BRAC UNIVERSITY

Inspiring Excellence

Algebriac Topology III (MAT484)

Lecture Notes

Contents

1 Singular Homology Groups

Let $\mathbb{R}^{\infty}$ denote the generalized Euclidean space $\mathbb{E}^{J}$, with J being the set of positive integers. An element of the vector space $\mathbb{R}^{\infty}$ is an infinite sequence of real numbers (functions from $\mathbb{N}$ to $\mathbb{R}$) with finitely many nonzero entries. Let Δ_p denote the p-simplex in $\mathbb{R}^{\infty}$ having vertices

$$\begin{aligned}
\varepsilon_0 &= (1,0,0,\dots,0,\dots)\ ,\\
\varepsilon_1 &= (0,1,0,\dots,0,\dots)\ ,\\
&\dots\\
\varepsilon_p &= (0,0,0,\dots,\underbrace{1}_{(p+1)\text{-th entry}},\dots)\,.
\end{aligned}$$

We call Δ_p the **standard p-simplex**. In this notation, Δ_{p-1} is a face of Δ_p.

> **Definition 1.1** (Singular p-simplex). Let X be a topological space. We define a **singular p-simplex** of X to be a continuous map $T:\Delta_p\to X$. The free abelian group generated by singular p-simplices of X is denoted by $S_p(X)$, and is called the **singular chain group** of X in dimension p. We shall denote an element of $S_p(X)$ by a $\mathbb{Z}$-linear combination of singular p-simplices of X.

Singular means that T could be a "bad" map, i.e. it may not be an imbedding. All we want that T is just continuous. Now, recall that

$$\Delta_p=\left\{(x_0,x_1,\dots,x_p,0,\dots)\in\mathbb{R}^{\infty}\,|\,0\le x_i\le 1 \text{ and } \sum_{i=0}^{p}x_i=1\right\}. \tag{1.1}$$

Given $a_0,a_1,\dots,a_p\in\mathbb{R}^{\infty}$, there is a unique affine map $l_{(a_0,\dots,a_p)}:\Delta_p\to\mathbb{R}^{\infty}$ that maps ε_i to a_i. It is defined by

$$\begin{aligned}
l_{(a_0,\dots,a_p)}(x_0,x_1,\dots,x_p,0,\dots) &= \sum_{i=0}^{p}x_ia_i=\sum_{i=0}^{p}x_ia_i+a_0-\sum_{i=0}^{p}x_ia_0\\
&= a_0+\sum_{i=0}^{p}x_i(a_i-a_0)\,.
\end{aligned} \tag{1.2}$$

We call this map the **linear singular simplex** determined by $a_0,a_1,\dots,a_p\in\mathbb{R}^{\infty}$. Now, what is $l_{(\varepsilon_0,\dots,\varepsilon_p)}$? Observe that

$$l_{(\varepsilon_0,\dots,\varepsilon_p)}\varepsilon_i=l_{(\varepsilon_0,\dots,\varepsilon_p)}(0,\dots,0,\underbrace{1}_{(i+1)\text{-th entry}},0,\dots)=\varepsilon_i. \tag{1.3}$$

Therefore, $l_{(\varepsilon_0,\dots,\varepsilon_p)}$ maps ε_i to itself, for every $i=0,1,\dots,p$. Also,

$$l_{(\varepsilon_0,\dots,\varepsilon_p)}(x_0,x_1,\dots,x_p,0,\dots)=\sum_{i=0}^{p}x_i\varepsilon_i=(x_0,x_1,\dots,x_p,0,\dots)\,. \tag{1.4}$$

Therefore, $l_{(\varepsilon_0,\dots,\varepsilon_p)}$ is just the inclusion map of Δ_p into $\mathbb{R}^{\infty}$. Now, suppose $(x_0,x_1,\dots,x_{p-1},0,\dots)\in\Delta_{p-1}$, so that $\sum_{i=0}^{p-1}x_i=1$. Then

$$\begin{aligned}
l_{(\varepsilon_0,\dots,\widehat{\varepsilon_i},\dots,\varepsilon_p)}(x_0,x_1,\dots,x_{p-1},0,\dots) &= x_0\varepsilon_0+\dots+x_{i-1}\varepsilon_{i-1}+0\cdot\varepsilon_i+x_{i+1}\varepsilon_{i+1}+\dots+x_{p-1}\varepsilon_p\\
&= (x_0,\dots,x_{i-1},0,x_{i+1},\dots,x_{p-1},0,\dots)\,,
\end{aligned} \tag{1.5}$$

which is a point on the face of Δ_p opposite to the vertex ε_i. In fact, $l_{(\varepsilon_0,\ldots,\widehat{\varepsilon_i},\ldots,\varepsilon_p)}$ is a linear homomorphism of Δ_{p-1} into the face of Δ_p that is opposite to the vertex ε_i. In other words,

$$l_{(\varepsilon_0,\ldots,\widehat{\varepsilon_i},\ldots,\varepsilon_p)} : \Delta_{p-1} \to \Delta_p$$

maps Δ_{p-1} to the face of Δ_p opposite to the vertex ε_i. Therefore, given a singular p-simplex $T : \Delta_p \to X$, one can form the composite

$$T \circ l_{(\varepsilon_0,\ldots,\widehat{\varepsilon_i},\ldots,\varepsilon_p)} : \Delta_{p-1} \to X,$$

which is a singular $(p-1)$-simplex. We think of it as the i-th face of the singular p-simplex T.

Definition 1.2 (Boundary homomorphism). We define $\partial : S_p(X) \to S_{p-1}(X)$ as follows. If $T : \Delta_p \to X$ is a singular p-simplex, we define ∂T to be

$$\partial T = \sum_{i=0}^{p} (-1)^i \, T \circ l_{(\varepsilon_0,\ldots,\widehat{\varepsilon_i},\ldots,\varepsilon_p)}. \tag{1.6}$$

In other words, ∂T is a formal sum of singular simplices of dimension $p-1$, which are the faces of T.

If $f : X \to Y$ is a continuous map, we define a group homomorphism $f_\# : S_p(X) \to S_p(Y)$ by defining it on singular p-simplices by the equation

$$f_\#(T) = f \circ T \tag{1.7}$$

for a singular p-simplex T.

$$\begin{tikzcd} \Delta_p \arrow[r, "T"] \arrow[rr, bend right, "f \circ T"'] & X \arrow[r, "f"] & Y \end{tikzcd}$$

Theorem 1.1

The homomorphism $f_\#$ commutes with ∂. Furthermore, $\partial^2 = 0$.

Proof. Given a singular p-simplex T,

$$\partial f_\#(T) = \partial(f \circ T) = \sum_{i=0}^{p} (-1)^i (f \circ T) \circ l_{(\varepsilon_0,\ldots,\widehat{\varepsilon_i},\ldots,\varepsilon_p)}. \tag{1.8}$$

$$f_\#(\partial T) = f_\#\left(\sum_{i=0}^{p} (-1)^i \, T \circ l_{(\varepsilon_0,\ldots,\widehat{\varepsilon_i},\ldots,\varepsilon_p)}\right) = \sum_{i=0}^{p} (-1)^i \, f \circ T \circ l_{(\varepsilon_0,\ldots,\widehat{\varepsilon_i},\ldots,\varepsilon_p)}. \tag{1.9}$$

Therefore, $\partial f_\#(T) = f_\#(\partial T)$. Now, to prove $\partial^2 = 0$, we first compute ∂ for linear singular simplices $l_{(a_0,\ldots,a_p)}$.

$$\partial l_{(a_0,\ldots,a_p)} = \sum_{i=0}^{p} (-1)^i \, l_{(a_0,\ldots,a_p)} \circ l_{(\varepsilon_0,\ldots,\widehat{\varepsilon_i},\ldots,\varepsilon_p)}. \tag{1.10}$$

Observe that

$$\begin{aligned} l_{(a_0,\ldots,a_p)} \circ l_{(\varepsilon_0,\ldots,\widehat{\varepsilon_i},\ldots,\varepsilon_p)}(x_0,\ldots,x_{p-1},0,\ldots) &= l_{(a_0,\ldots,a_p)}(x_0,\ldots,x_{i-1},0,x_i x_{p-1},0,) \\ &= x_0a_0 + \cdots + x_{i-1}a_{i-1} + 0 \cdot a_i + x_i a_{i+1} + \cdots x_{p-1}a_p \\ &= l_{(a_0,\ldots,\widehat{a_i},\ldots a_p)}(x_0,\ldots,x_{p-1},0,\ldots). \end{aligned} \tag{1.11}$$

Hence,

$$l_{(a_0,\ldots,a_p)} \circ l_{(\varepsilon_0,\ldots,\widehat{\varepsilon_i},\ldots,\varepsilon_p)} = l_{(a_0,\ldots,\widehat{a_i},\ldots a_p)}. \tag{1.12}$$

Therefore, from 1.10, it follows that

$$\partial l_{(a_0,\ldots,a_p)} = \sum_{i=0}^{p} (-1)^i \, l_{(a_0,\ldots,\widehat{a_i},\ldots a_p)}. \tag{1.13}$$

Let's now evaluate $\partial\partial l_{(a_0,\ldots,a_p)}$.

$$\begin{aligned}
\partial\partial l_{(a_0,\ldots,a_p)} &= \sum_{i=0}^{p} (-1)^i \, \partial l_{(a_0,\ldots,\widehat{a_i},\ldots a_p)} \\
&= \sum_{i=0}^{p} (-1)^i \sum_{j<i} (-1)^j \, l_{(a_0,\ldots,\widehat{a_j},\ldots\widehat{a_i},\ldots a_p)} + \sum_{i=0}^{p} (-1)^i \sum_{j>i} (-1)^{j-1} \, l_{(a_0,\ldots,\widehat{a_i},\ldots\widehat{a_j},\ldots a_p)} \\
&= \sum_{i=0}^{p} \sum_{j<i} (-1)^{i+j} \, l_{(a_0,\ldots,\widehat{a_j},\ldots\widehat{a_i},\ldots a_p)} - \sum_{i=0}^{p} \sum_{j>i} (-1)^{i+j} \, l_{(a_0,\ldots,\widehat{a_i},\ldots\widehat{a_j},\ldots a_p)}.
\end{aligned} \tag{1.14}$$

Now fix $0 \leq j_0 < i_0 \leq p$. In the first summand of 1.14, the contribution of $i = i_0, j = j_0$ is

$$(-1)^{i_0+j_0} \, l_{\left(a_0,\ldots,\widehat{a_{j_0}},\ldots\widehat{a_{i_0}},\ldots a_p\right)}. \tag{1.15}$$

On the other hand, in the second summand of 1.14, the contribution of $i = j_0, j = i_0$ is also

$$(-1)^{i_0+j_0} \, l_{\left(a_0,\ldots,\widehat{a_{j_0}},\ldots\widehat{a_{i_0}},\ldots a_p\right)}. \tag{1.16}$$

These two contributions cancel each other. This way, one finds that the RHS of 1.14 vanishes. Hence,

$$\partial\partial l_{(a_0,\ldots,a_p)} = 0. \tag{1.17}$$

In particular,

$$\partial\partial l_{(\varepsilon_0,\ldots,\varepsilon_p)} = 0. \tag{1.18}$$

Now, $l_{(\varepsilon_0,\ldots,\varepsilon_p)} : \Delta_p \to \Delta_p$ is continuous, so $l_{(\varepsilon_0,\ldots,\varepsilon_p)} \in S_p(\Delta_p)$. Furthermore, it is the identity map as we have seen in 1.4. Since $T : \Delta_p \to X$ is continuous, we can form $T_\# : S_p(\Delta_p) \to S_p(X)$.

$$T_\# \left(l_{(\varepsilon_0,\ldots,\varepsilon_p)}\right) = T \circ l_{(\varepsilon_0,\ldots,\varepsilon_p)} = T \circ \mathrm{id}_{\Delta_p} = T. \tag{1.19}$$

Therefore, using the fact that $T_\#$ commutes with ∂, we obtain

$$\partial\partial T = \partial\partial T_\# \left(l_{(\varepsilon_0,\ldots,\varepsilon_p)}\right) = T_\# \left(\partial\partial l_{(\varepsilon_0,\ldots,\varepsilon_p)}\right) = 0. \tag{1.20}$$

Hence, $\partial^2 T = 0$. ■