



Inspiring Excellence

## **Algebraic Topology III (MAT484)**

**Lecture Notes**

# Preface

This series of lecture notes has been prepared for aiding students who took the BRAC University course **Algebraic Topology III (MAT484)** in Spring 2023 semester. These notes were typeset under the supervision of mathematician **Dr. Syed Hasibul Hassan Chowdhury**. The main goal of this typeset is to have an organized digital version of the notes, which is easier to share and handle. Lecture notes of the previous Algebraic Topology courses can be found in the following links.

- **Algebraic Topology I (MAT431)**: [https://atonurc.github.io/assets/MAT431\\_AT1.pdf](https://atonurc.github.io/assets/MAT431_AT1.pdf)
- **Algebraic Topology II (MAT432)**: [https://atonurc.github.io/assets/MAT432\\_AT2.pdf](https://atonurc.github.io/assets/MAT432_AT2.pdf)

If you see any mistakes or typos, please send me an email at [atonuroychowdhury@gmail.com](mailto:atonuroychowdhury@gmail.com)

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## References:

- *Elements of Algebraic Topology*, by James R. Munkres
- *Foundations of Algebraic Topology*, by Samuel Eilenberg & Norman E. Steenrod
- *Axiomatic Approach to Homology Theory*, by Samuel Eilenberg & Norman E. Steenrod. Link to the paper: <https://www.pnas.org/content/pnas/31/4/117.full.pdf>

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# 1 Singular Homology Theory

## §1.1 Singular Homology Groups

Let  $\mathbb{R}^\infty$  denote the generalized Euclidean space  $\mathbb{E}^J$ , with  $J$  being the set of positive integers. An element of the vector space  $\mathbb{R}^\infty$  is an infinite sequence of real numbers (functions from  $\mathbb{N}$  to  $\mathbb{R}$ ) with finitely many nonzero entries. Let  $\Delta_p$  denote the  $p$ -simplex in  $\mathbb{R}^\infty$  having vertices

$$\begin{aligned}\varepsilon_0 &= (1, 0, 0, \dots, 0, \dots), \\ \varepsilon_1 &= (0, 1, 0, \dots, 0, \dots), \\ &\dots \\ \varepsilon_p &= (0, 0, 0, \dots, \underbrace{1}_{(p+1)\text{-th entry}}, \dots).\end{aligned}$$

We call  $\Delta_p$  the **standard  $p$ -simplex**. In this notation,  $\Delta_{p-1}$  is a face of  $\Delta_p$ .

**Definition 1.1** (Singular  $p$ -simplex). Let  $X$  be a topological space. We define a **singular  $p$ -simplex** of  $X$  to be a continuous map  $T : \Delta_p \rightarrow X$ . The free abelian group generated by singular  $p$ -simplices of  $X$  is denoted by  $S_p(X)$ , and is called the **singular chain group** of  $X$  in dimension  $p$ . We shall denote an element of  $S_p(X)$  by a  $\mathbb{Z}$ -linear combination of singular  $p$ -simplices of  $X$ .

Singular means that  $T$  could be a “bad” map, i.e. it may not be an imbedding. All we want that  $T$  is just continuous. Now, recall that

$$\Delta_p = \left\{ (x_0, x_1, \dots, x_p, 0, \dots) \in \mathbb{R}^\infty \mid 0 \leq x_i \leq 1 \text{ and } \sum_{i=0}^p x_i = 1 \right\}. \quad (1.1)$$

Given  $a_0, a_1, \dots, a_p \in \mathbb{R}^\infty$ , there is a unique affine map  $l_{(a_0, \dots, a_p)} : \Delta_p \rightarrow \mathbb{R}^\infty$  that maps  $\varepsilon_i$  to  $a_i$ . It is defined by

$$\begin{aligned}l_{(a_0, \dots, a_p)}(x_0, x_1, \dots, x_p, 0, \dots) &= \sum_{i=0}^p x_i a_i = \sum_{i=0}^p x_i a_i + a_0 - \sum_{i=0}^p x_i a_0 \\ &= a_0 + \sum_{i=0}^p x_i (a_i - a_0).\end{aligned} \quad (1.2)$$

We call this map the **linear singular simplex** determined by  $a_0, a_1, \dots, a_p \in \mathbb{R}^\infty$ . Now, what is  $l_{(\varepsilon_0, \dots, \varepsilon_p)}$ ? Observe that

$$l_{(\varepsilon_0, \dots, \varepsilon_p)} \varepsilon_i = l_{(\varepsilon_0, \dots, \varepsilon_p)}(0, \dots, 0, \underbrace{1}_{(i+1)\text{-th entry}}, 0, \dots) = \varepsilon_i. \quad (1.3)$$

Therefore,  $l_{(\varepsilon_0, \dots, \varepsilon_p)}$  maps  $\varepsilon_i$  to itself, for every  $i = 0, 1, \dots, p$ . Also,

$$l_{(\varepsilon_0, \dots, \varepsilon_p)}(x_0, x_1, \dots, x_p, 0, \dots) = \sum_{i=0}^p x_i \varepsilon_i = (x_0, x_1, \dots, x_p, 0, \dots). \quad (1.4)$$

Therefore,  $l_{(\varepsilon_0, \dots, \varepsilon_p)}$  is just the inclusion map of  $\Delta_p$  into  $\mathbb{R}^\infty$ . Now, suppose  $(x_0, x_1, \dots, x_{p-1}, 0, \dots) \in \Delta_{p-1}$ , so that  $\sum_{i=0}^{p-1} x_i = 1$ . Then

$$\begin{aligned}l_{(\varepsilon_0, \dots, \widehat{\varepsilon_i}, \dots, \varepsilon_p)}(x_0, x_1, \dots, x_{p-1}, 0, \dots) &= x_0 \varepsilon_0 + \dots + x_{i-1} \varepsilon_{i-1} + 0 \cdot \varepsilon_i + x_{i+1} \varepsilon_{i+1} + \dots + x_{p-1} \varepsilon_p \\ &= (x_0, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_{p-1}, 0, \dots),\end{aligned} \quad (1.5)$$

which is a point on the face of  $\Delta_p$  opposite to the vertex  $\varepsilon_i$ . In fact,  $l_{(\varepsilon_0, \dots, \widehat{\varepsilon}_i, \dots, \varepsilon_p)}$  is a linear homomorphism of  $\Delta_{p-1}$  into the face of  $\Delta_p$  that is opposite to the vertex  $\varepsilon_i$ . In other words,

$$l_{(\varepsilon_0, \dots, \widehat{\varepsilon}_i, \dots, \varepsilon_p)} : \Delta_{p-1} \rightarrow \Delta_p$$

maps  $\Delta_{p-1}$  to the face of  $\Delta_p$  opposite to the vertex  $\varepsilon_i$ . Therefore, given a singular  $p$ -simplex  $T : \Delta_p \rightarrow X$ , one can form the composite

$$T \circ l_{(\varepsilon_0, \dots, \widehat{\varepsilon}_i, \dots, \varepsilon_p)} : \Delta_{p-1} \rightarrow X,$$

which is a singular  $(p-1)$ -simplex. We think of it as the  $i$ -th face of the singular  $p$ -simplex  $T$ .

**Definition 1.2** (Boundary homomorphism). We define  $\partial : S_p(X) \rightarrow S_{p-1}(X)$  as follows. If  $T : \Delta_p \rightarrow X$  is a singular  $p$ -simplex, we define  $\partial T$  to be

$$\partial T = \sum_{i=0}^p (-1)^i T \circ l_{(\varepsilon_0, \dots, \widehat{\varepsilon}_i, \dots, \varepsilon_p)}. \quad (1.6)$$

In other words,  $\partial T$  is a formal sum of singular simplices of dimension  $p-1$ , which are the faces of  $T$ .

**Remark 1.1 (IMPORTANT!).** Note that only the singular  $p$ -simplices are maps, not the singular  $p$ -chains. The  $p$ -chains are just formal sum of continuous maps from  $\Delta_p$  to  $X$ . If  $T_1$  and  $T_2$  are two singular  $p$ -simplices, i.e. continuous maps  $\Delta_p \rightarrow X$ , then  $T_1 + T_2$  is **NOT** a map. The sum present here is nothing but a formal notation. So one cannot act  $T_1 + T_2$  on a point of  $\Delta_p$ . For the same reason,  $\partial T_1$  is not a map. It is merely a formal linear combination of the continuous maps  $T \circ l_{(\varepsilon_0, \dots, \widehat{\varepsilon}_i, \dots, \varepsilon_p)}$ .

If  $f : X \rightarrow Y$  is a continuous map, we define a group homomorphism  $f_{\#} : S_p(X) \rightarrow S_p(Y)$  by defining it on singular  $p$ -simplices by the equation

$$f_{\#}(T) = f \circ T \quad (1.7)$$

for a singular  $p$ -simplex  $T$ .

$$\begin{array}{ccccc} \Delta_p & \xrightarrow{T} & X & \xrightarrow{f} & Y \\ & \searrow & \text{f} \circ \text{T} & \nearrow & \\ & & & & \end{array}$$

### Theorem 1.1

The homomorphism  $f_{\#}$  commutes with  $\partial$ . Furthermore,  $\partial^2 = 0$ .

*Proof.* Given a singular  $p$ -simplex  $T$ ,

$$\partial f_{\#}(T) = \partial(f \circ T) = \sum_{i=0}^p (-1)^i (f \circ T) \circ l_{(\varepsilon_0, \dots, \widehat{\varepsilon}_i, \dots, \varepsilon_p)}. \quad (1.8)$$

$$f_{\#}(\partial T) = f_{\#}\left(\sum_{i=0}^p (-1)^i T \circ l_{(\varepsilon_0, \dots, \widehat{\varepsilon}_i, \dots, \varepsilon_p)}\right) = \sum_{i=0}^p (-1)^i f \circ T \circ l_{(\varepsilon_0, \dots, \widehat{\varepsilon}_i, \dots, \varepsilon_p)}. \quad (1.9)$$

Therefore,  $\partial f_{\#}(T) = f_{\#}(\partial T)$ . Now, to prove  $\partial^2 = 0$ , we first compute  $\partial$  for linear singular simplices  $l_{(a_0, \dots, a_p)}$ .

$$\partial l_{(a_0, \dots, a_p)} = \sum_{i=0}^p (-1)^i l_{(a_0, \dots, a_p)} \circ l_{(\varepsilon_0, \dots, \widehat{\varepsilon}_i, \dots, \varepsilon_p)}. \quad (1.10)$$

Observe that

$$\begin{aligned} l_{(a_0, \dots, a_p)} \circ l_{(\varepsilon_0, \dots, \widehat{\varepsilon_i}, \dots, \varepsilon_p)}(x_0, \dots, x_{p-1}, 0, \dots) &= l_{(a_0, \dots, a_p)}(x_0, \dots, x_{i-1}, 0, x_i x_{p-1}, 0, \dots) \\ &= x_0 a_0 + \dots + x_{i-1} a_{i-1} + 0 \cdot a_i + x_i a_{i+1} + \dots + x_{p-1} a_p \\ &= l_{(a_0, \dots, \widehat{a_i}, \dots, a_p)}(x_0, \dots, x_{p-1}, 0, \dots). \end{aligned} \quad (1.11)$$

Hence,

$$l_{(a_0, \dots, a_p)} \circ l_{(\varepsilon_0, \dots, \widehat{\varepsilon_i}, \dots, \varepsilon_p)} = l_{(a_0, \dots, \widehat{a_i}, \dots, a_p)}. \quad (1.12)$$

Therefore, from 1.10, it follows that

$$\partial l_{(a_0, \dots, a_p)} = \sum_{i=0}^p (-1)^i l_{(a_0, \dots, \widehat{a_i}, \dots, a_p)}. \quad (1.13)$$

Let's now evaluate  $\partial \partial l_{(a_0, \dots, a_p)}$ .

$$\begin{aligned} \partial \partial l_{(a_0, \dots, a_p)} &= \sum_{i=0}^p (-1)^i \partial l_{(a_0, \dots, \widehat{a_i}, \dots, a_p)} \\ &= \sum_{i=0}^p (-1)^i \sum_{j < i} (-1)^j l_{(a_0, \dots, \widehat{a_j}, \dots, \widehat{a_i}, \dots, a_p)} + \sum_{i=0}^p (-1)^i \sum_{j > i} (-1)^{j-1} l_{(a_0, \dots, \widehat{a_i}, \dots, \widehat{a_j}, \dots, a_p)} \\ &= \sum_{i=0}^p \sum_{j < i} (-1)^{i+j} l_{(a_0, \dots, \widehat{a_j}, \dots, \widehat{a_i}, \dots, a_p)} - \sum_{i=0}^p \sum_{j > i} (-1)^{i+j} l_{(a_0, \dots, \widehat{a_i}, \dots, \widehat{a_j}, \dots, a_p)}. \end{aligned} \quad (1.14)$$

Now fix  $0 \leq j_0 < i_0 \leq p$ . In the first summand of 1.14, the contribution of  $i = i_0, j = j_0$  is

$$(-1)^{i_0+j_0} l_{(a_0, \dots, \widehat{a_{j_0}}, \dots, \widehat{a_{i_0}}, \dots, a_p)}. \quad (1.15)$$

On the other hand, in the second summand of 1.14, the contribution of  $i = j_0, j = i_0$  is also

$$(-1)^{i_0+j_0} l_{(a_0, \dots, \widehat{a_{j_0}}, \dots, \widehat{a_{i_0}}, \dots, a_p)}. \quad (1.16)$$

These two contributions cancel each other. This way, one finds that the RHS of 1.14 vanishes. Hence,

$$\partial \partial l_{(a_0, \dots, a_p)} = 0. \quad (1.17)$$

In particular,

$$\partial \partial l_{(\varepsilon_0, \dots, \varepsilon_p)} = 0. \quad (1.18)$$

Now,  $l_{(\varepsilon_0, \dots, \varepsilon_p)} : \Delta_p \rightarrow \Delta_p$  is continuous, so  $l_{(\varepsilon_0, \dots, \varepsilon_p)} \in S_p(\Delta_p)$ . Furthermore, it is the identity map as we have seen in 1.4. Since  $T : \Delta_p \rightarrow X$  is continuous, we can form  $T_{\#} : S_p(\Delta_p) \rightarrow S_p(X)$ .

$$T_{\#}(l_{(\varepsilon_0, \dots, \varepsilon_p)}) = T \circ l_{(\varepsilon_0, \dots, \varepsilon_p)} = T \circ \text{id}_{\Delta_p} = T. \quad (1.19)$$

Therefore, using the fact that  $T_{\#}$  commutes with  $\partial$ , we obtain

$$\partial \partial T = \partial \partial T_{\#}(l_{(\varepsilon_0, \dots, \varepsilon_p)}) = T_{\#}(\partial \partial l_{(\varepsilon_0, \dots, \varepsilon_p)}) = 0. \quad (1.20)$$

Hence,  $\partial^2 T = 0$ . ■

**Definition 1.3** (Singular homology groups). The family of groups  $S_p(X)$  and homomorphisms  $\partial_p : S_p(X) \rightarrow S_{p-1}(X)$  is called **singular chain complex** of  $X$ , and is denoted by  $\mathcal{S}(X)$ .

$$\cdots \longrightarrow S_{p+1}(X) \xrightarrow{\partial_{p+1}} S_p(X) \xrightarrow{\partial_p} S_{p-1}(X) \longrightarrow \cdots$$

The homology groups of this chain complex are called the **singular homology groups** of  $X$ , and are denoted by  $H_p(X)$ .

**Definition 1.4** (Augmentation map). The chain complex  $\mathcal{S}(X)$  is augmented by the homomorphism  $\epsilon : S_0(X) \rightarrow \mathbb{Z}$  defined by setting  $\epsilon(T) = 1$  for each singular 0-simplex  $T : \Delta_0 \rightarrow X$ . (A generic singular 0-chain is a  $\mathbb{Z}$ -linear combination of singular 0-simplices.)

It's immediate that if  $T$  is a singular 1-simplex, then  $\epsilon(\partial T) = 0$ . Indeed,

$$\epsilon(\partial T) = \epsilon(T \circ l_{(\widehat{\epsilon}_0, \epsilon_1)}) - \epsilon(T \circ l_{(\epsilon_0, \widehat{\epsilon}_1)}) = 0. \quad (1.21)$$

**Definition 1.5** (Reduced homology groups). The homology groups of  $\{\mathcal{S}(X), \epsilon\}$  are called the **reduced singular homology groups** of  $X$ , and are denoted by  $\tilde{H}_p(X)$ .

Now, given continuous map  $f : X \rightarrow Y$  and  $T : \Delta_0 \rightarrow X$  a singular 0-simplex on  $X$ , then  $f_{\#}(T) = f \circ T : \Delta_0 \rightarrow Y$ .

$$\begin{array}{ccccc} \Delta_0 & \xrightarrow{T} & X & \xrightarrow{f} & Y \\ & \searrow & & \nearrow & \\ & & f \circ T & & \end{array}$$

Now, consider the augmented singular chain complexes  $\{\mathcal{S}(X), \epsilon^X\}$  and  $\{\mathcal{S}(Y), \epsilon^Y\}$ . Noting continuous  $T : \Delta_0 \rightarrow X$  and  $f_{\#}(T) : \Delta_0 \rightarrow Y$ , one obtains  $\epsilon^X(T) = 1$  and  $\epsilon^Y(f_{\#}(T)) = 1$ . In other words, the following diagram commutes

$$\begin{array}{ccc} S_0(X) & \xrightarrow{\epsilon^X} & \mathbb{Z} \\ (f_{\#})_0 \downarrow & & \downarrow \text{id} \\ S_0(Y) & \xrightarrow{\epsilon^Y} & \mathbb{Z} \end{array}$$

Therefore,  $f_{\#} : S_p(X) \rightarrow S_p(Y)$  is an **augmentation preserving chain map** between  $\{\mathcal{S}(X), \epsilon^X\}$  and  $\{\mathcal{S}(Y), \epsilon^Y\}$ . Thus,  $f_{\#}$  induces a homomorphism  $f_*$  in both ordinary and reduced singular homology.

In [Theorem 1.1](#), we saw that the chain map  $f_{\#}$  commutes with the boundary operator  $\partial$ . In other words,  $(f_{\#})_p : S_p(X) \rightarrow S_p(Y)$  takes cycles to cycles and boundaries to boundaries. Suppose  $c_p \in Z_p(X) = \text{Ker } \partial_p^X$ , so that  $\partial_p^X c_p = 0$ . Now,

$$\partial_p^Y \left( (f_{\#})_p c_p \right) = (f_{\#})_{p-1} (\partial_p^X c_p) = 0. \quad (1.22)$$

Hence,  $(f_{\#})_p c_p \in Z_p(Y)$ . On the other hand, let  $b_p \in B_p(X) = \text{Im } \partial_{p+1}^X$ . Then  $b_p = \partial_{p+1}^X d_{p+1}$  for some  $d_{p+1} \in S_{p+1}(X)$ . Then

$$(f_{\#})_p b_p = (f_{\#})_p (\partial_{p+1}^X d_{p+1}) = \partial_{p+1}^Y \left( (f_{\#})_{p+1} d_{p+1} \right). \quad (1.23)$$

In other words,  $(f_{\#})_p b_p \in B_p(Y)$ . This reflects the fact that  $(f_{\#})_p : S_p(X) \rightarrow S_p(Y)$  induces a homomorphism between the singular homology groups  $(f_*)_p : H_p(X) \rightarrow H_p(Y)$ .  $(f_*)_p$  is given by

$$(f_*)_p (c_p + B_p(X)) = (f_{\#})_p c_p + B_p(Y). \quad (1.24)$$

If the reduced homology groups of  $X$  vanishes in all dimensions, we say that  $X$  is **acyclic** (in singular homology).

**Theorem 1.2**

If  $i : X \rightarrow X$  is the identity, then so is  $(i_*)_p : H_p(X) \rightarrow H_p(X)$ . If  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  are continuous, then  $((g \circ f)_*)_p = (g_*)_p \circ (f_*)_p$ .

*Proof.* It is sufficient to show that the equations hold at the chain level. We know from the definition of  $(f_\#)_p : S_p(X) \rightarrow S_p(Y)$  that it maps  $T \in S_p(X)$  to  $f \circ T \in S_p(Y)$ . Since  $i : X \rightarrow X$  is the identity map,

$$(i_\#)_p(T) = i \circ T = T. \quad (1.25)$$

So  $(i_\#)_p : S_p(X) \rightarrow S_p(X)$  is the identity homomorphism. As a result,

$$(i_*)_p(c_p + B_p(X)) = (i_\#)_p c_p + B_p(X) = c_p + B_p(X). \quad (1.26)$$

Therefore,  $(i_*)_p = \text{id}_{H_p(X)}$ .

Given continuous  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$ ,  $((g \circ f)_\#)_p : S_p(X) \rightarrow S_p(Z)$  is defined by

$$((g \circ f)_\#)_p T = (g \circ f) \circ T = g \circ (f \circ T) = (g_\#)_p((f_\#)_p T). \quad (1.27)$$

Therefore,  $((g \circ f)_\#)_p = (g_\#)_p \circ (f_\#)_p$ . Now, at the homology level, for  $c_p + B_p(X) \in H_p(X) = Z_p(X) / B_p(X)$

$$((g \circ f)_*)_p(c_p + B_p(X)) = ((g \circ f)_\#)_p c_p + B_p(Z) = (g_\#)_p((f_\#)_p c_p) + B_p(Z). \quad (1.28)$$

Also,

$$(g_*)_p \circ (f_*)_p(c_p + B_p(X)) = (g_\#)_p((f_\#)_p c_p + B_p(Y)) = (g_\#)_p((f_\#)_p c_p) + B_p(Z). \quad (1.29)$$

From 1.28 and 1.29, we can deduce that  $((g \circ f)_*)_p = (g_*)_p \circ (f_*)_p$ . ■

**Corollary 1.3**

If  $h : X \rightarrow Y$  is a homeomorphism, then  $(h_*)_p : H_p(X) \rightarrow H_p(Y)$  is an isomorphism.

*Proof.* Both  $h : X \rightarrow Y$  and  $h^{-1} : Y \rightarrow X$  are continuous, and  $h \circ h^{-1} = \text{id}_Y$ . Therefore,

$$(h_*)_p \circ ((h^{-1})_*)_p = ((h \circ h^{-1})_*)_p = ((\text{id}_Y)_*)_p = \text{id}_{H_p(Y)}. \quad (1.30)$$

Similarly, starting with  $h^{-1} \circ h = \text{id}_X$ , we will get  $((h^{-1})_*)_p \circ (h_*)_p = \text{id}_{H_p(X)}$ . Therefore,  $((h^{-1})_*)_p$  is the inverse of  $(h_*)_p$ . In other words,  $(h_*)_p$  is an invertible homomorphism, i.e. an isomorphism. ■

**Theorem 1.4**

Let  $X$  be a topological space. Then  $H_0(X)$  is free abelian. If  $\{X_\alpha\}$  is the collection of path components of  $X$ , and if  $T_\alpha$  is a singular 0-simplex with image in  $X_\alpha$  for each  $\alpha$ , then the homology classes of the chains  $T_\alpha$  form a basis for  $H_0(X)$ . The group  $\tilde{H}_0(X)$  is also free abelian; it vanishes if  $X$  is path connected. Otherwise, let  $\alpha_0$  be a fixed index, then the homology classes of the chains  $T_\alpha - T_{\alpha_0}$  for  $\alpha \neq \alpha_0$  form a basis for  $\tilde{H}_0(X)$ .



*Proof.* Let  $x_\alpha = T_\alpha(\Delta_0) \in X_\alpha$ , with  $T_\alpha : \Delta_0 \rightarrow X$  being a singular 0-simplex. Here,  $\Delta_0$  consists of the point  $\varepsilon_0 = (1, 0, 0, \dots) \in \mathbb{R}^\infty$ . Also, let  $T : \Delta_0 \rightarrow X$  be any singular 0-simplex such that  $T(\Delta_0) \in X_\alpha$ . Since  $X_\alpha$  is path connected, there is a path connecting  $T(\Delta_0)$  and  $T_\alpha(\Delta_0)$ . In other words, there is a singular 1-simplex  $f : \Delta_1 \rightarrow X$  such that

$$f(1, 0, 0, \dots) = T(\Delta_0) \text{ and } f(0, 1, 0, \dots) = T_\alpha(\Delta_0). \quad (1.31)$$

Then we have

$$\partial_1 f = f \circ l_{(\widehat{\varepsilon}_0, \varepsilon_1)} - f \circ l_{(\varepsilon_0, \widehat{\varepsilon}_1)}. \quad (1.32)$$

Now,

$$f \circ l_{(\varepsilon_0, \widehat{\varepsilon}_1)}(1, 0, 0, \dots) = f(1, 0, 0, \dots) = T(\Delta_0) = T(1, 0, 0, \dots), \quad (1.33)$$

$$f \circ l_{(\widehat{\varepsilon}_0, \varepsilon_1)}(1, 0, 0, \dots) = f(0, 1, 0, \dots) = T_\alpha(\Delta_0) = T_\alpha(1, 0, 0, \dots). \quad (1.34)$$

Therefore,  $\partial_1 f = T_\alpha - T$ .

An arbitrary singular 0-chain is a  $\mathbb{Z}$ -linear combination of singular 0-simplices. Let's take  $c \in S_0(X)$ . Then  $c = \sum_\beta m_\beta T'_\beta$ , with  $m_\beta \in \mathbb{Z}$  and  $T'_\beta$  being singular 0-simplices. Each  $T'_\beta(\Delta_0)$  belongs to some  $X_\alpha$ , and hence homologous to  $T_\alpha$ . Therefore,  $c$  is homologous to some  $\mathbb{Z}$ -linear combination  $\sum_\alpha n_\alpha T_\alpha$  of the  $T_\alpha$ 's. We will now show that no such nontrivial 0-chain  $\sum_\alpha n_\alpha T_\alpha$  bounds.

Assume the contrary that  $\sum_\alpha n_\alpha T_\alpha = \partial_1 d$  for some  $d \in S_1(X)$ . Now, the singular 1-chain  $d$  is a formal linear combination of singular 1-simplices with path connected image, i.e. the image lies in one of the path components  $X_\alpha$ . Thus we can write  $d = \sum_\alpha d_\alpha$ , where  $d_\alpha$  consists of the terms whose images are in  $X_\alpha$ . Therefore,

$$\sum_\alpha n_\alpha T_\alpha = \partial_1 d = \sum_\alpha \partial_1 d_\alpha. \quad (1.35)$$

Hence, we get

$$n_\alpha T_\alpha = \partial_1 d_\alpha \quad (1.36)$$

for each  $\alpha$ . Applying  $\epsilon$  to both sides of 1.36, we get

$$\epsilon(n_\alpha T_\alpha) = \epsilon(\partial_1 d_\alpha) \implies n_\alpha = 0. \quad (1.37)$$

Therefore, no non-trivial 0-chain  $\sum_\alpha n_\alpha T_\alpha$  bounds. Since every 0-chain is automatically a 0-cycle, an element of  $H_0(X)$  is homologous to a 0-chain of the form  $\sum_\alpha n_\alpha T_\alpha$ . Hence, the homology classes of the singular 0-simplices  $\{T_\alpha\}$  form a basis for the free abelian group  $H_0(X)$ .

$$S_1(X) \xrightarrow{\partial_1} S_0(X) \xrightarrow{\epsilon} \mathbb{Z}$$

$\tilde{H}_0(X)$  is defined as  $\tilde{H}_0(X) = \text{Ker } \epsilon / \text{Im } \partial_1$ . Given a singular 0-chain  $T \in S_0(X)$ , we've seen that  $T$  is homologous to a 0-chain of the form  $T' = \sum_\alpha n_\alpha T_\alpha$ ; and  $T'$  bounds iff  $T' = 0$ , i.e.  $n_\alpha = 0$  for every  $\alpha$ . If further  $T \in \text{Ker } \epsilon$ , then  $\epsilon(T) = 0$ . Since  $T$  and  $T'$  are homologous,  $T = T' + \partial_1 d$  for some  $d \in S_1(X)$ . Therefore,

$$0 = \epsilon(T) = \epsilon(T') + \epsilon(\partial_1 d) = \epsilon\left(\sum_\alpha n_\alpha T_\alpha\right) = \sum_\alpha n_\alpha. \quad (1.38)$$

If  $X$  is path connected, there is only one component, and hence there is only one  $n_\alpha$  involved. Thus  $n_\alpha = 0$  from 1.38. This gives us  $T' = 0$ , leading to the fact that every  $T \in \text{Ker } \epsilon$  is homologous to 0, i.e.  $T = 0 + \partial_1 d$  for some  $d \in S_1(X)$ . So  $\text{Ker } \epsilon = \text{Im } \partial_1$ . Therefore,  $\tilde{H}_0(X) = 0$ , when  $X$  is path connected.

Now, suppose  $X$  has more than one path components. Fix  $\alpha_0$ . Then from 1.38, we get

$$0 = \sum_\alpha n_\alpha = n_{\alpha_0} + \sum_{\alpha \neq \alpha_0} n_\alpha \implies n_{\alpha_0} = - \sum_{\alpha \neq \alpha_0} n_\alpha. \quad (1.39)$$

Then  $T'$  is

$$T' = \sum_{\alpha} n_{\alpha} T_{\alpha} = \sum_{\alpha \neq \alpha_0} n_{\alpha} T_{\alpha} + n_{\alpha_0} T_{\alpha_0} = \sum_{\alpha \neq \alpha_0} n_{\alpha} T_{\alpha} - \sum_{\alpha \neq \alpha_0} n_{\alpha} T_{\alpha_0} = \sum_{\alpha \neq \alpha_0} n_{\alpha} (T_{\alpha} - T_{\alpha_0}). \quad (1.40)$$

1.40 suggests that  $T'$  is a linear combination of the singular 0-chains  $\{T_{\alpha} - T_{\alpha_0}\}_{\alpha \neq \alpha_0}$ . And  $T'$  bounds iff it is trivial, as shown earlier. Therefore, the homology classes of 0-chains  $\{T_{\alpha} - T_{\alpha_0}\}_{\alpha \neq \alpha_0}$  form a basis for  $\tilde{H}_0(X)$ . ■

Theorem 1.4 illustrates the following result:

$$H_p(X) = \begin{cases} \tilde{H}_p(X) & \text{if } p > 0 \\ \tilde{H}_0(X) \oplus \mathbb{Z} & \text{if } p = 0 \end{cases}. \quad (1.41)$$

## §1.2 Bracket Operation

**Definition 1.6** (Star convex set). A set  $X \subseteq \mathbb{E}^J$  is said to be **star convex** relative to the point  $w \in X$ , if for each  $x \in X$ , the line segment from  $x$  to  $w$  lies in  $X$ .

**Definition 1.7** (Bracket operation). Suppose  $X \in \mathbb{E}^J$  is star convex relative to  $w$ . We define bracket operation on singular chains of  $X$ . Let us first define it for singular  $p$ -simplices. Let  $T : \Delta_p \rightarrow X$  be a singular  $p$ -simplex of  $X$ . Define a singular  $(p+1)$ -simplex

$$[T, w] : \Delta_{p+1} \rightarrow X$$

by letting  $[T, w]$  carry the line segment from  $x$  to  $\varepsilon_{p+1}$ , for  $x \in \Delta_p$  (the collection of all such line segments as  $x$  varies in  $\Delta_p$  constitutes  $\Delta_{p+1}$ ), linearly onto the line segment  $T(x)$  to  $w$  in  $X$ . In other words,

$$[T, w](t\varepsilon_{p+1} + (1-t)x) = tw + (1-t)T(x), \quad (1.42)$$

for  $t \in [0, 1]$ . Now, extend the definition of bracket operation to arbitrary  $p$ -chains as follows: if  $c = \sum n_i T_i$  is a singular  $p$ -chain of  $X$  with each  $T_i$  being a singular  $p$ -simplex, then we define

$$[c, w] = \sum n_i [T_i, w]. \quad (1.43)$$

In other words,  $[\cdot, w] : S_p(X) \rightarrow S_{p+1}(X)$ ,  $c \mapsto [c, w]$  is a homomorphism.

From Figure 1.1, it's immediate that the restriction of  $[T, w]$  to the face  $\Delta_p$  of  $\Delta_{p+1}$  is just the map  $T$ . Now, consider the case when  $T$  is the linear singular simplex  $l_{(a_0, \dots, a_p)}$  for  $a_0, \dots, a_p \in \mathbb{R}^{\infty}$ . We want to calculate what  $[l_{(a_0, \dots, a_p)}, w]$  is.

Recall that  $l_{(a_0, \dots, a_p)} : \Delta_p \rightarrow \mathbb{R}^{\infty}$  is defined as

$$l_{(a_0, \dots, a_p)}(x_0, \dots, x_p) = \sum_{i=0}^p x_i a_i. \quad (1.44)$$

Consider a point  $(x_0, \dots, x_p, x_{p+1}, 0, \dots) \in \Delta_{p+1}$ . We want to see where  $[l_{(a_0, \dots, a_p)}, w]$  takes this point to. Since  $(x_0, \dots, x_p, x_{p+1}, 0, \dots) \in \Delta_{p+1}$ , each  $x_i$  is nonnegative with  $\sum_{i=0}^{p+1} x_i = 1$ . Now,

$$\sum_{i=0}^p \frac{x_i}{1-x_{p+1}} = 1, \quad (1.45)$$

so  $\left(\frac{x_0}{1-x_{p+1}}, \frac{x_1}{1-x_{p+1}}, \dots, \frac{x_p}{1-x_{p+1}}, 0, \dots\right) \in \Delta_p$ . Therefore,

$$(x_0, \dots, x_p, x_{p+1}, 0, \dots) = (1-x_{p+1}) \left(\frac{x_0}{1-x_{p+1}}, \frac{x_1}{1-x_{p+1}}, \dots, \frac{x_p}{1-x_{p+1}}, 0, \dots\right) + x_{p+1} \varepsilon_{p+1}. \quad (1.46)$$

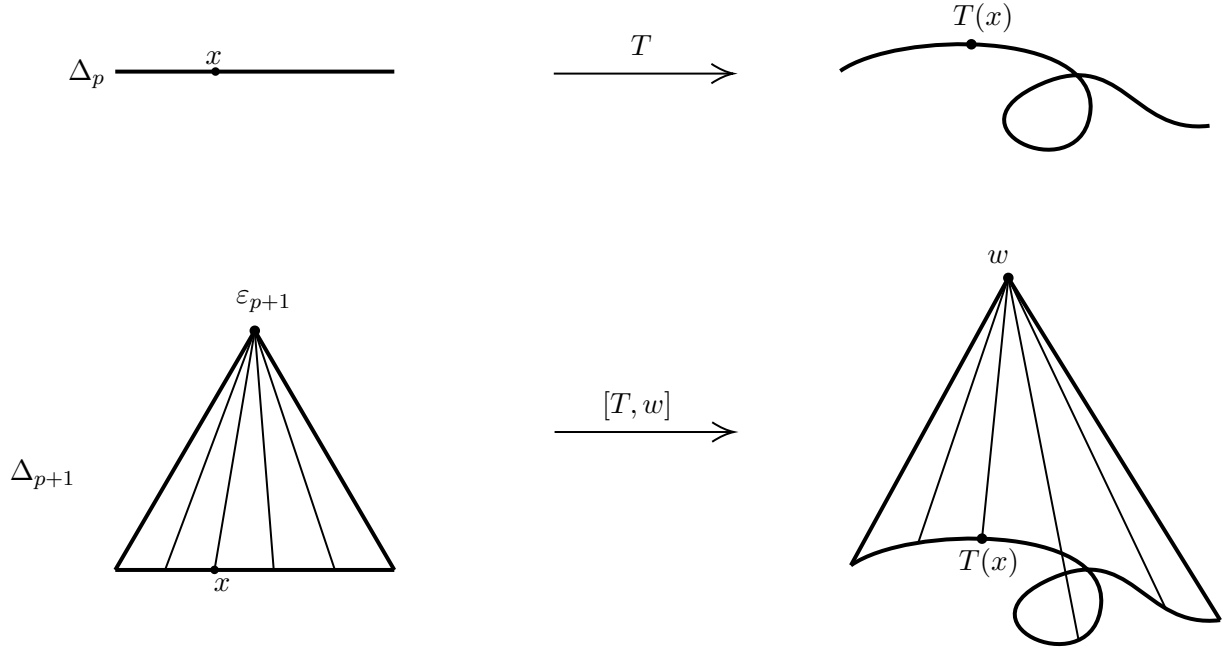


Figure 1.1

By the definition of bracket operation,

$$\begin{aligned}
 & [l_{(a_0, \dots, a_p)}, w] (x_0, \dots, x_p, x_{p+1}, 0, \dots) \\
 &= (1 - x_{p+1}) l_{(a_0, \dots, a_p)} \left( \frac{x_0}{1 - x_{p+1}}, \frac{x_1}{1 - x_{p+1}}, \dots, \frac{x_p}{1 - x_{p+1}}, 0, \dots \right) + x_{p+1} w \\
 &= (1 - x_{p+1}) \sum_{i=0}^p \frac{x_i}{1 - x_{p+1}} a_i + x_{p+1} w \\
 &= \sum_{i=0}^p x_i a_i + x_{p+1} w.
 \end{aligned} \tag{1.47}$$

Furthermore,

$$l_{(a_0, \dots, a_p, w)} (x_0, \dots, x_p, x_{p+1}, 0, \dots) = x_0 a_0 + \dots + x_p a_p + x_{p+1} w = \sum_{i=0}^p x_i a_i + x_{p+1} w. \tag{1.48}$$

Equating 1.47 and 1.48, we get

$$[l_{(a_0, \dots, a_p)}, w] = l_{(a_0, \dots, a_p, w)}. \tag{1.49}$$

Now we will show that  $[T, w] : \Delta_{p+1} \rightarrow X$  is continuous. We have seen earlier that given  $x \in \Delta_p$ , a point in  $\Delta_{p+1}$  is expressed as  $t\varepsilon_{p+1} + (1-t)x$ , with  $0 \leq t \leq 1$ . Hence, we are concerned with the following quotient map  $\pi : \Delta_p \times [0, 1] \rightarrow \Delta_{p+1}$  defined by

$$\pi(x, t) = t\varepsilon_{p+1} + (1-t)x. \tag{1.50}$$

If  $x = (x_0, \dots, x_p, 0, \dots) \in \Delta_p$ , then 1.50 takes the familiar form

$$\pi((x_0, \dots, x_p, 0, \dots), t) = ((1-t)x_0, \dots, (1-t)x_p, t, 0, \dots). \tag{1.51}$$

Observe that  $\pi|_{\Delta_p \times [0, 1]} : \Delta_p \times [0, 1] \rightarrow \Delta_{p+1}$  is 1-1, and  $\pi(\Delta_p \times \{1\}) = \{\varepsilon_{p+1}\}$ , showing that  $\pi$  collapses  $\Delta_p \times \{1\}$  to the  $(p+1)$ -th vertex  $\varepsilon_{p+1}$  of  $\Delta_{p+1}$ . Now, the continuous map  $f : \Delta_p \times [0, 1] \rightarrow X$  defined by

$$f(x, t) = tw + (1-t)T(x) \tag{1.52}$$

is constant on  $\Delta_p \times \{1\}$ . In fact,  $f(\Delta_p \times \{1\}) = \{w\}$ . Since  $\pi$  is 1-1 for other points,  $f$  is seen to be constant for  $\pi^{-1}(y)$  with  $y \in \Delta_{p+1} \setminus \{\varepsilon_{p+1}\}$ . In other words,  $f : \Delta_p \times [0, 1] \rightarrow X$  is constant for each  $\pi^{-1}(y)$  with  $y \in \Delta_{p+1}$ . Therefore,  $f$  induces a unique continuous map  $\tilde{f} : \Delta_{p+1} \rightarrow X$  such that the following diagram commutes

$$\begin{array}{ccc} \Delta_p \times [0, 1] & & \\ \pi \downarrow & \searrow f & \\ \Delta_{p+1} & \xrightarrow{\tilde{f}} & X \end{array}$$

This unique map  $\tilde{f}$  is precisely  $[T, w]$ , since

$$([T, w] \circ \pi)(x, t) = [T, w](t\varepsilon_{p+1} + (1-t)x) = tw + (1-t)T(x) = f(x, t). \quad (1.53)$$

Therefore,  $\tilde{f} = [T, w]$ , and hence it is continuous. So  $[T, w]$  is indeed a singular  $(p+1)$ -simplex.

### Lemma 1.5

Let  $X$  be a star convex set with respect to  $w$ ; let  $c$  be a singular  $p$ -chain of  $X$ . Then

$$\partial[c, w] = \begin{cases} [\partial c, w] + (-1)^{p+1} c & \text{if } p > 0 \\ \epsilon(c) T_w - c & \text{if } p = 0 \end{cases}, \quad (1.54)$$

where  $T_w$  is the singular 0-simplex mapping  $\Delta_0$  to  $w$ .

*Proof.* If  $T$  is a singular 0-simplex,  $[T, w]$  is a singular 1-simplex. Then

$$\partial[T, w] = [T, w] \circ l_{(\widehat{\varepsilon}_0, \varepsilon_1)} - [T, w] \circ l_{(\varepsilon_0, \widehat{\varepsilon}_1)}. \quad (1.55)$$

Now, recall  $[T, w] : \Delta_1 \rightarrow X$  maps the line joining  $\varepsilon_1$  to  $\varepsilon_0$  to the line joining  $w$  to  $T(\varepsilon_0)$ . So

$$[T, w](1-t, t, 0, \dots) = tw + (1-t)T(\varepsilon_0). \quad (1.56)$$

Now,

$$([T, w] \circ l_{(\widehat{\varepsilon}_0, \varepsilon_1)})(1, 0, \dots) = [T, w](0, 1, 0, \dots) = w = T_w(1, 0, \dots). \quad (1.57)$$

Therefore,  $([T, w] \circ l_{(\widehat{\varepsilon}_0, \varepsilon_1)}) = T_w$ .

$$([T, w] \circ l_{(\varepsilon_0, \widehat{\varepsilon}_1)})(1, 0, \dots) = [T, w](1, 0, \dots) = T(\varepsilon_0) = T(1, 0, \dots), \quad (1.58)$$

so  $[T, w] \circ l_{(\varepsilon_0, \widehat{\varepsilon}_1)} = T$ . By 1.55, we get

$$\partial[T, w] = T_w - T. \quad (1.59)$$

Now, let  $c = \sum_i n_i T_i$  be a singular 0-chain with  $T_i$  being singular 0-simplices. Then

$$\partial \left[ \sum_i n_i T_i, w \right] = \sum_i n_i \partial[T_i, w] = \sum_i n_i (T_w - T_i) = \left( \sum_i n_i \right) T_w - \sum_i n_i T_i. \quad (1.60)$$

Now, applying the augmentation map to  $c$ , we get

$$\epsilon(c) = \epsilon \left( \sum_i n_i T_i \right) = \sum_i n_i \epsilon(T_i) = \sum_i n_i. \quad (1.61)$$

Therefore, 1.60 gives us

$$\partial[c, w] = \epsilon(c) T_w - c. \quad (1.62)$$

Now we shall consider the case when  $T$  is a singular  $p$ -simplex, and we shall prove that  $\partial[T, w] = [\partial T, w] + (-1)^{p+1} T$ .

$$\begin{aligned} \partial[T, w] &= \sum_{i=0}^{p+1} (-1)^i [T, w] \circ l_{(\varepsilon_0, \dots, \widehat{\varepsilon}_i, \dots, \varepsilon_{p+1})} \\ &= \sum_{i=0}^p (-1)^i [T, w] \circ l_{(\varepsilon_0, \dots, \widehat{\varepsilon}_i, \dots, \varepsilon_{p+1})} + (-1)^{p+1} [T, w] \circ l_{(\varepsilon_0, \dots, \varepsilon_p, \widehat{\varepsilon}_{p+1})}. \end{aligned} \quad (1.63)$$

$l_{(\varepsilon_0, \dots, \varepsilon_p, \widehat{\varepsilon}_{p+1})}$  is the inclusion map of  $\Delta_p$  into  $\Delta_{p+1}$ . So  $[T, w] \circ l_{(\varepsilon_0, \dots, \varepsilon_p, \widehat{\varepsilon}_{p+1})}$  is nothing but the restriction of  $[T, w]$  to  $\Delta_p$ , which is the same as  $T$ . Now we want to show that

$$[T, w] \circ l_{(\varepsilon_0, \dots, \widehat{\varepsilon}_i, \dots, \varepsilon_{p+1})} = [T \circ l_{(\varepsilon_0, \dots, \widehat{\varepsilon}_i, \dots, \varepsilon_p)}, w]. \quad (1.64)$$

Both sides of 1.64 are maps from  $\Delta_p$  to  $X$ . Let  $(x_0, \dots, x_p, 0, \dots) \in \Delta_p$ . Then

$$([T, w] \circ l_{(\varepsilon_0, \dots, \widehat{\varepsilon}_i, \dots, \varepsilon_{p+1})})(x_0, \dots, x_p, 0, \dots) = [T, w](x_0, \dots, x_{i-1}, 0, x_i, \dots, x_{p-1}, x_p, 0, \dots). \quad (1.65)$$

Now,  $(x_0, \dots, x_{i-1}, 0, x_i, \dots, x_{p-1}, x_p, 0, \dots)$  is a point in  $\Delta_{p+1}$ . We can write it as

$$(x_0, \dots, x_{i-1}, 0, x_i, \dots, x_{p-1}, x_p, 0, \dots) = (1 - x_p) \left( \frac{x_0}{1 - x_p}, \dots, \frac{x_{i-1}}{1 - x_p}, 0, \frac{x_i}{1 - x_p}, \dots, \frac{x_{p-1}}{1 - x_p}, 0, \dots \right) + x_p \varepsilon_{p+1}. \quad (1.66)$$

Now,  $\left( \frac{x_0}{1 - x_p}, \dots, \frac{x_{i-1}}{1 - x_p}, 0, \frac{x_i}{1 - x_p}, \dots, \frac{x_{p-1}}{1 - x_p}, 0, \dots \right)$  is a point in  $\Delta_p$  since its nonzero components are all non-negative and they add to 1. Therefore,

$$\begin{aligned} &[T, w](x_0, \dots, x_{i-1}, 0, x_i, \dots, x_{p-1}, x_p, 0, \dots) \\ &= (1 - x_p) T \left( \frac{x_0}{1 - x_p}, \dots, \frac{x_{i-1}}{1 - x_p}, 0, \frac{x_i}{1 - x_p}, \dots, \frac{x_{p-1}}{1 - x_p}, 0, \dots \right) + x_p w. \end{aligned} \quad (1.67)$$

On the other hand, we can write  $(x_0, \dots, x_p, 0, \dots)$  as

$$(x_0, \dots, x_p, 0, \dots) = (1 - x_p) \left( \frac{x_0}{1 - x_p}, \dots, \frac{x_{p-1}}{1 - x_p}, 0, \dots \right) + x_p \varepsilon_p, \quad (1.68)$$

where  $\left( \frac{x_0}{1 - x_p}, \dots, \frac{x_{p-1}}{1 - x_p}, 0, \dots \right) \in \Delta_{p-1}$ . So

$$\begin{aligned} &[T \circ l_{(\varepsilon_0, \dots, \widehat{\varepsilon}_i, \dots, \varepsilon_p)}, w](x_0, \dots, x_p, 0, \dots) \\ &= x_p w + (1 - x_p) (T \circ l_{(\varepsilon_0, \dots, \widehat{\varepsilon}_i, \dots, \varepsilon_p)}) \left( \frac{x_0}{1 - x_p}, \dots, \frac{x_{p-1}}{1 - x_p}, 0, \dots \right) \\ &= x_p w + (1 - x_p) T \left( \frac{x_0}{1 - x_p}, \dots, \frac{x_{i-1}}{1 - x_p}, 0, \frac{x_i}{1 - x_p}, \dots, \frac{x_{p-1}}{1 - x_p}, 0, \dots \right). \end{aligned} \quad (1.69)$$

Combining 1.65, 1.67 and 1.69, we get that 1.64 indeed holds, i.e.

$$[T, w] \circ l_{(\varepsilon_0, \dots, \widehat{\varepsilon}_i, \dots, \varepsilon_{p+1})} = [T \circ l_{(\varepsilon_0, \dots, \widehat{\varepsilon}_i, \dots, \varepsilon_p)}, w].$$

Now, from 1.63, we then get

$$\begin{aligned} \partial[T, w] &= \sum_{i=0}^p (-1)^i [T \circ l_{(\varepsilon_0, \dots, \widehat{\varepsilon}_i, \dots, \varepsilon_p)}, w] + (-1)^{p+1} T \\ &= \left[ \sum_{i=0}^p (-1)^i T \circ l_{(\varepsilon_0, \dots, \widehat{\varepsilon}_i, \dots, \varepsilon_p)}, w \right] + (-1)^{p+1} T \\ &= [\partial T, w] + (-1)^{p+1} T. \end{aligned} \quad (1.70)$$

Now, if  $c = \sum_i n_i T_i$  is a singular  $p$ -chain with  $T_i$  being singular 0-simplices, then

$$\partial[c, w] = \sum_i n_i \partial[T_i, w] = \sum_i n_i [\partial T_i, w] + (-1)^{p+1} \sum_i n_i T_i = [\partial c, w] + (-1)^{p+1} c. \quad (1.71)$$

■

**Theorem 1.6**

Let  $X \subseteq \mathbb{E}^J$  be star convex with respect to  $w$ . Then  $X$  is acyclic in singular homology.

*Proof.* To show that  $\tilde{H}_0(X) = 0$ , let  $c \in \text{Ker } \epsilon$ .

$$S_1(X) \xrightarrow{\partial_1} S_0(X) \xrightarrow{\epsilon} \mathbb{Z}$$

So  $\epsilon(c) = 0$ . Now, by [Lemma 1.5](#),

$$\partial_1 [c, w] = \epsilon(c) T_w - c = -c. \quad (1.72)$$

Hence,  $c \in \text{Im } \partial_1$  leading to  $\text{Ker } \epsilon \subseteq \text{Im } \partial_1$ . We already know Hence,  $\text{Im } \partial_1 \subseteq \text{Ker } \epsilon$ . Therefore,  $\tilde{H}_0(X) = 0$ .

Now we shall show that  $H_p(X) = 0$  for  $p > 0$ . Let  $z \in \text{Ker } \partial_p$ . Then  $\partial_p z = 0$ . By [Lemma 1.5](#) again,

$$\partial_{p+1} [z, w] = [\partial_p z, w] + (-1)^{p+1} z = (-1)^{p+1} z. \quad (1.73)$$

Hence,  $z \in \text{Im } \partial_{p+1}$ . Therefore,  $H_p(X) = 0$ . In other words,  $\tilde{H}_p(X) = 0$  for all  $p$ , i.e.  $X$  is acyclic. ■

**Corollary 1.7**

Any simplex is acyclic in singular homology.

# 2 Axioms of Singular Homology

In this chapter, we shall verify that singular homology does, in fact, satisfy the Eilenberg-Steenrod axioms. The axioms can be found in chapter 6 of [AT2 lecture notes].

## §2.1 Relative Homology Groups

If  $X$  is a space and  $A$  is a subspace of  $X$ , there is a natural inclusion  $S_p(A) \hookrightarrow S_p(X)$ . The group of **relative singular chains** is defined by

$$S_p(X, A) = S_p(X) / S_p(A). \quad (2.1)$$

The boundary operator  $\partial_p^X : S_p(X) \rightarrow S_{p-1}(X)$  restricts to the boundary operator on  $S_p(A)$ , i.e.  $\partial_p^X|_{S_p(A)} : S_p(A) \rightarrow S_{p-1}(A)$ . It, therefore, induces a boundary operator at the relative singular chain level:

$$\begin{aligned} \partial_p^{(X,A)} : S_p(X, A) &\rightarrow S_{p-1}(X, A), \\ T + S_p(A) &\mapsto \partial_p^X T + S_{p-1}(A), \end{aligned} \quad (2.2)$$

with  $T = \sum_{\alpha} n_{\alpha} T_{\alpha}$  being a singular  $p$ -chain, where  $n_{\alpha} \in \mathbb{Z}$  and  $T_{\alpha}$  singular  $p$ -simplices. If any of the  $T_{\alpha}$ 's are such that  $T_{\alpha}(\Delta_p) \subseteq A$ , then  $T_{\alpha} \in S_p(A)$ . So, we can assume  $T_{\alpha}(\Delta_p) \setminus A \neq \emptyset$ . Such  $T_{\alpha}$ 's generate the group  $S_p(X, A)$ , and so  $S_p(X, A)$  is a free abelian group.

The family of groups  $S_p(X, A)$  and homomorphisms  $\partial_p^{(X,A)}$  is called **the singular chain complex** of the pair  $(X, A)$ , and is denoted by  $\mathcal{S}(X, A)$ . The homology groups of the chain complex  $\mathcal{S}(X, A)$  of the pair  $(X, A)$  are called the **singular homology groups** of the pair  $(X, A)$ , and are denoted by  $H_p(X, A)$ .

The chain complex  $\mathcal{S}(X, A)$  is free, i.e.  $S_p(X, A)$  is free for each  $p$ . The group  $S_p(X, A)$  has as basis all the cosets of the form  $T + S_p(A)$ , where  $T$  is a singular  $p$ -simplex with  $T(\Delta_p) \setminus A \neq \emptyset$ .

If  $f : (X, A) \rightarrow (Y, B)$  is a continuous map (recall that by the continuity of  $f$  between pairs  $(X, A)$  and  $(Y, B)$ , we actually mean that  $f : X \rightarrow Y$  is continuous, with  $f(A) \subseteq B$ ), then homomorphisms  $(f_{\#})_p : S_p(X) \rightarrow S_p(Y)$  carries singular  $p$ -chains of  $A$  into singular  $p$ -chains of  $B$ . So it induces a homomorphism (also denoted by  $(f_{\#})_p$ ) at the level of relative singular  $p$ -chains:

$$\begin{aligned} (f_{\#})_p : S_p(X, A) &\rightarrow S_p(Y, B), \\ T + S_p(A) &\mapsto (f_{\#})_p T + S_p(B) = f \circ T + S_p(B). \end{aligned} \quad (2.3)$$

where  $T$  is a singular  $p$ -simplex with  $T(\Delta_p) \setminus A \neq \emptyset$ . This map can be seen to commute with the boundary operator at the relative singular chain level. To be precise,

$$(f_{\#})_{p-1} \circ \partial_p^{(X,A)} = \partial_p^{(Y,B)} \circ (f_{\#})_p. \quad (2.4)$$

In other words, the following diagram commutes.

$$\begin{array}{ccc} S_p(X, A) & \xrightarrow{\partial_p^{(X,A)}} & S_{p-1}(X, A) \\ (f_{\#})_p \downarrow & & \downarrow (f_{\#})_{p-1} \\ S_p(Y, B) & \xrightarrow{\partial_p^{(Y,B)}} & S_{p-1}(Y, B) \end{array}$$

Therefore,  $f_{\#}$  induces a homomorphism

$$\begin{aligned} (f_{\#})_p : H_p(X, A) &\rightarrow H_p(Y, B), \\ c + \text{Im } \partial_{p+1}^{(X,A)} &\mapsto (f_{\#})_p c + \text{Im } \partial_{p+1}^{(Y,B)}. \end{aligned} \quad (2.5)$$

**Theorem 2.1**

If  $i : (X, A) \rightarrow (X, A)$  is the identity, then so is  $(i_*)_p : H_p(X, A) \rightarrow H_p(X, A)$ . If  $h : (X, A) \rightarrow (Y, B)$  and  $k : (Y, B) \rightarrow (Z, C)$  are continuous, then  $((k \circ h)_*)_p = (k_*)_p \circ (h_*)_p$ .

*Proof.* Since  $(i_\#)_p : S_p(X) \rightarrow S_p(X)$  is the identity map (as proven while proving [Theorem 1.2](#)), so is  $(i_\#)_p : S_p(X, A) \rightarrow S_p(X, A)$ . Then from [2.5](#), we get that  $(i_*)_p : H_p(X, A) \rightarrow H_p(X, A)$  is the identity, i.e.  $(i_*)_p = \text{id}_{H_p(X, A)}$ .

Now, let us prove  $((k \circ h)_\#)_p = (k_\#)_p \circ (h_\#)_p$ . The equality at the homology level will then follow from [2.5](#).

$$(h_\#)_p : S_p(X, A) \rightarrow S_p(Y, B), \quad (k_\#)_p : S_p(Y, B) \rightarrow S_p(Z, C).$$

We choose a singular  $p$ -simplex  $T$  such that  $T(\Delta_p) \setminus A \neq \emptyset$ . Then the cosets of the form  $T + S_p(A)$  form a basis of  $S_p(X, A)$ .

$$\Delta_p \xrightarrow{T} X \xrightarrow{h} Y \xrightarrow{k} Z$$

$\searrow \quad \nearrow$   
 $k \circ h$

Using [2.3](#), we get

$$(h_\#)_p(T + S_p(A)) = h \circ T + S_p(B), \quad (2.6)$$

$$(k_\#)_p((h_\#)_p(T + S_p(A))) = (k_\#)_p(h \circ T + S_p(B)) = k \circ h \circ T + S_p(C), \quad (2.7)$$

$$((k \circ h)_\#)_p(T + S_p(A)) = k \circ h \circ T + S_p(C). \quad (2.8)$$

Therefore, we can conclude that  $((k \circ h)_\#)_p = (k_\#)_p \circ (h_\#)_p$ . ■

**Theorem 2.2**

There is a homomorphism  $(\partial_*)_p : H_p(X, A) \rightarrow H_{p-1}(A)$ , defined for  $A \subset X$  and all  $p$ , such that the sequence

$$\cdots \longrightarrow H_p(A) \xrightarrow{(i_*)_p} H_p(X) \xrightarrow{(\pi_*)_p} H_p(X, A) \xrightarrow{(\partial_*)_p} H_{p-1}(A) \longrightarrow \cdots$$

is exact, where  $i$  and  $\pi$  are the inclusions

$$(A, \emptyset) \xhookrightarrow{i} (X, \emptyset) \xhookrightarrow{\pi} (X, A).$$

The same holds if reduced homology is used for  $X$  and  $A$ , provided  $A \neq \emptyset$ .

A continuous map  $f : (X, A) \rightarrow (Y, B)$  induces a homomorphism of the corresponding exact sequences in singular homology, either ordinary or reduced.

*Proof.* Let us recall the Zig-Zag lemma (Lemma 4.4.1 in the lecture note of [AT2](#)). Given a short exact sequence of chain complexes  $\mathcal{C} = \{C_p, \partial_p^C\}$ ,  $\mathcal{D} = \{D_p, \partial_p^D\}$  and  $\mathcal{E} = \{E_p, \partial_p^E\}$ , i.e.

$$0 \longrightarrow \mathcal{C} \xrightarrow{\phi} \mathcal{D} \xrightarrow{\psi} \mathcal{E} \longrightarrow 0$$

with  $\phi$  and  $\psi$  being chain maps, i.e. family of homomorphisms  $\{\phi_p\}$  and  $\{\psi_p\}$  such that

$$0 \longrightarrow C_p \xrightarrow{\phi_p} D_p \xrightarrow{\psi_p} E_p \longrightarrow 0$$

is exact for each  $p$ , then there is a long exact homology sequence



$$\begin{array}{ccccc}
& & \dots & & \dots \\
& & \swarrow & & \searrow \\
H_p(\mathcal{C}) & \xrightarrow{(\phi_p)_*} & H_p(\mathcal{D}) & \xrightarrow{(\psi_p)_*} & H_p(\mathcal{E}) \\
& \nwarrow & \nearrow & & \\
H_{p-1}(\mathcal{C}) & \xrightarrow{(\phi_{p-1})_*} & H_{p-1}(\mathcal{D}) & \longrightarrow & \dots
\end{array}$$

We shall use Zig-Zag lemma with  $C_p = S_p(A)$ ,  $D_p = S_p(X)$  and  $E_p = S_p(X, A)$ , with chain maps given as follows:

$$0 \longrightarrow S_p(A) \xrightarrow{(i_\#)_p} S_p(X) \xrightarrow{(\pi_\#)_p} S_p(X, A) \longrightarrow 0.$$

Then the above sequence is exact, since  $S_p(X, A) = S_p(X) / S_p(A)$ . Now, Zig-Zag lemma guarantees the existence of the homomorphism  $(\partial_*)_p : H_p(X, A) \rightarrow H_{p-1}(A)$  and the following long-exact sequence

$$\dots \longrightarrow H_p(A) \xrightarrow{(i_*)_p} H_p(X) \xrightarrow{(\pi_*)_p} H_p(X, A) \xrightarrow{(\partial_*)_p} H_{p-1}(A) \longrightarrow \dots$$

Now, given a continuous map  $f : (X, A) \rightarrow (Y, B)$ , we shall verify that the following diagram commutes:

$$\begin{array}{ccccccc}
0 & \longrightarrow & S_p(A) & \xrightarrow{(i_\#)_p} & S_p(X) & \xrightarrow{(\pi_\#)_p} & S_p(X, A) \longrightarrow 0 \\
& & \left( (f|_A)_\# \right)_p \downarrow & & \left( (f|_X)_\# \right)_p \downarrow & & \downarrow (f_\#)_p \\
0 & \longrightarrow & S_p(B) & \xrightarrow{(i'_\#)_p} & S_p(Y) & \xrightarrow{(\pi'_\#)_p} & S_p(Y, B) \longrightarrow 0
\end{array}$$

Here, by  $f|_X$ , we mean the map  $f : X \rightarrow Y$ . First, let's show the commutativity of the left hand square. Let's take a singular  $p$ -simplex  $T$  of  $A$ , i.e.  $T : \Delta_p \rightarrow A$  is continuous. Then

$$(i_\#)_p T = i \circ T = T, \quad (f_\#)_p \left( (i_\#)_p T \right) = f \circ T. \quad (2.9)$$

$$\left( (f|_A)_\# \right)_p T = f|_A \circ T = f \circ T, \quad (i'_\#)_p \left( \left( (f|_A)_\# \right)_p T \right) = i' \circ f \circ T = f \circ T. \quad (2.10)$$

$f|_A \circ T = f \circ T$  because the image of  $T$  lies entirely in  $A$ . Therefore, the left hand square commutes. Now we shall show that the right hand square commutes as well. Let's take a singular  $p$ -simplex  $T$  of  $X$ , i.e.  $T : \Delta_p \rightarrow X$  is continuous.

$$(\pi_\#)_p T = T + S_p(A), \quad (f_\#)_p \left( (\pi_\#)_p T \right) = (f_\#)_p T + S_p(B) = (\pi'_\#)_p \left( (f_\#)_p T \right). \quad (2.11)$$

Therefore, the right hand square commutes. So the diagram is commutative. Now, applying Theorem 5.1.1 from the lecture note of [AT2](#), we obtain that the following diagram commutes:

$$\begin{array}{ccccccc}
\dots & \longrightarrow & H_p(A) & \xrightarrow{(i_*)_p} & H_p(X) & \xrightarrow{(\pi_*)_p} & H_p(X, A) \xrightarrow{(\partial_*)_p} H_{p-1}(A) \longrightarrow \dots \\
& & \left( (f|_A)_* \right)_p \downarrow & & (f_*)_p \downarrow & & \downarrow (f_*)_p \downarrow \left( (f|_A)_* \right)_{p-1} \\
\dots & \longrightarrow & H_p(B) & \xrightarrow{(i'_*)_p} & H_p(Y) & \xrightarrow{(\pi'_*)_p} & H_p(Y, B) \xrightarrow{(\partial'_*)_p} H_{p-1}(B) \longrightarrow \dots
\end{array}$$

This establishes the induced homomorphisms between the respective long exact sequences of the singular homology. Following the same procedure, one can show that the same result holds in reduced homology. ■

**Theorem 2.3**

If  $P$  is a one-point space, then  $H_p(P) = 0$  for  $p \neq 0$ , and  $H_0(P) \cong \mathbb{Z}$ .

*Proof.* We provide a direct proof here. We first compute the chain complex  $\mathcal{S}(P)$ . Observe that there is exactly one singular  $p$ -simplex in each non-negative dimension  $p \geq 0$ :  $T_p : \Delta_p \rightarrow P$ , because  $P$  is a singleton. Therefore, the group of  $p$ -chains  $S_p(P) \cong \mathbb{Z}$ , which is infinite cyclic. Each of the “faces” of  $T_p : \Delta_p \rightarrow P$  is given

$$T_p \circ l_{(\varepsilon_0, \dots, \widehat{\varepsilon}_i, \dots, \varepsilon_p)} : \Delta_p \rightarrow P$$

and is precisely  $T_{p-1}$ . All  $(p+1)$  faces of  $T_p$  are just  $T_{p-1}$ . Therefore, if  $p$  is even, then the singular  $p$ -simplex  $(p+1)$  faces, which is an odd number. Hence, in the formula

$$\partial_p T_p = \sum_{i=0}^p (-1)^i T_p \circ l_{(\varepsilon_0, \dots, \widehat{\varepsilon}_i, \dots, \varepsilon_p)}, \quad (2.12)$$

only one term will survive, the others will cancel in pairs. Hence, we find that  $\partial_p T_p = T_{p-1}$ , when  $p$  is even.

On the other hand, when  $p$  is odd,  $T_p$  will have an even number of faces, and all the terms in 2.12 will cancel in pairs. Therefore,  $\partial_p T_p = 0$ , when  $p$  is odd. The chain complex  $\mathcal{S}(P)$  is, thus, of the following form:

$$\begin{aligned} \cdots &\longrightarrow S_{2k}(P) \longrightarrow S_{2k-1}(P) \longrightarrow \cdots \longrightarrow S_1(P) \longrightarrow S_0(P) \longrightarrow 0 \\ \cdots &\longrightarrow \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{\bar{0}} \cdots \longrightarrow \mathbb{Z} \xrightarrow{\bar{0}} \mathbb{Z} \longrightarrow 0 \end{aligned}$$

Here,  $\bar{0}$  maps everything to 0. In dimension  $(2k-1)$ , every  $(2k-1)$ -chain is a cycle, and every  $(2k-1)$ -chain can be seen to be a boundary of a  $2k$ -chain. Hence, there is no nontrivial  $(2k-1)$ -cycle that is not a  $(2k-1)$ -boundary. Therefore,  $H_{2k-1}(P) = 0$ .

In dimension  $2k$ , for  $k > 0$ , there is no nontrivial chain that is a cycle. Hence,  $H_{2k} = 0$ . In dimension 0, every chain is a cycle, and no nontrivial 0-chain is a boundary. Therefore,  $H_0(P) \cong \mathbb{Z}$ . ■

## §2.2 Compact Support Axiom

In this section, we shall verify that singular homology theory satisfies the compact support axiom.

**Definition 2.1** (Minimal carrier). If  $T : \Delta_p \rightarrow X$  is a singular  $p$ -simplex of  $X$ , then the **minimal carrier** of  $T$  is defined to be the image set  $T(\Delta_p)$ . If  $c = \sum n_i T_i$  is a singular  $p$ -chain, with  $T_i$  being singular  $p$ -simplices and each  $n_i$  nonzero, then the minimal carrier of  $c$  is defined to be the union of the minimal carriers of the singular  $p$ -simplices  $T_i$ .

A singular  $p$ -simplex  $T$  is a continuous map from  $\Delta_p$  to  $X$ . Since  $\Delta_p$  is compact, so is  $T(\Delta_p)$  since continuous map takes compact sets to compact sets. Now, a finite union of compact sets is also compact. Therefore, the minimal carrier of a singular  $p$ -chain is compact.

**Theorem 2.4**

Given  $\alpha \in H_p(X, A)$ , there is a compact pair  $(X_0, A_0) \subseteq (X, A)$ , with  $\iota : (X_0, A_0) \hookrightarrow (X, A)$  such that  $(\iota_*)_p(\beta) = \alpha$  for some  $\beta \in H_p(X_0, A_0)$ , where  $(\iota_*)_p : H_p(X_0, A_0) \rightarrow H_p(X, A)$  is the homomorphism induced by the inclusion  $\iota$ .

*Proof.* Given  $\alpha \in H_p(X, A) = Z_p(X, A)/B_p(X, A)$ ,  $\alpha$  is of the form  $C + B_p(X, A)$ , with  $C \in Z_p(X, A) \subset S_p(X, A) = S_p(X)/S_p(A)$ . Therefore,

$$\alpha = (c_p + S_p(A)) + B_p(X, A), \quad (2.13)$$

where  $c_p \in S_p(X)$  such that  $\partial_p c_p$  is carried by  $A$ . The minimal carrier of  $\partial_p c_p$  is a compact set contained in  $A$ . Let us denote this compact set by  $A_0$ . On the other hand,  $c_p$  is minimally carried by a compact set  $X_0$  contained in  $X$ . Now, we define

$$D = c_p + S_p(A_0) \in S_p(X_0, A_0). \quad (2.14)$$

Since  $\partial_p c_p$  is carried by  $A_0$ ,  $D \in Z_p(X_0, A_0)$ . Now, we claim that

$$\beta = D + B_p(X_0, A_0) = (c_p + S_p(A_0)) + B_p(X_0, A_0) \in H_p(X_0, A_0) \quad (2.15)$$

is the required element of  $H_p(X_0, A_0)$  whose image under  $(\iota_*)_p$  is  $\alpha$ . Now,

$$(\iota_*)_p(\beta) = (\iota_*)_p((c_p + S_p(A_0)) + B_p(X_0, A_0)) = ((\iota_{\#})_p c_p + S_p(A)) + B_p(X, A). \quad (2.16)$$

If  $c_p = \sum n_i T_i$ , with  $T_i$  being singular  $p$ -simplices, then

$$(\iota_{\#})_p c_p = \sum n_i (\iota_{\#})_p(T_i) = \sum n_i (\iota \circ T_i) = \sum n_i T_i = c_p. \quad (2.17)$$

Therefore,

$$(\iota_*)_p(\beta) = (c_p + S_p(A)) + B_p(X, A) = \alpha. \quad (2.18)$$

■

### Theorem 2.5

Let  $i : (X_0, A_0) \hookrightarrow (X, A)$  be inclusion, where  $(X_0, A_0)$  is a compact pair. If  $\alpha \in H_p(X_0, A_0)$  with  $(i_*)_p(\alpha) = 0$ , then there are a compact pair  $(X_1, A_1)$  and inclusions

$$(X_0, A_0) \xhookrightarrow{j} (X_1, A_1) \xhookrightarrow{k} (X, A)$$

such that  $(j_*)_p(\alpha) = 0$ .

*Proof.* Let  $\alpha = (c_p + S_p(A_0)) + B_p(X_0, A_0) \in H_p(X_0, A_0)$ , where  $c_p \in S_p(X_0)$  and  $\partial_p c_p$  is carried by  $A_0$ . Now,  $(i_*)_p : H_p(X_0, A_0) \rightarrow H_p(X, A)$ , so  $(i_*)_p(\alpha) = 0 + B_p(X, A)$ .

$$0 + B_p(X, A) = (i_*)_p(\alpha) = ((i_{\#})_p c_p + S_p(A)) + B_p(X, A). \quad (2.19)$$

Using a similar method as in 2.17, one can show that  $(i_{\#})_p c_p = c_p$ . So 2.19 reads

$$0 + B_p(X, A) = (c_p + S_p(A)) + B_p(X, A). \quad (2.20)$$

Therefore,  $c_p + S_p(A) \in B_p(X, A)$ . In other words, there exists a  $(p+1)$ -chain  $d_{p+1}$  such that  $c_p - \partial_{p+1} d_{p+1}$  is carried by  $A$ . Now,  $d_{p+1}$  is carried by

$$X_1 = X_0 \cup (\text{minimal carrier of } d_{p+1}),$$

and  $c_p - \partial_{p+1} d_{p+1}$  is carried by

$$A_1 = A_0 \cup (\text{minimal carrier of } c_p - \partial_{p+1} d_{p+1}).$$

Consider the inclusion maps

$$\begin{array}{ccccc} (X_0, A_0) & \xhookrightarrow{j} & (X_1, A_1) & \xhookrightarrow{k} & (X, A) \\ & \searrow & \text{ } & \nearrow & \\ & & i = k \circ j & & \end{array}$$

Then  $(j_*)_p(\alpha)$  is

$$(j_*)_p(\alpha) = (j_*)_p((c_p + S_p(A_0)) + B_p(X_0, A_0)) = ((j_\#)_p c_p + S_p(A_1)) + B_p(X_1, A_1). \quad (2.21)$$

Again, similarly as in 2.17, one can show that  $(j_\#)_p c_p = c_p$ .

$$(j_*)_p(\alpha) = (c_p + S_p(A_1)) + B_p(X_1, A_1). \quad (2.22)$$

$c_p - \partial_{p+1}d_{p+1}$  is carried by  $A_1$ , so  $c_p - \partial_{p+1}d_{p+1} \in S_p(A_1)$ . Therefore,

$$\begin{aligned} c_p + S_p(A_1) &= c_p - (c_p - \partial_{p+1}d_{p+1}) + S_p(A_1) = \partial_{p+1}d_{p+1} + S_p(A_1) \\ &= \partial_{p+1}(d_{p+1} + S_{p+1}(A_1)) \in B_p(X_1, A_1). \end{aligned} \quad (2.23)$$

Combining 2.22 and 2.23, we get

$$(j_*)_p(\alpha) = \partial_{p+1}(d_{p+1} + S_{p+1}(A_1)) + B_p(X_1, A_1) = 0 + B_p(X_1, A_1). \quad (2.24)$$

## §2.3 Chain Homotopy

**Definition 2.2.** Given chain complexes  $\mathcal{C} = \{C_p, \partial_p\}$  and  $\mathcal{C}' = \{C'_p, \partial'_p\}$  and chain maps  $\phi, \psi : \mathcal{C} \rightarrow \mathcal{C}'$ , a **chain homotopy** of  $\phi$  to  $\psi$  is a family of homomorphisms  $D_p : C_p \rightarrow C'_{p+1}$  such that the following holds

$$\partial'_{p+1}D_p + D_{p-1}\partial_p = \psi_p - \phi_p. \quad (2.25)$$

The following diagram might be useful for to understand the above formula in 2.25. Note that this is **NOT** a commutative diagram.

$$\begin{array}{ccc} & & C'_{p+1} \\ & \nearrow D_p & \downarrow \partial'_{p+1} \\ C_p & \xrightleftharpoons[\psi_p]{\phi_p} & C'_p \\ \partial_p \downarrow & \nearrow D_{p-1} & \\ C_{p-1} & & \end{array}$$

Now, consider the inclusions  $i, j : X \rightarrow X \times I$  ( $I$  is the unit interval  $[0, 1]$ ) given by

$$i(x) = (x, 0) \text{ and } j(x) = (x, 1). \quad (2.26)$$

The corresponding chain maps are denoted by  $(i_\#)_p, (j_\#)_p : S_p(X) \rightarrow S_p(X \times I)$ . Construct a chain homotopy  $D^X$  between the chain map  $i_\#$  and  $j_\#$  as follows:

$$\begin{aligned} D^X : \mathcal{S}(X) &\rightarrow \mathcal{S}(X \times I), \\ D^X_p : S_p(X) &\rightarrow S_p(X \times I). \end{aligned} \quad (2.27)$$

For  $D^X$  to be a chain homotopy, the following equation must hold:

$$\partial_{p+1}^{X \times I} \circ D^X_p + D^X_{p-1} \circ \partial_p^X = (j_\#)_p - (i_\#)_p. \quad (2.28)$$

$$\begin{array}{ccc} & & S_{p+1}(X \times I) \\ & \nearrow D^X_p & \downarrow \partial_{p+1}^{X \times I} \\ S_p(X) & \xrightleftharpoons[(j_\#)_p]{(i_\#)_p} & S_p(X \times I) \\ \partial_p^X \downarrow & \nearrow D^X_{p-1} & \\ S_{p-1}(X) & & \end{array}$$

One can now construct the following diagram to find that  $F_{\#} \circ D^X$  is a chain homotopy between the chain maps  $f_{\#}, g_{\#} : \mathcal{S}(X) \rightarrow \mathcal{S}(Y)$ , where  $X$  and  $Y$  are topological spaces and  $F$  is a homotopy between the maps  $f, g : X \rightarrow Y$ , i.e.  $F : X \times I \rightarrow Y$  is a continuous map such that

$$F(x, 0) = f(x) \text{ and } F(x, 1) = g(x).$$

Using 2.26, we then have

$$F \circ i = f \text{ and } F \circ j = g. \quad (2.29)$$

$F_{\#} : \mathcal{S}(X \times I) \rightarrow \mathcal{S}(Y)$ . In order to show that  $F_{\#} \circ D^X$  is a chain homotopy between  $f_{\#}$  and  $g_{\#}$ , one needs to prove that

$$\partial_{p+1}^Y \circ (F_{\#})_{p+1} \circ D_p^X + (F_{\#})_p \circ D_{p-1}^X \circ \partial_p^X = (g_{\#})_p - (f_{\#})_p. \quad (2.30)$$

$$\begin{array}{ccccc}
 & & & S_{p+1}(Y) & \\
 & & (F_{\#})_{p+1} \nearrow & \downarrow \partial_{p+1}^Y & \\
 & S_{p+1}(X \times I) & & & \\
 D_p^X \nearrow & & (f_{\#})_p \xrightarrow{\quad} & S_p(Y) & \\
 S_p(X) & \xrightarrow{\quad} & (g_{\#})_p \xrightarrow{\quad} & & \\
 \downarrow \partial_{p+1}^X & & S_p(X \times I) & \nearrow (F_{\#})_p & \\
 & D_{p-1}^X \nearrow & & & \\
 S_{p-1}(X) & & & & 
 \end{array}$$

Let us quickly see how 2.30 comes from 2.28. Since chain maps commute with the boundary operator, we have the following commutative diagram:

$$\begin{array}{ccc}
 S_{p+1}(X \times I) & \xrightarrow{(F_{\#})_{p+1}} & S_{p+1}(Y) \\
 \partial_{p+1}^{X \times I} \downarrow & & \downarrow \partial_{p+1}^Y \\
 S_p(X \times I) & \xrightarrow{(F_{\#})_p} & S_p(Y)
 \end{array}$$

i.e.  $\partial_{p+1}^Y \circ (F_{\#})_{p+1} = (F_{\#})_p \circ \partial_{p+1}^{X \times I}$ . Therefore, one obtains

$$\begin{aligned}
 \partial_{p+1}^Y \circ (F_{\#})_{p+1} \circ D_p^X &= (F_{\#})_p \circ \partial_{p+1}^{X \times I} \circ D_p^X \\
 &= (F_{\#})_p \circ \left[ (j_{\#})_p - (i_{\#})_p - D_{p-1}^X \circ \partial_p^X \right] \\
 &= \left( (F \circ j)_{\#} \right)_p - \left( (F \circ i)_{\#} \right)_p - (F_{\#})_p \circ D_{p-1}^X \circ \partial_p^X \\
 &= (g_{\#})_p - (f_{\#})_p - (F_{\#})_p \circ D_{p-1}^X \circ \partial_p^X, \quad (2.31)
 \end{aligned}$$

which can be rearranged to obtain 2.30. The existence of the chain map  $D^X : \mathcal{S}(X) \rightarrow \mathcal{S}(X \times I)$  is governed by the following lemma.

### Lemma 2.6

There exists, for each space  $X$ , and each non-negative integer  $p$ , a homomorphism  $D_p^X : S_p(X) \rightarrow S_{p+1}(X \times I)$  having the following properties:

(a) If  $T : \Delta_p \rightarrow X$  is a singular  $p$ -simplex then

$$\partial_{p+1}^{X \times I} D_p^X T + D_{p-1}^X \partial_p^X T = (j_{\#})_p T - (i_{\#})_p T. \quad (2.32)$$

Here, the map  $i : X \rightarrow X \times I$  carries  $x$  to  $(x, 0)$  and the map  $j : X \rightarrow X \times I$  carries  $x$  to

$(x, 1)$ .

(b)  $D_p^X$  is natural; i.e. given  $f : X \rightarrow Y$  continuous, the following diagram commutes:

$$\begin{array}{ccc} S_p(X) & \xrightarrow{D_p^X} & S_{p+1}(X \times I) \\ (f_\#)_p \downarrow & & \downarrow ((f \times \text{id}_I)_\#)_{p+1} \\ S_p(Y) & \xrightarrow{D_p^Y} & S_{p+1}(Y \times I) \end{array}$$

Note that continuous  $f : X \rightarrow Y$  induces a continuous map  $f \times \text{id}_I : X \times I \rightarrow Y \times I$  given by  $(x, t) \mapsto (f(x), t)$ . Hence there is a group homomorphism

$$\left( (f \times \text{id}_I)_\# \right)_p : S_p(X \times I) \rightarrow S_p(Y \times I)$$

for each non-negative integer  $p$ .

Proof of the lemma is omitted.

### Theorem 2.7

If  $f, g : (X, A) \rightarrow (Y, B)$  are homotopic, then  $(f_*)_p = (g_*)_p$  for all  $p$ , with  $(f_*)_p, (g_*)_p : H_p(X, A) \rightarrow H_p(Y, B)$  group homomorphisms. The same holds in the reduced homology if  $A = B = \emptyset$ .

*Proof.* Let  $F : (X \times I, A \times I) \rightarrow (Y \times I, B \times I)$  be the homotopy between  $f, g : (X, A) \rightarrow (Y, B)$ . Let  $i, j : (X, A) \rightarrow (X \times I, A \times I)$  be given by  $i(x) = (x, 0)$  and  $j(x) = (x, 1)$ , for  $x \in X$ . Let  $D_p^X : S_p(X) \rightarrow S_p(X \times I)$  be the group homomorphism associated with the chain homotopy  $D^X : \mathcal{S}(X) \rightarrow \mathcal{S}(X \times I)$  constructed in Lemma 2.6. Naturality of  $D^X$  with respect to the inclusion map  $\iota : A \hookrightarrow X$  dictates that the following diagram commutes:

$$\begin{array}{ccc} S_p(A) & \xrightarrow{D_p^A} & S_{p+1}(A \times I) \\ (\iota_\#)_p \downarrow & & \downarrow ((\iota \times \text{id}_I)_\#)_{p+1} \\ S_p(X) & \xrightarrow{D_p^X} & S_{p+1}(X \times I) \end{array}$$

Consider  $T \in S_{p+1}(A \times I)$  such that  $T$  is a  $(p+1)$ -singular simplex of  $A \times I$ , i.e.  $T : \Delta_{p+1} \rightarrow A \times I$  is continuous. For a given  $x \in \Delta_{p+1}$ , let  $T(x) = (a, t) \in A \times I$ . Now,

$$\left( (\iota \times \text{id}_I)_\# \right)_{p+1} T(x) = (\iota \times \text{id}_I) \circ T(x) = (\iota \times \text{id}_I)(a, t) = (a, t) = T(x). \quad (2.33)$$

Hence,  $\left( (\iota \times \text{id}_I)_\# \right)_{p+1} T = T$ . So, we have

$$\left( (\iota \times \text{id}_I)_\# \right)_{p+1} \circ D_p^A = D_p^A. \quad (2.34)$$

Now, commutativity of the above diagram yields

$$\left( (\iota \times \text{id}_I)_\# \right)_{p+1} \circ D_p^A = D_p^X \circ (\iota_\#)_p = D_p^X|_{S_p(A)}. \quad (2.35)$$

Therefore, combining 2.34 and 2.35, we get

$$D_p^X|_{S_p(A)} = D_p^A. \quad (2.36)$$

In other words,  $D_p^X : S_p(X) \rightarrow S_{p+1}(X \times I)$  carries  $S_p(A)$  into  $S_p(X \times I)$ , and thus induces a chain homotopy on the relative level. The constituent group homomorphisms are given by

$$D_p^{(X,A)} : S_p(X, A) \rightarrow S_{p+1}(X \times I, A \times I). \quad (2.37)$$

Now, 2.32 indeed holds for  $D_p^{(X,A)}$  as it is induced by  $D_p^X$ . Then we have

$$(F_{\#})_{p+1} \circ D_p^{(X,A)} : S_p(X, A) \rightarrow S_{p+1}(Y, B),$$

where the homomorphism  $(F_{\#})_{p+1}$  associated with the chain map  $F_{\#} : \mathcal{S}(X \times I, A \times I) \rightarrow \mathcal{S}(Y, B)$  is

$$(F_{\#})_{p+1} : S_{p+1}(X \times I, A \times I) \rightarrow S_{p+1}(Y, B).$$

Then

$$\begin{aligned} \partial_{p+1}^Y \circ (F_{\#})_{p+1} \circ D_p^{(X,A)} &= (F_{\#})_p \circ \partial_{p+1}^{X \times I} \circ D_p^{(X,A)} \\ &= (F_{\#})_p \circ \left[ (j_{\#})_p - (i_{\#})_p - D_{p-1}^{(X,A)} \circ \partial_p^X \right] \\ &= \left( (F \circ j)_{\#} \right)_p - \left( (F \circ i)_{\#} \right)_p - (F_{\#})_p \circ D_{p-1}^{(X,A)} \circ \partial_p^X \\ &= (g_{\#})_p - (f_{\#})_p - (F_{\#})_p \circ D_{p-1}^{(X,A)} \circ \partial_p^X. \end{aligned} \quad (2.38)$$

This proves that  $F_{\#} \circ D^{(X,A)} : \mathcal{S}(X, A) \rightarrow \mathcal{S}(Y, B)$  is a chain homotopy between  $f_{\#}, g_{\#} : \mathcal{S}(X, A) \rightarrow \mathcal{S}(Y, B)$ . It now remains to prove that  $(f_{\#})_p = (g_{\#})_p$  for all  $p$ .

Let  $\alpha \in Z_p(X, A)$ . It suffices to show that  $(f_{\#})_p(\alpha)$  and  $(g_{\#})_p(\alpha)$  differ by a boundary term. Given  $\alpha \in Z_p(X, A)$ ,  $\alpha = c_p + S_p(A)$  for some  $c_p \in S_p(X)$  such that  $\partial_p c_p$  is carried by  $A$ . By 2.38,

$$\begin{aligned} (g_{\#})_p(\alpha) - (f_{\#})_p(\alpha) &= \partial_{p+1}^Y \circ (F_{\#})_{p+1} \circ D_p^{(X,A)}(\alpha) + (F_{\#})_p \circ D_{p-1}^{(X,A)} \circ \partial_p^X(\alpha) \\ &= \partial_{p+1}^Y \circ (F_{\#})_{p+1} \circ D_p^{(X,A)}(\alpha), \end{aligned} \quad (2.39)$$

proving that  $(f_{\#})_p(\alpha)$  and  $(g_{\#})_p(\alpha)$  differ by a boundary term. Therefore,  $(f_{\#})_p(\alpha + B_p(X, A)) = (f_{\#})_p(\alpha + B_p(X, A))$ .

The result in reduced homology is left as an exercise. ■

## §2.4 Homotopy Equivalence

**Definition 2.3** (Retraction). Let  $A \subset X$ . A **retraction** of  $X$  onto  $A$  is a continuous map  $r : X \rightarrow A$  such that  $r(a) = a$  for every  $a \in A$ , i.e.  $r|_A = \text{id}_A$ . If there is a retraction of  $X$  onto  $A$ , we say that  $A$  is a retract of  $X$ ,

**Definition 2.4** (Deformation retraction). A **deformation retraction** of  $X$  onto  $A$  is a continuous map  $F : X \times I \rightarrow X$  such that

$$F(x, 0) = x, \quad F(x, 1) \in A, \quad \text{and} \quad F(a, t) = a \quad (2.40)$$

for all  $x \in X, a \in A, t \in I$ .

If  $F$  is a deformation retraction of  $X$  onto  $A$ , then one can define

$$r(x) = F(x, 1). \quad (2.41)$$

Then 2.40 tells us that  $r$  is a map from  $X$  to  $A$ , and  $r(a) = a$  for all  $a \in A$ . Hence,  $r$  is indeed a retraction of  $X$  onto  $A$ . Now, 2.40 also tells us that

$$F(x, 0) = x = \text{id}_X(x) \quad \text{and} \quad F(x, 1) = j \circ r(x), \quad (2.42)$$

where  $j : A \hookrightarrow X$  is the inclusion. Therefore,  $F$  is a homotopy between the identity map  $\text{id}_X : X \rightarrow X$  and  $j \circ r : X \rightarrow X$ .

**Definition 2.5.** Let  $f : (X, A) \rightarrow (Y, B)$  be continuous. If there is a continuous map  $g : (Y, B) \rightarrow (X, A)$  such that  $g \circ f$  is homotopic to the identity map  $\text{id}_{(X,A)} : (X, A) \rightarrow (X, A)$  and  $f \circ g$  is homotopic to the identity map  $\text{id}_{(Y,B)} : (Y, B) \rightarrow (Y, B)$ , then we call  $f$  a **homotopy equivalence**, and we call  $g$  a **homotopy inverse** for  $f$ .

### Theorem 2.8

Let  $f : (X, A) \rightarrow (Y, B)$  be continuous.

- (a) If  $f$  is a homotopy equivalence, then  $f_*$  is an isomorphism in relative homology.
- (b) More generally, if  $f : X \rightarrow Y$  and  $f|_A : A \rightarrow B$  are homotopy equivalences, then  $f_*$  is an isomorphism in relative homology.

*Proof.* Let  $f : (X, A) \rightarrow (Y, B)$  be a homotopy equivalence, and  $g : (Y, B) \rightarrow (X, A)$  its homotopy inverse. Then  $f \circ g \simeq \text{id}_{(Y,B)}$  and  $g \circ f \simeq \text{id}_{(X,A)}$ . Then by Theorem 2.7,

$$((f \circ g)_*)_p = ((\text{id}_{(Y,B)})_*)_p \quad \text{and} \quad ((g \circ f)_*)_p = ((\text{id}_{(X,A)})_*)_p.$$

In other words,

$$(f_*)_p \circ (g_*)_p = \text{id}_{H_p(Y,B)} \quad \text{and} \quad (g_*)_p \circ (f_*)_p = \text{id}_{H_p(X,A)}. \quad (2.43)$$

Therefore,  $(f_*)_p : H_p(X, A) \rightarrow H_p(Y, B)$  is an isomorphism.

Now we shall prove (b). Consider the long exact sequence of the pairs  $(X, A)$  and  $(Y, B)$ , separately with  $(f_*)_p$  being the respective connecting homomorphisms.

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & H_p(A) & \xrightarrow{(i_*)_p} & H_p(X) & \xrightarrow{(\pi_*)_p} & H_p(X, A) & \xrightarrow{(\partial_*)_p} & H_{p-1}(A) & \xrightarrow{(i_*)_{p-1}} & H_{p-1}(X) & \longrightarrow & \cdots \\ & & \left( (f|_A)_* \right)_p \downarrow & & (f_*)_p \downarrow & & \downarrow (f_*)_p & & \downarrow \left( (f|_A)_* \right)_{p-1} & & \downarrow (f_*)_{p-1} & & \\ \cdots & \longrightarrow & H_p(B) & \xrightarrow{(i_*)_p} & H_p(Y) & \xrightarrow{(\pi_*)_p} & H_p(Y, B) & \xrightarrow{(\partial_*)_p} & H_{p-1}(B) & \xrightarrow{(i_*)_{p-1}} & H_{p-1}(Y) & \longrightarrow & \cdots \end{array}$$

By hypothesis,  $f : (X, \emptyset) \rightarrow (Y, \emptyset)$  is a homotopy equivalence, and hence  $(f_*)_p : H_p(X) \rightarrow H_p(Y)$  is an isomorphism. Similarly, by hypothesis,  $f|_A : (A, \emptyset) \rightarrow (B, \emptyset)$  is a homotopy equivalence, and hence  $\left( (f|_A)_* \right)_p : H_p(A) \rightarrow H_p(B)$  is an isomorphism. Now, applying Steenrod five lemma to the diagram above, one obtains that

$$(f_*)_p : H_p(X, A) \rightarrow H_p(Y, B)$$

is an isomorphism. ■

**Remark 2.1.** If  $f : (X, A) \rightarrow (Y, B)$  is a homotopy equivalence, then  $f : X \rightarrow Y$  and  $f|_A : A \rightarrow B$  are automatically homotopy equivalences. However, the converse is not true. One counterexample is presented below.

### Example 2.1

Consider the inclusion map  $j : (B^n, S^{n-1}) \hookrightarrow (\mathbb{R}^n, \mathbb{R}^n \setminus \{\mathbf{0}\})$ .  $j : B^n \hookrightarrow \mathbb{R}^n$  has a homotopy inverse, so that  $B^n$  and  $\mathbb{R}^n$  are homotopy equivalent. The homotopy inverse is given by  $f : \mathbb{R}^n \rightarrow B^n$ ,

$$f(\mathbf{x}) = \begin{cases} \mathbf{x} & \text{if } \|\mathbf{x}\| \leq 1 \\ \frac{\mathbf{x}}{\|\mathbf{x}\|} & \text{if } \|\mathbf{x}\| > 1 \end{cases}. \quad (2.44)$$



Then  $f(j(\mathbf{x})) = \mathbf{x}$ , so  $f \circ j = \text{id}_{B^n}$ .  $j(f(\mathbf{x})) = f(\mathbf{x}) \in B^n$ . So  $F : \mathbb{R}^n \times I \rightarrow \mathbb{R}^n$  given by

$$F(\mathbf{x}, t) = (1 - t)\mathbf{x} + tj \circ f(\mathbf{x}) \quad (2.45)$$

is a homotopy between  $\text{id}_{\mathbb{R}^n}$  and  $j \circ f$ . Therefore,  $f$  is the homotopy inverse of  $j$ .

In a similar manner, one can show that  $j|_{S^{n-1}} : S^{n-1} \hookrightarrow \mathbb{R}^n \setminus \{\mathbf{0}\}$  also has a homotopy inverse. The homotopy inverse is  $h : \mathbb{R}^n \setminus \{\mathbf{0}\} \rightarrow S^{n-1}$  given by

$$h(\mathbf{x}) = \frac{\mathbf{x}}{\|\mathbf{x}\|}. \quad (2.46)$$

Then  $h \circ j|_{S^{n-1}} = \text{id}_{S^{n-1}}$ . Furthermore,  $G : (\mathbb{R}^n \setminus \{\mathbf{0}\}) \times I \rightarrow \mathbb{R}^n \setminus \{\mathbf{0}\}$  given by

$$G(\mathbf{x}, t) = (1 - t)\mathbf{x} + tj|_{S^{n-1}} \circ h(\mathbf{x}) = \left( (1 - t) + \frac{t}{\|\mathbf{x}\|} \right) \mathbf{x} \quad (2.47)$$

is a homotopy between  $\text{id}_{\mathbb{R}^n \setminus \{\mathbf{0}\}}$  and  $j|_{S^{n-1}} \circ h$ . Therefore,  $h$  is the homotopy inverse of  $j$ .

However,  $j : (B^n, S^{n-1}) \hookrightarrow (\mathbb{R}^n, \mathbb{R}^n \setminus \{\mathbf{0}\})$  has no homotopy inverse although both  $j : B^n \hookrightarrow \mathbb{R}^n$  and  $j|_{S^{n-1}} : S^{n-1} \hookrightarrow \mathbb{R}^n \setminus \{\mathbf{0}\}$  have homotopy inverses. To show this, assume the contrary that  $g : (\mathbb{R}^n, \mathbb{R}^n \setminus \{\mathbf{0}\}) \rightarrow (B^n, S^{n-1})$  is a homotopy inverse of  $j$ . Then  $g$  is continuous, and it maps  $\mathbb{R}^n \setminus \{\mathbf{0}\}$  into  $S^{n-1}$ . But  $\mathbf{0}$  is a limit point of  $\mathbb{R}^n \setminus \{\mathbf{0}\}$ , and  $S^{n-1}$  is closed. Therefore,  $g(\mathbf{0}) \in S^{n-1}$ . In other words,  $g$  maps all of  $\mathbb{R}^n$  into  $S^{n-1}$ . Hence, the composite

$$g \circ j : (B^n, S^{n-1}) \rightarrow (B^n, S^{n-1}) \quad (2.48)$$

maps all of  $B^n$  to  $S^{n-1}$ . If  $T : \Delta_p \rightarrow B^n$  is a singular  $p$ -simplex, then for  $T + S_p(S^{n-1}) \in S_p(B^n, S^{n-1})$ ,

$$\left( (g \circ j)_\# \right)_p (T + S_p(S^{n-1})) = g \circ j \circ T + S_p(S^{n-1}). \quad (2.49)$$

But the image of  $g \circ j \circ T$  lies entirely on  $S^{n-1}$ . So  $\left( (g \circ j)_\# \right)_p$  is the trivial chain map. Therefore,  $\left( (g \circ j)_* \right)_p : H_p(B^n, S^{n-1}) \rightarrow H_p(B^n, S^{n-1})$  is the trivial map. However, since  $g \circ j$  is homotopic with  $\text{id}_{(B^n, S^{n-1})}$ ,  $\left( (g \circ j)_* \right)_p$  is the identity homomorphism on  $H_p(B^n, S^{n-1})$ . This can only be true if  $H_p(B^n, S^{n-1}) = 0$ . We shall soon see this is not true.

## §2.5 Subdivision

**Definition 2.6.** Given a topological space  $X$  and a collection  $\mathcal{A}$  of subsets of  $X$  whose interiors cover  $X$ , a singular simplex of  $X$  is said to be  $\mathcal{A}$ -small if its image set lies in an element of  $\mathcal{A}$ .

Given a singular chain of  $X$ , we show how to “chop it up” so that all its simplices are  $\mathcal{A}$ -small.

**Definition 2.7** (Barycentric subdivision operator). Let  $X$  be a topological space, we define a homomorphism  $\text{sd}_X : S_p(X) \rightarrow S_p(X)$  by induction. If  $T : \Delta_0 \rightarrow X$  is a singular 0-simplex, we define

$$\text{sd}_X T = T. \quad (2.50)$$

Now suppose  $\text{sd}_X$  is defined in dimensions less than  $p$ . We will first take  $X : \Delta_p$  and choose the identity map  $i_p : \Delta_p \rightarrow \Delta_p$ , which is a singular  $p$ -simplex of  $\Delta_p$ , i.e.  $i_p \in S_p(\Delta_p)$ . Let us denote by  $\widehat{\Delta}_p$  the barycenter of  $\Delta_p$ . Then we define  $\text{sd}_{\Delta_p} i_p$  as follows:

$$\text{sd}_{\Delta_p} i_p = (-1)^p \left[ \text{sd}_{\Delta_p} \partial i_p, \widehat{\Delta}_p \right]. \quad (2.51)$$

Now, if  $T : \Delta_p \rightarrow X$  is any singular  $p$ -simplex on  $X$ , then we define

$$\text{sd}_X T = (T_\#)_p (\text{sd}_{\Delta_p} i_p). \quad (2.52)$$

Observe that  $\text{sd}_{\Delta_p} i_p$  is expected to be in  $S_p(\Delta_p)$ . Since  $\partial i_p \in S_{p-1}$  and  $\text{sd}_{\Delta_p}$  is assumed to be defined in dimension less than  $p$ ,  $\text{sd}_{\Delta_p} \partial i_p \in S_{p-1}(\Delta_p)$ . The bracket operation on the RHS of 2.51, therefore, yields  $[\text{sd}_{\Delta_p} \partial i_p, \widehat{\Delta_p}] \in S_p(\Delta_p)$  so that indeed by 2.51, one obtains  $\text{sd}_{\Delta_p} i_p \in S_p(\Delta_p)$ .

### Lemma 2.9

The homomorphism  $\text{sd}_X$  is an augmentation preserving chain map. Furthermore, it is natural in the sense that for any continuous map  $f : X \rightarrow Y$ , one has  $(f_\#)_p \circ \text{sd}_X = \text{sd}_Y \circ (f_\#)_p$ . In other words, the following diagram commutes:

$$\begin{array}{ccc} S_p(X) & \xrightarrow{(f_\#)_p} & S_p(Y) \\ \text{sd}_X \downarrow & & \downarrow \text{sd}_Y \\ S_p(X) & \xrightarrow{(f_\#)_p} & S_p(Y). \end{array}$$

*Proof.* Recall that in dimension 0, for  $T : \Delta_0 \rightarrow X$ , one has  $\text{sd}_X T = T$ . In other words,  $\text{sd}_X : S_0(X) \rightarrow S_0(X)$  is the identity map. Hence, in dimension 0,  $\text{sd}_X : S_0(X) \rightarrow S_0(X)$  is trivially augmentation preserving as the following diagram commutes:

$$\begin{array}{ccc} S_0(X) & \xrightarrow{\text{sd}_X} & S_0(X) \\ \epsilon \downarrow & & \downarrow \epsilon \\ \mathbb{Z} & \xrightarrow{\text{id}_{\mathbb{Z}}} & \mathbb{Z}. \end{array}$$

Let us immediately find that the naturality of  $\text{sd}_X$  in dimension 0 holds. It follows trivially from the following commutative diagram.

$$\begin{array}{ccc} S_0(X) & \xrightarrow{(f_\#)_0} & S_0(Y) \\ \text{sd}_X = \text{id} \downarrow & & \downarrow \text{sd}_Y = \text{id} \\ S_0(X) & \xrightarrow{(f_\#)_0} & S_0(Y). \end{array}$$

Now, let's verify naturality in positive dimensions. Let  $T : \Delta_p \rightarrow X$  be continuous. Then

$$(f_\#)_p (\text{sd}_X T) = (f_\#)_p \left[ (T_\#)_p (\text{sd}_{\Delta_p} i_p) \right] = \left( (f \circ T)_\# \right)_p (\text{sd}_{\Delta_p} i_p). \quad (2.53)$$

Now,  $f \circ T : \Delta_p \rightarrow Y$  is a singular  $p$ -simplex on  $Y$ . So we have

$$\text{sd}_Y (f \circ T) = \left( (f \circ T)_\# \right)_p (\text{sd}_{\Delta_p} i_p). \quad (2.54)$$

Now, 2.53 and 2.54 together imply

$$(f_\#)_p (\text{sd}_X T) = \text{sd}_Y (f \circ T) = \text{sd}_Y \left( (f_\#)_p T \right). \quad (2.55)$$

Therefore,  $(f_\#)_p \circ \text{sd}_X = \text{sd}_Y \circ (f_\#)_p$ .

Finally, we shall prove that  $\text{sd}_X$  is a chain map by induction. We need to verify that  $\text{sd}$  commutes with the boundary operator. The fact that  $\text{sd}$  commutes with the boundary homomorphism in dimension 0 follows trivially from the following commutative diagram.

$$\begin{array}{ccc} S_0(X) & \xrightarrow{\text{sd}_X = \text{id}_{S_0(X)}} & S_0(X) \\ \partial_0 \downarrow & & \downarrow \partial_0 \\ 0 & \xrightarrow{\text{id}} & 0. \end{array}$$

Now, assume that the result holds true in dimension less than  $p$ . Now,

$$\partial_p (\text{sd}_{\Delta_p} i_p) = (-1)^p \partial_p [\text{sd}_{\Delta_p} \partial i_p, \widehat{\Delta_p}], \quad (2.56)$$

where  $i_p : \Delta_p \rightarrow \Delta_p$  is the identity map.  $\Delta_p$  is star convex with respect to  $\widehat{\Delta_p}$ , and  $\text{sd}_{\Delta_p} \partial i_p$  is a  $(p-1)$ -chain of  $\Delta_p$ . Then by Lemma 1.5,

$$\begin{aligned} \partial_p [\text{sd}_{\Delta_p} \partial i_p, \widehat{\Delta_p}] &= \begin{cases} [\partial_{p-1} (\text{sd}_{\Delta_p} \partial_p i_p), \widehat{\Delta_p}] + (-1)^p \text{sd}_{\Delta_p} \partial_p i_p & \text{if } p-1 > 0 \\ \epsilon (\text{sd}_{\Delta_p} \partial_p i_p) T_0 - \text{sd}_{\Delta_p} \partial i_p & \text{if } p-1 = 0 \end{cases} \\ &= \begin{cases} [\partial_{p-1} (\text{sd}_{\Delta_p} \partial_p i_p), \widehat{\Delta_p}] + (-1)^p \text{sd}_{\Delta_p} \partial_p i_p & \text{if } p > 1 \\ \epsilon (\text{sd}_{\Delta_1} \partial_1 i_1) T_0 - \text{sd}_{\Delta_1} \partial_1 i_1 & \text{if } p = 1 \end{cases}, \end{aligned} \quad (2.57)$$

where  $T_0$  is the singular 0-simplex whose image point is  $\widehat{\Delta_1}$ , the barycenter of  $\Delta_1$ . If  $p = 1$ , since  $\text{sd}$  is augmentation preserving, the following diagram commutes:

$$\begin{array}{ccc} S_0(\Delta_1) & \xrightarrow{\text{sd}_{\Delta_1} = \text{id}_{S_0(\Delta_1)}} & S_0(\Delta_1) \\ \epsilon \downarrow & & \downarrow \epsilon \\ \mathbb{Z} & \xrightarrow{\text{id}_{\mathbb{Z}}} & \mathbb{Z}. \end{array}$$

So we get for  $\partial_1 i_1 \in S_0(\Delta_1)$ ,

$$\epsilon (\text{sd}_{\Delta_1} \partial_1 i_1) = \epsilon (\partial_1 i_1) = 0. \quad (2.58)$$

For  $p > 1$ , by the inductive hypothesis, the following diagram commutes:

$$\begin{array}{ccc} S_{p-1}(\Delta_p) & \xrightarrow{\text{sd}_{\Delta_p}} & S_{p-1}(\Delta_p) \\ \partial_{p-1} \downarrow & & \downarrow \partial_{p-1} \\ S_{p-2}(\Delta_p) & \xrightarrow{\text{sd}_{\Delta_p}} & S_{p-2}(\Delta_p). \end{array}$$

Hence, for  $\partial_p i_p \in S_{p-1}$ ,

$$\partial_{p-1} (\text{sd}_{\Delta_p} \partial_p i_p) = \text{sd}_{\Delta_p} \partial_{p-1} \partial_p i_p = 0. \quad (2.59)$$

Now, combining 2.58, 2.59 and plugging them into 2.57, we get

$$\partial_p [\text{sd}_{\Delta_p} \partial i_p, \widehat{\Delta_p}] = (-1)^p \text{sd}_{\Delta_p} \partial_p i_p \quad (2.60)$$

in both cases. Therefore, 2.56 gives us

$$\partial_p (\text{sd}_{\Delta_p} i_p) = \text{sd}_{\Delta_p} \partial_p i_p, \quad \forall p. \quad (2.61)$$

Now, in general, for  $T : \Delta_p \rightarrow X$  continuous,

$$\partial_p (\text{sd}_X T) = \partial_p [(T_{\#})_p (\text{sd}_{\Delta_p} i_p)] = (T_{\#})_{p-1} [\partial_p (\text{sd}_{\Delta_p} i_p)], \quad (2.62)$$

since  $T_{\#}$  is a chain map and hence the following diagram commutes.

$$\begin{array}{ccc} S_p(\Delta_p) & \xrightarrow{(T_{\#})_p} & S_p(X) \\ \partial_p \downarrow & & \downarrow \partial_p \\ S_{p-1}(\Delta_p) & \xrightarrow{(T_{\#})_{p-1}} & S_{p-1}(X) \end{array}$$

So

$$\partial_p (\text{sd}_X T) = (T_\#)_{p-1} [\partial_p (\text{sd}_{\Delta_p} i_p)] = (T_\#)_{p-1} (\text{sd}_{\Delta_p} \partial_p i_p) = \text{sd}_X (T_\#)_{p-1} (\partial_p i_p), \quad (2.63)$$

using the naturality of  $\text{sd}$ . Hence,

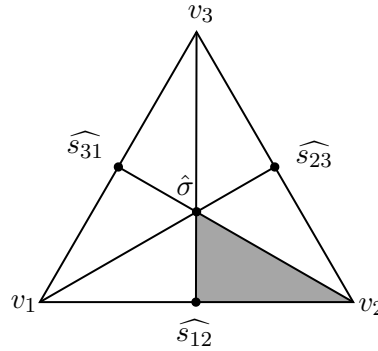
$$\partial_p (\text{sd}_X T) = \text{sd}_X (T_\#)_{p-1} (\partial_p i_p) = \text{sd}_X \partial_p (\text{sd}_X (T_\#)_p i_p). \quad (2.64)$$

Now,  $(T_\#)_p i_p = T \circ i_p = T$ . Therefore,

$$\partial_p (\text{sd}_X T) = \text{sd}_X \partial_p T. \quad (2.65)$$

So  $\text{sd}_X$  indeed commutes with the boundary operator, and hence is a chain map.  $\blacksquare$

Consider  $\sigma = \Delta_2$  and its first barycentric subdivision.



Denote  $v_1v_2$ ,  $v_2v_3$  and  $v_3v_1$  by  $s_{12}$ ,  $s_{23}$  and  $s_{31}$ , respectively. Denote the barycenter of  $\sigma$  by  $\hat{\sigma}$ , barycenter of  $s_{12}$  by  $\hat{s}_{12}$  and so on. Observe that, for 0-simplices  $v_1, v_2, v_3$ , their barycenters are just themselves, i.e.  $\hat{v}_i = v_i$  for  $i = 1, 2, 3$ . Then we have a natural ordering. For example,  $\sigma \succ s_{12} \succ v_2$ , meaning  $s_{12}$  is a proper face of  $\sigma$ ,  $v_2$  is a proper face of  $s_{12}$ . Then we have a distinct 2-simplex  $\hat{\sigma}\hat{s}_{12}\hat{v}_2$  (colored gray in the above image) by joining the 3 barycenters  $\hat{\sigma}, \hat{s}_{12}, \hat{v}_2$ . This 2-simplex belongs to the first barycentric subdivision of  $\Delta_2$ , which we denote by  $\text{Sd } \Delta_2$ <sup>1</sup>.

The first barycentric subdivision of  $\Delta_2$  contains also the following 2-simplices:  $\hat{\sigma}\hat{s}_{12}\hat{v}_1$ ,  $\hat{\sigma}\hat{s}_{23}\hat{v}_2$ ,  $\hat{\sigma}\hat{s}_{23}\hat{v}_3$ ,  $\hat{\sigma}\hat{s}_{31}\hat{v}_1$ ,  $\hat{\sigma}\hat{s}_{31}\hat{v}_3$ . It contains the following 1-simplices:  $\hat{s}_{12}\hat{v}_1$ ,  $\hat{s}_{12}\hat{v}_2$ ,  $\hat{s}_{23}\hat{v}_2$ ,  $\hat{s}_{23}\hat{v}_3$ ,  $\hat{s}_{31}\hat{v}_1$ ,  $\hat{s}_{31}\hat{v}_3$  and the 0-simplices  $\hat{v}_1$ ,  $\hat{v}_2$ ,  $\hat{v}_3$ ,  $\hat{s}_{12}$ ,  $\hat{s}_{23}$ ,  $\hat{s}_{31}$ ,  $\hat{\sigma}$ . We then have the following result:

### Lemma 2.10

Let  $K$  be a simplicial complex. The complex  $\text{Sd } K$  equals the collection of all simplices of the form

$$\hat{\sigma}_1 \hat{\sigma}_2 \cdots \hat{\sigma}_n,$$

where  $\hat{\sigma}_1 \succ \hat{\sigma}_2 \succ \cdots \succ \hat{\sigma}_n$ .

The proof of this lemma is omitted.

### Lemma 2.11

Let  $T : \Delta_p \rightarrow \sigma$  be a linear homeomorphism of  $\Delta_p$  with the  $p$ -simplex  $\sigma$ . Then each term of  $\text{sd}_\sigma T$  is a linear homeomorphism of  $\Delta_p$  with a simplex in the first barycentric subdivision of  $\sigma$ .

*Proof.* When  $p = 0$ ,  $\sigma$  is a 0-simplex and the first barycentric subdivision of  $\sigma$  contains just the 0-simplex  $\sigma$ . And, given linear homeomorphism  $T : \Delta_0 \rightarrow \sigma$ ,  $\text{sd}_\sigma T = T$  is the same linear homeomorphism of  $\Delta_0$  with the only simplex  $\sigma$  in the first barycentric subdivision of  $\sigma$ .

<sup>1</sup>Note that, the subdivision operator  $\text{sd}_X : S_p(X) \rightarrow S_p(X)$  is written  $\text{sd}$ , and the barycentric subdivision of a simplicial complex (which we studied in AT2) is denoted by  $\text{Sd}$ , to avoid confusion.

Now, suppose the lemma is true in dimension less than  $p$ . Consider the identity homeomorphism  $i_p : \Delta_p \rightarrow \Delta_p$ . Now,

$$\text{sd}_{\Delta_p} i_p = (-1)^p \left[ \text{sd}_{\Delta_p} \partial i_p, \widehat{\Delta_p} \right].$$

Note that

$$\partial i_p = \sum_{j=0}^p (-1)^j i_p \circ l_{(\varepsilon_0, \dots, \widehat{\varepsilon_j}, \dots, \varepsilon_p)}$$

so that each term in this sum is a linear homeomorphism of  $\Delta_{p-1}$  with a  $(p-1)$ -simplex in  $\text{Bd } \Delta_p$ .

$$\text{sd}_{\Delta_p} \partial i_p = \sum_{j=0}^p (-1)^j \text{sd}_{\Delta_p} \left( i_p \circ l_{(\varepsilon_0, \dots, \widehat{\varepsilon_j}, \dots, \varepsilon_p)} \right).$$

By the inductive hypothesis, each term of  $\text{sd}_{\Delta_p} \left( i_p \circ l_{(\varepsilon_0, \dots, \widehat{\varepsilon_j}, \dots, \varepsilon_p)} \right)$  is a linear homeomorphism of  $\Delta_{p-1}$  with a  $(p-1)$ -simplex  $\widehat{s_1} \widehat{s_2} \cdots \widehat{s_p}$  in the first barycentric subdivision of  $\text{Bd } \Delta_p$ .

$$\text{sd}_{\Delta_p} \left( i_p \circ l_{(\varepsilon_0, \dots, \widehat{\varepsilon_j}, \dots, \varepsilon_p)} \right) = \sum_k \pm T_{jk}, \quad (2.66)$$

where  $T_{jk}$  is a linear homeomorphism of  $\Delta_{p-1}$  with a  $(p-1)$ -simplex  $\widehat{s_1} \widehat{s_2} \cdots \widehat{s_p}$  in the first barycentric subdivision of  $\text{Bd } \Delta_p$ . So

$$\text{sd}_{\Delta_p} \partial i_p = \sum_{j=0}^p \sum_k \pm T_{jk}. \quad (2.67)$$

Then  $\left[ T_{jk}, \widehat{\Delta_p} \right]$  is by definition a linear homeomorphism of  $\Delta_p$  with the  $p$ -simplex  $\widehat{\Delta_p} \widehat{s_1} \widehat{s_2} \cdots \widehat{s_p}$ , which belongs to the first barycentric subdivision of  $\Delta_p$ . Now,

$$\text{sd}_{\Delta_p} i_p = \sum_{j=0}^p \sum_k \pm \left[ T_{jk}, \widehat{\Delta_p} \right]. \quad (2.68)$$

Therefore, each term of  $\text{sd}_{\Delta_p} i_p$  is a linear homeomorphism of  $\Delta_p$  with a  $p$ -simplex in the first barycentric subdivision of  $\Delta_p$ .

Now consider a general linear homeomorphism  $T : \Delta_p \rightarrow \sigma$ . It's clear that  $T$  defines a linear homeomorphism between the first barycentric subdivision of  $\Delta_p$  with that of  $\sigma$ , because  $T$  takes barycenter of  $\Delta_p$  to the barycenter of  $\sigma$  (since  $T$  is linear).

$$\text{sd}_\sigma T = (T_\#)_p (\text{sd}_{\Delta_p} i_p),$$

with  $T : \Delta_p \rightarrow \sigma$  being a linear homeomorphism. Using 2.68,

$$\text{sd}_\sigma T = \sum_{j=0}^p \sum_k \pm T \circ \left[ T_{jk}, \widehat{\Delta_p} \right]. \quad (2.69)$$

By construction,  $\left[ T_{jk}, \widehat{\Delta_p} \right] : \Delta_p \rightarrow \text{Sd}(\Delta_p)$  is a linear homeomorphism onto its image, and  $T : \Delta_p \rightarrow \sigma$  is a given linear homeomorphism. Hence, the composite  $T \circ \left[ T_{jk}, \widehat{\Delta_p} \right] : \Delta_p \rightarrow \sigma$  is a linear homeomorphism.

$\left[ T_{jk}, \widehat{\Delta_p} \right]$  takes  $\Delta_p$  linear homeomorphically to a  $p$ -simplex in the first barycentric subdivision of  $\Delta_p$  and we have seen that  $T$  is a linear homeomorphism between the first barycentric subdivision of  $\Delta_p$  with that of  $\sigma$ . Hence,  $T \circ \left[ T_{jk}, \widehat{\Delta_p} \right]$  takes  $\Delta_p$  linear homeomorphically to a  $p$ -simplex in the first barycentric subdivision of  $\sigma$ . So the terms of  $\text{sd}_\sigma T$  are linear homeomorphisms of  $\Delta_p$  with a  $p$ -simplex in the first barycentric subdivision of  $\sigma$ . ■

**Theorem 2.12**

Let  $\mathcal{A}$  be a collection of subsets of  $X$  whose interiors cover  $X$ . Given  $T : \Delta_p \rightarrow X$ , there is an  $m$  such that each term of  $\text{sd}_X^m T$  is  $\mathcal{A}$ -small.

*Proof.* Apply Lemma 2.11 to each term of  $\text{sd}_\sigma L$ , where  $L : \Delta_p \rightarrow \sigma$  is a linear homeomorphism of  $\Delta_p$  with a  $p$ -simplex  $\sigma$ . Each term of  $\text{sd}_\sigma L$  is a linear homeomorphism of  $\Delta_p$  with a simplex in  $\text{Sd } \sigma$ . Then each term of  $\text{sd}_\sigma^2 L$  is a linear homeomorphism of  $\Delta_p$  with a simplex in  $\text{Sd}^2 \sigma$ . More generally, each term of  $\text{sd}_\sigma^m L$  is a linear homeomorphism of  $\Delta_p$  with a simplex in the  $m$ -th barycentric subdivision of  $\sigma$ , i.e.  $\text{Sd}^m \sigma$ .

Now,  $\{\text{Int } A \mid A \in \mathcal{A}\}$  covers  $X$ . Let us first cover  $\Delta_p$  by open sets  $T^{-1}(\text{Int } A)$  with  $A \in \mathcal{A}$ .  $\Delta_p$  is a compact metric space. Let  $\lambda$  be the Lebesgue number associated with this cover  $\{T^{-1}(\text{Int } A) \mid A \in \mathcal{A}\}$  of  $\Delta_p$ . So every subset of  $\Delta_p$  with diameter less than  $\lambda$  must be contained in  $T^{-1}(\text{Int } A)$  for some  $A \in \mathcal{A}$ .

Now, choose  $m$  large enough such that each simplex in the  $m$ -th barycentric subdivision has diameter less than  $\lambda$ . Now, in the opening paragraph of the proof, take  $L = i_p : \Delta_p \rightarrow \Delta_p$ , the identity map from  $\Delta_p$  to itself. Then each term of  $\text{sd}_{\Delta_p}^m i_p$  is a linear homeomorphism of  $\Delta_p$  with a  $p$ -simplex in the  $m$ -th barycentric subdivision of  $\Delta_p$ , each of which has diameter smaller than  $\lambda$ .

Then by *Lebesgue number lemma*, the image of each term of  $\text{sd}_{\Delta_p}^m i_p$  is contained in  $T^{-1}(\text{Int } A)$  for some  $A \in \mathcal{A}$ . So,  $T$  composed with each term of  $\text{sd}_{\Delta_p}^m i_p$  is contained in  $\text{Int } A$  for some  $A \in \mathcal{A}$ . But  $T$  composed with each term of  $\text{sd}_{\Delta_p}^m i_p$  is nothing but each term of

$$(T_\#)_p \left( \text{sd}_{\Delta_p}^m i_p \right) = \text{sd}_X^m T. \quad (2.70)$$

Hence, each term of  $\text{sd}_X^m T$  has its image set contained in  $\text{Int } A$ . In other words, each term of  $\text{sd}_X^m T$  is  $\mathcal{A}$ -small. ■

**Remark 2.2.**  $\text{sd}_X^m : S_p(X) \rightarrow S_p(X)$  is of course a map. In fact, it is a group homomorphism. But we can't talk about the image set of  $\text{sd}_X^m T$  even when  $T : \Delta_p \rightarrow X$  is a singular  $p$ -simplex of  $X$ , as  $\text{sd}_X^m T$  is, in general, a  $p$ -chain, not a singular  $p$ -simplex.

Having shown how to chop up singular chains so that they are  $\mathcal{A}$ -small, we now show that these  $\mathcal{A}$ -small singular chains suffice to generate the homology of  $X$ . We first need a lemma.

**Lemma 2.13**

Let  $m$  be given. For each space  $X$ , there is a homomorphism  $D_p^X : S_p(X) \rightarrow S_{p+1}(X)$  such that for each singular  $p$ -simplex  $T$  of  $X$ ,

$$\partial_{p+1} D_p^X T + D_{p-1}^X \partial_p T = \text{sd}_X^m T - \text{id}_{S_p(X)} T. \quad (2.71)$$

Furthermore,  $D^X$  is natural; i.e., for continuous  $f : X \rightarrow Y$ , the following diagram commutes

$$\begin{array}{ccc} S_p(X) & \xrightarrow{(f_\#)_p} & S_p(Y) \\ D_p^X \downarrow & & \downarrow D_p^Y \\ S_{p+1}(X) & \xrightarrow{(f_\#)_{p+1}} & S_{p+1}(Y). \end{array}$$

In other words,  $D_p^Y \circ (f_\#)_p = (f_\#)_{p+1} \circ D_p^X$ .

**Remark 2.3.** The above lemma guarantees that there is a chain homotopy  $D^X$  between the chain maps  $\text{sd}_X^m, \text{id}_{\mathcal{S}(X)} : \mathcal{S}(X) \rightarrow \mathcal{S}(X)$ . Also, note that the naturality of  $\text{sd}_X^m$  and  $D^X$  shows that if  $A$  is a subspace of  $X$ , then  $\text{sd}_X^m$  and  $D^X$  carry  $S_p(A)$  into  $S_p(A)$  and  $S_{p+1}(A)$ , respectively. Thus they induce a chain map and a chain homotopy, respectively, on the relative chain complex  $\mathcal{S}(X, A)$  as well.