+1} \setminus \{0\}$, where the value of the nonzero real t is $\frac{1}{\|x\|}$. Furthermore, for $[x] \in S^n/\sim$, $x \in S^n$, so $\|x\| = 1$.

$$\left(\overline{f} \circ \overline{g}\right)[x] = \overline{f}[x] = \left[\frac{x}{\|x\|}\right] = [x] \ .$$

Hence, $\overline{g}$ is indeed the inverse of $\overline{f}$. Therefore, $\overline{f} : \mathbb{R}P^n \to S^n/\sim$ is a homeomorphism. ■

Proposition 5.1.2

The equivalence relation $\sim$ on $\mathbb{R}^{n+1} \setminus \{0\}$ is an open equivalence relation.

Proof. For an open set $U \subseteq \mathbb{R}^{n+1} \setminus \{0\}$, $\pi(U)$ is open in $\mathbb{R}P^n$ if and only if $\pi^{-1}(\pi(U))$ is open in $\mathbb{R}^{n+1} \setminus \{0\}$ (definition of Quotient Topology). Now, if we take an arbitrary point of U and then take its nonzero multiple, both the points will belong to the same equivalence class in $\mathbb{R}P^n$. In other words, for $x \in U$, tx and x will be mapped to the same point in $\pi(U)$. Hence,

$$\pi^{-1}(\pi(U)) = \bigcup_{t \in \mathbb{R}^\times} tU = \bigcup_{t \in \mathbb{R}^\times} \{tx \mid x \in U\} \ ,$$

where $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$. The map of multiplication by a nonzero real t is a homeomorphism from $\mathbb{R}^{n+1} \setminus \{0\}$ to itself. Hence, tU is open in $\mathbb{R}^{n+1} \setminus \{0\}$ for any nonzero t. Therefore, their union

$$\bigcup_{t \in \mathbb{R}^\times} tU = \pi^{-1}(\pi(U))$$

is also open in $\mathbb{R}^{n+1} \setminus \{0\}$. ■

Corollary 5.1.3
The real projective space $\mathbb{R}P^n$ is second countable.

Proof. Follows from the fact that $\mathbb{R}^{n+1} \setminus \{0\}$ is second countable and Corollary 1.10.4. ■

Proposition 5.1.4
$\mathbb{R}P^n$ is Hausdorff.

Proof. Let $S = \mathbb{R}^{n+1} \setminus \{0\}$. Now, consider the set

$$R = \left\{(x, y) \in S \times S \mid y = tx \text{ for some } t \in \mathbb{R}^\times\right\} = \{(x, y) \in S \times S \mid x \sim y\} .$$

R is the graph of $\sim$. We want to show that R is closed in $S \times S$. Consider the real valued function $f : S \times S \to \mathbb{R}$ defined by

$$f(x, y) = f\left(x^0, \ldots, x^n, y^0, \ldots, y^n\right) = \sum_{i \neq j} \left(x^i y^j - x^j y^i\right)^2 .$$

Note that f is continuous and vanishes if and only if $y = tx$ for some $t \in \mathbb{R}^\times$, since

$$\begin{aligned} f(x, y) = 0 &\iff \left(x^i y^j - x^j y^i\right)^2 \text{ for every } i \neq j \\ &\iff x^i y^j = x^j y^i \text{ for every } i \neq j \\ &\iff \frac{x^i}{y^i} = \frac{x^j}{y^j} \text{ for every } i \neq j \\ &\iff y = tx \text{ for some } t \in \mathbb{R}^\times \end{aligned}$$

Therefore, $R = f^{-1}(\{0\})$. $\{0\}$ is closed in $\mathbb{R}$ and f is continuous. Hence, R is closed in $S \times S$. Therefore, by Theorem 1.10.1, $S/{\sim} = \mathbb{R}P^n$ is Hausdorff. ■

The Standard Atlas on Real Projective Space

Let $\left[a^0, a^1, \ldots, a^n\right]$ be homogenous coordinates on projective space $\mathbb{R}P^n$. Consider the set

$$U_0 = \left\{\left[a^0, a^1, \ldots, a^n\right] \in \mathbb{R}P^n \mid a^0 \neq 0\right\} .$$

Let us denote by $\widetilde{U_0}$ the following set

$$\widetilde{U_0} = \left\{\left(a^0, a^1, \ldots, a^n\right) \in \mathbb{R}^{n+1} \setminus \{0\} \mid a^0 \neq 0\right\} .$$

The projection map $p_0 : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}$ onto first coordinate is continuous, and $\widetilde{U_0} = p_0^{-1}(\mathbb{R}^\times)$. $\mathbb{R}^\times$ is open in $\mathbb{R}$, so $\widetilde{U_0}$ is open in $\mathbb{R}^{n+1} \setminus \{0\}$. Note that $U_0 = \pi\left(\widetilde{U_0}\right)$. Since $\sim$ is an open equivalence relation and $\widetilde{U_0}$ is open, U_0 is also open in $\mathbb{R}P^n$. In a similar manner, we can also define the following open subsets of $\mathbb{R}P^n$ for each $i = 1, \ldots, n$.

$$U_i = \left\{\left[a^0, a^1, \ldots, a^n\right] \in \mathbb{R}P^n \mid a^i \neq 0\right\} .$$

It is trivial that

$$\bigcup_{i=0}^{n} U_0 = \mathbb{R}P^n .$$

Now, define $\widetilde{\varphi_0} : \widetilde{U_0} \to \mathbb{R}^n$ by

$$\varphi_0\left(a^0, a^1, \ldots, a^n\right) = \left(\frac{a^1}{a^0}, \frac{a^2}{a^0}, \ldots, \frac{a^n}{a^0}\right) .$$

$\widetilde{\varphi_0}$ is continuous since $a^0 \neq 0$. This induces a map $\varphi_0 : U_0 \to \mathbb{R}^n$ by

$$\varphi_0\left(\left[a^0, a^1, \ldots, a^n\right]\right) = \varphi_0\left(a^0, a^1, \ldots, a^n\right) = \left(\frac{a^1}{a^0}, \frac{a^2}{a^0}, \ldots, \frac{a^n}{a^0}\right).$$

This map is well-defined since

$$\varphi_0\left(\left[ta^0, ta^1, \ldots, ta^n\right]\right) = \left(\frac{ta^1}{ta^0}, \frac{ta^2}{ta^0}, \ldots, \frac{ta^n}{ta^0}\right) = \varphi_0\left(\left[a^0, a^1, \ldots, a^n\right]\right).$$

Due to Proposition 1.7.1, continuity of $\widetilde{\varphi_0}$ implies continuity of φ_0. φ_0 has a continuous inverse $\varphi_0^{-1} : \mathbb{R}^n \to U_0$ given by

$$\varphi_0^{-1}\left(b^1, b^2, \ldots, b^n\right) = \pi\left(1, b^1, b^2, \ldots, b^n\right) = \left[1, b^1, b^2, \ldots, b^n\right].$$

φ_0^{-1} is continuous because it is the composition of π and the continuous map $\left(b^1, \ldots, b^n\right) \mapsto \left(1, b^1, b^2, \ldots, b^n\right)$. Now we shall check that φ_0^{-1} is indeed the inverse of φ_0.

$$\left(\varphi_0 \circ \varphi_0^{-1}\right)\left(b^1, b^2, \ldots, b^n\right) = \varphi_0\left(\left[1, b^1, b^2, \ldots, b^n\right]\right) = \left(b^1, b^2, \ldots, b^n\right).$$

$$\left(\varphi_0^{-1} \circ \varphi_0\right)\left[a^0, a^1, \ldots, a^n\right] = \varphi_0^{-1}\left(\frac{a^1}{a^0}, \frac{a^2}{a^0}, \ldots, \frac{a^n}{a^0}\right) = \left[1, \frac{a^1}{a^0}, \frac{a^2}{a^0}, \ldots, \frac{a^n}{a^0}\right] = \left[a^0, a^1, \ldots, a^n\right].$$

Hence, φ_0^{-1} is indeed the inverse of φ. Therefore, φ_0 is a homeomorphism. Similarly, there are homeomorphisms $\varphi_i : U_i \to \mathbb{R}^n$ for each $i = 1, \ldots, n$.

$$\varphi_i\left(\left[a^0, \ldots, a^n\right]\right) = \left(\frac{a^0}{a^i}, \ldots, \widehat{\frac{a^i}{a^i}}, \ldots, \frac{a^n}{a^i}\right),$$

where the caret sign $\widehat{\ }$ over $\frac{a^i}{a^i}$ means that this entry is to be omitted. This proves that $\mathbb{R}P^n$ is locally Euclidean with $(U_i . \varphi_i)$ as charts.

Now, on the intersection $U_0 \cap U_1$, there are two charts. For $\left[a^0, a^1, \ldots, a^n\right] \in U_0 \cap U_1$, we have $a_0 \neq 0$ and $a_1 \neq 0$.

$$\begin{tikzcd}
 & \left[a^0, a^1, \ldots, a^n\right] \arrow[dl, "\varphi_0"'] \arrow[dr, "\varphi_1"] & \\
\left(\frac{a^1}{a^0}, \frac{a^2}{a^0}, \ldots, \frac{a^n}{a^0}\right) & & \left(\frac{a^0}{a^1}, \frac{a^2}{a^1}, \ldots, \frac{a^n}{a^1}\right)
\end{tikzcd}$$

On $U_0 \cap U_1$, one has $\varphi_0\left(U_0 \cap U_1\right) = \left\{\left(b^1, b^2, \ldots, b^n\right) \in \mathbb{R}^n \mid b^1 \neq 0\right\}$. Given $\left(b^1, b^2, \ldots, b^n\right) \in \varphi_0\left(U_0 \cap U_1\right)$, one obtains

$$\left(\varphi_1 \circ \varphi_0^{-1}\right)\left(b^1, b^2, \ldots, b^n\right) = \varphi_1\left(\left[1, b^1, b^2, \ldots, b^n\right]\right) = \left(\frac{1}{b^1}, \frac{b^2}{b^1}, \ldots, \frac{b^n}{b^1}\right).$$

This is a C^∞ map between open subsets of $\mathbb{R}^n$ since $b^1 \neq 0$ for $\left(b^1, b^2, \ldots, b^n\right) \in \varphi_0\left(U_0 \cap U_1\right)$. In a similar manner, one can show that $\varphi_i \circ \varphi_j^{-1}$ is C^∞ for every i, j. Therefore,

$$\left\{\left(U_i, \varphi_i\right) \mid 0 \leq i \leq n\right\}$$

is a C^∞ atlas on $\mathbb{R}P^n$, called the **standard atlas**. So we have shown that $\mathbb{R}P^n$ is second countable, Hausdorff locally Euclidean space equipped with a C^∞ atlas. Therefore, $\mathbb{R}P^n$ is a smooth manifold.

§5.2 The Grassmannian

The Grassmannianin $G(k,n)$ is the set of all k-planes through the origin in $\mathbb{R}^n$. Such a k-plane is a linear subspace of dimension k of $\mathbb{R}^n$ and has a basis consisting of k linearly independent vectors $a_1, a_2, \ldots, a_k$ in $\mathbb{R}^n$. It is therefore completely specified by an $n \times k$ matrix

$$A = \begin{bmatrix} a_1 & a_2 & \cdots & a_k \end{bmatrix} \text{ such that } \operatorname{rank} A = k\,.$$

This matrix is called a matrix representative of the k-plane. Two bases $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$ determine the same k-plane if there is a change-of-basis matrix $g = [g_{ij}] \in \mathrm{GL}(k, \mathbb{R})$ such that

$$b_j = \sum_i a_i g_{ij}\,. \text{ In matrix notation, } B = Ag\,.$$

Let $F(k,n)$ be the set of all $n \times k$ matrices of rank k, topologized as a subspace of $\mathbb{R}^{n\times k}$. We define an equivalence relation $\sim$ on $F(k,n)$ as follows:

$$A \sim B \iff \text{there is a matrix } g \in \mathrm{GL}(k,\mathbb{R}) \text{ such that } B = Ag\,.$$

There is a bijection between $G(k,n)$ and the quotient space $F(k,n)/\!\sim$. We give the Grassmannian $G(k,n)$ the quotient topology on $F(k,n)/\sim$. Let $\pi : F(k,n) \to F(k,n)/\sim$ be the quotient map.

Lemma 5.2.1

Let A be an $m \times n$ matrix (not necessarily square), and k a positive integer. Then $\operatorname{rank} A \geq k$ if and only if A has a nonsingular $k \times k$ submatrix. Equivalently, $\operatorname{rank} A \leq k-1$ if and only if all $k \times k$ minors of A vanish. (A $k \times k$ minor of a matrix A is the determinant of a $k \times k$ submatrix of A.)

Proof. **($\Rightarrow$):** Suppose $\operatorname{rank} A \geq k$. Then one can find k linearly independent columns, which we call $a_1, \ldots, a_k$. Since the $m \times k$ matrix $B = \begin{bmatrix} a_1 & \cdots & a_k \end{bmatrix}$ has rank k, it has k linearly independent rows $b_1, \ldots, b_k$. The submatrix C of B whose rows are $b_1, \ldots, b_k$ is a $k \times k$ submatrix of A, and $\operatorname{rank} C = k$. In other words, C is a nonsingular $k \times k$ submatrix of A.

($\Leftarrow$): Suppose A has a nonsingular $k \times k$ submatrix B. Let $a_1, \ldots, a_k$ be the columns of A such that the submatrix $\begin{bmatrix} a_1 & \cdots & a_k \end{bmatrix}$ contains B. Since $\begin{bmatrix} a_1 & \cdots & a_k \end{bmatrix}$ has k linearly independent rows, it also has k linearly independent columns. Thus, $\operatorname{rank} A \geq k$. ■

Lemma 5.2.2

If $A \in F(k,n)$, then $Ag \in F(k,n)$ for every $g \in \mathrm{GL}(k,\mathbb{R})$.

Proof. Using the Linear Algebra fact that $\operatorname{rank}(XY) \leq \min\{\operatorname{rank} X, \operatorname{rank} Y\}$, we get that

$$\operatorname{rank}(Ag) \leq \min\{\operatorname{rank} A, \operatorname{rank} g\} = k\,.$$

Using the same result on $A = Ag\,g^{-1}$, we get

$$k = \operatorname{rank}(A) \leq \min\left\{\operatorname{rank}(Ag), \operatorname{rank} g^{-1}\right\} = \min\{\operatorname{rank}(Ag), k\} = \operatorname{rank}(Ag)\,.$$

Therefore, $\operatorname{rank}(Ag) = k$. ■

So we get, the multiplication-by-g-map m_g that maps $A \in F(k,n)$ to Ag is truly a map from $F(k,n)$ to itself. m_g is continuous, because the components are nothing but polynomials in the entries. The inverse map of m_g is $m_{g^{-1}}$, since

$$\left(m_g \circ m_{g^{-1}}\right)(A) = m_g\left(Ag^{-1}\right) = Ag^{-1}g = A\,,$$

$$\left(m_{g^{-1}} \circ m_g\right)(A) = m_{g^{-1}}(Ag) = Agg^{-1} = A\,.$$

Therefore, $m_{g^{-1}} = m_g^{-1}$. It is also continuous by a similar reasoning. Therefore, m_g is a homeomorphism from $F(k,n)$ to itself.

Proposition 5.2.3

The equivalence relation $\sim$ on $F(k,n)$ is an open equivalence relation.

Proof. We shall mimic the proof of Proposition 5.1.2. For an open set $U \subseteq F(k,n)$, $\pi(U)$ is open in $G(k,n)$ if and only if $\pi^{-1}(\pi(U))$ is open in $F(k,n)$ (definition of Quotient Topology).

$$\pi^{-1}(\pi(U)) = \bigcup_{A\in U}[A] = \bigcup_{A\in U}\{Ag \mid g \in \mathrm{GL}(k,\mathbb{R})\} = \bigcup_{g\in \mathrm{GL}(k,\mathbb{R})}\{Ag \mid A \in U\} = \bigcup_{g\in \mathrm{GL}(k,\mathbb{R})} m_g(U)\,.$$

The map $m_g : F(k,n) \to F(k,n)$ is a homeomorphism, as shown above. Therefore, it is an open map. So $m_g(U)$ is open in $F(k,n)$ for every $g \in \mathrm{GL}(k,\mathbb{R})$. Hence, their union

$$\bigcup_{g\in \mathrm{GL}(k,\mathbb{R})} m_g(U) = \pi^{-1}(\pi(U))$$

is also open in $F(k,n)$. ∎

Corollary 5.2.4

The Grassmannian $G(k,n)$ is second countable.

Proof. $F(k,n)$ is a subspace of the second countable space $\mathbb{R}^{n\times k}$, hence it is also second countable. $\sim$ is an open equivalence relation. Therefore, by Corollary 1.10.4, $F(k,n)/\sim\, = G(k,n)$ is second countable. ∎

Proposition 5.2.5

$G(k,n)$ is Hausdorff.

Proof. Let $S = F(k,n)$. Now, consider the set

$$R = \{(A,B) \in S\times S \mid B = Ag \text{ for some } g \in \mathrm{GL}(k,\mathbb{R})\} = \{(A,B)\in S\times S \mid A\sim B\}\,.$$

R is the graph of $\sim$. We want to show that R is closed. Take $A = \begin{bmatrix} a_1 & \cdots & a_k\end{bmatrix}$ and $B = \begin{bmatrix} b_1 & \cdots & b_k\end{bmatrix}$ from S. $A \sim B$ if and only if the columns of B can be expressed as a linear combination of the columns of A. In that case, we would have that the $n\times 2k$ matrix $M_{A,B} = \begin{bmatrix} A & B\end{bmatrix}$ has k linearly independent columns. In other words, $\operatorname{rank} M_{A,B} = k \le k$. By Lemma 5.2.1, this is equivalent to **all** $(k+1)\times(k+1)$ minors of $M_{A,B}$ being 0.

Let $I = (i_0, i_1, \ldots, i_k)$ and $J = (j_0, j_1, \ldots, j_k)$ be $(k+1)$-tuples of integers such that

$$1 \le i_0 < i_1 < \cdots < i_k \le n \quad\text{and}\quad 1 \le j_0 < j_1 < \cdots < j_k \le 2k\,.$$

Define $f_{I,J} : S\times S \to \mathbb{R}$ such that it takes two matrices A and B and returns the $(k+1)\times(k+1)$ minor of $M_{A,B}$ corresponding to rows $i_0, i_1, \ldots, i_k$ and columns $j_0, j_1, \ldots, j_k$.

We have seen before that $A \sim B$ if and only if **all** the $(k+1)\times(k+1)$ minors of $M_{A,B}$ are 0. Therefore,

$$A\sim B \iff f_{I,J}(A,B) = 0 \text{ for every } I, J\,.$$

So, we can write R as

$$R = \bigcap_{I,J} f_{I,J}^{-1}(\{0\})\,.$$

$f_{I,J} : S\times S \to \mathbb{R}$ is continuous since determinant is nothing but a polynomial of the entries. $\{0\}$ is closed in $\mathbb{R}$. Therefore, $f_{I,J}^{-1}(\{0\})$ is closed in $S\times S$. There are only finitely many choices for I, J. In fact, there are total

$$\binom{n}{k+1}\binom{2k}{k+1}$$

ways one can choose I, J. Intersection of finitely many closed sets is closed. Therefore, R is closed in $S\times S$. Hence, by Theorem 1.10.1, $S/\sim\, = G(k,n)$ is Hausdorff. ∎

Next we want to find a C^∞ atlas on the Grassmannian $G(k,n)$. For simplicity, we specialize to $G(2,4)$. For any 4×2 matrix A, let A_{ij} be the 2×2 submatrix consisting of its i-th row and j-th row. Define

$$V_{ij} = \{A \in F(2,4) \mid A_{ij} \text{ is nonsingular }\} = \{A \in F(2,4) \mid \det A_{ij} \neq 0\} .$$

The map $A \mapsto \det A_{ij}$ is a continuous real-valued function. V_{ij} is the pre-image of $\mathbb{R}\setminus\{0\}$ under this continuous function. So, we can conclude that V_{ij} is an open subset of $F(2,4)$.

Lemma 5.2.6

If $A \in V_{ij}$ then $Ag \in V_{ij}$ for every $g \in \mathrm{GL}(2,\mathbb{R})$.

Proof. Let A_i be the i-th row of A.

$$A = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} \implies Ag = \begin{bmatrix} A_1 g \\ A_2 g \\ A_3 g \\ A_4 g \end{bmatrix} \implies (Ag)_{ij} = \begin{bmatrix} A_i g \\ A_j g \end{bmatrix} = A_{ij} g .$$

Therefore, $\det(Ag)_{ij} = \det A_{ij} \det g \neq 0$. So $Ag \in V_{ij}$. ■

Define $U_{ij} = V_{ij}/\!\sim\, = \pi(V_{ij})$. Since $\sim$ is an open equivalence relation, U_{ij} is an open subset of $G(2,4)$. Also, the collection $\{V_{ij}\}$ covers $F(2,4)$, so $\{U_{ij}\}$ covers $G(2,4)$. Now, for $A \in V_{12}$, $A_{12} \in \mathrm{GL}(2,\mathbb{R})$.

$$A \sim AA_{12}^{-1} = \begin{bmatrix} \mathbb{I}_2 \\ A_{34}A_{12}^{-1} \end{bmatrix}$$

So we define a map $\widetilde{\varphi_{12}} : V_{12} \to \mathbb{R}^{2\times 2}$ by $\widetilde{\varphi_{12}}(A) = A_{34}A_{12}^{-1}$. This induces a map $\varphi_{12} : U_{12} \to \mathbb{R}^{2\times 2}$, $\varphi_{12}[A] = \widetilde{\varphi_{12}}(A) = A_{34}A_{12}^{-1}$. This map is well-defined, since

$$\varphi_{12}[Ag] = (Ag)_{34}(Ag)_{12}^{-1} = A_{34}g(A_{12}g)^{-1} = A_{34}gg^{-1}A_{12}^{-1} = A_{34}A_{12}^{-1} = \varphi_{12}[A] ,$$

for any $g \in \mathrm{GL}(2,\mathbb{R})$. $\widetilde{\varphi_{12}}$ is continuous since it is just rational function on the entries. In the light of Proposition 1.7.1, continuity of $\widetilde{\varphi_{12}}$ implies the continuity of φ_{12}. φ_{12} has a continuous inverse $\varphi_{12}^{-1} : \mathbb{R}^{2\times 2} \to U_{12}$ given by

$$\varphi_{12}^{-1}(g) = \pi\left(\begin{bmatrix} \mathbb{I}_2 \\ g \end{bmatrix}\right) = \left[\begin{bmatrix} \mathbb{I}_2 \\ g \end{bmatrix}\right]$$

φ_{12}^{-1} is continuous because it is the composition of π and the continuous map that takes g to $\begin{bmatrix} \mathbb{I}_2 \\ g \end{bmatrix}$. Now we shall check that φ_{12}^{-1} is indeed the inverse of φ_{12}.

$$\left(\varphi_{12} \circ \varphi_{12}^{-1}\right)(g) = \varphi_{12}\left[\begin{bmatrix} \mathbb{I}_2 \\ g \end{bmatrix}\right] = g\mathbb{I}_2^{-1} = g .$$

$$\left(\varphi_{12}^{-1} \circ \varphi_{12}\right)[A] = \varphi_{12}^{-1}\left(A_{34}A_{12}^{-1}\right) = \left[\begin{bmatrix} \mathbb{I}_2 \\ A_{34}A_{12}^{-1} \end{bmatrix}\right] = [A] .$$

Hence, φ_{12}^{-1} is indeed the inverse of φ_{12}. Therefore, φ_{12} is a homeomorphism. Similarly, there are homeomorphisms $\varphi_{ij} : U_{ij} \to \mathbb{R}^{2\times 2}$ for every i, j. This proves $G(2,4)$ is locally Euclidean with (U_{ij}, φ_{ij}) as charts.

Now, on the intersection $U_{12} \cap U_{23}$, there are two charts. For $[A] \in U_{12} \cap U_{23}$, both A_{12} and A_{23} are invertible. Take $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \varphi_{23}(U_{12} \cap U_{23})$. Then we have

$$\left(\varphi_{12} \circ \varphi_{23}^{-1}\right)\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \varphi_{12}\left[\begin{bmatrix} a & b \\ 1 & 0 \\ 0 & 1 \\ c & d \end{bmatrix}\right]$$

Since $\left[\begin{bmatrix} a & b \\ 1 & 0 \\ 0 & 1 \\ c & d \end{bmatrix}\right] \in U_{12} \cap U_{23}$, $\begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix}$ is invertible. So b cannot be 0.

$$\left(\varphi_{12} \circ \varphi_{23}^{-1}\right) \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{b} & -\frac{a}{b} \\ \frac{d}{b} & c - \frac{da}{b} \end{bmatrix}$$

This is a C^∞ map between open subsets of $\mathbb{R}^{2\times 2} \cong \mathbb{R}^4$ since $b \neq 0$. In a similar manner, one can show that $\varphi_{ij} \circ \varphi_{pq}^{-1}$ is smooth for every i, j, p, q. Therefore,

$$\{(U_{ij}, \varphi_{ij}) \mid 1 \leq i < j \leq 4\}$$

is a C^∞ atlas on $G(2,4)$. So we have shown that $G(2,4)$ is second countable, Hausdorff locally Euclidean space equipped with a C^∞ atlas. Therefore, $G(2,4)$ is a smooth manifold.

Now it can be generalized in a similar manner. Let I be a strictly ascending multi-index $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. Let A_I be the $k \times k$ submatrix of A consisting of i_1-th, i_2-th, …, i_k-th rows of A. Define

$$V_I = \{A \in F(k,n) \mid \det A_I \neq 0\} .$$

V_I is an open subset of $F(k,n)$ because it is the pre-image of $\mathbb{R} \setminus \{0\}$ under the continuos real-valued function $A \mapsto \det A_I$. As before, it can be easily seen that $A \in V_I$ implies $Ag \in V_I$ for every $g \in \mathrm{GL}(k,\mathbb{R})$.

$$(Ag)_I = A_I g \implies \det (Ag)_I = \det A_I \det g \neq 0 .$$

Let $U_I = V_I/{\sim} = \pi(V_I)$. Since $\sim$ is an open equivalence relation, $U_I = \pi(V_I)$ is open in $G(k,n)$. Also, the collection $\{V_I\}$ covers $F(k,n)$, so $\{U_I\}$ covers $G(k,n)$.

Next, we define $\widetilde{\varphi_I} : V_I \to \mathbb{R}^{(n-k)\times k}$ as follows

$$\widetilde{\varphi_I}(A) = \left(A A_I^{-1}\right)_{I'} ,$$

where $()_{I'}$ denotes the $(n-k) \times k$ submatrix obtained from the complement I' of the multi-index I. This induces a map $\varphi_I : U_I \to \mathbb{R}^{(n-k)\times k}$, $\varphi_I[A] = \widetilde{\varphi_I}(A) = \left(A A_I^{-1}\right)_{I'}$. One can easily check the well-definedness of φ_I.

$$\varphi_I[Ag] = \left(Ag\,(Ag)_I^{-1}\right)_{I'} = \left(Ag\,(A_I g)^{-1}\right)_{I'} = \left(Agg^{-1}A_I^{-1}\right)_{I'} = \left(AA_I^{-1}\right)_{I'} ,$$

for every $g \in \mathrm{GL}(k,\mathbb{R})$. $\widetilde{\varphi_I}$ is continuous. Therefore, by Proposition 1.7.1, φ_I is continuous.

Let $\varphi_I^{-1} : \mathbb{R}^{(n-k)\times k} \to U_I$ be defined as follows: for $X \in \mathbb{R}^{(n-k)\times k}$, $\varphi_I^{-1}(X) = [A]$, where $A_I = \mathbb{I}_k$ and $A_{I'} = X$. Then φ_I^{-1} is easily seen to be continuous. Also, one can easily check that φ_I^{-1} is indeed the inverse of φ_I.

$$\left(\varphi_I \circ \varphi_I^{-1}\right)(X) = \varphi_I[A] = \left(AA_I^{-1}\right)_{I'} = \left(A\mathbb{I}_k^{-1}\right)_{I'} = A_{I'} = X .$$

$$\left(\varphi_I^{-1} \circ \varphi_I\right)[A] = \varphi_I^{-1}\left(AA_I^{-1}\right)_{I'} = [B] ,$$

where $B_I = \mathbb{I}_k$ and $B_{I'} = \left(AA_I^{-1}\right)_{I'}$. Therefore, $B = AA_I^{-1}$.

$$\left(\varphi_I^{-1} \circ \varphi_I\right)[A] = \left[AA_I^{-1}\right] = \left[AA_I^{-1}A_I\right] = [A] .$$

Therefore, φ_I^{-1} is indeed the inverse of φ_I. This completes the proof that φ_I is a homeomorphism. Hence, $G(k,n)$ is a locally Euclidean space with charts (U_I, φ_I).

Now, we want to show that $\varphi_I \circ \varphi_J^{-1}$ is a smooth map between open subsets of Euclidean space. For $[A] \in U_I \cap U_J$, both A_I and A_J are invertible. Now, take some $X \in \varphi_J(U_I \cap U_J)$. Then we have

$$\left(\varphi_I \circ \varphi_J^{-1}\right)(X) = \varphi_I[A] ,$$

with $A_J = \mathbb{I}_k$ and $A_{J'} = X$. Since $[A] \in U_I \cap U_J$, $\det A_I \neq 0$.

$$\left(\varphi_I \circ \varphi_J^{-1}\right)(X) = \left(AA_I^{-1}\right)_{I'} .$$

Now, the entries of $\left(AA_I^{-1}\right)_{I'}$ can be expressed as rational functions on the entries of X, with the denominator being $\det A_I \neq 0$. Therefore, we can conclude that $\varphi_I \circ \varphi_J^{-1}$ is a smooth map between open subsets of Euclidean space. Therefore,

$$\{(U_I, \varphi_I) \mid I \text{ is strictly asecnding multi-index } 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}$$

is a C^∞ atlas on $G(k,n)$. So we have shown that $G(k,n)$ is second countable, Hausdorff locally Euclidean space equipped with a C^∞ atlas. Therefore, $G(k,n)$ is a smooth manifold.

6 The Tangent Space

§6.1 The Tangent Space at a Point

We define a germ of a C^∞ function at $p \in M$ to be an equivalence class of C^∞ functions defined in some neighborhood of $p \in M$. Two functions of this sort are called equivalent if they agree on some, possibly smaller, neighborhood of p. In other words, two smooth functions $f : U \to \mathbb{R}$ and $g : V \to \mathbb{R}$ defined on some neighborhood of p are equivalent if there exists an open set $W \subseteq U \cap V$ containing p such that

$$f\big|_W = g\big|_W .$$

One can easily verify that this relation is indeed an equivalence relation. The set of germs of real-valued C^∞ functions at $p \in M$ is denoted by $C_p^\infty(M)$.

$C_p^\infty(M)$ is actually a set of equivalence classes. We denote the equivalence class of f as $[f]_p$. We can define addition, multiplication on $C_p^\infty(M)$ as follows:

$$[f]_p + [g]_p := [f+g]_p \ , [f]_p \cdot [g]_p := [f \cdot g]_p \ ,$$

where $f+g$ and $f \cdot g$ are defined pointwise, *i.e.* $(f+g)(x) = f(x) + g(x)$ and $(f \cdot g)(x) = f(x)\, g(x)$.

Now, we can check the well-definedness of addition and multiplication. Let $f \sim f'$ and $g \sim g'$. Then there exists neighborhoods U and V of p such that

$$f\big|_U = f'\big|_U \quad \text{and} \quad g\big|_V = g'\big|_V .$$

Now, on $U \cap V$ (which is also a neighborhood of p),

$$(f+g)(x) = f(x) + g(x) = f'(x) + g'(x) = \left(f' + g'\right)(x) \implies f + g \sim f' + g' .$$

Therefore, addition in $C_p^\infty(M)$ is well-defined. One can similarly check that multiplication is also well-defined. With this addition and multiplication, $C_p^\infty(M)$ forms a ring. We can also define scalar multiplication by $\alpha \in \mathbb{R}$.

$$\alpha\, [f]_p := [\alpha f]_p \ ,$$

where $(\alpha f)(x) = \alpha f(x)$. The well-definedness of scalar multiplication can be checked in a similar manner. Let $f \sim f'$. So there exists a neighborhood U of p such that

$$f\big|_U = f'\big|_U .$$

For $x \in U$,

$$(\alpha f)(x) = \alpha f(x) = \alpha f'(x) = \left(\alpha f'\right)(x) \implies \alpha f \sim \alpha f' .$$

Hence, scalar multiplication by a real number is well-defined. Therefore, with respect to the addition and scalar multiplication, $C_p^\infty(M)$ forms a vector space over $\mathbb{R}$. Since $C_p^\infty(M)$ is a ring and a vector space, and one can easily check that the following holds

$$\alpha\left([f]_p \cdot [g]_p\right) = \left(\alpha\, [f]_p\right) \cdot [g]_p = [f]_p \cdot \left(\alpha\, [g]_p\right) ,$$

$C_p^\infty(M)$ becomes an $\mathbb{R}$-algebra.

Abuse of Notation. Oftentimes we don't distinguish between an element of $C_p^\infty(M)$ (which is an equivalence class) and a representative of the class.

Definition 6.1.1 (Point-Derivation). A point-derivation of $C_p^\infty(M)$ is a linear map $D_p : C_p^\infty(M) \to \mathbb{R}$ such that

$$D_p(fg) = (D_p f)\, g(p) + f(p)\,(D_p g) \ .$$

Definition 6.1.2 (Tangent Vector). A tangent vector X_p at $p \in M$ is a point-derivation at p. The tangent vectors at $p \in M$ form a vector space denoted by T_pM.

Remark 6.1.1. If U is an open set containing $p \in M$, then the algebra $C_p^\infty(U)$ of germs of C^∞ functions in U at p is the same as $C_p^\infty(M)$. Hence, $T_pU = T_pM$.

We learned in chapter 4 that given the standard coordinates $r^1, r^2, \ldots, r^n$ on $\mathbb{R}^n$ and given a coordinate neighborhood $(U, \varphi) = \left(U, x^1, x^2, \ldots, x^n\right)$ about $p \in M$ with $x^i = r^i \circ \varphi$, one defines partial derivatives $\frac{\partial}{\partial x^i}\Big|_p$ at $p \in M$ as

$$\frac{\partial}{\partial x^i}\bigg|_p f := \frac{\partial}{\partial r^i}\bigg|_{\varphi(p)} \left(f \circ \varphi^{-1}\right) \in \mathbb{R}$$

with $f \in C_p^\infty(M)$. One can verify that $\frac{\partial}{\partial x^i}\Big|_p$ is a point-derivation.

$$\frac{\partial}{\partial x^i}\bigg|_p (f \cdot g) = \frac{\partial}{\partial r^i}\bigg|_{\varphi(p)} \left((f \cdot g) \circ \varphi^{-1}\right) .$$

Here, $f, g \in C_p^\infty(M)$. Now, for $a \in \varphi(U)$ with $\varphi^{-1}(a)$ belonging in the domain of both f and g,

$$\left((f \cdot g) \circ \varphi^{-1}\right)(a) = (f \cdot g)\left(\varphi^{-1}(a)\right) = f\left(\varphi^{-1}(a)\right) g\left(\varphi^{-1}(a)\right) = \left(\left(f \circ \varphi^{-1}\right) \cdot \left(g \circ \varphi^{-1}\right)\right)(a) \ .$$

Therefore, $(f \cdot g) \circ \varphi^{-1} = \left(f \circ \varphi^{-1}\right) \cdot \left(g \circ \varphi^{-1}\right)$. As a result,

$$\begin{aligned}
\frac{\partial}{\partial x^i}\bigg|_p (f \cdot g) &= \frac{\partial}{\partial r^i}\bigg|_{\varphi(p)} \left(\left(f \circ \varphi^{-1}\right) \cdot \left(g \circ \varphi^{-1}\right)\right) \\
&= \left[\frac{\partial}{\partial r^i}\bigg|_{\varphi(p)} \left(f \circ \varphi^{-1}\right)\right] \left(g \circ \varphi^{-1}\right)(\varphi(p)) + \left(f \circ \varphi^{-1}\right)(\varphi(p)) \left[\frac{\partial}{\partial r^i}\bigg|_{\varphi(p)} \left(g \circ \varphi^{-1}\right)\right] \\
&= \left(\frac{\partial}{\partial x^i}\bigg|_p f\right) g(p) + f(p) \left(\frac{\partial}{\partial x^i}\bigg|_p g\right)
\end{aligned}$$

Furthermore, $\frac{\partial}{\partial x^i}\Big|_p$ is indeed linear. Therefore, it is a point-derivation.

Definition 6.1.3 (Differential of a Smooth Map). Let $F : N \to M$ be a C^∞ map between manifolds. At $p \in N$, F induces a linear map between tangent spaces T_pN and $T_{F(p)}M$, called the **differential** of F at p, denoted by $F_{*,p} : T_pN \to T_{F(p)}M$ as follows. If $X_p \in T_pN$, then $F_{*,p}(X_p) \in T_{F(p)}M$ defined as

$$F_{*,p}(X_p)\, f := X_p(f \circ F) \ ,$$

where $f \in C_{F(p)}^\infty(M)$, so $f \circ F \in C_p^\infty(N)$.

Example 6.1.1 (Differential of a map between Euclidean Spaces)

Suppose $F : \mathbb{R}^n \to \mathbb{R}^m$ is smooth at $p \in \mathbb{R}^n$. Let $x^1, x^2, \ldots, x^n$ be the coordinates on $\mathbb{R}^n$ and $y^1, y^2, \ldots, y^m$ be the coordinates on $\mathbb{R}^m$. Then the tangent vectors

$$\frac{\partial}{\partial x^1}\bigg|_p , \frac{\partial}{\partial x^2}\bigg|_p , \ldots , \frac{\partial}{\partial x^n}\bigg|_p$$

form a basis for the tangent space $T_p\mathbb{R}^n$, and the tangent vectors

$$\left.\frac{\partial}{\partial y^1}\right|_{F(p)}, \left.\frac{\partial}{\partial y^2}\right|_{F(p)}, \ldots, \left.\frac{\partial}{\partial y^m}\right|_{F(p)}$$

form a basis for the tangent space $T_{F(p)}\mathbb{R}^m$. The linear map $F_{*,p} : T_p\mathbb{R}^n \to T_{F(p)}\mathbb{R}^m$ is represented by a matrix $\left[a^i_j(p)\right]$ relative to these bases as follows:

$$F_{*,p}\left(\left.\frac{\partial}{\partial x^j}\right|_p\right) = \sum_{k=1}^{m} a^k_j(p) \left.\frac{\partial}{\partial y^k}\right|_{F(p)}, a^k_j(p) \in \mathbb{R}\,.$$

Let's evaluate both sides of the above equation on y^i.

$$\begin{aligned} \text{RHS} &= \sum_{k=1}^{m} a^k_j(p) \left.\frac{\partial}{\partial y^k}\right|_{F(p)} y^i = \sum_{k=1}^{m} a^k_j(p)\,\delta^i_k = a^i_j(p) \\ \text{LHS} &= F_{*,p}\left(\left.\frac{\partial}{\partial x^j}\right|_p\right) y^i = \left.\frac{\partial}{\partial x^j}\right|_p \left(y^i \circ F\right) = \left.\frac{\partial F^i}{\partial x^j}\right|_p \end{aligned}$$

where $F^i = y^i \circ F : \mathbb{R}^n \to \mathbb{R}$ is the i-th component of F. Therefore, the matrix representation of $F_{*,p}$ relative to the bases $\left\{\left.\frac{\partial}{\partial x^j}\right|_p\right\}$ and $\left\{\left.\frac{\partial}{\partial y^i}\right|_{F(p)}\right\}$ is $\left[\frac{\partial F^i}{\partial x^j}(p)\right]$. This is precisely the Jacobian matrix of the derivative DF_p of F at p as discussed in chapter 2.

Theorem 6.1.1 (The Chain Rule)
Let $F : N \to M$ and $G : M \to P$ be smooth maps of manifolds, and $p \in N$. Then,

$$(G \circ F)_{*,p} = G_{*,F(p)} \circ F_{*,p}\,.$$

In other words, the following diagram commutes:

$$\begin{tikzcd} T_pN \arrow[r, "F_{*,p}"] \arrow[rr, bend right, "(G\circ F)_{*,p}"'] & T_{F(p)}M \arrow[r, "G_{*,F(p)}"] & T_{G(F(p))}P \end{tikzcd}$$

Proof. Let $X_p \in T_pN$ and $f \in C^\infty_{G(F(p))}(P)$. Then

$$\left((G \circ F)_{*,p} X_p\right) f = X_p\,(f \circ G \circ F)\,.$$

Now, let $Y_{F(p)} = F_{*,p}X_p \in T_{F(p)}M$.

$$\begin{aligned} \left(\left(G_{*,F(p)} \circ F_{*,p}\right) X_p\right) f &= \left(G_{*,F(p)}\left(F_{*,p}X_p\right)\right) f = \left(G_{*,F(p)}\left(Y_{F(p)}\right)\right) f \\ &= Y_{F(p)}\,(f \circ G) = F_{*,p}X_p\,(f \circ G) \\ &= X_p\,(f \circ G \circ F) \end{aligned}$$

Therefore, $\left(G_{*,F(p)} \circ F_{*,p}\right) X_p = (G \circ F)_{*,p} X_p$, $\forall\, X_p \in T_pN$. Hence, $(G \circ F)_{*,p} = G_{*,F(p)} \circ F_{*,p}$. ■

Remark 6.1.2. Consider the identity map $\mathbb{1}_M : M \to M$ as a smooth map from M to itself. Then for a given $p \in M$, the differential

$$(\mathbb{1}_M)_{*,p} : T_pM \to T_pM$$

of $\mathbb{1}_M$ is the usual identity map on the tangent space T_pM, because for $X_p \in T_pM$ and $f \in C^\infty_p(M)$,

$$\left((\mathbb{1}_M)_{*,p} X_p\right) f = X_p\,(f \circ \mathbb{1}_M) = X_p f\,.$$

Therefore, $(\mathbb{1}_M)_{*,p} X_p = X_p$. Hence,

$$(\mathbb{1}_M)_{*,p} = \mathbb{1}_{T_pM} : T_pM \to T_pM\,.$$

Corollary 6.1.2

If $F : N \to M$ is a diffeomorphism of manifolds and $p \in N$, then $F_{*,p} : T_pN \to T_{F(p)}M$ is an isomorphism of vector spaces.

Proof. Since F is a diffeomorphism, it has a smooth inverse $G : M \to N$ such that $G \circ F = \mathbb{1}_N$ and $F \circ G = \mathbb{1}_M$. By Theorem 6.1.1 and the remark above,

$$G_{*,F(p)} \circ F_{*,p} = (G \circ F)_{*,p} = (\mathbb{1}_N)_{*,p} = \mathbb{1}_{T_pN}\,.$$

$$F_{*,p} \circ G_{*,F(p)} = F_{*,G(F(p))} \circ G_{*,F(p)} = (F \circ G)_{*,F(p)} = (\mathbb{1}_M)_{*,F(p)} = \mathbb{1}_{T_{F(p)}M}\,.$$

$G_{*,F(p)} \circ F_{*,p} = \mathbb{1}_{T_pN}$ and $F_{*,p} \circ G_{*,F(p)} = \mathbb{1}_{T_{F(p)}M}$ together imply that $F_{*,p} : T_pN \to T_{F(p)}M$ is an isomorphism of vector spaces. ■

Corollary 6.1.3 (Invariance of Dimension)

If an open set $U \subseteq \mathbb{R}^n$ is diffeomorphic to an open set $V \subseteq \mathbb{R}^m$, then $n = m$.

Proof. Let $F : U \to V$ be a diffeomorphism and $p \in U$. By Corollary 6.1.2, $F_{*,p} : T_pU \to T_{F(p)}V$ is an isomorphism of vector spaces. There are vector space isoomorphisms $T_pU \simeq \mathbb{R}^n$ and $T_{F(p)}V \simeq \mathbb{R}^m$. Therefore, $\mathbb{R}^n$ is isomorphic to $\mathbb{R}^m$. This happens only when $n = m$. ■

§6.2 Bases of the Tangent Space at a Point

We denote by $r^1, r^2, \ldots, r^n$ the standard coordinates on $\mathbb{R}^n$. Let (U, φ) be a chart about $p \in M$, where M is a smooth manifold of dimension n. We set $x^i = r^i \circ \varphi$. Since $\varphi : U \to \mathbb{R}^n$ is a diffeomorphism onto its image (Proposition 4.2.4), by Corollary 6.1.2,

$$\varphi_{*,p} : T_pM \to T_{\varphi(p)}\varphi(U) = T_{\varphi(p)}\mathbb{R}^n$$

is a vector space isomorphism. In particular, the tangent space T_pM is isomorphic to the vector space $T_{\varphi(p)}\mathbb{R}^n \simeq \mathbb{R}^n$, and hence T_pM has the same dimension n as the manifold.

Proposition 6.2.1

Let $(U, \varphi) = \left(U, x^1, x^2, \ldots, x^n\right)$ be a chart about a point $p \in M$. Then

$$\varphi_{*,p}\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = \left.\frac{\partial}{\partial r^i}\right|_{\varphi(p)}\,.$$

Proof. Let $f \in C^\infty_{\varphi(p)}(\mathbb{R}^n)$. Then

$$\varphi_{*,p}\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) f = \left.\frac{\partial}{\partial x^i}\right|_p (f \circ \varphi) = \left.\frac{\partial}{\partial r^i}\right|_{\varphi(p)} \left(f \circ \varphi \circ \varphi^{-1}\right) = \left.\frac{\partial}{\partial r^i}\right|_{\varphi(p)} f$$

Hence, $\varphi_{*,p}\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = \left.\frac{\partial}{\partial r^i}\right|_{\varphi(p)}$. ■

Lemma 6.2.2

Let $T : V \to W$ be an isomorphism between the n-dimensional $\mathbb{F}$-vector spaces V and W. Let $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be a basis for V. Then $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \dots, T(\mathbf{v}_n)\}$ is a basis for W.

Proof. First, we want to show that $\mathcal{B} = \{T(\mathbf{v}_1), T(\mathbf{v}_2), \dots, T(\mathbf{v}_n)\}$ is a linearly independent set. Suppose for $c_1, c_2, \dots, c_n \in \mathbb{F}$,

$$c_1 T(\mathbf{v}_1) + c_2 T(\mathbf{v}_2) + \cdots + c_n T(\mathbf{v}_n) = \mathbf{0}_W \,.$$

Applying linearity on the LHS, we get

$$T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n) = \mathbf{0}_W = T(\mathbf{0}_V) \implies c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}_V \,.$$

Since $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for V, it's a linearly independent set. Therefore, $c_i = 0$ for each $i = 1, 2, \dots, n$. Therefore, $\mathcal{B}$ is a linearly independent set of vectors.

Now, since T is surjective, for $\mathbf{w} \in W$, $\mathbf{w} = T(\mathbf{v})$ for some $\mathbf{v} \in V$. We can write $\mathbf{v}$ as a linear combination of the basis vectors.

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n \,,$$

for $c_i \in \mathbb{F}$.

$$\mathbf{w} = T(\mathbf{v}) = T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n) = c_1 T(\mathbf{v}_1) + c_2 T(\mathbf{v}_2) + \cdots + c_n T(\mathbf{v}_n) \,.$$

Hence, $\mathcal{B}$ spans W. Therefore, $\mathcal{B}$ is a basis for W. ■

Proposition 6.2.3

If $(U, \varphi) = (U, x^1, x^2, \dots, x^n)$ is a chart containing $p \in M$, then the tangent space T_pM has basis

$$\left.\frac{\partial}{\partial x^1}\right|_p, \left.\frac{\partial}{\partial x^2}\right|_p, \dots, \left.\frac{\partial}{\partial x^n}\right|_p \,.$$

Proof. We have seen earlier that since $\varphi : U \to \mathbb{R}^n$ is a diffeomorphism onto its image, by Corollary 6.1.2,

$$\varphi_{*,p} : T_pM \to T_{\varphi(p)}\varphi(U) = T_{\varphi(p)}\mathbb{R}^n$$

is an isomorphism of vector spaces. By Proposition 6.2.1,

$$\varphi_{*,p}\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = \left.\frac{\partial}{\partial r^i}\right|_{\varphi(p)} \implies \varphi_{*,p}^{-1}\left(\left.\frac{\partial}{\partial r^i}\right|_{\varphi(p)}\right) = \left.\frac{\partial}{\partial x^i}\right|_p \,.$$

Since $\left\{ \left.\frac{\partial}{\partial r_1}\right|_{\varphi(p)}, \left.\frac{\partial}{\partial r_2}\right|_{\varphi(p)}, \dots, \left.\frac{\partial}{\partial r_n}\right|_{\varphi(p)} \right\}$ is a basis for $T_{\varphi(p)}\mathbb{R}^n$, by Lemma 6.2.2,

$$\left\{ \left.\frac{\partial}{\partial x^1}\right|_p, \left.\frac{\partial}{\partial x^2}\right|_p, \dots, \left.\frac{\partial}{\partial x^n}\right|_p \right\}$$

is a basis for T_pM. ■

Proposition 6.2.4 (Transition Matrix for Coordinate Vectors)

If $(U, x^1, x^2, \dots, x^n)$ and $(V, y^1, y^2, \dots, y^n)$ are two coordinate charts on a manifold M, then

$$\left.\frac{\partial}{\partial x^j}\right|_p = \sum_{i=1}^{n} \frac{\partial y^i}{\partial x^j}(p) \left.\frac{\partial}{\partial y^i}\right|_p \quad \text{for } p \in U \cap V \,.$$

Proof. At each point $p \in U \cap V$, the sets $\left\{ \frac{\partial}{\partial x^j}\Big|_p \right\}$ and $\left\{ \frac{\partial}{\partial y^i}\Big|_p \right\}$ are both bases for the tangent space T_pM. So there exists a change of basis matrix $\left[a_j^i\left(p\right)\right]$ of real numbers such that

$$\frac{\partial}{\partial x^j}\bigg|_p = \sum_{k=1}^{n} a_j^k\left(p\right)\frac{\partial}{\partial y^k}\bigg|_p .$$

Applying both sides of the equation to y^i, we get

$$\frac{\partial}{\partial x^j}\bigg|_p y^i = \sum_{k=1}^{n} a_j^k\left(p\right)\frac{\partial}{\partial y^k}\bigg|_p y^i = \sum_{k=1}^{n} a_j^k\left(p\right)\delta_k^i = a_j^i\left(p\right) .$$

Therefore, $a_j^i\left(p\right) = \frac{\partial}{\partial x^j}\Big|_p y^i = \frac{\partial y^i}{\partial x^j}\left(p\right)$. As a result,

$$\frac{\partial}{\partial x^j}\bigg|_p = \sum_{k=1}^{n} \frac{\partial y^k}{\partial x^j}\left(p\right)\frac{\partial}{\partial y^k}\bigg|_p .$$

■

Proposition 6.2.5 (Local Expression for the Differential)
Given a smooth map $F : N \to M$ of manifolds and a point $p \in N$, let $\left(U, x^1, x^2, \ldots, x^n\right)$ and $\left(V, y^1, y^2, \ldots, y^m\right)$ be coordinate charts about $p \in N$ and $F\left(p\right) \in M$, respectively. Relative to the bases $\left\{ \frac{\partial}{\partial x^j}\Big|_p \right\}$ for T_pN and $\left\{ \frac{\partial}{\partial y^i}\Big|_{F(p)} \right\}$ for $T_{F(p)}M$, the differential $F_{*,p} : T_pN \to T_{F(p)}M$ is represented by the matrix $\left[\frac{\partial F^i}{\partial x^j}\left(p\right)\right]$, where $F^i = y^i \circ F$ is the i-th component of F

Proof. By Proposition 6.2.3, $\left\{ \frac{\partial}{\partial x^j}\Big|_p \right\}$ is a basis for T_pN and $\left\{ \frac{\partial}{\partial y^i}\Big|_{F(p)} \right\}$ is a basis for $T_{F(p)}M$. $F_{*,p}\left(\frac{\partial}{\partial x^j}\Big|_p\right) \in T_{F(p)}M$, so it can be written as a linear combination of the basis vectors.

$$F_{*,p}\left(\frac{\partial}{\partial x^j}\bigg|_p\right) = \sum_{k=1}^{m} a_j^k\left(p\right)\frac{\partial}{\partial y^k}\bigg|_{F(p)} .$$

Applying both sides of the equation to y^i, we obtain

$$F_{*,p}\left(\frac{\partial}{\partial x^j}\bigg|_p\right) y^i = \sum_{k=1}^{m} a_j^k\left(p\right)\frac{\partial}{\partial y^k}\bigg|_{F(p)} y^i = \sum_{k=1}^{m} a_j^k\left(p\right)\delta_k^i = a_j^i\left(p\right) .$$

Using the definition of $F_{*,p}$,

$$F_{*,p}\left(\frac{\partial}{\partial x^j}\bigg|_p\right) y^i = \frac{\partial}{\partial x^j}\bigg|_p \left(y^i \circ F\right) = \frac{\partial F^i}{\partial x^j}\left(p\right) \implies a_j^i\left(p\right) = \frac{\partial F^i}{\partial x^j}\left(p\right) .$$

■

Remark 6.2.1. In terms of differentials, the inverse function theorem for manifolds can be described in the following coordinate free way: a C^∞ map $F : N \to M$ between two manifolds of same dimension is locally invertible at a point $p \in N$ if and only if the differential $F_{*,p} : T_pN \to T_{F(p)}M$ is an isomorphism of vector spaces.

Example 6.2.1 (The Chain Rule in Calculus Notation)
Suppose $w = G(x, y, z)$ is a C^∞ function $w : \mathbb{R}^3 \to \mathbb{R}$ and $F(t) = (\widetilde{x}(t), \widetilde{y}(t), \widetilde{z}(t))$ is a C^∞ function $F : \mathbb{R} \to \mathbb{R}^3$. Under composition

$$w = (G \circ F)(t) = G(x(t), y(t), z(t))$$

becomes a C^∞ function of $t \in \mathbb{R}$. (Here we abused notation $\widetilde{x}(t) = x, \widetilde{y}(t) = y, \widetilde{z}(t) = z$.) Now, let t_0 be fixed in $\mathbb{R}$. By Proposition 6.2.5, the matrix representation of $F_{*,t_0} : T_{t_0}\mathbb{R} \to T_{F(t_0)}\mathbb{R}^3$ is

$$\begin{bmatrix} \frac{\partial F^1}{\partial t}(t_0) \\ \frac{\partial F^2}{\partial t}(t_0) \\ \frac{\partial F^2}{\partial t}(t_0) \end{bmatrix} = \begin{bmatrix} \frac{\mathrm{d}\widetilde{x}}{\mathrm{d}t}(t_0) \\ \frac{\mathrm{d}\widetilde{y}}{\mathrm{d}t}(t_0) \\ \frac{\mathrm{d}\widetilde{z}}{\mathrm{d}t}(t_0) \end{bmatrix}.$$

Similarly, for $\mathbf{p}_0 = (x_0, y_0, z_0) \in \mathbb{R}^3$, the matrix representation of $G_{*,\mathbf{p}_0} : T_{\mathbf{p}_0}\mathbb{R}^3 \to T_{G(\mathbf{p}_0)}\mathbb{R}$ is

$$\begin{bmatrix} \frac{\partial G}{\partial x}(\mathbf{p}_0) & \frac{\partial G}{\partial y}(\mathbf{p}_0) & \frac{\partial G}{\partial z}(\mathbf{p}_0) \end{bmatrix} = \begin{bmatrix} \frac{\partial w}{\partial x}(\mathbf{p}_0) & \frac{\partial w}{\partial y}(\mathbf{p}_0) & \frac{\partial w}{\partial z}(\mathbf{p}_0) \end{bmatrix}.$$

In a similar manner, the matrix representation of $(G \circ F)_{*,t_0} : T_{t_0}\mathbb{R} \to T_{G(F(t_0))}\mathbb{R}$ is

$$\begin{bmatrix} \frac{\partial (G \circ F)}{\partial t}(t_0) \end{bmatrix} = \begin{bmatrix} \frac{\mathrm{d}w}{\mathrm{d}t}(t_0) \end{bmatrix}.$$

Now, the chain rule $(G \circ F)_{*,t_0} = G_{*,F(t_0)} \circ F_{*,t_0}$ is equivalent to

$$\begin{bmatrix} \frac{\mathrm{d}w}{\mathrm{d}t}(t_0) \end{bmatrix} = \begin{bmatrix} \frac{\partial w}{\partial x}(F(t_0)) & \frac{\partial w}{\partial y}(F(t_0)) & \frac{\partial w}{\partial z}(F(t_0)) \end{bmatrix} \begin{bmatrix} \frac{\mathrm{d}\widetilde{x}}{\mathrm{d}t}(t_0) \\ \frac{\mathrm{d}\widetilde{y}}{\mathrm{d}t}(t_0) \\ \frac{\mathrm{d}\widetilde{z}}{\mathrm{d}t}(t_0) \end{bmatrix}$$

$$\iff \frac{\mathrm{d}w}{\mathrm{d}t}(t_0) = \frac{\partial w}{\partial x}(F(t_0)) \frac{\mathrm{d}x}{\mathrm{d}t}(t_0) + \frac{\partial w}{\partial y}(F(t_0)) \frac{\mathrm{d}y}{\mathrm{d}t}(t_0) + \frac{\partial w}{\partial z}(F(t_0)) \frac{\mathrm{d}z}{\mathrm{d}t}(t_0)$$

(Here also we abused notation $\widetilde{x}(t) = x, \widetilde{y}(t) = y, \widetilde{z}(t) = z$.) This is the usual form of chain rule taught in Calculus classes.

§6.3 Curves in a Manifold

Definition 6.3.1 (Smooth Curve). A **smooth curve** in a manifold M is a smooth map $c : (a, b) \to M$ from some open interval (a, b) into M. Usually we assume $0 \in (a, b)$ and say that c is a curve starting at p if $c(0) = p$. The **velocity vector** $c'(t_0)$ of the curve c at time $t_0 \in (a, b)$ is defined to be

$$c'(t_0) := c_{*,t_0}\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0} \right) \in T_{c(t_0)}M.$$

Notational Confusion

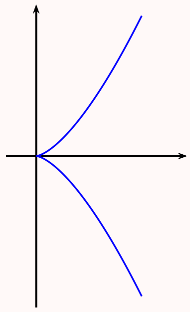
In the special case $c : (a, b) \to \mathbb{R}$, *i.e.*, when the target manifold is $\mathbb{R}$, by $c'(t)$ we mean a tangent vector at $c(t)$. Hence, $c'(t)$ is a real multiple of $\left.\frac{\mathrm{d}}{\mathrm{d}x}\right|_{c(t)}$. On the other hand, in calculus notation, $c'(t)$ is the derivative of the real valued function $c(t)$ and hence it is a scalar. In order to resolve the issue, we write $\dot{c}(t)$ for the calculus derivative (scalar).

Example 6.3.1

Define $c : \mathbb{R} \to \mathbb{R}^2$ by $c(t) = \left(t^2, t^3\right)$. This curve is known as a cuspidal cube.

$$c'(t_0) = c_{*,t_0}\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0}\right) \in T_{c(t_0)}\mathbb{R}^2 .$$

Then $c'(t_0)$ can be written as a linear combination of the basis vectors $\left.\frac{\partial}{\partial x}\right|_{c(t_0)}$ and $\left.\frac{\partial}{\partial y}\right|_{c(t_0)}$ of $T_{c(t_0)}\mathbb{R}^2$.



$$c_{*,t_0}\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0}\right) = a \left.\frac{\partial}{\partial x}\right|_{c(t_0)} + b \left.\frac{\partial}{\partial y}\right|_{c(t_0)} ,$$

where $a, b \in \mathbb{R}$. We now evaluate both sides on x to obtain:

$$c_{*,t_0}\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0}\right) x = a \implies a = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0} (x \circ c) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0} t^2 = 2t_0 .$$

Similarly, we evaluate both sides on y to obtain:

$$c_{*,t_0}\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0}\right) y = b \implies b = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0} (y \circ c) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0} t^3 = 3t_0^2 .$$

Therefore,

$$c_{*,t_0}\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0}\right) = 2t_0 \left.\frac{\partial}{\partial x}\right|_{c(t_0)} + 3t_0^2 \left.\frac{\partial}{\partial y}\right|_{c(t_0)} .$$

This means that in terms of the basis $\left\{ \left.\frac{\partial}{\partial x}\right|_{c(t_0)}, \left.\frac{\partial}{\partial y}\right|_{c(t_0)} \right\}$ for $T_{c(t_0)}\mathbb{R}^2$,

$$c'(t_0) = \begin{bmatrix} 2t_0 \\ 3t_0^2 \end{bmatrix} .$$

The example above inspires the following proposition.

Proposition 6.3.1 (Velocity of a curve in local coordinates)

Let $c : (a, b) \to M$ be a smooth curve, and let $\left(U, x^1, x^2, \ldots, x^n\right)$ be a coordinate chart about $c(t)$. Let $c^i = x^i \circ c$ be the i-th component of the curve c in the chart. Then $c'(t)$ is given by

$$c'(t) = \sum_{i=1}^{n} \dot{c}^i(t) \left.\frac{\partial}{\partial x^i}\right|_{c(t)} .$$

Thus, relative to the basis $\left\{ \frac{\partial}{\partial x^i}\Big|_{c(t)} \right\}$ for $T_{c(t)}M$, the velocity $c'(t)$ is represented by the column vector

$$\begin{bmatrix} \dot{c}^1(t) \\ \dot{c}^2(t) \\ \cdots \\ \dot{c}^n(t) \end{bmatrix}$$

Note: $\dot{c}^i(t) = \frac{\mathrm{d}c^i}{\mathrm{d}t}(t)$ is the derivative of the real valued function $c^i(t)$.

Proof. $c'(t) = c_{*,t}\left(\frac{\mathrm{d}}{\mathrm{d}t}\big|_t\right) \in T_{c(t)}M$. Since $\left\{ \frac{\partial}{\partial x^j}\Big|_{c(t)} \right\}$ is a basis for $T_{c(t)}M$, we can write $c'(t)$ as a linear combination of basis vectors.

$$c'(t) = \sum_{i=1}^{n} a_j \left.\frac{\partial}{\partial x^j}\right|_{c(t)} .$$

Evaluating both sides on x^i, we get

$$c'(t)\, x^i = \sum_{i=1}^{n} a_j \left.\frac{\partial}{\partial x^j}\right|_{c(t)} x^i = \sum_{i=1}^{n} a_j \delta_j^i = a^i ,$$

because Proposition 4.3.1 tells us that $\frac{\partial x^i}{\partial x^j} = \delta_j^i$. Now, using the definition of $c'(t)$,

$$a^i = c'(t)\, x^i = c_{*,t}\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_t\right) x^i = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_t \left(x^i \circ c\right) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_t c^i = \frac{\mathrm{d}c^i}{\mathrm{d}t}(t) = \dot{c}^i(t) .$$

Therefore,

$$c'(t) = \sum_{i=1}^{n} \dot{c}^i(t) \left.\frac{\partial}{\partial x^i}\right|_{c(t)} .$$

■

Proposition 6.3.2 (Existence of a curve with a given initial vector)

For any point p in a manifold M and any tangent vector $X_p \in T_pM$, there are $\varepsilon > 0$ and a smooth curve $c : (-\varepsilon, \varepsilon) \to M$ such that $c(0) = p$ and $c'(0) = X_p$.

Proof. Let $(U, \varphi) = \left(U, x^1, x^2, \ldots, x^n\right)$ be a chart centered at p, i.e. $\varphi(p) = \mathbf{0} \in \mathbb{R}^n$. Since $X_p \in T_pM$, it can be written as a linear combination of the basis vectors.

$$X_p = \sum_{i=1}^{n} a^i \left.\frac{\partial}{\partial x^i}\right|_p .$$

Claim — There exists a curve $\alpha : (-\varepsilon, \varepsilon) \to \varphi(U)$ such that $\alpha(0) = \mathbf{0}$ and $\alpha'(0) = \sum_{i=1}^{n} a^i \frac{\partial}{\partial r^i}\big|_{\mathbf{0}}$.

Proof. We define $\alpha : \mathbb{R} \to \mathbb{R}^n$ by $\alpha(t) = \left(a^1 t, a^2 t, \ldots, a^n t\right)$. Then $\alpha(0) = \mathbf{0}$. We can choose ε sufficiently small such that $\alpha(t) \in \varphi(U)$ for $-\varepsilon < t < \varepsilon$. By Proposition 6.3.1,

$$\alpha'(0) = \sum_{i=1}^{n} \dot{\alpha}^i(0) \left.\frac{\partial}{\partial r^i}\right|_{\alpha(0)} = \sum_{i=1}^{n} a^i \left.\frac{\partial}{\partial r^i}\right|_{\mathbf{0}} ,$$

since $\dot{\alpha}^i(0) = \frac{\mathrm{d}\alpha^i}{\mathrm{d}t}(0) = \frac{\mathrm{d}\left(a^i t\right)}{\mathrm{d}t}(0) = a^i$. □

We define $c = \varphi^{-1} \circ \alpha : (-\varepsilon, \varepsilon) \to U \subseteq M$. Then $c(0) = \varphi^{-1}(\alpha(0)) = \varphi^{-1}(\mathbf{0}) = p$. Furthermore, using The Chain Rule,

$$c'(0) = c_{*,0}\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0\right) = \left(\left(\varphi^{-1}\right)_{*,\alpha(0)} \circ \alpha_{*,0}\right)\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0\right) = \left(\varphi^{-1}\right)_{*,\mathbf{0}} \alpha'(0) \ .$$

Using the expression for $\alpha'(0)$ and Proposition 6.2.1, we get

$$\begin{aligned}\left(\varphi^{-1}\right)_{*,\mathbf{0}} \alpha'(0) &= \left(\varphi^{-1}\right)_{*,\mathbf{0}}\left(\sum_{i=1}^n a^i \left.\frac{\partial}{\partial r^i}\right|_{\mathbf{0}}\right) = \sum_{i=1}^n a^i \left(\varphi^{-1}\right)_{*,\mathbf{0}}\left(\left.\frac{\partial}{\partial r^i}\right|_{\mathbf{0}}\right)\\ &= \sum_{i=1}^n a^i \left.\frac{\partial}{\partial x^i}\right|_{\varphi(0)} = \sum_{i=1}^n a^i \left.\frac{\partial}{\partial x^i}\right|_p = X_p\end{aligned}$$

Therefore, $c'(0) = X_p$. ■

Proposition 6.3.3

Suppose X_p is a tangent vector at a point p of a manifold M and $f \in C_p^\infty(M)$. If $c : (-\varepsilon, \varepsilon) \to M$ is a smooth curve starting at p with $c'(0) = X_p$, then

$$X_p f = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 (f \circ c) \ .$$

Proof. By definition of $c'(0)$ and $c_{*,0}$,

$$X_p f = c'(0) f = c_{*,0}\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0\right) f = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 (f \circ c) \ .$$

■

Computing the Differential Using Curves

Proposition 6.3.4

Let $F : N \to M$ be a smooth map of manifolds, $p \in N$, and $X_p \in T_p N$. If c is a smooth curve starting at $p \in N$ and with velocity X_p at p, then

$$F_{*,p}(X_p) = (F \circ c)'(0) \ .$$

In other words, $F_{*,p}(X_p)$ is the velocity vector of the curve $F \circ c$ at $(F \circ c)(0) = F(p)$.

Proof. By hypothesis, $c(0) = p$ and $c'(0) = X_p$. Now,

$$\begin{aligned}F_{*,p}(X_p) &= F_{*,p}\left(c'(0)\right) = F_{*,p}\left(c_{*,0}\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0\right)\right)\\ &= \left(F_{*,c(0)} \circ c_{*,0}\right)\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0\right) = (F \circ c)_{*,0}\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0\right)\\ &= (F \circ c)'(0)\end{aligned}$$

which is the velocity vector to the curve $F \circ c$ at $(F \circ c)(0) = F(p)$. ■

Example 6.3.1 (Differential of Left Multiplication)**.** $\mathrm{GL}(n, \mathbb{R})$ stands for the group of all $n \times n$ invertible matrices over $\mathbb{R}$. It is called the general linear group. Let $g \in \mathrm{GL}(n, \mathbb{R})$. Also, let $l_g : \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$ denote the left multiplication by the matrix g. In other words, for $B \in \mathrm{GL}(n, \mathbb{R})$, $l_g(B) = gB \in \mathrm{GL}(n, \mathbb{R})$.

Since we've seen earlier that $\mathrm{GL}(n,\mathbb{R})$ is an open subset of $\mathbb{R}^{n\times n} \simeq \mathbb{R}^{n^2}$, $T_g\,\mathrm{GL}(n,\mathbb{R})$ can be identified with $\mathbb{R}^{n\times n} \simeq \mathbb{R}^{n^2}$ for any $g \in \mathrm{GL}(n,\mathbb{R})$. Now, if I is the $n \times n$ identity matrix, then show that

$$(l_g)_{*,I} : T_I\,\mathrm{GL}(n,\mathbb{R}) \to T_g\,\mathrm{GL}(n,\mathbb{R})$$

is also left multiplication by g.

Solution. Since $\mathrm{GL}(n,\mathbb{R})$ is an open subset of the Euclidean space $\mathbb{R}^{n\times n} \equiv \mathbb{R}^{n^2}$, the only coordinate chart on $\mathrm{GL}(n,\mathbb{R})$ is $\left(\mathrm{GL}(n,\mathbb{R}), r^{11}, r^{12}, \dots, r^{nn}\right)$, where r^{ij}'s are the usual coordinates on $\mathbb{R}^{n\times n}$. There are vector space isomorphisms $\psi : T_I\,\mathrm{GL}(n,\mathbb{R}) \to \mathbb{R}^{n\times n}$ and $\varphi : T_g\,\mathrm{GL}(n,\mathbb{R}) \to \mathbb{R}^{n\times n}$.

$$\psi\left(\sum_{i,j=1}^{n} a^{ij} \left.\frac{\partial}{\partial r^{ij}}\right|_I\right) = \left[a^{ij}\right]_{i,j=1}^{n} \quad \text{and} \quad \varphi\left(\sum_{i,j=1}^{n} a^{ij} \left.\frac{\partial}{\partial r^{ij}}\right|_g\right) = \left[a^{ij}\right]_{i,j=1}^{n} .$$

Let $X \in T_I\,\mathrm{GL}(n,\mathbb{R})$. Then $(l_g)_{*,I} X \in T_g\,\mathrm{GL}(n,\mathbb{R})$. So we need to prove that

$$\varphi\left((l_g)_{*,I} X\right) = g\psi(X) .$$

Let us now compute $(l_g)_{*,I} X$. Choose a curve $c(t)$ in $\mathrm{GL}(n,\mathbb{R})$ with $c(0) = I$ and $c'(0) = X$. Let $\alpha = l_g \circ c$. By Proposition 6.3.4,

$$(l_g)_{*,I}(X) = (l_g \circ c)_{*,0}\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0\right) = (l_g \circ c)'(0) = \alpha'(0) .$$

Now, α is a map from an open interval to $\mathrm{GL}(n,\mathbb{R})$, and it is the composition of two smooth maps. Hence, α is a smooth curve in $\mathrm{GL}(n,\mathbb{R})$. $\alpha(0) = l_g(I) = g$. By Proposition 6.3.1,

$$\alpha'(0) = \sum_{i,j=1}^{n} \dot{\alpha}^{ij}(0) \left.\frac{\partial}{\partial r^{ij}}\right|_g \implies \varphi\left((l_g)_{*,I}(X)\right) = \varphi\left(\sum_{i,j=1}^{n} \dot{\alpha}^{ij}(0) \left.\frac{\partial}{\partial r^{ij}}\right|_g\right) = \left[\dot{\alpha}^{ij}(0)\right]_{i,j=1}^{n} .$$

Now, since $\alpha(t) = g\,c(t)$,

$$\begin{aligned}\alpha^{ij}(t) = \sum_{k=1}^{n} g^{ik} c^{kj}(t) \implies \dot{\alpha}^{ij}(0) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \left(\sum_{k=1}^{n} g^{ik} c^{kj}(t)\right) = \sum_{k=1}^{n} g^{ik} \left.\frac{\mathrm{d}c^{kj}(t)}{\mathrm{d}t}\right|_0 = \sum_{k=1}^{n} g^{ik} \dot{c}^{kj}(0) \\ &\implies \left[\dot{\alpha}^{ij}(0)\right]_{i,j=1}^{n} = g\left[\dot{c}^{ij}(0)\right]_{i,j=1}^{n}\end{aligned}$$

Now, since $X = c'(0)$, by Proposition 6.3.1,

$$X = c'(0) = \sum_{i,j=1}^{n} \dot{c}^{ij}(0) \left.\frac{\partial}{\partial r^{ij}}\right|_I \implies \psi(X) = \left[\dot{c}^{ij}(0)\right]_{i,j=1}^{n} .$$

$$\therefore \left[\dot{\alpha}^{ij}(0)\right]_{i,j=1}^{n} = g\left[\dot{c}^{ij}(0)\right]_{i,j=1}^{n} \implies \varphi\left((l_g)_{*,I}(X)\right) = g\psi(X) ,$$

as desired. ■

§6.4 Immersions and Submersions

> **Definition 6.4.1** (Immersion and Submersion). A C^∞ map $F : N \to M$ is said to be an **immersion** at $p \in N$ if its differential $F_{*,p} : T_pN \to T_{F(p)}M$ is injective, and a **submersion** at p if $F_{*,p}$ is surjective. We call F an immersion if it is an immersion at every $p \in N$, and a submersion if it is a submersion ar every $p \in N$.

Remark 6.4.1. Suppose N and M are manifolds of dimension n and m, respectively. Then $\dim(T_pN) = n$ and $\dim\left(T_{F(p)}M\right) = m$. The injectivity of the differential $F_{*,p} : T_pN \to T_{F(p)}M$ immediately implies $m \geq n$. Similarly, surjectivity of $F_{*,p} : T_pN \to T_{F(p)}M$ implies that $n \geq m$. Thus, if $F : N \to M$ is an immersion at a point of N, then $m \geq n$ and if F is a submersion at a point of N, then $n \geq m$.

Example 6.4.1

The prototype of an immersion is the inclusion of $\mathbb{R}^n$ into a higher dimensional $\mathbb{R}^m$:

$$i\left(x^1, x^2, \ldots, x^n\right) = \left(x^1, x^2, \ldots, x^n, 0, \ldots, 0\right) .$$

The prototype of a submersion is the projection of $\mathbb{R}^n$ onto a lower dimensional $\mathbb{R}^m$:

$$\pi\left(x^1, x^2, \ldots, x^m, x^{m+1}, \ldots, x^n\right) = \left(x^1, x^2, \ldots, x^m\right) .$$

Example 6.4.2

Let U be an open subset of a manifold M and hence a manifold. The inclusion map $i : U \hookrightarrow M$ is injective and in genral, not surjective. The differential of i at $p \in U$, denoted by $i_{*,p} : T_pU \to T_pM$, is bijective. Indeed $\dim(T_pU) = \dim(T_pM)$ as $T_pU \simeq T_pM$ as vector spaces. Hence, the inclusion map $i : U \to M$ is both an immersion and a submersion. This is an example exhibiting the fact that a submersion need not be onto.

Rank, and Critical and Regular Points

Consider a smooth map $F : N \to M$ of manifolds. Its rank at a point $p \in N$, denoted by $\operatorname{rank} F(p)$, is defined as the rank of the differential $F_{*,p} : T_pN \to T_{F(p)}M$. Relative to the coordinate neighborhoods $\left(U, x^1, x^2, \ldots, x^n\right)$ at p and $\left(V, y^1, y^2, \ldots, y^m\right)$ at $F(p)$, the differential $F_{*,p}$ is represented by the Jacobian matrix $\left[\frac{\partial F^i}{\partial x^j}(p)\right]$ (Proposition 6.2.5), so

$$\operatorname{rank} F(p) = \operatorname{rank}\left[\frac{\partial F^i}{\partial x^j}(p)\right] .$$

Since the differential of a map is independent of coordinate charts, so is the rank of a Jacobian matrix.

Definition 6.4.2 (Critical and Regular Points)**.** A point $p \in N$ is a **critical point** of $F : N \to M$ if the differential $F_{*,p} : T_pN \to T_{F(p)}M$ is not surjective. It is a **regular point** of F if the differential $F_{*,p}$ is surjective. In other words, p is a regular point of F if and only if F is a submersion at p. A point in M is a **critical value** if it is the image of a critical point; otherwise it is a **regular value**.

Couple of important aspects of this definitions:

(i) We *do not* define regular value to be the image of a regular point. Any point of M that is not a critical value is defined as a regular value. Therefore, any point of M that is not in the image of F, or which doesn't have a preimage in N under F, is automarically a regular value.

(ii) A point $c \in M$ is a critical value if and only if some point in the preimage $F^{-1}(\{c\})$ is a critical point. A point c in the image of F is a regular value if and only if every point in the preimage $F^{-1}(\{c\})$ is a regular point.

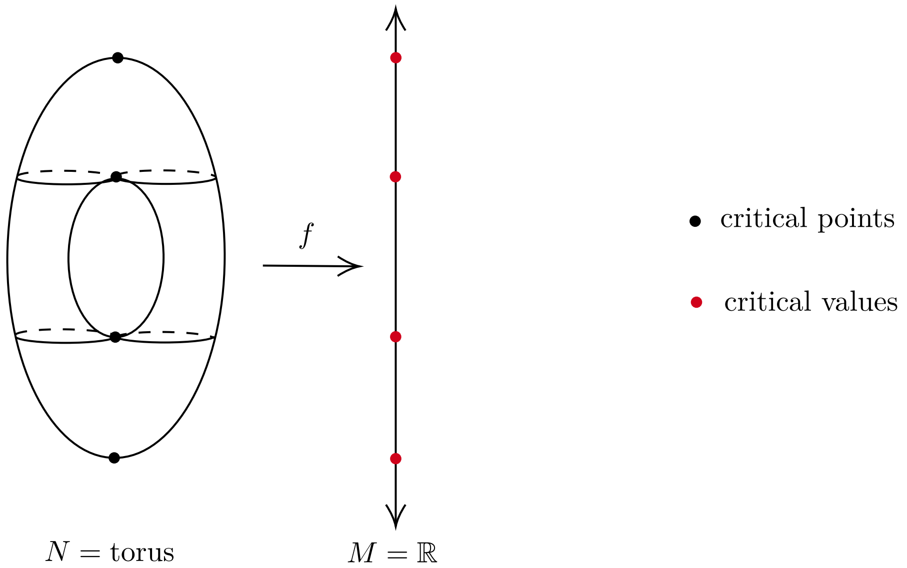


Figure 6.1: Critical points and critical values of the height function $f(x, y, z) = z$ of the 2-torus

Proposition 6.4.1

For a real valued function $f : M \to \mathbb{R}$, a point $p \in M$ is a critical point if and only if relative to some chart $\left(U, x^1, x^2, \dots, x^n\right)$ contaning p, all the partial derivatives satisfy

$$\frac{\partial f}{\partial x^j}(p) = 0\ ,\ \ j = 1, 2, \dots, n\,.$$

Proof. By Proposition 6.2.5, the differential $f_{*,p} : T_pM \to T_{f(p)}\mathbb{R} \simeq \mathbb{R}$ is represented by the matrix

$$\left[\frac{\partial f}{\partial x^1}(p) \quad \frac{\partial f}{\partial x^2}(p) \quad \cdots \quad \frac{\partial f}{\partial x^n}(p)\right]$$

with respect to the basis $\left\{ \left.\frac{\partial}{\partial x^1}\right|_p, \dots, \left.\frac{\partial}{\partial x^n}\right|_p \right\}$ of T_pM and $\left\{ \left.\frac{\partial}{\partial y}\right|_{f(p)} \right\}$ of $T_{f(p)}\mathbb{R}$.

Since the image of $f_{*,p}$ is a vector subspace of $T_{f(p)}\mathbb{R} \simeq \mathbb{R}$, it is either 0-dimensional or 1-dimensional. In other words, $f_{*,p} : T_pM \to T_{f(p)}\mathbb{R} \simeq \mathbb{R}$ is either the 0-map (everything is mapped to the 0-vector of the codomain) or a surjective map. Therefore, $f_{*,p}$ fails to be surjective if and only if the matrix representing it is the 0-matrix. In other words, all the partial derivatives $\frac{\partial f}{\partial x^j}(p)$ vanish. ∎

7 Submanifolds

§7.1 Regular Submanifolds

Definition 7.1.1 (Regular Submanifold)**.** A subset S of a manifold N of dimension n is a **regular submanifold** of dimension k if for every $p \in S$, there is a coordinate neighborhood $\left(U, x^1, x^2, \ldots, x^n\right)$ of p in the maximal atlas of N such that $U \cap S$ is defined by the vanishing of $n-k$ coordinate functions.

By renumbering the coordinates, we may assyme that these vanishing $n-k$ coordinate functions are $x^{k+1}, \ldots, x^n$. We call such a chart (U, φ) in N an **adapted chart** relative to S.

It is important to note that $U \cap S$ is supposed to be the maximal subset of U where $n-k$ coordinate functions vanish. By maximal, we mean that it is not contained in any other subset of U other than itself where $n-k$ coordinate functions vanish. This fact can be mathematically described as

$$U \cap S = \varphi^{-1}(*, \ldots, *, \underbrace{0, 0, \ldots, 0}_{(n-k) \text{ many}}).$$

On $U \cap S$, $\varphi = \left(x^1, x^2, \ldots, x^k, 0, 0, \ldots, 0\right)$. Let $\varphi_S : U \cap S \to \mathbb{R}^k$ be the restriction of the first k components of φ to $U \cap S$. In other words,

$$\varphi_S = \left(x^1, x^2, \ldots, x^k\right).$$

Note that, $(U \cap S, \varphi_S)$ is a chart for S in the subspace topology.

Definition 7.1.2 (Codimension)**.** If S is a regular submanifold of dimension k in a manifold N of dimension n, then $n-k$ is said to be the **codimension** of S in N.

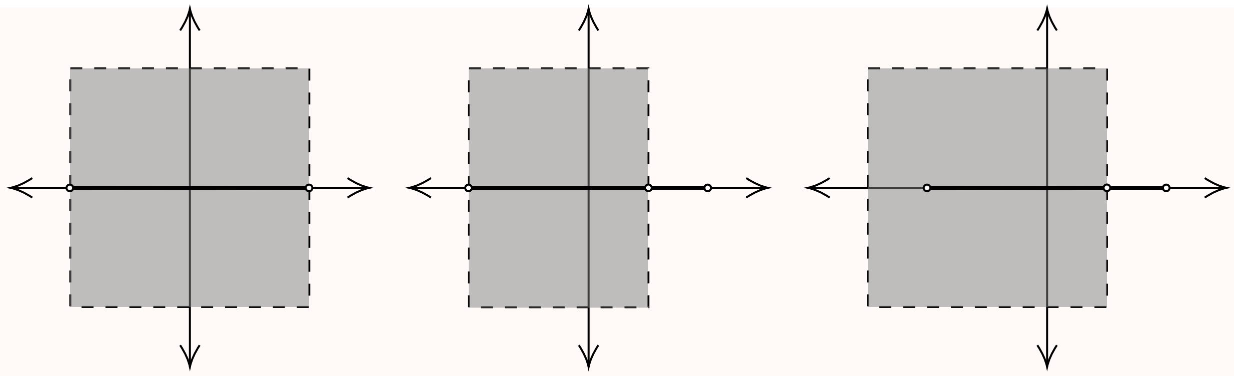
Remark 7.1.1. As a topological space, a regular submanifold of N is required to have the subspace topology. It's also noteworthy that the dimensioon k of the regular submanifold S may be equal to the dimension n of the manifold N. In this case, $U \cap S$ is defined by the vanishing of none of the coordinate functions, so $U \cap S = U$. Hence, $\varphi\big|_{U\cap S} = \varphi$. Therefore, an open subset of a manifold is a regular submanifold of the same dimension.

Example 7.1.1

The interval $S := (-1, 1)$ on the x-axis is a regular submanifold of the x-y plane ($\mathbb{R}^2$). As an adapted chart (for any point $p \in S$), we can choose the open square $U = (-1, 1) \times (-1, 1)$ with coordinates x, y of the plane $\mathbb{R}^2$. Then one immediately finds that $U \cap S$ is precisely the zero set of y on U.

Similarly, the open rectangle $V = (-1, 0.5) \times (-1, 1)$ with coordinates x, y of $\mathbb{R}^2$ is also an adapted chart relative to S. Because, the zero set of y on V is $(-1, 0.5)$, which is precisely $V \cap S$.

However, if we take $W = (-1.5, 0.5) \times (-1, 1)$, then (W, x, y) is not an adapted chart relative to S for any point $p \in (-1, 0.5) = W \cap S$. This is because the zero set of y on W is $(-1.5, 0.5) \neq W \cap S$.



Exercise 7.1. Let Γ be the graph of the function $f(x) = \sin\left(\frac{1}{x}\right)$ on $(0, 1)$, and I be the open interval on y axis:

$$I = \{(0, y) \mid -1 < y < 1\} .$$

Then show that $S = \Gamma \cup I$ is not a regular submanifold of $\mathbb{R}^2$.

Proposition 7.1.1

Let S be a regular submanifold of N and let $\mathscr{U} = \{(U, \varphi)\}$ be a collection of compatible adapted charts of N that covers S. Then $\mathscr{U}_S = \{(U \cap S, \varphi_S)\}$ is an atlas for S. Therefore, a regular submanifold is itself a manifold. If N has dimension n and S is locally defined by the vanishing of $n - k$ coordinate functions, then $\dim S = k$.

Proof. Let $(U, \varphi) = \left(U, x^1, x^2, \ldots, x^n\right)$ and $(V, \psi) = \left(V, y^1, y^2, \ldots, y^n\right)$ be two adapted charts in the given collection $\mathscr{U}$. Assume that they intersect. From the definition of adapted chart relative to a submanifold S, it is possible to renumber the coordinates such that the last $n-k$ coordinate functions vanish on points of S intersected with the open set of the pertaining adapted chart. Therefore, for $p \in U \cap V \cap S$,

$$\varphi(p) = \left(x^1, x^2, \ldots, x^k, 0, 0, \ldots, 0\right) \quad \text{and} \quad \psi(p) = \left(y^1, y^2, \ldots, y^k, 0, 0, \ldots, 0\right) .$$

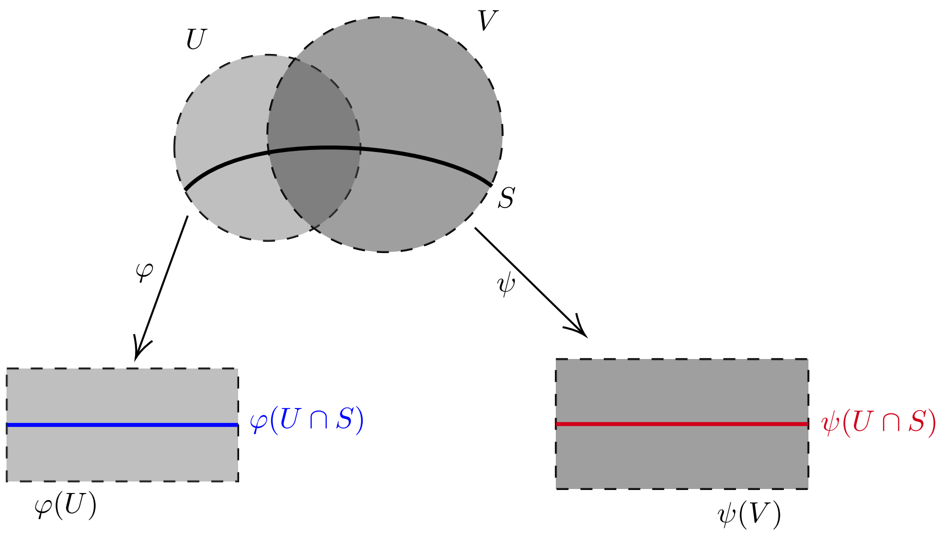


Figure 7.1: Overlapping adapted charts relative to a regular submanifold S.

So $\varphi_S(p) = (x^1, x^2, \ldots, x^k)$ and $\psi_S(p) = (y^1, y^2, \ldots, y^k)$. Therefore,

$$\left(\psi_S \circ \varphi_S^{-1}\right)\left(x^1, x^2, \ldots, x^k\right) = \left(y^1, y^2, \ldots, y^k\right),$$

where $\psi_S \circ \varphi_S^{-1} : \varphi_S(U \cap V \cap S) \subseteq \mathbb{R}^k \to \psi_S(U \cap V \cap S) \subseteq \mathbb{R}^k$. Note that since $\mathscr{U}$ is a C^∞ compatible collection of adapted charts of N, $\psi \circ \varphi^{-1} : \varphi(U \cap V) \subseteq \mathbb{R}^n \to \psi(U \cap V) \subseteq \mathbb{R}^n$ is C^∞. Let $i : \mathbb{R}^k \hookrightarrow \mathbb{R}^n$ be the inclusion map and $\pi : \mathbb{R}^n \to \mathbb{R}^k$ be the projection onto first k coordinates. Now,

$$\left(\psi \circ \varphi^{-1}\right)\left(x^1, x^2, \ldots, x^k, 0, 0, \ldots, 0\right) = \left(y^1, y^2, \ldots, y^k, 0, 0, \ldots, 0\right)$$

gives us the following identity:

$$\psi_S \circ \varphi_S^{-1} = \pi\big|_{\psi(U \cap V \cap S)} \circ \left(\psi \circ \varphi^{-1}\right)\Big|_{\varphi(U \cap V \cap S)} \circ i\big|_{\varphi_S(U \cap V \cap S)}.$$

Therefore, as a composition of smooth maps, $\psi_S \circ \varphi_S^{-1}$ is also smooth. Hence, any two charts in $\mathscr{U}_S$ are C^∞ compatible. Now, since the open sets in the collection $\mathscr{U}$ covers S, the open sets in $\mathscr{U}_S$ also covers S, proving that the collection $\mathscr{U}_S$ is a C^∞ atlas on S. ■

§7.2 Level Sets of a Map

Definition 7.2.1. A **level set** of a map $F : N \to M$ is a subset

$$F^{-1}(\{c\}) = \{p \in N \mid F(p) = c\}$$

for some $c \in M$. The value $c \in M$ is called the **level** of the level set $F^{-1}(\{c\})$. If $F : N \to \mathbb{R}^m$, then $Z(F) := F^{-1}(\{\mathbf{0}\})$ is the **zero set** of F.

Recall that c is a regular value of F if either c is not in the image of F or at every point $p \in F^{-1}(\{c\})$, the differential $F_{*,p} : T_pN \to T_{F(p)}M$ is surjective.

Definition 7.2.2. The preimage $F^{-1}(\{c\})$ of a regular value c is called a **regular level set**. If $F : N \to \mathbb{R}^m$ and $\mathbf{0} \in \mathbb{R}^m$ is a regular value of F, then $F^{-1}(\{\mathbf{0}\})$ is called a **regular zero set**.

Remark 7.2.1. If a regular level set $F^{-1}(\{c\})$ is nonempty, then for $p \in F^{-1}(\{c\})$, the map $F : N \to M$ is a submersion at p. By Remark 6.4.1, $\dim N \geq \dim M$.

Example 7.2.1. The unit 2-sphere $S^2 = \left\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\right\}$ is the zero set of the function $f : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$f(x, y, z) = x^2 + y^2 + z^2 - 1.$$

In other words, $S^2 = f^{-1}(\{0\})$. Show that S^2 is a regular submanifold of $\mathbb{R}^3$.

Solution. Note that

$$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = 2y, \quad \frac{\partial f}{\partial z} = 2z.$$

By Proposition 6.4.1, $\mathbf{p} \in \mathbb{R}^3$ is a critical point of f if and only if

$$\left.\frac{\partial f}{\partial x}\right|_{\mathbf{p}} = \left.\frac{\partial f}{\partial y}\right|_{\mathbf{p}} = \left.\frac{\partial f}{\partial z}\right|_{\mathbf{p}} = 0.$$

Hence, the only critical point of f is $\mathbf{0} \equiv (0,0,0)$. Since $\mathbf{0} \notin S^2 = f^{-1}(\{0\})$, 0 is a regular value of f.

Let us choose $p \in S^2$ such that $\frac{\partial f}{\partial x}(p) = 2x(p) \neq 0$. Then the Jacobian matrix of the map $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $\varphi(x, y, z) = (f(x, y, z), y, z)$ is as follows:

$$\begin{bmatrix} \frac{\partial \varphi^1}{\partial x} & \frac{\partial \varphi^1}{\partial y} & \frac{\partial \varphi^1}{\partial z} \\ \frac{\partial \varphi^2}{\partial x} & \frac{\partial \varphi^2}{\partial y} & \frac{\partial \varphi^2}{\partial z} \\ \frac{\partial \varphi^2}{\partial x} & \frac{\partial \varphi^2}{\partial y} & \frac{\partial \varphi^2}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The Jacobian determinant is $\frac{\partial f}{\partial x}$, which is nonzero at the point p. Then by Corollary 4.3.3, there is a neighborhood U_p of $p \in \mathbb{R}^3$ such that $\left(U_p, \varphi^1, \varphi^2, \varphi^3\right)$ is a chart in the maximal atlas of $\mathbb{R}^3$. Now, in the chart $\left(U_p, \varphi^1, \varphi^2, \varphi^3\right) = (U_p, f, y, z)$, the set $U_p \cap S^2$ is defined by the vanishing of the first coordinate $\varphi^1 = f$. Because S^2 is the zero set of f, so the maximal subset of U_p where f vanish must be $U_p \cap S^2$. Thus (U_p, f, y, z) is an adapted chart relative to S^2, and $\left(U_p \cap S^2, y, z\right)$ is a chart for S^2 (applies to those points $p \in S^2$ for which $\frac{\partial f}{\partial x}(p) \neq 0$).

Similarly, if one chooses $p \in S^2$ with $\frac{\partial f}{\partial y}(p) \neq 0$, then there is an adapted chart (V_p, x, f, z) containing p in which $V_p \cap S^2$ is defined by the vanishing of the second coordinate f. Then $\left(V_p \cap S^2, x, z\right)$ is a chart for S^2 (applies to those points $p \in S^2$ for which $\frac{\partial f}{\partial y}(p) \neq 0$).

In a similar manner, if now one chooses $p \in S^2$ with $\frac{\partial f}{\partial z}(p) \neq 0$, then there is an adapted chart (W_p, x, y, f) containing p in which $W_p \cap S^2$ is defined by the vanishing of the third coordinate f. Then $\left(W_p \cap S^2, x, y\right)$ is a chart for S^2 (applies to those points $p \in S^2$ for which $\frac{\partial f}{\partial z}(p) \neq 0$).

Now, for every $p \in S^2$, at least one of the partial derivatives $\frac{\partial f}{\partial x}(p)$, $\frac{\partial f}{\partial y}(p)$, $\frac{\partial f}{\partial z}(p)$ is nonzero. Hence, as p varies on all over S^2, one obtains a collection of adapted charts of $\mathbb{R}^3$ that covers S^2. Therefore, S^2 is a regular submanifold of $\mathbb{R}^3$. By Proposition 7.1.1, S^2 is a manifold of dimension $3 - 1 = 2$. ■

Lemma 7.2.1

Let $g : N \to \mathbb{R}$ be a C^∞ function. A regular level set $g^{-1}(\{c\})$ of level c of the function g is the regular zero level set $f^{-1}(\{0\})$ of the function $f = g - c$.

Proof. For $p \in N$,

$$g(p) = c \iff f(p) = g(p) - c = 0 .$$

Hence, $g^{-1}(\{c\}) = f^{-1}(\{0\})$. Call this set S. Note that

$$f_{*,p} = g_{*,p} , \qquad \forall\, p \in N .$$

Hence, critical points of f and g are exactly the same. Since S is a regular level set of the function g, it does not contain any critical point of the function g. Hence, S does not contain any critical point of the function f either. In other words, $f^{-1}(\{0\}) = S$ is a regular zero set of the function f. ■

Theorem 7.2.2

Let $g : N \to \mathbb{R}$ be a C^∞ function on the manifold N. Then a non-empty regular level set $S = g^{-1}(\{c\})$ iis a regular submanifold of N of codimension 1.

Proof. Let $f = g - c$. By Lemma 7.2.1, $S = f^{-1}(\{0\})$ is a regular zero set of f. Let $p \in S$. Since p is a regular point of f, relative to any chart $\left(U, x^1, x^2, \ldots, x^n\right)$ containing p,

$$\frac{\partial f}{\partial x^i}(p) \neq 0 \quad \text{for some } i.$$

Otherwise, p would be a critical point of f according to Proposition 6.4.1. By renumberiing the coordinate functions, we may assyme that $\frac{\partial f}{\partial x^1}(p) \neq 0$. Now, the Jacobian matrix of the C^∞ map $\varphi : U \to \mathbb{R}^n$ defined by

$$\varphi(p) = \left(f(p), x^2(p), \ldots, x^n(p)\right)$$

is given by

$$\begin{bmatrix} \frac{\partial \varphi^1}{\partial x^1} & \frac{\partial \varphi^1}{\partial x^2} & \cdots & \frac{\partial \varphi^1}{\partial x^n} \\ \frac{\partial \varphi^2}{\partial x^1} & \frac{\partial \varphi^2}{\partial x^2} & \cdots & \frac{\partial \varphi^2}{\partial x^n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \varphi^n}{\partial x^1} & \frac{\partial \varphi^n}{\partial x^2} & \cdots & \frac{\partial \varphi^n}{\partial x^n} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x^1} & \frac{\partial f}{\partial x^2} & \cdots & \frac{\partial f}{\partial x^n} \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} .$$

So, the Jacobian determinant $\frac{\partial(f,x^2,\dots,x^n)}{\partial(x^1,x^2,\dots,x^n)}$ at p is $\frac{\partial f}{\partial x^1}(p) \neq 0$. Therefore, by Corollary 4.3.3, there is a neighborhood U_p of $p \in N$ on which $\varphi^1, \varphi^2, \dots, \varphi^n$ form a coordinate system. In other words, $\left(U_p, f, x^2, \dots, x^n\right)$ is a coordinate neighborhood containing p. In this chart, the set $U_p \cap S$ is defined by the vanishing of the first coordinate $\varphi^1 = f$. Because S is the zero set of f, so the maximal subset of U_p where f vanish must be $U_p \cap S$. Thus $\left(U_p, f, x^2, \dots, x^n\right)$ is an adapted chart relative to S, and $\left(U_p \cap S, x^2, \dots, x^n\right)$ is a chart for S containing p. Since p is arbitrary, S is a regular submanifold of dimension $n-1$ by Proposition 7.1.1. ■

The next step is to extend the result of Theorem 7.2.2 to a regular level set of a smooth map between manifolds.

Theorem 7.2.3 (Regular Level Set Theorem)
Let $F : N \to M$ be a C^∞ map of manifolds, with $\dim N = n$ and $\dim M = m$. Then a non-empty regular level set $S = F^{-1}(\{c\})$, with $c \in M$, is a regular submanifold of N of dimension $n - m$.

Proof. Choose a chart $(V, \psi) = \left(V, y^1, y^2, \dots, y^m\right)$ of M centered at c, *i.e.* $\psi(c) = \mathbf{0} \in \mathbb{R}^m$. Then $F^{-1}(V)$ is open in N containing $F^{-1}(\{c\})$. Also, in $F^{-1}(V)$,

$$F^{-1}(\{c\}) = F^{-1}\left(\psi^{-1}(\{\mathbf{0}\})\right) = (\psi \circ F)^{-1}(\{\mathbf{0}\}) .$$

Hence, the level set $F^{-1}(\{c\})$ is the zero set of $\psi \circ F$.

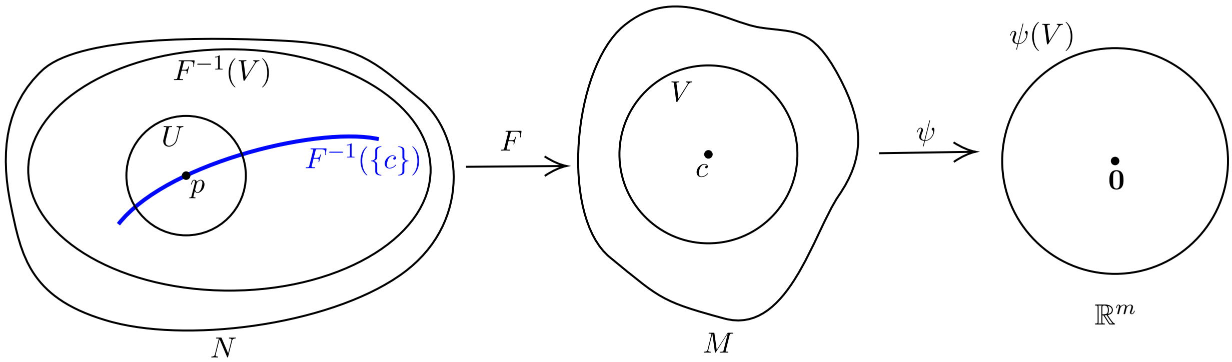


Figure 7.2: The level set $F^{-1}(\{c\})$ of F is the zero set of $\psi \circ F$.

Denote by $F^i = y^i \circ F = r^i \circ \psi \circ F = r^i \circ (\psi \circ F)$, with each $F^i : F^{-1}(V) \to \mathbb{R}$. Now for each $i \in \{1, 2, \dots, m\}$,

$$F^i\left(F^{-1}(\{c\})\right) = r^i \circ \psi \circ F\left(F^{-1}(\{c\})\right) = \left(r^i \circ \psi\right)(c) = r^i(\mathbf{0}) = 0 .$$

Now we claim that, $F^{-1}(\{c\})$ is the maximal common zero set of the functions $F^1, F^2, \dots, F^m$ on $F^{-1}(V)$. In other words, we want to show that

$$F^{-1}(\{c\}) = \bigcap_{i=1}^{m} \left(F^i\right)^{-1}(\{0\}) .$$

Assume for the sake of contradiction that there exists some $b \notin F^{-1}(\{c\})$ such that $F^i(b) = 0$ for every i. Then we have

$$F^i(b) = 0 \implies r^i(\psi(F(b))) = 0 \quad \forall i \implies \psi(F(b)) = \mathbf{0} = \psi(c) \implies F(b) = c .$$

This implies that $b \in F^{-1}(\{c\})$, contradiction! Therefore, $F^{-1}(\{c\})$ is the maximal common zero set of the functions $F^1, F^2, \dots, F^m$ on $F^{-1}(V)$.

By hypothesis, the regular level set $F^{-1}(\{c\})$ is nonempty. So, by Remark 7.2.1, $n \geq m$. Now fix a point $p \in F^{-1}(\{c\})$ and let $(U, \varphi) = \left(U, x^1, x^2, \dots, x^n\right)$ be a coordinate neighborhood of $p \in N$

contained in $F^{-1}(V)$. In other words, $p \in U \subseteq F^{-1}(V)$ (See Figure 7.2). Since $F^{-1}(\{c\})$ is a regular level set and $p \in F^{-1}(\{c\})$, p is a regular point of F. In other words, the differential $F_{*,p} : T_pN \to T_{F(p)}M$ is surjective. In other words, $\operatorname{rank} F_{*,p} = \dim T_{F(p)}M = m$. So, the matrix representation of $F_{*,p}$ also has rank m.

$$\operatorname{rank}\left[\frac{\partial F^i}{\partial x^j}(p)\right]_{1\le i\le m, 1\le j\le n} = m\,.$$

By renumbering F^i's and x^j's, we may assume that the first $m \times m$ block $\left[\frac{\partial F^i}{\partial x^j}(p)\right]_{1\le i,j\le m}$ is nonsingular. Now we replace the first m coordinates of the chart $(U, \varphi) = \left(U, x^1, x^2, \dots, x^n\right)$ by $F^1, F^2, \dots, F^m$.

Now we claim that there is a neighborhood U_p of p such that $\left(U_p, F^1, F^2, \dots, F^m, x^{m+1}, \dots, x^n\right)$ is a chart in the maximal atlas of N. It suffices to compute the pertinent Jacobian matrix at p. We write the $n \times n$ Jacobian matrix as block form:

$$J = \begin{bmatrix} \frac{\partial F^i}{\partial x^j} & \frac{\partial F^i}{\partial x^\beta} \\ \frac{\partial x^\alpha}{\partial x^j} & \frac{\partial x^\alpha}{\partial x^\beta} \end{bmatrix},$$

where $1 \le i, j \le m$ and $m + 1 \le \alpha, \beta \le n$. Therefore,

$$J = \begin{bmatrix} \frac{\partial F^i}{\partial x^j} & * \\ 0_{(n-m)\times m} & I_{(n-m)\times(n-m)} \end{bmatrix}.$$

The determinant of J is $\det\left[\frac{\partial F^i}{\partial x^j}\right]_{1\le i,j\le m} \neq 0$ since $\left[\frac{\partial F^i}{\partial x^j}\right]_{1\le i,j\le m}$ is nonsingular. Therefore, by Corollary 4.3.3, there exists a neighborhood U_p of $p \in N$ such that there exists a coordinate neighborhood $\left(U_p, F^1, F^2, \dots, F^m, x^{m+1}, \dots, x^n\right)$ in the maximal atlas of N. In this chart, the set $U_p \cap S$ is defined by the vanishing of the first m coordinates $F^1, F^2, \dots, F^m$. Because we proved earlier that S is the maximal common zero set of these m coordinate functions $F^1, F^2, \dots, F^m$. So the maximal subset of U_p where all these m coordinates vanish must be $U_p \cap S$. Thus $\left(U_p, F^1, F^2, \dots, F^m, x^{m+1}, \dots, x^n\right)$ is an adapted chart relative to S, and $\left(U_p \cap S, x^{m+1}, \dots, x^n\right)$ is a chart for S containing p. Since p is arbitrary, S is a regular submanifold of dimension $n - m$ by Proposition 7.1.1. ■

Following is a useful lemma that follows from the proof of the regular level set theorem.

Lemma 7.2.4

Let $F : N \to \mathbb{R}^m$ be a C^∞ map on a manifold N of dimension n and let S be the level set $F^{-1}(\{\mathbf{0}\})$. If relative to some coordinate chart $\left(U, x^1, x^2, \dots, x^n\right)$ about $p \in S$, the Jacobian determinant

$$\frac{\partial\left(F^1, F^2, \dots, F^m\right)}{\partial\left(x^{j_1}, x^{j_2}, \dots, x^{j_m}\right)}(p)$$

is nonzero with $j_1, j_2, \dots, j_m \in \{1, 2, \dots, n\}$, then in some neighborhood of p, one may replace $x^{j_1}, x^{j_2}, \dots, x^{j_m}$ by $F^1, F^2, \dots, F^m$ to obtain an adapted chart of N relative to S.

Remark 7.2.2. The regular level set theorem gives a sufficient but not necessary condition for a subset of a manifold to be a regular submanifold — if the subset is a regular level set of some smooth map, then it is a regular submanifold. But there can be a regular submanifold of a manifold that fails to be a regular level set of some smooth map. Here is an example elucidating the fact: take $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = y^2$. This map is C^∞ and the zero set $Z(f)$ is the x-axis, a regular submanifold of $\mathbb{R}^2$. However, both $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ on the x-axis. In other words, every point of the x-axis is a critical point of f. Thus, although $Z(f)$ is a regular submanifold of $\mathbb{R}^2$, it is not a regular level set of f.

An Example of a Regular Submanifold

As a set, the special linear group $\mathrm{SL}(n, \mathbb{R})$ is the subset of $\mathrm{GL}(n, \mathbb{R})$ consisting of $n \times n$ real matrices of determinant 1. The product of two matrices with unit determmninant is again a matrix with unit determinant. Furthermore, the inverse of a matrix with unit determminant is also a matrix with unit determminant. They follow from the following properties of the determinant function:

$$\det(AB) = (\det A)(\det B) \quad \text{and} \quad \det\left(A^{-1}\right) = \frac{1}{\det A}.$$

Hence, $\mathrm{SL}(n, \mathbb{R})$ is a subgroup of $\mathrm{GL}(n, \mathbb{R})$. Now let $f : \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}$ be given by $f(A) = \det A$. Notice that f is a C^∞ map and $\mathrm{SL}(n, \mathbb{R}) = f^{-1}(\{1\})$. We will now check that 1 is a regular value of the C^∞ map f, *i.e.* the matrices in $f^{-1}(\{1\})$ are all regular points of f.

Let a_{ij}, $1 \le i, j \le n$, be the standard coordinates on $\mathbb{R}^{n\times n}$, and let S_{ij} be the submatrix of $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ formed by deleting the i-th row and the j-th column of A. Then denote by $m_{ij}(A) := \det S_{ij}$, the (i, j) minor of A. Then, we have

$$f(A) = \det A = (-1)^{i+1} a_{i1} m_{i1}(A) + (-1)^{i+2} a_{i2} m_{i2}(A) + \cdots + (-1)^{i+n} a_{in} m_{in}(A) .$$

This is obtained by expanding along the i-th row. Therefore,

$$\frac{\partial f}{\partial a_{ij}}(A) = (-1)^{i+j} m_{ij}(A) .$$

By Proposition 6.4.1, a matrix B will be a critical point of f if and only if all the partial derivatives $\frac{\partial f}{\partial a_{ij}}(B)$ vanish. In other words,

$$\frac{\partial f}{\partial a_{ij}}(B) = (-1)^{i+j} m_{ij}(B) = 0 , \qquad \text{for every } i, j .$$

Hence, a matrix $B = [b_{ij}] \in \mathrm{GL}(n, \mathbb{R})$ is a critical point of f if and only if all the $(n-1) \times (n-1)$ minors $m_{ij}(B)$ of B are 0. Then we have,

$$\det B = (-1)^{i+1} b_{i1} m_{i1}(B) + (-1)^{i+2} b_{i2} m_{i2}(B) + \cdots + (-1)^{i+n} b_{in} m_{in}(B) = 0 .$$

Since every matrix in $\mathrm{SL}(n, \mathbb{R})$ has determinant 1, no matrix in $\mathrm{SL}(n, \mathbb{R})$ can be a critical point of f. In other words, all the matrices in $\mathrm{SL}(n, \mathbb{R})$ are regular points of f, and thus $\mathrm{SL}(n, \mathbb{R}) = f^{-1}(\{1\})$ is a regular level set. Then by Theorem 7.2.2, $\mathrm{SL}(n, \mathbb{R})$ is a regular submanifold of $\mathrm{GL}(n, \mathbb{R})$ of codimension 1. In other words,

$$\dim \mathrm{SL}(n, \mathbb{R}) = \dim \mathrm{GL}(n, \mathbb{R}) - 1 = n^2 - 1 .$$

§7.3 Rank of a Smooth Map

Recall from the previous chapter that the rank of a smooth map $F : N \to M$ at $p \in N$ is defined as the rank of its differential at p, *i.e.* the rank of the linear map $F_{*,p} : T_pN \to T_{F(p)}M$. Here, $n = \dim N$ and $m = \dim M$. Now, we will study two situations related to rank of a smooth map:

(i) when F has maximal rank at $p \in N$,

(ii) F has constant rank in a neighborhood of $p \in N$.

If $F : N \to M$ has maximal rank at $p \in N$, then there are 3 mutually not exclusive possibilities:

(a) If $n = m$, then since $F_{*,p} : T_pN \to T_{F(p)}M$ is a linear map with maximal rank, one must have

$$\operatorname{rank} F_{*,p} = n = \dim T_pN .$$

By rank-nullity theorem,

$$\operatorname{rank} F_{*,p} + \operatorname{nullity} F_{*,p} = \dim T_pN = n \implies \operatorname{nullity} F_{*,p} = 0 .$$

Hence, $\operatorname{Ker} F_{*,p} = \{\mathbf{0}\}$, yielding $F_{*,p}$ is non-singular. Since $\dim T_pN = \dim T_{F(p)}M$, and $F_{*,p}$ is non-singular, $F_{*,p}$ is bijective. Therefore, $F_{*,p}$ is a bijective linear transformation, *i.e.* an isomorphism between T_pN and $T_{F(p)}M$. Therefore, by Remark 6.2.1, $F : N \to M$ is locally invertible or a local diffeomorphism at $p \in N$.

(b) If $n \leq m$, then using the fact that $\operatorname{rank} F_{*,p} \leq \min\{m,n\}$, one obtains $\operatorname{rank} F_{*,p} \leq n$. Since $\operatorname{rank} F_{*,p}$ is maximal, $\operatorname{rank} F_{*,p} = n$. By rank-nullity theorem,

$$\operatorname{rank} F_{*,p} + \operatorname{nullity} F_{*,p} = \dim T_p N = n \implies \operatorname{nullity} F_{*,p} = 0\,.$$

In other words, $\operatorname{Ker} F_{*,p} = \{\mathbf{0}\}$, so $F_{*,p} : T_p N \to T_{F(p)} M$ is injective. Therefore, $F : N \to M$ is an immersion at $p \in N$.

(c) If $n \geq m$, then again using $\operatorname{rank} F_{*,p} \leq \min\{m,n\}$, one obtains $\operatorname{rank} F_{*,p} \leq m$. Since $\operatorname{rank} F_{*,p}$ is maximal, $\operatorname{rank} F_{*,p} = m$. In other words, $\operatorname{im} F_{*,p}$ is a m-dimensional vector subspace of $T_{F(p)} M$, so $\operatorname{im} F_{*,p}$ must be $T_{F(p)} M$ itself. Hence, $F_{*,p} : T_p N \to T_{F(p)} M$ is surjective. Therefore, $F : N \to M$ is a submersion at $p \in N$.

Lemma 7.3.1

Suppose $T : V \to W$ is a linear transformation between finite dimensional vector spaces V and W. Let $I_W : W \to X$ and $I_V : V \to Y$ be vector space isomorphisms. Then $\operatorname{rank}(I_W \circ T) = \operatorname{rank} T$ and $\operatorname{rank}\left(T \circ I_V^{-1}\right) = \operatorname{rank} T$.

Proof. $I_W \circ T : V \to X$. $T(V)$ is a vector subspace of W. Then restricting I_W on $T(V)$ gives us an isomorphism $I_W\big|_{T(V)} : T(V) \to I_W(T(V))$. Therefore,

$$\operatorname{rank}(I_W \circ T) = \dim(I_W(T(V))) = \dim T(V) = \operatorname{rank} T\,.$$

Furthermore, $T \circ I_V^{-1} : X \to W$. Since $I_V^{-1} : Y \to V$ is a vector space isomorphism, $I_V^{-1}(Y) = V$. Therefore,

$$\operatorname{rank}\left(T \circ I_V^{-1}\right) = \dim\left(T\left(I_V^{-1}(Y)\right)\right) = \dim T(V) = \operatorname{rank} T\,.$$

■

Theorem 7.3.2 (Constant Rank Theorem for Manifolds)

Let N and M be manifolds of dimension n and m, respectively. Suppose $F : N \to M$ has constant rank k in a neighborhood of a point $p \in N$. Then there are charts (U, φ) centered at $p \in N$ ($\varphi(p) = \mathbf{0} \in \mathbb{R}^n$) and (V, ψ) centered at $F(p) \in M$ ($\psi(F(p)) = \mathbf{0} \in \mathbb{R}^m$) such that for $\left(r^1, r^2, \ldots, r^n\right) \in \varphi(U)$,

$$\left(\psi \circ F \circ \varphi^{-1}\right)\left(r^1, r^2, \ldots, r^n\right) = (r^1, r^2, \ldots, r^k, \underbrace{0, 0, \ldots, 0}_{(m-k) \text{ many } 0\text{s}})\,.$$

Remark 7.3.1. Same as Remark 2.6.1, we can add $F(U) \subseteq V$ in the statement of Constant Rank Theorem for Manifolds, because otherwise we can always find a smaller U such that $F(U)$ is contained in V. Here, too, the notation $\psi \circ F \circ \varphi^{-1}$ is a bit sloppy. What this composition actually means is the following:

$$\psi\big|_{F(U)} \circ F\big|_U \circ \varphi^{-1} : \varphi(U) \to \psi(F(U))\,.$$

Proof of Constant Rank Theorem for Manifolds. Choose a chart $\left(\overline{U}, \overline{\varphi}\right)$ about $p \in N$ and $\left(\overline{V}, \overline{\psi}\right)$ about $F(p) \in M$ with $F\left(\overline{U}\right) \subseteq \overline{V}$. Then $\overline{\psi} \circ F \circ \overline{\varphi}^{-1} : \overline{\varphi}\left(\overline{U}\right) \subseteq \mathbb{R}^n \to \overline{\psi}\left(\overline{V}\right) \subseteq \mathbb{R}^m$ is a map between open subsets of Euclidean spaces. Because $\overline{\psi}$ and $\overline{\varphi}^{-1}$ are diffeomorphisms onto the respective images, the pertaining differentials are isomorphisms by Corollary 6.1.2. In other words, $\overline{\psi}_{*,F(p)}$ and $\left(\overline{\varphi}^{-1}\right)_{*,\overline{\varphi}(p)}$ are both vector space isomorphisms. Also, by The Chain Rule,

$$\left(\overline{\psi} \circ F \circ \overline{\varphi}^{-1}\right)_{*,\overline{\varphi}(p)} = \overline{\psi}_{*,F(p)} \circ F_{*,p} \circ \left(\overline{\varphi}^{-1}\right)_{*,\overline{\varphi}(p)}\,.$$

By Lemma 7.3.1, composition with isomorphism does not change the rank of a linear map. Therefore,

$$\operatorname{rank}\left(\overline{\psi} \circ F \circ \overline{\varphi}^{-1}\right)_{*,\overline{\varphi}(p)} = \operatorname{rank} F_{*,p}\,.$$

rank $F_{*,p}$ is the rank of the smooth map $F : N \to M$ at $p \in N$, which is constant k in a neighborhood of p. Therefore, the map $\overline{\psi} \circ F \circ \overline{\varphi}^{-1}$ between open subsets of Euclidean spaces also has constant rank k in a neighborhood of $\overline{\varphi}(p) \in \mathbb{R}^n$. By Constant Rank Theorem for Euclidean Spaces, there are a diffeomorphism G of a neighborhood of $\overline{\varphi}(p) \in \mathbb{R}^n$ and a diffeomorphism H of a neighborhood of $\overline{\psi}(F(p)) \in \mathbb{R}^m$ such that

$$\left(H \circ \overline{\psi} \circ F \circ \overline{\varphi}^{-1} \circ G^{-1}\right)\left(r^1, r^2, \ldots, r^n\right) = (r^1, r^2, \ldots, r^k, \underbrace{0, 0, \ldots, 0}_{(m-k) \text{ many } 0\text{s}}).$$

Setting $H \circ \overline{\psi} = \psi$ and $G \circ \overline{\varphi} = \varphi$, we obtain

$$\left(\psi \circ F \circ \varphi^{-1}\right)\left(r^1, r^2, \ldots, r^n\right) = (r^1, r^2, \ldots, r^k, \underbrace{0, 0, \ldots, 0}_{(m-k) \text{ many } 0\text{s}}).$$

Furthermore, Constant Rank Theorem for Euclidean Spaces guarantees that G sends $\overline{\varphi}(p)$ to $\mathbf{0} \in \mathbb{R}^n$ and H sends $\overline{\psi}(F(p))$ to $\mathbf{0} \in \mathbb{R}^m$. Therefore,

$$\varphi(p) = G(\overline{\varphi}(p)) = \mathbf{0} \in \mathbb{R}^n \quad \text{and} \quad \psi(F(p)) = H\left(\overline{\psi}(F(p))\right) = \mathbf{0} \in \mathbb{R}^m.$$

■

By a neighborhood of a subset A of a manifold M we mean an open set containing A.

Theorem 7.3.3 (Constant-Rank Level Set Theorem)
Let $f : N \to M$ be a C^∞ map of manifolds and $c \in M$. If f has constant rank k in a neighborhood of the level set $f^{-1}(\{c\}) \subseteq N$, then $f^{-1}(\{c\})$ is a regular submanifold of N of codimension k.

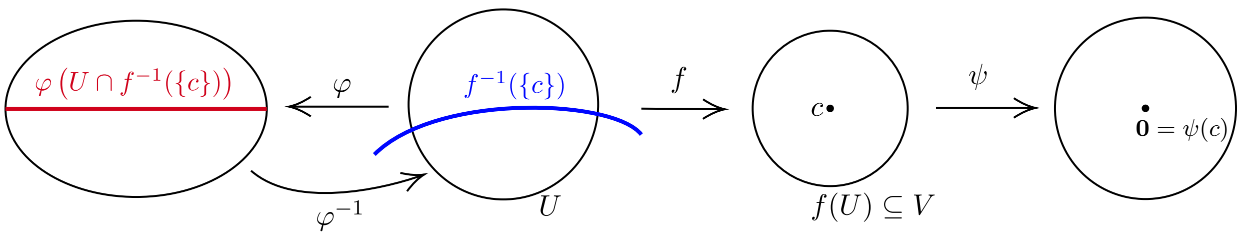
Proof. Let $p \in f^{-1}(\{c\})$. By hypothesis, there is a neighborhood p of N where f has constant rank. By Constant Rank Theorem for Manifolds, there are a coordinate chart $(U, \varphi) = \left(U, x^1, x^2, \ldots, x^n\right)$ centered at $p \in N$ $(\varphi(p) = \mathbf{0} \in \mathbb{R}^n)$ and a coordinate chart $(V, \psi) = \left(V, y^1, y^2, \ldots, y^m\right)$ centered at $f(p) = c \in N$ $(\psi(c) = \mathbf{0} \in \mathbb{R}^m)$ such that

$$\left(\psi\big|_{f(U)} \circ f\big|_U \circ \varphi^{-1}\right)\left(r^1, r^2, \ldots, r^n\right) = (r^1, r^2, \ldots, r^k, \underbrace{0, 0, \ldots, 0}_{(m-k) \text{ many } 0\text{s}}).$$

Let $\psi\big|_{f(U)} \circ f\big|_U \circ \varphi^{-1} = \widehat{f}$. Then we have,

$$\widehat{f}\left(r^1, r^2, \ldots, r^n\right) = \left(r^1, r^2, \ldots, r^k, 0, 0, \ldots, 0\right) = \mathbf{0} \iff r^1 = r^2 = \cdots = r^k = 0.$$

Hence, $\widehat{f}^{-1}(\{\mathbf{0}\})$ is defined by the vanishing of $r^1, r^2, \ldots, r^k$.



Now, observe that

$$\begin{aligned}\varphi\left(U \cap f^{-1}(\{c\})\right) &= \varphi\left(f\big|_U^{-1}(\{c\})\right) = \varphi\left(f\big|_U^{-1}\left(\psi\big|_{f(U)}^{-1}(\{\mathbf{0}\})\right)\right) \\ &= \left(\psi\big|_{f(U)} \circ f\big|_U \circ \varphi^{-1}\right)^{-1}(\{\mathbf{0}\}) = \widehat{f}^{-1}(\{\mathbf{0}\}).\end{aligned}$$

Hence, the image of the set $U \cap f^{-1}(\{c\})$ under φ is the level set $\widehat{f}^{-1}(\{\mathbf{0}\})$. One, therefore, obtains that a generic element of $\varphi\left(U \cap f^{-1}(\{c\})\right)$ has the first k coordinates vanishing.

Recall that $(U,\varphi) = \left(U, x^1, x^2, \ldots, x^n\right)$ is a chart in N centered at $p \in N$, where $x^i = r^i \circ \varphi$ with $i \in \{1, 2, \ldots, n\}$. Since $r^1, r^2, \ldots, r^k$ vanishes on $\varphi\left(U \cap f^{-1}(\{c\})\right)$, on $U \cap f^{-1}(\{c\})$, φ is given by the vanishing of the first k coordinates. In other words,

$$x^1 = r^1 \circ \varphi = 0\,,\ x^2 = r^2 \circ \varphi = 0\,,\ \ldots\,,\ x^k = r^k \circ \varphi = 0\,.$$

On the other hand, if $q \in U \setminus f^{-1}(\{c\})$, then since $\varphi\left(U \cap f^{-1}(\{c\})\right) = \widehat{f}^{-1}(\{\mathbf{0}\})$, $\varphi(q) \notin \widehat{f}^{-1}(\{\mathbf{0}\})$. Hence, not all the first k coordinate functions of $\varphi(q)$ are vanishing. Hence, the maximal subset of U where all the first k coordinates are vanishing is $U \cap f^{-1}(\{c\})$. Therefore, U is an adapted chart of N relative to $f^{-1}(\{c\})$ containing p. Since $p \in f^{-1}(\{c\})$ was an arbitrary point, $f^{-1}(\{c\})$ is a regular submanifold of N of codimension k. ■

Example 7.3.1. The orthogonal group $\mathrm{O}(n)$ is defined to be the subgroup of $\mathrm{GL}(n,\mathbb{R})$ consisting of matrices A with $A^T A = AA^T = I$, where I is the $n \times n$ identity matrix. Using the Constant-Rank Level Set Theorem, prove that $\mathrm{O}(n)$ is a regular submanifold of $\mathrm{GL}(n,\mathbb{R})$.

Solution. Define $f : \mathrm{GL}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R})$ by $f(A) = A^T A$. Then $\mathrm{O}(n) = f^{-1}(\{I\})$.

For $A, B \in \mathrm{GL}(n,\mathbb{R})$, there exists a unique matrix $C \in \mathrm{GL}(n,\mathbb{R})$ such that $B = AC$. Denote by $l_C, r_C : \mathrm{GL}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R})$, the left and right multiplication by C, respectively:

$$l_C(B) = CB \quad \text{and} \quad r_C(B) = BC\,.$$

Now, since $f(AC) = (AC)^T AC = C^T A^T AC = C^T f(A)\, C$, one has

$$(f \circ r_C)(A) = f(AC) = C^T f(A)\, C\,.$$

On the other hand,

$$(l_{C^T} \circ r_C \circ f)(A) = (l_{C^T} \circ r_C)(f(A))\, C^T f(A)\, C\,.$$

Therefore, $(f \circ r_C)(A) = (l_{C^T} \circ r_C \circ f)(A)$. Since this is true for every $A \in \mathrm{GL}(n,\mathbb{R})$,

$$f \circ r_C = l_{C^T} \circ r_C \circ f\,.$$

Now, by The Chain Rule,

$$(f \circ r_C)_{*,A} = (l_{C^T} \circ r_C \circ f)_{*,A} \implies f_{*,AC} \circ (r_C)_{*,A} = (l_{C^T})_{*,A^TAC} \circ (r_C)_{*,A^TA} \circ f_{*,A}\,.$$

Since left and right multiplications are local diffeomorphisms, the pertaining differentials are isomorphisms by Remark 6.2.1. Composition with isomorphism does not change the rank of a linear map (Lemma 7.3.1). Therefore,

$$\operatorname{rank} f_{*,A} = \operatorname{rank} f_{*,AC} = \operatorname{rank} f_{*,B}\,,$$

since $AC = B$. Since A and B are two arbitrary points of $\mathrm{GL}(n,\mathbb{R})$, we can conclude that f has constant rank on $\mathrm{GL}(n,\mathbb{R})$. Then by Constant-Rank Level Set Theorem, the level set $f^{-1}(\{I\}) = \mathrm{O}(n)$ is a regular submanifold of $\mathrm{GL}(n,\mathbb{R})$. ■

§7.4 The Immersion and Submersion Theorems

Consider a C^∞ map $f : N \to M$. Let $(U,\varphi) = \left(U, x^1, x^2, \ldots, x^n\right)$ be a chart about $p \in N$ and $(V,\psi) = \left(V, y^1, y^2, \ldots, y^m\right)$ be a chart about $f(p) \in M$. Write $f^i = y^i \circ f$. By Proposition 6.2.5, relative to the charts (U,φ) and (V,ψ), *i.e.*, relative to the basis $\left\{ \frac{\partial}{\partial x^i}\Big|_p \right\}_{i=1}^n$ of T_pN and the basis $\left\{ \frac{\partial}{\partial y^j}\Big|_{f(p)} \right\}_{j=1}^m$ of $T_{f(p)}M$, the linear map $f_{*,p} : T_pN \to T_{f(p)}M$ is represented by the matrix

$$\left[\frac{\partial f^i}{\partial x^j}(p)\right]_{1 \le i \le m; 1 \le j \le n}\,.$$

Hence,

$$
\begin{aligned}
f_{*,p} \text{ is injective} &\iff n \le m \text{ and } \operatorname{rank}\left[\frac{\partial f^i}{\partial x^j}(p)\right] = n\,, \\
f_{*,p} \text{ is surjective} &\iff m \le n \text{ and } \operatorname{rank}\left[\frac{\partial f^i}{\partial x^j}(p)\right] = m\,.
\end{aligned}
$$

The rank of a matrix is the number of linearly independent rows/columns of the matrix. Since the matrix $\left[\frac{\partial f^i}{\partial x^j}(p)\right]$ is of size $m \times n$, $\operatorname{rank}\left[\frac{\partial f^i}{\partial x^j}(p)\right] \le \min\{m, n\}$. Therefore, one finds that whenever $f : N \to M$ is an immersion or a submersion at $p \in N$, the matrix $\left[\frac{\partial f^i}{\partial x^j}(p)\right]$ is of maximal rank.

Having maximal rank at a point is an *open condition* in the following sense: the set

$$
D_{\max}(f) := \{p \in U \mid f_{*,p} \text{ has maximal rank at } p\}
$$

is an open subset of U.

Claim — $D_{\max}(f)$ is an open subset of U.

Proof. Suppose k is the maximal rank of f. Then

$$
\begin{aligned}
\operatorname{rank} f_{*,p} = k &\iff \operatorname{rank}\left[\frac{\partial f^i}{\partial x^j}(p)\right] = k \\
&\iff \operatorname{rank}\left[\frac{\partial f^i}{\partial x^j}(p)\right] \ge k
\end{aligned}
$$

The last $\Rightarrow$ is obvious and $\Leftarrow$ holds since k is the maximal rank, so

$$
\operatorname{rank}\left[\frac{\partial f^i}{\partial x^j}(p)\right] \ge k \implies k \ge \operatorname{rank}\left[\frac{\partial f^i}{\partial x^j}(p)\right] \ge k \implies \operatorname{rank}\left[\frac{\partial f^i}{\partial x^j}(p)\right] = k\,.
$$

Hence, the complement $U \setminus D_{\max}(f)$ is defined by

$$
U \setminus D_{\max}(f) = \left\{p \in U \mid \operatorname{rank}\left[\frac{\partial f^i}{\partial x^j}(p)\right] < k\right\}.
$$

By Lemma 5.2.1, $\operatorname{rank}\left[\frac{\partial f^i}{\partial x^j}(p)\right] < k$ is equivalent to the vanishing of **all** $k \times k$ minors of $\left[\frac{\partial f^i}{\partial x^j}(p)\right]$.

Now we shall use the fact that the common zero set of finitely many continuous functions is closed. In other words, if $f_1, f_2, \ldots, f_n : X \to \mathbb{R}$ are continuous functions, then

$$
\bigcap_{i=1}^{n} f_i^{-1}(\{0\})
$$

is closed in X. This is because each $f_i^{-1}(\{0\})$ is closed in X, and intersection of finitely many closed sets is also closed.

Now, $U \setminus D_{\max}(f)$ is the collection of all $p \in U$ such that all the determinant functions on $k \times k$ submatrices of $\left[\frac{\partial f^i}{\partial x^j}(p)\right]$ vanish simultaneously. So we are looking at the common zero set of all the determinant functions on all $k \times k$ submatrices of $\left[\frac{\partial f^i}{\partial x^j}(p)\right]$. Since there are only finitely many $k \times k$ submatrices, we can conclude that $U \setminus D_{\max}(f)$ is closed in U. Therefore, $D_{\max}(f)$ is open in U. $\square$

In particular, if f has maximal rank at p, it has maximal rank at all points in some neighborhood of p, which is denoted by $D_{\max}(f)$ here. We summarize all these results formally by means of the following proposition.

Proposition 7.4.1

Let N and M be manifolds of dimension n and m, respectively. If a C^∞ map $f : N \to M$ is an immersion at a point $p \in N$, then it has a constant rank n in a neighborhood of p (in this case $m \geq n$). If a C^∞ map $f : N \to M$ is a submersion at $p \in N$ (in which case $n \geq m$), then it has a constant rank m in a neighborhood of p.

Example 7.4.1

Although maximal rank at a point implies constant rank in a neighborhood, the converse is not true. The map $f : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $f(x, y) = (x, 0, 0)$ has Jacobian matrix

$$J = \begin{bmatrix} \frac{\partial f^1}{\partial x} & \frac{\partial f^1}{\partial y} \\ \frac{\partial f^2}{\partial x} & \frac{\partial f^2}{\partial y} \\ \frac{\partial f^3}{\partial x} & \frac{\partial f^3}{\partial y} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then J has rank 1 everywhere in $\mathbb{R}^2$. But this is not maximal since $\min\{2, 3\} = 2 \neq \operatorname{rank} J$.

Proposition 7.4.1 and Constant Rank Theorem for Manifolds imply the following theorem.

Theorem 7.4.2

Let N and M be manifolds of dimension n and m, respectively.

(i) **(Immersion theorem)** Suppose $f : N \to M$ is an immersion at $p \in N$ (then $n \leq m$). Then there are charts (U, φ) centered at $p \in N$ and (V, ψ) centered at $f(p) \in M$ such that in a neighborhood of $\varphi(p) = \mathbf{0} \in \mathbb{R}^n$,

$$\left(\psi \circ f \circ \varphi^{-1}\right)\left(r^1, r^2, \ldots, r^n\right) = (r^1, r^2, \ldots, r^n, \underbrace{0, 0, \ldots, 0}_{(m-n)\ 0\text{s}}).$$

(ii) **(Submersion theorem)** Suppose $f : N \to M$ is a submersion at $p \in N$ (then $n \geq m$). Then there are charts (U, φ) centered at $p \in N$ and (V, ψ) centered at $f(p) \in M$ such that in a neighborhood of $\varphi(p) = \mathbf{0} \in \mathbb{R}^n$,

$$\left(\psi \circ f \circ \varphi^{-1}\right)\left(r^1, r^2, \ldots, r^m, r^{m+1}, \ldots, r^n\right) = \left(r^1, r^2, \ldots, r^m\right).$$

Corollary 7.4.3

A submersion $f : N \to M$ is an open map.

Proof. Let $W \subseteq N$ be open and $p \in W$ so that $f(p) \in f(W)$. Now, $f : N \to M$ is a submersion at p by hypothesis. Then by submersion theorem, there are charts (U, φ) centered at $p \in W \subseteq N$ and (V, ψ) centered at $f(p) \in f(W) \subseteq M$ such that in a neighborhood of $\varphi(p) = \mathbf{0} \in \mathbb{R}^n$, one has

$$\left(\psi \circ f \circ \varphi^{-1}\right)\left(r^1, r^2, \ldots, r^m, r^{m+1}, \ldots, r^n\right) = \left(r^1, r^2, \ldots, r^m\right).$$

We can take U small enough such that $U \subseteq W$.

Now, consider the subset $f(W) \subseteq M$. Any point in $f(W)$ is of the form $f(p)$ with $p \in W$. But from the previous argument, it follows that for $p \in W \subseteq N$ with W open, one can find an open set $U \subseteq W$ with respect to which f acts as a local projection. In other words,

$$\widehat{f} = \psi\big|_{f(U)} \circ f\big|_U \circ \varphi^{-1} : \varphi(U) \subseteq \mathbb{R}^n \to \psi(V) \subseteq \mathbb{R}^m$$

is projection onto the first m coordinates. By Proposition 1.6.5, projection is an open map. Therefore,

$$\widehat{f}(\varphi(U)) = \psi|_{f(U)}(f(U)) = \psi(f(U))$$

is open in $\psi(V) \subseteq \mathbb{R}^m$. Since ψ is a diffeomorphism, $\psi^{-1} : \psi(V) \subseteq \mathbb{R}^m \to V$ is an open map. Therefore,

$$\psi^{-1}(\psi(f(U))) = f(U)$$

is open in V. Since V is open in M, $f(U)$ is open in M. Hence, for every $f(p) \in f(W) \subseteq M$, one can find $f(U)$ open in M such that

$$f(p) \in f(U) \subseteq f(W) \; .$$

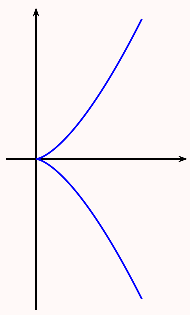
This proves that $f(W)$ is open in M. ■

Example 7.4.2

Let $f : \mathbb{R} \to \mathbb{R}^2$ be defined by $f(t) = (t^2, t^3)$. Observe that

$$f(t_1) = f(t_2) \iff (t_1^2, t_1^3) = (t_2^2, t_2^3) \iff t_1 = t_2 \, .$$

The equality of the second component $t_1^3 = t_2^3$ forces $t_1 = t_2$, although $t_1^2 = t_2^2$ has 2 solutions $t_1 = \pm t_2$. So, the injectivity of the function t^3 forces the injectivity of $f(t) = (t^2, t^3)$. This f is represented by a cuspidal cubic.



We've seen in Example 6.3.1 that

$$f_{*,t_0}\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0}\right) = 2t_0 \left.\frac{\partial}{\partial x}\right|_{f(t_0)} + 3t_0^2 \left.\frac{\partial}{\partial y}\right|_{f(t_0)} \, .$$

Therefore, for $t_0 = 0$, we find that the differential $f_{*,0} : T_0\mathbb{R} \to T_{(0,0)}\mathbb{R}^2$ is the zero map, and hence it's not injective. Therefore, despite f being an injective map, it's not an immersion at 0.

8 Immersed vs Regular Submanifold

§8.1 Embedding

Example 8.1.1

Consider the smooth map $f : \mathbb{R} \to \mathbb{R}^2$ defined by $f(t) = \left(t^2 - 1, t^3 - t\right)$. This map is not injective as $f(1) = f(-1) = (0,0)$. The matrix representation of f_{*,t_0} with respect to the standard coordinate of $\mathbb{R}$ and $\mathbb{R}^2$ is

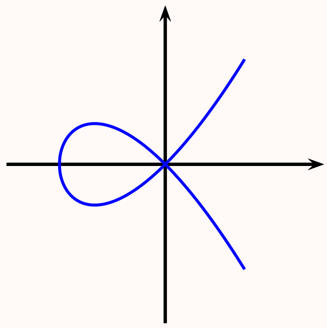
$$\begin{bmatrix} 2t_0 \\ 3t_0^2 - 1 \end{bmatrix} .$$

There is no t_0 such that $2t_0 = 3t_0^2 - 1 = 0$. So, $\operatorname{rank} f_{*,t_0} = 1$ for every t_0. Therefore, f is an immersion, but it's not injective.

To find an equation for the image $f(\mathbb{R})$, let $x = t^2 - 1$ and $y = t^3 - t$. Then

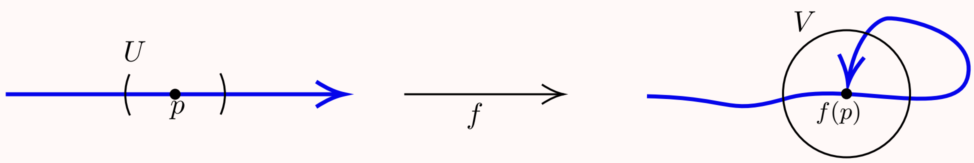
$$y = t\left(t^2 - 1\right) = tx \implies y^2 = t^2x^2 = x^2(x+1) .$$

Thus, the image of f is the nodal cubic $y^2 = x^2(x+1)$.



Example 8.1.2

The map f shown in the following figure is an injective immersion but it's image, with respect to the subspace topology inherited from $\mathbb{R}^2$ is not homeomorphic to the domain $\mathbb{R}$, because there are points near $f(p)$ in the image that corresponds to points in $\mathbb{R}$ far away from p. Let us try to see it mathematically.



By Lemma 1.5.3, f^{-1} is continuous if and only if for every $f(p) \in f(\mathbb{R})$ and for every neighborhood U of $f^{-1}(f(p)) = p$, there exists a neighborhood V of $f(p)$ such that $f^{-1}(V) \subseteq U$. If we choose U to be an interval around p, then there is no neighborhood $V \subseteq \mathbb{R}^2$ of $f(p)$ such that $f^{-1}(V) \subseteq U$. Because, no matter how small V is, it will contain points whose image under f^{-1} will be far away from p, let alone be contained in U. Therefore, f^{-1} is not continuous. In other words, $f : \mathbb{R} \to f(\mathbb{R})$ is not a homeomorphism.

Definition 8.1.1 (Embedding)**.** A C^∞ map $f : N \to M$ is called an **embedding** if

(i) it is an injective immersion, and

(ii) the image $f(N)$ with respect to the subspace topology is homeomorphic to N under f.

Remark 8.1.1. Here $f : N \to M$ is injective. In fact, the injectivity condition is redundant as the condition that $f : N \to f(N)$ is a homeomorphism already demands that f is injective.

One can equip the image $f(N)$ of N under f with not the subspace topology inherited from M, but the topology inherited from f. That is, a subset $f(U)$ of $f(N)$ is said to be open if and only if U is open in N. With this topology, $f : N \to f(N)$ is a homeomorphism.

Let $f(U) \subseteq f(N)$ be open. Then by the definition of the topology inherited from f, $U = f^{-1}(f(U))$ is open in N. Therefore, f is continuous. Now, consider $f^{-1} : f(N) \to N$. Let $U \subseteq N$ be open. Then $f(U) = \left(f^{-1}\right)^{-1}(U)$ is open. Therefore, f^{-1} is also continuous. Hence, with respect to the topology inherited by f, $f : N \to f(N)$ is a homeomorphism.

The image $f(N)$ of an injective immersion is called an **immersed submanifold**. The topology of an immersed submanifold is the one inherited from f. If the underlying set of an immersed submanifold is given the subspace topology, then the resulting space need not be a manifold at all. Contrary to an immersed submanifold, a regular submanifold of a manifold M us a subset S of M with the subspace topology such that every point of S has a neighborhood $U \cap S$ that is defined by the vanishing of the coordinate functions on U, where U is a chart in M.

Example 8.1.3. The figure eight is the image of the injective immersion

$$f : \left(-\frac{\pi}{2}, 3\frac{\pi}{2}\right) \to \mathbb{R}^2 \ , \ f(t) = (\cos t, \sin 2t) \ .$$

It is easily seen to be injective, since

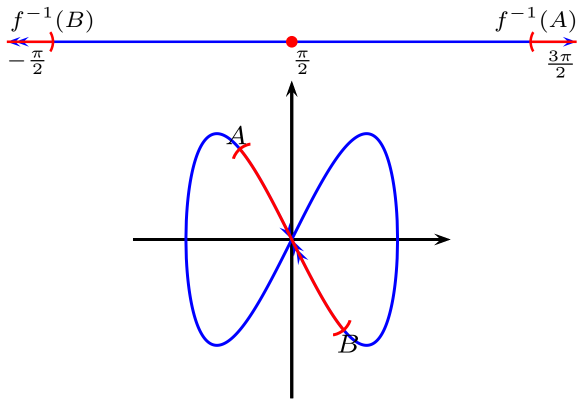
$$f(t_1) = f(t_2) \implies \cos t_1 = \cos t_2 \text{ and } \sin 2t_1 = \sin 2t_2 \implies \sin t_1 = \sin t_2 \, .$$

Therefore, either $\cos t_1 = \cos t_2 = 0$ or $t_1 - t_2 = 2n\pi$ for some integer n. There are only one $t \in \left(-\frac{\pi}{2}, 3\frac{\pi}{2}\right)$ with $\cos t = 0$. And, $t_1 - t_2 = 2n\pi$ is also not possible since the length of the interval is 2π, and the endpoints are not included. So f is injective.

To see that f is an immersion, the matrix representation of f_{*,t_0} is

$$\begin{bmatrix} -\sin t_0 \\ 2\cos 2t_0 \end{bmatrix} .$$

$\cos 2_{t_0} = 1 - 2\sin^2 t_0$, so $-\sin t_0$ and $2\cos 2t_0$ can't both be 0 simultaneously. Therefore, rank $f_{*,t_0} = 1$ for every t_0. So, f is an immersion.



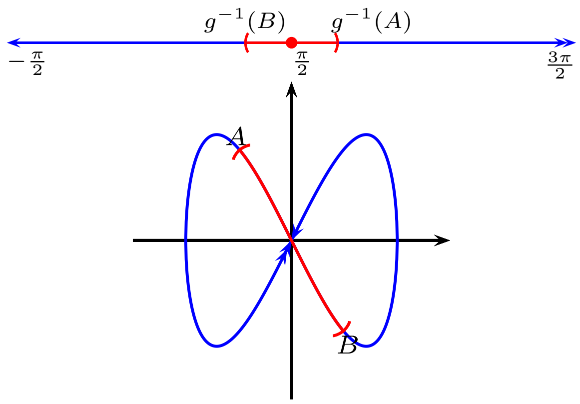
The set denoted by the segment AB in the x-y plane is **not** an open set relative to the topology inherited from f, as

$$f^{-1}(AB) = \left(-\frac{\pi}{2}, f^{-1}(B)\right) \cup \left(f^{-1}(A), 3\frac{\pi}{2}\right) \cup \left\{\frac{\pi}{2}\right\} .$$

$f^{-1}(AB)$ contains an isolated point and hence it's not open.

Figure eight is also the image of the injective immersion

$$g : \left(-\frac{\pi}{2}, 3\frac{\pi}{2}\right) \to \mathbb{R}^2 \ , \ g(t) = (\cos t, -\sin 2t) \ .$$



For this injective immersion, $g^{-1}(AB) = \left(g^{-1}(B), g^{-1}(A)\right)$, which is open in $\left(-\frac{\pi}{2}, 3\frac{\pi}{2}\right)$.

Hence, these two injective immersions f and g are distinct. The segment AB is not open in the topology inherited from f, while it is open in the topology inherited from g.

Why is the image of figure 8 not a manifold in subspace topology?

Let S be the image of figure 8. Assume the contrary that S is a manifold in subspace topology. Take $(0,0) \in S$. Since S is a manifold, there is an open set (in the subspace topology) U around $(0,0)$ such that U is homeomorphic to an open ball $B(\mathbf{a}, \varepsilon)$ of radius ε centered at $\mathbf{a} \in \mathbb{R}^n$. Denote this homeomorphism by φ, *i.e.* $\varphi(U) = B(\mathbf{a}, \varepsilon)$. Then $U \setminus \{(0,0)\}$ is homeomorphic to $B(\mathbf{a}, \varepsilon) \setminus \{\varphi(0,0)\}$. Therefore, they must have the same number of connected components. But $U \setminus \{(0,0)\}$ has 4 connected components, whereas $B(\mathbf{a}, \varepsilon) \setminus \{\varphi(0,0)\}$ has 1 or 2 connected components (depending on the dimension n[1]). Thus we arrive at a contradiction!

Theorem 8.1.1

If $f : N \to M$ is an embedding, then its image $f(N)$ is a regular submanifold of M.

Proof. Let $p \in N$. By the immersion theorem (Theorem 7.4.2), there are charts (U, φ) centered at $p \in N$ and (V, ψ) centered at $f(p) \in M$ such that in **any** neighborhood of $\varphi(p) = \mathbf{0} \in \mathbb{R}^n$ contained in $\varphi(U)$,

$$\left(\psi \circ f \circ \varphi^{-1}\right)\left(r^1, r^2, \ldots, r^n\right) = \left(r^1, r^2, \ldots, r^n, 0, \ldots, 0\right) \ .$$

In fact, this is an abuse of notation as discussed in the previous chapter:

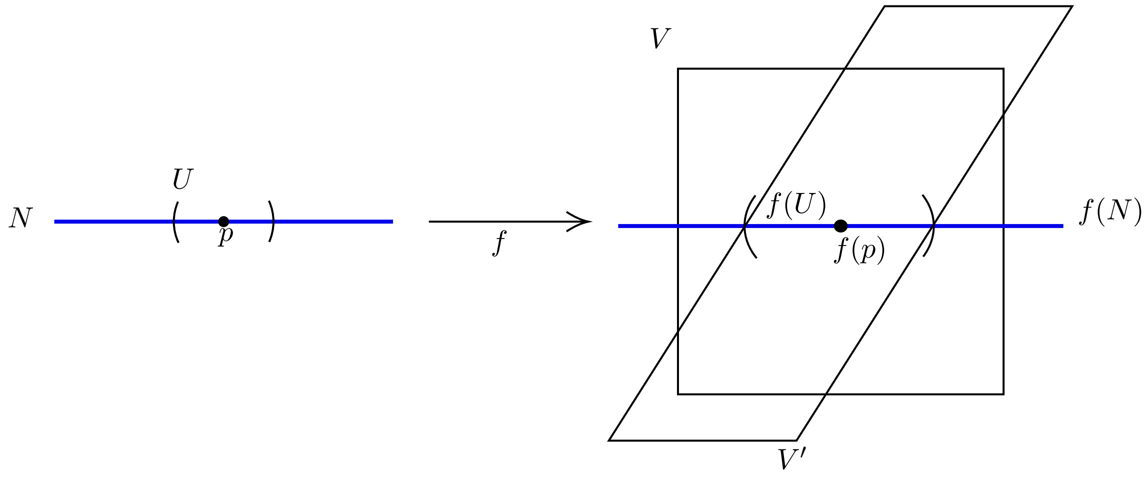
$$\psi\big|_{f(U)} \circ f\big|_U \circ \varphi^{-1}\left(r^1, r^2, \ldots, r^n\right) = \left(r^1, r^2, \ldots, r^n, 0, \ldots, 0\right) \ .$$

So, evidently, we want $f(U) \subseteq V$. This means any element of $\psi(f(U))$ is of the form $\left(r^1, r^2, \ldots, r^n, 0, \ldots, 0\right)$. Thus, $f(U)$ is defined in V by the vanishing of the last $m-n$ coordinate functions:

$$r^{n+1} \circ \psi = y^{n+1} = 0 \, , \ldots \, , r^m \circ \psi = y^m = 0 \, .$$

Now, $V \cap f(N)$ can be larger than $f(U)$, and on $f(U)$ only the last $m-n$ coordinate functions vanish identically. It might be the case that on $V \cap f(N)$, the last $m-n$ corrdinate functions don't vanish, and hence V can't be an adapted chart about $f(p)$ relative to $f(N)$.

[1]If $n = 1$, the number of connected components is 2. For $n \geq 2$, it is 1.



We need to show that in some neighborhood of $f(p)$ in V, the set $f(N)$ is defined by the vanishing of the last $m-n$ coordinate functions.

Since $f : N \to M$ is an embedding, N is homeomorphic to $f(N)$ in the subspace topology. Hence, given an open subset U of N, $f(U)$ is open in $f(N)$. By the definition of subspace topology, there is an open set V' in M such that $f(U) = V' \cap f(N)$. Now,

$$V \cap V' \cap f(N) = V \cap f(U) = f(U) \ ,$$

and $f(U)$ is defined by the vanishing of the last $m-n$ coordinate functions on V. Therefore, $V \cap V'$ is a neighborhood of $f(p)$ in M such that $f(N)$ is defined by the vanishing of the last $m-n$ coordinate functions on $(V \cap V') \cap f(N)$. Thus,

$$\left(V \cap V', \psi|_{V \cap V'}\right) = \left(V \cap V', y^1, y^2, \ldots, y^m\right)$$

is an adapted chart containing $f(p)$ relative to $f(N)$. Since $f(p)$ is an arbitrary point of $f(N)$, this proves that $f(N)$ is a regular submanifold. ■

Theorem 8.1.2

If N is a regular submanifold of M, then the inclusion $i : N \hookrightarrow M$, $i(p) = p$ is an embedding.

Proof. Note that $i : N \to i(N) = N$ is the identity map, so it is a homeomorphism since both the domain and codomain space N are equipped with the same topology, *i.e.* subspace topology inherited from M. Furthermore, $N \subseteq M$ implies that $\dim N \leq \dim M$. So, in order to show that i is an embedding, it suffices to show that $i_{*,p}$ is of rank $n = \dim N$ for every $p \in N$.

Since N is a regular submanifold of M, choose an adapted chart $\left(V, y^1, y^2, \ldots, y^n, y^{n+1}, \ldots, y^m\right)$ for M about p relative to N such that $V \cap N$ is the zero set of the last $m-n$ coordinate functions $y^{n+1}, \ldots, y^m$. Hence, $\left(V \cap N, y^1, y^2, \ldots, y^n\right)$ is going to be a chart of the manifold N about p. Relative to these two charts, $i : N \hookrightarrow M$ is given by

$$\left(y^1, y^2, \ldots, y^n\right) \mapsto \left(y^1, y^2, \ldots, y^n, 0, \ldots, 0\right) \ .$$

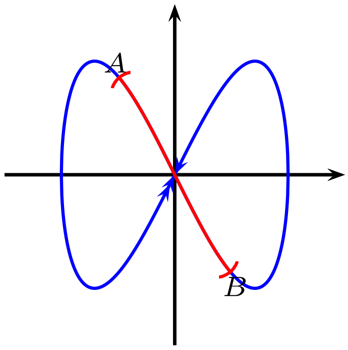
The corresponding differential $i_{*,p} : T_pN \to T_pM$ is represented by the following $m \times n$ matrix relative to the abovementioned charts:

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} I_{n \times n} \\ 0_{m \times n} \end{bmatrix} .$$

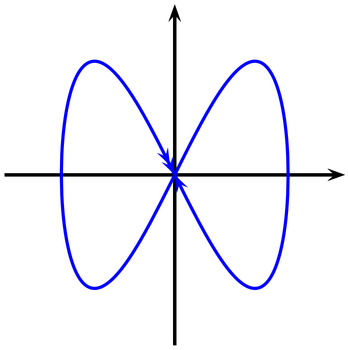
Hence, $\operatorname{rank} i_{*,p} = n$. This is true for every p. Hence, $i : N \to M$ is an immersion. We have already proved that $i : N \to i(N) = N$ is a homeomorphism. Therefore, i is an embedding. ■

§8.2 Smooth Maps into a Submanifold

We shall start with an observation. Consider the injective immersion $g : \left(-\frac{\pi}{2}, 3\frac{\pi}{2}\right) \to \mathbb{R}^2$ given by $g(t) = (\cos t, -\sin 2t)$. The image of g in $\mathbb{R}^2$ is the figure 8 given by the following image:



Let us denote by S the image of g, *i.e.* $S = \left\{g(t) \mid -\frac{\pi}{2} < t < 3\frac{\pi}{2}\right\}$. Now, consider the other injective immersion $f : \left(-\frac{\pi}{2}, 3\frac{\pi}{2}\right) \to \mathbb{R}^2$ given by $f(t) = (\cos t, \sin 2t)$, the image of which is given by the following image:



Now, one can show that $\operatorname{im} f \subseteq S$. In can be seen by proving that for every $t_1 \in \left(-\frac{\pi}{2}, 3\frac{\pi}{2}\right)$, there exists some $t_2 \in \left(-\frac{\pi}{2}, 3\frac{\pi}{2}\right)$ such that $f(t_1) = g(t_2)$. In fact, this can be achieved by choosing

$$t_2 = \begin{cases} -t_1 & \text{if } t_1 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \\ 2\pi - t_1 & \text{if } t_1 \in \left(\frac{\pi}{2}, 3\frac{\pi}{2}\right) \\ t_1 & \text{if } t_1 = \frac{\pi}{2} \end{cases}.$$

So there is an inclusion $\iota : \operatorname{im} f \to S \subseteq \mathbb{R}^2$. Denote by $\widetilde{f} = \iota \circ f$, so that $\widetilde{f} : \left(-\frac{\pi}{2}, 3\frac{\pi}{2}\right) \to S$. The map $\widetilde{f}$ is induced from the map $f : \left(-\frac{\pi}{2}, 3\frac{\pi}{2}\right) \to \mathbb{R}^2$ given by $f(t) = (\cos t, \sin 2t)$, which is a C^∞ map. However, the induced map $\widetilde{f}$ is not even continuous, let alone C^∞. Here we equipped S with the immersed submanifold topology inherited from the injective immersion g. In this topology, the segment AB is open in S. But

$$\widetilde{f}^{-1}(AB) = (\iota \circ f)^{-1}(AB) = f^{-1}\left(\iota^{-1}(AB)\right) = f^{-1}(AB)$$

is not open in $\left(-\frac{\pi}{2}, 3\frac{\pi}{2}\right)$ because it contains an isolated point as discussed earlier.

To summarize, although f is C^∞, the induced map $\widetilde{f}$ is not even continuous, let alone being C^∞. This is so because the set S containing the image of f is not a regular submanifold of $\mathbb{R}^2$. In fact, we have the following result.

Theorem 8.2.1

Suppose $f : N \to M$ is C^∞ and the image of f lies in a subset S of M. If S is a regular submanifold of M, then the induced map $\widetilde{f} : N \to S$ is C^∞.

Proof. Let $p \in N$. Also, let $n = \dim N$ and $m = \dim N$. By hypothesis, $\operatorname{im} f \subseteq S$. So $f(p) \in S \subseteq M$. Since S is a regular submanifold of M, there is an adopted chart $(V, \psi) = \left(V, y^1, y^2, \ldots, y^m\right)$ of M about $f(p)$ such that $S \cap V$ is the zero set of the last $m - s$ coordinate functions $y^{s+1}, \ldots, y^m$. If one denotes $\psi_S = \left(y^1, y^2, \ldots, y^s\right)$, then $(S \cap V, \psi_S)$ is a chart for the regular submanifold S. Now, take a chart (U, φ) about p. By choosing U sufficiently small, we can assume $f(U) \subseteq V$. This is possible since f is continuous. Since $\operatorname{im} f \subseteq S$, $f(U) \subseteq S \cap V$. Then, with respect to the charts $(S \cap V, \psi_S)$ about $f(p)$ and (U, φ) about p,

$$\psi_S \circ \widetilde{f} \circ \varphi^{-1} : \varphi(U) \subseteq \mathbb{R}^n \to \mathbb{R}^s$$

is given by

$$\left(\psi_S \circ \widetilde{f} \circ \varphi^{-1}\right)(a) = \left(y^1\left(\widetilde{f}\left(\varphi^{-1}(a)\right)\right), y^2\left(\widetilde{f}\left(\varphi^{-1}(a)\right)\right), \ldots, y^s\left(\widetilde{f}\left(\varphi^{-1}(a)\right)\right)\right)$$

for $a \in \varphi(U)$. Now, $\widetilde{f}\left(\varphi^{-1}(a)\right) = f\left(\varphi^{-1}(a)\right)$. So

$$\left(\psi_S \circ \widetilde{f} \circ \varphi^{-1}\right)(a) = \left(y^1\left(f\left(\varphi^{-1}(a)\right)\right), y^2\left(f\left(\varphi^{-1}(a)\right)\right), \ldots, y^s\left(f\left(\varphi^{-1}(a)\right)\right)\right) .$$

Since $f : N \to M$ is smooth, relative to the charts (U, φ) of N and (V, ψ) of M, $\psi \circ f \circ \varphi^{-1} : \varphi(U) \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is smooth. The components of $\psi \circ f \circ \varphi^{-1}$ are

$$r^i \circ \left(\psi \circ f \circ \varphi^{-1}\right) = y^i \circ f \circ \varphi^{-1} ,$$

and they are smooth. We've seen that the components of $\psi_S \circ \widetilde{f} \circ \varphi^{-1}$ are $y^i \circ f \circ \varphi^{-1}$ for $i = 1, 2, \ldots, s$. Therefore, $\psi_S \circ \widetilde{f} \circ \varphi^{-1}$ is smooth. This proves that $\widetilde{f}$ is smooth at p. This is true for every $p \in N$, so $\widetilde{f}$ is smooth. ■

Example 8.2.1 (Multiplication map of $\mathrm{SL}(n, \mathbb{R})$)**.** The multiplication map $\mu : \mathrm{GL}(n, \mathbb{R}) \times \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$ is given by

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

where a_{ik}, b_{kj} are the matrix entries of $A, B \in \mathrm{GL}(n, \mathbb{R})$, respectively. Since $\mathrm{GL}(n, \mathbb{R})$ is an open subset of $\mathbb{R}^{n^2}$, it is an n^2-dimensional manifold with a single chart $\left(\mathrm{GL}(n, \mathbb{R}), \mathbb{1}_{\mathrm{GL}(n,\mathbb{R})}\right)$. Similarly, $\mathrm{GL}(n, \mathbb{R}) \times \mathrm{GL}(n, \mathbb{R})$ is an open subset of $\mathbb{R}^{n^2+n^2} = \mathbb{R}^{2n^2}$, so it is a manifold with a single chart $\left(\mathrm{GL}(n, \mathbb{R}) \times \mathrm{GL}(n, \mathbb{R}), \mathbb{1}_{\mathrm{GL}(n,\mathbb{R}) \times \mathrm{GL}(n,\mathbb{R})}\right)$. Now, the map μ is smooth because each component of $\mu(AB)$ is a polynomial in the entries of the matrices A and B.

Now, since $\mathrm{SL}(n, \mathbb{R})$ is a regular submanifold of $\mathrm{GL}(n, \mathbb{R})$, $\mathrm{SL}(n, \mathbb{R}) \times \mathrm{SL}(n, \mathbb{R})$ is a regular submanifold of $\mathrm{GL}(n, \mathbb{R}) \times \mathrm{GL}(n, \mathbb{R})$. Hence, the inclusion map

$$i : \mathrm{SL}(n, \mathbb{R}) \times \mathrm{SL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R}) \times \mathrm{GL}(n, \mathbb{R})$$

is an embedding and hence C^∞ by Theorem 8.1.2. Hence,

$$\mu \circ i : \mathrm{SL}(n, \mathbb{R}) \times \mathrm{SL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$$

is also C^∞ as a composition of smooth maps. Since $\mathrm{SL}(n, \mathbb{R})$ is a subgroup of $\mathrm{GL}(n, \mathbb{R})$, the product of two matrices in $\mathrm{SL}(n, \mathbb{R})$ is also in $\mathrm{SL}(n, \mathbb{R})$. Hence, the image of $\mu \circ i$ lies in $\mathrm{SL}(n, \mathbb{R})$, which is a regular submanifold of $\mathrm{GL}(n, \mathbb{R})$. Therefore, the induced map

$$\widetilde{\mu} : \mathrm{SL}(n, \mathbb{R}) \times \mathrm{SL}(n, \mathbb{R}) \to \mathrm{SL}(n, \mathbb{R})$$

is C^∞ by Theorem 8.2.1.

9 The Tangent Bundle

§9.1 The Topology of the Tangent Bundle

Definition 9.1.1 (Tangent Bundle). Let M be a smooth manifold. The **tangent bundle** TM of M is the *disjoint union* of all the tangent spaces

$$TM := \bigsqcup_{p \in M} T_p M \, .$$

In general, if $\{A_i\}_{i \in I}$ is a collection of subsets of a set S, then their *disjoint union* is defined as

$$\bigsqcup_{i \in I} A_i = \bigcup_{i \in I} \{p\} \times A_i \, .$$

So, a generic element of TM is of the form (p, X_p) where $X_p \in T_p M$. There is a natural map $\pi : TM \to M$ given by

$$\pi \left(p, X_p\right) = p \, .$$

TM as a set consists of ordered pairs (p, X_p) such that $p \in M$ and $X_p \in T_p M$.

Remark 9.1.1. $T_p M$ consists of all the point-derivations at p. A point-derivation at p is certainly not a point-derivation at q, for $p \neq q$. Therefore, $T_p M$ and $T_q M$ are disjoint. Therefore, the union $\bigcup\limits_{p \in M} T_p M$ is (up to notation) the same as the disjoint union $\bigsqcup\limits_{p \in M} T_p M$, since for distinct points p and q in M, the tangent spaces $T_p M$ and $T_q M$ are already disjoint. That's why we sometimes write $TM = \bigcup\limits_{p \in M} T_p M$.

We now give the set TM a topology. Let $(U, \varphi) = \left(U, x^1, x^2, \ldots, x^n\right)$ be a chart on M. Since U is an open subset of M, by Remark 6.1.1, $T_p U = T_p M$. Let

$$TU = \bigsqcup_{p \in U} T_p U = \bigsqcup_{p \in U} T_p M \, .$$

At $p \in U$, let $X_p \in T_p M$ so that

$$X_p = \sum_{i=1}^{n} c^i \left(X_p\right) \left.\frac{\partial}{\partial x^i}\right|_p \, .$$

Now, we define the map $\widetilde{\varphi} : TU \to \varphi\left(U\right) \times \mathbb{R}^n$ by

$$\left(p, X_p\right) \mapsto \left(x^1 \left(p\right), x^2 \left(p\right), \ldots, x^n \left(p\right), c^1 \left(X_p\right), c^2 \left(X_p\right), \ldots, c^n \left(X_p\right)\right) \, .$$

It is easy to see that $\widetilde{\varphi}$ has an inverse given by

$$\left(\varphi\left(p\right), c^1, c^2, \ldots, c^n\right) \mapsto \left(p, \sum_{i=1}^{n} c^i \left.\frac{\partial}{\partial x^i}\right|_p\right) \, .$$

$\widetilde{\varphi}$ given as above is a bijection. We use $\widetilde{\varphi}$ to transfer the topology of $\varphi\left(U\right) \times \mathbb{R}^n$ to TU: a set $A \subseteq TU$ is open if and only if $\widetilde{\varphi}\left(A\right)$ is open in $\varphi\left(U\right) \times \mathbb{R}^n$, where $\varphi\left(U\right) \times \mathbb{R}^n$ is given its standard topology as an open subset of $\mathbb{R}^{2n}$. With this topology induced from $\widetilde{\varphi}$, TU and $\varphi\left(U\right) \times \mathbb{R}^n$ are homeomorphic to each other.

Lemma 9.1.1

Let $V \subseteq U$ be open in U. Then the topology on TV as a subspace of TU is the same as the one induced by the bijection $\widetilde{\varphi}\big|_{TV} : TV \to \varphi(V) \times \mathbb{R}^n \subseteq \varphi(U) \times \mathbb{R}^n$.

Proof. φ is a homeomorphism, and hence an open map. Therefore, $\varphi(V)$ is open in $\varphi(U)$. As a result, $\varphi(V) \times \mathbb{R}^n \subseteq \varphi(U) \times \mathbb{R}^n$ is open in the subspace topology inherited from $\mathbb{R}^{2n}$.

Now, consider the subspace topology on TV inherited from TU. In this topology, let $A \subseteq TV$ be open. Then there exists $B \subseteq TU$ open such that $A = B \cap TV$.

$$\widetilde{\varphi}(A) = \widetilde{\varphi}(B \cap TV) = \widetilde{\varphi}(B) \cap \widetilde{\varphi}(TV) = \widetilde{\varphi}(B) \cap (\varphi(V) \times \mathbb{R}^n) .$$

Since B is open in TU, $\widetilde{\varphi}(B)$ is open in $\varphi(U) \times \mathbb{R}^n$. Therefore, $\widetilde{\varphi}\big|_{TV}(A) = \widetilde{\varphi}(B) \cap (\varphi(V) \times \mathbb{R}^n)$ is open in $\varphi(V) \times \mathbb{R}^n$. Therefore, A is open in TV in the topology induced by the bijection $\widetilde{\varphi}\big|_{TV} : TV \to \varphi(V) \times \mathbb{R}^n$.

Now, let A be open in TV in the topology induced by the bijection $\widetilde{\varphi}\big|_{TV}$. Then $\widetilde{\varphi}\big|_{TV}(A) = \widetilde{\varphi}(A)$ is open in $\varphi(V) \times \mathbb{R}^n$. We have shown that $\varphi(V) \times \mathbb{R}^n$ is open in $\varphi(U) \times \mathbb{R}^n$. Therefore, $\widetilde{\varphi}(A)$ is open in $\varphi(U) \times \mathbb{R}^n$. This means that A is open in TU. Then $A = A \cap TV$ as $A \subseteq TV$. So A is open in TV in the subspace topology inherited from TU.

Therefore, one can conclude that the subspace topology on TV inherited from TU is the same as the one induced by the bijection $\widetilde{\varphi}\big|_{TV} : TV \to \varphi(V) \times \mathbb{R}^n \subseteq \varphi(U) \times \mathbb{R}^n$. ■

Now, let $\mathcal{B}$ be the collection of all open subsets of TU_α as U_α runs over all coordinate open sets in M. In other words, if $\mathscr{A} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ is the maximal atlas of M,

$$\begin{aligned}\mathcal{B} &= \bigcup_{\alpha \in I} \{A \mid A \subseteq TU_\alpha \text{ is open in } TU_\alpha\} \\ &= \{A \mid A \subseteq TU_\alpha \text{ is open in } TU_\alpha\,,\ \alpha \in I\}\end{aligned}$$

Now we shall show that $\mathcal{B}$ forms a basis for topology.

Lemma 9.1.2 (i) For any manifold M, the set M is the union of all $A \in \mathcal{B}$.

(ii) Let U and V be coordinate open sets in a manifold M. If A is open in TU and B is open in TV, then $A \cap B$ is open in $T(U \cap V)$.

Proof. (i) Let $\mathscr{A} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ be the maximal atlas for M. Then for every $\alpha \in I$, $T_pU_\alpha = T_pM$ since U_α is open in M for each α. Now,

$$\begin{aligned}TM &= \bigsqcup_{p \in M} T_pM = \bigcup_{p \in M} \{p\} \times T_pM = \bigcup_{p \in M} \{p\} \times T_pU_\alpha \\ &= \bigcup_{\alpha \in I} \left(\bigcup_{p \in M} \{p\} \times T_pU_\alpha \right) = \bigcup_{\alpha \in I} TU_\alpha\end{aligned}$$

Now, TU_α is open in itself. So $TU_\alpha \in \mathcal{B}$. Therefore,

$$TM = \bigcup_{\alpha \in I} TU_\alpha \subseteq \bigcup_{A \in \mathcal{B}} A\,.$$

Also, each $A \in \mathcal{B}$ is contained in some TU_α. Therefore, the union of all A's is contained in the union of all TU_α's. In other words,

$$\bigcup_{A \in \mathcal{B}} A \subseteq \bigcup_{\alpha \in I} TU_\alpha = TM\,.$$

Therefore, we can conclude that $TM = \bigcup_{\alpha \in I} TU_\alpha = \bigcup_{A \in \mathcal{B}} A$.

(ii) Note that since $U \cap V \subseteq U$, $T(U \cap V)$ is endowed with the subspace topology inherited from TU. Since $A \subseteq TU$ is open, $A \cap T(U \cap V)$ is open in $T(U \cap V)$ in the subspace topology. Similarly, for $B \subseteq TV$ open, $B \cap T(U \cap V)$ is open in $T(U \cap V)$ in the subspace topology.

Now, we want to show that $TU \cap TV = T(U \cap V)$.

$$\begin{aligned} TU \cap TV &= \left(\bigsqcup_{p \in U} T_p U\right) \cap \left(\bigsqcup_{q \in V} T_q V\right) = \left(\bigcup_{p \in U} \{p\} \times T_p U\right) \cap \left(\bigcup_{q \in V} \{q\} \times T_q V\right) \\ &= \bigcup_{p \in U \cap V} \left((\{p\} \times T_p U) \cap (\{p\} \times T_p V)\right) \\ &= \bigcup_{p \in U \cap V} \{p\} \times (T_p U \cap T_p V) \\ &= \bigcup_{p \in U \cap V} \{p\} \times T_p(U \cap V) = T(U \cap V) \end{aligned}$$

Since $A \subseteq TU$ and $B \subseteq TV$, $A \cap B \subseteq TU \cap TV = T(U \cap V)$. Hence,

$$A \cap B = A \cap B \cap T(U \cap V) = (A \cap T(U \cap V)) \cap (B \cap T(U \cap V)) .$$

We have previously shown that both $A \cap T(U \cap V)$ and $B \cap T(U \cap V)$ are open in $T(U \cap V)$. Therefore, their intersection $A \cap B$ is also open in $T(U \cap V)$.

■

Lemma 9.1.2 implies that $\mathcal{B}$ is a basis for some topology on TM. This is because of Proposition 1.3.2. Now, we give TM the topology generated by the basis $\mathcal{B}$. We declare $A \subseteq TM$ to be open if and only if there exists $\{B_\lambda\} \subseteq \mathcal{B}$ such that

$$A = \bigcup_{\lambda} B_\lambda .$$

Lemma 9.1.3

A manifold M has a countable basis consisting of coordinate open sets.

Proof. Let $\mathscr{A} = \{(U_\alpha, \varphi_\alpha)\}_\alpha$ be the maximal atlas on M. M is second countable, so there exists a countable basis. Let $\mathfrak{B} = \{B_i\}_i$ be a countable basis for M.

For each U_α and $p \in U_\alpha$, choose $B_{p,\alpha} \in \mathfrak{B}$ such that

$$p \in B_{p,\alpha} \subseteq U_\alpha .$$

Such $B_{p,\alpha}$ exists because $\mathfrak{B}$ is a basis. Then $\{B_{p,\alpha}\}$ is a subcollection of $\mathfrak{B}$, and hence it is countable. Also for any open set $U \subseteq M$ and $p \in U$, let U_β be a coordinate open set about p. Then any open subset of U_β is also a coordinate open set. If we take $U_\alpha = U \cap U_\beta$, U_α is a coordinate open set. So

$$p \in B_{p,\alpha} \subseteq U_\alpha \subseteq U .$$

Therefore, $\{B_{p,\alpha}\}$ is a countable basis. Now, since any open subset of a coordinate open set is again a coordinate open set, and $B_{p,\alpha}$ is an open subset of U_α, we can conclude that $\{B_{p,\alpha}\}$ is a countable basis consisting of coordinate open sets. ■

Proposition 9.1.4

The tangent bundle TM is second countable.

Proof. Let $\{U_i\}_i$ be a countable basis for M consisting of coordinate open sets. Let φ_i be the coordinate map on U_i. We have shown that TU_i is homeomorphic to $\varphi_i(U_i) \times \mathbb{R}^n$, which is an open subset of $\mathbb{R}^n$. Hence, $\varphi_i(U_i) \times \mathbb{R}^n$ is secoond countable. Now, homeomorphism preserves second countability, so TU_i is also second countable.

For each i, choose a countable basis $\{B_{i,j}\}_j$ for TU_i. Then $\{B_{i,j}\}_{i,j}$ is also countable. Now we need to show that $\{B_{i,j}\}_{i,j}$ is a basis for TM. Let $A \subseteq TM$ be open and take $(p, X_p) \in A$. We need to show the existence of $B_{i,j}$ such that $(p, X_p) \in B_{i,j} \subseteq A$.

Since $\{U_i\}$ is a basis for M, $p \in U_i$ for some i. Then

$$(p, X_p) \in \{p\} \times T_pM = \{p\} \times T_pU_i \subseteq \bigcup_{p \in U_i} \{p\} \times T_pU_i = TU_i \, .$$

Therefore, $(p, X_p) \in A \cap TU_i$. A is open in TM, and TU_i is open in TM. Therefore, $A \cap TU_i$ is also open in TM. Let $\widetilde{A} = A \cap TU_i$. We want to show that $\widetilde{A}$ is open in TU_i.

Since $\widetilde{A}$ is open in TM, it can be expressed as

$$\widetilde{A} = \bigcup_{\alpha \in J \subseteq I} \widetilde{A}_\alpha \, ,$$

where $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ is the maximal atlas of M, and $\widetilde{A}_\alpha$ is open in TU_α.

$$\widetilde{A}_\alpha \subseteq \widetilde{A} \subseteq TU_i \, .$$

$\widetilde{A}_\alpha$ is open in TU_α, TU_i is open in TU_i. Therefore, $\widetilde{A}_\alpha \cap TU_i = \widetilde{A}_\alpha$ is open in $T(U_i \cap U_\alpha)$. Since $U_i \cap U_\alpha$ is open in U_i, $T(U_i \cap U_\alpha)$ is open in TU_i. Therefore, $\widetilde{A}_\alpha$ is open in TU_i. This is true for each $\alpha \in J$. Hence, $\widetilde{A}$ is open in TU_i.

Now, $\widetilde{A}$ is open in TU_i and $(p, X_p) \in \widetilde{A} = A \cap TU_i$. Since $\{B_{i,j}\}_j$ is a basis for TU_i, there exists some $B_{i,j}$ such that

$$(p, X_p) \in B_{i,j} \subseteq \widetilde{A} = A \cap TU_i \subseteq A \implies (p, X_p) \in B_{i,j} \subseteq A \, .$$

Therefore, the countable collection $\{B_{i,j}\}_{i,j}$ is a basis for TM. ■

Proposition 9.1.5

TM is Hausdorff.

Proof. Let (p, X_p) and (q, Y_q) be distinct points of TM.

Case 1: $p \neq q$.

Since M ia Hausdorff, there exists disjoint open subsets U_1 and V_1 of M that contain p and q, respectively. Furthermore, there exists coordinate open sets U_2 and V_2 around p and q, respectively. Then $U = U_1 \cap U_2$ and $V = V_1 \cap V_2$ are disjoint coordinate open sets that contain p and q, respectively.

$$(p, X_p) \in \{p\} \times T_pM = \{p\} \times T_pU \subseteq \bigcup_{p \in U} \{p\} \times T_pU = TU \, .$$

Similarly, $(q, Y_q) \in TV$. We have shown that $TU \cap TV = T(U \cap V)$. Since $U \cap V = \varnothing$, $TU \cap TV = \varnothing$. Therefore, TU and TV are the disjoint open subsets of TM that contain (p, X_p) and (q, Y_q), respectively.

Case 2: $p = q$.

Let (U, φ) be a coordinate chart containing p. Then (p, X_p) and (p, Y_p) are distinct points on TU, which is homeomorphic to $\varphi(U) \times \mathbb{R}^n$. $\varphi(U) \times \mathbb{R}^n$ is Hausdorff, hence so is TU. Therefore, (p, X_p) and (p, Y_p) can be separated by open subsets of TU, which are also open subset of TM.

Therefore, TM is Hausdoeff. ■

The Manifold Structure on TM

Proposition 9.1.6

Let $\{U_\alpha, \varphi_\alpha\}_{\alpha\in I}$ be an atlas for M. Then $\{TU_\alpha, \widetilde{\varphi}_\alpha\}_{\alpha\in I}$ is an atlas for TM.

Let's begin with an observation. Let $\left(U, x^1, x^2, \ldots, x^n\right)$ and $\left(V, y^1, y^2, \ldots, y^n\right)$ be two charts on M. Then for any $p \in U \cap V$, there are two bases for T_pM:

$$\left\{ \left.\frac{\partial}{\partial x^j}\right|_p \right\}_{j=1}^n \quad \text{and} \quad \left\{ \left.\frac{\partial}{\partial y^i}\right|_p \right\}_{i=1}^n .$$

So $X_p \in T_pM$ has two basis expansions

$$X_p = \sum_{j=1}^n a^j \left.\frac{\partial}{\partial x^j}\right|_p = \sum_{i=1}^n \left.\frac{\partial}{\partial y^i}\right|_p .$$

By applying y^k on both sides, one obtains

$$b^k = \sum_{j=1}^n a^j \frac{\partial y^k}{\partial x^j}(p) .$$

Proof of Proposition 9.1.6. We have already shown that $\widetilde{\varphi}_\alpha : TU_\alpha \to \varphi_\alpha(U_\alpha) \times \mathbb{R}^n \subseteq \mathbb{R}^{2n}$ is a homeomorphism from an open subset of TM to an open subset of the Euclidean space $\mathbb{R}^{2n}$. Also, we have shown that

$$TM = \bigcup_{\alpha\in I} TU_\alpha .$$

So, it remains to check that on $TU_\alpha \cap TU_\beta$, $\widetilde{\varphi}_\alpha$ and $\widetilde{\varphi}_\beta$ are C^∞-compatible.

Let $U_{\alpha\beta} = U_\alpha \cap U_\beta$. Then $\widetilde{\varphi}_\beta \circ \widetilde{\varphi}_\alpha^{-1} : \varphi_\alpha(U_{\alpha\beta}) \times \mathbb{R}^n \to \varphi_\beta(U_{\alpha\beta}) \times \mathbb{R}^n$ is given by

$$\left(\varphi_\alpha(p), a^1, \ldots, a^n\right) \xmapsto{\widetilde{\varphi}_\alpha^{-1}} \left(\varphi_\alpha^{-1}(\varphi_\alpha(p)), \sum_{j=1}^n a^j \left.\frac{\partial}{\partial x^j}\right|_p\right) \xmapsto{\widetilde{\varphi}_\beta} \left(\left(\varphi_\beta \circ \varphi_\alpha^{-1}\right)\varphi_\alpha(p), b^1, \ldots, b^n\right) ,$$

where $b^k = \sum\limits_{j=1}^n a^j \frac{\partial y^k}{\partial x^j}(p)$. Since $\{U_\alpha, \varphi_\alpha\}_\alpha$ is an atlas for M, $\varphi_\beta \circ \varphi_\alpha^{-1}$ is smooth. Now,

$$\begin{aligned} b^i &= \sum_{j=1}^n a^j \frac{\partial y^i}{\partial x^j}(p) = \sum_{j=1}^n a^j \frac{\partial\left(y^i \circ \varphi_\alpha^{-1}\right)}{\partial r^j}(\varphi_\alpha(p)) \\ &= \sum_{j=1}^n a^j \frac{\partial\left(r^i \circ \varphi_\beta \circ \varphi_\alpha^{-1}\right)}{\partial r^j}(\varphi_\alpha(p)) \\ &= \sum_{j=1}^n a^j \frac{\partial\left(\varphi_\beta \circ \varphi_\alpha^{-1}\right)^i}{\partial r^j}(\varphi_\alpha(p)) \end{aligned}$$

$\frac{\partial\left(\varphi_\beta \circ \varphi_\alpha^{-1}\right)^i}{\partial r^j}$ is C^∞ as $\varphi_\beta \circ \varphi_\alpha^{-1}$ is C^∞. Therefore, the map $\widetilde{\varphi}_\beta \circ \widetilde{\varphi}_\alpha^{-1}$ given by

$$\left(\varphi_\alpha(p), a^1, \ldots, a^n\right) \mapsto \left(\left(\varphi_\beta \circ \varphi_\alpha^{-1}\right)\varphi_\alpha(p), \sum_{j=1}^n a^j \frac{\partial y^1}{\partial x^j}(p), \sum_{j=1}^n a^j \frac{\partial y^2}{\partial x^j}(p), \ldots, \sum_{j=1}^n a^j \frac{\partial y^n}{\partial x^j}(p)\right)$$

is C^∞. ■

§9.2 Vector Bundle

On the tangent bundle TM is a smooth manifold M, there is a natural projection map $\pi : TM \to M$ with $\pi(p, X_p) = p$. This makes the tangent bundle into a C^∞ vector bundle that we will define now.

Definition 9.2.1. Given any map $\pi : E \to M$ between two smooth manifolds, we call the preimage $\pi^{-1}(p) := \pi^{-1}(\{p\})$ of a point $p \in M$ the **fibre** at p. The fibre at p is often written E_p, i.e. $E_p = \pi^{-1}(p)$.

Definition 9.2.2. For any two maps $\pi : E \to M$ and $\pi' : E' \to M$ with the same target space M, a map $\phi : E \to E'$ is said to be **fibre-preserving** if $\phi(E_p) \subseteq E'_p$ for every $p \in M$.

Exercise 9.1. Given two maps $\pi : E \to M$ and $\pi' : E' \to M$, a map $\phi : E \to E'$ is fibre-preserving if and only if the following diagram commutes:

$$\begin{tikzcd} E \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & E' \arrow[dl, "\pi'"] \\ & M & \end{tikzcd}$$

In other words, $\pi = \pi' \circ \phi$.

Solution. **(⇒):** Let $\phi : E \to E'$ be fibre-preserving. Then $\phi(E_p) \subseteq E'_p$ for every $p \in M$. Given $x \in E$, $x \in E_p$ for some $p \in M$.

$$x \in E_p = \pi^{-1}(p) \implies \pi(x) = p\,.$$

Since $\phi(E_p) \subseteq E'_p$, $\phi(x) \in E'_p$. As a result, $\pi'(\phi(x)) = p$. So we obtain

$$\pi(x) = p = \pi'(\phi(x)) \implies \pi = \pi' \circ \phi\,.$$

(⇐): Now, let $\pi = \pi' \circ \phi$. Take $x \in E_p$. Then $\pi(x) = p$. As a result,

$$\pi'(\phi(x)) = p \implies \phi(x) \in E'_p\,.$$

This is true for every $x \in E_p$. Therefore, $\phi(E_p) \subseteq E'_p$. ■

Definition 9.2.3. A surjective smooth map $\pi : E \to M$ of manifolds is said to be **locally trivial of rank r** if

(i) Each fibre $\pi^{-1}(p)$ has the structure if a vector space of dimension r.

(ii) For each $p \in M$, there are open neighborhood U of p and a fibre-preversing diffeomorphism

$$\phi : \pi^{-1}(U) \to U \times \mathbb{R}^r\,,$$

such that for every $q \in U$, the restriction

$$\phi\big|_{\pi^{-1}(q)} : \pi^{-1}(q) \to \{q\} \times \mathbb{R}^r$$

is a vector space isomorphism. Such an open set U is called a **trivializing open set** for E, and ϕ is called a **trivialization** of E over U.

The collection $\{(U_\alpha, \phi_\alpha)\}_\alpha$ with

$$\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^r$$

being the diffeomorphism discussed above (called the trivialization of E over U_α) and $\{U_\alpha\}_\alpha$ an open cover of M, is called a **local trivilization** for E and $\{U_\alpha\}_\alpha$ is called a **trivializing open cover** of M for E.

Definition 9.2.4. A C^∞ **vector bundle of rank** r is a triple (E, M, π) consisting of manifolds E and M and a surjective smooth map $\pi : E \to M$ that is locally trivial of rank r. The manifold E is called the total space of the vector bundle and M the base space.

Abuse of Notation. We sometimes say that E is a vector bundle over M. We also call the surjective smooth map $\pi : E \to M$ the vector bundle.

The tangent bundle of a manifold is the triple (TM, M, π) where TM is the total space of the tangent bundle.

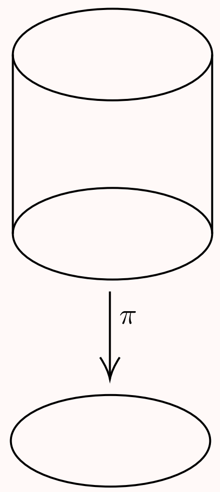
Example 9.2.1 (Product Bundle)

Given a manifold M, let $\pi : M \times \mathbb{R}^r \to M$ be the projection onto the first factor. Then $M \times \mathbb{R}^r$ is a vector bundle of rank r, called the product bundle of rank r over M. The vector space structure on the fibre $\pi^{-1}(p) = \{(p, \mathbf{v}) \mid \mathbf{v} \in \mathbb{R}^r\}$ is the obvious one:

$$(p, \mathbf{u}) + (p, \mathbf{v}) = (p, \mathbf{u} + \mathbf{v}) \ , \quad \text{and } \alpha (p, \mathbf{v}) = (p, \alpha\mathbf{v}) \ \text{ for } \alpha \in \mathbb{R} \, .$$

A local trivialization on $M \times \mathbb{R}^r$ is given by the identity map $\mathbb{1}_{M\times\mathbb{R}^r}$.

The infinite cylinder $S^1 \times \mathbb{R}$ is the product bundle of rank 1 over the unit circle S^1.



Remark 9.2.1. In general, a generic element of the vector bundle E belongs in some E_p where p is a point in the base space M. Since E_p has a vector space structure, we can say that a generic element of E is $\mathbf{e}_p$, where $\mathbf{e}_p$ is a vector in E_p for some $p \in M$. However, in the case of tangent bundle, we saw that a generic element of TM is an ordered pair (p, X_p) where $X_p \in T_pM$. This is because in this case $\pi^{-1}(p) = E_p$ is not the same as T_pM. Rather, we have $E_p = \{p\} \times T_pM$.

Let $\pi : E \to M$ be a C^∞ vector bundle. Suppose $\left(W, \widetilde{\psi}\right) = \left(W, x^1, x^2, \ldots, x^n\right)$ is a chart on M and

$$\widetilde{\phi} : \pi^{-1}(V) \to V \times \mathbb{R}^r$$

is a trivialization of E over V, with $V \cap W \neq 0$. Sometimes we write $E\big|_V = \pi^{-1}(V)$. Then $U = V \cap W$ is a coordinate open set, with the chart $\left(U, \widetilde{\psi}\big|_U\right)$. Furthermore,

$$\widetilde{\phi}\big|_U : \pi^{-1}(U) \to U \times \mathbb{R}^r$$

is a trivialization of E over U. We write $\widetilde{\psi}\big|_U = \psi$, and $\widetilde{\phi}\big|_U = \phi$. Then (U, ψ) is a chart in the maximal atlas of M, and

$$\phi : \pi^{-1}(U) \to U \times \mathbb{R}^r$$

is a trivialization of E over U.

Since a generic element of $\pi^{-1}(U) \subseteq E$ is a vector $\mathbf{e}_p$ for some $p \in U \subseteq M$, the map ϕ is given by

$$\phi(\mathbf{e}_p) = \left(p, c^1(\mathbf{e}_p), c^2(\mathbf{e}_p), \ldots, c^r(\mathbf{e}_p)\right)$$

where $\mathbf{e}_p = \sum_{i=1}^{r} c^i(\mathbf{e}_p)\,\widehat{e}_i$ in terms of an ordered basis $\{\widehat{e}_i\}_{i=1}^r$ for E_p. Now, consider the map

$$(\psi \times \mathbb{1}_{\mathbb{R}^r}) \circ \phi : \pi^{-1}(U) \to \psi(U) \times \mathbb{R}^r \subseteq \mathbb{R}^{n+r}\,.$$

This map is given by

$$\mathbf{e}_p \mapsto \left(x^1(p), x^2(p), \ldots, x^n(p), c^1(\mathbf{e}_p), c^2(\mathbf{e}_p), \ldots, c^r(\mathbf{e}_p)\right)\,.$$

$\psi \times \mathbb{1}_{\mathbb{R}^r}$ is a diffeomorphism since both ψ and $\mathbb{1}_{\mathbb{R}^r}$ are diffoemorphisms. Also, ϕ is a diffeomorphism. Therefore, $(\psi \times \mathbb{1}_{\mathbb{R}^r}) \circ \phi$ is a diffeomorphism from $E\big|_U$ onto its image. Hence, $\left(E\big|_U, (\psi \times \mathbb{1}_{\mathbb{R}^r}) \circ \phi\right)$ is a chart on the total space E of the C^∞ vector bundle $\pi : E \to M$. We call $x^1, x^2, \ldots, x^n$ the **base coordinates** and $c^1, c^2, \ldots, c^r$ the **fibre coordinates** of the chart $\left(E\big|_U, (\psi \times \mathbb{1}_{\mathbb{R}^r}) \circ \phi\right)$ on E. Note that the fibre coordinates c^i depend on the trivialization ϕ of the bundle. c^i's don't depend on the coordinate map ψ on the base U.

Definition 9.2.5 (Bundle Map)**.** Let $\pi_E : E \to M$ and $\pi_F : F \to N$ be two vector bundles. A **bundle map** from E to F is a pair of maps $\left(f, \widetilde{f}\right)$, $f : M \to N$ and $\widetilde{f} : E \to F$ such that

(i) The following diagram commutes

$$\begin{tikzcd} E \arrow[r, "\widetilde{f}"] \arrow[d, "\pi_E"'] & F \arrow[d, "\pi_F"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

In other words, $\pi_F \circ \widetilde{f} = f \circ \pi_E$.

(ii) $\widetilde{f} : E \to F$ is linear on each fibre, i.e. for every $p \in M$, $\widetilde{f}\big|_{E_p} : E_p \to F_{f(p)}$ is a linear map of vector spaces.

The collection of all vector bundles (as objects) together with bundle maps between them (as morphisms) forms a category[1].

Example 9.2.2

A smooth map $f : N \to M$ of manifolds induces a bundle map $\left(f, \widetilde{f}\right)$, where $\widetilde{f} : TN \to TM$ is given by

$$\widetilde{f}(p, X_p) = \left(f(p), f_{*,p}(X_p)\right) \in \{f(p)\} \times T_{f(p)}M \subseteq TM\,.$$

This gives rise to a *covariant functor* from the category of smooth manifolds and smooth maps to the category of vector bundles and bundle maps. To each manifold M, we associate its tangent bundle TM, and to each smooth map $f : N \to M$, we aasociate the bundle map $Tf = \left(f : N \to M, \widetilde{f} : TN \to TM\right)$.

If E and F are two C^∞ vector bundles over the same manifold M, then a bundle map $\left(\mathbb{1}_M, \widetilde{f}\right)$ from E to F over M is a bundle map in which the base map is the identity $\mathbb{1}_M$. Then we have the following commutative diagram:

[1]See Appendix A if you're not familiar with the definition of categories and functors.

$$\begin{tikzcd} E \arrow[r, "\widetilde{f}"] \arrow[d, "\pi_E"'] & F \arrow[d, "\pi_F"] \\ M \arrow[r, "\mathbb{1}_M"'] & M \end{tikzcd}$$

The commutativity of this diagram implies that $\widetilde{f}$ is a fibre-preserving map (Exercise 9.1).

For a fixed manifold M, we can also consider the category of all C^∞ vector bundles over M and C^∞ bundle maps (of the form $\left(\mathbb{1}_M, \widetilde{f}\right)$) over M. In this category, it makes sense to speak of an **isomorphism** of vector bundles over M. In this case, the linear map $\widetilde{f} : E_p \to F_{\mathbb{1}(p)} = F_p$ is an isomorphism of vector spaces. Any vector bundle over M isomorphic over M to the product bundle $M \times \mathbb{R}^r$ is called a trivial bundle.

§9.3 Smooth Sections

Definition 9.3.1 (Section). A **section** of a vector bundle $\pi : E \to M$ is a map $s : M \to M$ such that $\pi \circ s = \mathbb{1}_M$, the identity map on M. We say that a section $s : M \to E$ is smooth if it is smooth as a map from M to E. A smooth section of E over U is a smooth map $s : U \to E\big|_U$ such that $\pi\big|_{E|_U} \circ s = \mathbb{1}_U$.

Definition 9.3.2 (Vector Field). A **vector field** X on a manifold M is a map that assigns a tangent vector $X_p \in T_pM$ to each point $p \in M$. In terms of tangent bundle, a vector field on M is simply a section $X : M \to TM$ of the tangent bundle $\pi : TM \to M$. The vector field is smooth if X is a smooth map between manifolds.

Remark 9.3.1. When we say $X : M \to TM$ is a section of the tangent bundle $\pi : TM \to M$, we consider TM to be the union of T_pM across all $p \in M$, not disjoint union. Since there is a one-to-one correspondence between $\bigsqcup_{p \in M} T_pM$ and $\bigcup_{p \in M} T_pM$, we give $\bigcup_{p \in M} T_pM$ the topology inherited from $\bigsqcup_{p \in M} T_pM$ via the one-to-one correspondence. In other words, if $i : \bigcup_{p \in M} T_pM \to \bigsqcup_{p \in M} T_pM$ is the bijection given by $i(\mathbf{v}_p) = (p, \mathbf{v}_p)$, then a set $X \subseteq \bigcup_{p \in M} T_pM$ is open if and only if $i(X) \subseteq \bigsqcup_{p \in M} T_pM$ is open.

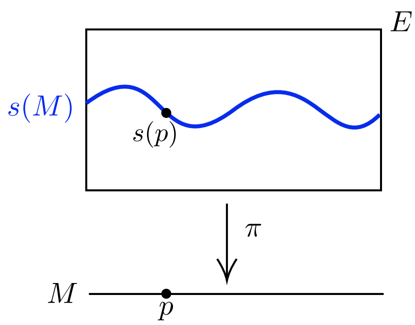


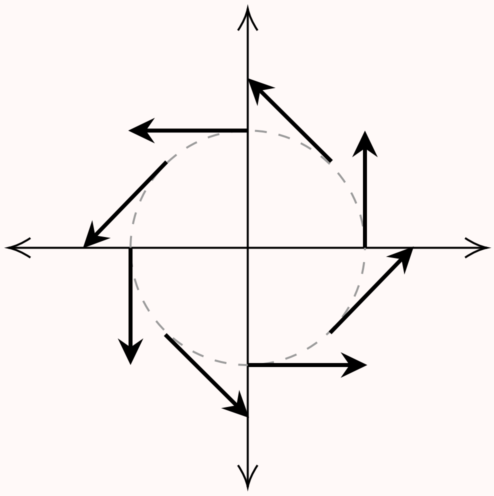
Figure 9.1: A section of a vector bundle $\pi : E \to M$ is a map $s : M \to E$.

Example 9.3.1

Consider the following vector field on $\mathbb{R}^2$

$$X_{(x,y)} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} = \begin{bmatrix} -y \\ x \end{bmatrix} .$$

It is a smooth vector field on $\mathbb{R}^2$. Visually, it's described as the following image:



Proposition 9.3.1

Let s and t be C^∞ sections of a C^∞ vector bundle $\pi : E \to M$, and let f be a C^∞ real-valued function on M. Then

(i) The sum $s + t : M \to E$ defined by

$$(s + t)(p) = s(p) + t(p) \in E_p \text{ for } p \in M ,$$

is a C^∞ section of E.

(ii) The product $fs : M \to E$ defined by

$$(fs)(p) = f(p)\, s(p) \in E_p \text{ for } p \in M ,$$

is a C^∞ section of E.

Proof. (i) It is clear that $s + t$ is a section of E. Indeed,

$$\pi \circ (s + t)(p) = \pi(s(p) + t(p)) = \pi(\mathbf{e}_p + \mathbf{v}_p) = p ,$$

so that $\pi \circ (s + t) = \mathbb{1}_M$. Now it remains to show that $s + t$ is smooth. For this purpose, let $p \in M$ and let V be a trivializing open set for E containing $p \in M$, with the trivialization

$$\phi : \pi^{-1}(V) \to V \times \mathbb{R}^r .$$

Choose a chart $(U, \varphi) = \left(U, x^1, \ldots, x^n\right)$ about $p \in M$ such that $U \subseteq V$. Then $\left(E\big|_U, (\varphi \times \mathbb{1}_{\mathbb{R}^r}) \circ \phi\big|_U\right)$ is a chart on E. Let $q \in U$ and $s(q) = \mathbf{e}_q$ with $\mathbf{e}_q \in E_q$. If $\{\widehat{e}_i\}_{i=1}^r$ is a basis of E_q, and $\mathbf{e}_q = \sum\limits_{i=1}^r c^i(\mathbf{e}_q)\,\widehat{e}_i$. Then we have

$$(\phi \circ s)(q) = \phi(\mathbf{e}_q) = \left(q, c^1(\mathbf{e}_q), \ldots, c^r(\mathbf{e}_q)\right) = \left(q, \left(c^1 \circ s\right)(q), \ldots, (c^r \circ s)(q)\right) .$$

Therefore, the map $(\varphi \times \mathbb{1}_{\mathbb{R}^r}) \circ \phi\big|_U \circ s \circ \varphi^{-1} : \varphi(U) \subseteq \mathbb{R}^n \to \varphi(U) \times \mathbb{R}^r \subseteq \mathbb{R}^{n+r}$ is given by

$$\varphi(q) = \left(x^1(q), x^2(q), \ldots, x^n(q)\right) \mapsto \left(x^1(q), x^2(q), \ldots, x^n(q), \left(c^1 \circ s\right)(q), \ldots, (c^r \circ s)(q)\right) .$$

Since s is smooth, this map is smooth. In particular, all of it components are smooth. Therefore, $c^i \circ s$ is a smooth function on U. Similarly, let $t(q) = \mathbf{v}_q \in E_q$. Then $\mathbf{v}_q = \sum_{i=1}^{r} d^i(\mathbf{v}_q)\, \widehat{e}_i$. Then in a similar manner as above, the map $(\varphi \times \mathbb{1}_{\mathbb{R}^r}) \circ \phi\big|_U \circ t \circ \varphi^{-1} : \varphi(U) \subseteq \mathbb{R}^n \to \varphi(U) \times \mathbb{R}^r \subseteq \mathbb{R}^{n+r}$ is given by

$$\varphi(q) = \left(x^1(q), x^2(q), \ldots, x^n(q)\right) \mapsto \left(x^1(q), x^2(q), \ldots, x^n(q), \left(d^1 \circ t\right)(q), \ldots, (d^r \circ t)(q)\right) .$$

Since t is smooth, this map is smooth. In particular, all of it components are smooth. Therefore, $d^i \circ t$ is a smooth function on U. Hence, $c^i \circ s + d^i \circ t$ is a smooth function on U.

Now, the map $(\varphi \times \mathbb{1}_{\mathbb{R}^r}) \circ \phi\big|_U \circ (s+t) \circ \varphi^{-1} : \varphi(U) \subseteq \mathbb{R}^n \to \varphi(U) \times \mathbb{R}^r \subseteq \mathbb{R}^{n+r}$ is given by

$$\begin{aligned}\varphi(q) = \left(x^1(q), x^2(q), \ldots, x^n(q)\right) \mapsto \big(&x^1(q), x^2(q), \ldots, x^n(q),\\ &\left(c^1 \circ s\right)(q) + \left(d^1 \circ t\right)(q), \ldots, (c^r \circ s)(q) + (d^r \circ t)(q)\big) .\end{aligned}$$

Since all the components of this map are smooth, this map is smooth. Therefore, $s+t$ is smooth on U. In patricular, $s+t$ is smooth at p. Since p is chosen arbitrarily, $s+t$ is smooth on all of M.

(ii) We shall use the same setup as above. f is a smooth function on M, so it is smooth on U. $c^i \circ s$ is also smooth on U. Therefore, their product is smooth on U. Now, the map $(\varphi \times \mathbb{1}_{\mathbb{R}^r}) \circ \phi\big|_U \circ (fs) \circ \varphi^{-1} : \varphi(U) \subseteq \mathbb{R}^n \to \varphi(U) \times \mathbb{R}^r \subseteq \mathbb{R}^{n+r}$ is given by

$$\begin{aligned}\varphi(q) = \left(x^1(q), x^2(q), \ldots, x^n(q)\right) \mapsto \big(&x^1(q), x^2(q), \ldots, x^n(q),\\ &f(q)\left(c^1 \circ s\right)(q), \ldots, f(q)(c^r \circ s)(q)\big) .\end{aligned}$$

Since all the components of this map are smooth, this map is smooth. Therefore, fs is smooth on U. In patricular, fs is smooth at p. Since p is chosen arbitrarily, fs is smooth on all of M.

■

Denote the set of all C^∞ sections of E by $\Gamma(E)$. Proposition 9.3.1 shows that $\Gamma(E)$ is not only a vector space over $\mathbb{R}$, but also a module over the ring $C^\infty(M)$ of C^∞ functions on M. For any open subset $U \subseteq M$, one can also consider the vector space $\Gamma(U, E)$ of C^∞ sections of E over U. Then $\Gamma(U, E)$ is both an $\mathbb{R}$-vector space and a $C^\infty(U)$-module. A section over the whole manifold is called a **global section**.

Exercise 9.2. Show that the image of a smooth section $s : M \to E$ is a regular submanifold of E.

Solution. This follows readily from Theorem 8.1.1. It suffices to show that s is an embedding. Firstly, $\pi\big|_{s(M)} : s(M) \to M$ is the inverse of s. Since π is continuous, so is its restriction $\pi\big|_{s(M)}$. Therefore, s is a homeomorphism onto its image. Now we need to show that s is an immersion. For $p \in M$,

$$\pi \circ s = \mathbb{1}_M \implies \pi_{*,s(p)} \circ s_{*,p} = (\mathbb{1}_M)_{*,p} = \mathbb{1}_{T_pM} .$$

Hence, $s_{*,p}$ is injective for every $p \in M$. Therefore, s is an embedding, and consequently, $s(M)$ is a regular submanifold of E. ■

§9.4 Smooth Frames

Definition 9.4.1 (Frame)**.** A **frame** for a vector bundle $\pi : E \to M$ over an open set U is a collection of sections $s_1, \ldots, s_r$ of E over U such that at each point $p \in U$, the elements $s_1(p), \ldots, s_r(p)$ form a basis for the r-dimensional vector space $E_p = \pi^{-1}(p)$. A frame $s_1, \ldots, s_r$ is said to be smooth if $s_1, \ldots, s_r$ are C^∞ as sections of E over U. A frame for the tangent bundle $TM \to M$ over an open set U is simply called a frame on U.

Example 9.4.1

The collection of vector fields

$$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$$

is a smooth frame on $\mathbb{R}^3$.

Example 9.4.2

Let M be a manifold and $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_r$ the standard basis on $\mathbb{R}^r$. In other words, $\mathbf{e}_i \in \mathbb{R}^r$ whose i-th component is 1 and rest of the components are all 0. Define $\bar{e}_i : M \to M \times \mathbb{R}^r$ by

$$\bar{e}_i(p) = (p, \mathbf{e}_i) \ .$$

Then $\bar{e}_1, \ldots, \bar{e}_r$ is a C^∞ frame for the product bundle $M \times \mathbb{R}^r \to M$.

Example 9.4.3 (The frame of a trivialization)

Let $\pi : E \to M$ be a smooth vector bundle of rank r. If $\phi : \pi^{-1}(U) \xrightarrow{\sim} U \times \mathbb{R}^r$ is a trivialization of E over an open set $U \subseteq M$, then ϕ^{-1} carries the frame $\bar{e}_1, \ldots, \bar{e}_r$ of the product bundle $U \times \mathbb{R}^r$ to a C^∞ frame $t_1, t_2, \ldots, t_r$ for E over U:

$$t_i(p) = \phi^{-1}(\bar{e}_i(p)) = \phi^{-1}(p, \mathbf{e}_i) \ ,$$

for $p \in U$. We call $t_1, \ldots, t_r$ the C^∞ frame over U of the trivialization ϕ.

Lemma 9.4.1

Let $\phi : E\big|_U \to U \times \mathbb{R}^r$ be a trivialization over an open set U of a C^∞ vector bundle $E \to M$, and $t_1, \ldots, t_r$ the C^∞ frame over U of the trivilization. Then a section $s = \sum\limits_{i=1}^{r} b^i t_i$ of E over U is C^∞ if and only if its coefficients b^i relative to the frame $t_1, \ldots, t_r$ are C^∞. (Here $b^i : U \to \mathbb{R}$.)

Proof. ($\Leftarrow$) According to Proposition 9.3.1, each $b^i t_i$ is a C^∞ section, and hence their sum $s = \sum\limits_{i=1}^{r} b^i t_i$ is also a C^∞ section.

($\Rightarrow$) Suppose the section $s = \sum\limits_{i=1}^{r} b^i t_i$ of E over U is C^∞. s is a map $s : U \to E\big|_U$, and so $\phi \circ s : U \to U \times \mathbb{R}^r$ is C^∞ as it's the composition of two C^∞ maps. Now, note that

$$(\phi \circ s)(p) = \phi\left(\sum_{i=1}^{r} b^i(p)\, t_i(p)\right) = \sum_{i=1}^{r} b^i(p)\, \phi(t_i(p)) \ ,$$

since ϕ is linear at each E_p. By Example 9.4.3, $\phi(t_i(p)) = (p, \mathbf{e}_i)$. Hence,

$$(\phi \circ s)(p) = \sum_{i=1}^{r} b^i(p)\,(p, \mathbf{e}_i) = \left(p, \sum_{i=1}^{r} b^i(p)\, \mathbf{e}_i\right) = \left(p, b^1(p), \ldots, b^r(p)\right) \ .$$

Let $P : U \times \mathbb{R}^r \to \mathbb{R}^r$ be the projection map. It is a smooth map. Therefore, $P \circ \phi \circ s$ is a smooth map on U.

$$(P \circ \phi \circ s)(p) = \left(b^1(p), \ldots, b^r(p)\right) \ .$$

So, b^i's are the components of $P \circ \phi \circ s$. Hence, by Proposition 4.2.7, b^i is smooth on U for every i. ■

Proposition 9.4.2 (Characterization of C^∞ sections)
Let $\pi : E \to M$ be a C^∞ vector bundle and U an open subset of M. Suppose $s_1, \ldots, s_r$ is a C^∞ frame for E over U. Then a section $s = \sum_{j=1}^{r} c^j s_j$ of E over U is C^∞ if and only if the coefficients c^j are C^∞ functions on U.

Proof. If $s_1, \ldots, s_r$ is the frame of a trivialization over U, then this proposition is exactly Lemma 9.4.1. We prove the general result by reducing it to this case.

($\Leftarrow$) According to Proposition 9.3.1, each $c^j s_j$ is a C^∞ section, and hence their sum $s = \sum_{i=1}^{r} c^j s_j$ is also a C^∞ section.

($\Rightarrow$) Suppose $s = \sum_{i=1}^{r} c^j s_j$ is a C^∞ section of E over U. Fix a point $p \in U$ and choose a trivializing open set $V \subseteq U$ for E containing p with trivialization $\phi : \pi^{-1}(V) \to V \times \mathbb{R}^r$. (There exists a trivializing open set V' containing p, and a trivialization $\Phi : \pi^{-1}(V') \to V' \times \mathbb{R}^r$. Then $V = U \cap V'$, and $\phi = \Phi\big|_{\pi^{-1}(V)}$.)

By Example 9.4.3, let $t_1, \ldots, t_r$ denote the C^∞ frame of the trivialization ϕ. Now, we write the sections s and s_j by means of the frame $t_1, \ldots, t_r$.

$$s\big|_V = \sum_{i=1}^{r} b^i t_i \quad \text{and} \quad s_j\big|_V = \sum_{i=1}^{r} a^i_j t_i\,.$$

Here, we need to restrict s and s_j's on V, because t_i's are sections of E over V. The coefficients b^i and a^i_j are C^∞ functions on V by Lemma 9.4.1. Now,

$$\sum_{i=1}^{r} b^i t_i = s\big|_V = \sum_{j=1}^{r} c^j\big|_V s_j\big|_V = \sum_{i,j=1}^{r} c^j\big|_V a^i_j t_i\,.$$

Comparing the coefficients of t_i yields

$$b^i = \sum_{j=1}^{r} c^j\big|_V a^i_j\,.$$

In matrix notation, if we denote $\left[a^i_j\right]_{i,j=1}^{r} = A$,

$$b = \begin{bmatrix} b^1 \\ \vdots \\ b^r \end{bmatrix} = A \begin{bmatrix} c^1\big|_V \\ \vdots \\ c^r\big|_V \end{bmatrix} = Ac\,.$$

At each point of V, being the transition matrix between two bases (t_i's and s_j's), A is invertible. By Cramer's rule for matrix inverse,

$$\left(A^{-1}\right)_{ij} = \frac{1}{\det A} (-1)^{i+j} \left((j,i) \text{ minor of } A \right).$$

The (j,i) minor of A is a smooth function of a^i_j's, and a^i_j's are smooth function on V. Therefore, the entries of the inverse A^{-1} are C^∞ functions on V.

Now, $c = A^{-1}b$. We have already shown that b^i's are C^∞ functions on V. Hence, $c = A^{-1}b$ is a column vector of C^∞ functions on V. This proves that $c^1, \ldots, c^r$ are smooth at $p \in V \subseteq U$. Since p is an arbitrary point of U, $c^1, \ldots, c^r$ are smooth on all of U. ■

10 Partition of Unity

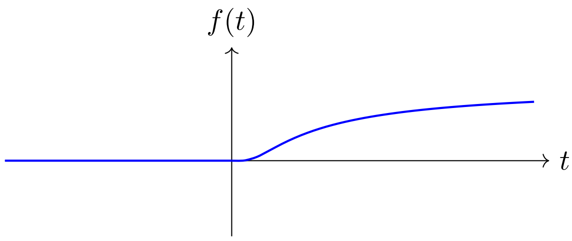
A partition of unity on a manifold is a collection of non-negative functions that sum to 1 (subjected to some other conditions that we will specify later). Usually, one demands, in addition, that the partition of unity be *subordinate* to an open cover $\{U_\alpha\}_{\alpha\in A}$ of the manifold M. What this means is that the partition of unity $\{\rho_\alpha\}_{\alpha\in A}$ is indexed by the same set as the open cover $\{U_\alpha\}_{\alpha\in A}$ and for each α in the index set A, the support of ρ_α (to be defined shortly) is contained in U_α. In particular, ρ_α vanishes outside U_α.

§10.1 Smooth Bump Functions

We introduce the function $f : \mathbb{R} \to \mathbb{R}$, defined by

$$f(t) = \begin{cases} e^{-\frac{1}{t}} & \text{for } t > 0 \\ 0 & \text{for } t \leq 0 \end{cases},$$

the graph of which looks like



Lemma 10.1.1

$f : \mathbb{R} \to \mathbb{R}$ is smooth.

Proof. It is clearly smooth in $\mathbb{R} \setminus \{0\}$ because of the exponential nature. So, one only needs to show that all the derivatives of f exist and are continuous at 0.

One first verifies that for any $k \geq 0$,

$$\lim_{t\to 0^+} \frac{e^{-\frac{1}{t}}}{t^k} = 0\,.$$

In fact, $\lim_{t\to 0^+} \frac{e^{-\frac{1}{t}}}{t^k} = \lim_{t\to 0^+} \frac{t^{-k}}{e^{\frac{1}{t}}}$. So it suffices to show that $\lim_{t\to 0^+} \frac{t^{-k}}{e^{\frac{1}{t}}} = 0$. We shall prove it by inuction. The base case is $k = 0$.

$$\lim_{t\to 0^+} \frac{t^0}{e^{\frac{1}{t}}} = \lim_{t\to 0^+} \frac{1}{e^{\frac{1}{t}}} = 0\,.$$

By inductive hypothesis, this statement is true for some $k \geq 0$. Now we shall show it for $k+1$.

$$\lim_{t\to 0^+} \frac{t^{-(k+1)}}{e^{\frac{1}{t}}} \overset{\text{L'Hôpital}}{=} \lim_{t\to 0^+} \frac{-(k+1)\,t^{-k-2}}{-t^{-2}e^{\frac{1}{t}}} = (k+1) \lim_{t\to 0^+} \frac{t^{-k}}{e^{\frac{1}{t}}} = 0\,.$$

Hence, for any $k \geq 0$, one has $\lim_{t\to 0^+} \frac{e^{-\frac{1}{t}}}{t^k} = 0$. Now, we show by induction that for $t > 0$, the k-th derivative of f is of the form

$$f^{(k)}(t) = \frac{P_k(t)}{t^{2k}} e^{-\frac{1}{t}}\,,$$

for some polynomial P_k. It's clearly true for $k=0$, i.e. $f(t)=\frac{P_0(t)}{1}e^{-\frac{1}{t}}$. Here, $P_0(t)=1$. Suppose this statement is true for some $k\geq 0$. Now we shall show it for $k+1$.

$$\begin{aligned} f^{(k+1)}(t) &= \left(\frac{P_k(t)}{t^{2k}}e^{-\frac{1}{t}}\right)' = \frac{P_k'(t)}{t^{2k}}e^{-\frac{1}{t}} + \frac{P_k(t)}{t^{2k}}\frac{1}{t^2}e^{-\frac{1}{t}} - 2k\frac{P_k(t)}{t^{2k+1}}e^{-\frac{1}{t}} \\ &= \frac{t^2P_k'(t)+P_k(t)-2ktP_k(t)}{t^{2k+2}}e^{-\frac{1}{t}} = \frac{\widetilde{P_{k+1}}(t)}{t^{2k+2}}e^{-\frac{1}{t}} \end{aligned}$$

where $\widetilde{P_{k+1}} = t^2P_k'(t)+P_k(t)-2ktP_k(t)$ is a polynomial. So, we have proved by induction that $f^{(k)}(t)=\frac{P_k(t)}{t^{2k}}e^{-\frac{1}{t}}$. Now,

$$\lim_{t\to 0^+} f^{(k)}(t) = \lim_{t\to 0^+}\frac{P_k(t)}{t^{2k}}e^{-\frac{1}{t}} = P_k(0)\lim_{t\to 0^+}\frac{e^{-\frac{1}{t}}}{t^{2k}} = 0\,.$$

We now show that for each $k\geq 0$, $f^{(k)}(0)=0$. Again, we use induction. For $k=0$, $f(0)=0$. Let us assume that $f^{(k)}(0)=0$ for some $k\geq 0$. Now, we shall show that $f^{(k+1)}(0)=0$.

$$\begin{aligned} f^{(k+1)}(0) &= \lim_{t\to 0^+}\frac{f^{(k)}(t)-f^{(k)}(0)}{t} = \lim_{t\to 0^+}\frac{\frac{P_k(t)}{t^{2k}}e^{-\frac{1}{t}}-0}{t} \\ &= \lim_{t\to 0^+}\frac{P_k(t)}{t^{2k+1}}e^{-\frac{1}{t}} = P_k(0)\lim_{t\to 0^+}\frac{e^{-\frac{1}{t}}}{t^{2k+1}} \\ &= 0 \end{aligned}$$

Therefore, we have shown that

$$\lim_{t\to 0^+} f^{(k)}(t) = 0 = f^{(k)}(0)\ ,$$

proving that each $f^{(k)}$ is continuous at 0, and hence f is smooth at $t=0$. ■

We now construct a smooth version of a step function denoted by $g(t)$ by dividing $f(t)$ by a positive function $l(t)$. The quotient will then be zero for $t\leq 0$ as follows from the definition of $f(t)$. We want the denominator function $l(t)$ to be equal to $f(t)$ for $t\geq 1$, which will then mean that $g(t)=1$ for $t\geq 1$. This suggests that we choose $l(t)=f(t)+f(1-t)$. So we define

$$g(t)=\frac{f(t)}{f(t)+f(1-t)}\,.$$

Clearly, g is 0 when $t\leq 0$, and 1 when $t\geq 1$. Now, for $0<t<1$, $0<1-t<1$. In this case, g is

$$g(t)=\frac{f(t)}{f(t)+f(1-t)}=\frac{e^{-1/t}}{e^{-1/t}+e^{-1/(1-t)}}=\frac{1}{1+\frac{e^{1/t}}{e^{1/(1-t)}}}\,.$$

Thus, we obtain a piecewise formula for g.

$$g(t)=\begin{cases} 0 & \text{if } t\leq 0 \\ \frac{1}{1+\frac{e^{1/t}}{e^{1/(1-t)}}} & \text{if } 0<t<1 \\ 1 & \text{if } t\geq 1 \end{cases}$$

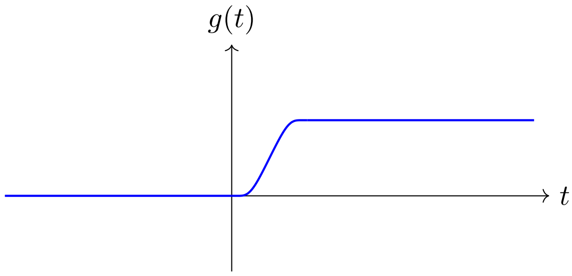
The function g is C^∞ on $\mathbb{R}$ since it is a quotient of C^∞ functions and the denominator is never 0. Next, we want to show that g is strictly increasing on $(0,1)$. For $t\in[0,1]$,

$$g(t)=\frac{1}{1+\frac{e^{1/t}}{e^{1/(1-t)}}}=\frac{e^{1/(1-t)}}{e^{1/(1-t)}+e^{1/t}}\,.$$

Taking derivative, we get

$$\begin{aligned}\frac{\mathrm{d}g}{\mathrm{d}t}(t) &= \frac{\frac{1}{(1-t)^2}e^{1/(1-t)}\left(e^{1/(1-t)}+e^{1/t}\right)-e^{1/(1-t)}\left(\frac{1}{(1-t)^2}e^{1/(1-t)}-\frac{1}{t^2}e^{1/t}\right)}{\left(e^{1/(1-t)}+e^{1/t}\right)^2}\\ &= \frac{e^{\frac{1}{1-t}+\frac{1}{t}}}{\left(e^{1/(1-t)}+e^{1/t}\right)^2}\left(\frac{1}{(1-t)^2}+\frac{1}{t^2}\right)\end{aligned}$$

which is positive in $(0,1)$. Hence, g is strictly increasing on $(0,1)$. The graph of g is as follows:



We will now make a linear change of variables. Choose two positive real numbers $a<b$, and make a linear change of variables to map $\left[a^2,b^2\right]$ to $[0,1]$.

$$x \mapsto \widetilde{x} = \frac{x}{b^2-a^2} - \frac{a^2}{b^2-a^2}.$$

Then we have,

$$\begin{aligned} x \in \left(-\infty, a^2\right) &\implies \widetilde{x} \in (-\infty,0)\\ x \in \left[a^2,b^2\right] &\implies \widetilde{x} \in [0,1]\\ x \in \left(b^2,\infty\right) &\implies \widetilde{x} \in (1,\infty)\end{aligned}$$

Now, set $h(x) = g(\widetilde{x}) = g\left(\frac{x-a^2}{b^2-a^2}\right)$. Since

$$g(\widetilde{x}) = \begin{cases} 0 & \text{for } \widetilde{x} \leq 0 \\ 1 & \text{for } \widetilde{x} \geq 1 \end{cases},$$

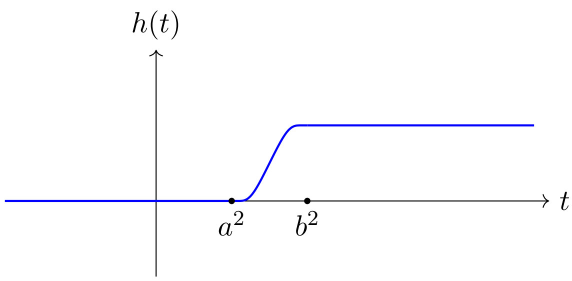
one has

$$h(x) = \begin{cases} 0 & \text{for } x \leq a^2 \\ 1 & \text{for } x \geq b^2 \end{cases}.$$

Furthermore,

$$\frac{\mathrm{d}h}{\mathrm{d}x}(x) = \frac{\mathrm{d}g}{\mathrm{d}\widetilde{x}}(\widetilde{x})\frac{\mathrm{d}\widetilde{x}}{\mathrm{d}x} = \frac{g'(\widetilde{x})}{b^2-a^2}.$$

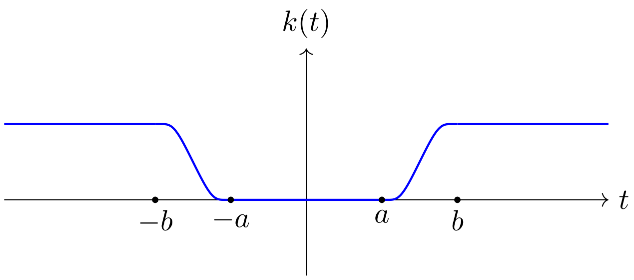
We have seen that $g'(\widetilde{x}) > 0$ when $\widetilde{x} \in (0,1)$. Hence, $h'(x) > 0$ when $x \in \left(a^2,b^2\right)$. So, the smooth function h is strrictly increasing on $\left(a^2,b^2\right)$. The graph of h is as follows:



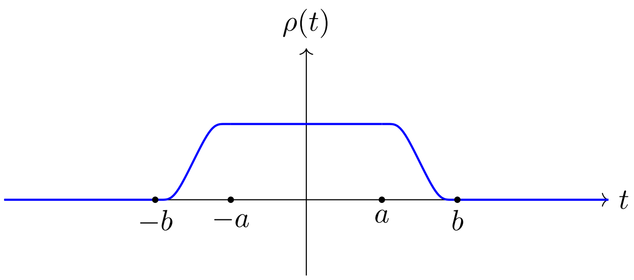
Now, let $k(x) = h(x^2)$, which makes k and even function of x and hence is symmetric about the x axis.

$$k(x) = h(x^2) = g\left(\frac{x^2 - a^2}{b^2 - a^2}\right).$$

One finds that whenever $x \in (-\infty, -b] \cup [b, \infty)$, $k(x) = 1$, and whenever $x \in [-a, a]$, $k(x) = 0$. One can verify as above that $k(x)$ is strictly decreasing in $(-b, -a)$, and strictly increasing in (a, b). Hence, the graph of k is as follows:



Finally, set $\rho(x) = 1 - k(x) = 1 - g\left(\frac{x^2-a^2}{b^2-a^2}\right)$, the graph of which is as follows:



This $\rho(x)$ is a C^∞ bump function at $0 \in \mathbb{R}$. We now give the definition of a bump function. Recall that $\mathbb{R}^\times$ denotes the set of nonzero real numbers.

Definition 10.1.1 (Support)**.** The **support** of a real-valued function $f : M \to \mathbb{R}$ on a manifold M is defined to be the closure in M of the subset on which $f \neq 0$.

$$\operatorname{supp} f = \overline{\{q \in M \mid f(q) \neq 0\}} = \operatorname{cl}_M\left(f^{-1}(\mathbb{R}^\times)\right).$$

Definition 10.1.2 (Bump Function)**.** Let $q \in M$ and U a neighborhood of q. By a **bump function** at q supported in U, we mean any continuous non-negative function $\rho : M \to \mathbb{R}$ that is 1 is a neighborhood of q with $\operatorname{supp} \rho \subseteq U$. We call it a **smooth bump function** if it is C^∞ as a map between manifolds.

We have previously constructed a C^∞ bump function at 0 in $\mathbb{R}$ that is identically 1 on $[-a, a]$ and has support in $[-b, b]$. By shifting the graph to the right, for any $q \in \mathbb{R}$, $\rho(x - q)$ is a C^∞ bump function at q.

One can easily extend this construction for a bump function on $\mathbb{R}^n$. To get a C^∞ bump function at $\mathbf{0} \in \mathbb{R}^n$ that is 1 on the closed ball $\overline{B}(\mathbf{0}, a)$ and has support in the closed ball $\overline{B}(\mathbf{0}, b)$, set

$$\sigma(\mathbf{x}) = \rho(\|\mathbf{x}\|) = 1 - g\left(\frac{\|\mathbf{x}\|^2 - a^2}{b^2 - a^2}\right).$$

σ is smooth, because

$$\mathbf{x} \mapsto \frac{\|\mathbf{x}\|^2 - a^2}{b^2 - a^2} \text{ and } t \mapsto 1 - g(t)$$

are both C^∞, and their composition is σ. To get a C^∞ bump function at $\mathbf{q} \in \mathbb{R}^n$, take $\sigma(\mathbf{x} - \mathbf{q})$.

Construction of a Smooth Bump Function on a Manifold

We have constructed a C^∞ bump function $\sigma(\mathbf{x}-\mathbf{q})$ at $\mathbf{q}\in\mathbb{R}^n$ from a C^∞ bump function $\sigma(\mathbf{x})$ at $\mathbf{0}\in\mathbb{R}^n$ whose supprt is contained in the closed ball $\overline{B}(\mathbf{0},b)$. Now we want to extend this idea from $\mathbb{R}^n$ to a manifold M.

Exercise 10.1. Let M be an n-dimensional manifold and $q\in M$. Suppose U is any neighborhood of q. Construct a smooth bump function at q supprted in U.

Solution. There exists a coordinate chart (V,ψ) in the maximal atlas of M such that $q\in V\subseteq U$. Such a coordinate open set exists because if V' is a coordinate open set about q, we can just take $V=V'\cap U$.

Now, there exists a bump function $\rho:\mathbb{R}^n\to\mathbb{R}$ at $\psi(q)$ supported in $\psi(V)\subseteq\mathbb{R}^n$ that is identically 1 in $\overline{B}(\psi(q),a)$. Suppose the support of ρ is $\overline{B}(\psi(q),b)$ for some $b>0$. Then

$$\overline{B}(\psi(q),a)\subseteq\overline{B}(\psi(q),b)\subseteq\psi(V)\subseteq\mathbb{R}^n\,.$$

Now, define a function $f:M\to\mathbb{R}$ by

$$f(p)=\begin{cases}\rho(\psi(p)) & \text{if } p\in V\\ 0 & \text{if } p\notin V\end{cases}.$$

For $p\in V$, f is the composition of two smooth maps ρ and ψ, and hence f is smooth at p. Now we need to check that f is smooth at $p\notin V$.

Note that, by the construction of $\rho:\mathbb{R}^n\to\mathbb{R}$, $\operatorname{supp}\rho=\overline{B}(\psi(q),b)\subseteq\psi(V)\subseteq\mathbb{R}^n$. Being a closed and bounded subset of $\mathbb{R}^n$, $\operatorname{supp}\rho$ is compact. Since ψ^{-1} is continuous, $\psi^{-1}(\operatorname{supp}\rho)$ is also a compact subspace of M. M is a manifold, hence it is Hausforff. Therefore, $\psi^{-1}(\operatorname{supp}\rho)$ is closed in M. As a result,

$$\begin{aligned}\operatorname{supp} f&=\operatorname{cl}_M\left(\psi^{-1}\left(\rho^{-1}\left(\mathbb{R}^\times\right)\right)\right)=\psi^{-1}\left(\operatorname{cl}_{\mathbb{R}^n}\left(\rho^{-1}\left(\mathbb{R}^\times\right)\right)\right)\\&=\psi^{-1}(\operatorname{supp}\rho)\subseteq V\end{aligned}$$

Since $\operatorname{supp} f\subseteq V$, $p\notin V$ gives us

$$p\in M\setminus V\subseteq M\setminus\operatorname{supp} f\implies p\in M\setminus\operatorname{supp} f\,.$$

$\operatorname{supp} f$ is closed, hence $M\setminus\operatorname{supp} f$ is open. If (W',φ') is a chart about p, $W=W'\cap(M\setminus\operatorname{supp} f)$ is a coordinate open set about p contained in $M\setminus\operatorname{supp} f$. If we write $\varphi=\varphi'|_W$, then (W,φ) is a chart about p.

$$f\circ\varphi^{-1}:\varphi(W)\to\mathbb{R}\,.$$

For $\varphi(x)\in\varphi(W)$, $x\in W\subseteq M\setminus\operatorname{supp} f$. So $f(x)=0$. Therefore, the map $f\circ\varphi^{-1}$ is identically 0, and hence smooth in $\varphi(W)$. As a result, f is smooth on $W\ni p$. In particular, f is smooth at p. Since $p\notin V$ is arbitrary, f is smooth at all $p\notin V$. Therefore, f is smooth on all of M. ■

In general, a C^∞ function on an open subset U of a manifold M cannot be extended to a C^∞ function on the whole of M. For instance, take $\sec(x)$ as a C^∞ function on $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\subseteq\mathbb{R}$. We can't find a C^∞ function on $\mathbb{R}$ that agrees with $\sec(x)$ globally, i.e. on the whole of $\mathbb{R}$. However, if we require the C^∞ function on M (to be found) to agree with a given C^∞ function only on some neighborhood of a point in U, then such a C^∞ extension is possible.

Proposition 10.1.2 (C^∞ extension of a function)

Suppose f is a C^∞ function defined on a neighborhood U of a point $p\in M$. Then there is a C^∞ function $\widetilde{f}$ on M that agrees with f in some possibly smaller neighborhood of p (i.e., contained in U).

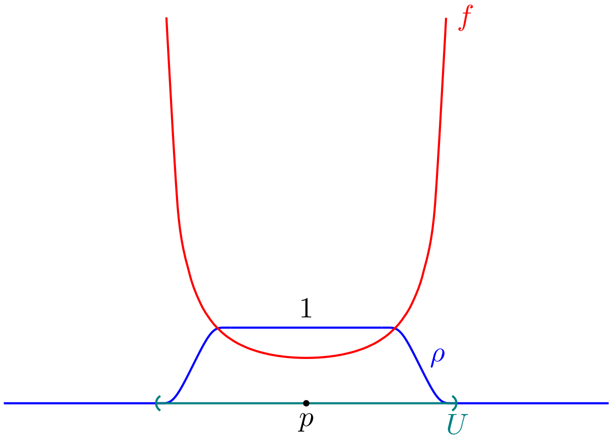


Figure 10.1: Extending the domain of a function by multiplying by a bump function.

Proof. Choose a C^∞ bump function $\rho : M \to \mathbb{R}$ that is identically 1 in a neighborhood V of p with $V \subseteq U$ (see the previous construction) and $\operatorname{supp}\rho \subseteq U$. Now, define

$$\widetilde{f}(q) = \begin{cases} \rho(q) f(q) & \text{for } q \in U \\ 0 & \text{for } q \notin U \end{cases} .$$

By the definition, $\widetilde{f}$ agrees with f on $V \subset U$. As the product of two smooth functions on U, $\widetilde{f}$ is smooth on U. Now, using the definition of $\widetilde{f}$,

$$\widetilde{f}(p) \neq 0 \implies \rho(q) \neq 0 \text{ and } f(q) \neq 0 \implies \widetilde{f}^{-1}\left(\mathbb{R}^\times\right) \subseteq \rho^{-1}\left(\mathbb{R}^\times\right) .$$

$$\therefore \operatorname{cl}_M\left(\widetilde{f}^{-1}\left(\mathbb{R}^\times\right)\right) \subseteq \operatorname{cl}_M\left(\rho^{-1}\left(\mathbb{R}^\times\right)\right) \implies \operatorname{supp}\widetilde{f} \subseteq \operatorname{supp}\rho .$$

If $q \notin U$, then $q \notin \operatorname{supp}\widetilde{f}$ (since $\operatorname{supp}\widetilde{f} \subseteq \operatorname{supp}\rho \subseteq U$). Since $\operatorname{supp}\widetilde{f}$ is closed in M, one can find an coordinate neighborhood of q that is disjoint from $\operatorname{supp}\widetilde{f}$. On this open set, $\widetilde{f}$ is identically 0. Therefore, similarly as in the solution of previous exercise, $\widetilde{f}$ is smooth at q. Since $q \notin U$ is arbitrary, $\widetilde{f}$ is smooth at every $q \notin U$. ■

§10.2 Partitions of Unity

If $\{U_i\}_{i\in I}$, I being finite, is a finite open cover of M, a C^∞ partition of unity subordinate to $\{U_i\}_{i\in I}$ is a collection of non-negative C^∞ functions $\{\rho_i : M \to \mathbb{R}\}_{i\in I}$ such that $\operatorname{supp}\rho_i \subseteq U_i$, and

$$\sum_{i\in I} \rho_i = 1 .$$

When I is an infinite set, for the sum to make sense, we'll impose *local finiteness* condition.

Definition 10.2.1 (Local Finiteness)**.** A collection $\{A_\alpha\}_\alpha$ of subsets of a topological space S is said to be **locally finite** for every point $q \in S$ has a neighborhood that intersects only finitely many of the A_α's. In particular, every $q \in S$ is contained in only finitely many of the A_α's.

Example 10.2.1 (An open cover that is not locally finite)
Let $U_{r,n}$ be the open interval $\left(r - \frac{1}{n}, r + \frac{1}{n}\right)$ on the real line $\mathbb{R}$. The open cover $\{U_{r,n} \mid r \in \mathbb{Q},\ n \in \mathbb{Z}^+\}$ of $\mathbb{R}$ is not locally finite.

Definition 10.2.2 (Partition of Unity). A C^∞ **partition of unity** on a manifold M is a collection of non-negative C^∞ functions $\{\rho_\alpha : M \to \mathbb{R}\}_{\alpha \in A}$ such that

(i) The collection of supports $\{\operatorname{supp} \rho_\alpha\}_{\alpha \in A}$ is locally finite.

(ii) $\sum \rho_\alpha = 1$.

Given an open cover $\{U_\alpha\}_{\alpha \in A}$ of M, we say that a partition of unity $\{\rho_\alpha\}_{\alpha \in A}$ is **subordinate** to the open cover $\{U_\alpha\}_{\alpha \in A}$ if $\operatorname{supp} \rho_\alpha \subseteq U_\alpha$ for every $\alpha \in A$.

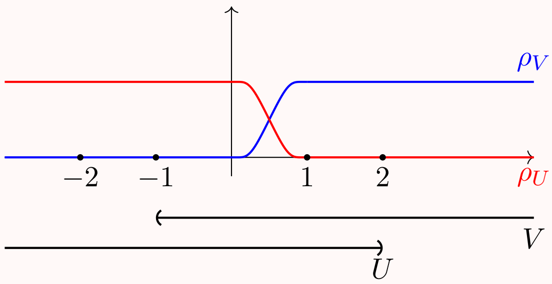
Remark 10.2.1. Since the collection of supports $\{\operatorname{supp} \rho_\alpha\}_{\alpha \in A}$ is locally finite, every point $q \in M$ lies in finitely many sets $\operatorname{supp} \rho_\alpha$. Hence, $\rho_\alpha(q) \neq 0$ for only finitely many α. Hence, the sum $\sum \rho_\alpha$ is a finite sum at every point.

Example 10.2.2

Let U and V be the open intervals $(-\infty, 2)$ and $(1, \infty)$ in $\mathbb{R}$, respectively. Define

$$\rho_V(t) = g(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ \frac{e^{1/(1-t)}}{e^{1/(1-t)} + e^{1/t}} & \text{if } 0 < t < 1 \\ 1 & \text{if } t \geq 1 \end{cases}.$$

Define $\rho_U = 1 - \rho_V$.



Then $\{\rho_U, \rho_V\}$ is a partition of unity subordinate to the open cover $\{U, V\}$.

Existence of Partition of Unity

Lemma 10.2.1

If $\rho_1, \ldots, \rho_m$ are real-valued function on a manifold M, then

$$\operatorname{supp}\left(\sum_{i=1}^{m} \rho_i\right) \subseteq \bigcup_{i=1}^{m} \operatorname{supp} \rho_i .$$

Proof. Let $\rho = \sum_{i=1}^{m} \rho_m$, $A_i = \rho_i^{-1}(\mathbb{R}^\times)$ and $A = \rho^{-1}(\mathbb{R}^\times)$. If $p \in A$, $\rho(p) \neq 0$. Then $\rho_i(p) \neq 0$ for some i.

$$p \in A \implies p \in A_i \subseteq \bigcup_{i=1}^{m} A_i \implies A \subseteq \bigcup_{i=1}^{m} A_i .$$

Now, by Proposition 1.2.5 and Proposition 1.2.7,

$$\operatorname{cl}_M(A) \subseteq \operatorname{cl}_M\left(\bigcup_{i=1}^{m} A_i\right) = \bigcup_{i=1}^{m} \operatorname{cl}_M(A_i) .$$

$\mathrm{cl}_M(A) = \operatorname{supp} \rho = \operatorname{supp}\left(\sum_{i=1}^m \rho_m\right)$, and $\mathrm{cl}_M(A_i) = \operatorname{supp} \rho_i$. Therefore,

$$\operatorname{supp}\left(\sum_{i=1}^m \rho_i\right) \subseteq \bigcup_{i=1}^m \operatorname{supp} \rho_i\,.$$

■

Proposition 10.2.2

Let M be a compact manifold and $\{U_\alpha\}_{\alpha\in A}$ and open cover of M. There exists a C^∞ partition of unity $\{\rho_\alpha\}_{\alpha\in A}$ subordinate to $\{U_\alpha\}_{\alpha\in A}$.

Proof. Let $q \in M$, find an open set U_α containing q fro the given cover, and let ψ_q be a C^∞ bump function at q supported in U_α. Since $\psi_q(q) > 0$, we can find a neighborhood W_q of q such that $\psi_q(p) > 0$ for every $p \in W_q$.

By the compactness of M, the open cover $\{W_q \mid q \in M\}$ has a finite subcover, say $\{W_{q_1}, \ldots, W_{q_m}\}$. Let $\psi_{q_1}, \ldots, \psi_{q_m}$ be the corresponding bump functions (eacg of these functions is supported in some U_α). Since $\{W_{q_1}, \ldots, W_{q_m}\}$ is a finite open cover of M, for any point $q \in M$, $q \in W_{q_i}$ for some $i \in \{1, 2, \ldots, m\}$. Hence,

$$\psi(q) := \sum_{i=1}^m \psi_{q_i}(q) > 0\,.$$

Define $\varphi_i = \frac{\psi_{q_i}}{\psi}$. In other words, for $q \in M$,

$$\varphi_i(q) = \frac{\psi_{q_i}(q)}{\psi(q)}\,.$$

This division is well-defined, since $\psi(q) > 0$.

$$\sum_{i=1}^m \varphi_i(q) = \sum_{i=1}^m \frac{\psi_{q_i}(q)}{\psi(q)} = \frac{\sum\limits_{i=1}^m \psi_{q_i}(q)}{\psi(q)} = \frac{\psi(q)}{\psi(q)} = 1\,.$$

Since $\psi > 0$, $\varphi_i(q) \neq 0$ if and only if $\psi_{q_i}(q)$. Therefore,

$$\operatorname{supp} \varphi_i \subseteq \operatorname{supp} \psi_{q_i} \subseteq U_\alpha \text{ for some } \alpha \in A\,.$$

Hence, $\{\varphi_i\}_{i=1}^m$ is a partition of unity such that for every $i \in \{1, 2, \ldots, m\}$, $\operatorname{supp} \varphi_i \subseteq U_\alpha$ for some $\alpha \in A$.

The next step is to make the index set of the partition of unity the same as that of the open cover. Now we shall define a function $\tau : \{1, 2, \ldots, m\} \to A$. For each $i \in \{1, 2, \ldots, m\}$, we define $\tau(i)$ to be an index $\alpha \in A$ such that

$$\operatorname{supp} \varphi_i \subseteq U_\alpha\,.$$

Note that, there might be multiple choices for $\tau(i)$. In other words, it might be the case that $\operatorname{supp} \varphi_i$ is contained in both U_α and U_β. In that case, we define $\tau(i)$ to be either of α or β.

Now, we group the collection of functions $\{\varphi_i\}$ into subcollections according to $\tau(i)$, i.e., all the φ_i's will be in the same subcollection if $\tau(i) = \alpha$ for some $\alpha \in A$. Let us define ρ_α as

$$\rho_\alpha = \sum_{\tau(i)=\alpha} \varphi_i\,.$$

If there is no $i \in \{1, 2, \ldots, m\}$ for which $\tau(i) = \alpha$, we simply define $\rho_\alpha = 0$. Then we have,

$$\sum_{\alpha\in A} \rho_\alpha = \sum_{\alpha\in A} \sum_{\tau(i)=\alpha} \varphi_i = \sum_{i=1}^m \varphi_i = 1\,.$$

If $\rho_\alpha = 0$, then $\operatorname{supp}\rho_\alpha = \varnothing \subseteq U_\alpha$. Otherwise, by Lemma 10.2.1,

$$\operatorname{supp}(\rho_\alpha) = \operatorname{supp}\left(\sum_{\tau(i)=\alpha} \varphi_i\right) \subseteq \bigcup_{\tau(i)=\alpha} \operatorname{supp}\varphi_i \subseteq \bigcup_{\tau(i)=\alpha} U_\alpha = U_\alpha\,.$$

τ is a map from a finite set $\{1, 2, \ldots, m\}$ to A. Therefore, $\operatorname{im}\tau$ is a finite subset of A. This means that there are only finitely many α for which ρ_α is not identically 0 on all of M. Hence, there are only finitely many α's for which $\operatorname{supp}\rho_\alpha \neq \varnothing$. Since there are only finitely many nonempty sets in the collection $\{\operatorname{supp}\rho_\alpha\}_{\alpha\in A}$, the collection is locally finite. Therefore, $\{\rho_\alpha\}_{\alpha\in A}$ is a partition of unity subordinate to the open cover $\{U_\alpha\}_{\alpha\in A}$. ■

Now we shall state a generalization of Proposition 10.2.2 without proof.

Theorem 10.2.3 (Existence of a C^∞ partition of unity)
Let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of a manifold M.

(i) There is a C^∞ partition of unity $\{\varphi_k\}_{k=1}^{\infty}$ with every φ_k having compact support, such that for each k, $\operatorname{supp}\varphi_k \subseteq U_\alpha$ for some $\alpha \in A$.

(ii) If we relax the condition of having compact supprt, then there is a C^∞ partition of unity $\{\rho_\alpha\}_{\alpha\in A}$ subordinate to the open cover $\{U_\alpha\}_{\alpha\in A}$.

11 Vector Field

§11.1 Smoothness of a Vector Field

Recall from the definition of Vector Field that a vector field X on a manifold M is smooth if the map $X : M \to TM$ is smooth as a section of the tangent bundle $\pi : TM \to M$[1]. In a coordinate chart $(U, \varphi) = \left(U, x^1, \ldots, x^n\right)$ on M about p, the value of the vector field X at $p \in U$ is a linear combination

$$X_p = \sum_{i=1}^{n} a^i(p) \left.\frac{\partial}{\partial x^i}\right|_p .$$

As p varies in U, the coefficients a^i become functions on U. We've seen that the chart (U, φ) on the manifold M induces a chart $(TU, \widetilde{\varphi})$ on the tangent bundle TM. Here,

$$TU = \bigsqcup_{p \in U} T_pU\,, \text{ and } \widetilde{\varphi} : TU \to \varphi(U) \times \mathbb{R}^n$$

is a homeomorphism. The homeomorphism $\widetilde{\varphi}$ is given by

$$(p, \mathbf{v}_p) \mapsto \left(x^1(p), \ldots, x^n(p), c^1(\mathbf{v}_p), \ldots, c^n(\mathbf{v}_p)\right) ,$$

where $\mathbf{v}_p \in T_pM$ with $\mathbf{v}_p = \sum\limits_{i=1}^{n} c^i(\mathbf{v}_p) \left.\frac{\partial}{\partial x^i}\right|_p$. Therefore,

$$X_p = \sum_{i=1}^{n} a^i(p) \left.\frac{\partial}{\partial x^i}\right|_p = \sum_{i=1}^{n} c^i(X_p) \left.\frac{\partial}{\partial x^i}\right|_p .$$

Writing X_p as $X(p)$ and equating the coefficients, we get

$$a^i(p) = \left(c^i \circ X\right)(p) \implies a^i = c^i \circ X$$

as functions on U. Since $X(U) \subseteq TU$, and $a^i = c^i \circ X$, one finds that c^i's are smooth functions on TU. Thus, if X is smooth and $\left(U, x^1, \ldots, x^n\right)$ is any chart on M, then the coefficients a^i of $X = \sum\limits_{i=1}^{n} a^i \frac{\partial}{\partial x^i}$ relative to the frame $\frac{\partial}{\partial x^i}$ are smooth functions on U (By Proposition 9.4.2). The converse is also true as provided by the following lemma.

Lemma 11.1.1 (Smoothness of a vector field on a chart)

Let $(U, \varphi) = \left(U, x^1, \ldots, x^n\right)$ be a chart on a manifold M. A vector field $X = \sum\limits_{i=1}^{n} a^i \frac{\partial}{\partial x^i}$ on U is smooth (i.e., the section $X : U \to \pi^{-1}(U) = TU$ of the tangent bundle $\pi : TM \to M$ over U is smooth) if and only if the coefficient functions a^i are all smooth on U.

Proof. This lemma is a special case of Proposition 9.4.2, where we take E to be the tangent bundle TM of M and $\{s_i\}_{i=1}^{r}$, the C^∞ frame for E over U to be the coordinate vector fields $\left\{\frac{\partial}{\partial x^i}\right\}_{i=1}^{n}$. ■

Proposition 11.1.2 (Smoothness of a vector field in terms of coefficients)

Let X be a vector field on a manifold M. The following are equivalent:

(i) The vector field X is smooth on M.

(ii) The manifold M has an atlas $\mathscr{U}$ such that on any chart $(U, \varphi) = \left(U, x^1, \ldots, x^n\right)$ of $\mathscr{U}$, the

[1] As discussed in Remark 9.3.1, we are considering $TM = \bigcup\limits_{p \in M} T_pM$.

coefficients a^i of $X = \sum_{i=1}^{n} a^i \frac{\partial}{\partial x^i}$ relative to the frame $\frac{\partial}{\partial x^i}$ are all smooth.

(iii) On any chart $(U, \varphi) = \left(U, x^1, \ldots, x^n\right)$ on the manifold M, the coefficients a^i of $X = \sum_{i=1}^{n} a^i \frac{\partial}{\partial x^i}$ relative to the frame $\frac{\partial}{\partial x^i}$ are all smooth.

Proof. **(ii)⇒(i):** Since (ii) holds, X is smooth on every chart (U, φ) of the atlas $\mathscr{U}$. Since $\mathscr{U}$ covers all of M, X is smooth on M.

(i)⇒(iii): If X is smooth on M, then it is smooth on U for any chart (U, φ). Then Lemma 11.1.1 implies (iii).

(iii)⇒(ii): Just take $\mathscr{U}$ to be the maximal atlas of M. ■

A vector field X on a manifold M induces a linear map on the algebra $C^\infty(M)$ of C^∞ functions on M: for $f \in C^\infty(M)$, define Xf to be the function

$$(Xf)(p) := X_p f\,, \qquad p \in M\,.$$

Proposition 11.1.3 (Smoothness of a vector field in terms of functions)

A vector field X on M is smooth if and only if for every smooth function f on M, the function Xf is smooth on M.

Proof. **(⇒):** Suppose X is smooth and $f \in C^\infty(M)$. By Proposition 11.1.2, on any chart $\left(U, x^1, \ldots, x^n\right)$ on M, the coefficients a^i of $X = \sum_{i=1}^{n} a^i \frac{\partial}{\partial x^i}$ relative to the frame $\frac{\partial}{\partial x^i}$ are all smooth.

$$Xf = \sum_{i=1}^{n} a^i \frac{\partial f}{\partial x^i}$$

$a^i \frac{\partial f}{\partial x^i}$ is smooth on U as the product of two smooth functions. Hence, their sum Xf is smooth on U. Since M can be covered by charts, Xf is C^∞ on M.

(⇐): Let $(U, \varphi) = \left(U, x^1, \ldots, x^n\right)$ be any chart on M. Suppose $X = \sum_{i=1}^{n} a^i \frac{\partial}{\partial x^i}$ is C^∞ on U and $p \in U$. Each x^k is a smooth function on U. Hence, by Proposition 10.1.2, each x^k can be extended to a C^∞ function $\widetilde{x}^k$ on M that agrees with x^k in a neighborhood V of p that is contained in U. Xf is smooth for every $f \in C^\infty(M)$, taking $f = \widetilde{x}^k$ we get that $X\widetilde{x}^k$ is also smooth on M. Now, on $V \subseteq U$,

$$X\widetilde{x}^k = \left(\sum_{i=1}^{n} a^i \frac{\partial}{\partial x^i}\right) \widetilde{x}^k = \left(\sum_{i=1}^{n} a^i \frac{\partial}{\partial x^i}\right) x^k = a^k\,.$$

Therefore a^k is smooth at $p \in U$. Since p is an arbitrary point in U, a^k is smooth on U for any chart (U, φ). Therefore, by Proposition 11.1.2, X is smooth on M. ■

Proposition 11.1.4 (Smooth extension of a vector field)

Suppose X is a C^∞ vector field defined on a neighborhood U of a point p in a manifold M. Then there is a C^∞ vector field $\widetilde{X}$ on X that agrees with X on some possibly smaller neighborhood of p, say $V \subseteq U$.

Proof. Choose a C^∞ bump function $\rho : M \to \mathbb{R}$ supported in U that is identically 1 in a neighborhood V of p. Define

$$\widetilde{X}(q) = \begin{cases} \rho(q)\, X(q) & \text{for } q \in U \\ \mathbf{0}_{T_q M} & \text{for } q \notin U \end{cases},$$

where $\mathbf{0}_{T_qM}$ is the zero vector of T_qM. By the definition of $\widetilde{X}$, it agrees with X on V. By Proposition 9.3.1(ii), $\widetilde{X}$ is smooth on U. Now, let $q \notin U$. We want to show that $\widetilde{X}$ is smooth at q.

Since $\operatorname{supp}\rho \subseteq U$, $q \notin U$ implies $q \in M \setminus U \subseteq M \setminus \operatorname{supp}\rho$. Since $\operatorname{supp}\rho$ is closed, $M \subseteq \operatorname{supp}\rho$ is open. Hence, we can find a coordinate chart (W, φ) about q such that $W \subseteq M \setminus \operatorname{supp}\rho$. Then, for $r \in W$, $\widetilde{X}(r) = \mathbf{0}_{T_rM}$. Also, $(TW, \widetilde{\varphi})$ is a chart on TM about $\mathbf{0}_{T_rM}$.

$$\left(\widetilde{\varphi} \circ \widetilde{X}\right)(r) = (\varphi(r), \underbrace{0, 0, \ldots, 0}_{n \text{ 0s}}) \,.$$

φ is smooth. Therefore, by Proposition 4.2.8, $\widetilde{X}$ is smooth on W. In particular, $\widetilde{X}$ is smooth at q. Since $q \notin U$ was arbitrary, $\widetilde{X}$ is smooth at every $q \notin U$. Therefore, $\widetilde{X}$ is smooth on all of M. ■

§11.2 Integral Curves

Definition 11.2.1 (Integral Curve). Let X be a C^∞ vector field on M, and $p \in M$. An **integral curve** of X is a smooth curve $c : (a, b) \to M$ such that $c'(t) = X_{c(t)}$ for every $t \in (a, b)$.

Usually, we assume that $0 \in (a, b)$. In this case, if $c(0) = p$, then we say that c is the integral curve starting at p, and call p the initial point of c.

To show the dependence of an integral curve on the initial point p, one also writes $c_t(p)$ instead of $c(t)$.

Definition 11.2.2. An integral curve is **maximal** if its domain can't be extended to a larger domain.

Example 11.2.1. Recall the vector field $X_{(x,y)} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ on $\mathbb{R}^2$ (Example 9.3.1). We will find an integral curve $c(t)$ of X starting at the point $(1, 0) \in \mathbb{R}^2$. The condition for $c(t) = (x(t), y(t))$ to be an integral curve of X is

$$c'(t) = X_{c(t)} \implies \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} -y(t) \\ x(t) \end{bmatrix} \,.$$

We need to solve the system of first order ODEs

$$\dot{x}(t) = -y(t) \ , \ \dot{y}(t) = x(t)$$

with the initial conditoin $(x(0), y(0)) = (1, 0)$.

$$y(t) = -\dot{x}(t) \implies \dot{y}(t) = -\ddot{x}(t) \implies x(t) = -\ddot{x}(t) \,.$$

It's well-known that the general solution to this equation is

$$x(t) = A\cos t + B\sin t \,.$$

Hence, $y(t) = -\dot{x}(t) = A\sin t - B\cos t$. By plugging in the initial condition $(x(0), y(0)) = (1, 0)$, one obtains

$$x(0) = A = 1 \text{ and } y(0) = -B = 0 \,.$$

So, the integral curve starting at $(1, 0)$ is

$$c(t) \equiv (x(t), y(t)) = (\cos t, \sin t) \ ,$$

which parametrizes the unit circle.

More generally, if the initial point of the integral curve, corresponding to $t = 0$, is $\mathbf{p} = (x_0, y_0)$, then

$$x_0 = x(0) = A \text{ and } y_0 = y(0) = -B \,.$$

In that case, a general solution for $x(t)$ and $y(t)$ would be

$$x(t) = x_0 \cos t - y_0 \sin t\,,\ \ y(t) = x_0 \sin t + y_0 \cos t\,.$$

This can be written in matrix notation as

$$c(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \mathbf{p}\,,$$

where $\mathbf{p} \in \mathbb{R}^2$ is the given initial point. This shows that the integral curve $c(t)$ of X starting at $\mathbf{p} \in \mathbb{R}^2$, i.e. $c(0) = \mathbf{p}$ can be obtained by rotating the point $\mathbf{p} \in \mathbb{R}^2$ counterclockwise about the origin through an angle t. Notice that

$$c_s(c_t(\mathbf{p})) = c_{s+t}(\mathbf{p})\ ,$$

since a rotation through an angle t followed by a rotation through an angle s is the same as a rotation through an angle $s+t$. Also, notice that for each $t \in \mathbb{R}$, $c_t : \mathbb{R}^2 \to \mathbb{R}^2$ is a diffeomorphism with inverse c_{-t}. Indeed, for a fixed $t_0 \in \mathbb{R}$,

$$c_{t_0} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos t_0 & -\sin t_0 \\ \sin t_0 & \cos t_0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \cos t_0 - y \sin t_0 \\ x \sin t_0 + y \cos t_0 \end{bmatrix}\,.$$

$c_{t_0} : \mathbb{R}^2 \to \mathbb{R}^2$ is easily seen to be a smooth map, with the inverse $c_{-t_0} : \mathbb{R}^2 \to \mathbb{R}^2$, which is also smooth.

$\mathrm{Diff}(M)$ stands for the group of diffeomorphisms of a manifold M with itself, with the group operation being composition. A homomorphism $c : \mathbb{R} \to \mathrm{Diff}(M)$ is called a **1-parameter group of diffeomorphisms** of M. In Example 11.2.1, the integral curves of the vector field $X_{(x,y)} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ on $\mathbb{R}^2$ gives rise to a 1-parameter group of diffeomorphisms of $\mathbb{R}^2$.

Exercise 11.1. Let $X = x^2 \frac{\mathrm{d}}{\mathrm{d}x}$ be a vector field on $\mathbb{R}$. Find the maximal integral curve of X starting at $x = 2$.

Solution. Denote the integral curve by $x(t)$. Then

$$x'(t) = X_{x(t)} \implies \dot{x}(t) \left.\frac{\mathrm{d}}{\mathrm{d}x}\right|_{x(t)} = x^2 \left.\frac{\mathrm{d}}{\mathrm{d}x}\right|_{x(t)}\,.$$

Therefore, it follows that

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) = x^2 \implies \frac{\mathrm{d}x}{x^2} = \mathrm{d}t \implies -\frac{1}{x} = t + C\,.$$

Now, using the condition that $x(0) = 2$, we get

$$-\frac{1}{2} = 0 + C \implies -\frac{1}{x} = t - \frac{1}{2} = \frac{2t-1}{2} \implies \boxed{x = \frac{2}{1-2t}}\,.$$

The maximal interval containing 0 on which $x(t)$ is defined is $\left(-\infty, \frac{1}{2}\right)$. This example exhibits the fact that it may not be possible to extend the domain of definition of an integral curce to the entire real line. ■

§11.3 Local Flows

In the previous two examples, we've seen that locally, finding an integral curve requires solving a system of first order ODEs with initial conditions. In general, if X is a smooth vector field on M, to find an integral curve $c(t)$ of X starting at $p \in M$, choose first a coordinate chart $(U, \varphi) = \left(U, x^1, \ldots, x^n\right)$ about p. In local coordinates,

$$X_{c(t)} = \sum_{i=1}^{n} a^i(c(t)) \left.\frac{\partial}{\partial x^i}\right|_{c(t)}\,.$$

By Proposition 6.3.1,

$$c'(t) = \sum_{i=1}^{n} \dot{c}^i(t) \left.\frac{\partial}{\partial x^i}\right|_{c(t)} ,$$

where $c^i = x^i \circ c$ is the i-th component of $c(t)$ in the chart (U, φ). The condition $X_{c(t)} = c'(t)$ is, thus, equivalent to

$$\dot{c}^i(t) = a^i(c(t))$$

for $i = 1, \ldots, n$. This is a system of ODSs. The initial condition $c(0) = p$ translates to

$$\left(x^i \circ c\right)(0) = x^i(p) \implies \left(c^1(0), c^2(0), \ldots, c^n(0)\right) = \left(p^1, p^2, \ldots, p^n\right) ,$$

where $p^i = x^i(p)$. By an existence and uniqueness theorem from the theory of ODE, such a system has a unique solution in the following sense.

Theorem 11.3.1

Let V be an open subset of $\mathbb{R}^n$, $p_0 \in V$, and $f : V \to \mathbb{R}^n$ a C^∞ map. Then the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(y) , \quad y(0) = p_0 ,$$

has a unique C^∞ solution $y : (a(p_0), b(p_0)) \to V$, where $(a(p_0), b(p_0))$ is the maximal interval containing 0 on which y is defined.

Remark 11.3.1. The uniqueness of the solution means that if $z : (\delta, \varepsilon) \to V$ satisfies the same ODE

$$\frac{\mathrm{d}z}{\mathrm{d}t} = f(z) , \quad z(0) = p_0 ,$$

then the domain (δ, ε) of the equation of z is a subset of $(a(p_0), b(p_0))$, and $z(t) = y(t)$ on the interval (δ, ε).

The map $y : (a(p_0), b(p_0)) \to V$ can actually be thought of as a map with two arguments: t and q, and the condition for y to be an integral curve starting at the point q is

$$\frac{\mathrm{d}y}{\mathrm{d}t}(t, q) = f(y(t, q)) , \quad y(0, q) = q .$$

Theorem 11.3.2 (Smooth dependece of solution on the initial point)

Let V be an open subset of $\mathbb{R}^n$ and $f : V \to \mathbb{R}^n$ a C^∞ map on V. For each point $p_0 \in V$, there are a neighborhood W of p_0 in V and a number $\varepsilon > 0$, and a C^∞ map

$$y : (-\varepsilon, \varepsilon) \times W \to V$$

such that

$$\frac{\mathrm{d}y}{\mathrm{d}t}(t, q) = f(y(t, q)) , \quad y(0, q) = q ,$$

for all $(t, q) \in (-\varepsilon, \varepsilon) \times W$.

It follows from Theorem 11.3.2 that if X is any C^∞ vector field on a chart (U, φ) and $p \in U$, then there are a neighborhood W of p in U, an $\varepsilon > 0$, and a C^∞ map

$$F : (-\varepsilon, \varepsilon) \times W \to U$$

such that for every $q \in W$, the map $F(t, q)$ is an integral curve of X starting at q. In particular, $F(0, q) = q$. We usually write $F_t(q)$ for $F(t, q)$.

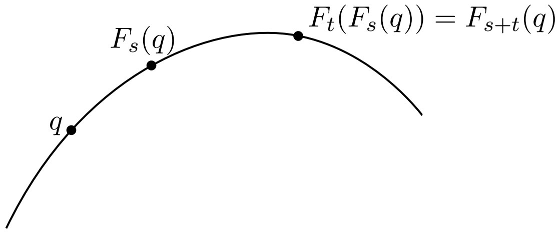


Figure 11.1: The flow line through q of a local flow.

Suppose $s, t \in (-\varepsilon, \varepsilon)$ are such that $F_t\left(F_s\left(q\right)\right)$ and $F_{t+s}\left(q\right)$ are defined. Then $F_t\left(F_s\left(q\right)\right)$ as a map of argument t is an integral curve of X starting at $F_s\left(q\right)$ (due to $t = 0$). By the uniqueness of the integral curve for a given vector field starting at a point,

$$F_t\left(F_s\left(q\right)\right) = F_{t+s}\left(q\right) \ .$$

The map F is called a **local flow** generated by the vector field X. For each $q \in U$, the map $F_t\left(q\right)$ of t is called the **flow line** of the local flow. Each flow line is an integral curve of X. If a local flow F is defined on $\mathbb{R} \times M$, then it's called a **global flow**.

Every smooth vector field has a local flow about any point, but not necessarily a global flow. A vector field having a global flow is called a **complete** vector field.

A Category Theory Basics

Lecture videos and lecture notes of Category Theory course.

§A.1 What is a Category?

Definition A.1.1 (Category)**.** A category $\mathcal{C}$ consists of

- A collection $\mathcal{C}_0$ whose elements are called the objects of $\mathcal{C}$. Elements of $\mathcal{C}_0$ are denoted by uppercase letters $X, Y, Z, \ldots$
- A collection $\mathcal{C}_1$ whose elements are called the morphisms or arrows of $\mathcal{C}$. Elements of $\mathcal{C}_1$ are denoted by lowercase letters $f, g, h, \ldots$

such that the following hold:

(i) Each morphism assigns two objects called source (or domain) and target (or codomain). We denote them by $s(f)$ and $t(f)$, respectively for a given arrow f. If $s(f) = X \in \mathcal{C}_0$ and $t(f) = Y \in \mathcal{C}_0$ for a given $f \in \mathcal{C}_1$, we write

$$f : X \to Y \ , \ \text{ or } \ X \xrightarrow{f} Y$$

(ii) Each object $X \in \mathcal{C}_0$ has a distinguished morphism $\mathrm{id}_X : X \to X$.

(iii) For each pair of morphisms $f, g \in \mathcal{C}_1$ such that $t(f) = s(g)$, there exist specified morphisms $g \circ f$ called composite morphisms such that

$$s\,(g \circ f) = s(f) \quad \text{and} \quad t\,(g \circ f) = t(g)$$

In other words, $X \xrightarrow{f} Y \xrightarrow{g} Z$ implying $g \circ f : X \to Z$.

These structures need to satisfy the following axioms:

(a) (Unitality) For each morphism $f : X \to Y$,

$$f \circ \mathrm{id}_X = \mathrm{id}_Y \circ f = f$$

Warning: $\mathrm{id}_X \circ f$ doesn't make sense, so doesn't $f \circ \mathrm{id}_Y$.

(b) (Associativity) For $X, Y, Z, W \in \mathcal{C}_0$ and $f, g, h \in \mathcal{C}_1$ satisfying

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$$

one must have

$$h \circ (g \circ f) = (h \circ g) \circ f$$

The two morphisms that are set equal using associativity axiom can be understood more clearly using the following diagrams:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] \arrow[rrr, bend right=50, "h \circ (g \circ f)"'] & Y \arrow[r, "g"] & Z \arrow[r, "h"] & W
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "f"] \arrow[rrr, bend right=50, "(h \circ g) \circ f"'] & Y \arrow[r, "g"] \arrow[rr, bend right, "h \circ g"'] & Z \arrow[r, "h"] & W
\end{tikzcd}$$

Example A.1.1

We can take a collection of groups in $\mathcal{C}_0$ and the group homomorphisms between them in $\mathcal{C}_1$. In other words, the objects are groups and the morphisms are group homomorphisms. This forms a category.

Similarly, one can form a category of topological spaces too. In that case, the morphisms will be continuous functions between the spaces.

§A.2 Functor

Definition A.2.1 (Functor). Let $\mathcal{C} = (\mathcal{C}_0, \mathcal{C}_1)$ and $\mathcal{D} = (\mathcal{D}_0, \mathcal{D}_1)$ be two categories. A (covariant) **functor** $\mathcal{F}$ from $\mathcal{C}$ to $\mathcal{D}$, denoted by $\mathcal{F} : \mathcal{C} \to \mathcal{D}$, is a map that has the following properties:

i. $\mathcal{F}$ maps objects of $\mathcal{C}_0$ to objects of $\mathcal{D}_0$.

ii. $\mathcal{F}$ maps morphisms of $\mathcal{C}_1$ to morphisms of $\mathcal{D}_1$, such that for $f \in \mathcal{C}_1$ and $X, Y \in \mathcal{C}_0$

$$f : X \to Y \implies \mathcal{F}(f) : \mathcal{F}(X) \to \mathcal{F}(Y) \ .$$

iii. For every $X \in \mathcal{C}_0$, $\mathcal{F}(\mathrm{id}_X) = \mathrm{id}_{\mathcal{F}(X)}$.

iv. For $f, g \in \mathcal{C}_1$ with $t(f) = s(g)$ (in other words, $g \circ f$ makes sense), condition ii guarantees that $t(\mathcal{F}(f)) = s(\mathcal{F}(g))$ (so $\mathcal{F}(g) \circ \mathcal{F}(f)$ makes sense). Then we must have

$$\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f)$$

The definition of functor can be visualized using the following diagram:

Category $\mathcal{C}$

$$\begin{tikzcd} \cdots \arrow[r] & X \arrow[loop above, "\mathrm{id}_X"] \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r] & \cdots \end{tikzcd}$$

$$\Big\downarrow \mathcal{F}$$

Category $\mathcal{D}$

$$\begin{tikzcd} \cdots \arrow[r] & \mathcal{F}(X) \arrow[loop above, "\mathrm{id}_{\mathcal{F}(X)}"] \arrow[r, "\mathcal{F}(f)"] & \mathcal{F}(Y) \arrow[r, "\mathcal{F}(g)"] & \mathcal{F}(Z) \arrow[r] & \cdots \end{tikzcd}$$

Definition A.2.2 (Contravariant Functor). Let $\mathcal{C} = (\mathcal{C}_0, \mathcal{C}_1)$ and $\mathcal{D} = (\mathcal{D}_0, \mathcal{D}_1)$ be two categories. A **contravariant functor** $\mathcal{F}$ from $\mathcal{C}$ to $\mathcal{D}$, denoted by $\mathcal{F} : \mathcal{C} \to \mathcal{D}$, is a map such that

i. $\mathcal{F}$ maps objects of $\mathcal{C}_0$ to objects of $\mathcal{D}_0$.

ii. $\mathcal{F}$ maps morphisms of $\mathcal{C}_1$ to morphisms of $\mathcal{D}_1$, such that for $f \in \mathcal{C}_1$ and $X, Y \in \mathcal{C}_0$

$$f : X \to Y \implies \mathcal{F}(f) : \mathcal{F}(Y) \to \mathcal{F}(X) \ .$$

iii. For every $X \in \mathcal{C}_0$, $\mathcal{F}(\mathrm{id}_X) = \mathrm{id}_{\mathcal{F}(X)}$.

iv. For $f, g \in \mathcal{C}_1$, $\mathcal{F}(g \circ f) = \mathcal{F}(f) \circ \mathcal{F}(g)$ whenever these compositions make sense.

One can think of a contravariant functor as a functor that alters the direction of the morphisms, as opposed to a covariant functor. Here is a diagram for visualization purposes:

Category $\mathcal{C}$

$$\begin{tikzcd}
\cdots \arrow[r] & X \arrow["\mathrm{id}_X"', loop above] \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r] & \cdots
\end{tikzcd}$$

$$\Big\downarrow \mathcal{F}$$

Category $\mathcal{D}$

$$\begin{tikzcd}
\cdots & \mathcal{F}(X) \arrow[l] \arrow["\mathrm{id}_{\mathcal{F}(X)}"', loop above] & \mathcal{F}(Y) \arrow[l, "\mathcal{F}(f)"'] & \mathcal{F}(Z) \arrow[l, "\mathcal{F}(g)"'] & \cdots \arrow[l]
\end{tikzcd}$$