Exercise sheet sol^n

$\boxed{1}$ $F := \mathrm{Hom}_{\mathcal{C}}(-,-) : \mathcal{C}^{op} \times \mathcal{C} \to \mathrm{Sets}$ is a functor.

Given $(X,Y) \in (\mathcal{C}^{op} \times \mathcal{C})_0$, $F(X,Y) = \mathrm{Hom}_{\mathcal{C}}(X,Y)$

Given $(f^{op}, g) \in (\mathcal{C}^{op} \times \mathcal{C})_1$,

$$(f^{op}, g) : (X,Y) \longrightarrow (A,B)$$

$f : A \to X$, $g : Y \to B$ arrows in $\mathcal{C}$

$$F(f^{op}, g) : \mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(A,B)$$

$$x \longmapsto g \circ x \circ f$$

Suppose $(f^{op}, g) = 1\!\!1_{(X,Y)} = (1\!\!1_X^{\,op}, 1\!\!1_Y)$

Then $F(1\!\!1_{(X,Y)})(x) = 1\!\!1_Y \circ x \circ 1\!\!1_X = x$.

$$\begin{tikzcd}
(X,Y) \arrow[r, "(f_1^{op}, g_1)"] \arrow[rr, bend right, "(f_2^{op} \circ f_1^{op},\, g_2 \circ g_1) = ((f_1 \circ f_2)^{op},\, g_2 \circ g_1)"'] & (A,B) \arrow[r, "(f_2^{op}, g_2)"] & (P,Q)
\end{tikzcd}$$

In $\mathcal{C}$,

$A \xrightarrow{f_1} X$ $\qquad$ $Y \xrightarrow{g_1} B$

$P \xrightarrow{f_2} A$ $\qquad$ $B \xrightarrow{g_2} Q$

$$F\left((f_2^{op}, g_2) \circ (f_1^{op}, g_1)\right) : Hom_{\mathcal{C}}(X,Y) \longrightarrow Hom_{\mathcal{C}}(P,Q)$$

$$= F\left((f_1 \circ f_2)^{op}, g_2 \circ g_1\right)$$

$$x \longmapsto g_2 \circ g_1 \circ x \circ f_1 \circ f_2$$

$$\left[F(f_2^{op}, g_2) \circ F(f_1^{op}, g_1)\right](x)$$

$$= F(f_2^{op}, g_2)\left(g_1 \circ x \circ f_1\right)$$

$$= g_2 \circ g_1 \circ x \circ f_1 \circ f_2 .$$

Recall: $F: V \to W$ $\quad F^T: W^* \to V^*$

sat sun mon tue wed thu fri Date: $F^T(\alpha)(\bar{v}) = \alpha(F(\bar{v}))$

$\boxed{2}$ Given linear $f: V \to W$, TFDC

$$\begin{tikzcd}
V \arrow[r, "f"] \arrow[d, "\eta_V"'] \arrow[dr, phantom, "\checkmark"] & W \arrow[d, "\eta_W"] \\
V^{**} \arrow[r, "(f^T)^T"'] & W^{**}
\end{tikzcd}$$

$\eta_V(\bar{v})(\varphi) = \varphi(\bar{v})$

$(f^T)^T \circ \eta_V(\bar{v}) \in W^{**}$

$(f^T)^T(\eta_V(\bar{v}))\cancel{(W)} \Rightarrow (\psi)$

$= \eta_V(\bar{v})(f^T(\psi))$

$= f^T(\psi)(\bar{v})$

$= \psi(f(\bar{v}))$

$(\eta_W \circ f)(\bar{v}) \in W^{**}$

$(\eta_W \circ f)(\bar{v})(\psi)$

$= \eta_W(f(\bar{v}))(\psi)$

$= \psi(f(\bar{v}))$

Hom-Tensor Adjunction:

Here, all Hom are Hom vectk

First, define the isomorphism

$$\eta_{(\mathcal{U},W)} : Hom(\mathcal{U}\otimes V, W) \longrightarrow Hom(\mathcal{U}, Hom(V,W))$$

$$\begin{tikzcd}
\mathcal{U}\otimes V \arrow[r, "f"] \arrow[d, leftrightarrow, "\theta_{\mathcal{U}}"] & W \\
\mathcal{U}\times V \arrow[ur, "f\circ\theta_{\mathcal{U}}"'] &
\end{tikzcd}$$

✓

$$\{\text{linear } \mathcal{U}\otimes V \to W\} \longleftrightarrow \{\text{bilinear } \mathcal{U}\times V \to W\}$$

$$f \longmapsto f\circ\theta_{\mathcal{U}}$$

$$\{\text{bilinear } \mathcal{U}\times V \to W\} \longleftrightarrow \{\text{linear } \mathcal{U} \to Hom(V,W)\}$$

$$f \longmapsto f(\bar{u}, -)$$

$$\eta_{(\mathcal{U},W)} : Hom(\mathcal{U}\otimes V, W) \longrightarrow Hom(\mathcal{U}, Hom(V,W))$$

$$f \longmapsto (f\circ\theta_{\mathcal{U}})(\bar{u}, -)$$

$$
\begin{tikzcd}
\mathrm{Hom}(U \otimes V, W) \arrow[r, "\eta_{(U,W)}"] \arrow[d, "F(\alpha_1^{op}, \alpha_2)"'] & \mathrm{Hom}(U, \mathrm{Hom}(V, W)) \arrow[d, "G(\alpha_1^{op}, \alpha_2)"] \\
\mathrm{Hom}(U' \otimes V, W') \arrow[r, "\eta_{(U', W')}"'] & \mathrm{Hom}(U', \mathrm{Hom}(V, W'))
\end{tikzcd}
$$

$F = \mathrm{Hom}(\_ \otimes V, -)$

$G = \mathrm{Hom}(-, \mathrm{Hom}(V, -))$

$\alpha_1 : U' \longrightarrow U, \quad \alpha_2 : W \longrightarrow W'.$

$$
\begin{tikzcd}
f \arrow[r, mapsto] \arrow[d, mapsto] & (\bar{u} \mapsto (f \circ \theta_U)(\bar{u}, -)) \arrow[d, mapsto] \\
\alpha_2 \circ f \circ (\alpha_1 \otimes 1_V) \arrow[r, mapsto] & \bar{u}' \mapsto (\alpha_2 \circ f \circ (\alpha_1 \otimes 1_V)) \circ \theta_{U'} (\bar{u}', -)
\end{tikzcd}
$$

$U' \otimes V \xrightarrow{\alpha_1 \otimes 1_V} U \otimes V$

$\bar{u}' \longmapsto \alpha_2[(f \circ \theta_U)(\alpha_1(\bar{u}'), \_\_)]$

$\alpha_2 \, f(\alpha_1(\bar{u}') \otimes -)$

$= \alpha_2 \, f(\alpha_1(\bar{u}') \otimes -)$

$\boxed{4}$ $\operatorname{im} f \cong G/\operatorname{Ker} f$, for $f: G \to H$ a homomorphism.

Consider functors

$$\begin{tikzcd} \mathrm{Arr}(\mathrm{Groups}) \arrow[r, bend left, "F:\, f \mapsto \operatorname{dom} f/\operatorname{Ker} f"] \arrow[r, bend right, "G:\, f \mapsto \operatorname{im} f"'] \arrow[r, phantom, "\Downarrow \eta"] & \mathrm{Groups} \end{tikzcd}$$

$$\begin{tikzcd} G_1 \arrow[r, "x"] \arrow[d, "f_1"'] \arrow[dr, phantom, "\checkmark"] & G_2 \arrow[d, "f_2"] \\ H_1 \arrow[r, "y"'] & H_2 \end{tikzcd}$$

$G\big((x,y): f_1 \to f_2\big) : \operatorname{im} f_1 \longrightarrow \operatorname{im} f_2$

$h \longmapsto y(h)$

$$\begin{tikzcd} G_1 \arrow[r, "x"] \arrow[d, "f_1"'] & G_2 \arrow[d, "f_2"] \\ H_1 \arrow[r, "y"'] & H_2 \end{tikzcd}$$

$F\big((x,y): f_1 \to f_2\big) : \operatorname{dom} f_1/\operatorname{Ker} f_1 \to \operatorname{dom} f_2/\operatorname{Ker} f_2$

$[g] \longmapsto [x(g)]$

$(x,y) : f_1 \to f_2$

$$
\begin{tikzcd}
\mathrm{dom} f_1/\mathrm{Ker} f_1 \arrow[r, "\eta_{f_1}"] \arrow[d, "\alpha"'] & \mathrm{im} f_1 \arrow[d, "\beta"] \\
\mathrm{dom} f_2/\mathrm{Ker} f_2 \arrow[r, "\eta_{f_2}"'] & \mathrm{im} f_2
\end{tikzcd}
$$

$$
\begin{tikzcd}
{[g]} \arrow[r, mapsto] \arrow[d, mapsto] & f_1(g) \arrow[d, mapsto] \\
{[x(g)]} \arrow[r, mapsto] & y f_1(g) = f_2 x(g)
\end{tikzcd}
$$

$\boxed{5}$

$$\begin{tikzcd}
\mathcal{C}^{op} \times \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Sets}) \arrow[rr, bend left, "\alpha : (X,F) \mapsto \mathrm{Nat}(\mathrm{Hom}_{\mathcal{C}}(-,X),\, F)"] \arrow[rr, bend right, "\beta : (X,F) \mapsto F(X)"'] \arrow[rr, phantom, "\Downarrow"] & & \mathrm{Sets}
\end{tikzcd}$$

$(f^{op}, \eta) : (X,F) \longrightarrow (Y,G)$ arrow.

$f^{op} : X \longrightarrow Y$ arrow in $\cancel{\mathcal{C}^{op}}$ $\mathcal{C}^{op}$

$\eta : F \Rightarrow G$

$$\begin{tikzcd}
F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f^{op})"'] & G(X) \arrow[d, "G(f^{op})"] \\
F(Y) \arrow[r, "\eta_Y"'] & G(Y)
\end{tikzcd}$$

$$\beta(f^{op}, \eta) = G(f^{op}) \circ \eta_X = \eta_Y \circ F(f^{op})$$

$$\alpha(f^{op}, \eta) : \mathrm{Nat}(\mathrm{Hom}_{\mathcal{C}}(-,X), F) \longrightarrow \mathrm{Nat}(\mathrm{Hom}_{\mathcal{C}}(-,Y), G)$$

$$\mathrm{Hom}_{\mathcal{C}}(-,Y) \overset{f_*^{op}}{\Rightarrow} \cancel{T} \cdot \mathrm{Hom}_{\mathcal{C}}(-,X) \overset{\tau}{\Rightarrow} F \overset{\eta}{\Rightarrow} G$$

$$\tau \longmapsto \cancel{f_*}\ \eta \circ \tau \circ f_* .$$

sat sun mon tue wed thu fri Date: / /

$$y : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Sets})$$

$$X \longmapsto \mathrm{Hom}_{\mathcal{C}}(-, X)$$

$$f : X \to Y \longmapsto f_* : \mathrm{Hom}_{\mathcal{C}}(-, X) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(-, Y)$$

Suppose $\alpha^{op} : A \to B$ is an arrow in $\mathcal{C}^{op}$.

$\alpha : B \to A$ is in $\mathcal{C}$

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(A, X) \arrow[r, "(f_*)_A"] \arrow[d] & \mathrm{Hom}_{\mathcal{C}}(A, Y) \arrow[d] \\
\mathrm{Hom}_{\mathcal{C}}(B, X) \arrow[r, "(f_*)_B"] & \mathrm{Hom}_{\mathcal{C}}(B, Y)
\end{tikzcd}$$

$$\begin{tikzcd}
h \arrow[r, mapsto] \arrow[d] & f \circ h \arrow[d] \\
h \circ \alpha \arrow[r, mapsto] & f \circ h \circ \alpha
\end{tikzcd}$$

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

$$\mathrm{Hom}_{\mathcal{C}}(-, X) \xrightarrow[f_*]{} \mathrm{Hom}_{\mathcal{C}}(-, Y) \xrightarrow[g_*]{} \mathrm{Hom}_{\mathcal{C}}(-, Z)$$

$$- \longmapsto f \circ - \longmapsto g \circ f \circ -$$
$$= (g \circ -)(f \circ -).$$

sat sun mon tue wed thu fri Date : / /

$\boxed{7}$ Show that

$$y_{X,Y} : Hom_{\mathcal{C}}(X,Y) \longrightarrow Nat(Hom_{\mathcal{C}}(-,X), Hom_{\mathcal{C}}(-,Y))$$

is bijection.

$\boxed{\text{inj:}}$ $f, g \in Hom_{\mathcal{C}}(X,Y)$, $f_* = g_*$.

$(f_*)_A = (g_*)_A$ for any object A

$$(f_*)_A, (g_*)_A : Hom_{\mathcal{C}}(A,X) \to Hom_{\mathcal{C}}(A,Y)$$

$$x \longmapsto f \circ x = g \circ x$$

Take $A = X$, $x = 1\!\!1_X$

$$x \mapsto f \circ x = g \circ x \Rightarrow f = g.$$

$\boxed{\text{surj}}$ Given any natural transformation

$$\eta : Hom_{\mathcal{C}}(-,X) \longrightarrow Hom_{\mathcal{C}}(-,Y),$$

$\eta = f_*$ for some $f : X \to Y$

Let $f = \eta_X(1\!\!1_X) : X \to Y$.

For any $g : A \to B$ in $\mathscr{C}$, TFDC

$$
\begin{tikzcd}
\mathrm{Hom}_{\mathscr{C}}(A,X) \arrow[r, "\eta_A"] & \mathrm{Hom}_{\mathscr{C}}(A,Y) \\
\mathrm{Hom}_{\mathscr{C}}(B,X) \arrow[u, "g^*"] \arrow[r, "\eta_B"'] & \mathrm{Hom}_{\mathscr{C}}(B,Y) \arrow[u, "g^*"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
h \circ g \arrow[r, mapsto] & \eta_A(h \circ g) = \eta_B(h) \circ g \\
h \arrow[u, mapsto] \arrow[r, mapsto] & \eta_B(h) \arrow[u, mapsto]
\end{tikzcd}
$$

~~fo~~ because of naturality

$$\eta_A(h \circ g) = \eta_B(h) \circ g$$

Choose $B = X$, $h = 1\!\!1_X$

$$\eta_A(g) = f \circ g.$$