



Inspiring Excellence

## **Differential Geometry II (MAT401)**

**Lecture Notes**

# Preface

This series of lecture notes has been prepared for aiding students who took the BRAC University course **Differential Geometry II (MAT401)** in Fall 2023 semester. These notes were typeset under the supervision of mathematician **Dr. Syed Hasibul Hassan Chowdhury**. The main goal of this typeset is to have an organized digital version of the notes, which is easier to share and handle. If you see any mistakes or typos, please send me an email at [atonuroychowdhury@gmail.com](mailto:atonuroychowdhury@gmail.com)

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## References:

- *An Introduction to Manifolds*, by **Loring W. Tu**
- *An Introduction to Differentiable Manifolds and Riemannian Geometry*, by **William Boothby**
- *Introduction to Smooth Manifolds*, by **John M. Lee**
- *Lectures on Differential geometry*, by **S.S Chern, W.H. Chen and K.S. Lam**
- *Geometry of Differential Forms*, by **Shigeyuki Morita**
- *From Calculus to Cohomology: De Rham Cohomology and Characteristic Classes*, by **Ib Madsen and Jxrgen Tornehave**.

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# 1 Review of Multilinear Algebra

## §1.1 Dual Space

Let  $V$  and  $W$  be real vector spaces. We denote by  $\text{Hom}(V, W)$  the vector space of all linear maps  $f : V \rightarrow W$ . In particular, if we choose  $W = \mathbb{R}$ , we get the **dual space**  $V^*$ .

$$V^* = \text{Hom}(V, \mathbb{R}).$$

The elements of  $V^*$  are called covectors on  $V$ . In the rest of the lecture, we will assume  $V$  to be a finite dimensional vector space. Let  $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$  be a basis for  $V$ . Then every  $\mathbf{v} \in V$  is a unique linear combination

$$\mathbf{v} = \sum_{i=1}^n v^i \mathbf{e}_i, \quad (1.1)$$

with  $v^i \in \mathbb{R}$ .  $v^i$ 's are called the coordinates of  $\mathbf{v}$  relative to the basis  $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ . Let  $\hat{\alpha}^i$  be the linear function on  $V$  that picks up the  $i$ -th coordinate of the vector, i.e.

$$\hat{\alpha}^i(\mathbf{v}) = \hat{\alpha}^i\left(\sum_{i=1}^n v^i \mathbf{e}_i\right) = v^i. \quad (1.2)$$

When  $\mathbf{v}$  is one of the basis vectors,

$$\hat{\alpha}^i(\mathbf{e}_j) = \delta^i_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \quad (1.3)$$

### Proposition 1.1

The functions  $\hat{\alpha}^1, \dots, \hat{\alpha}^n$  form a basis for  $V^*$ .

*Proof.* Suppose  $f \in V^*$ . Then for any  $\mathbf{v} = \sum_{i=1}^n v^i \mathbf{e}_i \in V$ ,

$$f(\mathbf{v}) = f\left(\sum_{i=1}^n v^i \mathbf{e}_i\right) = \sum_{i=1}^n v^i f(\mathbf{e}_i) = \sum_{i=1}^n f(\mathbf{e}_i) \hat{\alpha}^i(\mathbf{v}).$$

Since this holds for any  $\mathbf{v} \in V$ ,

$$f = \sum_{i=1}^n f(\mathbf{e}_i) \hat{\alpha}^i. \quad (1.4)$$

Therefore,  $\hat{\alpha}^1, \dots, \hat{\alpha}^n$  span  $V^*$ . As for linear independence, suppose

$$\sum_{i=1}^n c_i \hat{\alpha}^i = \mathbf{0}, \quad (1.5)$$

where  $\mathbf{0}$  is the function that takes all of  $V$  to 0 in  $\mathbb{R}$ . If we evaluate (1.5) at  $\mathbf{e}_j$ , we get

$$0 = \sum_{i=1}^n c_i \hat{\alpha}^i(\mathbf{e}_j) = \sum_{i=1}^n c_i \delta^i_j = c_j. \quad (1.6)$$

So  $c_j = 0$ , and this holds for each  $j = 1, 2, \dots, n$ . Therefore,  $\{\hat{\alpha}^1, \dots, \hat{\alpha}^n\}$  is a linearly independent set that spans  $V^*$ , i.e. a basis. ■

### Corollary 1.2

The dual space  $V^*$  of a finite dimensional vector space has the same dimension as  $V$ .

The basis  $\{\hat{\alpha}^1, \dots, \hat{\alpha}^n\}$  for  $V^*$  is said to be dual to the basis  $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$  for  $V$ .

## §1.2 Permutations

Fix a positive integer  $k$ . A permutation of the set  $A = \{1, 2, \dots, k\}$  is a bijection  $\sigma : A \rightarrow A$ . The product of two permutations  $\tau$  and  $\sigma$  is the composition  $\tau \circ \sigma : A \rightarrow A$ . The **cyclic permutation**  $(a_1 a_2 \cdots a_r)$  is the permutation  $\sigma$  such that

$$\sigma(a_1) = a_2, \sigma(a_2) = a_3, \dots, \sigma(a_{r-1}) = a_r, \text{ and } \sigma(a_r) = 1,$$

leaving all other elements unchanged, i.e.  $\sigma(j) = j$  if  $j$  is not one of the  $a_i$ 's. A cyclic permutation  $(a_1 a_2 \cdots a_r)$  is also called a **cycle** of length  $r$  or an  $r$ -cycle. A **transposition** is a permutation of the form  $(a b)$  that interchanges  $a$  and  $b$ , leaving all other elements of  $A$  fixed.

A permutation  $\sigma : A \rightarrow A$  can be described by

$$\begin{bmatrix} 1 & 2 & \cdots & k \\ \sigma(1) & \sigma(2) & \cdots & \sigma(k) \end{bmatrix}.$$

We also write it as

$$[1 \ 2 \ \cdots \ k] \xrightarrow{\sigma} [\sigma(1) \ \sigma(2) \ \cdots \ \sigma(k)].$$

**Example 1.1.** Suppose  $\sigma : \{1, 2, 3, 4, 5\} \rightarrow \{1, 2, 3, 4, 5\}$  is the permutation given by

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 5 & 1 & 3 \end{bmatrix}.$$

In other words,  $\sigma(1) = 2$ ,  $\sigma(2) = 4$ ,  $\sigma(3) = 5$ ,  $\sigma(4) = 1$ , and  $\sigma(5) = 3$ .

$$[1 \ 2 \ 3 \ 4 \ 5] \xrightarrow{\sigma} [2 \ 4 \ 5 \ 1 \ 3].$$

Observe that the cyclic permutation  $\sigma' = (1 \ 2 \ 4)$  acts as  $\sigma'(1) = 2$ ,  $\sigma'(2) = 4$  and  $\sigma'(4) = 1$ , keeping 3 and 5 unchanged, i.e.  $\sigma'(3) = 3$  and  $\sigma'(5) = 5$ .

$$[1 \ 2 \ 3 \ 4 \ 5] \xrightarrow{(1 \ 2 \ 4)} [2 \ 4 \ 3 \ 1 \ 5].$$

Now the transposition  $\sigma'' = (3 \ 5)$  acts as  $\sigma''(3) = 5$  and  $\sigma''(5) = 3$ , keeping 1, 2, 4 unchanged. Therefore,

$$\begin{array}{ccccc} [1 \ 2 \ 3 \ 4 \ 5] & \xrightarrow{(1 \ 2 \ 4)} & [2 \ 4 \ 3 \ 1 \ 5] & \xrightarrow{(3 \ 5)} & [2 \ 4 \ 5 \ 1 \ 3] \\ & \searrow & & \nearrow & \\ & & (3 \ 5)(1 \ 2 \ 4) & & \end{array}$$

so that  $\sigma = (3 \ 5)(1 \ 2 \ 4)$ .

Let  $S_k$  be the group of permutations of the set  $\{1, 2, \dots, k\}$ . The order of this group is  $k!$ . A permutation is *even* or *odd* depending on whether it is the product of an even or an odd number of transpositions. The sign of a permutation  $\sigma$  is 1 if the permutation is even, and  $-1$  otherwise. It is denoted by  $\text{sgn } \sigma$ . For example, in [Example 1.1](#),  $\sigma = (3 \ 5)(1 \ 2 \ 4)$ . Note that we can write  $(1 \ 2 \ 4)$  as a product of two transpositions:

$$\begin{array}{ccccc} [1 \ 2 \ 3 \ 4 \ 5] & \xrightarrow{(1 \ 2)} & [2 \ 1 \ 3 \ 4 \ 5] & \xrightarrow{(1 \ 4)} & [2 \ 4 \ 3 \ 1 \ 5] \\ & \searrow & & \nearrow & \\ & & (1 \ 4)(1 \ 2) = (1 \ 2 \ 4) & & \end{array}$$

In other words,  $\sigma = (3 \ 5)(1 \ 4)(1 \ 2)$ . Hence,  $\text{sgn } \sigma = -1$ . One can easily check that

$$\text{sgn } (\sigma\tau) = (\text{sgn } \sigma)(\text{sgn } \tau). \quad (1.7)$$

So  $\text{sgn} : S_k \rightarrow \{1, -1\}$  is a group homomorphism.

**Example 1.2.** Observe that the 5-cycle  $(1\ 2\ 3\ 4\ 5)$  can be written as

$$(1\ 2\ 3\ 4\ 5) = (1\ 5)(1\ 4)(1\ 3)(1\ 2).$$

Indeed,

$$\begin{array}{ccccccc} [1\ 2\ 3\ 4\ 5] & \xrightarrow{(1\ 2)} & [2\ 1\ 3\ 4\ 5] & \xrightarrow{(1\ 3)} & [2\ 3\ 1\ 4\ 5] & \xrightarrow{(1\ 4)} & [2\ 3\ 4\ 1\ 5] & \xrightarrow{(1\ 5)} & [2\ 3\ 4\ 5\ 1] \\ & & & & & & \nearrow & & \\ & & & & & & (1\ 2\ 3\ 4\ 5) = (1\ 5)(1\ 4)(1\ 3)(1\ 2) & & \end{array}$$

Therefore,  $\text{sgn}(1\ 2\ 3\ 4\ 5) = 1$ .

An **inversion** in a permutation  $\sigma$  is an ordered pair  $(\sigma(i), \sigma(j))$  such that  $i < j$  but  $\sigma(i) > \sigma(j)$ . In [Example 1.1](#),  $\sigma(1) = 2$ ,  $\sigma(2) = 4$ ,  $\sigma(3) = 5$ ,  $\sigma(4) = 1$ , and  $\sigma(5) = 3$ . So, the inversions in this permutation are

$$(\sigma(1), \sigma(4)), (\sigma(2), \sigma(4)), (\sigma(2), \sigma(5)), (\sigma(3), \sigma(4)), (\sigma(3), \sigma(5)).$$

Hence, there are 5 inversions associated with the permutation  $\sigma$ . There is an efficient way of determining the sign of a permutation.

### Proposition 1.3

A permutation is even if and only if it has an even number of inversions.

*Proof.* Let  $\sigma \in S_k$  with  $n$  inversions. We shall prove that we can multiply  $\sigma$  by  $n$  transpositions and get the identity permutation. This will prove that  $\text{sgn } \sigma = (-1)^n$ .

Suppose  $\sigma(j_1) = 1$ . Then for each  $i < j_1$ ,  $(\sigma(i), \sigma(j_1))$  is an inversion, and there are  $j_1 - 1$  many of them. These are all the inversions with 1 in the second slot of the ordered pair of inversion. If we now multiply  $\sigma$  by the  $j_1$ -cycle

$$(\sigma(1)\ 1)(\sigma(2)\ 1) \cdots (\sigma(j_1 - 1)\ 1)$$

to the left of  $\sigma$ , the resulting permutation  $\sigma_1$  would be

$$\begin{bmatrix} 1 & 2 & 3 & \cdots & j_1 & j_1 + 1 & \cdots & k \\ 1 & \sigma(1) & \sigma(2) & \cdots & \sigma(j_1 - 1) & \sigma(j_1 + 1) & \cdots & \sigma(k) \end{bmatrix}$$

This permutation has no inversion with 1 in the second slot of the ordered pair of inversion. Suppose now that  $\sigma(j_2) = 2$ . Now observe that if  $(\sigma_1(i), 2)$  is an inversion in  $\sigma_1$ , then either  $(\sigma(i), 2)$  (if  $i \geq j_1 + 1$ ) or  $\sigma(i - 1), 2$  (if  $i \leq j_1 - 1$ ) is an inversion in  $\sigma$ . Therefore, the number of inversions in  $\sigma_1$  ending in 2 is precisely the same as the number of inversions in  $\sigma$  ending in 2. So following a similar procedure as above, we can multiply  $\sigma_1$  by  $i_2$ -many transpositions to the left ( $i_2$  is the number of transpositions ending in 2) and get

$$\begin{bmatrix} 1 & 2 & 3 & \cdots & k \\ 1 & 2 & \sigma(1) & \cdots & \sigma(k) \end{bmatrix}$$

We can continue these steps for each  $j = 1, 2, \dots, k$ , and the number of transpositions required to move  $j$  to its natural position is the same as the number of inversions ending in  $j$ . In the end we achieve the identity permutation. Therefore,  $\text{sgn } \sigma = (-1)^n$ , where  $n$  is the number of inversions. ■

### §1.3 Multilinear Functions

**Definition 1.1.** Let  $V^k$  be the cartesian product of  $k$ -copies of a real vector space  $V$ .

$$V^k = \underbrace{V \times V \times \cdots \times V}_{k\text{-copies}}$$

A function  $f : V^k \rightarrow \mathbb{R}$  is called  $k$ -linear if it is linear in each of its  $k$  arguments:

$$f(\dots, a\mathbf{v} + b\mathbf{w}, \dots) = af(\dots, \mathbf{v}, \dots) + bf(\dots, \mathbf{w}, \dots), \quad (1.8)$$

for  $a, b \in \mathbb{R}$  and  $\mathbf{v}, \mathbf{w} \in V$ .

Instead of 2-linear and 3-linear, it's customary to call “bilinear” and “trilinear”, respectively. A  $k$ -linear function on  $V$  is called a  **$k$ -tensor** on  $V$ . We will denote the vector space of all  $k$ -tensors on  $V$  by  $L_k(V)$ . The vector addition and scalar multiplication of the real vector space  $L_k(V)$  is the straightforward pointwise operation.

**Example 1.3.** The dot product  $f(\mathbf{v}, \mathbf{w}) = \mathbf{v} \cdot \mathbf{w}$  on  $\mathbb{R}^n$  is bilinear: if  $\mathbf{v} = \sum_{i=1}^n v^i \mathbf{e}_i$  and  $\mathbf{w} = \sum_{i=1}^n w^i \mathbf{e}_i$ , then

$$f(\mathbf{v}, \mathbf{w}) = \mathbf{v} \cdot \mathbf{w} = \sum_{i=1}^n v^i w^i.$$

**Example 1.4.** The determinant

$$f(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n) = \det \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}$$

viewed as a function of the  $n$  column vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  is  $n$ -linear.

**Definition 1.2** (Symmetric and alternating function). A  $k$ -linear function  $f : V^k \rightarrow \mathbb{R}$  is **symmetric** if

$$f(\mathbf{v}_{\sigma(1)}, \mathbf{v}_{\sigma(2)}, \dots, \mathbf{v}_{\sigma(k)}) = f(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k), \quad (1.9)$$

for all permutations  $\sigma \in S_k$ . It is **alternating** if

$$f(\mathbf{v}_{\sigma(1)}, \mathbf{v}_{\sigma(2)}, \dots, \mathbf{v}_{\sigma(k)}) = (\text{sgn } \sigma) f(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k), \quad (1.10)$$

for all permutations  $\sigma \in S_k$ .

The dot product function on  $\mathbb{R}^n$  in [Theorem 1.3](#) is symmetric, and the determinant function on  $\mathbb{R}^n$  in [Theorem 1.4](#) is alternating.

We are especially interested in the vector space  $A_k(V)$  of all alternating  $k$ -linear functions on a vector space  $V$ , for  $k > 0$ . The elements of  $A_k(V)$  are called alternating  $k$ -tensors (also known as  $k$ -covectors). We define  $A_0(V)$  to be  $\mathbb{R}$ . The elements of  $A_0(V)$  are simply constants, which we call 0-covectors. The elements of  $A_1(V)$  are simply covectors, i.e. the elements of  $V^*$ .

#### Permutation action on $k$ -linear functions

If  $f \in L_k(V)$  and  $\sigma \in S_k$ , define  $\sigma f \in L_k(V)$  as follows:

$$(\sigma f)(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k) = f(\mathbf{v}_{\sigma(1)}, \mathbf{v}_{\sigma(2)}, \dots, \mathbf{v}_{\sigma(k)}). \quad (1.11)$$

Thus,  $f$  is symmetric if and only if  $f = \sigma f$  for all  $\sigma \in S_k$ ; and  $f$  is alternating if and only if  $\sigma f = (\text{sgn } \sigma) f$  for all  $\sigma \in S_k$ . When  $k = 1$ ,  $S_k$  only has the identity permutation. In that case, a 1-linear function or simply linear function on  $V$  is both symmetric and alternating. In particular,

$$A_1(V) = L_1(V) = V^*.$$

**Lemma 1.4**

If  $\sigma, \tau \in S_k$  and  $f \in L_k(V)$ , then  $\tau(\sigma f) = (\tau\sigma)f$ .

*Proof.* For any  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in V$ ,

$$\begin{aligned}
 (\tau(\sigma f))(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k) &= (\sigma f)(\mathbf{v}_{\tau(1)}, \mathbf{v}_{\tau(2)}, \dots, \mathbf{v}_{\tau(k)}) \\
 &= (\sigma f)(\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_k) && [\mathbf{w}_i = \mathbf{v}_{\tau(i)}] \\
 &= f(\mathbf{w}_{\sigma(1)}, \mathbf{w}_{\sigma(2)}, \dots, \mathbf{w}_{\sigma(k)}) \\
 &= f(\mathbf{v}_{\tau(\sigma(1))}, \mathbf{v}_{\tau(\sigma(2))}, \dots, \mathbf{v}_{\tau(\sigma(k))}) \\
 &= ((\tau\sigma)f)(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k).
 \end{aligned}$$

Therefore,  $\tau(\sigma f) = (\tau\sigma)f$ . ■

**Definition 1.3.** If  $G$  is a group and  $X$  is a set, a map

$$\begin{aligned}
 G \times X &\rightarrow X \\
 (g, x) &\mapsto g \cdot x
 \end{aligned}$$

is called a **left action** of  $G$  on  $X$  if

- (i)  $e \cdot x = x$ , where  $e$  is the identity element in  $G$  and  $x$  is any element in  $X$ ; and
- (ii)  $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$ , for all  $g_1, g_2 \in G$  and  $x \in X$ .

Similarly, a **right action** of  $G$  on  $X$  is a map

$$\begin{aligned}
 X \times G &\rightarrow X \\
 (x, g) &\mapsto x \cdot g
 \end{aligned}$$

such that

- (i)  $x \cdot e = x$ , for all  $x \in X$ ; and
- (ii)  $(x \cdot g_1) \cdot g_2 = x \cdot (g_1 g_2)$ , for all  $g_1, g_2 \in G$  and  $x \in X$ .

**Symmetrizing and alternating operators**

Given  $f \in L_k(V)$ , there is a way to make it a symmetric  $k$ -linear function  $\mathcal{S}f$  from it:

$$(\mathcal{S}f)(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k) = \sum_{\sigma \in S_k} f(\mathbf{v}_{\sigma(1)}, \mathbf{v}_{\sigma(2)}, \dots, \mathbf{v}_{\sigma(k)}). \quad (1.12)$$

In other words,

$$\mathcal{S}f = \sum_{\sigma \in S_k} \sigma f. \quad (1.13)$$

Similarly, there is a way to make an alternating  $k$ -linear function from  $f$ :

$$\mathcal{A}f = \sum_{\sigma \in S_k} (\text{sgn } \sigma) \sigma f. \quad (1.14)$$

**Proposition 1.5** (i) The  $k$ -linear function  $\mathcal{S}f$  is symmetric.

(ii) The  $k$ -linear function  $\mathcal{A}f$  is alternating.



*Proof.* (i) Let  $\tau \in S_k$ . Then

$$\tau(\mathcal{S}f) = \tau\left(\sum_{\sigma \in S_k} \sigma f\right). \quad (1.15)$$

The group action of  $S_k$  on  $L_k(V)$  is distributive over the vector space addition. Therefore,

$$\tau(\mathcal{S}f) = \sum_{\sigma \in S_k} \tau(\sigma f) = \sum_{\sigma \in S_k} (\tau\sigma)f. \quad (1.16)$$

As  $\sigma$  varies over  $S_k$ ,  $\tau\sigma$  also varies over  $S_k$ . Therefore,  $\sum_{\sigma \in S_k} (\tau\sigma)f = \mathcal{S}f$ . In other words,

$$\tau(\mathcal{S}f) = \mathcal{S}f, \quad (1.17)$$

i.e.  $\mathcal{S}f$  is symmetric.

(ii) Let  $\tau \in S_k$ . Then

$$\tau(\mathcal{A}f) = \tau\left(\sum_{\sigma \in S_k} (\text{sgn } \sigma) \sigma f\right) = \sum_{\sigma \in S_k} (\text{sgn } \sigma) \tau(\sigma f) = \sum_{\sigma \in S_k} (\text{sgn } \sigma) (\tau\sigma)f. \quad (1.18)$$

Since  $(\text{sgn } \tau)^2 = 1$ ,

$$\begin{aligned} \tau(\mathcal{A}f) &= \sum_{\sigma \in S_k} (\text{sgn } \tau)^2 (\text{sgn } \sigma) (\tau\sigma)f \\ &= (\text{sgn } \tau) \sum_{\sigma \in S_k} (\text{sgn } \tau) (\text{sgn } \sigma) (\tau\sigma)f \\ &= (\text{sgn } \tau) \sum_{\sigma \in S_k} (\text{sgn } (\tau\sigma)) (\tau\sigma)f. \end{aligned} \quad (1.19)$$

As  $\sigma$  varies over  $S_k$ ,  $\tau\sigma$  also varies over  $S_k$ . Therefore,  $\sum_{\sigma \in S_k} (\text{sgn } (\tau\sigma)) (\tau\sigma)f = \mathcal{A}f$ . In other words,

$$\tau(\mathcal{A}f) = \mathcal{A}f, \quad (1.20)$$

i.e.  $\mathcal{A}f$  is alternating. ■

### Lemma 1.6

If  $f \in A_k(V)$ , then  $\mathcal{A}f = (k!)f$ .

*Proof.* Since  $f$  is alternating,

$$\mathcal{A}f = \sum_{\sigma \in S_k} (\text{sgn } \sigma) \sigma f = \sum_{\sigma \in S_k} (\text{sgn } \sigma)^2 f = \sum_{\sigma \in S_k} f = (k!)f, \quad (1.21)$$

because the order of  $S_k$  is  $k!$ . ■

## §1.4 Tensor Product and Wedge Product

**Definition 1.4** (Tensor Product). Let  $f$  be a  $k$ -linear function and  $g$  an  $l$ -linear function on a vector space  $V$ . Their tensor product  $f \otimes g$  is the  $(k+l)$ -linear function defined by

$$(f \otimes g)(\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{v}_{k+1}, \dots, \mathbf{v}_{k+l}) = f(\mathbf{v}_1, \dots, \mathbf{v}_k) g(\mathbf{v}_{k+1}, \dots, \mathbf{v}_{k+l}). \quad (1.22)$$

$(k+l)$ -linearity of  $f \otimes g$  follows from  $k$ -linearity of  $f$  and  $l$ -linearity of  $g$ .

**Lemma 1.7** (Associativity of Tensor Product)

Let  $f \in L_k(V)$ ,  $g \in L_l(V)$  and  $h \in L_m(V)$ . Then

$$(f \otimes g) \otimes h = f \otimes (g \otimes h).$$

*Proof.* For  $\mathbf{v}_1, \dots, \mathbf{v}_{k+l+m}$ ,

$$\begin{aligned} [(f \otimes g) \otimes h](\mathbf{v}_1, \dots, \mathbf{v}_{k+l+m}) &= (f \otimes g)(\mathbf{v}_1, \dots, \mathbf{v}_{k+l}) h(\mathbf{v}_{k+l+1}, \dots, \mathbf{v}_{k+l+m}) \\ &= f(\mathbf{v}_1, \dots, \mathbf{v}_k) g(\mathbf{v}_{k+1}, \dots, \mathbf{v}_{k+l}) h(\mathbf{v}_{k+l+1}, \dots, \mathbf{v}_{k+l+m}). \end{aligned} \quad (1.23)$$

$$\begin{aligned} [f \otimes (g \otimes h)](\mathbf{v}_1, \dots, \mathbf{v}_{k+l+m}) &= f(\mathbf{v}_1, \dots, \mathbf{v}_k) (g \otimes h)(\mathbf{v}_{k+1}, \dots, \mathbf{v}_{k+l+m}) \\ &= f(\mathbf{v}_1, \dots, \mathbf{v}_k) g(\mathbf{v}_{k+1}, \dots, \mathbf{v}_{k+l}) h(\mathbf{v}_{k+l+1}, \dots, \mathbf{v}_{k+l+m}). \end{aligned} \quad (1.24)$$

Therefore,  $(f \otimes g) \otimes h = f \otimes (g \otimes h)$ , i.e. tensor product is associative.  $\blacksquare$

**Example 1.5.** Let  $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$  be the standard basis for  $\mathbb{R}^n$ , and  $\{\hat{\alpha}^1, \dots, \hat{\alpha}^n\}$  its dual basis. The Euclidean inner product on  $\mathbb{R}^n$  is the bilinear function

$$\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$$

defined by

$$\langle \cdot, \cdot \rangle(\mathbf{v}, \mathbf{w}) = \langle \mathbf{v}, \mathbf{w} \rangle = \sum_{i=1}^n v^i w^i,$$

for  $\mathbf{v} = \sum_{i=1}^n v^i \mathbf{e}_i$  and  $\mathbf{w} = \sum_{i=1}^n w^i \mathbf{e}_i$ . We can express  $\langle \cdot, \cdot \rangle$  in terms of tensor product as follows:

$$\langle \cdot, \cdot \rangle(\mathbf{v}, \mathbf{w}) = \sum_{i=1}^n v^i w^i = \sum_{i=1}^n \hat{\alpha}^i(\mathbf{v}) \hat{\alpha}^i(\mathbf{w}) = \sum_{i=1}^n (\hat{\alpha}^i \otimes \hat{\alpha}^i)(\mathbf{v}, \mathbf{w}).$$

Since  $\mathbf{v}, \mathbf{w}$  are arbitrary,

$$\langle \cdot, \cdot \rangle = \sum_{i=1}^n (\hat{\alpha}^i \otimes \hat{\alpha}^i). \quad (1.25)$$

If  $f \in A_k(V)$  and  $g \in A_l(V)$ , then it's not true that  $f \otimes g \in A_{k+l}(V)$ , in general. We need to construct a product that is also alternating.

**Definition 1.5** (Wedge Product). For  $f \in A_k(V)$  and  $g \in A_l(V)$ , the wedge product of  $f$  and  $g$  is defined as follows:

$$f \wedge g = \frac{1}{k!l!} \mathcal{A}(f \otimes g). \quad (1.26)$$

Explicitly,

$$\begin{aligned} (f \wedge g)(\mathbf{v}_1, \dots, \mathbf{v}_{k+l}) &= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (\text{sgn } \sigma) \sigma(f \otimes g)(\mathbf{v}_1, \dots, \mathbf{v}_{k+l}) \\ &= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (\text{sgn } \sigma) (f \otimes g)(\mathbf{v}_{\sigma(1)}, \dots, \mathbf{v}_{\sigma(k+l)}) \\ &= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (\text{sgn } \sigma) f(\mathbf{v}_{\sigma(1)}, \dots, \mathbf{v}_{\sigma(k)}) g(\mathbf{v}_{\sigma(k+1)}, \dots, \mathbf{v}_{\sigma(k+l)}). \end{aligned} \quad (1.27)$$

When  $k = 0$ , the element  $f \in A_0(V)$  is simply a constant  $c \in \mathbb{R}$  as discussed earlier. In this case, the wedge product  $c \wedge g$  is just scalar multiplication as is evident from (1.27).

$$\begin{aligned}
 (c \wedge g)(\mathbf{v}_1, \dots, \mathbf{v}_l) &= \frac{1}{l!} \sum_{\sigma \in S_l} (\text{sgn } \sigma) c g(\mathbf{v}_{\sigma(1)}, \dots, \mathbf{v}_{\sigma(l)}) \\
 &= \frac{1}{l!} \sum_{\sigma \in S_l} (\text{sgn } \sigma) c (\text{sgn } \sigma) g(\mathbf{v}_1, \dots, \mathbf{v}_l) \\
 &= \frac{1}{l!} \sum_{\sigma \in S_l} c g(\mathbf{v}_1, \dots, \mathbf{v}_l) \\
 &= \frac{1}{l!} l! c g(\mathbf{v}_1, \dots, \mathbf{v}_l) \\
 &= c g(\mathbf{v}_1, \dots, \mathbf{v}_l).
 \end{aligned}$$

Thus  $c \wedge g = cg$ , for  $c \in \mathbb{R}$  and  $g \in A_l(V)$ .

**Example 1.6.** For  $f \in A_2(V)$  and  $g \in A_1(V)$ ,

$$\begin{aligned}
 \mathcal{A}(f \otimes g)(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) &= f(\mathbf{v}_1, \mathbf{v}_2)g(\mathbf{v}_3) - f(\mathbf{v}_1, \mathbf{v}_3)g(\mathbf{v}_2) - f(\mathbf{v}_2, \mathbf{v}_1)g(\mathbf{v}_3) \\
 &\quad - f(\mathbf{v}_3, \mathbf{v}_2)g(\mathbf{v}_1) + f(\mathbf{v}_2, \mathbf{v}_3)g(\mathbf{v}_1) + f(\mathbf{v}_3, \mathbf{v}_1)g(\mathbf{v}_2).
 \end{aligned}$$

Among these 6 terms, there are 3 pairs of equal terms due to the alternating nature of  $f$ .

$$\begin{aligned}
 f(\mathbf{v}_1, \mathbf{v}_2)g(\mathbf{v}_3) &= -f(\mathbf{v}_2, \mathbf{v}_1)g(\mathbf{v}_3), \\
 f(\mathbf{v}_3, \mathbf{v}_1)g(\mathbf{v}_2) &= -f(\mathbf{v}_1, \mathbf{v}_3)g(\mathbf{v}_2), \\
 f(\mathbf{v}_2, \mathbf{v}_3)g(\mathbf{v}_1) &= -f(\mathbf{v}_3, \mathbf{v}_2)g(\mathbf{v}_1).
 \end{aligned}$$

Therefore,

$$\mathcal{A}(f \otimes g)(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = 2f(\mathbf{v}_1, \mathbf{v}_2)g(\mathbf{v}_3) + 2f(\mathbf{v}_3, \mathbf{v}_1)g(\mathbf{v}_2) + 2f(\mathbf{v}_2, \mathbf{v}_3)g(\mathbf{v}_1). \quad (1.28)$$

Hence,

$$\begin{aligned}
 (f \wedge g)(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) &= \frac{1}{2!1!} \mathcal{A}(f \otimes g)(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \\
 &= f(\mathbf{v}_1, \mathbf{v}_2)g(\mathbf{v}_3) + f(\mathbf{v}_3, \mathbf{v}_1)g(\mathbf{v}_2) + f(\mathbf{v}_2, \mathbf{v}_3)g(\mathbf{v}_1).
 \end{aligned} \quad (1.29)$$

**Example 1.7** (Wedge product of 2 covectors). If  $f, g \in A_1(V)$ , and  $\mathbf{v}_1, \mathbf{v}_2 \in V$ , then

$$(f \wedge g)(\mathbf{v}_1, \mathbf{v}_2) = \frac{1}{1!1!} \mathcal{A}(f \otimes g)(\mathbf{v}_1, \mathbf{v}_2).$$

$S_2$  has 2 elements: the identity element  $e$  and  $(1\ 2)$ . Therefore,

$$(f \wedge g)(\mathbf{v}_1, \mathbf{v}_2) = \mathcal{A}(f \otimes g)(\mathbf{v}_1, \mathbf{v}_2) = f(\mathbf{v}_1)g(\mathbf{v}_2) - f(\mathbf{v}_2)g(\mathbf{v}_1).$$

### Proposition 1.8 (Anticommutativity of wedge product)

The wedge product is anticommutative: if  $f \in A_k(V)$  and  $g \in A_l(V)$ , then

$$f \wedge g = (-1)^{kl} g \wedge f.$$

*Proof.* Define  $\tau \in S_{k+l}$  to be the following permutation:

$$\begin{bmatrix} 1 & 2 & \cdots & l & l+1 & l+2 & \cdots & l+k \\ k+1 & k+2 & \cdots & k+l & 1 & 2 & \cdots & k \end{bmatrix}.$$

In other words,

$$\tau(i) = \begin{cases} k+i & \text{if } 1 \leq i \leq l, \\ i-l & \text{if } l+1 \leq i \leq l+k. \end{cases}$$

Then for any  $\sigma \in S_{k+l}$ ,

$$\sigma(j) = \begin{cases} \sigma(\tau(l+j)) & \text{if } 1 \leq j \leq k, \\ \sigma(\tau(j-k)) & \text{if } k+1 \leq j \leq k+l. \end{cases} \quad (1.30)$$

Now, for any  $\mathbf{v}_1, \dots, \mathbf{v}_{k+l} \in V$ ,

$$\begin{aligned} \mathcal{A}(f \otimes g)(\mathbf{v}_1, \dots, \mathbf{v}_{k+l}) &= \sum_{\sigma \in S_{k+l}} (\text{sgn } \sigma) f(\mathbf{v}_{\sigma(1)}, \dots, \mathbf{v}_{\sigma(k)}) g(\mathbf{v}_{\sigma(k+1)}, \dots, \mathbf{v}_{\sigma(k+l)}) \\ &= \sum_{\sigma \in S_{k+l}} (\text{sgn } \sigma) f(\mathbf{v}_{\sigma(\tau(l+1))}, \dots, \mathbf{v}_{\sigma(\tau(l+k))}) g(\mathbf{v}_{\sigma(\tau(1))}, \dots, \mathbf{v}_{\sigma(\tau(l))}) \\ &= (\text{sgn } \tau) \sum_{\sigma \in S_{k+l}} (\text{sgn } \sigma) (\text{sgn } \tau) g(\mathbf{v}_{\sigma(\tau(1))}, \dots, \mathbf{v}_{\sigma(\tau(l))}) f(\mathbf{v}_{\sigma(\tau(l+1))}, \dots, \mathbf{v}_{\sigma(\tau(l+k))}) \\ &= (\text{sgn } \tau) \sum_{\sigma \in S_{k+l}} (\text{sgn } \sigma \tau) g(\mathbf{v}_{\sigma(\tau(1))}, \dots, \mathbf{v}_{\sigma(\tau(l))}) f(\mathbf{v}_{\sigma(\tau(l+1))}, \dots, \mathbf{v}_{\sigma(\tau(l+k))}). \end{aligned}$$

Again, as  $\sigma$  varies over  $S_{k+l}$ ,  $\sigma\tau$  also varies over  $S_{k+l}$ . Therefore,

$$\mathcal{A}(f \otimes g)(\mathbf{v}_1, \dots, \mathbf{v}_{k+l}) = (\text{sgn } \tau) \mathcal{A}(g \otimes f)(\mathbf{v}_1, \dots, \mathbf{v}_{k+l}). \quad (1.31)$$

Now, let us evaluate the sign of the permutation  $\tau$ . Let  $(\tau(i), \tau(j))$  be an inversion of  $\tau$ . Then it's not possible that  $1 \leq i < j \leq l$ , or  $l+1 \leq i < j \leq l+k$ ; because if we have  $1 \leq i < j \leq l$  or  $l+1 \leq i < j \leq l+k$ , then  $\tau(i) < \tau(j)$ . Therefore,  $i$  must be in between 1 and  $l$  (inclusive), and  $j$  must be in between  $l+1$  and  $l+k$  (inclusive). So there are  $l$  options for  $i$ , and  $k$  options for  $j$ . Therefore,  $\tau$  has  $kl$  many inversions. So  $\text{sgn } \tau = (-1)^{kl}$ . Using (1.31),

$$\mathcal{A}(f \otimes g) = (-1)^{kl} \mathcal{A}(g \otimes f). \quad (1.32)$$

Dividing by  $k!l!$ , we obtain

$$f \wedge g = (-1)^{kl} g \wedge f. \quad (1.33)$$

■

### Corollary 1.9

If  $f$  is a  $k$ -covector on  $V$ , i.e.  $f \in A_k(V)$ , and  $k$  is odd, then  $f \wedge f = 0$ .

*Proof.* By anticommutativity of wedge product,

$$f \wedge f = (-1)^{k^2} f \wedge f = -f \wedge f.$$

Therefore,  $f \wedge f = 0$ . ■

If  $f$  is a  $k$ -covector and  $g$  is an  $l$ -covector, i.e.  $f \in A_k(V)$  and  $g \in A_l(V)$ , then we have defined their wedge product to be the  $(k+l)$ -covector

$$f \wedge g = \frac{1}{k!l!} \mathcal{A}(f \otimes g). \quad (1.34)$$

We have the following lemmas associated with the alternating operator  $\mathcal{A}$ .

**Lemma 1.10**

Suppose  $f \in L_k(V)$  and  $g \in L_l(V)$ . Then

- (i)  $\mathcal{A}(\mathcal{A}(f) \otimes g) = k! \mathcal{A}(f \otimes g)$ .
- (ii)  $\mathcal{A}(f \otimes \mathcal{A}(g)) = l! \mathcal{A}(f \otimes g)$ .

*Proof.* (i) By definition,

$$\begin{aligned} \mathcal{A}(\mathcal{A}(f) \otimes g) &= \sum_{\sigma \in S_{k+l}} (\text{sgn } \sigma) \sigma(\mathcal{A}(f) \otimes g) \\ &= \sum_{\sigma \in S_{k+l}} (\text{sgn } \sigma) \sigma \left[ \sum_{\tau \in S_k} (\text{sgn } \tau) (\tau f) \otimes g \right]. \end{aligned} \quad (1.35)$$

We can view  $\tau \in S_k$  as a permutation in the following way: define  $\tau' \in S_{k+l}$  as follows

$$\tau'(i) = \begin{cases} \tau(i) & \text{if } i \leq k, \\ i & \text{if } i > k. \end{cases} \quad (1.36)$$

Then for  $\mathbf{v}_1, \dots, \mathbf{v}_{k+l}$ , we have

$$\begin{aligned} [(\tau f) \otimes g](\mathbf{v}_1, \dots, \mathbf{v}_{k+l}) &= (\tau f)(\mathbf{v}_1, \dots, \mathbf{v}_k) g(\mathbf{v}_{k+1}, \dots, \mathbf{v}_{k+l}) \\ &= f(\mathbf{v}_{\tau(1)}, \dots, \mathbf{v}_{\tau(k)}) g(\mathbf{v}_{k+1}, \dots, \mathbf{v}_{k+l}) \\ &= f(\mathbf{v}_{\tau'(1)}, \dots, \mathbf{v}_{\tau'(k)}) g(\mathbf{v}_{\tau'(k+1)}, \dots, \mathbf{v}_{\tau'(k+l)}) \\ &= [\tau'(f \otimes g)](\mathbf{v}_1, \dots, \mathbf{v}_{k+l}). \end{aligned}$$

Therefore,  $(\tau f) \otimes g = \tau'(f \otimes g)$ . Furthermore,  $\text{sgn } \tau = \text{sgn } \tau'$  since the inversions  $(\tau'(i), \tau'(j))$  occur only when  $1 \leq i < j \leq k$ , so that the  $\tau$  and  $\tau'$  has the same number of inversions.

Let us abuse notation a bit and denote by  $S_k$  the subgroup of permutations in  $S_{k+l}$  by keeping the last  $l$  arguments fixed. This subgroup of  $S_{k+l}$  is indeed isomorphic to  $S_k$ , so we will denote both these groups by  $S_k$ . Therefore, from (1.35),

$$\begin{aligned} \mathcal{A}(\mathcal{A}(f) \otimes g) &= \sum_{\sigma \in S_{k+l}} (\text{sgn } \sigma) \sigma \left[ \sum_{\tau' \in S_k \subseteq S_{k+l}} (\text{sgn } \tau') \tau'(f \otimes g) \right] \\ &= \sum_{\sigma \in S_{k+l}} \sum_{\tau' \in S_k \subseteq S_{k+l}} (\text{sgn } \sigma) (\text{sgn } \tau') \sigma \tau'(f \otimes g) \\ &= \sum_{\tau' \in S_k \subseteq S_{k+l}} \sum_{\sigma \in S_{k+l}} (\text{sgn } \sigma \text{sgn } \tau') ((\sigma \tau')(f \otimes g)). \end{aligned}$$

For a fixed  $\tau'$ , as  $\sigma$  varies over  $S_{k+l}$ ,  $\sigma \tau'$  also varies over  $S_{k+l}$ . Therefore,

$$\mathcal{A}(\mathcal{A}(f) \otimes g) = \sum_{\tau' \in S_k \subseteq S_{k+l}} \mathcal{A}(f \otimes g) = k! \mathcal{A}(f \otimes g). \quad (1.37)$$

(ii) By (1.32),

$$\begin{aligned} \mathcal{A}(f \otimes \mathcal{A}(g)) &= \mathcal{A}((-1)^{kl} \mathcal{A}(g) \otimes f) \\ &= (-1)^{kl} \mathcal{A}(\mathcal{A}(g) \otimes f) \\ &= (-1)^{kl} l! \mathcal{A}(g \otimes f) \\ &= l! \mathcal{A}((-1)^{kl} g \otimes f) \\ &= l! \mathcal{A}(f \otimes g). \end{aligned} \quad (1.38)$$

■

**Proposition 1.11** (Associativity of wedge product)

Let  $V$  be a real vector space and  $f, g, h$  be alternating multilinear functions on  $V$  of degree  $k, l, m$ , respectively. Then

$$(f \wedge g) \wedge h = f \wedge (g \wedge h).$$

*Proof.* Using the definition of wedge product,

$$\begin{aligned} (f \wedge g) \wedge h &= \frac{1}{(k+l)!m!} \mathcal{A}[(f \wedge g) \otimes h] \\ &= \frac{1}{(k+l)!m!} \mathcal{A}\left[\frac{1}{k!l!} \mathcal{A}(f \otimes g) \otimes h\right] \\ &= \frac{1}{(k+l)!k!l!m!} \mathcal{A}[\mathcal{A}(f \otimes g) \otimes h] \\ &= \frac{(k+l)!}{(k+l)!k!l!m!} \mathcal{A}[(f \otimes g) \otimes h] \\ &= \frac{1}{k!l!m!} \mathcal{A}[(f \otimes g) \otimes h]. \end{aligned}$$

On the other hand,

$$\begin{aligned} f \wedge (g \wedge h) &= \frac{1}{k!(l+m)!} \mathcal{A}[f \otimes (g \wedge h)] \\ &= \frac{1}{k!(l+m)!} \mathcal{A}\left[f \otimes \left(\frac{1}{l!m!} \mathcal{A}(g \otimes h)\right)\right] \\ &= \frac{1}{k!(l+m)!l!m!} \mathcal{A}[f \otimes \mathcal{A}(g \otimes h)] \\ &= \frac{(l+m)!}{k!(l+m)!l!m!} \mathcal{A}[f \otimes (g \otimes h)] \\ &= \frac{1}{k!l!m!} \mathcal{A}[f \otimes (g \otimes h)]. \end{aligned}$$

Since tensor product is associative (by [Lemma 1.7](#)), we conclude that

$$(f \wedge g) \wedge h = f \wedge (g \wedge h). \quad (1.39)$$

■

By associativity, we can omit the parenthesis and write univocally  $f \wedge g \wedge h$  instead of  $(f \wedge g) \wedge h$  or  $f \wedge (g \wedge h)$ .

**Corollary 1.12**

Under the hypothesis of [Proposition 1.11](#),

$$f \wedge g \wedge h = \frac{1}{k!l!m!} \mathcal{A}[f \otimes g \otimes h]. \quad (1.40)$$

This easily generalizes to an arbitrary number of factors: if  $f_i \in A_{d_i}(V)$  for  $i = 1, 2, \dots, r$ , i.e.  $f_i$  is an alternating  $d_i$ -linear function on  $V$ , then

$$f_1 \wedge \dots \wedge f_r = \frac{1}{d_1! \dots d_r!} \mathcal{A}(f_1 \otimes \dots \otimes f_r). \quad (1.41)$$

**Proposition 1.13**

Let  $\hat{\alpha}^1, \hat{\alpha}^2, \dots, \hat{\alpha}^k$  be linear functions on a real vector space  $V$  (i.e.  $\hat{\alpha}^i : V \rightarrow \mathbb{R}$ ) and  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in V$ . Then

$$\begin{aligned} (\hat{\alpha}^1 \wedge \dots \wedge \hat{\alpha}^k)(\mathbf{v}_1, \dots, \mathbf{v}_k) &= \det [\hat{\alpha}^i(\mathbf{v}_j)] \\ &= \det \begin{bmatrix} \hat{\alpha}^1(\mathbf{v}_1) & \hat{\alpha}^1(\mathbf{v}_2) & \dots & \hat{\alpha}^1(\mathbf{v}_k) \\ \hat{\alpha}^2(\mathbf{v}_1) & \hat{\alpha}^2(\mathbf{v}_2) & \dots & \hat{\alpha}^2(\mathbf{v}_k) \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\alpha}^k(\mathbf{v}_1) & \hat{\alpha}^k(\mathbf{v}_2) & \dots & \hat{\alpha}^k(\mathbf{v}_k) \end{bmatrix}. \end{aligned}$$

*Proof.* By 1.41,

$$(\hat{\alpha}^1 \wedge \dots \wedge \hat{\alpha}^k)(\mathbf{v}_1, \dots, \mathbf{v}_k) = \mathcal{A}(\hat{\alpha}^1 \otimes \dots \otimes \hat{\alpha}^k)(\mathbf{v}_1, \dots, \mathbf{v}_k).$$

By the definition of the action of alternating operator,

$$\mathcal{A}(\hat{\alpha}^1 \otimes \dots \otimes \hat{\alpha}^k)(\mathbf{v}_1, \dots, \mathbf{v}_k) = \sum_{\sigma \in S_k} (\text{sgn } \sigma) \hat{\alpha}^1(\mathbf{v}_{\sigma(1)}) \dots \hat{\alpha}^k(\mathbf{v}_{\sigma(k)}). \quad (1.42)$$

By the definition of determinant of a  $k \times k$  matrix  $A = [a_{ij}]$ ,

$$\det A = \sum_{\sigma \in S_k} (\text{sgn } \sigma) a_{1\sigma(1)} a_{2\sigma(2)} \dots a_{k\sigma(k)}. \quad (1.43)$$

Using (1.43) in (1.42), we get

$$\mathcal{A}(\hat{\alpha}^1 \otimes \dots \otimes \hat{\alpha}^k)(\mathbf{v}_1, \dots, \mathbf{v}_k) = \det [\hat{\alpha}^i(\mathbf{v}_j)]. \quad (1.44)$$

■

**§1.5 A Basis for  $A_k(V)$** 

Let  $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$  be a basis for a real vector space  $V$ , and let  $\{\hat{\alpha}^1, \dots, \hat{\alpha}^n\}$  be the dual basis for  $V^*$ . Introduce the multi-index notation

$$I = (i_1, i_2, \dots, i_k)$$

and write  $\mathbf{e}_I$  for  $(\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \dots, \mathbf{e}_{i_k})$  and  $\hat{\alpha}^I$  for  $\hat{\alpha}^{i_1} \wedge \hat{\alpha}^{i_2} \wedge \dots \wedge \hat{\alpha}^{i_k}$ .

A  $k$ -linear function  $f$  on  $V$  is completely determined by its values on all  $k$ -tuples  $(\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \dots, \mathbf{e}_{i_k})$ . If  $f$  is alternating, then  $f$  is completely determined by its values on all  $k$ -tuples  $(\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \dots, \mathbf{e}_{i_k})$  with

$$1 \leq i_1 < i_2 < \dots < i_k \leq n.$$

In other words, it's sufficient to consider  $\mathbf{e}_I$  with  $I$  in ascending order.

**Lemma 1.14**

Suppose  $I$  and  $J$  are ascending multi-indices of length  $k$ . Then

$$\hat{\alpha}^I(\mathbf{e}_J) = \delta^I_J := \begin{cases} 1 & \text{if } I = J, \\ 0 & \text{if } I \neq J. \end{cases}$$

*Proof.* Suppose  $I = (i_1, \dots, i_k)$  and  $J = (j_1, \dots, j_k)$ . Using (1.42), we get

$$\begin{aligned} \hat{\alpha}^I(\mathbf{e}_J) &= (\hat{\alpha}^{i_1} \wedge \hat{\alpha}^{i_2} \wedge \dots \wedge \hat{\alpha}^{i_k})(\mathbf{e}_{j_1}, \mathbf{e}_{j_2}, \dots, \mathbf{e}_{j_k}) \\ &= \sum_{\sigma \in S_k} (\text{sgn } \sigma) \hat{\alpha}^{i_1}(\mathbf{e}_{j_{\sigma(1)}}) \dots \hat{\alpha}^{i_k}(\mathbf{e}_{j_{\sigma(k)}}) \\ &= \sum_{\sigma \in S_k} (\text{sgn } \sigma) \delta^{i_1}_{j_{\sigma(1)}} \dots \delta^{i_k}_{j_{\sigma(k)}}. \end{aligned} \quad (1.45)$$

The terms in the sum (1.45) contribute  $\text{sgn } \sigma$  if and only if

$$(i_1, i_2, \dots, i_k) = (j_{\sigma(1)}, j_{\sigma(2)}, \dots, j_{\sigma(k)});$$

otherwise they contribute 0 to the sum. Both  $I$  and  $J$  are ascending multi-indices. Permuting the elements of  $J$  no longer gives an ascending multi-index (unless the permutation  $\sigma$  is the identity permutation). Therefore, in (1.45), all the summands corresponding to  $\sigma$  being a non-identity permutation contribute 0.

$$\hat{\alpha}^I(\mathbf{e}_J) = \sum_{\sigma \in S_k} (\text{sgn } \sigma) \delta^{i_1}_{j_{\sigma(1)}} \cdots \delta^{i_k}_{j_{\sigma(k)}} = \delta^{i_1}_{j_1} \cdots \delta^{i_k}_{j_k} = \begin{cases} 1 & \text{if } I = J, \\ 0 & \text{if } I \neq J. \end{cases} \quad (1.46)$$

■

### Proposition 1.15

The alternating  $k$ -linear functions  $\hat{\alpha}^I$ ,  $I = (i_1, \dots, i_k)$ , with  $1 \leq i_1 < \dots < i_k \leq n$  form a basis for the space  $A_k(V)$  of alternating  $k$ -linear functions on  $V$ .

*Proof.* Let us first show linear independence. Suppose

$$\sum_I c_I \hat{\alpha}^I = \mathbf{0}, \quad (1.47)$$

$c_I \in \mathbb{R}$  with  $I$  running over ascending multi-indices of length  $k$ . Applying  $\mathbf{e}_J$  to both sides, we get

$$0 = \sum_I c_I \hat{\alpha}^I(\mathbf{e}_J) = \sum_I c_I \delta^I_J = c_J. \quad (1.48)$$

Therefore,  $\{\hat{\alpha}^I \mid I \text{ is ascending multi-index of length } k\}$  is a linearly independent set. Now let us prove that this set spans  $A_k(V)$ . Let  $f \in A_k(V)$ . We claim that

$$f = \sum_I f(\mathbf{e}_I) \hat{\alpha}^I. \quad (1.49)$$

Let  $g = \sum_I f(\mathbf{e}_I) \hat{\alpha}^I$ . We need to prove that  $f = g$ . By  $k$ -linearity and alternating property, if two  $k$ -covectors agree on all  $\mathbf{e}_J$  where  $J$  is an ascending multi-index, then they are equal. Now,

$$g(\mathbf{e}_J) = \sum_I f(\mathbf{e}_I) \hat{\alpha}^I(\mathbf{e}_J) = \sum_I f(\mathbf{e}_I) \delta^I_J = f(\mathbf{e}_J). \quad (1.50)$$

Therefore,  $f = g = \sum_I f(\mathbf{e}_I) \hat{\alpha}^I$ . ■

### Corollary 1.16

If the vector space  $V$  has dimension  $n$ , then the vector space  $A_k(V)$  of  $k$ -covectors on  $V$  has dimension  $\binom{n}{k}$ .

*Proof.* An ascending multi-index  $I = (i_1, i_2, \dots, i_k)$ ,  $1 \leq i_1 < i_2 < \dots < i_k \leq n$  is obtained by choosing a  $k$ -element subset of  $\{1, 2, \dots, n\}$ . This can be done in  $\binom{n}{k}$  ways. ■

### Corollary 1.17

If  $k > \dim V$ , then  $A_k(V) = 0$ .

*Proof.* If  $k > \dim V = n$ , then in the expression

$$\hat{\alpha}^{i_1} \wedge \hat{\alpha}^{i_2} \wedge \dots \wedge \hat{\alpha}^{i_k}$$

with each  $i \in \{1, 2, \dots, n\}$ , there must be a repeated  $i_j$ 's, say  $\hat{\alpha}^r$ . Then  $\hat{\alpha}^r \wedge \hat{\alpha}^r$  arises in the expression  $\hat{\alpha}^{i_1} \wedge \hat{\alpha}^{i_2} \wedge \dots \wedge \hat{\alpha}^{i_k}$ . But  $\hat{\alpha}^r \wedge \hat{\alpha}^r = 0$  by Corollary 1.9. Hence,  $\hat{\alpha}^{i_1} \wedge \hat{\alpha}^{i_2} \wedge \dots \wedge \hat{\alpha}^{i_k} = 0$ . Therefore, the basis set of  $A_k(V)$  is empty, meaning  $A_k(V) = 0$ . ■



# 2 Differential Forms on $\mathbb{R}^n$

Given an open set  $U \subseteq \mathbb{R}^n$  and  $p \in U$ ,  $T_p U$  is the set of tangent vectors at  $p \in U$  is identified with the point derivations of  $C_p^\infty$  (germs of smooth functions at  $p$ ), i.e. a tangent vector  $X_p \in T_p U$  is a map  $X_p : C_p^\infty \rightarrow \mathbb{R}$  such that  $X_p$  is  $\mathbb{R}$ -linear:

$$X_p(\alpha f + g) = \alpha(X_p f) + X_p g; \quad (2.1)$$

and satisfies the Leibniz condition:

$$X_p(fg) = (X_p f)g(p) + f(p)(X_p g).$$

In contrast to the notion of point derivation, there is this notion of derivation of aa algebra. If  $X$  is a  $C^\infty$  vector field on an open subset  $U \subseteq \mathbb{R}^n$ , i.e.  $X \in \mathfrak{X}(U)$ , and  $f$  is a  $C^\infty$  function on  $U$ , i.e.  $f \in C^\infty(U)$ , then  $Xf \in C^\infty(U)$  defined by

$$(Xf)(p) = X_p f.$$

Remember that  $f$  in (2.1) and (2) is a representative of an equivalence class, the equivalence class of germs of  $C^\infty$  functions at  $p \in U$ . These equivalence classes constitute  $C_p^\infty(U)$ . It is of course an  $\mathbb{R}$ -algebra. While in (2),  $f \in C^\infty(U)$ , the algebra of  $C^\infty$  functions on  $U$  with no reference of  $p$  whatsoever.

From the discussion above, a  $C^\infty$  vector field  $X$  gives rise to an  $\mathbb{R}$ -linear map  $C^\infty(U) \rightarrow C^\infty(U)$  by  $f \mapsto Xf$  that additionally has to satisfy the following Leibniz condition:

$$X(fg) = (Xf)g + f(Xg). \quad (2.2)$$

Note that a derivation at  $p$  is not a derivation of the algebra  $C_p^\infty$ . A derivation at  $p$  is a map from  $C_p^\infty \rightarrow \mathbb{R}$  that satisfies (2), while a derivation of the algebra  $C_p^\infty$  is supposed to be a map from  $C_p^\infty$  to itself obeying Leibniz condition.

## §2.1 1 form

From any  $C^\infty$  function  $f : U \rightarrow \mathbb{R}$ , one can construct a 1-form (dual notion of  $C^\infty$  vector field)  $df$ , the restriction of which to a given point  $p \in U$  yields a covector  $(df)_p \in T_p^* U$ , the dual space of  $T_p U$ , in the following way:

$$(df)_p(X_p) = X_p f. \quad (2.3)$$

### Proposition 2.1

If  $x^1, x^2, \dots, x^n$  are the standard coordinates on  $\mathbb{R}^n$ , then at each point  $p \in \mathbb{R}^n$ ,

$$\left\{ (dx^1)_p, (dx^2)_p, \dots, (dx^n)_p \right\}$$

is the basis for the cotangent space  $T_p^* \mathbb{R}^n$  dual to the basis  $\left\{ \frac{\partial}{\partial x^1} \Big|_p, \dots, \frac{\partial}{\partial x^n} \Big|_p \right\}$  for the tangent space  $T_p \mathbb{R}^n$ .

*Proof.*  $(dx^i)_p : T_p^* \mathbb{R}^n \rightarrow \mathbb{R}$  is a linear map for each  $i$ . Now,

$$(dx^i)_p \left( \frac{\partial}{\partial x^j} \Big|_p \right) = \frac{\partial}{\partial x^j} \Big|_p (x^i) = \delta^i_j. \quad (2.4)$$

Therefore,  $\{(\mathrm{d}x^1)_p, (\mathrm{d}x^2)_p, \dots, (\mathrm{d}x^n)_p\}$  is the basis of  $T_p^*\mathbb{R}^n$  dual to the basis  $\left\{\frac{\partial}{\partial x^1}\Big|_p, \dots, \frac{\partial}{\partial x^n}\Big|_p\right\}$  for  $T_p\mathbb{R}^n$ .  $\blacksquare$

If  $\omega$  is a 1-form on an open subset  $U \subseteq \mathbb{R}^n$ , then by [Proposition 2.1](#), there is a linear combination

$$\omega_p = \sum_{i=1}^n a_i(p) (\mathrm{d}x^i)_p, \quad (2.5)$$

for some  $a_i(p) \in \mathbb{R}$ . As  $p$  varies over  $U$ , the coefficients  $a_i$  become functions on  $U$ , and we may write

$$\omega = \sum_{i=1}^n a_i \mathrm{d}x^i. \quad (2.6)$$

The 1-form  $\omega$  is said to be  $C^\infty$  on  $U$  if the coefficient functions  $a_i$  are all  $C^\infty$  functions on  $U$ .

**Proposition 2.2 (The differential in terms of coordinates)**

If  $f : U \rightarrow \mathbb{R}$  is a  $C^\infty$  function on an open set  $U \subseteq \mathbb{R}^n$ , then

$$\mathrm{d}f = \sum_{i=1}^n \frac{\partial f}{\partial x^i} \mathrm{d}x^i.$$

*Proof.* By [Proposition 2.1](#), at each point  $p \in U$ ,

$$(\mathrm{d}f)_p = \sum_{i=1}^n a_i(p) (\mathrm{d}x^i)_p, \quad (2.7)$$

for some constants  $a_i(p)$  depending on  $p$ . Thus

$$\mathrm{d}f = \sum_{i=1}^n a_i \mathrm{d}x^i, \quad (2.8)$$

for some functions  $a_i$  on  $U$ . To evaluate  $a_j$ , apply both sides of (2.8) to the coordinate vector field  $\frac{\partial}{\partial x^j}$ :

$$\mathrm{d}f\left(\frac{\partial}{\partial x^j}\right) = \sum_{i=1}^n a_i \mathrm{d}x^i\left(\frac{\partial}{\partial x^j}\right) = \sum_{i=1}^n a_i \delta^i_j = a_j. \quad (2.9)$$

On the other hand, using  $(\mathrm{d}f)_p(X_p) = X_p f = (Xf)(p)$ , we get  $(\mathrm{d}f)(X) = Xf$ . So

$$\mathrm{d}f\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial f}{\partial x^j}. \quad (2.10)$$

Therefore,  $a_j = \frac{\partial f}{\partial x^j}$ . Hence,

$$\mathrm{d}f = \sum_{i=1}^n \frac{\partial f}{\partial x^i} \mathrm{d}x^i. \quad (2.11)$$

$\blacksquare$

(2.11) tells us that  $\mathrm{d}f$  will be a  $C^\infty$  1-form if  $\frac{\partial f}{\partial x^i}$  is  $C^\infty$  on  $U$ . Hence, it is sufficient to have  $f$  as a  $C^\infty$  function on  $U$  in order to have  $\mathrm{d}f$  as a  $C^\infty$  1-form.

## §2.2 Differential $k$ -forms

A differential form  $\omega$  of degree  $k$  (or a  $k$ -form) on an open subset  $U \subseteq \mathbb{R}^n$  is a map that assigns to each point  $p \in U$ , an alternating  $k$ -linear function on the tangent space  $T_p\mathbb{R}^n$ , i.e.

$$\omega_p \in A_k(T_p\mathbb{R}^n).$$

By Proposition 1.15, a basis for  $A_k(T_p\mathbb{R}^n)$  is

$$\left(dx^I\right)_p = \left(dx^{i_1}\right)_p \wedge \left(dx^{i_2}\right)_p \wedge \cdots \wedge \left(dx^{i_k}\right)_p,$$

where  $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ . Therefore, at each point  $p \in U$ ,  $\omega_p$  is a linear combination

$$\omega_p = \sum_I a_I(p) \left(dx^I\right)_p, \quad (2.12)$$

and a  $k$ -form  $\omega$  on  $U$  is a linear combination

$$\omega = \sum_I a_I dx^I, \quad (2.13)$$

with function coefficients  $a_I : U \rightarrow \mathbb{R}$ . We say that a  $k$ -form  $\omega$  is **smooth** on  $U$  if all the coefficients  $a_I$  are  $C^\infty$  functions on  $U$ .

Denote by  $\Omega^k(U)$  the vector space of  $C^\infty$   $k$ -forms on  $U$ . A 0-form on  $U$  assigns to each point  $p \in U$  an element of  $A_0(T_p\mathbb{R}^n) = \mathbb{R}$ . Thus a 0-form on  $U$  is simply a real-valued function on  $U$ , and  $\Omega^0(U) = C^\infty(U)$ .

Since one can multiply a  $C^\infty$   $k$ -form by a  $C^\infty$  function on  $U$  from the left, the set  $\Omega^k(U)$  of  $C^\infty$   $k$ -forms on  $U$  is both a real vector space and a  $C^\infty(U)$ -module. With the wedge product as multiplication, the direct sum

$$\Omega^*(U) = \bigoplus_{k=0}^n \Omega^k(U)$$

becomes an algebra over  $\mathbb{R}$  as well as a module over  $C^\infty(U)$ . As an algebra, it is anticommutative and associative.

**Example 2.1.** Let  $x, y, z$  be the coordinates on  $\mathbb{R}^3$ . The  $C^\infty$  1-forms on  $\mathbb{R}^3$  are

$$P(x, y, z) dx + Q(x, y, z) dy + R(x, y, z) dz,$$

where  $P, Q, R$  range over all  $C^\infty$  functions on  $\mathbb{R}^3$ . The  $C^\infty$  2-forms are

$$A(x, y, z) dy \wedge dz + B(x, y, z) dx \wedge dz + C(x, y, z) dx \wedge dy;$$

and the  $C^\infty$  1-forms are

$$a(x, y, z) dx \wedge dy \wedge dz.$$

**Example 2.2** (A basis for 3-covectors). Let  $x^1, x^2, x^3, x^4$  be the standard coordinates on  $\mathbb{R}^4$ , and  $p \in \mathbb{R}^4$ . A basis for  $A_3(T_p\mathbb{R}^4)$  is

$$\left\{ \left(dx^1\right)_p \wedge \left(dx^2\right)_p \wedge \left(dx^3\right)_p, \left(dx^1\right)_p \wedge \left(dx^2\right)_p \wedge \left(dx^4\right)_p, \right. \\ \left. \left(dx^1\right)_p \wedge \left(dx^3\right)_p \wedge \left(dx^4\right)_p, \left(dx^2\right)_p \wedge \left(dx^3\right)_p \wedge \left(dx^4\right)_p \right\}.$$

So  $\dim(A_3(T_p\mathbb{R}^n)) = 4$ .

### §2.3 Exterior Derivative

Before defining exterior derivative of a  $C^\infty$   $k$ -form on an open subset  $U \subseteq \mathbb{R}^n$ , we first define it on 0-forms. The exterior derivative of a  $C^\infty$  function  $f \in C^\infty(U)$  is its differential:

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i \in \Omega^1(U).$$

**Definition 2.1** (Exterior Derivative). If  $\omega = \sum_I a_I dx^I \in \omega^K(U)$ , then its exterior derivative is defined as follows:

$$d\omega = \sum_I da_I \wedge dx^I = \sum_I \left( \sum_{j=1}^n \frac{\partial a_I}{\partial x^j} dx^j \right) \wedge dx^I \in \Omega^{k+1}(U). \quad (2.14)$$

**Example 2.3.** Let  $\omega$  be the 1 form  $f dx + g dy$  on  $\mathbb{R}^2$ , where  $f$  and  $g$  are  $C^\infty$  functions on  $\mathbb{R}^2$ . Let us write  $f_x = \frac{\partial f}{\partial x}$  and  $f_y = \frac{\partial f}{\partial y}$ . Then

$$\begin{aligned} d\omega &= df \wedge dx + dg \wedge dy \\ &= (f_x dx + f_y dy) \wedge dx + (g_x dx + g_y dy) \wedge dy \\ &= -f_y dx \wedge dy + g_x dx \wedge dy \\ &= (g_x - f_y) dx \wedge dy. \end{aligned}$$

**Definition 2.2** (Graded Algebra). An algebra  $A$  over a field  $\mathbb{K}$  is said to be **graded** if it can be written as a direct sum

$$A = \bigoplus_{k=0}^{\infty} A^k$$

of vector spaces over  $\mathbb{K}$  so that the multiplication map sends  $A^k \times A^l$  to  $A^{k+l}$ .

The notation  $A = \bigoplus_{k=0}^{\infty} A^k$  means that each element of  $A$  is uniquely a **finite sum**

$$a = a_{i_1} + a_{i_2} + \cdots + a_{i_m},$$

where  $a_{i_j} \in A^{i_j}$ .

**Example 2.4.** The polynomial algebra

$$\mathbb{R}[x, y] = \bigoplus_{k=0}^{\infty} A^k$$

with  $A^k$  being the vector space of homogenous polynomials of degree  $k$  in  $x$  and  $y$ . Observe that the 0 polynomial is trivially homogenous of any degree, and hence belongs to  $A^k$  for all  $k \geq 0$ . Multiplication of degree  $k$  homogenous polynomial with a degree  $l$  homogenous polynomial in  $x$  and  $y$  will result in a homogenous polynomial of degree  $k + l$  in  $x$  and  $y$ .

**Example 2.5.** The algebra  $\Omega^*(U)$  of  $C^\infty$  differential forms on  $U$  is also graded by the degree of differential forms. Each  $\Omega^k(U)$  is a vector space. Multiplication of differential forms is defined by wedge product between them. The wedge product of a degree  $k$  differential form on  $U$  with a degree  $l$  differential form results in a degree  $k + l$  differential form.

**Definition 2.3** (Anti-derivation). Let  $A = \bigoplus_{k=0}^{\infty} A^k$  be a graded algebra over a field  $\mathbb{K}$ . An **anti-derivation** of the graded algebra  $A$  is a  $\mathbb{K}$ -linear map  $D : A \rightarrow A$  such that for  $\omega \in A^k$  and  $\tau \in A^l$ , one has

$$D(\omega\tau) = (D\omega)\tau + (-1)^k \omega(D\tau). \quad (2.15)$$

If the antiderivation  $D$  sends  $\omega \in A^k$  to  $D\omega \in A^{k+m}$ , we say that it is an antiderivation of degree  $m$ .

**Proposition 2.3** (i) The exterior derivative  $d : \Omega^*(U) \rightarrow \Omega^*(U)$  is an antiderivation of degree 1:

$$d(\omega \wedge \tau) = (d\omega) \wedge \tau + (-1)^{\deg \omega} \omega \wedge d\tau. \quad (2.16)$$

(ii)  $d^2 = 0$ .

(iii) If  $f \in \Omega^0(U) = C^\infty(U)$  and  $X \in \mathfrak{X}(U)$  (the space of  $C^\infty$  vector fields), then  $(df)(X) = Xf$ .

*Proof.* (i) Since the exterior derivative operator  $d : \Omega^k(U) \rightarrow \Omega^{k+1}(U)$  is linear, it suffices to check the equality (2.16) for  $\omega = f dx^I$  and  $\tau = g dx^J$  with  $I = (i_1, \dots, i_k)$  and  $J = (j_1, \dots, j_l)$  being strictly ascending multi-indices.

$$\begin{aligned} d(\omega \wedge \tau) &= d(fg dx^I \wedge dx^J) \\ &= \sum_{i=1}^n \frac{\partial(fg)}{\partial x^i} dx^i \wedge dx^I \wedge dx^J \\ &= \sum_{i=1}^n \frac{\partial f}{\partial x^i} \cdot g dx^i \wedge dx^I \wedge dx^J + \sum_{i=1}^n f \cdot \frac{\partial g}{\partial x^i} dx^i \wedge dx^I \wedge dx^J \\ &= \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i \wedge dx^I \wedge g dx^J + \sum_{i=1}^n \frac{\partial g}{\partial x^i} dx^i \wedge f dx^I \wedge dx^J. \end{aligned} \quad (2.17)$$

Now, in the second sum in (2.17), one has to push  $\frac{\partial g}{\partial x^i} dx^i$  through the  $k$ -fold wedge product  $dx^I$  and hence in the process picks out a sign  $(-1)^k$ . Therefore,

$$\sum_{i=1}^n \frac{\partial g}{\partial x^i} dx^i \wedge f dx^I \wedge dx^J = (-1)^k f dx^I \wedge \frac{\partial g}{\partial x^i} dx^i \wedge dx^J. \quad (2.18)$$

Now, observe that

$$d\omega = d(f dx^I) = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i \wedge dx^I, \text{ and} \quad (2.19)$$

$$d\tau = d(g dx^J) = \sum_{i=1}^n \frac{\partial g}{\partial x^i} dx^i \wedge dx^J. \quad (2.20)$$

Therefore,

$$d(\omega \wedge \tau) = d\omega \wedge \tau + (-1)^k \omega \wedge d\tau. \quad (2.21)$$

(ii) Again, by  $\mathbb{R}$ -linearity of  $d$ , it suffices to show that  $d^2\omega = 0$  for  $\omega = f dx^I$ .

$$\begin{aligned} d^2(f dx^I) &= d\left(\sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i \wedge dx^I\right) \\ &= \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 f}{\partial x^j \partial x^i} dx^j \wedge dx^i \wedge dx^I. \end{aligned} \quad (2.22)$$

If  $i = j$ , then  $dx^j \wedge dx^i = 0$ . If  $i \neq j$ , then  $\frac{\partial^2 f}{\partial x^j \partial x^i}$  is symmetric in  $i$  and  $j$ , but  $dx^j \wedge dx^i$  is alternating in  $i$  and  $j$ . Therefore, the terms with  $i \neq j$  pair up and cancel out.

(iii) Let  $X = \sum_{i=1}^n a^i \frac{\partial}{\partial x^i}$ . Then

$$\begin{aligned}
 (df)(X) &= \left( \sum_{j=1}^n \frac{\partial f}{\partial x^j} dx^j \right) \left( \sum_{i=1}^n a^i \frac{\partial}{\partial x^i} \right) \\
 &= \sum_{j=1}^n \sum_{i=1}^n a^i \frac{\partial f}{\partial x^j} dx^j \left( \frac{\partial}{\partial x^i} \right) \\
 &= \sum_{j=1}^n \sum_{i=1}^n a^i \frac{\partial f}{\partial x^j} \delta^j_i \\
 &= \sum_{i=1}^n a^i \frac{\partial f}{\partial x^i} = Xf.
 \end{aligned} \tag{2.23}$$

■

### Proposition 2.4 (Characterization of exterior derivative)

The 3 properties of Proposition 2.3 uniquely characterize exterior derivative on an open set  $U \subseteq \mathbb{R}^n$ . In other words, if  $D : \Omega^*(U) \rightarrow \Omega^*(U)$  is an antiderivation of degree 1 such that  $D^2 = 0$  and for  $f \in C^\infty(U)$  and  $X \in \mathfrak{X}(U)$ ,  $(Df)(X) = Xf$ , then  $D = d$ .

*Proof.* Since every  $k$ -form on  $U$  is a sum of terms such as  $f dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}$ , by linearity of  $d$ , it suffices to show that  $D = d$  on a  $k$ -form of this type. Applying property (iii) for  $f = x^i$ , one has

$$Dx^i(X) = X(x^i).$$

Writing  $X = \sum_{j=1}^n a^j \frac{\partial}{\partial x^j}$ , we get

$$Dx^i \left( \sum_{j=1}^n a^j \frac{\partial}{\partial x^j} \right) = \sum_{j=1}^n a^j \frac{\partial}{\partial x^j} (x^i) = a^i = dx^i \left( \sum_{j=1}^n a^j \frac{\partial}{\partial x^j} \right).$$

Therefore,

$$Dx^i = dx^i. \tag{2.24}$$

Now,

$$\begin{aligned}
 D(f dx^{i_1} \wedge \cdots \wedge dx^{i_k}) &= D(f Dx^{i_1} \wedge \cdots \wedge Dx^{i_k}) \\
 &= Df \wedge (Dx^{i_1} \wedge \cdots \wedge Dx^{i_k}) + (-1)^0 f D(Dx^{i_1} \wedge \cdots \wedge Dx^{i_k}).
 \end{aligned} \tag{2.25}$$

Now, since  $df(X) = Xf = Df(X)$  for any  $X \in \mathfrak{X}(U)$ ,  $df = Df$ . Furthermore,  $D(Dx^{i_1}) = 0$ , and

$$\begin{aligned}
 D(Dx^{i_1} \wedge \cdots \wedge Dx^{i_k}) &= D^2 x^{i_1} \wedge Dx^{i_2} \wedge \cdots \wedge Dx^{i_k} - Dx^{i_1} \wedge D(Dx^{i_2} \wedge \cdots \wedge Dx^{i_k}) \\
 &= -Dx^{i_1} \wedge D(Dx^{i_2} \wedge \cdots \wedge Dx^{i_k}).
 \end{aligned} \tag{2.26}$$

Therefore, by induction on  $k$ ,

$$D(Dx^{i_1} \wedge \cdots \wedge Dx^{i_k}) = 0. \tag{2.27}$$

Hence, from (2.25),

$$\begin{aligned}
 D(f dx^{i_1} \wedge \cdots \wedge dx^{i_k}) &= Df \wedge (Dx^{i_1} \wedge \cdots \wedge Dx^{i_k}) \\
 &= df \wedge (dx^{i_1} \wedge \cdots \wedge dx^{i_k}) \\
 &= d(f dx^{i_1} \wedge \cdots \wedge dx^{i_k}).
 \end{aligned} \tag{2.28}$$

So  $D = d$  on  $\Omega^*(U)$ .

■

### Closed Forms and Exact Forms

A  $k$ -form  $\omega$  on  $U$  is **closed** if  $d\omega = 0$ ; it's **exact** if there is a  $(k-1)$ -form  $\tau$  on  $U$  such that  $\omega = d\tau$ . Since  $d^2 = 0$ , every exact form is closed. But in general, a closed form may fail to be exact. We will see how non-exact closed forms capture the geometry of a manifold when we do de Rham cohomology on a manifold.

**Example 2.6.** Define a 1-form  $\omega$  on  $\mathbb{R}^2 \setminus \{0\}$  by

$$\omega = \frac{1}{x^2 + y^2} (-ydx + xdy). \quad (2.29)$$

Then  $\omega$  is closed.

A collection of vector spaces  $\{V^k\}_{k=0}^\infty$  with linear maps  $d_k : V^k \rightarrow V^{k+1}$  such that  $d_{k+1} \circ d_k = 0$  is called a **differential complex** or a **cochain complex**. For any open set  $U \subseteq \mathbb{R}^n$ , the exterior derivative  $d$  makes the vector space  $\Omega^*(U)$  of  $C^\infty$  forms on  $U$  into a cochain complex, called the **de Rham complex** on  $U$ :

$$\Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \Omega^2(U) \xrightarrow{d} \dots$$

The closed forms are precisely the elements of the kernel of  $d$  and the exact forms are the elements of the image of  $d$ . In the language of cohomology,  $d$  is also called the coboundary operator.

## §2.4 Applications to Vector Calculus

The theory of differential forms unifies many theorems in vector calculus on  $\mathbb{R}^3$ . A vector valued function on  $\mathbb{R}^3$  is the same as a vector field. Recall the 3 operators on scalar and vector-valued functions on  $\mathbb{R}^3$ .

$$\{\text{scalar function}\} \xrightarrow{\text{grad}} \{\text{vector function}\} \xrightarrow{\text{curl}} \{\text{vector function}\} \xrightarrow{\text{div}} \{\text{scalar function}\}.$$

Let  $f$  be a scalar function and  $\begin{bmatrix} P \\ Q \\ R \end{bmatrix}$  be a vector field on  $\mathbb{R}^3$ , where each of  $P, Q, R$  is a scalar function on  $\mathbb{R}^3$ . Then

$$\begin{aligned} \text{grad } f &= \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix}, \\ \text{curl} \begin{bmatrix} P \\ Q \\ R \end{bmatrix} &= \begin{bmatrix} R_y - Q_z \\ P_z - R_x \\ Q_x - P_y \end{bmatrix}, \\ \text{div} \begin{bmatrix} P \\ Q \\ R \end{bmatrix} &= P_x + Q_y + R_z. \end{aligned} \quad (2.30)$$

Then one has the following results.

### Proposition 2.5

$$\text{curl}(\text{grad } f) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \quad (2.31)$$

**Proposition 2.6**

$$\operatorname{div} \left( \operatorname{curl} \begin{bmatrix} P \\ Q \\ R \end{bmatrix} \right) = 0. \quad (2.32)$$

**Proposition 2.7**

On  $\mathbb{R}^3$ , a vector field  $\mathbf{F}(x, y, z)$  is the gradient of some scalar function if and only if  $\operatorname{curl} \mathbf{F} = \mathbf{0}$ .

A 1-form on  $\mathbb{R}^3$  can be written as

$$P(x, y, z) dx + Q(x, y, z) dy + R(x, y, z) dz.$$

This 1-form on  $\mathbb{R}^3$  can be identified with the vector field  $\begin{bmatrix} P \\ Q \\ R \end{bmatrix}$ .

Similarly, the 2-forms on  $\mathbb{R}^3$  given by

$$A(x, y, z) dy \wedge dz + B(x, y, z) dz \wedge dx + C(x, y, z) dx \wedge dy$$

can be identified with the vector field  $\begin{bmatrix} A \\ B \\ C \end{bmatrix}$  on  $\mathbb{R}^3$ .

In terms of these identifications, the exterior derivative of a 0-form  $f$  (scalar function) is

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz,$$

which can be identified with the vector field

$$\begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \operatorname{grad} f.$$

The exterior derivative of a 1-form on  $\mathbb{R}^3$  is

$$\begin{aligned} & d(Pdx + Qdy + Rdz) \\ &= \frac{\partial P}{\partial y} dy \wedge dx + \frac{\partial P}{\partial z} dz \wedge dx + \frac{\partial Q}{\partial x} dx \wedge dy + \frac{\partial Q}{\partial z} dz \wedge dy + \frac{\partial R}{\partial x} dx \wedge dz + \frac{\partial R}{\partial y} dy \wedge dz \\ &= (R_y - Q_z) dy \wedge dz + (P_z - R_x) dz \wedge dx + (Q_x - P_y) dx \wedge dy, \end{aligned}$$

which corresponds to

$$\begin{bmatrix} R_y - Q_z \\ P_z - R_x \\ Q_x - P_y \end{bmatrix} = \operatorname{curl} \begin{bmatrix} P \\ Q \\ R \end{bmatrix}.$$

The exterior derivative of a 2-form is

$$\begin{aligned} & d(A dy \wedge dz + B dz \wedge dx + C dx \wedge dy) \\ &= A_x dx \wedge dy \wedge dz + B_y dy \wedge dz \wedge dx + C_z dz \wedge dx \wedge dy \\ &= (A_x + B_y + C_z) dx \wedge dy \wedge dz, \end{aligned}$$

which corresponds to

$$A_x + B_y + C_z = \operatorname{div} \begin{bmatrix} A \\ B \\ C \end{bmatrix}.$$



In summary, exterior derivative  $d$  on 0-forms is identified with **gradient**; exterior derivative  $d$  on 1-forms is identified with **curl**; exterior derivative  $d$  on 2-forms is identified with **divergence**. Using de Rham complex on  $U$ :

$$\Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \Omega^2(U) \xrightarrow{d} \Omega^3(U).$$

Using vector calculus language,

$$C^\infty(U) \xrightarrow{\text{grad}} \mathfrak{X}(U) \xrightarrow{\text{curl}} \mathfrak{X}(U) \xrightarrow{\text{div}} C^\infty(U).$$

**Remark 2.1.** Proposition 2.5 and Proposition 2.6 express the property  $d^2 = 0$  of exterior deriva-

tive. A vector field  $\begin{bmatrix} P \\ Q \\ R \end{bmatrix}$  is the gradient of a  $C^\infty$  function  $f$  if and only if the corresponding 1-form  $Pdx + Qdy + Rdz$  is  $df$ . Proposition 2.7 expresses the fact that a 1-form on  $\mathbb{R}^3$  is exact if and only if it is closed. It's worth remarking at this stage that Proposition 2.7 need not hold true on a region other than  $\mathbb{R}^3$ , as the following well-known example from calculus suggests.

**Example 2.7.** Suppose  $U = \mathbb{R}^3 \setminus \{z\text{-axis}\}$ , and  $\mathbf{F}(x, y, z)$  is the vector field

$$\mathbf{F} = \begin{bmatrix} \frac{-y}{x^2+y^2} \\ \frac{x}{x^2+y^2} \\ 0 \end{bmatrix}$$

on  $U$ . Then  $\text{curl } \mathbf{F} = \mathbf{0}$ . Indeed,

$$\begin{aligned} \text{curl } \mathbf{F} &= \begin{bmatrix} \frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}\left(\frac{x}{x^2+y^2}\right) \\ \frac{\partial}{\partial z}\left(\frac{-y}{x^2+y^2}\right) - \frac{\partial}{\partial x}(0) \\ \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ \frac{(x^2+y^2)-x \cdot 2x}{(x^2+y^2)^2} - \frac{-(x^2+y^2)+y \cdot 2y}{(x^2+y^2)^2} \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0}. \end{aligned}$$

But  $\mathbf{F}$  is not the gradient of a  $C^\infty$  function on  $U$ . Recall the theorem from vector calculus that the line integral of the gradient of a function along a curve gives the total change in the value of the function from the start to the end of the curve. In other words, if  $\mathbf{r} : [a, b] \rightarrow \mathbb{R}^3$  is a curve and  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$  is a scalar function, then

$$\int_a^b (\nabla f) \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)). \quad (2.33)$$

Then if  $\mathbf{F}$  is the gradient of a smooth scalar function, then the line integral

$$\oint_C \frac{-y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy$$

over any closed curve would become 0. Let us take the closed curve to be the unit circle:  $x = \cos t$ ,

$y = \sin t$ ,  $z = 0$  for  $t \in [0, 2\pi]$ . Then

$$\begin{aligned} & \oint_C \frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy \\ &= \int_0^{2\pi} -\sin t \, d(\cos t) + \int_0^{2\pi} \cos t \, d(\sin t) \\ &= \int_0^{2\pi} \sin^2 t \, dt + \int_0^{2\pi} \cos^2 t \, dt \\ &= 2\pi. \end{aligned}$$

Hence, although  $\text{curl } \mathbf{F} = \mathbf{0}$ , there is no  $C^\infty$  function  $f$  on  $U$  such that  $\mathbf{F} = \text{grad } f$ . In the language of differential forms, the 1-form

$$\omega = \frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy$$

is closed but not exact.

It turns out that whether [Proposition 2.7](#) is true for a region  $U \subseteq \mathbb{R}^3$  depends on the topology of  $U$ . One measure of the failure of a closed  $k$ -form to be exact is the quotient vector space

$$H^k(U) = \frac{\{\text{closed } k\text{-forms on } U\}}{\{\text{exact } k\text{-forms on } U\}},$$

called the  $k$ -th de Rham cohomology of  $U$ . The generalization of [Proposition 2.7](#) to any differential form on  $\mathbb{R}^n$  is called the **Poincaré lemma**:

For  $k \geq 1$ , every closed  $k$ -form on  $\mathbb{R}^n$  is exact.

This statement is equivalent to the vanishing of the  $k$ -th de Rham cohomology  $H^k(\mathbb{R}^n)$  for  $k \geq 1$ .

# 3 Differential Forms on Manifold

## §3.1 Definition and Local Expression

Let  $M$  be a smooth manifold and  $p \in M$ . The **cotangent space** of  $M$  at  $p$ , denoted by  $T_p^*M$  is the dual space of the tangent space  $T_pM$ . An element in  $T_p^*M$  is called a covector at  $p$ . Thus, a covector  $\omega_p \in T_p^*M$  is a linear function

$$\omega_p : T_pM \rightarrow \mathbb{R}.$$

A 1-form on  $M$  is a function that assigns to each  $p \in M$ , a covector at  $p$ .

**Definition 3.1** (Differential of a function). Let  $f : M \rightarrow \mathbb{R}$  be a  $C^\infty$  function on a manifold  $M$ . Its **differential** is defined to be the 1-form  $df$  on  $M$  such that for any  $p \in M$  and  $X_p \in T_pM$ ,

$$(df)_p(X_p) = X_p f. \quad (3.1)$$

### Proposition 3.1

If  $f : M \rightarrow \mathbb{R}$  is a  $C^\infty$  function, then for  $p \in M$  and  $X_p \in T_pM$ ,

$$f_{*,p}(X_p) = (df)_p(X_p) \left. \frac{\partial}{\partial x} \right|_{f(p)}.$$

*Proof.* Since  $f_{*,p}(X_p) \in T_{f(p)}\mathbb{R}$ , there is a real number  $c$  such that

$$f_{*,p}(X_p) = c \left. \frac{\partial}{\partial x} \right|_{f(p)}. \quad (3.2)$$

(Here the chart chosen on  $\mathbb{R}$  is  $(\mathbb{R}, \mathbb{1}_{\mathbb{R}})$  so that  $x$  is the coordinate of this chart, i.e.  $x = \mathbb{1}_{\mathbb{R}}$ .) To evaluate  $c$ , apply both sides of (3.2) to the function  $x \in C^\infty(\mathbb{R})$ . Then

$$f_{*,p}(X_p)(x) = c \left. \frac{\partial}{\partial x} \right|_{f(p)}(x) = c.$$

Therefore,

$$c = f_{*,p}(X_p)(x) = X_p(x \circ f) = X_p f = (df)_p(X_p), \quad (3.3)$$

since  $x = \mathbb{1}_{\mathbb{R}}$ . Therefore, substituting the value of  $c$  into (3.2),

$$f_{*,p}(X_p) = (df)_p(X_p) \left. \frac{\partial}{\partial x} \right|_{f(p)}. \quad (3.4)$$

■

Let  $(U, \varphi) \equiv (U, x^1, x^2, \dots, x^n)$  be a coordinate chart on  $M$ . Here  $x^i = r^i \circ \varphi$ , where  $r^i$  is the  $i$ -th coordinate function of a vector in  $\mathbb{R}^n$ . Then the differentials  $dx^1, dx^2, \dots, dx^n$  are 1-forms on  $U$ .

### Proposition 3.2

At each point  $p \in U$ , the covectors  $(dx^1)_p, \dots, (dx^n)_p$  form a basis for the cotangent space  $T_p^*M$ , dual to the basis  $\left\{ \left. \frac{\partial}{\partial x^1} \right|_p, \dots, \left. \frac{\partial}{\partial x^n} \right|_p \right\}$  for the tangent space  $T_pM$ .

*Proof.* Observe that

$$\left(dx^i\right)_p \left(\frac{\partial}{\partial x^j}\Big|_p\right) = \frac{\partial}{\partial x^j}\Big|_p \left(x^i\right) = \delta^i_j. \quad (3.5)$$

So  $\{(dx^1)_p, \dots, (dx^n)_p\}$  is the dual basis to  $\left\{\frac{\partial}{\partial x^1}\Big|_p, \dots, \frac{\partial}{\partial x^n}\Big|_p\right\}$ . ■

Thus, every 1-form  $\omega$  on  $U$  can be written as a linear combination

$$\omega = \sum_{i=1}^n a_i dx^i,$$

where  $a_i$  are functions on  $U$ . In particular, if  $f$  is a  $C^\infty$  function on  $M$ , then the 1-form  $df$ , when restricted to  $U$ , must be a linear combination

$$df = \sum_{i=1}^n a_i dx^i. \quad (3.6)$$

If we evaluate both sides of (3.6) on  $\frac{\partial}{\partial x^j}$ ,

$$(df) \left(\frac{\partial}{\partial x^j}\right) = \sum_{i=1}^n a_i dx^i \left(\frac{\partial}{\partial x^j}\right) = \sum_{i=1}^n a_i \delta^i_j = a_j.$$

Then

$$a_j = (df) \left(\frac{\partial}{\partial x^j}\right) = \frac{\partial f}{\partial x^j}.$$

Therefore,

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i. \quad (3.7)$$

## §3.2 The Cotangent Bundle

The underlying set of the **cotangent bundle** is the disjoint union of the cotangent spaces at all points of  $M$ :

$$T^*M = \bigsqcup_{p \in M} T_p^*M = \bigcup_{p \in M} \{p\} \times T_p^*M. \quad (3.8)$$

Let us give  $T^*M$  a topology in the following way: let  $(U, x^1, \dots, x^n)$  be a chart on  $M$  and  $p \in U$ . Then each  $\omega_p \in T_p^*M$  can be written uniquely as a linear combination

$$\omega_p = \sum_{i=1}^n c_i(\omega_p) (dx^i)_p,$$

with  $c_i(\omega_p) \in \mathbb{R}$ . This gives rise to a bijection

$$\begin{aligned} \tilde{\varphi} : T^*U &\rightarrow \varphi(U) \times \mathbb{R}^n \\ (p, \omega_p) &\mapsto (\varphi(p), c_1(\omega_p), c_2(\omega_p), \dots, c_n(\omega_p)). \end{aligned}$$

We use this bijection  $\tilde{\varphi}$  to transfer the topology of  $\varphi(U) \times \mathbb{R}^n$  to  $T^*U$ : a set  $A \subseteq T^*U$  is said to be open if and only if  $\tilde{\varphi}(A)$  is open in  $\varphi(U) \times \mathbb{R}^n$ , where  $\varphi(U) \times \mathbb{R}^n$  is given the subspace topology of  $\mathbb{R}^{2n}$ . Now, let  $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$  be the maximal atlas of  $M$ . Now, let

$$\begin{aligned} \mathcal{B} &= \bigcup_{\alpha \in I} \{A \mid A \subseteq T^*U_\alpha \text{ is open in } A \subseteq T^*U_\alpha\} \\ &= \{A \mid A \subseteq T^*U_\alpha \text{ is open in } A \subseteq T^*U_\alpha, \alpha \in I\}. \end{aligned}$$

It can be shown using the same technique of tangent bundle that  $\mathcal{B}$  forms a basis for topology. We give  $T^*M$  the topology generated by the basis  $\mathcal{B}$ . We declare  $A \subseteq T^*M$  to be open if and only if there exists a subfamily  $\{B_\lambda\}_\lambda \subseteq \mathcal{B}$  such that

$$A = \bigcup_\lambda B_\lambda.$$

Furthermore,  $T^*M$  has the structure of a  $C^\infty$  manifold. An atlas for  $T^*M$  is

$$\{(T^*U_\alpha, \tilde{\varphi}_\alpha)\}_{\alpha \in I}.$$

If two coordinate open sets  $U_\alpha$  and  $U_\beta$  intersect, suppose  $U_{\alpha\beta} = U_\alpha \cap U_\beta$ . Then for  $p \in U_{\alpha\beta}$ , each  $\omega_p \in T_p^*M$  has two basis expansions:

$$\omega_p = \sum_{i=1}^n a_i (dx^i)_p = \sum_{j=1}^n b_j (dy^j)_p. \quad (3.9)$$

(Here  $(U_\alpha, x^1, \dots, x^n)$  and  $(U_\beta, y^1, \dots, y^n)$  are charts.) Now applying  $\left. \frac{\partial}{\partial y^k} \right|_p$  to both sides of (3.9),

$$b_k = \sum_{i=1}^n a_i (dx^i)_p \left( \left. \frac{\partial}{\partial y^k} \right|_p \right) = \sum_{i=1}^n a_i \left. \frac{\partial x^i}{\partial y^k} \right|_p.$$

Therefore,  $\tilde{\varphi}_\beta \circ \tilde{\varphi}_\alpha^{-1} : \varphi_\alpha(U_{\alpha\beta}) \times \mathbb{R}^n \rightarrow \varphi_\beta(U_{\alpha\beta}) \times \mathbb{R}^n$  is given by

$$(\varphi_\alpha(p), a_1, \dots, a_n) \mapsto \left( (\varphi_\beta \circ \varphi_\alpha^{-1})(\varphi_\alpha(p)), \sum_{i=1}^n a_i \left. \frac{\partial x^i}{\partial y^1} \right|_p, \dots, \sum_{i=1}^n a_i \left. \frac{\partial x^i}{\partial y^n} \right|_p \right).$$

$\varphi_\beta \circ \varphi_\alpha^{-1}$  is smooth, and each  $\left. \frac{\partial x^i}{\partial y^j} \right|_p$  is smooth. Therefore, the transition map  $\tilde{\varphi}_\beta \circ \tilde{\varphi}_\alpha^{-1}$  is smooth, making  $T^*M$  a smooth manifold.

$T^*M$  is, in fact, a **vector bundle** of rank  $n$  over  $M$ . It has a natural projection  $\pi : T^*M \rightarrow M$  given by  $(p, \omega_p) \mapsto p$ . In terms of cotangent bundle, a 1-form on  $M$  is simply a section of the cotangent bundle  $T^*M$ , i.e. it is a map  $\omega : M \rightarrow T^*M$  such that  $\pi \circ \omega = \mathbb{1}_M$ . We say that a 1-form is **smooth** if it is  $C^\infty$  as a map  $\omega : M \rightarrow T^*M$  between two manifolds.

### §3.3 Characterization of Smooth 1-forms

By definition, a 1-form on an open set  $U \subseteq M$  is  $C^\infty$  if it is  $C^\infty$  as a section of the cotangent bundle  $T^*M$  over  $U$ .

#### Lemma 3.3

Let  $(U, \varphi) \equiv (U, x^1, \dots, x^n)$  be a chart on a manifold  $M$ . A 1-form  $\omega = \sum a_i dx^i$  on  $U$  is smooth if and only if the coefficient functions  $a_i$  are all smooth on  $U$ .

*Proof.* This is a special case of *Proposition 9.4.2* of [DG1](#) which states that:

Let  $\pi : E \rightarrow M$  be a  $C^\infty$  vector bundle and  $U$  an open subset of  $M$ . Suppose  $s_1, \dots, s_r$  is a  $C^\infty$  frame for  $E$  over  $U$ . Then a section  $s = \sum_{j=1}^r c^j s_j$  of  $E$  over  $U$  is  $C^\infty$  if and only if the coefficients  $c^j$  are  $C^\infty$  functions on  $U$ .

Here we take  $E$  to be the cotangent bundle  $T^*M$ , and  $\{s_i\}_{i=1}^r$  the  $C^\infty$  frame for  $E$  over  $U$  to be the coordinate 1-forms  $\{(dx^i)\}_{i=1}^n$ . ■

**Proposition 3.4**

Let  $\omega$  be a 1-form on a manifold  $M$ . Then the following are equivalent:

- (i)  $\omega$  is  $C^\infty$ .
- (ii) For every point  $p \in M$ , there is a chart  $(U, x^1, \dots, x^n)$  about  $p$  such that if  $\omega = \sum_{i=1}^n a_i dx^i$  on  $U$ , then the functions  $a_i$  are  $C^\infty$  on  $U$ .
- (iii) For any chart  $(U, x^1, \dots, x^n)$  on  $M$ , if  $\omega = \sum_{i=1}^n a_i dx^i$  on  $U$ , then the functions  $a_i$  are  $C^\infty$  on  $U$ .

*Proof.* (ii) $\Rightarrow$ (i): By Lemma 3.3, for every point  $p \in M$ , there is a chart  $(U, x^1, \dots, x^n)$  about  $p$  such that  $\omega$  is smooth on  $U$ . In particular, the section  $\omega : M \rightarrow T^*M$  is smooth at  $p$ , for every  $p \in M$ . Therefore,  $\omega : M \rightarrow T^*M$  is a smooth map between manifolds.

(i) $\Rightarrow$ (iii): If  $\omega : M \rightarrow T^*M$  is a smooth map between manifolds,  $\omega$  is smooth at every chart of  $M$ . Therefore, by Lemma 3.3, if  $\omega = \sum_{i=1}^n a_i dx^i$  on a chart  $(U, x^1, \dots, x^n)$ , each  $a_i$  is smooth on  $U$ .

(iii) $\Rightarrow$ (ii): Obvious. ■

**Proposition 3.5**

A 1-form  $\omega$  on a manifold  $M$  is  $C^\infty$  if and only if for every  $C^\infty$  vector field  $X$ , the function  $\omega(X)$  is  $C^\infty$  on  $M$ .

*Proof.* ( $\Rightarrow$ ) Suppose  $\omega$  is a  $C^\infty$  1-form and  $X$  is a  $C^\infty$  vector field. Let  $(U, x^1, \dots, x^n)$  be a chart on  $M$ . Then

$$\omega = \sum_{i=1}^n a_i dx^i \quad \text{and} \quad X = \sum_{j=1}^n b^j \frac{\partial}{\partial x^j}, \quad (3.10)$$

for  $C^\infty$  functions  $a_i$  and  $b^j$  on  $U$ . Then on  $U$ , one has

$$\omega(X) = \left( \sum_{i=1}^n a_i dx^i \right) \left( \sum_{j=1}^n b^j \frac{\partial}{\partial x^j} \right) = \sum_{i=1}^n \sum_{j=1}^n a_i b^j \delta^i_j = \sum_{i=1}^n a_i b^i, \quad (3.11)$$

which is a  $C^\infty$  function on  $U$ . Since  $U$  was chosen to be an arbitrary coordinate open set,  $\omega(X)$  is a smooth function on all of  $M$ .

( $\Leftarrow$ ) Suppose  $\omega$  is a 1-form on  $M$  such that for every  $C^\infty$  vector field  $X$  on  $M$ , the function  $\omega(X)$  is smooth on  $M$ . For a given  $p \in M$ , choose a coordinate neighborhood  $(U, \varphi) \equiv (U, x^1, \dots, x^n)$  about  $p$ . Then one has

$$\omega = \sum_{i=1}^n a_i dx^i$$

on  $U$ . Now fix an integer  $j \in \{1, 2, \dots, n\}$ . We can extend the  $C^\infty$  vector field  $\frac{\partial}{\partial x^j}$  on  $U$  to a  $C^\infty$  vector field  $X$  on the whole of  $M$  that agrees with  $\frac{\partial}{\partial x^j}$  in a neighborhood  $V$  of  $p$  (not necessarily the whole of  $U$ , but possibly a smaller neighborhood) contained in  $U$  (Proposition 11.1.4 of DG1). The extended vector field is defined in the following way: let  $\sigma : M \rightarrow \mathbb{R}$  be a  $C^\infty$  bump function which is identically 1 on a neighborhood  $V$  of  $p$  and which has support contained in  $U$ . Now, define the vector field  $q \mapsto X_q \in T_q M$ , denoted by  $X$ , in terms of the bump function  $\sigma$  in the following way:

$$X_q = \begin{cases} \sigma(q) \frac{\partial}{\partial x^j} \Big|_q & \text{if } q \in U, \\ \mathbf{0} & \text{if } q \notin U. \end{cases} \quad (3.12)$$

The vector field  $X$  is smooth in the whole of  $M$ , as proved in Proposition 11.1.4 of DG1. Now, by the hypothesis,  $\omega(X)$  is  $C^\infty$  on  $M$ . In particular,  $\omega(X)$  is smooth on  $V$ . Therefore,

$$\omega(X) = \left( \sum_{i=1}^n a_i dx^i \right) \left( \frac{\partial}{\partial x^j} \right) = \sum_{i=1}^n a_i \delta^i_j = a_j$$

is smooth on  $V$ . We, therefore, see that the coefficient functions  $a_i$ 's appearing in  $\omega = \sum_{i=1}^n a_i dx^i$  are smooth on  $V \subseteq U$ . It means that for a given point  $p$ , we can find a chart  $(V, \tilde{x}^1, \dots, \tilde{x}^n)$ , where

$$\tilde{x}^i = r^i \circ \varphi|_V,$$

such that  $\omega = \sum_{i=1}^n a_i|_V d\tilde{x}^i$  on  $V$ , with each  $a_i|_V$  smooth on  $V$ . Therefore, by [Proposition 3.4](#),  $\omega$  is  $C^\infty$ . ■

### §3.4 Pullback of 1-forms

Recall that the differential of a smooth map  $F : N \rightarrow M$  at  $p \in N$  is a linear map  $F_{*,p} : T_p N \rightarrow T_{F(p)} M$  defined by

$$[F_{*,p}(X_p)](f) = X_p(f \circ F), \quad (3.13)$$

where  $f \in C_{F(p)}^\infty(M)$ . Indeed,  $f \circ F \in C_p^\infty(N)$ . Analogously, the **codifferential** (the dual of a differential) at  $F(p) \in M$  is a linear map

$$F^{*,p} : T_{F(p)}^* M \rightarrow T_p^* N.$$

One observes that the differential  $F_{*,p}$  pushes forward a tangent vector at  $p \in N$  while the codifferential  $F^{*,p}$  pulls back a covector from  $T_{F(p)}^* M$  at  $F(p) \in M$  to  $T_p^* N$ .

**Remark 3.1.** Note that a vector field, in general, cannot be pushed forward under a smooth map  $F : N \rightarrow M$ . Suppose  $F : N \rightarrow M$  is a smooth map of manifolds. Also suppose  $F(p) = F(q) = z \in M$  so that  $F$  is not injective. Now, the differentials

$$F_{*,p} : T_p N \rightarrow T_z M \text{ and } F_{*,q} : T_q N \rightarrow T_z M$$

are linear maps. Now, let  $X \in \mathfrak{X}(N)$  be a  $C^\infty$  vector field on  $N$  so that  $X_p$  under  $F_{*,p}$  is pushed forward to  $F_{*,p}(X_p) \in T_z M$  and  $X_q$  is pushed forward to  $F_{*,q}(X_q) \in T_z M$  under  $F_{*,q}$ . There is no reason for  $F_{*,p}(X_p)$  and  $F_{*,q}(X_q)$  to be the same tangent vector in  $T_z M$ . In other words, in general,

$$F_{*,p}(X_p) \neq F_{*,q}(X_q),$$

so that  $z \mapsto F_{*,p}(X_p) := Y_z \in T_z M$  and  $z \mapsto F_{*,q}(X_q) := Y'_z \in T_z M$  are distinct vector fields on  $M$ , denoted by  $Y$  and  $Y'$ , respectively. Therefore, if there were push forward of vector fields  $F_* : \mathfrak{X}(N) \rightarrow \mathfrak{X}(M)$  associated with the non-injective smooth map  $F : N \rightarrow M$ , there is an ambiguity regarding which vector field  $X$  gets mapped to.

Furthermore, if  $F$  is not surjective, there is  $z \in M$  such that  $z \neq F(p)$  for any  $p \in N$ . In that case as well, defining the push forward vector field  $F_*(X)$  at the point  $z$  is impossible. However, when  $F : N \rightarrow M$  is a diffeomorphism, one can define the push forward of a vector field.

Contrary to the non-existence of push forward of a vector field associated with a generic smooth map  $F : N \rightarrow M$ , one can always talk about pullback of a 1-form  $\omega$  on  $M$ :

$$(F^*\omega)_p(X_p) = \omega_{F(p)}(F_{*,p}(X_p)). \quad (3.14)$$

Here,  $\omega \in \Omega^1(M)$ ,  $X_p \in T_p N$ ,  $p \in N$ . Note that  $(F^*\omega)_p$  is simply the image of the covector  $\omega_{F(p)} \in T_{F(p)}^* M$  under the codifferential  $F^{*,p} : T_{F(p)}^* M \rightarrow T_p^* N$ . In other words,

$$(F^*\omega)_p = F^{*,p}(\omega_{F(p)}). \quad (3.15)$$

# 4 Differential $k$ -forms

## §4.1 Definition and Local Expression

We denoted by  $A_k(V)$  the vector space of alternating  $k$ -tensors on  $V$ . We have also seen that if  $\{\hat{\alpha}^1, \dots, \hat{\alpha}^n\}$  is a basis for 1-tensors on  $V$ , then a basis element of  $A_k(V)$  is

$$\hat{\alpha}^{i_1} \wedge \hat{\alpha}^{i_2} \wedge \dots \wedge \hat{\alpha}^{i_k},$$

where  $1 \leq i_1 < i_2 < \dots < i_k \leq n$ . We apply this construction to the tangent space  $T_p M$  of a manifold  $M$  at a point  $p \in M$ . The vector space  $A_k(T_p M)$ , usually denoted by  $\Lambda^k(T_p^* M)$ , is the space of all alternating  $k$ -tensors on the tangent space  $T_p M$ .

**Definition 4.1** (Differential  $k$ -form). A **differential  $k$ -form** on a manifold  $M$  is a function  $\omega$  that assigns to each point  $p \in M$ , a  $k$ -covector  $\omega_p \in \Lambda^k(T_p^* M)$ . An  $n$ -form on a manifold of dimension  $n$  is called a **top degree form**.

**Example 4.1.** On  $\mathbb{R}^n$ , at each point  $p$ , there is a standard basis for the tangent space  $T_p \mathbb{R}^n$ :

$$\left\{ \left. \frac{\partial}{\partial r^1} \right|_p, \left. \frac{\partial}{\partial r^2} \right|_p, \dots, \left. \frac{\partial}{\partial r^n} \right|_p \right\}.$$

Let  $\{(dr^1)_p, \dots, (dr^n)_p\}$  be the dual basis of  $T_p^* \mathbb{R}^n$ .

$$(dr^i)_p \left( \left. \frac{\partial}{\partial r^j} \right|_p \right) = \delta^i_j.$$

As  $p$  varies over  $\mathbb{R}^n$ , we get differential forms  $dr^1, \dots, dr^n$  on  $\mathbb{R}^n$ . By [Proposition 1.15](#), a basis element of alternating  $k$ -tensors  $\Lambda^k(T_p^* \mathbb{R}^n)$  is

$$(dr^{i_1})_p \wedge (dr^{i_2})_p \wedge \dots \wedge (dr^{i_k})_p,$$

where  $1 \leq i_1 < i_2 < \dots < i_k \leq n$ . If  $\omega$  is a  $k$ -form on  $\mathbb{R}^n$ , then at each point  $p \in \mathbb{R}^n$ ,  $\omega_p$  is the following linear combination:

$$\omega_p = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} a_{i_1 \dots i_k} (dr^{i_1})_p \wedge (dr^{i_2})_p \wedge \dots \wedge (dr^{i_k})_p. \quad (4.1)$$

Omitting the point  $p$ , we write

$$\omega = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} a_{i_1 \dots i_k} dr^{i_1} \wedge dr^{i_2} \wedge \dots \wedge dr^{i_k}. \quad (4.2)$$

In the expression above,  $a_{i_1 \dots i_k}$  are functions on  $\mathbb{R}^n$ . To simplify the notations, we use multi-indices to write (4.2) as

$$\omega = \sum_I a_I dr^I, \quad (4.3)$$

where  $dr^I = dr^{i_1} \wedge dr^{i_2} \wedge \dots \wedge dr^{i_k}$ , and  $I = (i_1, i_2, \dots, i_k)$  is a strictly ascending multi-index.



Suppose  $(U, x^1, \dots, x^n)$  is a coordinate chart on a manifold  $M$ . We have already defined the 1-forms  $dx^1, \dots, dx^n$  on  $U$ . Since at each point  $p \in U$ ,  $\{(dx^1)_p, \dots, (dx^n)_p\}$  is a basis for  $T_p^*M$ , by [Proposition 1.15](#), a basis for  $\Lambda^k(T_p^*\mathbb{R}^n)$  is

$$(dx^{i_1})_p \wedge (dx^{i_2})_p \wedge \dots \wedge (dx^{i_k})_p,$$

where  $1 \leq i_1 < i_2 < \dots < i_k \leq n$ . Thus, locally a  $k$ -form on  $U$  will be a linear combination

$$\omega = \sum_I a_I dx^I, \quad (4.4)$$

where  $dx^I = dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_k}$ ,  $I = (i_1, i_2, \dots, i_k)$  is a strictly ascending multi-index, and  $a_I$  are functions on  $U$ .

## §4.2 The Bundle Point of View

Let  $V$  be a real vector space. Another common notation for the vector space  $A_k(V)$  of alternating  $k$ -linear functionos on  $V$  is  $\Lambda^k(V^*)$ .

$$\begin{aligned} \Lambda^0(V^*) &= A_0(V) = \mathbb{R}, \\ \Lambda^1(V^*) &= A_1(V) = V^*, \\ \Lambda^2(V^*) &= A_2(V), \end{aligned}$$

and so on. Now,  $\Lambda^k(T^*M)$  is defined to be the disjoint union of the vector spaces  $\Lambda^k(T_p^*M)$  as  $p$  varies over  $M$ . So

$$\begin{aligned} \Lambda^k(T^*M) &= \bigsqcup_{p \in M} \Lambda^k(T_p^*M) = \bigsqcup_{p \in M} A_k(T_pM) \\ &= \bigcup_{p \in M} \{p\} \times A_k(T_pM), \end{aligned} \quad (4.5)$$

which is the set of all alternating  $k$ -tensors at all points of  $M$ . This set is called the  $k$ -th **exterior power** of the cotangent bundle  $T^*M$ .

If  $(U, \varphi)$  is a coordinate chart on  $M$ , then there is a bijection  $\bar{\varphi} : \Lambda^k(T^*U) \rightarrow \varphi(U) \times \mathbb{R}^{\binom{n}{k}}$  defined as follows: a generic element of  $\Lambda^k(T^*U)$  is  $(p, \omega_p)$ , where  $\omega_p \in \Lambda^k(T_p^*U)$ . Then  $\omega_p$  is a unique linear combination

$$\omega_p = \sum_I a_I(p) dx^I,$$

where  $I$  runs over the set of strictly ascending multi-indices of length  $k$ . There are  $\binom{n}{k}$  many such multi-indices. If we fix a labeling of the multi-indices once and for all, then we have a  $\binom{n}{k}$ -tuple  $(a_I)_I$ . Then we define

$$\bar{\varphi}(p, \omega_p) = (\varphi(p), (a_I)_I) \in \varphi(U) \times \mathbb{R}^{\binom{n}{k}}.$$

Thus,  $\Lambda^k(T^*U)$  is in a bijective correspondence with  $\varphi(U) \times \mathbb{R}^{\binom{n}{k}}$ . Using this bijective correspondence, one transfers the topology of  $\varphi(U) \times \mathbb{R}^{\binom{n}{k}}$  to  $\Lambda^k(T^*U)$ . By varying the open set  $U$  in the charts contained in the maximal atlas of  $M$ , one can obtain a basis that generates the topology on the whole of  $\Lambda^k(T^*M)$ .

$\Lambda^k(T^*M)$  can, in fact, be shown to be a  $C^\infty$  vector bundle of rank  $\binom{n}{k}$  over  $M$ , i.e.  $\pi : \Lambda^k(T^*U) \rightarrow M$  is a  $C^\infty$  vector bundle of rank  $\binom{n}{k}$  over  $M$ . A differential  $k$ -form is a section of this vector bundle. We define a  $k$ -form to be  $C^\infty$  if it is  $C^\infty$  as a section of the vector bundle  $\Lambda^k(T^*M)$ .

**Notation.** If  $\pi : E \rightarrow M$  is a  $C^\infty$  vector bundle, then the vector space of  $C^\infty$  sections of  $E$  is denoted by  $\Gamma(E)$ , or  $\Gamma(M, E)$ . The vector space of all  $C^\infty$   $k$ -forms, i.e. all  $C^\infty$  sections of the bundle  $\Lambda^k(T^*M)$  is usually denoted by  $\Omega^k(M)$ . Thus,

$$\Omega^k(M) = \Gamma(\Lambda^k(T^*M)) = \Gamma(M, \Lambda^k(T^*M)).$$