Category Theory (MAT434)

Lecture Notes

Preface

This series of lecture notes has been prepared for aiding students who took the BRAC University course **Category Theory (MAT434)** in Summer 2023 semester. These notes were typeset under the supervision of mathematician **Dr. Syed Hasibul Hassan Chowdhury**. The main goal of this typeset is to have an organized digital version of the notes, which is easier to share and handle. If you see any mistakes or typos, please send me an email at atonuroychowdhury@gmail.com

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References:

- *Category Theory*, by **Steve Awodey**.
- *Category Theory for Scientists*, by **David Spivak**.
- *Categories for the Working Mathematician*, by **Saunders Mac Lane**.
- *Basic Category Theory*, by **Tom Leinster**.

Contents

1 Categories

§1.1 Definition of a Category

Category theory arises from the idea of a system of "functions" among some objects.

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "g\circ f"'] & B \arrow[d, "g"] \\
& C
\end{tikzcd}
$$

A category consists of objects $A, B, C, \ldots$ and arrows $f : A \to B, g : B \to C, \ldots$ that are closed under composition and satisfy certain conditions typical of composition of functions. Before formally defining what a category is, let us begin our discussion with the setting where the objects are sets and arrows are functions between sets.

Let f be a function from a set A to another set B. This is mathematically expressed as $f : A \to B$. Explicitly, it refers to the fact that f is defined for all of A, and all the values of f are contained in B. In other words, $\mathrm{range}\,(f) \subseteq B$.

Now suppose we have another function $g : B \to C$. Then there is a unique function $g \circ f : A \to C$, given by

$$(g \circ f)\,(a) = g\,(f\,(a))\,, \qquad \text{for } a \in A. \tag{1.1}$$

This unique function is called the composite of g and f, or g after f.

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "g\circ f"'] & B \arrow[d, "g"] \\
& C
\end{tikzcd}
$$

Now, this operation $\circ$ of composition of functions is associative. In other words, the two arrows from A to D in the following diagram are the same:

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] \arrow[rrr, bend left=60, "(h\circ g)\circ f"] \arrow[rrr, bend right=60, "h\circ(g\circ f)"'] & B \arrow[r, "g"] \arrow[rr, bend left, "h\circ g"] & C \arrow[r, "h"] & D
\end{tikzcd}
$$

Given $f : A \to B$, $g : B \to C$ and $h : C \to D$, one has unique compositions $h \circ g : B \to D$ and $g \circ f : A \to C$. These two composed functions can be further composed with f (from the left) and with h (from the right), respectively, to yield a unique function

$$(h \circ g) \circ f = h \circ (g \circ f)\,, \tag{1.2}$$

from A to D as demanded by the associativity law. Using the definition of composition of functions, one verifies that this is indeed the case:

$$
\begin{aligned}
((h \circ g) \circ f)\,(a) &= (h \circ g)\,(f\,(a)) = h\,(g\,(f\,(a)))\,, \\
(h \circ (g \circ f))\,(a) &= h\,((g \circ f)\,(a)) = h\,(g\,(f\,(a)))\,.
\end{aligned}
$$

Therefore, $(h \circ g) \circ f = h \circ (g \circ f)$.

Finally, note that for every set A, there is an identity function $1_A : A \to A$ given by

$$1_A(a) = a. \tag{1.3}$$

These identity functions act as units for composition, i.e. given $f : A \to B$, we have

$$\begin{aligned}(f \circ 1_A)(a) &= f(1_A(a)) = f(a),\\ (1_B \circ f)(a) &= 1_B(f(a)) = f(a),\end{aligned}$$

for each $a \in A$. Therefore,

$$f \circ 1_A = 1_B \circ f = f. \tag{1.4}$$

The equality above is equivalent to the following commutative diagram:

$$\begin{tikzcd} A \arrow[r, "1_A"] \arrow[dr, "f\circ 1_A"'] & A \arrow[d, "f"'] \arrow[dr, "1_B\circ f"] & \\ & B \arrow[r, "1_B"'] & B \end{tikzcd}$$

We have the following abstract version of sets and functions between sets called a **category**.

Definition 1.1 (Category)**.** A **category** $\mathcal{C}$ consists of the following data:

- **Objects:** $A, B, C, \ldots$. The collection of objects of $\mathcal{C}$ is denoted by $\mathrm{Ob}(\mathcal{C})$.
- **Arrows:** $f, g, h, \ldots$. Given two objects A and B, the set of arrows from A to B is denoted by $\mathrm{Hom}_{\mathcal{C}}(A, B)$.
- For each arrow f, there are given objects $\mathrm{dom}(f), \mathrm{cod}(f)$ called the **domain** and **codomain** of f. We write $f : A \to B$ to indicate that $A = \mathrm{dom}(f)$ and $B = \mathrm{cod}(f)$.
- Given arrows $f : A \to B$ and $g : B \to C$, i.e. with $\mathrm{cod}(f) = \mathrm{dom}(g)$, there is a unique arrow $g \circ f : A \to C$, i.e. $g \circ f \in \mathrm{Hom}_{\mathcal{C}}(A, C)$ called the **composite** of f and g. This fact can be rephrased as the following: given $A, B, C \in \mathrm{Ob}(\mathcal{C})$, there is a function

  $$\circ : \mathrm{Hom}_{\mathcal{C}}(B, C) \times \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(A, C), \tag{1.5}$$

  with $(g, f) \mapsto g \circ f$. The well-definedness of $\circ$ is synonymous to claiming that $g \circ f \in \mathrm{Hom}_{\mathcal{C}}(A, C)$ is unique for given $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$ and $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$.
- For each $A \in \mathrm{Ob}(\mathcal{C})$, there exists an unique arrow $1_A \in \mathrm{Hom}_{\mathcal{C}}(A, A)$.

The above data are required to satisfy the following laws:

- **Associativity:** For any $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$, $h \in \mathrm{Hom}_{\mathcal{C}}(C, D)$ with $A, B, C, D \in \mathrm{Ob}(\mathcal{C})$,

  $$(h \circ g) \circ f = h \circ (g \circ f), \tag{1.6}$$
- **Unit:** For any $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ with $A, B \in \mathrm{Ob}(\mathcal{C})$,

  $$f \circ 1_A = 1_B \circ f = f. \tag{1.7}$$

Remark 1.1. Suppose we have the following commutative diagram:

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[dr, "h\circ f=h\circ g"'] & B \arrow[d, "h"] \\
& C
\end{tikzcd}$$

Commutativity of this diagram doesn't violate the uniqueness of the composition $\circ$. It just means that the map $\circ$ in (1.5) is a many-to-one function.

§1.2 Examples of Categories

- Sets and functions between sets. This category is called **Sets**.
- Groups and group homomorphisms
- Vector spaces and linear mappings between them
- Graphs and graph isomorphisms
- The set of real numbers $\mathbb{R}$ as an object, and continuous functions $f : \mathbb{R} \to \mathbb{R}$ as arrows
- Open subsets $U \subseteq \mathbb{R}$ and continuous functions $f : U \to V \subseteq \mathbb{R}$ defined on them
- Differentiable manifolds and smooth (C^∞) mappings
- Posets and monotone functions.

Let us discuss the last category at length.

Definition 1.2. A **partially ordered set** or **poset** is a set A equipped with a binary relation (a subset of $A \times A$) $a \leq_A b$ (in other words, $(a, b) \in R \subset A \times A$) such that the following conditions hold for all $a, b, c \in A$:

(i) **Reflexivity:** $a \leq_A a$.

(ii) **Transitivity:** if $a \leq_A b$ and $b \leq_A c$, then $a \leq_A c$.

(iii) **Antisymmetry:** if $a \leq_A b$ and $b \leq_A a$, then $a = b$.

Remark 1.2. The antisymmetry condition tells us that if both $a \leq_A b$ and $b \leq_A a$ hold, then a and b cannot be distinct. Contrapositively, for distinct a and b in A, noth both $a \leq_A b$ and $b \leq_A a$ hold true. Also, note that if $(A, \leq_A)$ is a partially ordered set, there can be elements $a, b \in A$ such that neither (a, b) nor (b, a) is in R. If it happens that given any $a, b \in A$, either (a, b) or (b, a) is in R, i.e. either $a \leq_A b$ or $b \leq_A a$, then we call A a **totally ordered set**.

Example 1.1. $(\mathbb{R}, \leq)$, the set of real numbers with the usual ordering $\leq$ is a totally ordered set.

Now we define an arrow from a poset $(A, \leq_A)$ to another poset $(B, \leq_B)$ to be a function $m : A \to B$ that is **monotone**, in the sense that for all $a, a' \in A$,

$$\text{whenever } a \leq_A a', \text{ one has } m(a) \leq_B m(a').$$

We need to verify that under this definition of arrows, we have a category. First of all, we must have $1_A : A \to A$, defined by $1_A(a) = a$ for each $a \in A$, to be monotone. Indeed, if $a \leq a'$ in A, then we automatically have $1_A(a) \leq 1_A(a')$. Therefore, 1_A is monotone.

Given monotone functions $f : A \to B$ bewteen posets $(A, \leq_A)$ and $(B, \leq_B)$, and $g : B \to C$ bewteen posets $(B, \leq_B)$ and $(C, \leq_C)$, we need to verify that the composition $g \circ f : A \to C$ is also monotone. Indeed, given $a \leq_A a'$, since f is monotone, we have

$$f(a) \leq_B f\left(a'\right). \tag{1.8}$$

Since g is monotone, this gives us

$$g(f(a)) \leq_C g\left(f\left(a'\right)\right). \tag{1.9}$$

In other words, $(g \circ f)(a) \leq_C (g \circ f)(a')$ given $a \leq_A a'$. Therefore, $g \circ f : A \to C$ is monotone.

The category thus formed is called the category of posets and monotone functions, and is denoted by **Pos**.

Finite Categories

A finite category consists of finitely many objects and finitely many arrows between them.

- The category **1** looks as follows:

$$\begin{tikzcd} * \arrow[loop right] \end{tikzcd}$$

  It has one object $*$ and its identity arrow.

- The category **2** looks as follows:

$$\begin{tikzcd} \star \arrow[loop left] \arrow[r] & * \arrow[loop right] \end{tikzcd}$$

  It has two objects $\star$ and $*$, their identity arrows, and exactly one arrow $\star \to *$.

- The categort **3** looks as follows:

$$\begin{tikzcd} \star \arrow[loop left] \arrow[r] \arrow[dr] & * \arrow[loop right] \arrow[d] \\ & \bullet \arrow[loop below] \end{tikzcd}$$

  It has three objects $\star, *, \bullet$, their respective identity arrows, and the other arrows are $\star \to *$, $* \to \bullet$, and $\star \to \bullet$ (which is the composition of the previous two arrows).

- The category **0** looks as follows:

  It has no objects or arrows.

§1.3 Functor

Definition 1.3 (Functor)**.** A **functor** $F : \mathcal{C} \to \mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$ is an assignment of $\mathrm{Ob}(\mathcal{C})$ to $\mathrm{Ob}(\mathcal{D})$ and a mapping of arrows in $\mathcal{C}$ to arrows in $\mathcal{D}$, i.e. for any $A, B \in \mathrm{Ob}(\mathcal{C})$, a mapping

$$\mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F(A), F(B)),$$

where $F(A), F(B) \in \mathrm{Ob}(\mathcal{D})$ are the assigned objects of $\mathcal{D}$ under F. In other words, for given $A, B \in \mathrm{Ob}(\mathcal{C})$ and an arrow $f : A \to B$, one has $F(A), F(B) \in \mathrm{Ob}(\mathcal{D})$ and an arrow $F(f) : F(A) \to F(B)$ such that the following hold:

(a) $F(1_A) = 1_{F(A)}$.

(b) $F(g \circ f) = F(g) \circ F(f)$.

> In other words, F preserves domains and codomains, identity arrows and composition.

$$\begin{tikzcd}
\mathcal{C} \arrow[dd, "F"'] & & A \arrow[r, "f"] \arrow[dr, "g\circ f"'] & B \arrow[d, "g"] \\
& & & C \\
\mathcal{D} & & F(A) \arrow[r, "F(f)"] \arrow[dr, "F(g\circ f)=F(g)\circ F(f)"'] & F(B) \arrow[d, "F(g)"] \\
& & & F(C)
\end{tikzcd}$$

Now, one can see that functors compose in the expected way and that every category $\mathcal{C}$ has a distinguished functor called the identity functor $1_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$. Thus we have a category, namely **Cat**, the category of all categories and functors between them.

Preorder

A **preorder** is a set P equipped with a binary relation $\leq$ that is both reflexive and transitive: $a \leq a$; and if $a \leq b$ and $b \leq c$, then $a \leq c$ for any $a, b, c \in P$. Any preorder $(P, \leq)$ can be regarded as a category by taking the objects of the category to be the elements of P and taking a unique arrow

$$a \to b \text{ if and only if } a \leq b \text{ in } (P, \leq)\,. \tag{1.10}$$

Remark 1.3. Reflexivity and transitivity property ensures that the preorder $(P, \leq)$ is indeed a category. Note that the above condition implies that there is at most one arrow from an object of $(P, \leq)$ to another. In the other direction, any category with at most one arrow from an object to another determines a preorder simply by defininig a binary relation $\leq$ on the objects by (1.10).

Remark 1.4 (On the similarities between a poset and a preorder). A poset $(P, \leq)$ is evidently a preorder with the additional condition of antisymmetry. Hence, a poset is also a category. Examples of poset include the power set $\mathscr{P}(X)$ of a given set X under the usual subset relation: $U \subseteq V$ between the subsets U, V of X.

There can be preorders that are not posets. For instance, $(\mathbb{Z}, |)$ is a preorder on the set of integers, where "$|$" is the usual divides binary relation: given $a, b \in \mathbb{Z}$, we have $a \mid b$ (read a divides b) if and only if $b = ca$ for some $c \in \mathbb{Z}$. It is clearly reflexive and transitive. Note that $a \mid b$ and $b \mid a$ imply $a = \pm b$ which is not the same as $a = b$. Hence, "$|$" is not antisymmetric. Therefore, $(\mathbb{Z}, |)$ is a preorder that is not a poset.

§1.4 Monoid

> **Definition 1.4** (Monoid). A **monoid** is a set M equipped with a binary operation $\cdot : M \times M \to M$ and a distinguished "unit" element $u \in M$ such that for each $x, y, z \in M$,
>
> $$x \cdot (y \cdot z) = (x \cdot y) \cdot z \text{ and } u \cdot x = x \cdot u = x. \tag{1.11}$$

Equivalently, a monoid is a category with just one object. The arrows of the category are the elements of the monoid. In particular, the identity arrow on the object is the unit element u. Composition of arrows is the binary operation $x \cdot y$ of the monoid.

For example, $\mathbb{N}$ (we are adopting the convention that $0 \in \mathbb{N}$), $\mathbb{Q}$, $\mathbb{R}$ with addition and 0 as the unit element. Also, $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$ with multiplication and 1 as unit are monoids. For any set X, the set of functions from X to itself, written as

$$\mathrm{Hom}_{\mathbf{Sets}}(X, X),$$

is a monoid under the operation of composition. Here **Sets** is the category of sets and functions between sets. More generally, for any object $C \in \mathrm{Ob}(\mathcal{C})$ in a category $\mathcal{C}$, the set of arrows from C to itself, written as

$$\mathrm{Hom}_{\mathcal{C}}(C, C),$$

is a monoid under the composition of arrows in $\mathcal{C}$.

Since monoids are structured sets (sets equipped with a binary operation fulfilling associativity, unitality etc.), there is a category **Mon** whose objects are monoids and arrows are functions that preserve the monoid structure, namely monoid homomorphisms. In detail, a **monoid homomorphism** from a monoid $(M, \cdot_M)$ to a monoid $(N, \cdot_N)$ is a function $f : M \to N$ such that for all $m, n \in M$,

$$h(m \cdot_M n) = h(m) \cdot_N h(n) \text{ and } h(u_M) = u_N. \tag{1.12}$$

Here u_M and u_N are unit elements of M and N, respectively.

§1.4.i Isomorphisms

Definition 1.5. In any category $\mathcal{C}$, an arrow $f : A \to B$ is called an **isomorphism** if there is an arrow $g : B \to A$ such that

$$g \circ f = 1_A \text{ and } f \circ g = 1_B. \tag{1.13}$$

Suppose there is another arrow $\widetilde{g} : B \to A$ with

$$\widetilde{g} \circ f = 1_A \text{ and } f \circ \widetilde{g} = 1_B. \tag{1.14}$$

Then we have

$$g = g \circ 1_B = g \circ (f \circ \widetilde{g}) = (g \circ f) \circ \widetilde{g} = 1_A \circ \widetilde{g} = \widetilde{g}. \tag{1.15}$$

Hence, if an arrow $g : B \to A$ exists satisfying (1.13), then it is unique. Such unique arrow $g : B \to A$ is called the inverse of $f : A \to B$, and we write $g = f^{-1}$. When such an arrow $f : A \to B$ exists, we say that A is isomorphic to B, written $A \cong B$.

Definition 1.6 (Group). A **group** G is a monoid with an inverse g^{-1} for every element $g \in G$. Thus G is a category with one object in which every arrow is an isomorphism.

The natural numbers $\mathbb{N}$ do not form a group either under addition or multiplication. But the integers $\mathbb{Z}$ form a group under addition. So do the positive ratuonals $\mathbb{Q}^+$ under multiplication. For any set X, we have the group $\mathrm{Aut}(X)$ of all the automorphisms of X, i.e. isomorphisms $f : X \to X$. A **group of permutations** is a subgroup $G \subseteq \mathrm{Aut}(X)$ for some X. Thus the set G must satisfy the following:

1. The identity function 1_X on X is in G.
2. If $g, g' \in G$, then $g \circ g' \in G$.
3. If $g \in G$, $g^{-1} \in G$.

We now have the following theorem due to Arthur Cayley.

Theorem 1.1 (Cayley's theorem)
Every group G is isomorphic to a group of permutations.

Sketch of proof. First, define the Cayley representation $\overline{G}$ of G to be the following group of permutations on a set: the set is G itself, and for each $g \in G$, one has the permutation $\overline{g} : G \to G$ defined as

$$\overline{g}\,(h) = g \cdot h \text{ for each } h \in G. \tag{1.16}$$

Indeed, $\overline{g}$ has an inverse $\overline{g}^{-1} = \overline{g^{-1}}$:

$$\overline{g}^{-1}\,(h) = g^{-1}h. \tag{1.17}$$

One, thus, verifies that $\overline{g} : G \to G$ is indeed an isomorphism, and hence a permutation on G.

Now define homomorphisms $i : G \to \overline{G}$ by $i\,(g) = \overline{g}$, and $j : \overline{G} \to G$ by $j\,(\overline{g}) = \overline{g}\,(u) = g$, with u being the identity element of the group G.

Observe that $i \circ j = 1_{\overline{G}}$ and $j \circ i = 1_G$. Indeed, for $g \in G$ and $\overline{g} \in \overline{G}$,

$$\begin{aligned}(j \circ i)\,(g) &= j\,(i\,(g)) = j\,(\overline{g}) = g,\\ (i \circ j)\,(\overline{g}) &= i\,(j\,(\overline{g})) = i\,(g) = \overline{g},\end{aligned}$$

establishing that $i : G \to \overline{G}$ is an isomorphism. ■

Remark 1.5. There are two different types of isomorphisms involved in this proof. For each $g \in G$, one defines an isomorphism $\overline{g} : G \to G$. This is an isomorphism in the category **Sets**. Later, we defined an isomorphism $i : G \to \overline{G}$, which is an isomorphism in the categorty **Groups** of groups and group homomorphisms.

Remark 1.6. The group $\overline{G}$ is the group of permutations (automorphisms) on the group G which is a subgroup of the automorphism group on G itself. This subgroup has the same unit element of that of the automorphism group on G, i.e. 1_G, the identity function on the group G. Note that this is also the unit of the group $\overline{G}$ which is not the same as $1_{\overline{G}}$. This identity function $1_{\overline{G}}$ on $\overline{G}$ was used to establish the required isomorphism in Cayley's theorem.

Cayley's theorem can be generalized to prove that any category not "too big" (which has the collection of objects to be a set) is isomorphic to a category in which the objects are sets and the arrows are functions between those sets. In other words, any not "too big" category is isomorphic to a subcategory of **Sets**.

§1.5 Construction on Categories

1. The product of 2 categories $\mathcal{C}$ and $\mathcal{D}$, written as $\mathcal{C} \times \mathcal{D}$ has objects of the form (C, D) for $C \in \mathrm{Ob}\,(\mathcal{C})$ and $D \in \mathrm{Ob}\,(\mathcal{D})$, and arrows of the form

$$(f, g) : (C, D) \to \left(C', D'\right),$$

with $C, C' \in \mathrm{Ob}\,(\mathcal{C})$, $D, D' \in \mathrm{Ob}\,(\mathcal{D})$, $f \in \mathrm{Hom}_{\mathcal{C}}\,(C, C')$ and $g \in \mathrm{Hom}_{\mathcal{D}}\,(D, D')$.

Composition and units are defined componentwise, i.e.

$$\left(f', g'\right) \circ (f, g) = \left(f' \circ f, g' \circ g\right) \text{ and } 1_{(C,D)} = (1_C, 1_D)\,, \tag{1.18}$$

with $C, C', C'' \in \mathrm{Ob}\,(\mathcal{C})$ and $D, D', D'' \in \mathrm{Ob}\,(\mathcal{D})$ and

$$\begin{tikzcd} C \arrow[loop left, "1_C"] \arrow[r, "f"] \arrow[rr, bend right, "f' \circ f"'] & C' \arrow[r, "f'"] & C'' \end{tikzcd} \qquad \begin{tikzcd} D \arrow[loop above, "1_D"] \arrow[r, "g"] \arrow[rr, bend right, "g' \circ g"'] & D' \arrow[r, "g'"] & D'' \end{tikzcd}$$

Then in $\mathcal{C} \times \mathcal{D}$, we have

$$\begin{tikzcd}
(C,D) \arrow[loop left, "1_{(C,D)}=(1_C,1_D)"] \arrow[r, "(f,g)"] \arrow[rr, bend right, "(f',g')\circ(f,g)=(f'\circ f,g'\circ g)"'] & (C',D') \arrow[r, "(f',g')"] & (C'',D'')
\end{tikzcd}$$

Then there are two **projection functors**:

$$\mathcal{C} \xleftarrow{\pi_1} \mathcal{C} \times \mathcal{D} \xrightarrow{\pi_2} \mathcal{D}$$

Given $(C,D) \in \mathrm{Ob}\,(\mathcal{C} \times \mathcal{D})$ and $(f,g) : (C,D) \to (C',D,)$,

$$\pi_1\,(C,D) = C,\ \ \pi_1\,(f,g) = f. \tag{1.19}$$

Similarly,

$$\pi_2\,(C,D) = D,\ \ \pi_2\,(f,g) = g. \tag{1.20}$$

2. The opposite category $\mathcal{C}^{\mathrm{op}}$ has objects that are in a one-to-one correspondence with the objects of $\mathcal{C}$. Let $C^* \in \mathrm{Ob}\,(\mathcal{C}^{\mathrm{op}})$ be the object in $\mathcal{C}^{\mathrm{op}}$ that corresponds to $C \in \mathrm{Ob}\,(\mathcal{C})$. Then an arrow $f : C \to D$ in $\mathcal{C}$ corresponds to an arrow $f^* : D^* \to C^*$. With this notation, one can define composition and units in $\mathcal{C}^{\mathrm{op}}$ in terms of the corresponding operations in $\mathcal{C}$, namely

$$1_{C^*} = (1_C)^*\,. \tag{1.21}$$

For $f : C \to D$, $g : D \to E$ in $\mathcal{C}$, we have $f^* : D^* \to C^*$ and $g^* : E^* \to D^*$ in $\mathcal{C}^{\mathrm{op}}$. Then their composition is defined as

$$f^* \circ g^* = (g \circ f)^*\,. \tag{1.22}$$

$$\begin{tikzcd}
C \arrow[loop left, "1_C"] \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & D \arrow[r, "g"] & E & {} \arrow[r, leftrightarrow, "\text{Duality}"] & {} & C^* \arrow[loop above, "1_{C^*}=(1_C)^*"] & D^* \arrow[l, "f^*"'] & E^* \arrow[l, "g^*"'] \arrow[ll, bend left, "f^*\circ g^*=(g\circ f)^*"]
\end{tikzcd}$$

3. The slice category $\mathcal{C}/C$ of a category $\mathcal{C}$ over an object $C \in \mathrm{Ob}\,(\mathcal{C})$ has
   - **Objects:** all arrows f in $\mathcal{C}$ such that $\mathrm{cod}\,(f) = C$. In other words, all arrows $f \in \mathrm{Hom}_{\mathcal{C}}\,(X,C)$ with some $X \in \mathrm{Ob}\,(\mathcal{C})$.
   - **Arrows:** an arrow a from $f : X \to C$ to $f' : X' \to C$ is precisely an arrow in $\mathrm{Hom}_{\mathcal{C}}\,(X,X')$ such that $f' \circ a = f$. In othe words, the following diagram commutes:

$$\begin{tikzcd}
X \arrow[rr, "a"] \arrow[dr, "f"'] & & X' \arrow[dl, "f'"] \\
& C &
\end{tikzcd}$$

Now suppose $f,g,h \in \mathrm{Ob}\,\mathcal{C}/C$ and $a \in \mathrm{Hom}_{\mathcal{C}/C}(f,g), b \in \mathrm{Hom}_{\mathcal{C}/C}(g,h)$. Then there are objects $X, X', X'' \in \mathrm{Ob}\,(\mathcal{C})$ such that the two triangles in the following diagram commute:

$$\begin{tikzcd}
X \arrow[r, "a"'] \arrow[rr, bend left, "b\circ a"] \arrow[dr, "f"'] & X' \arrow[r, "b"'] \arrow[d, "g"'] & X'' \arrow[dl, "h"] \\
& C &
\end{tikzcd}$$

In other words, $g \circ a = f$ and $h \circ b = g$, so that one obtains

$$f = g \circ a = (h \circ b) \circ a = h \circ (b \circ a)\,. \tag{1.23}$$

Therefore, we have the following commutative diagram:

$$\begin{tikzcd} X \arrow[rr, "b\circ a"] \arrow[dr, "f"'] & & X'' \arrow[dl, "h"] \\ & C & \end{tikzcd}$$

Hence, $b \circ a \in \mathrm{Hom}_{\mathcal{C}/C}(f, h)$, using the definition of arrows in $\mathcal{C}/C$. For a given $f \in \mathrm{Ob}(\mathcal{C}/C)$, 1_f is precisely the identity arrow on $\mathrm{dom}(f)$ in the category $\mathcal{C}$, which is evident from the following commutative diagram:

$$\begin{tikzcd} \mathrm{dom}(f) \arrow[rr, "1_{\mathrm{dom}(f)}"] \arrow[dr, "f"'] & & \mathrm{dom}(f) \arrow[dl, "f"] \\ & C & \end{tikzcd}$$

If $g : C \to D$ is any arrow in $\mathcal{C}$, then there is a functor called the **composition functor**:

$$g_* : \mathcal{C}/C \to \mathcal{C}/D,$$

defined on $\mathrm{Ob}(\mathcal{C}/C)$ as

$$g_*(f) = g \circ f. \tag{1.24}$$

$$\begin{tikzcd} X \arrow[d, "f"'] \arrow[dr, "g\circ f"] & \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

Commutativity of the above diagram dictates that $g \circ f \in \mathrm{Ob}(\mathcal{C}/D)$. Now suppose $f, f' \in \mathrm{Ob}(\mathcal{C}/C)$, and consider $a \in \mathrm{Hom}_{\mathcal{C}/C}(f, f')$ so that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "f"'] \arrow[drr, "g\circ f"] & X' \arrow[dl, "f'"'] \arrow[dr, "g\circ f'"] & \\ C \arrow[rr, "g"'] & & D \end{tikzcd}$$

$$g \circ f = g \circ \left(f' \circ a\right) = \left(g \circ f'\right) \circ a, \tag{1.25}$$

so the diagram indeed commutes. So we have the following commutative diagram:

$$\begin{tikzcd} X \arrow[rr, "a"] \arrow[dr, "g\circ f"'] & & X' \arrow[dl, "g\circ f'"] \\ & D & \end{tikzcd}$$

The commutativity of this diagram dictates that $g_*(a) = a$. In fact, the whole construction above is a functor $\mathcal{C}/(-) : \mathcal{C} \to \mathbf{Cat}$.

$$\boxed{\begin{matrix} \mathcal{C} \\ C \xrightarrow{g} D \end{matrix}} \xrightarrow{\mathcal{C}/(-)} \boxed{\begin{matrix} \mathbf{Cat} \\ \mathcal{C}/C \xrightarrow{g_*} \mathcal{C}/D \end{matrix}}$$

4. The coslice category $C/\mathcal{C}$ of a category $\mathcal{C}$ under an object $C \in \mathrm{Ob}(\mathcal{C})$ has as objects all arrows f of $\mathcal{C}$ such that $\mathrm{dom}(f) = C$. An arrow in $\mathrm{Hom}_{C/\mathcal{C}}(f, f')$ is an arrow $h \in \mathrm{Hom}_{\mathcal{C}}(X, X')$ (where $X = \mathrm{cod}(X)$ and $X' = \mathrm{cod}(f')$) such that the diagram below commutes:

$$\begin{tikzcd} & C \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ X \arrow[rr, "h"'] & & X' \end{tikzcd}$$

In other words,

$$h \circ f = f'. \tag{1.26}$$

Question. How can the coslice category be defined in terms of the slice category and the opposite construction?

Example 1.2. The category $\mathbf{Sets}_*$ of pointed sets consists of sets A with a distinguished element $a \in A$, and arrows $f : (A, a) \to (B, b)$ are functions $f : A \to B$ that preserves the distinguished elements $f(a) = b$. Now, $\mathbf{Sets}_*$ is isomorphic to the coslice categroy $1/\mathbf{Sets}$ of sets under any singleton $1 = \{\star\}$.

$$\mathbf{Sets}_* \cong 1/\mathbf{Sets}. \tag{1.27}$$

Indeed, functions $\overline{a} : 1 \to A$ are uniquely determined by $\overline{a}(\star) = a \in A$, and are objects in $1/\mathbf{Sets}$. Now we define a functor $\mathcal{F} : \mathbf{Sets}_* \to 1/\mathbf{Sets}$ by

$$\mathcal{F}(A, a) = \overline{a} \text{ and } \mathcal{F}(f) = f. \tag{1.28}$$

Then we define $\mathcal{G} : 1/\mathbf{Sets} \to \mathbf{Sets}_*$ by

$$\mathcal{G}(\overline{a}) = (A, \overline{a}(*)) \text{ and } \mathcal{G}(f) = f. \tag{1.29}$$

One can easily verify that $\mathcal{G}$ and $\mathcal{F}$ are functors, and

$$\mathcal{G} \circ \mathcal{F} = 1_{\mathbf{Sets}_*} \text{ and } \mathcal{F} \circ \mathcal{G} = 1_{1/\mathbf{Sets}}. \tag{1.30}$$

Therefore, $1/\mathbf{Sets}$ and $\mathbf{Sets}_*$ are isomorphic categories.

2 Free Categories

§2.1 Free Monoid

Start with an "alphabet" A of "letters" $\mathtt{a}, \mathtt{b}, \mathtt{c}, \ldots$, i.e. a set

$$A = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \ldots\} . \tag{2.1}$$

A word over A is a finite sequence of letters:

$$\mathtt{thisword}, \mathtt{categoriesarefun}, \mathtt{asdfghjkl}, \ldots$$

We write "-" for empty word. The **Kleene closure** of A is defined to be the set

$$A^* = \{\text{words over } A\} . \tag{2.2}$$

Define a binary operation $*$ on A^* by $w * w' = ww'$ for words $w, w' \in A^*$. Thus, the binary operation $*$ on A^* is just concatenation. The operation can easily be seen to be associative, and the empty word "-" is a unit. Therefore, A^* is a monoid– called the **free monoid** on the set A.

The number of letters in a word is called its **length**. The elements $\mathtt{a} \in A$ can be regarded as words of length 1. One has a function $i : A \to A^*$ defined by $i(\mathtt{a}) = \mathtt{a}$, and called the " insertion of generators". The elements of A generate the free monoid, in the sense that every $w \in A^*$ can be written as a $*$ products of elements of A, i.e.,

$$w = \mathtt{a}_1 * \mathtt{a}_2 * \cdots * \mathtt{a}_\mathtt{n},$$

for some $\mathtt{a}_1, \ldots, \mathtt{a}_\mathtt{n} \in A$.
A monoid M is **freely generated** by a subset A of M, if the following conditions hold:

(a) Every element $m \in M$ can be written as a product of elements of A:

$$m = a_1 \cdot_M a_2 \cdot_M \cdots \cdot_M a_n , \quad \text{where } a_i \in A.$$

(b) No "nontrivial" relations hold in M. In other words, if

$$a_1 \cdot_M \cdots \cdot_M a_n = a'_1 \cdot_M \cdots \cdot_M a'_k,$$

for $a_i, a'_j \in A$, then this is required by the axioms of monoids.

The second condition of the definition of a free monoid is made more precise in the following way:

First, every monoid N has an underlying set $|N|$, and every monoid homomorphism $f : N \to M$ has an underlying function $|f| : |N| \to |M|$. This way, one has a functor $\mathsf{\mathbf{Mon}} \to \mathsf{\mathbf{Sets}}$. This functor is called the **forgetful functor**.

The free monoid $M(A)$ on a set A is by definition "the" monoid with the following universal mapping property or UMP:

> **Universal mapping property (UMP) of $M(A)$:**
> There is a function $i : A \to |M(A)|$; and given any monoid N and any function $f : A \to |N|$, there is a **unique** monoid homomorphism $\overline{f} : M(A) \to N$ such that $|\overline{f}| \circ i = f$, as indicated in the following diagram:

$$\text{in } \mathbf{Mon} : \qquad M(A) \xrightarrow{\overline{f}} N$$

$$\text{in } \mathbf{Sets} : \qquad \begin{tikzcd} {|M(A)|} \arrow[r, "|\overline{f}|"] & {|N|} \\ A \arrow[u, "i"] \arrow[ur, "f"'] & \end{tikzcd}$$

Proposition 2.1

A^* has the UMP of the free monoid on A.

Proof. Given any monoid N and any function $f : A \to |N|$, define $\overline{f} : A^* \to N$ by $\overline{f}(\text{-}) = u_N$, and

$$\overline{f}(\mathtt{a}_1\mathtt{a}_2\cdots\mathtt{a}_n) = \overline{f}(\mathtt{a}_1 * \mathtt{a}_2 * \cdots * \mathtt{a}_n) := f(\mathtt{a}_1) \cdot_N \cdots \cdot_N f(\mathtt{a}_n)\,. \tag{2.3}$$

$\overline{f} : A^* \to N$ is clearly a monoid homomorphism, with $\overline{f}(\mathtt{a}) = f(\mathtt{a})$, so that

$$\left(|\overline{f}| \circ i\right)(a) = f(a)\,. \tag{2.4}$$

Therefore, the following diagram commutes:

$$\begin{tikzcd} {|A^*|} \arrow[r, "|\overline{f}|"] & {|N|} \\ A \arrow[u, "i"] \arrow[ur, "f"'] & \end{tikzcd}$$

This proves the existence of $\overline{f} : A^* \to N$ with the required universal mapping property. Let us now prove the uniqueness. Suppose there is another monoid homomorphism $g : A^* \to N$ so that $g(\mathtt{a}) = f(\mathtt{a})$, which in turn will give us the commutative diagram exhibiting UMP. Therefore, for all $\mathtt{a}_1, \ldots, \mathtt{a}_n \in A$,

$$\begin{aligned} g(\mathtt{a}_1\mathtt{a}_2\cdots\mathtt{a}_n) &= g(\mathtt{a}_1 * \mathtt{a}_2 * \cdots * \mathtt{a}_n) \\ &= g(\mathtt{a}_1) \cdot_N \cdots \cdot_N g(\mathtt{a}_n) \\ &= f(\mathtt{a}_1) \cdot_N \cdots \cdot_N f(\mathtt{a}_n) \\ &= \overline{f}(\mathtt{a}_1\mathtt{a}_2\cdots\mathtt{a}_n)\,. \end{aligned}$$

Therefore, $g = \overline{f}$, proving the uniqueness of $\overline{f} : A^* \to N$. ■

Remark 2.1. Existence of a monoid homomorphism $\overline{f} : M(A) \to N$ implies that if there is an additional equality (sometimes referred to as "noise") besides the ones imposed by associativity law and unitality law in $M(A)$, then the additional equality is transported to the monoid N. But N is **any** monoid to which the free monoid $M(A)$ is supposed to be mapped to via the monoid homomorphism $\overline{f} : M(A) \to N$. Hence, the existence of monoid homomorphism $\overline{f} : M(A) \to N$ is equivalent to the second condition of absense of any "noise" in $M(A)$.

Proposition 2.2

Given monoids M and N with functions $i : A \to |M|$ and $j : A \to |N|$, each with the UMP of the free monoid on A, there is a unique monoid isomorphism $h : M \xrightarrow{\cong} N$ such that

$$|h| \circ i = j \text{ and } \left|h^{-1}\right| \circ j = i.$$

Proof. From $j : A \to |N|$ and the UMP of M, one has $\bar{j} : M \to N$ with $\left|\bar{j}\right| \circ i = j$.

$$\text{in } \mathbf{Mon} : \qquad M \overset{\bar{j}}{\dashrightarrow} N$$

$$\text{in } \mathbf{Sets} : \qquad \begin{tikzcd} {|M|} \arrow[r, "|\bar{j}|"] & {|N|} \\ A \arrow[u, "i"] \arrow[ur, "j"'] & \end{tikzcd}$$

From $i : A \to |M|$ and the UMP of N, one has $\bar{i} : N \to M$ with $\left|\bar{i}\right| \circ j = i$.

$$\text{in } \mathbf{Mon} : \qquad N \overset{\bar{i}}{\dashrightarrow} M$$

$$\text{in } \mathbf{Sets} : \qquad \begin{tikzcd} {|N|} \arrow[r, "|\bar{i}|"] & {|M|} \\ A \arrow[u, "j"] \arrow[ur, "i"'] & \end{tikzcd}$$

Combining these two, we get the following commutative diagram:

$$\text{in } \mathbf{Mon} : \qquad \begin{tikzcd} M \arrow[r, dashed, "\bar{j}"'] \arrow[rr, bend left, "\bar{i}\circ\bar{j}"] & N \arrow[r, dashed, "\bar{i}"'] & M \end{tikzcd}$$

$$\text{in } \mathbf{Sets} : \qquad \begin{tikzcd} {|M|} \arrow[r, "|\bar{j}|"] & {|N|} \arrow[r, "|\bar{i}|"] & {|M|} \\ & A \arrow[ul, "i"] \arrow[u, "j"] \arrow[ur, "i"'] & \end{tikzcd}$$

From $i : A \to |M|$ and the UMP of M, we have the existence of a unique homomorphism $f : M \to M$ such that $|f| \circ i = i$. From the above commutative diagram, we get that $f = \bar{i} \circ \bar{j}$ satisfies $|f| \circ i = i$. Furthermore, $f = 1_M : M \to M$ also satisfies $|f| \circ i = i$. Therefore,

$$\bar{i} \circ \bar{j} = 1_M. \tag{2.5}$$

Similarly, exchanging M and N, we get

$$\bar{j} \circ \bar{i} = 1_N. \tag{2.6}$$

Now, $\bar{j} : M \to N$ is the required monoid isomorphism h, i.e. $\bar{j} = h$ and $\bar{i} = h^{-1}$, so that we have $|h| \circ i = j$ and $\left|h^{-1}\right| \circ j = i$. ■

In light of Proposition 2.1 and Proposition 2.2, we can say that if $M(A)$ has the UMP of a free monoid on A, then $M(A)$ is isomorphic to A^*.

§2.2 Free Category

Just as a monoid has an underlying set, a category has an underlying graph. A directed graph consists of vertices and edges, each of which has a "source" and a "target" vertex. Figure 2.1 is an example of a graph.

$$
\begin{tikzcd}
A \arrow[r, "z"] \arrow[dr, "u"'] & B \\
C \arrow[u, "x"] & D \arrow[u, "y"']
\end{tikzcd}
$$

Figure 2.1: A graph

> **Definition 2.1.** A **(directed) graph** consists of two sets: a set E of edges, and a set V of vertices, and two functions $s : E \to V$ (called source) adn $t : E \to V$ (called target). We denote a directed graph G by a quadruple (V, E, s, t). A **path** in a graph G is a finite sequence of edges $e_1, \dots, e_n$ such that $t(e_i) = s(e_{i+1})$ for each $i = 1, \dots, n-1$.

Suppose we have a path $e_1, \dots, e_n$ in G.

$$
v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} v_2 \xrightarrow{e_3} \cdots \xrightarrow{e_n} v_n
$$

Put $\operatorname{dom}(e_n \cdots e_1) = s(e_1)$ and $\operatorname{cod}(e_n \cdots e_1) = t(e_n)$, and define composition by concatenation:

$$
e_n \cdots e_1 \circ e'_m \cdots e'_1 = e_n \cdots e_1 e'_m \cdots e'_1, \tag{2.7}
$$

where $\operatorname{dom}(e_n \cdots e_1) = \operatorname{cod}(e'_m \cdots e'_1)$.

For each vertex v, we have an "empty path" denoted by 1_v which is to be the identity arrow at v. With all of these terminologies at our disposal, we see that every graph G generates a category $\mathcal{C}(G)$ called the **free category** on G. It is defined by taking vertices of G as objects and paths in G as arrows. For example, take the graph given in Figure 2.1 with 4 vertices A, B, C, D. The corresponding free category on G is given by:

$$
G: \quad
\begin{tikzcd}
A \arrow[r, "z"] \arrow[dr, "u"'] & B \\
C \arrow[u, "x"] & D \arrow[u, "y"']
\end{tikzcd}
$$

$$
\mathcal{C}(G) \quad
\begin{tikzcd}
C \arrow[r, "x"] \arrow[rr, bend left=60, "z \circ x"] \arrow[rr, bend right=30, "y \circ u \circ x"'] \arrow[rrr, bend right=50, "u \circ x"'] & A \arrow[r, dashed, "z"] \arrow[r, bend left, "y \circ u"] \arrow[rr, bend right, "u"'] & B & D \arrow[l, "y"']
\end{tikzcd}
$$

> **Definition 2.2** (Graph Homomorphism)**.** Let $G \equiv (V, E, s, t)$ and $G' \equiv (V', E', s', t')$ be two graphs. A **graph homomorphism** f from G to G', denoted by $f : G \to G'$ consists of two functions $f_0 : V \to V'$ and $f_1 : E \to E'$ such that the following diagrams commute:
>
> $$
> \begin{tikzcd}
> E \arrow[r, "f_1"] \arrow[d, "s"'] & E' \arrow[d, "s'"] \\
> V \arrow[r, "f_0"'] & V'
> \end{tikzcd}
> \qquad
> \begin{tikzcd}
> E \arrow[r, "f_1"] \arrow[d, "t"'] & E' \arrow[d, "t'"] \\
> V \arrow[r, "f_0"'] & V'
> \end{tikzcd}
> $$

> **Remark 2.2.** Note that if G has only one vertex, then $\mathcal{C}(G)$ is just the free monoid on the set of edges of G. If, on the other hand, G has only vertices with no edges, then $\mathcal{C}(G)$ is the discrete category on the set of vertices of G.

Let us now see that $\mathcal{C}(G)$ has a UMP (universal mapping property). Define a "forgetful functor" $\mathcal{U} : \mathbf{Cat} \to \mathbf{Graphs}$ in the following way: the underlying graph of a (small) category has the collection of arrows as the set of edges E and the collection of objects as the set of vertices V, with $s = \mathrm{dom}$ and $t = \mathrm{cod}$.

Also, observe that we can describe a category $\mathcal{C}$ with a diagram as below:

$$\begin{tikzcd} C_2 \arrow[r, "\circ"] & C_1 \arrow[r, shift left=2, "\mathrm{cod}"] \arrow[r, shift right=2, "\mathrm{dom}"'] & C_0 \arrow[l, "i"' description] \end{tikzcd},$$

where C_0 is the collection of objects of $\mathcal{C}_1$, C_1 is the collection of arrows, i is the identity arrow operation, and C_2 is the collection

$$C_2 = \{(f, g) \in C_1 \times C_1 \mid \mathrm{cod}\, f = \mathrm{dom}\, g\}.$$

Then a functor $F : \mathcal{C} \to \mathcal{D}$ from $\mathcal{C}$ to another category $\mathcal{D}$ (with D_2, D_1, D_0 as given above) is a pair of assignments $F_0 : C_0 \to D_0$ and $F_1 : C_1 \to D_1$ such that each similarly labeled square in the following diagram commutes:

$$\begin{tikzcd} C_2 \arrow[r, "\circ"] \arrow[d, "F_2"'] & C_1 \arrow[r, shift left=2, "\mathrm{cod}"] \arrow[r, shift right=2, "\mathrm{dom}"'] \arrow[d, "F_1"'] & C_0 \arrow[l, "i" description] \arrow[d, "F_0"] \\ D_2 \arrow[r, "\circ"] & D_1 \arrow[r, shift left=2, "\mathrm{cod}"] \arrow[r, shift right=2, "\mathrm{dom}"'] & D_0 \arrow[l, "i" description] \end{tikzcd},$$

where $F_2(f, g) = (F_1(f), F_1(g))$. Commutativity of the first square tells us that

$$F_1(g \circ f) = F_1(g) \circ F_1(f). \tag{2.8}$$

Commutativity of the second square is reminiscent of graph homomorphism if one removes the identity arrow operation.