Category Theory (MAT434)

Lecture Notes

Preface

This series of lecture notes has been prepared for aiding students who took the BRAC University course **Category Theory (MAT434)** in Summer 2023 semester. These notes were typeset under the supervision of mathematician **Dr. Syed Hasibul Hassan Chowdhury**. The main goal of this typeset is to have an organized digital version of the notes, which is easier to share and handle. If you see any mistakes or typos, please send me an email at atonuroychowdhury@gmail.com

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References:

- *Category Theory*, by **Steve Awodey**.
- *Category Theory for Scientists*, by **David Spivak**.
- *Categories for the Working Mathematician*, by **Saunders Mac Lane**.
- *Basic Category Theory*, by **Tom Leinster**.

Contents

1 Basic Definitions

Category theory arises from the idea of a system of "functions" among some objects.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g\circ f"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

A category consists of objects $A, B, C, \ldots$ and arrows $f : A \to B, g : B \to C, \ldots$ that are closed under composition and satisfy certain conditions typical of composition of functions. Before formally defining what a category is, let us begin our discussion with the setting where the objects are sets and arrows are functions between sets.

Let f be a function from a set A to another set B. This is mathematically expressed as $f : A \to B$. Explicitly, it refers to the fact that f is defined for all of A, and all the values of f are contained in B. In other words, $\mathrm{range}\,(f) \subseteq B$.

Now suppose we have another function $g : B \to C$. Then there is a unique function $g \circ f : A \to C$, given by

$$(g \circ f)(a) = g(f(a)), \qquad \text{for } a \in A. \tag{1.1}$$

This unique function is called the composite of g and f, or g after f.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g\circ f"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

Now, this operation $\circ$ of composition of functions is associative. In other words, the two arrows from A to D in the following diagram are the same:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] \arrow[rrr, bend left=50, "(h\circ g)\circ f"] \arrow[rrr, bend right=50, "h\circ(g\circ f)"'] & B \arrow[r, "g"] \arrow[rr, bend left, "h\circ g"] & C \arrow[r, "h"] & D \end{tikzcd}$$

Given $f : A \to B$, $g : B \to C$ and $h : C \to D$, one has unique compositions $h \circ g : B \to D$ and $g \circ f : A \to C$. These two composed functions can be further composed with f (from the left) and with h (from the right), respectively, to yield a unique function

$$(h \circ g) \circ f = h \circ (g \circ f), \tag{1.2}$$

from A to D as demanded by the associativity law. Using the definition of composition of functions, one verifies that this is indeed the case:

$$\begin{aligned} ((h \circ g) \circ f)(a) &= (h \circ g)(f(a)) = h(g(f(a))), \\ (h \circ (g \circ f))(a) &= h((g \circ f)(a)) = h(g(f(a))). \end{aligned}$$

Therefore, $(h \circ g) \circ f = h \circ (g \circ f)$.

Finally, note that for every set A, there is an identity function $1_A : A \to A$ given by

$$1_A(a) = a. \tag{1.3}$$

These identity functions act as units for composition, i.e. given $f : A \to B$, we have

$$\begin{aligned}(f \circ 1_A)(a) &= f(1_A(a)) = f(a), \\ (1_B \circ f)(a) &= 1_B(f(a)) = f(a),\end{aligned}$$

for each $a \in A$. Therefore,

$$f \circ 1_A = 1_B \circ f = f. \tag{1.4}$$

The equality above is equivalent to the following commutative diagram:

$$\begin{tikzcd} A \arrow[r, "1_A"] \arrow[dr, "f \circ 1_A"'] & A \arrow[d, "f"'] \arrow[dr, "1_B \circ f"] & \\ & B \arrow[r, "1_B"'] & B \end{tikzcd}$$

We have the following abstract version of sets and functions between sets called a **category**.

Definition 1.1 (Category). A **category** $\mathcal{C}$ consists of the following data:

- **Objects:** $A, B, C, \ldots$. The collection of objects of $\mathcal{C}$ is denoted by $\mathrm{Ob}(\mathcal{C})$.
- **Arrows:** $f, g, h, \ldots$. Given two objects A and B, the set of arrows from A to B is denoted by $\mathrm{Hom}_{\mathcal{C}}(A, B)$.
- For each arrow f, there are given objects $\mathrm{dom}(f), \mathrm{cod}(f)$ called the **domain** and **codomain** of f. We write $f : A \to B$ to indicate that $A = \mathrm{dom}(f)$ and $B = \mathrm{cod}(f)$.
- Given arrows $f : A \to B$ and $g : B \to C$, i.e. with $\mathrm{cod}(f) = \mathrm{dom}(g)$, there is a unique arrow $g \circ f : A \to C$, i.e. $g \circ f \in \mathrm{Hom}_{\mathcal{C}}(A, C)$ called the **composite** of f and g. This fact can be rephrased as the following: given $A, B, C \in \mathrm{Ob}(\mathcal{C})$, there is a function

$$\circ : \mathrm{Hom}_{\mathcal{C}}(B, C) \times \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(A, C), \tag{1.5}$$

  with $(g, f) \mapsto g \circ f$. The well-definedness of $\circ$ is synonymous to claiming that $g \circ f \in \mathrm{Hom}_{\mathcal{C}}(A, C)$ is unique for given $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$ and $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$.
- For each $A \in \mathrm{Ob}(\mathcal{C})$, there exists an unique arrow $1_A \in \mathrm{Hom}_{\mathcal{C}}(A, A)$.

The above data are required to satisfy the following laws:

- **Associativity:** For any $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$, $h \in \mathrm{Hom}_{\mathcal{C}}(C, D)$ with $A, B, C, D \in \mathrm{Ob}(\mathcal{C})$,

$$(h \circ g) \circ f = h \circ (g \circ f), \tag{1.6}$$

- **Unit:** For any $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ with $A, B \in \mathrm{Ob}(\mathcal{C})$,

$$f \circ 1_A = 1_B \circ f = f. \tag{1.7}$$

Remark 1.1. Suppose we have the following commutative diagram:

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[dr, "h \circ f = h \circ g"'] & B \arrow[d, "h"] \\ & C \end{tikzcd}$$

Commutativity of this diagram doesn't violate the uniqueness of the composition $\circ$. It just means that the map $\circ$ in (1.5) is a many-to-one function.