On the Category Theoretic Implications on Induced Representation of Finite Groups

Part 1: Categories

Atonu Roy Chowdhury

BRAC University

atonuroychowdhury@gmail.com

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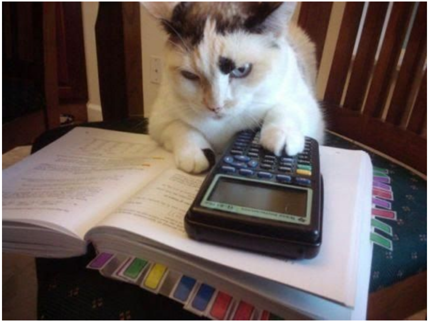

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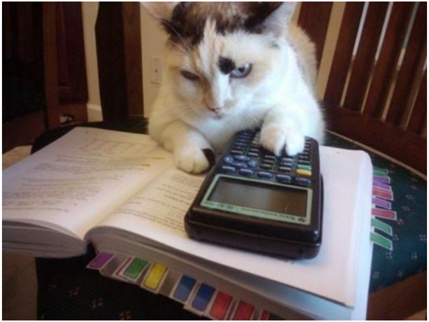

Sadly, no! :(

What do we do in math?

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In Linear Algebra

we study vector spaces and linear transformations between them

$$
\begin{tikzcd}
U \arrow[d, "g"'] \arrow[dr, shift left, "j"] \arrow[dr, shift right, "h"'] & \\
V \arrow[r, "f"'] & W \arrow[loop below, "i"]
\end{tikzcd}
$$

What do we do in math?

In Group Theory

we study groups and group homomorphisms between them

$$
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K \arrow[d, "g"'] \arrow[dr, shift right, "h"'] \arrow[dr, shift left, "j"] & \\
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\end{tikzcd}
$$

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In Topology

we study topological spaces and conotinuous between them

$$\begin{tikzcd}
Z \arrow[d, "g"'] \arrow[dr, shift left, "j"] \arrow[dr, shift right, "h"'] & \\
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  - Composition of linear maps is a linear map:

$$\begin{tikzcd} U \arrow[r, "f_1"] \arrow[rr, bend right, "f_2 \circ f_1"'] & V \arrow[r, "f_2"] & W \end{tikzcd}$$

  - and so on ...

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  - Every vector space has an identity transformation: $\mathbb{1}_V$

$$\begin{tikzcd} V \arrow[loop left, "\mathbb{1}_V"] \arrow[r, "f"] & W \arrow[loop right, "\mathbb{1}_W"] \end{tikzcd}$$

$$f = f \circ \mathbb{1}_V = \mathbb{1}_W \circ f.$$

  - and so on ...

Now we strip down all the details, and we have Category Theory!

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> *Category theory is the bird's eye view of mathematics.*
>
> — *Tom Leinster*

Definition 1

A **category** $\mathcal{C}$ consists of

- a collection of **objects**, often denoted as $\mathcal{C}_0$;
- a collection **arrows** from one object to another, often denoted as $\mathcal{C}_1$; if f is an arrow from A to B, we write $f : A \to B$ and call $A = \operatorname{dom} f$ and $B = \operatorname{cod} f$; $\operatorname{Hom}_{\mathcal{C}}(A, B)$ is the collection of all arrows from the object A to the object B.

such that

- given any two arrows f, g with $\operatorname{cod} f = \operatorname{dom} g$, there is a composition $g \circ f : \operatorname{dom} f \to \operatorname{cod} g$;
- for every $A \in \mathcal{C}_0$, there is a unique identity arrow $\mathbb{1}_A \in \operatorname{Hom}_{\mathcal{C}}(A, A)$.

These data have the following properties:

- composition is associative: given $f : A \to B, g : B \to C, h : C \to D$, $h \circ (g \circ f) = (h \circ g) \circ f$.
- identity arrow is the identity of composition: for any $f : A \to B$, $\mathbb{1}_B \circ f = f = f \circ \mathbb{1}_A$.

Caution!

Objects need not be sets, and arrows need not be functions!

Examples of Categories

Example 1

Any structured sets (groups, vector spaces, topological spaces, etc) and structure preserving maps between them. For instance,

1. **Groups** is the category of all groups. $\mathbf{Groups}_0$ is the collection of all groups, $\mathbf{Groups}_1$ is the collection of all group homomorphisms.
2. $\mathbf{Vect}_{\mathbb{C}}$ is the category of all vector spaces over the field $\mathbb{C}$. $(\mathbf{Vect}_{\mathbb{C}})_0$ is the collection of all vector spaces over $\mathbb{C}$, $(\mathbf{Vect}_{\mathbb{C}})_1$ is the collection of all linear maps between them.
3. **Sets** is the category of all sets. $\mathbf{Sets}_0$ is the collection of all sets, $\mathbf{Sets}_1$ is the collection of all functions between sets.
4. and so on ...

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$\mathcal{C}(G)_0 = \{G\}$, $\mathcal{C}(G) = G$. In other words, there is only one object, G itself. The arrows are the elements of G.

$$\begin{tikzcd} G \arrow[loop above, "g'"] \arrow[loop right, "g"] \arrow[loop below, "e"] \arrow[loop left, "\cdots"] \end{tikzcd}$$

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The identity arrow $\mathbb{1}_G$ is the identity element e of G. The composition of arrows is given by the group operation: $g_1 \circ g_2 = g_1 g_2$.

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The only structure on the catrgories we have are composition of arrows and the identity arrow.

Definition 2

A **functor** $F : \mathcal{C} \to \mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$ is two mappings $F_0 : \mathcal{C}_0 \to \mathcal{D}_0$, and $F_1 : \mathcal{C}_1 \to \mathcal{D}_1$ such that

- if $f : A \to B$ is an arrow in $\mathcal{C}$, then $F_1(f) : F_0(A) \to F_0(B)$ in $\mathcal{D}$, i.e. F preserves domains and codomains;
- for every $A \in \mathcal{C}_0$, $F_1(\mathbb{1}_A) = \mathbb{1}_{F_0(A)}$;
- given two composable arrows $f : A \to B$ and $g : B \to C$ in $\mathcal{C}$,

$$F_1(g \circ f) = F_1(g) \circ F_1(f).$$

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We shall often abuse the notation by writing $F(f) : F(A) \to F(B)$.

Category of Categories

$$
\begin{tikzcd}
\mathcal{C} \arrow[dd, "F"'] \\
\\
\mathcal{D}
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "g \circ f"'] & B \arrow[d, "g"] \\
& C
\end{tikzcd}
$$

$$
\begin{tikzcd}
F_0(A) \arrow[r, "F_1(f)"] \arrow[dr, "F_1(g \circ f) = F_1(g) \circ F_1(f)"'] & F_0(B) \arrow[d, "F_1(g)"] \\
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Category of Categories

In order to make a category of categories, we need identity functors $\mathbb{1}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$. This is easy, we can just map everything to itself.

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What about composition?

$$\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}$$

Given $A \in \mathcal{C}_0$, we can just define $(G \circ F)_0 (A) = G_0 (F_0 (A))$. For an arrow $f : A \to B$ in $\mathcal{C}$, we can define $(G \circ F)_1 (f) = G_1 (F_1 (f))$.

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Then we have the category of all categories, **Cat**.

Example 3

The opposite category $\mathcal{C}^{\mathrm{op}}$ has the same objects as $\mathcal{C}$, but the arrows are reversed.

$\mathcal{C}$

$$\begin{tikzcd}
C \arrow[loop left, "\mathbb{1}_C"] \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & D \arrow[r, "g"] & E
\end{tikzcd}$$

$\mathcal{C}^{\mathrm{op}}$

$$\begin{tikzcd}
C \arrow[loop above, "(\mathbb{1}_C)^{\mathrm{op}}"] & D \arrow[l, "f^{\mathrm{op}}"'] & E \arrow[l, "g^{\mathrm{op}}"'] \arrow[ll, bend left, "f^{\mathrm{op}}\circ g^{\mathrm{op}}=(g\circ f)^{\mathrm{op}}"]
\end{tikzcd}$$

Often we use this interpretation that an arrow $f : X \to Y$ in $\mathcal{C}^{\mathrm{op}}$ is really an arrow $f : Y \to X$ in $\mathcal{C}$. This is an abuse of notation since we are dropping the superscript op from the arrows in $\mathcal{C}^{\mathrm{op}}$.

Making New categories From Old Ones

Example 4

The product of 2 categories $\mathcal{C}$ and $\mathcal{D}$, written as $\mathcal{C} \times \mathcal{D}$.

- the objects are (C, D) for $C \in \mathcal{C}_0$ and $D \in \mathcal{D}_0$,
- and the arrows are
$$(f, g) : (C, D) \to (C', D'),$$
where $f : C \to C'$ and $g : D \to D'$ are arrows in $\mathcal{C}$ and $\mathcal{D}$, respectively.

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$$\begin{tikzcd} C \arrow[loop left, "\mathbb{1}_C"] \arrow[r, "f"] \arrow[rr, bend right, "f' \circ f"'] & C' \arrow[r, "f'"] & C'' \end{tikzcd} \qquad \begin{tikzcd} D \arrow[loop left, "\mathbb{1}_D"] \arrow[r, "g"] \arrow[rr, bend right, "g' \circ g"'] & D' \arrow[r, "g'"] & D'' \end{tikzcd}$$

$$\begin{tikzcd} (C, D) \arrow[loop left, "\mathbb{1}_{(C,D)} = (\mathbb{1}_C, \mathbb{1}_D)"] \arrow[r, "(f,g)"] \arrow[rr, bend right, "(f',g') \circ (f,g) = (f' \circ f, g' \circ g)"'] & (C', D') \arrow[r, "(f',g')"] & (C'', D'') \end{tikzcd}$$

Example 5

We can also form a category with the arrows of a category $\mathcal{C}$. It is known as the **arrow category** of $\mathcal{C}$, and is denoted as $\text{Arr}(\mathcal{C})$. The objects are arrows of $\mathcal{C}$. What are the arrows of $\text{Arr}(\mathcal{C})$?

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We can also form a category with the arrows of a category $\mathcal{C}$. It is known as the **arrow category** of $\mathcal{C}$, and is denoted as $\mathsf{Arr}(\mathcal{C})$. The objects are arrows of $\mathcal{C}$. What are the arrows of $\mathsf{Arr}(\mathcal{C})$?

Suppose $f : A \to B, g : C \to D \in \mathsf{Arr}(\mathcal{C})_0$. An arrow $x : f \to g$ in the arrow category is a pair of arrows $x_1 : A \to C$ and $x_2 : B \to D$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "x_1"] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "x_2"'] & D \end{tikzcd} \quad \text{, i.e. } x_2 \circ f = g \circ x_1.$$

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Identity and composition are defined in the obvious way.

Hom-sets

The categories we see in our everyday life are **locally small** categories, i.e. $\mathrm{Hom}_{\mathcal{C}}(A, B)$ are sets. We shall not worry about it anymore and assume that all categories are locally small.

Hom-sets

For a locally small category $\mathcal{C}$, and $X \in \mathcal{C}_0$, we have a functor

$$\mathrm{Hom}_{\mathcal{C}}(X, -) : \mathcal{C} \to \mathbf{Sets},$$

called the Hom functor.

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Given an arrow $f : A \to B$ in $\mathcal{C}$, the Hom functor is supposed to take it to a set function $f^* : \mathrm{Hom}_{\mathcal{C}}(X, A) \to \mathrm{Hom}_{\mathcal{C}}(X, B)$. How does f^* work?

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X \arrow[d, "g"'] \arrow[dr, "f^*(g) = f \circ g"] & \\
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$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[dl, "f_*(g)=???"] \\ X & \end{tikzcd}$$

Given an arrow $f : A \to B$, seemingly there's an issue with defining $f_* : \mathrm{Hom}_{\mathcal{C}}(A, X) \to \mathrm{Hom}_{\mathcal{C}}(B, X)$.

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So $\mathrm{Hom}_{\mathcal{C}}(-, X) : \mathcal{C} \to \mathbf{Sets}$ reverses the direction of arrows?

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The solution is tweaking the domain category to be the opposite category of $\mathcal{C}$.

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The objects of $\mathcal{C}^{\mathrm{op}}$ are the same as $\mathcal{C}$. The arrows of $\mathcal{C}^{\mathrm{op}}$ are $f^{\mathrm{op}} : B \to A$, corresponding to an arrow $f : A \to B$ in $\mathcal{C}$.

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The functor $\mathrm{Hom}_{\mathcal{C}}(-, X)$ takes the arrow $f^{\mathrm{op}} : B \to A$ to a set function $f_* : \mathrm{Hom}_{\mathcal{C}}(B, X) \to \mathrm{Hom}_{\mathcal{C}}(A, X)$.

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A \arrow[r, "f"] \arrow[d, "f_*(g) = g \circ f"'] & B \arrow[dl, "g"] \\
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Generalizing further, we define the functor

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In the arrow level, let (f^{op}, g) be an arrow in $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$. Then $f : A \to B$ and $g : X \to Y$ are arrows in $\mathcal{C}$.

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$$\mathrm{Hom}_{\mathcal{C}}(-,-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Sets}.$$

In the object level, it takes (A, B) to the set $\mathrm{Hom}_{\mathcal{C}}(A, B)$.

In the arrow level, let (f^{op}, g) be an arrow in $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$. Then $f : A \to B$ and $g : X \to Y$ are arrows in $\mathcal{C}$.
$\mathrm{Hom}_{\mathcal{C}}(f^{\mathrm{op}}, g)$ will be a set function $\mathrm{Hom}_{\mathcal{C}}(B, X) \to \mathrm{Hom}_{\mathcal{C}}(A, Y)$.

$$\begin{tikzcd} A \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[ur, "x"] & Y \end{tikzcd}$$

Given $x \in \mathrm{Hom}_{\mathcal{C}}(B, X)$, we define $\mathrm{Hom}_{\mathcal{C}}(f^{\mathrm{op}}, g) = g \circ x \circ f$.

Isomorphisms

We define

- isomorphism between groups,
- isomorphism between vector spaces,
- homeomorphism between topological spaces,
- diffeomorphisms between smooth manifolds,
- and so on . . .

Category theory captures this pattern of "sameness" as well.

Isomorphisms

Definition 3

An arrow $f : A \to B$ in a category $\mathcal{C}$ is called an **isomorphism** if there exists another arrow $g : B \to A$ such that $f \circ g = \mathbb{1}_B$ and $g \circ f = \mathbb{1}_A$.
If there is an isomorphism from A to B, we call A and B **isomorphic objects**.

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If there is an isomorphism from A to B, we call A and B **isomorphic objects**.

This definition aligns with our definition of isomorphisms in other categories.

Isomorphism and Functors

Suppose A and B are isomorphic objects in a category $\mathcal{C}$. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then are $F(A)$ and $F(B)$ isomorphic objects in the category $\mathcal{D}$?

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$\mathcal{C}$ $\qquad\qquad\qquad\qquad$ $\mathcal{D}$

$$A \underset{g}{\overset{f}{\rightleftarrows}} B \qquad\qquad F(A) \underset{F(g)}{\overset{F(f)}{\rightleftarrows}} F(B)$$

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$$F(f) \circ F(g) = F(f \circ g) = F(\mathbb{1}_A) = \mathbb{1}_{F(A)}.$$
$$F(g) \circ F(f) = F(g \circ f) = F(\mathbb{1}_B) = \mathbb{1}_{F(B)}.$$

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Therefore, functors preserve isomorphisms.

Isomorphism and Hom-sets

Suppose A and B are isomorphic objects in a category. Then for any other object X, since functors preserve isomorphisms, $\mathrm{Hom}_{\mathcal{C}}(X, A)$ and $\mathrm{Hom}_{\mathcal{C}}(X, B)$ are isomorphic objects in **Sets**.

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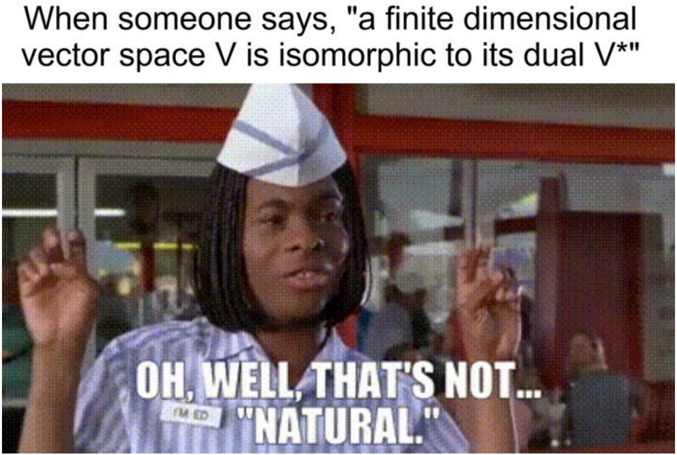
Similarly, $\mathrm{Hom}_{\mathcal{C}}(A, X)$ and $\mathrm{Hom}_{\mathcal{C}}(B, X)$ are also isomorphic objects in **Sets**.

Does the converse hold? If $\mathrm{Hom}_{\mathcal{C}}(A, X)$ and $\mathrm{Hom}_{\mathcal{C}}(B, X)$ are also isomorphic objects in **Sets**, then can we say that A and B are isomorphic objects in $\mathcal{C}$?


When someone says, "a finite dimensional vector space V is isomorphic to its dual V*"
OH, WELL, THAT'S NOT...
"NATURAL."

Natural Transformation

Given two categories $\mathcal{C}$ and $\mathcal{D}$, we can form the functor category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ whose objects are all the functors from $\mathcal{C}$ to $\mathcal{D}$. What should be the arrows in this category?

This is similar to the construction of arrow category.

Natural Transformation

Definition 4

Let $F, G : \mathcal{C} \to \mathcal{D}$ be two functors. Then a **natural transformation** $\eta : F \Rightarrow G$ is a family of arrows

$$\{\eta_X : F(X) \to G(X)\}_{X \in \mathcal{C}_0}$$

in $\mathcal{D}$ such that for every arrow $f : X \to Y$ in $\mathcal{C}$, the following diagram commutes:

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\eta_X"'] & F(Y) \arrow[d, "\eta_Y"] \\ G(X) \arrow[r, "G(f)"'] & G(Y) \end{tikzcd}$$

In other words, $\eta_Y \circ F(f) = G(f) \circ \eta_X$.

Given such a natural transformation $\eta : F \Rightarrow G$, the arrow η_X is called the component of η at X.

Natural Transformation

If $F, G : \mathcal{C} \to \mathcal{D}$ are functors, and $\eta : F \Rightarrow G$ is a natural transformation, it is denoted as follows

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "F"{name=U}] \arrow[r, bend right=50, "G"'{name=D}] & \mathcal{D} \arrow[Rightarrow, from=U, to=D, "\eta"] \end{tikzcd}$$

Why do we care about natural transformations?

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Because they take commutative diagrams to commutative diagrams!!!

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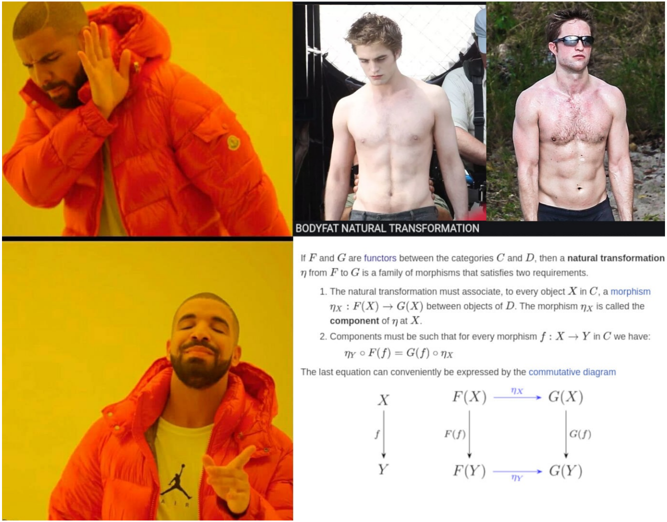
\begin{tikzcd}
F(X) \arrow[rr, "\eta_X"] \arrow[dr, "F(f)"] \arrow[dd, "F(g\circ f)"'] & & G(X) \arrow[dr, "G(f)"] \arrow[dd, "G(g\circ f)"' near start] & \\
& F(Y) \arrow[rr, "\eta_Y"' near start] \arrow[dl, "F(g)"] & & G(Y) \arrow[dl, "G(g)"] \\
F(Z) \arrow[rr, "\eta_Z"'] & & G(Z) &
\end{tikzcd}
$$

Natural Transformation



Natural Isomorphism

Definition 5

Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A natural transformation $\eta : F \Rightarrow G$ is called a **natural isomorphism** if all its components

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are isomorphisms in $\mathcal{D}$.

Since natural transformations are arrows in the functor category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, natural isomorphisms are just isomorphisms in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

Natural Isomorphism

What do we mean when we say V and V^{**} are "naturally isomorphic" (when V is a finite dimensional vector space)? Where is the natural isomorphism here?

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What do we mean when we say V and V^{**} are "naturally isomorphic" (when V is a finite dimensional vector space)? Where is the natural isomorphism here?

Let $\mathbf{Vect}_{\mathbb{K}}^{\text{fin}}$ be the category of all finite dimensional vector spaces over the field $\mathbb{K}$. Consider the functors $\mathbb{1}_{\mathbf{Vect}_{\mathbb{K}}^{\text{fin}}}, (-)^{**} : \mathbf{Vect}_{\mathbb{K}}^{\text{fin}} \to \mathbf{Vect}_{\mathbb{K}}^{\text{fin}}$.

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The second functor sends a vector space V to its double dual V^{**}, and a linear map $f : V \to W$ to $\left(f^T\right)^T : V^{**} \to W^{**}$.

Natural Isomorphism

Then we can define an isomorphism $\eta_V : V \to V^{**}$ such that

$$\eta_V(\mathbf{v})(\varphi) = \varphi(\mathbf{v}),$$

for $\mathbf{v} \in V$ and $\varphi \in V^*$.

Natural Isomorphism

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for $\mathbf{v} \in V$ and $\varphi \in V^*$.

Then $\{\eta_V : V \to V^{**}\}_{V \in \left(\mathbf{Vect}_{\mathbb{K}}^{\text{fin}}\right)_0}$ is a natural isomorphism, because the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\eta_V"'] & W \arrow[d, "\eta_W"] \\ V^{**} \arrow[r, "\left(f^T\right)^T"'] & W^{**} \end{tikzcd} \qquad \begin{tikzcd} \mathbf{Vect}_{\mathbb{K}}^{\text{fin}} \arrow[r, bend left=60, "\mathbb{1}_{\mathbf{Vect}_{\mathbb{K}}^{\text{fin}}}"{name=U}] \arrow[r, bend right=60, "(-)^{**}"'{name=D}] & \mathbf{Vect}_{\mathbb{K}}^{\text{fin}} \arrow[Rightarrow, from=U, to=D, "\eta"] \end{tikzcd}$$

Isomorphism	Equivalence	Adjoints
$F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$, s.t. $F \circ G = \mathbb{1}_{\mathcal{D}}$ and $G \circ F = \mathbb{1}_{\mathcal{C}}$	$F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$, s.t. **natural isomorphisms** $F \circ G \Rightarrow \mathbb{1}_{\mathcal{D}}$ and $G \circ F \Rightarrow \mathbb{1}_{\mathcal{C}}$	$F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$, s.t. **natural transformations** $F \circ G \Rightarrow \mathbb{1}_{\mathcal{D}}$ and $G \circ F \Leftarrow \mathbb{1}_{\mathcal{C}}$

Adjoints

Definition 6

Two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are called **adjoints** to each other if

$$\mathrm{Hom}_{\mathcal{D}}\left(F\left(C\right), D\right) \cong \mathrm{Hom}_{\mathcal{C}}\left(C, G\left(D\right)\right) \text{ \textit{naturally}.}$$

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In other words, there exists a natural isomorphism η:

$$\begin{tikzcd}
\mathcal{C}^{\mathrm{op}} \times \mathcal{D} \arrow[r, bend left=50, "\mathrm{Hom}_{\mathcal{D}}(F(-),-)"{name=U}] \arrow[r, bend right=50, "\mathrm{Hom}_{\mathcal{C}}(-,G(-))"'{name=L}] & \textbf{Sets} \arrow[from=U, to=L, Rightarrow, "\eta"]
\end{tikzcd}$$

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If this happens, we call F a **left adjoint** of G; and we call G a **right adjoint** of F. We write this as $F \dashv G$.

Tensor product of vector spaces

The **tensor product** of two $\mathbb{K}$-vector spaces U and V is another $\mathbb{K}$-vector space $U \otimes V$ equipped with a bilinear map $\theta : U \times V \to U \otimes V$ that is *universal*:

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$$\begin{tikzcd} U \times V \arrow[r, "\theta"] \arrow[d, "\forall \text{ bilinear } \beta"'] & U \otimes V \arrow[dl, dashed, "\exists!\, \alpha"] \\ W & \end{tikzcd}$$

In other words, $\beta = \alpha \circ \theta$.

Adjoints Example

So we have a 1-1 correspondence

$$\{\text{linear maps } U \otimes V \to W\} \leftrightarrow \{\text{bilinear maps } U \times V \to W\}.$$

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((\mathbf{u}, \mathbf{v}) \mapsto g\,(\mathbf{u})\,(\mathbf{v})) &\leftarrow\!\shortmid (g : U \to \mathrm{Hom}(V, W))\,.
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In other words,

$$\operatorname{Hom}\left(U \otimes V, W\right) \cong \operatorname{Hom}\left(U, \operatorname{Hom}\left(V, W\right)\right).$$

Does that mean $- \otimes V$ and $\operatorname{Hom}(V, -) : \mathbf{Vect}_{\mathbb{K}} \to \mathbf{Vect}_{\mathbb{K}}$ are adjoint functors?

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Does that mean $- \otimes V$ and $\mathrm{Hom}(V, -) : \mathbf{Vect}_{\mathbb{K}} \to \mathbf{Vect}_{\mathbb{K}}$ are adjoint functors? Well, not yet. We need to show the naturality of this isomorphism.

$$\begin{tikzcd}
\mathbf{Vect}_{\mathbb{K}}^{\mathrm{op}} \times \mathbf{Vect}_{\mathbb{K}} \arrow[rr, bend left, "\mathrm{Hom}_{\mathbf{Vect}_{\mathbb{K}}}(- \otimes V, -)"] \arrow[rr, bend right, "\mathrm{Hom}_{\mathbf{Vect}_{\mathbb{K}}}(-, \mathrm{Hom}(V, -))"'] \arrow[rr, phantom, "\Downarrow \eta"] & & \mathbf{Sets}
\end{tikzcd}$$

Adjoints Example

Given an arrow $(\alpha_1^{\text{op}}, \alpha_2) : (U, W) \to (U', W)$ in $\mathbf{Vect}_{\mathbb{K}}^{\text{op}} \times \mathbf{Vect}_{\mathbb{K}}$, we need to show the commutativity of the following diagram in the category **Sets**:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{Vect}_{\mathbb{K}}}(U \otimes V, W) \arrow[r, "\eta_{(U,W)}"] \arrow[d, "F(\alpha_1^{\text{op}}, \alpha_2)"'] & \mathrm{Hom}_{\mathbf{Vect}_{\mathbb{K}}}(U, \mathrm{Hom}(V, W)) \arrow[d, "G(\alpha_1^{\text{op}}, \alpha_2)"] \\
\mathrm{Hom}_{\mathbf{Vect}_{\mathbb{K}}}(U' \otimes V, W') \arrow[r, "\eta_{(U',W')}"'] & \mathrm{Hom}_{\mathbf{Vect}_{\mathbb{K}}}(U', \mathrm{Hom}(V, W'))
\end{tikzcd}$$

where $F = \mathrm{Hom}_{\mathbf{Vect}_{\mathbb{K}}}(- \otimes V, -)$ and $G = \mathrm{Hom}_{\mathbf{Vect}_{\mathbb{K}}}(-, \mathrm{Hom}(V, -))$.

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\end{tikzcd}$$

where $F = \mathrm{Hom}_{\mathbf{Vect}_{\mathbb{K}}}(- \otimes V, -)$ and $G = \mathrm{Hom}_{\mathbf{Vect}_{\mathbb{K}}}(-, \mathrm{Hom}(V, -))$.
And this diagram indeed commutes! So $- \otimes V$ is the left adjoint of $\mathrm{Hom}(V, -)$.

Adjoints

Are adjoints unique? Can a functor $F : \mathcal{C} \to \mathcal{D}$ have two right adjoints $G_1, G_2 : \mathcal{D} \to \mathcal{C}$?

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$$\mathrm{Hom}_{\mathcal{C}} \left(C, G_1 \left(D \right) \right) \cong \mathrm{Hom}_{\mathcal{D}} \left(F \left(C \right), D \right)$$

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$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(C, G_1(D)) &\cong \mathrm{Hom}_{\mathcal{D}}(F(C), D) \\ &\cong \mathrm{Hom}_{\mathcal{C}}(C, G_2(D)). \end{aligned}$$

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$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}}(C, G_1(D)) &\cong \mathrm{Hom}_{\mathcal{D}}(F(C), D)\\ &\cong \mathrm{Hom}_{\mathcal{C}}(C, G_2(D)).\end{aligned}$$

Therefore,

$$\mathrm{Hom}_{\mathcal{C}}(C, G_1(D)) \cong \mathrm{Hom}_{\mathcal{C}}(C, G_2(D)).$$

Does this mean G_1 and G_2 are isomorphic functors?

Yoneda Lemma

Theorem 6 (Yoneda Lemma)

For any functor $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Sets}$ and any $X \in \mathcal{C}_0$, the natural transformations $\mathrm{Hom}_{\mathcal{C}}(-, X) \Rightarrow F$ are in bijection with the elements of the set $F(X)$. In other words,

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Sets})}\left(\mathrm{Hom}_{\mathcal{C}}(-, X), F\right) \cong F(X),$$

and this isomorphism is natural in both F and X.

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and this isomorphism is natural in both F and X.

The last line means that we have a natural isomorphism

$$\begin{tikzcd}
\mathcal{C}^{\mathrm{op}} \times \mathrm{Fun}\left(\mathcal{C}^{\mathrm{op}}, \mathbf{Sets}\right) \arrow[rr, bend left, "{(X,F) \mapsto \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Sets})}(\mathrm{Hom}_{\mathcal{C}}(-,X),F)}"] \arrow[rr, bend right, "{(X,F) \mapsto F(X)}"'] \arrow[rr, phantom, "\Downarrow \eta"] & & \mathbf{Sets}
\end{tikzcd}$$

What does it even mean?

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Definition 7

A functor $F : \mathcal{C} \to \mathcal{D}$ is *fully faithful* if the set-functions

$$F_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$$

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are bijective for all $X, Y \in \mathcal{C}_0$.

Theorem 7

Let $F : \mathcal{C} \to \mathcal{D}$ be a fully faithful functor. Then $X \cong Y$ in $\mathcal{C}$ if and only if $F(X) \cong F(Y)$ in $\mathcal{D}$.

Yoneda Lemma

Corollary 8

The **Yoneda embedding**

$$\begin{aligned} \mathscr{Y} : \mathcal{C} &\to \mathrm{Fun}\left(\mathcal{C}^{\mathrm{op}}, \mathbf{Sets}\right) \\ X &\mapsto \mathrm{Hom}\left(-, X\right) \\ (f : X \to Y) &\mapsto \left(f_* : \mathrm{Hom}_{\mathcal{C}}\left(-, X\right) \to \mathrm{Hom}_{\mathcal{C}}\left(-, Y\right)\right) \end{aligned}$$

is *fully faithful*.

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is *fully faithful*.

Proof.

Take $F = \mathrm{Hom}_{\mathcal{C}}\left(-, Y\right)$ in Yoneda Lemma. This gives us

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Sets})}\left(\mathrm{Hom}_{\mathcal{C}}(-, X), \mathrm{Hom}_{\mathcal{C}}(-, Y)\right) \cong \mathrm{Hom}_{\mathcal{C}}\left(X, Y\right).$$

Isomorphism in sets is bijection!! ■

Yoneda Lemma

Corollary 9

In a category $\mathcal{C}$, $X \cong Y$ if and only if $\mathrm{Hom}_{\mathcal{C}}(-, X)$ and $\mathrm{Hom}_{\mathcal{C}}(-, Y)$ are naturally isomorhic functors.

Yoneda Lemma

Corollary 9

In a category $\mathcal{C}$, $X \cong Y$ if and only if $\mathrm{Hom}_{\mathcal{C}}(-, X)$ and $\mathrm{Hom}_{\mathcal{C}}(-, Y)$ are naturally isomorhic functors.

Proof.

Because Yoneda embedding $X \mapsto \mathrm{Hom}_{\mathcal{C}}(-, X)$ is fully faithful!!! ∎

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Proof.

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We usually use this variant of Yoneda lemma to prove isomorphisms.

Yoneda Lemma Summarized

Interchanging $\mathcal{C}$ and $\mathcal{C}^{\mathsf{op}}$ throughout, we get the following result:

Corollary 10

In a category $\mathcal{C}$, $X \cong Y$ if and only if $\mathrm{Hom}_{\mathcal{C}}(X, -)$ and $\mathrm{Hom}_{\mathcal{C}}(Y, -)$ are naturally isomorhic functors.

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$\blacksquare$

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Proof.

$$X \cong Y \text{ in } \mathcal{C} \iff X \cong Y \text{ in } \mathcal{C}^{\text{op}}$$

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Proof.

$$\begin{aligned} X \cong Y \text{ in } \mathcal{C} &\iff X \cong Y \text{ in } \mathcal{C}^{\text{op}} \\ &\iff \text{Hom}_{\mathcal{C}^{\text{op}}}(-, X) \cong \text{Hom}_{\mathcal{C}^{\text{op}}}(-, Y) \end{aligned}$$

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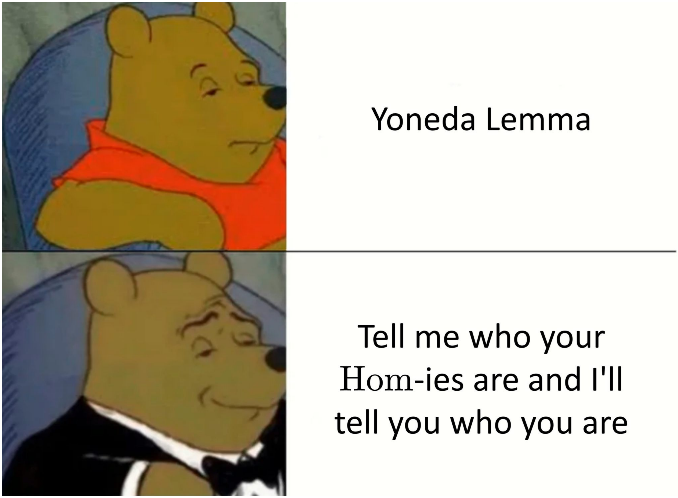
$$\begin{aligned} X \cong Y \text{ in } \mathcal{C} &\iff X \cong Y \text{ in } \mathcal{C}^{\mathrm{op}} \\ &\iff \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(-, X) \cong \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(-, Y) \\ &\iff \mathrm{Hom}_{\mathcal{C}}(X, -) \cong \mathrm{Hom}_{\mathcal{C}}(Y, -). \end{aligned}$$

■

Yoneda Lemma Summarized



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2. *Category Theory in Context*, by Emily Riehl
3. *Basic Category Theory*, by Tom Leinster
4. *Categories for the Working Mathematician*, by Saunders Mac Lane
5. Math3ma blog: `https://www.math3ma.com/blog/the-yoneda-lemma`
6. Ncatlab: `https://ncatlab.org/nlab/show/Yoneda+lemma`

Thank you for joining!

The slides are available in my webpage
https://atonurc.github.io/assets/catrep_talk_1.pdf