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Algebriac Topology III (MAT484)

Lecture Notes

Preface

This series of lecture notes has been prepared for aiding students who took the BRAC University course **Algebraic Topology III (MAT484)** in Spring 2023 semester. These notes were typeset under the supervision of mathematician **Dr. Syed Hasibul Hassan Chowdhury**. The main goal of this typeset is to have an organized digital version of the notes, which is easier to share and handle. Lecture notes of the previous Algebraic Topology courses can be found in the following links.

- **Algebraic Topology I (MAT431)**: https://atonurc.github.io/assets/MAT431_AT1.pdf
- **Algebraic Topology II (MAT432)**: https://atonurc.github.io/assets/MAT432_AT2.pdf

If you see any mistakes or typos, please send me an email at atonuroychowdhury@gmail.com

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References:

- *Elements of Algebraic Topology*, by James R. Munkres
- *Topology*, by Klaus Jänich, translated by Silvio Levy.
- Note on CW Complexes, by Soren Hansen. Link: https://www.math.ksu.edu/~hansen/CWcomplexes.pdf

Contents

1 Singular Homology Theory

§1.1 Singular Homology Groups

Let $\mathbb{R}^\infty$ denote the generalized Euclidean space $\mathbb{E}^J$, with J being the set of positive integers. An element of the vector space $\mathbb{R}^\infty$ is an infinite sequence of real numbers (functions from $\mathbb{N}$ to $\mathbb{R}$) with finitely many nonzero entries. Let Δ_p denote the p-simplex in $\mathbb{R}^\infty$ having vertices

$$\begin{aligned}
\varepsilon_0 &= (1,0,0,\ldots,0,\ldots)\,, \\
\varepsilon_1 &= (0,1,0,\ldots,0,\ldots)\,, \\
&\cdots \\
\varepsilon_p &= (0,0,0,\ldots,\underbrace{1}_{(p+1)\text{-th entry}},\ldots)\,.
\end{aligned}$$

We call Δ_p the **standard p-simplex**. In this notation, Δ_{p-1} is a face of Δ_p.

> **Definition 1.1** (Singular p-simplex)**.** Let X be a topological space. We define a **singular p-simplex** of X to be a continuous map $T:\Delta_p \to X$. The free abelian group generated by singular p-simplices of X is denoted by $S_p(X)$, and is called the **singular chain group** of X in dimension p. We shall denote an element of $S_p(X)$ by a $\mathbb{Z}$-linear combination of singular p-simplices of X.

Singular means that T could be a "bad" map, i.e. it may not be an imbedding. All we want that T is just continuous. Now, recall that

$$\Delta_p = \left\{(x_0,x_1,\ldots,x_p,0,\ldots)\in\mathbb{R}^\infty \middle| 0\le x_i\le 1 \text{ and } \sum_{i=0}^p x_i = 1\right\}. \tag{1.1}$$

Given $a_0,a_1,\ldots,a_p\in\mathbb{R}^\infty$, there is a unique affine map $l_{(a_0,\ldots,a_p)}:\Delta_p\to\mathbb{R}^\infty$ that maps ε_i to a_i. It is defined by

$$\begin{aligned}
l_{(a_0,\ldots,a_p)}(x_0,x_1,\ldots,x_p,0,\ldots) &= \sum_{i=0}^p x_i a_i = \sum_{i=0}^p x_i a_i + a_0 - \sum_{i=0}^p x_i a_0 \\
&= a_0 + \sum_{i=0}^p x_i(a_i - a_0)\,.
\end{aligned} \tag{1.2}$$

We call this map the **linear singular simplex** determined by $a_0,a_1,\ldots,a_p\in\mathbb{R}^\infty$. Now, what is $l_{(\varepsilon_0,\ldots,\varepsilon_p)}$? Observe that

$$l_{(\varepsilon_0,\ldots,\varepsilon_p)}\varepsilon_i = l_{(\varepsilon_0,\ldots,\varepsilon_p)}(0,\ldots,0,\underbrace{1}_{(i+1)\text{-th entry}},0,\ldots) = \varepsilon_i. \tag{1.3}$$

Therefore, $l_{(\varepsilon_0,\ldots,\varepsilon_p)}$ maps ε_i to itself, for every $i=0,1,\ldots,p$. Also,

$$l_{(\varepsilon_0,\ldots,\varepsilon_p)}(x_0,x_1,\ldots,x_p,0,\ldots) = \sum_{i=0}^p x_i\varepsilon_i = (x_0,x_1,\ldots,x_p,0,\ldots)\,. \tag{1.4}$$

Therefore, $l_{(\varepsilon_0,\ldots,\varepsilon_p)}$ is just the inclusion map of Δ_p into $\mathbb{R}^\infty$. Now, suppose $(x_0,x_1,\ldots,x_{p-1},0,\ldots)\in\Delta_{p-1}$, so that $\sum_{i=0}^{p-1}x_i = 1$. Then

$$\begin{aligned}
l_{(\varepsilon_0,\ldots,\widehat{\varepsilon_i},\ldots,\varepsilon_p)}(x_0,x_1,\ldots,x_{p-1},0,\ldots) &= x_0\varepsilon_0+\cdots+x_{i-1}\varepsilon_{i-1}+0\cdot\varepsilon_i+x_{i+1}\varepsilon_{i+1}+\cdots+x_{p-1}\varepsilon_p \\
&= (x_0,\ldots,x_{i-1},0,x_{i+1},\ldots,x_{p-1},0,\ldots)\,,
\end{aligned} \tag{1.5}$$

which is a point on the face of Δ_p opposite to the vertex ε_i. In fact, $l_{(\varepsilon_0,\ldots,\widehat{\varepsilon_i},\ldots,\varepsilon_p)}$ is a linear homomorphism of Δ_{p-1} into the face of Δ_p that is opposite to the vertex ε_i. In other words,

$$l_{(\varepsilon_0,\ldots,\widehat{\varepsilon_i},\ldots,\varepsilon_p)} : \Delta_{p-1} \to \Delta_p$$

maps Δ_{p-1} to the face of Δ_p opposite to the vertex ε_i. Therefore, given a singular p-simplex $T : \Delta_p \to X$, one can form the composite

$$T \circ l_{(\varepsilon_0,\ldots,\widehat{\varepsilon_i},\ldots,\varepsilon_p)} : \Delta_{p-1} \to X,$$

which is a singular $(p-1)$-simplex. We think of it as the i-th face of the singular p-simplex T.

Definition 1.2 (Boundary homomorphism). We define $\partial : S_p(X) \to S_{p-1}(X)$ as follows. If $T : \Delta_p \to X$ is a singular p-simplex, we define ∂T to be

$$\partial T = \sum_{i=0}^{p} (-1)^i \, T \circ l_{(\varepsilon_0,\ldots,\widehat{\varepsilon_i},\ldots,\varepsilon_p)}. \tag{1.6}$$

In other words, ∂T is a formal sum of singular simplices of dimension $p-1$, which are the faces of T.

Remark 1.1 (IMPORTANT!). Note that only the singular p-simplices are maps, not the singular p-chains. The p-chains are just formal sum of continuous maps from Δ_p to X. If T_1 and T_2 are two singular p-simplices, i.e. continuous maps $\Delta_p \to X$, then $T_1 + T_2$ is **NOT** a map. The sum present here is nothing but a formal notation. So one cannot act $T_1 + T_2$ on a point of Δ_p. For the same reason, ∂T_1 is not a map. It is merely a formal linear combination of the continuous maps $T \circ l_{(\varepsilon_0,\ldots,\widehat{\varepsilon_i},\ldots,\varepsilon_p)}$.

If $f : X \to Y$ is a continuous map, we define a group homomorphism $f_\# : S_p(X) \to S_p(Y)$ by defining it on singular p-simplices by the equation

$$f_\#(T) = f \circ T \tag{1.7}$$

for a singular p-simplex T.

$$\begin{tikzcd} \Delta_p \arrow[r, "T"] \arrow[rr, bend right, "f\circ T"'] & X \arrow[r, "f"] & Y \end{tikzcd}$$

Theorem 1.1

The homomorphism $f_\#$ commutes with ∂. Furthermore, $\partial^2 = 0$.

Proof. Given a singular p-simplex T,

$$\partial f_\#(T) = \partial(f \circ T) = \sum_{i=0}^{p} (-1)^i (f \circ T) \circ l_{(\varepsilon_0,\ldots,\widehat{\varepsilon_i},\ldots,\varepsilon_p)}. \tag{1.8}$$

$$f_\#(\partial T) = f_\#\left(\sum_{i=0}^{p} (-1)^i \, T \circ l_{(\varepsilon_0,\ldots,\widehat{\varepsilon_i},\ldots,\varepsilon_p)}\right) = \sum_{i=0}^{p} (-1)^i f \circ T \circ l_{(\varepsilon_0,\ldots,\widehat{\varepsilon_i},\ldots,\varepsilon_p)}. \tag{1.9}$$

Therefore, $\partial f_\#(T) = f_\#(\partial T)$. Now, to prove $\partial^2 = 0$, we first compute ∂ for linear singular simplices $l_{(a_0,\ldots,a_p)}$.

$$\partial l_{(a_0,\ldots,a_p)} = \sum_{i=0}^{p} (-1)^i \, l_{(a_0,\ldots,a_p)} \circ l_{(\varepsilon_0,\ldots,\widehat{\varepsilon_i},\ldots,\varepsilon_p)}. \tag{1.10}$$

Observe that

$$\begin{aligned} l_{(a_0,\dots,a_p)} \circ l_{(\varepsilon_0,\dots,\widehat{\varepsilon}_i,\dots,\varepsilon_p)} (x_0,\dots,x_{p-1},0,\dots) &= l_{(a_0,\dots,a_p)} (x_0,\dots,x_{i-1},0,x_i x_{p-1},0,) \\ &= x_0 a_0 + \cdots + x_{i-1}a_{i-1} + 0 \cdot a_i + x_i a_{i+1} + \cdots x_{p-1}a_p \\ &= l_{(a_0,\dots,\widehat{a}_i,\dots a_p)} (x_0,\dots,x_{p-1},0,\dots). \end{aligned} \tag{1.11}$$

Hence,

$$l_{(a_0,\dots,a_p)} \circ l_{(\varepsilon_0,\dots,\widehat{\varepsilon}_i,\dots,\varepsilon_p)} = l_{(a_0,\dots,\widehat{a}_i,\dots a_p)}. \tag{1.12}$$

Therefore, from 1.10, it follows that

$$\partial l_{(a_0,\dots,a_p)} = \sum_{i=0}^{p} (-1)^i \, l_{(a_0,\dots,\widehat{a}_i,\dots a_p)}. \tag{1.13}$$

Let's now evaluate $\partial\partial l_{(a_0,\dots,a_p)}$.

$$\begin{aligned} \partial\partial l_{(a_0,\dots,a_p)} &= \sum_{i=0}^{p} (-1)^i \, \partial l_{(a_0,\dots,\widehat{a}_i,\dots a_p)} \\ &= \sum_{i=0}^{p} (-1)^i \sum_{j<i} (-1)^j \, l_{(a_0,\dots,\widehat{a}_j,\dots\widehat{a}_i,\dots a_p)} + \sum_{i=0}^{p} (-1)^i \sum_{j>i} (-1)^{j-1} \, l_{(a_0,\dots,\widehat{a}_i,\dots\widehat{a}_j,\dots a_p)} \\ &= \sum_{i=0}^{p} \sum_{j<i} (-1)^{i+j} \, l_{(a_0,\dots,\widehat{a}_j,\dots\widehat{a}_i,\dots a_p)} - \sum_{i=0}^{p} \sum_{j>i} (-1)^{i+j} \, l_{(a_0,\dots,\widehat{a}_i,\dots\widehat{a}_j,\dots a_p)}. \end{aligned} \tag{1.14}$$

Now fix $0 \le j_0 < i_0 \le p$. In the first summand of 1.14, the contribution of $i = i_0, j = j_0$ is

$$(-1)^{i_0+j_0} \, l_{\left(a_0,\dots,\widehat{a_{j_0}},\dots\widehat{a_{i_0}},\dots a_p\right)}. \tag{1.15}$$

On the other hand, in the second summand of 1.14, the contribution of $i = j_0, j = i_0$ is also

$$(-1)^{i_0+j_0} \, l_{\left(a_0,\dots,\widehat{a_{j_0}},\dots\widehat{a_{i_0}},\dots a_p\right)}. \tag{1.16}$$

These two contributions cancel each other. This way, one finds that the RHS of 1.14 vanishes. Hence,

$$\partial\partial l_{(a_0,\dots,a_p)} = 0. \tag{1.17}$$

In particular,

$$\partial\partial l_{(\varepsilon_0,\dots,\varepsilon_p)} = 0. \tag{1.18}$$

Now, $l_{(\varepsilon_0,\dots,\varepsilon_p)} : \Delta_p \to \Delta_p$ is continuous, so $l_{(\varepsilon_0,\dots,\varepsilon_p)} \in S_p(\Delta_p)$. Furthermore, it is the identity map as we have seen in 1.4. Since $T : \Delta_p \to X$ is continuous, we can form $T_\# : S_p(\Delta_p) \to S_p(X)$.

$$T_\# \left(l_{(\varepsilon_0,\dots,\varepsilon_p)} \right) = T \circ l_{(\varepsilon_0,\dots,\varepsilon_p)} = T \circ \mathrm{id}_{\Delta_p} = T. \tag{1.19}$$

Therefore, using the fact that $T_\#$ commutes with ∂, we obtain

$$\partial\partial T = \partial\partial T_\# \left(l_{(\varepsilon_0,\dots,\varepsilon_p)} \right) = T_\# \left(\partial\partial l_{(\varepsilon_0,\dots,\varepsilon_p)} \right) = 0. \tag{1.20}$$

Hence, $\partial^2 T = 0$. ■

Definition 1.3 (Singular homology groups)**.** Th family of groups $S_p(X)$ and homomorphisms $\partial_p : S_p(X) \to S_{p-1}(X)$ is called **singular chain complex** of X, and is denoted by $\mathcal{S}(X)$.

$$\cdots \longrightarrow S_{p+1}(X) \xrightarrow{\partial_{p+1}} S_p(X) \xrightarrow{\partial_p} S_{p-1}(X) \longrightarrow \cdots$$

The homology groups of this chain complex are called the **singular homology groups** of X, and are denoted by $H_p(X)$.

Definition 1.4 (Augmentation map)**.** The chain complex $\mathcal{S}(X)$ is augmented by the homomorphism $\epsilon : S_0(X) \to \mathbb{Z}$ defined by setting $\epsilon(T) = 1$ for each singular 0-simplex $T : \Delta_0 \to X$. (A generic singular 0-chain is a $\mathbb{Z}$-linear combination of singular 0-simplices.)

It's immediate that if T is a singular 1-simplex, then $\epsilon(\partial T) = 0$. Indeed,

$$\epsilon(\partial T) = \epsilon\left(T \circ l_{(\widehat{\varepsilon_0}, \varepsilon_1)}\right) - \epsilon\left(T \circ l_{(\varepsilon_0, \widehat{\varepsilon_1})}\right) = 0. \tag{1.21}$$

Definition 1.5 (Reduced homology groups)**.** The homology groups of $\{\mathcal{S}(X), \epsilon\}$ are called the **reduced singular homology groups** of X, and are denoted by $\widetilde{H}_p(X)$.

Now, given continuous map $f : X \to Y$ and $T : \Delta_0 \to X$ a singular 0-simplex on X, then $f_\#(T) = f \circ T : \Delta_0 \to Y$.

$$\begin{tikzcd} \Delta_0 \arrow[r, "T"] \arrow[rr, bend right, "f\circ T"'] & X \arrow[r, "f"] & Y \end{tikzcd}$$

Now, consider the augmented singular chain complexes $\left\{\mathcal{S}(X), \epsilon^X\right\}$ and $\left\{\mathcal{S}(Y), \epsilon^Y\right\}$. Noting continuous $T : \Delta_0 \to X$ and $f_\#(T) : \Delta_0 \to Y$, one obtains $\epsilon^X(T) = 1$ and $\epsilon^Y(f_\#(T)) = 1$. In other words, the following diagram commutes

$$\begin{tikzcd} S_0(X) \arrow[r, "\epsilon^X"] \arrow[d, "(f_\#)_0"'] & \mathbb{Z} \arrow[d, "\mathrm{id}"] \\ S_0(Y) \arrow[r, "\epsilon^Y"] & \mathbb{Z} \end{tikzcd}$$

Therefore, $f_\# : S_p(X) \to S_p(Y)$ is an **augmentation preserving chain map** between $\left\{\mathcal{S}(X), \epsilon^X\right\}$ and $\left\{\mathcal{S}(Y), \epsilon^Y\right\}$. Thus, $f_\#$ induces a homomorphism f_* in both ordinary and reduced singular homology.

In Theorem 1.1, we saw that the chain map $f_\#$ commutes with the boundary operator ∂. In other words, $(f_\#)_p : S_p(X) \to S_p(Y)$ takes cycles to cycles and boundaries to boundaries. Suppose $c_p \in Z_p(X) = \operatorname{Ker} \partial_p^X$, so that $\partial_p^X c_p = 0$. Now,

$$\partial_p^Y \left((f_\#)_p c_p\right) = (f_\#)_{p-1}\left(\partial_p^X c_p\right) = 0. \tag{1.22}$$

Hence, $(f_\#)_p c_p \in Z_p(Y)$. On the other hand, let $b_p \in B_p(X) = \operatorname{Im} \partial_{p+1}^X$. Then $b_p = \partial_{p+1}^X d_{p+1}$ for some $d_{p+1} \in S_{p+1}(X)$. Then

$$(f_\#)_p b_p = (f_\#)_p \left(\partial_{p+1}^X d_{p+1}\right) = \partial_{p+1}^Y \left((f_\#)_{p+1} d_{p+1}\right). \tag{1.23}$$

In other words, $(f_\#)_p b_p \in B_p(Y)$. This reflects the fact that $(f_\#)_p : S_p(X) \to S_p(Y)$ induces a homomorphism between the singular homology groups $(f_*)_p : H_p(X) \to H_p(Y)$. $(f_*)_p$ is given by

$$(f_*)_p (c_p + B_p(X)) = (f_\#)_p c_p + B_p(Y). \tag{1.24}$$

If the reduced homology groups of X vanishes in all dimensions, we say that X is **acyclic** (in singular homology).

Theorem 1.2

If $i : X \to X$ is the identity, then so is $(i_*)_p : H_p(X) \to H_p(X)$. If $f : X \to Y$ and $g : Y \to Z$ are continuous, then $((g \circ f)_*)_p = (g_*)_p \circ (f_*)_p$.

Proof. It is sufficient to show that the equations hold at the chain level. We know from the definition of $(f_\#)_p : S_p(X) \to S_p(Y)$ that it maps $T \in S_p(X)$ to $f \circ T \in S_p(Y)$. Since $i : X \to X$ is the identity map,

$$(i_\#)_p(T) = i \circ T = T. \tag{1.25}$$

So $(i_\#)_p : S_p(X) \to S_p(X)$ is the identity homomorphism. As a result,

$$(i_*)_p(c_p + B_p(X)) = (i_\#)_p c_p + B_p(X) = c_p + B_p(X). \tag{1.26}$$

Therefore, $(i_*)_p = \mathrm{id}_{H_p(X)}$.

Given continuous $f : X \to Y$ and $g : Y \to Z$, $\left((g \circ f)_\#\right)_p : S_p(X) \to S_p(Z)$ is defined by

$$\left((g \circ f)_\#\right)_p T = (g \circ f) \circ T = g \circ (f \circ T) = (g_\#)_p\left((f_\#)_p T\right). \tag{1.27}$$

Therefore, $\left((g \circ f)_\#\right)_p = (g_\#)_p \circ (f_\#)_p$. Now, at the homology level, for $c_p + B_p(X) \in H_p(X) = Z_p(X)/B_p(X)$

$$((g \circ f)_*)_p(c_p + B_p(X)) = \left((g \circ f)_\#\right)_p c_p + B_p(Z) = (g_\#)_p\left((f_\#)_p c_p\right) + B_p(Z). \tag{1.28}$$

Also,

$$(g_*)_p \circ (f_*)_p(c_p + B_p(X)) = (g_*)_p\left((f_\#)_p c_p + B_p(Y)\right) = (g_\#)_p\left((f_\#)_p c_p\right) + B_p(Z). \tag{1.29}$$

From 1.28 and 1.29, we can deduce that $((g \circ f)_*)_p = (g_*)_p \circ (f_*)_p$. ■

Corollary 1.3

If $h : X \to Y$ is a homeomorphims, then $(h_*)_p : H_p(X) \to H_p(Y)$ is an isomorphism.

Proof. Both $h : X \to Y$ and $h^{-1} : Y \to X$ are continuous, and $h \circ h^{-1} = \mathrm{id}_Y$. Therefore,

$$(h_*)_p \circ \left((h^{-1})_*\right)_p = \left((h \circ h^{-1})_*\right)_p = ((\mathrm{id}_Y)_*)_p = \mathrm{id}_{H_p(Y)}. \tag{1.30}$$

Similarly, starting with $h^{-1} \circ h = \mathrm{id}_X$, we will get $\left((h^{-1})_*\right)_p \circ (h_*)_p = \mathrm{id}_{H_p(X)}$. Therefore, $\left((h^{-1})_*\right)_p$ is the inverse of $(h_*)_p$. In other words, $(h_*)_p$ is an invertible homomorphism, i.e. an isomorphism. ■

Theorem 1.4

Let X be a topological space. Then $H_0(X)$ is free abelian. If $\{X_\alpha\}$ is the collection of path components of X, and if T_α is a singular 0-simplex with image in X_α for each α, then the homology classes of the chains T_α form a basis for $H_0(X)$. The group $\widetilde{H}_0(X)$ is also free abelian; it vanishes if X is path connected. Otherwise, let α_0 be a fixed index, then the homology classes of the chains $T_\alpha - T_{\alpha_0}$ for $\alpha \neq \alpha_0$ form a basis for $\widetilde{H}_0(X)$.

Proof. Let $x_\alpha = T_\alpha(\Delta_0) \in X_\alpha$, with $T_\alpha : \Delta_0 \to X$ being a singular 0-simplex. Here, Δ_0 consists of the point $\varepsilon_0 = (1,0,0,\ldots) \in \mathbb{R}^\infty$. Also, let $T : \Delta_0 \to X$ be any singular 0-simplex such that $T(\Delta_0) \in X_\alpha$. Since X_α is path connected, there is a path connecting $T(\Delta_0)$ and $T_\alpha(\Delta_0)$. In other words, there is a singular 1-simplex $f : \Delta_1 \to X$ such that

$$f(1,0,0\ldots) = T(\Delta_0) \text{ and } f(0,1,0\ldots) = T_\alpha(\Delta_0). \tag{1.31}$$

Then we have

$$\partial_1 f = f \circ l_{(\widehat{\varepsilon_0},\varepsilon_1)} - f \circ l_{(\varepsilon_0,\widehat{\varepsilon_1})}. \tag{1.32}$$

Now,

$$f \circ l_{(\varepsilon_0,\widehat{\varepsilon_1})}(1,0,0,\ldots) = f(1,0,0,\ldots) = T(\Delta_0) = T(1,0,0,\ldots), \tag{1.33}$$

$$f \circ l_{(\widehat{\varepsilon_0},\varepsilon_1)}(1,0,0,\ldots) = f(0,1,0,\ldots) = T_\alpha(\Delta_0) = T_\alpha(1,0,0,\ldots). \tag{1.34}$$

Therefore, $\partial_1 f = T_\alpha - T$.

An arbitrary singular 0-chain is a $\mathbb{Z}$-linear combination of singular 0-simplices. Let's take $c \in S_0(X)$. Then $c = \sum_\beta m_\beta T'_\beta$, with $m_\beta \in \mathbb{Z}$ and T'_β being singular 0-simplices. Each $T'_\beta(\Delta_0)$ belongs to some X_α, and hence homologous to T_α. Therefore, c is homologous to some $\mathbb{Z}$-linear combination $\sum_\alpha n_\alpha T_\alpha$ of the T_α's. We will now show that no such nontrivial 0-chain $\sum_\alpha n_\alpha T_\alpha$ bounds.

Assume the contrary that $\sum_\alpha n_\alpha T_\alpha = \partial_1 d$ for some $d \in S_1(X)$. Now, the singular 1-chain d is a formal linear combination of singular 1-simplices with path connected image, i.e. the image lies in one of the path components X_α. Thus we can write $d = \sum_\alpha d_\alpha$, where d_α consists of the terms whose images are in X_α. Therefore,

$$\sum_\alpha n_\alpha T_\alpha = \partial_1 d = \sum_\alpha \partial_1 d_\alpha. \tag{1.35}$$

Hence, we get

$$n_\alpha T_\alpha = \partial_1 d_\alpha \tag{1.36}$$

for each α. Applying ϵ to both sides of 1.36, we get

$$\epsilon(n_\alpha T_\alpha) = \epsilon(\partial_1 d_\alpha) \implies n_\alpha = 0. \tag{1.37}$$

Therefore, no non-trivial 0-chain $\sum_\alpha n_\alpha T_\alpha$ bounds. Since every 0-chain is automatically a 0-cycle, an element of $H_0(X)$ is homologous to a 0-chain of the form $\sum_\alpha n_\alpha T_\alpha$. Hence, the homology classes of the singular 0-simplices $\{T_\alpha\}$ form a basis for the free abelian group $H_0(X)$.

$$S_1(X) \xrightarrow{\partial_1} S_0(X) \xrightarrow{\epsilon} \mathbb{Z}$$

$\widetilde{H}_0(X)$ is defined as $\widetilde{H}_0(X) = \operatorname{Ker}\epsilon / \operatorname{Im}\partial_1$. Given a singular 0-chain $T \in S_0(X)$, we've seen that T is homologous to a 0-chain of the form $T' = \sum_\alpha n_\alpha T_\alpha$; and T' bounds iff $T' = 0$, i.e. $n_\alpha = 0$ for every α. If further $T \in \operatorname{Ker}\epsilon$, then $\epsilon(T) = 0$. Since T and T' are homologous, $T = T' + \partial_1 d$ for some $d \in S_1(X)$. Therefore,

$$0 = \epsilon(T) = \epsilon\left(T'\right) + \epsilon(\partial_1 d) = \epsilon\left(\sum_\alpha n_\alpha T_\alpha\right) = \sum_\alpha n_\alpha. \tag{1.38}$$

If X is path connected, there is only one component, and hence there is only one n_α involved. Thus $n_\alpha = 0$ from 1.38. This gives us $T' = 0$, leading to the fact that every $T \in \operatorname{Ker}\epsilon$ is homologous to 0, i.e. $T = 0 + \partial_1 d$ for some $d \in S_1(X)$. So $\operatorname{Ker}\epsilon = \operatorname{Im}\partial_1$. Therefore, $\widetilde{H}_0(X) = 0$, when X is path connected.

Now, suppose X has more than one path components. Fix α_0. Then from 1.38, we get

$$0 = \sum_\alpha n_\alpha = n_{\alpha_0} + \sum_{\alpha \neq \alpha_0} n_\alpha \implies n_{\alpha_0} = -\sum_{\alpha \neq \alpha_0} n_\alpha. \tag{1.39}$$

Then T' is

$$T' = \sum_{\alpha} n_\alpha T_\alpha = \sum_{\alpha \neq \alpha_0} n_\alpha T_\alpha + n_{\alpha_0} T_{\alpha_0} = \sum_{\alpha \neq \alpha_0} n_\alpha T_\alpha - \sum_{\alpha \neq \alpha_0} n_\alpha T_{\alpha_0} = \sum_{\alpha \neq \alpha_0} n_\alpha \left(T_\alpha - T_{\alpha_0}\right). \tag{1.40}$$

1.40 suggests that T' is a linear combination of the singular 0-chains $\{T_\alpha - T_{\alpha_0}\}_{\alpha \neq \alpha_0}$. And T' bounds iff it is trivial, as shown earlier. Therefore, the homology classes of 0-chains $\{T_\alpha - T_{\alpha_0}\}_{\alpha \neq \alpha_0}$ form a basis for $\widetilde{H}_0(X)$. ■

Theorem 1.4 illustrates the following result:

$$H_p(X) = \begin{cases} \widetilde{H}_p(X) & \text{if } p > 0 \\ \widetilde{H}_0(X) \oplus \mathbb{Z} & \text{if } p = 0 \end{cases}. \tag{1.41}$$

§1.2 Bracket Operation

Definition 1.6 (Star convex set)**.** A set $X \subseteq \mathbb{E}^J$ is said to be **star convex** relative to the point $w \in X$, if for each $x \in X$, the line segment from x to w lies in X.

Definition 1.7 (Bracket operation)**.** Suppose $X \in \mathbb{E}^J$ is star convex relative to w. We define bracket operation on singular chains of X. Let us first define it for singular p-simplices. Let $T : \Delta_p \to X$ be a singular p-simplex of X. Define a singular $(p+1)$-simplex

$$[T, w] : \Delta_{p+1} \to X$$

by letting $[T, w]$ carry the line segment from x to ε_{p+1}, for $x \in\in \Delta_p$ (the collection of all such line segments as x varies in Δ_p constitutes Δ_{p+1}), linearly onto the line segment $T(x)$ to w in X. In other words,

$$[T, w]\left(t\varepsilon_{p+1} + (1-t)x\right) = tw + (1-t)T(x), \tag{1.42}$$

for $t \in [0, 1]$. Now, extend the definition of bracket operation to arbitrary p-chains as follows: if $c = \sum n_i T_i$ is a singular p-chain of X with each T_i being a singular p-simplex, then we define

$$[c, w] = \sum n_i [T_i, w]. \tag{1.43}$$

In other words, $[\cdot\,, w] : S_p(X) \to S_{p+1}(X)$, $c \mapsto [c, w]$ is a homomorphism.

From Figure 1.1, it's immediate that the restriction of $[T, w]$ to the face Δ_p of Δ_{p+1} is just the map T. Now, consider the case when T is the linear singular simplex $l_{(a_0,\ldots,a_p)}$ for $a_0, \ldots, a_p \in \mathbb{R}^\infty$. We want to calculate what $\left[l_{(a_0,\ldots,a_p)}, w\right]$ is.

Recall that $l_{(a_0,\ldots,a_p)} : \Delta_p \to \mathbb{R}^\infty$ is defined as

$$l_{(a_0,\ldots,a_p)}(x_0, \ldots, x_p) = \sum_{i=0}^{p} x_i a_i. \tag{1.44}$$

Consider a point $(x_0, \ldots, x_p, x_{p+1}, 0, \ldots) \in \Delta_{p+1}$. We want to see where $\left[l_{(a_0,\ldots,a_p)}, w\right]$ takes this point to. Since $(x_0, \ldots, x_p, x_{p+1}, 0, \ldots) \in \Delta_{p+1}$, each x_i is nonnegative with $\sum_{i=0}^{p+1} x_i = 1$. Now,

$$\sum_{i=0}^{p} \frac{x_i}{1 - x_{p+1}} = 1, \tag{1.45}$$

so $\left(\frac{x_0}{1-x_{p+1}}, \frac{x_1}{1-x_{p+1}}, \ldots, \frac{x_p}{1-x_{p+1}}, 0, \ldots\right) \in \Delta_p$. Therefore,

$$(x_0, \ldots, x_p, x_{p+1}, 0, \ldots) = (1 - x_{p+1})\left(\frac{x_0}{1-x_{p+1}}, \frac{x_1}{1-x_{p+1}}, \ldots, \frac{x_p}{1-x_{p+1}}, 0, \ldots\right) + x_{p+1}\varepsilon_{p+1}. \tag{1.46}$$

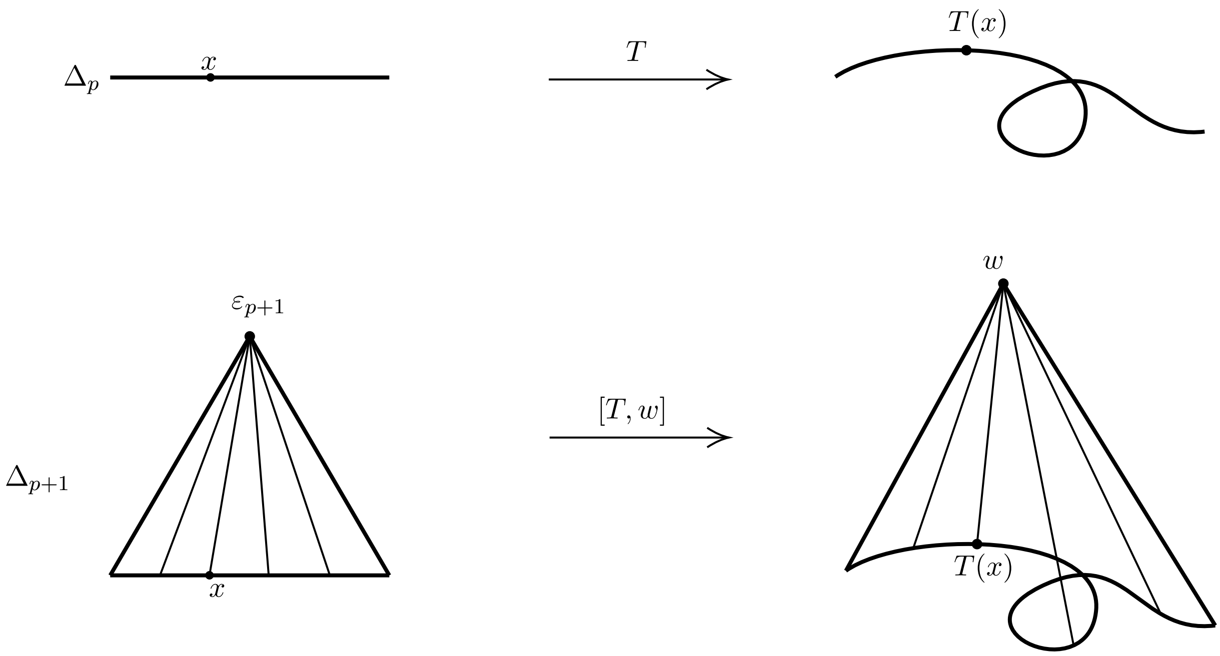


Figure 1.1

By the definition of bracket operation,

$$\begin{aligned}
&\left[l_{(a_0,\ldots,a_p)}, w\right](x_0,\ldots,x_p,x_{p+1},0,\ldots) \\
=\ &(1-x_{p+1})\, l_{(a_0,\ldots,a_p)}\left(\frac{x_0}{1-x_{p+1}}, \frac{x_1}{1-x_{p+1}}, \ldots, \frac{x_p}{1-x_{p+1}}, 0, \ldots\right) + x_{p+1}w \\
=\ &(1-x_{p+1})\sum_{i=0}^{p}\frac{x_i}{1-x_{p+1}}a_i + x_{p+1}w \\
=\ &\sum_{i=0}^{p} x_i a_i + x_{p+1}w. \qquad (1.47)
\end{aligned}$$

Furthermore,

$$l_{(a_0,\ldots,a_p,w)}(x_0,\ldots,x_p,x_{p+1},0,\ldots) = x_0a_0 + \cdots + x_pa_p + x_{p+1}w = \sum_{i=0}^{p} x_ia_i + x_{p+1}w. \qquad (1.48)$$

Equating 1.47 and 1.48, we get

$$\left[l_{(a_0,\ldots,a_p)}, w\right] = l_{(a_0,\ldots,a_p,w)}. \qquad (1.49)$$

Now we will showo that $[T, w] : \Delta_{p+1} \to X$ is continuous. We have seen earlier that given $x \in \Delta_p$, a point in Δ_{p+1} is expressed as $t\varepsilon_{p+1} + (1-t)\,x$, with $0 \le t \le 1$. Hence, we are concerned with the following quotient map $\pi : \Delta_p \times [0,1] \to \Delta_{p+1}$ defined by

$$\pi(x,t) = t\varepsilon_{p+1} + (1-t)\,x. \qquad (1.50)$$

If $x = (x_0,\ldots,x_p,0,\ldots) \in \Delta_p$, then 1.50 takes the familiar form

$$\pi((x_0,\ldots,x_p,0,\ldots), t) = ((1-t)\,x_0,\ldots,(1-t)\,x_p, t, 0,\ldots). \qquad (1.51)$$

Observe that $\pi\big|_{\Delta_p\times[0,1)} : \Delta_p \times [0,1) \to \Delta_{p+1}$ is 1-1, and $\pi(\Delta_p \times \{1\}) = \{\varepsilon_{p+1}\}$, showing that π collapses $\Delta_p \times \{1\}$ to the $(p+1)$-th vertex ε_{p+1} of Δ_{p+1}. Now, the continuous map $f : \Delta_p \times [0,1] \to X$ defined by

$$f(x,t) = tw + (1-t)\,T(x) \qquad (1.52)$$

is constant on $\Delta_p \times \{1\}$. In fact, $f(\Delta_p \times \{1\}) = \{w\}$. Since π is $1-1$ for other points, f is seen to be constant for $\pi^{-1}(y)$ with $y \in \Delta_{p+1} \setminus \{\varepsilon_{p+1}\}$. In other words, $f : \Delta_p \times [0,1] \to X$ is constant for each $\pi^{-1}(y)$ with $y \in \Delta_{p+1}$. Therefore, f induces a unique continuous map $\widetilde{f} : \Delta_{p+1} \to X$ such that the following diagram commutes

$$\begin{tikzcd}
\Delta_p \times [0,1] \arrow[d, "\pi"'] \arrow[dr, "f"] & \\
\Delta_{p+1} \arrow[r, "\widetilde{f}"] & X
\end{tikzcd}$$

This unique map $\widetilde{f}$ is precisely $[T,w]$, since

$$([T,w] \circ \pi)(x,t) = [T,w](t\varepsilon_{p+1} + (1-t)x) = tw + (1-t)T(x) = f(x,t). \tag{1.53}$$

Therefore, $\widetilde{f} = [T,w]$, and hence it is continuous. So $[T,w]$ is indeed a singular $(p+1)$-simplex.

Lemma 1.5

Let X be a star convex set with respect to w; let c be a singular p-chain of X. Then

$$\partial[c,w] = \begin{cases} [\partial c, w] + (-1)^{p+1} c & \text{if } p > 0 \\ \epsilon(c) T_w - c & \text{if } p = 0 \end{cases}, \tag{1.54}$$

where T_w is the singular 0-simplex mappting Δ_0 to w.

Proof. If T is a singular 0-simplex, $[T,w]$ is a singular 1-simplex. Then

$$\partial[T,w] = [T,w] \circ l_{(\widehat{\varepsilon}_0, \varepsilon_1)} - [T,w] \circ l_{(\varepsilon_0, \widehat{\varepsilon}_1)}. \tag{1.55}$$

Now, recall $[T,w] : \Delta_1 \to X$ maps the line joining ε_1 to ε_0 to the line joining w to $T(\varepsilon_0)$. So

$$[T,w](1-t,t,0,\ldots) = tw + (1-t)T(\varepsilon_0). \tag{1.56}$$

Now,

$$\left([T,w] \circ l_{(\widehat{\varepsilon}_0, \varepsilon_1)}\right)(1,0,\ldots) = [T,w](0,1,0,\ldots) = w = T_w(1,0,\ldots). \tag{1.57}$$

Therefore, $\left([T,w] \circ l_{(\widehat{\varepsilon}_0, \varepsilon_1)}\right) = T_w$.

$$\left([T,w] \circ l_{(\varepsilon_0, \widehat{\varepsilon}_1)}\right)(1,0,\ldots) = [T,w](1,0,\ldots) = T(\varepsilon_0) = T(1,0,\ldots), \tag{1.58}$$

so $[T,w] \circ l_{(\varepsilon_0, \widehat{\varepsilon}_1)} = T$. By 1.55, we get

$$\partial[T,w] = T_w - T. \tag{1.59}$$

Now, let $c = \sum_i n_i T_i$ be a singular 0-chain with T_i being singular 0-simplices. Then

$$\partial\left[\sum_i n_i T_i, w\right] = \sum_i n_i \partial[T_i, w] = \sum_i n_i (T_w - T_i) = \left(\sum_i n_i\right) T_w - \sum_i n_i T_i. \tag{1.60}$$

Now, applying the augmentation map to c, we get

$$\epsilon(c) = \epsilon\left(\sum_i n_i T_i\right) = \sum_i n_i \epsilon(T_i) = \sum_i n_i. \tag{1.61}$$

Therefore, 1.60 gives us

$$\partial[c,w] = \epsilon(c) T_w - c. \tag{1.62}$$

Now we shall consider the case when T is a singular p-simplex, and we shall prove that $\partial\left[T,w\right] = \left[\partial T, w\right] + (-1)^{p+1}\, T$.

$$\begin{aligned}\partial\left[T,w\right] &= \sum_{i=0}^{p+1} (-1)^i \left[T,w\right] \circ l_{(\varepsilon_0,\ldots\widehat{\varepsilon}_i,\ldots,\varepsilon_{p+1})} \\ &= \sum_{i=0}^{p} (-1)^i \left[T,w\right] \circ l_{(\varepsilon_0,\ldots\widehat{\varepsilon}_i,\ldots,\varepsilon_{p+1})} + (-1)^{p+1} \left[T,w\right] \circ l_{(\varepsilon_0,\ldots,\varepsilon_p,\widehat{\varepsilon}_{p+1})}. \end{aligned} \tag{1.63}$$

$l_{(\varepsilon_0,\ldots,\varepsilon_p,\widehat{\varepsilon}_{p+1})}$ is the inclusion map of Δ_p into Δ_{p+1}. So $[T,w]\circ l_{(\varepsilon_0,\ldots,\varepsilon_p,\widehat{\varepsilon}_{p+1})}$ is nothing but the restriction of $[T,w]$ to Δ_p, which is the same as T. Now we want to show that

$$\left[T,w\right] \circ l_{(\varepsilon_0,\ldots\widehat{\varepsilon}_i,\ldots,\varepsilon_{p+1})} = \left[T \circ l_{(\varepsilon_0,\ldots\widehat{\varepsilon}_i,\ldots,\varepsilon_p)}, w\right]. \tag{1.64}$$

Both sides of 1.64 are maps from Δ_p to X. Let $(x_0,\ldots,x_p,0,\ldots) \in \Delta_p$. Then

$$\left(\left[T,w\right] \circ l_{(\varepsilon_0,\ldots\widehat{\varepsilon}_i,\ldots,\varepsilon_{p+1})}\right)(x_0,\ldots,x_p,0,\ldots) = \left[T,w\right](x_0,\ldots,x_{i-1},0,x_i,\ldots,x_{p-1},x_p,0,\ldots). \tag{1.65}$$

Now, $(x_0,\ldots,x_{i-1},0,x_i,\ldots,x_{p-1},x_p,0,\ldots)$ is a point in Δ_{p+1}. We can write it as

$$(x_0,\ldots,x_{i-1},0,x_i,\ldots,x_{p-1},x_p,0,\ldots) = (1-x_p)\left(\frac{x_0}{1-x_p},\ldots,\frac{x_{i-1}}{1-x_p},0,\frac{x_i}{1-x_p},\ldots,\frac{x_{p-1}}{1-x_p},0,\ldots\right) + x_p\varepsilon_{p+1}. \tag{1.66}$$

Now, $\left(\frac{x_0}{1-x_p},\ldots,\frac{x_{i-1}}{1-x_p},0,\frac{x_i}{1-x_p},\ldots,\frac{x_{p-1}}{1-x_p},0,\ldots\right)$ is a point in Δ_p since its nonzero components are all non-negative and they add to 1. Therefore,

$$\begin{aligned} &\left[T,w\right](x_0,\ldots,x_{i-1},0,x_i,\ldots,x_{p-1},x_p,0,\ldots) \\ =\ &(1-x_p)\,T\left(\frac{x_0}{1-x_p},\ldots,\frac{x_{i-1}}{1-x_p},0,\frac{x_i}{1-x_p},\ldots,\frac{x_{p-1}}{1-x_p},0,\ldots\right) + x_p w. \end{aligned} \tag{1.67}$$

On the other hand, we can write $(x_0,\ldots,x_p,0,\ldots)$ as

$$(x_0,\ldots,x_p,0,\ldots) = (1-x_p)\left(\frac{x_0}{1-x_p},\ldots,\frac{x_{p-1}}{1-x_p},0,\ldots\right) + x_p\varepsilon_p, \tag{1.68}$$

where $\left(\frac{x_0}{1-x_p},\ldots,\frac{x_{p-1}}{1-x_p},0,\ldots\right) \in \Delta_{p-1}$. So

$$\begin{aligned} &\left[T \circ l_{(\varepsilon_0,\ldots\widehat{\varepsilon}_i,\ldots,\varepsilon_p)}, w\right](x_0,\ldots,x_p,0,\ldots) \\ &= x_p w + (1-x_p)\left(T \circ l_{(\varepsilon_0,\ldots\widehat{\varepsilon}_i,\ldots,\varepsilon_p)}\right)\left(\frac{x_0}{1-x_p},\ldots,\frac{x_{p-1}}{1-x_p},0,\ldots\right) \\ &= x_p w + (1-x_p)\,T\left(\frac{x_0}{1-x_p},\ldots,\frac{x_{i-1}}{1-x_p},0,\frac{x_i}{1-x_p},\ldots,\frac{x_{p-1}}{1-x_p},0,\ldots\right). \end{aligned} \tag{1.69}$$

Combining 1.65, 1.67 and 1.69, we get that 1.64 indeed holds, i.e.

$$\left[T,w\right] \circ l_{(\varepsilon_0,\ldots\widehat{\varepsilon}_i,\ldots,\varepsilon_{p+1})} = \left[T \circ l_{(\varepsilon_0,\ldots\widehat{\varepsilon}_i,\ldots,\varepsilon_p)}, w\right].$$

Now, from 1.63, we then get

$$\begin{aligned}\partial\left[T,w\right] &= \sum_{i=0}^{p} (-1)^i \left[T \circ l_{(\varepsilon_0,\ldots\widehat{\varepsilon}_i,\ldots,\varepsilon_p)}, w\right] + (-1)^{p+1}\,T \\ &= \left[\sum_{i=0}^{p} (-1)^i\, T \circ l_{(\varepsilon_0,\ldots\widehat{\varepsilon}_i,\ldots,\varepsilon_p)}, w\right] + (-1)^{p+1}\,T \\ &= \left[\partial T, w\right] + (-1)^{p+1}\,T. \end{aligned} \tag{1.70}$$

Now, if $c = \sum_i n_i T_i$ is a singular p-chain with T_i being singular 0-simplices, then

$$\partial\left[c,w\right] = \sum_i n_i \partial\left[T_i,w\right] = \sum_i n_i\left[\partial T_i, w\right] + (-1)^{p+1}\sum_i n_i T_i = \left[\partial c, w\right] + (-1)^{p+1}\,c. \tag{1.71}$$

■

Theorem 1.6

Let $X \subseteq \mathbb{E}^J$ be star convex with respect to w. Then X is acyclic in singular homology.

Proof. To show that $\widetilde{H}_0(X) = 0$, let $c \in \operatorname{Ker} \epsilon$.

$$S_1(X) \xrightarrow{\partial_1} S_0(X) \xrightarrow{\epsilon} \mathbb{Z}$$

So $\epsilon(c) = 0$. Now, by Lemma 1.5,

$$\partial_1 [c, w] = \epsilon(c) T_w - c = -c. \tag{1.72}$$

Hence, $c \in \operatorname{Im} \partial_1$ leading to $\operatorname{Ker} \epsilon \subseteq \operatorname{Im} \partial_1$. We already know Hence, $\operatorname{Im} \partial_1 \subseteq \operatorname{Ker} \epsilon$. Therefore, $\widetilde{H}_0(X) = 0$.

Now we shall show that $H_p(X) = 0$ for $p > 0$. Let $z \in \operatorname{Ker} \partial_p$. Then $\partial_p z = 0$. By Lemma 1.5 again,

$$\partial_{p+1} [z, w] = [\partial_p z, w] + (-1)^{p+1} z = (-1)^{p+1} z. \tag{1.73}$$

Hence, $z \in \operatorname{Im} \partial_{p+1}$. Therefore, $H_p(X) = 0$. In other words, $\widetilde{H}_p(X) = 0$ for all p, i.e. X is acyclic. ■

Corollary 1.7

Any simplex is acyclic in singular homology.

2 Axioms of Singular Homology

In this chapter, we shall verify that singular homology does, in fact, satisfy the Eilenberg-Steenrod axioms. The axioms can be found in chapter 6 of [AT2 lecture notes].

§2.1 Relative Homology Groups

If X is a space and A is a subspace of X, there is a natural inclusion $S_p(A) \hookrightarrow S_p(X)$. The group of **relative singular chains** is defined by

$$S_p(X, A) = S_p(X) / S_p(A). \tag{2.1}$$

The boundary operator $\partial_p^X : S_p(X) \to S_{p-1}(X)$ restricts to the boundary operator on $S_p(A)$, i.e. $\partial_p^X|_{S_p(A)} : S_p(A) \to S_{p-1}(A)$. It, therefore, induces a boundary operator at the relative singular chain level:

$$\begin{aligned} \partial_p^{(X,A)} &: S_p(X, A) \to S_{p-1}(X, A), \\ T + S_p(A) &\mapsto \partial_p^X T + S_{p-1}(A), \end{aligned} \tag{2.2}$$

with $T = \sum_\alpha n_\alpha T_\alpha$ being a singular p-chain, where $n_\alpha \in \mathbb{Z}$ and T_α singular p-simplices. If any of the T_α's are such that $T_\alpha(\Delta_p) \subseteq A$, then $T_\alpha \in S_p(A)$. So, we can assume $T_\alpha(\Delta_p) \setminus A \neq \varnothing$. Such T_α's generate the group $S_p(X, A)$, and so $S_p(X, A)$ is a free abelian group.

The family of groups $S_p(X, A)$ and homomorphisms $\partial_p^{(X,A)}$ is called **the singular chain complex** of the pair (X, A), and is denoted by $\mathcal{S}(X, A)$. The homology groups of the chain complex $\mathcal{S}(X, A)$ of the pair (X, A) are called the **singular homology groups** of the pair (X, A), and are denoted by $H_p(X, A)$.

The chain complex $\mathcal{S}(X, A)$ is free, i.e. $S_p(X, A)$ is free for each p. The group $S_p(X, A)$ has as basis all the cosets of the form $T + S_p(A)$, where T is a singular p-simplex with $T(\Delta_p) \setminus A \neq \varnothing$.

If $f : (X, A) \to (Y, B)$ is a continuous map (recall that by the continuity of f between pairs (X, A) and (Y, B), we actually mean that $f : X \to Y$ is continuous, with $f(A) \subseteq B$), then homomorphisms $(f_\#)_p : S_p(X) \to S_p(Y)$ carries singular p-chains of A into singular p-chains of B. So it induces a homomorphism (also denoted by $(f_\#)_p$) at the level of relative singular p-chains:

$$\begin{aligned} (f_\#)_p &: S_p(X, A) \to S_p(Y, B), \\ T + S_p(A) &\mapsto (f_\#)_p T + S_p(B) = f \circ T + S_p(B). \end{aligned} \tag{2.3}$$

where T is a singular p-simplex with $T(\Delta_p) \setminus A \neq \varnothing$. This map can be seen to commute with the boundary oeprator at the relative singular chain level. To be precise,

$$(f_\#)_{p-1} \circ \partial_p^{(X,A)} = \partial_p^{(Y,B)} \circ (f_\#)_p. \tag{2.4}$$

In other words, the following diagram commutes.

$$\begin{tikzcd} S_p(X,A) \arrow[r, "\partial_p^{(X,A)}"] \arrow[d, "(f_\#)_p"'] & S_{p-1}(X,A) \arrow[d, "(f_\#)_{p-1}"] \\ S_p(Y,B) \arrow[r, "\partial_p^{(Y,B)}"'] & S_{p-1}(Y,B) \end{tikzcd}$$

Therefore, $f_\#$ induces a homomorphism

$$\begin{aligned} (f_*)_p &: H_p(X, A) \to H_p(Y, B), \\ c + \operatorname{Im} \partial_{p+1}^{(X,A)} &\mapsto (f_\#)_p c + \operatorname{Im} \partial_{p+1}^{(Y,B)}. \end{aligned} \tag{2.5}$$

Theorem 2.1

If $i : (X, A) \to (X, A)$ is the identity, then so is $(i_*)_p : H_p(X, A) \to H_p(X, A)$. If $h : (X, A) \to (Y, B)$ and $k : (Y, B) \to (Z, C)$ are continuous, then $((k \circ h)_*)_p = (k_*)_p \circ (h_*)_p$.

Proof. Since $(i_\#)_p : S_p(X) \to S_p(X)$ is the identity map (as proven while proving Theorem 1.2), so is $(i_\#)_p : S_p(X, A) \to S_p(X, A)$. Then from 2.5, we get that $(i_*)_p : H_p(X, A) \to H_p(X, A)$ is the identity, i.e. $(i_*)_p = \mathrm{id}_{H_p(X,A)}$.

Now, let us prove $\left((k \circ h)_\#\right)_p = (k_\#)_p \circ (h_\#)_p$. The equality at the homology level will then follow from 2.5.

$$(h_\#)_p : S_p(X, A) \to S_p(Y, B)\ ,\ (k_\#)_p : S_p(Y, B) \to S_p(Z, C)\,.$$

We choose a singular p-simplex T such that $T(\Delta_p) \setminus A \neq \varnothing$. Then the cosets of the form $T + S_p(A)$ form a basis of $S_p(X, A)$.

$$\begin{tikzcd} \Delta_p \arrow[r, "T"] & X \arrow[r, "h"] \arrow[rr, bend right, "k\circ h"'] & Y \arrow[r, "k"] & Z \end{tikzcd}$$

Using 2.3, we get

$$(h_\#)_p (T + S_p(A)) = h \circ T + S_p(B)\,, \tag{2.6}$$

$$(k_\#)_p \left((h_\#)_p (T + S_p(A))\right) = (k_\#)_p (h \circ T + S_p(B)) = k \circ h \circ T + S_p(C)\,, \tag{2.7}$$

$$\left((k \circ h)_\#\right)_p (T + S_p(A)) = k \circ h \circ T + S_p(C)\,. \tag{2.8}$$

Therefore, we can conclude that $\left((k \circ h)_\#\right)_p = (k_\#)_p \circ (h_\#)_p$. ■

Theorem 2.2

There is a homomorphism $(\partial_*)_p : H_p(X, A) \to H_{p-1}(A)$, defined for $A \subset X$ and all p, such that the sequence

$$\cdots \longrightarrow H_p(A) \xrightarrow{(i_*)_p} H_p(X) \xrightarrow{(\pi_*)_p} H_p(X, A) \xrightarrow{(\partial_*)_p} H_{p-1}(A) \longrightarrow \cdots$$

is exact, where i and π are the inclusions

$$(A, \varnothing) \xhookrightarrow{i} (X, \varnothing) \xhookrightarrow{\pi} (X, A).$$

The same holds if reduced homology is used for X and A, provided $A \neq \varnothing$.

A continuous map $f : (X, A) \to (Y, B)$ induces a homomorphism of the corresponding exact sequences in singular homology, either ordinary or reduced.

Proof. Let us recall the Zig-Zag lemma (Lemma 4.4.1 in the lecture note of AT2). Given a short exact sequence of chain complexes $\mathcal{C} = \left\{C_p, \partial_p^C\right\}$, $\mathcal{D} = \left\{D_p, \partial_p^D\right\}$ and $\mathcal{E} = \left\{E_p, \partial_p^E\right\}$, i.e.

$$0 \longrightarrow \mathcal{C} \xrightarrow{\phi} \mathcal{D} \xrightarrow{\psi} \mathcal{E} \longrightarrow 0$$

with ϕ and ψ being chain maps, i.e. family of homomorphisms $\{\phi_p\}$ and $\{\psi_p\}$ such that

$$0 \longrightarrow C_p \xrightarrow{\phi_p} D_p \xrightarrow{\psi_p} E_p \longrightarrow 0$$

is exact for each p, then there is a long exact homology sequence

$$\begin{tikzcd}
& \cdots & \cdots \arrow[dll] \\
H_p\left(\mathcal{C}\right) \arrow[r, "\left(\phi_p\right)_*"'] & H_p\left(\mathcal{D}\right) \arrow[r, "\left(\psi_p\right)_*"] & H_p\left(\mathcal{E}\right) \arrow[dll, "\left(\partial_*\right)_p"] \\
H_{p-1}\left(\mathcal{C}\right) \arrow[r, "\left(\phi_{p-1}\right)_*"'] & H_{p-1}\left(\mathcal{D}\right) \arrow[r] & \cdots
\end{tikzcd}$$

We shall use Zig-Zag lemma with $C_p = S_p\left(A\right)$, $D_p = S_p\left(X\right)$ and $E_p = S_p\left(X, A\right)$, with chain maps given as follows:

$$0 \longrightarrow S_p\left(A\right) \xrightarrow{\left(i_\#\right)_p} S_p\left(X\right) \xrightarrow{\left(\pi_\#\right)_p} S_p\left(X, A\right) \longrightarrow 0.$$

Then the above sequence is exact, since $S_p\left(X, A\right) = S_p\left(X\right)/S_p\left(A\right)$. Now, Zig-Zag lemma guarantees the existence of the homomorphism $\left(\partial_*\right)_p : H_p\left(X, A\right) \to H_{p-1}\left(A\right)$ and the following long-exact sequence

$$\cdots \longrightarrow H_p(A) \xrightarrow{\left(i_*\right)_p} H_p(X) \xrightarrow{\left(\pi_*\right)_p} H_p(X, A) \xrightarrow{\left(\partial_*\right)_p} H_{p-1}(A) \qquad \cdots.$$

Now, given a continuous map $f : (X, A) \to (Y, B)$, we shall verify that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & S_p(A) \arrow[r, "\left(i_\#\right)_p"] \arrow[d, "\left(\left(f\big|_A\right)_\#\right)_p"'] & S_p(X) \arrow[r, "\left(\pi_\#\right)_p"] \arrow[d, "\left(\left(f\big|_X\right)_\#\right)_p"'] & S_p(X, A) \arrow[r] \arrow[d, "\left(f_\#\right)_p"] & 0 \\
0 \arrow[r] & S_p(B) \arrow[r, "\left(i'_\#\right)_p"'] & S_p(Y) \arrow[r, "\left(\pi'_\#\right)_p"'] & S_p(Y, B) \arrow[r] & 0
\end{tikzcd}$$

Here, by $f\big|_X$, we mean the map $f : X \to Y$. First, let's show the commutativity of the left hand square. Let's take a singular p-simplex T of A, i.e. $T : \Delta_p \to A$ is continuous. Then

$$\left(i_\#\right)_p T = i \circ T = T\,,\ \left(f_\#\right)_p \left(\left(i_\#\right)_p T\right) = f \circ T\,. \tag{2.9}$$

$$\left(\left(f\big|_A\right)_\#\right)_p T = f\big|_A \circ T = f \circ T\,,\ \left(i'_\#\right)_p \left(\left(\left(f\big|_A\right)_\#\right)_p T\right) = i' \circ f \circ T = f \circ T. \tag{2.10}$$

$f\big|_A \circ T = f \circ T$ because the image of T lies entirely in A. Therefore, the left hand square commutes. Now we shall show that the right hand square commutes as well. Let's take a singular p-simplex T of X, i.e. $T : \Delta_p \to X$ is continuous.

$$\left(\pi_\#\right)_p T = T + S_p\left(A\right)\,,\ \left(f_\#\right)_p \left(\left(\pi_\#\right)_p T\right) = \left(f_\#\right)_p T + S_p\left(B\right) = \left(\pi'_\#\right)_p \left(\left(f_\#\right)_p T\right). \tag{2.11}$$

Therefore, the right hand square commutes. So the diagram is commutative. Now, applying Theorem 5.1.1 from the lecture note of AT2, we obtain that the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r] & H_p(A) \arrow[r, "\left(i_*\right)_p"] \arrow[d, "\left(\left(f\big|_A\right)_*\right)_p"'] & H_p(X) \arrow[r, "\left(\pi_*\right)_p"] \arrow[d, "\left(f_*\right)_p"'] & H_p(X, A) \arrow[r, "\left(\partial_*\right)_p"] \arrow[d, "\left(f_*\right)_p"] & H_{p-1}(A) \arrow[r] \arrow[d, "\left(\left(f\big|_A\right)_*\right)_{p-1}"] & \cdots \\
\cdots \arrow[r] & H_p(B) \arrow[r, "\left(i'_*\right)_p"'] & H_p(Y) \arrow[r, "\left(\pi'_*\right)_p"'] & H_p(Y, B) \arrow[r, "\left(\partial'_*\right)_p"'] & H_{p-1}(B) \arrow[r] & \cdots
\end{tikzcd}$$

This establishes the induced homomorphisms between the respective long exact sequences of the singular homology. Following the same procedure, one can show that the same result holds in reduced homology. ■

Theorem 2.3

If P is a one-point space, then $H_p(P) = 0$ for $p \neq 0$, and $H_0(P) \cong \mathbb{Z}$.

Proof. We provide a direct proof here. We first compute the chain complex $\mathcal{S}(P)$. Observe that there is exactly one singular p-simplex in each non-negative dimention $p \geq 0$: $T_p : \Delta_p \to P$, because P is a singleton. Therefore, the group of p-chains $S_p(P) \cong \mathbb{Z}$, which is infinite cyclic. Each of the "faces" of $T_p : \Delta_p \to P$ is given

$$T_p \circ l_{(\varepsilon_0,\ldots,\widehat{\varepsilon}_i,\ldots,\varepsilon_p)} : \Delta_p \to P$$

and is precisely T_{p-1}. All $(p+1)$ faces of T_p are just T_{p-1}. Therefore, if p is even, then the singular p-simplex $(p+1)$ faces, which is an odd number. Hence, in the formula

$$\partial_p T_p = \sum_{i=0}^{p} (-1)^i \, T_p \circ l_{(\varepsilon_0,\ldots,\widehat{\varepsilon}_i,\ldots,\varepsilon_p)}, \tag{2.12}$$

only one term will survive, the others will cancel in pairs. Hence, we find that $\partial_p T_p = T_{p-1}$, when p is even.

On the other hand, when p is odd, T_p will have an even number of faces, and all the terms in 2.12 will cancel in pairs. Therefore, $\partial_p T_p = 0$, when p is odd. The chain complex $\mathcal{S}(P)$ is, thus, of the following form:

$$\begin{array}{ccccccccccccc} \cdots & \longrightarrow & S_{2k}(P) & \longrightarrow & S_{2k-1}(P) & \longrightarrow & \cdots & \longrightarrow & S_1(P) & \longrightarrow & S_0(P) & \longrightarrow & 0 \\ \\ \cdots & \longrightarrow & \mathbb{Z} & \xrightarrow{\cong} & \mathbb{Z} & \xrightarrow{\bar{0}} & \cdots & \longrightarrow & \mathbb{Z} & \xrightarrow{\bar{0}} & \mathbb{Z} & \longrightarrow & 0 \end{array}$$

Here, $\bar{0}$ maps everything to 0. In dimension $(2k-1)$, every $(2k-1)$-chain is a cycle, and every $(2k-1)$-chain can be seen to be a boundary of a $2k$-chain. Hence, there is no nontrivial $(2k-1)$-cycle that is not a $(2k-1)$-boundary. Therefore, $H_{2k-1}(P) = 0$.

In dimension $2k$, for $k > 0$, there is no nontrivial chain that is a cycle. Hence, $H_{2k} = 0$. In dimension 0, every chain is a cycle, and no nontrivial 0-chain is a bounday. Therefore, $H_0(P) \cong \mathbb{Z}$. ■

§2.2 Compact Support Axiom

In this section, we shall verify that singular homology theory satisfies the compact support axiom.

Definition 2.1 (Minimal carrier). If $T : \Delta_p \to X$ is a singular p-simplex of X, then the **minimal carrier** of T is defined to be the image set $T(\Delta_p)$. If $c = \sum n_i T_i$ is a singular p-chain, with T_i being singular p-simplices and each n_i nonzero, then the minimal carrier of c is defined to be the union of the minimal carriers of the singular p-simplices T_i.

A singular p-simplex T is a continuous map from Δ_p to X. Since Δ_p is compact, so is $T(\Delta_p)$ since continuous map takes compact sets to compact sets. Now, a finite union of compact sets is also compact. Therefore, the minimal carrier of a singular p-chain is compact.

Theorem 2.4

Given $\alpha \in H_p(X, A)$, there is a compact pair $(X_0, A_0) \subseteq (X, A)$, with $\iota : (X_0, A_0) \hookrightarrow (X, A)$ such that $(\iota_*)_p(\beta) = \alpha$ for some $\beta \in H_p(X_0, A_0)$, where $(\iota_*)_p : H_p(X_0, A_0) \to H_p(X, A)$ is the homomorphism induced by the inclusion ι.

Proof. Given $\alpha \in H_p(X, A) = Z_p(X, A) / B_p(X, A)$, α is of the form $C + B_p(X, A)$, with $C \in Z_p(X, A) \subset S_p(X, A) = S_p(X) / S_p(A)$. Therefore,

$$\alpha = (c_p + S_p(A)) + B_p(X, A), \tag{2.13}$$

where $c_p \in S_p(X)$ such that $\partial_p c_p$ is carried by A. The minimal carrier of $\partial_p c_p$ is a compact set contained in A. Let us denote this compact set by A_0. On the other hand, c_p is minimally carried by a compact set X_0 contained in X. Now, we define

$$D = c_p + S_p(A_0) \in S_p(X_0, A_0). \tag{2.14}$$

Since $\partial_p c_p$ is carried by A_0, $D \in Z_p(X_0, A_0)$. Now, we claim that

$$\beta = D + B_p(X_0, A_0) = (c_p + S_p(A_0)) + B_p(X_0, A_0) \in H_p(X_0, A_0) \tag{2.15}$$

is the required element of $H_p(X_0, A_0)$ whose image under $(\iota_*)_p$ is α. Now,

$$(\iota_*)_p(\beta) = (\iota_*)_p((c_p + S_p(A_0)) + B_p(X_0, A_0)) = \left((\iota_\#)_p c_p + S_p(A)\right) + B_p(X, A). \tag{2.16}$$

If $c_p = \sum n_i T_i$, with T_i being singular p-simplices, then

$$(\iota_\#)_p c_p = \sum n_i (\iota_\#)_p(T_i) = \sum n_i(\iota \circ T_i) = \sum n_i T_i = c_p. \tag{2.17}$$

Therefore,

$$(\iota_*)_p(\beta) = (c_p + S_p(A)) + B_p(X, A) = \alpha. \tag{2.18}$$

■

Theorem 2.5

Let $i : (X_0, A_0) \hookrightarrow (X, A)$ be inclusion, where (X_0, A_0) is a compact pair. If $\alpha \in H_p(X_0, A_0)$ with $(i_*)_p(\alpha) = 0$, then there are a compact pais (X_1, A_1) and inclusions

$$(X_0, A_0) \xhookrightarrow{j} (X_1, A_1) \xhookrightarrow{k} (X, A)$$

such that $(j_*)_p(\alpha) = 0$.

Proof. Let $\alpha = (c_p + S_p(A_0)) + B_p(X_0, A_0) \in H_p(X_0, A_0)$, where $c_p \in S_p(X_0)$ and $\partial_p c_p$ is carried by A_0. Now, $(i_*)_p : H_p(X_0, A_0) \to H_p(X, A)$, so $(i_*)_p(\alpha) = 0 + B_p(X, A)$.

$$0 + B_p(X, A) = (i_*)_p(\alpha) = \left((i_\#)_p c_p + S_p(A)\right) + B_p(X, A). \tag{2.19}$$

Using a similar method as in 2.17, one can show that $(i_\#)_p c_p = c_p$. So 2.19 reads

$$0 + B_p(X, A) = (c_p + S_p(A)) + B_p(X, A). \tag{2.20}$$

Therefore, $c_p + S_p(A) \in B_p(X, A)$. In other words, there exists a $(p+1)$-chain d_{p+1} such that $c_p - \partial_{p+1} d_{p+1}$ is carried by A. Now, d_{p+1} is carried by

$$X_1 = X_0 \cup (\text{minimal carrier of } d_{p+1}),$$

and $c_p - \partial_{p+1} d_{p+1}$ is carried by

$$A_1 = A_0 \cup (\text{minimal carrier of } c_p - \partial_{p+1} d_{p+1}).$$

Consider the inclusion maps

$$\begin{tikzcd} (X_0, A_0) \arrow[r, hook, "j"] \arrow[rr, bend right, "i=k\circ j"'] & (X_1, A_1) \arrow[r, hook, "k"] & (X, A). \end{tikzcd}$$

Then $(j_*)_p(\alpha)$ is

$$(j_*)_p(\alpha) = (j_*)_p((c_p + S_p(A_0)) + B_p(X_0, A_0)) = \Big((j_\#)_p c_p + S_p(A_1)\Big) + B_p(X_1, A_1). \tag{2.21}$$

Again, similarly as in 2.17, one can show that $(j_\#)_p c_p = c_p$.

$$(j_*)_p(\alpha) = (c_p + S_p(A_1)) + B_p(X_1, A_1). \tag{2.22}$$

$c_p - \partial_{p+1} d_{p+1}$ is carried by A_1, so $c_p - \partial_{p+1} d_{p+1} \in S_p(A_1)$. Therefore,

$$\begin{aligned} c_p + S_p(A_1) &= c_p - (c_p - \partial_{p+1} d_{p+1}) + S_p(A_1) = \partial_{p+1} d_{p+1} + S_p(A_1) \\ &= \partial_{p+1}(d_{p+1} + S_{p+1}(A_1)) \in B_p(X_1, A_1). \end{aligned} \tag{2.23}$$

Combining 2.22 and 2.23, we get

$$(j_*)_p(\alpha) = \partial_{p+1}(d_{p+1} + S_{p+1}(A_1)) + B_p(X_1, A_1) = 0 + B_p(X_1, A_1). \tag{2.24}$$

■

§2.3 Chain Homotopy

Definition 2.2. Given chain complexes $\mathcal{C} = \{C_p, \partial_p\}$ and $\mathcal{C}' = \{C'_p, \partial'_p\}$ and chain maps $\phi, \psi : \mathcal{C} \to \mathcal{C}'$, a **chain homotopy** of ϕ to ψ is a family of homomorphisms $D_p : C_p \to C'_{p+1}$ such that the following holds

$$\partial'_{p+1} D_p + D_{p-1}\partial_p = \psi_p - \phi_p. \tag{2.25}$$

The following diagram might be useful for to understand the above formula in 2.25. Note that this is **NOT** a commutative diagram.

$$\begin{tikzcd} & C'_{p+1} \arrow[d, "\partial'_{p+1}"] \\ C_p \arrow[ur, "D_p"] \arrow[r, shift left, "\phi_p"] \arrow[r, shift right, "\psi_p"'] \arrow[d, "\partial_p"'] & C'_p \\ C_{p-1} \arrow[ur, "D_{p-1}"'] & \end{tikzcd}$$

Now, consider the inclusions $i, j : X \to X \times I$ (I is the unit interval $[0,1]$) given by

$$i(x) = (x, 0) \text{ and } j(x) = (x, 1). \tag{2.26}$$

The corresponding chain maps are denoted by $(i_\#)_p, (j_\#)_p : S_p(X) \to S_p(X \times I)$. Construct a chain homotopy D^X between the chain map $i_\#$ and $j_\#$ as follows:

$$\begin{aligned} D^X &: \mathcal{S}(X) \to \mathcal{S}(X \times I), \\ D^X_p &: S_p(X) \to S_p(X \times I). \end{aligned} \tag{2.27}$$

For D^X to be a chain homotopy, the following equation must hold:

$$\partial^{X \times I}_{p+1} \circ D^X_p + D^X_{p-1} \circ \partial^X_p = (j_\#)_p - (i_\#)_p. \tag{2.28}$$

$$\begin{tikzcd} & S_{p+1}(X \times I) \arrow[d, "\partial^{X\times I}_{p+1}"] \\ S_p(X) \arrow[ur, "D^X_p"] \arrow[r, shift left, "(i_\#)_p"] \arrow[r, shift right, "(j_\#)_p"'] \arrow[d, "\partial^X_p"'] & S_p(X \times I) \\ S_{p-1}(X) \arrow[ur, "D^X_{p-1}"'] & \end{tikzcd}$$

One can now construct the following diagram to find that $F_{\#} \circ D^X$ is a chain homotopy between the chain maps $f_{\#}, g_{\#} : \mathcal{S}(X) \to \mathcal{S}(Y)$, where X and Y are topological spaces and F is a homotopy between the maps $f, g : X \to Y$, i.e. $F : X \times I \to Y$ is a continuous map such that

$$F(x,0) = f(x) \text{ and } F(x,1) = g(x).$$

Using 2.26, we then have

$$F \circ i = f \text{ and } F \circ j = g. \tag{2.29}$$

$F_{\#} : \mathcal{S}(X \times I) \to \mathcal{S}(Y)$. In order to show that $F_{\#} \circ D^X$ is a chain homotopy between $f_{\#}$ and $g_{\#}$, one needs to prove that

$$\partial^Y_{p+1} \circ (F_{\#})_{p+1} \circ D^X_p + (F_{\#})_p \circ D^X_{p-1} \circ \partial^X_p = (g_{\#})_p - (f_{\#})_p. \tag{2.30}$$

$$\begin{tikzcd}
 & & & S_{p+1}(Y) \arrow[dd, "\partial^Y_{p+1}"] \\
 & S_{p+1}(X \times I) \arrow[urr, "(F_{\#})_{p+1}"] & & \\
S_p(X) \arrow[ur, "D^X_p"] \arrow[rrr, shift left, "(f_{\#})_p"] \arrow[rrr, shift right, "(g_{\#})_p"'] \arrow[dd, "\partial^X_{p+1}"'] & & & S_p(Y) \\
 & S_p(X \times I) \arrow[urr, "(F_{\#})_p"'] & & \\
S_{p-1}(X) \arrow[ur, "D^X_{p-1}"'] & & &
\end{tikzcd}$$

Let us quickly see how 2.30 comes from 2.28. Since chain maps commute with the boundary operator, we have the following commutative diagram:

$$\begin{tikzcd}
S_{p+1}(X \times I) \arrow[r, "(F_{\#})_{p+1}"] \arrow[d, "\partial^{X \times I}_{p+1}"'] & S_{p+1}(Y) \arrow[d, "\partial^Y_{p+1}"] \\
S_p(X \times I) \arrow[r, "(F_{\#})_p"'] & S_p(Y)
\end{tikzcd}$$

i.e. $\partial^Y_{p+1} \circ (F_{\#})_{p+1} = (F_{\#})_p \circ \partial^{X \times I}_{p+1}$. Therefore, one obtains

$$\begin{aligned}
\partial^Y_{p+1} \circ (F_{\#})_{p+1} \circ D^X_p &= (F_{\#})_p \circ \partial^{X \times I}_{p+1} \circ D^X_p \\
&= (F_{\#})_p \circ \left[(j_{\#})_p - (i_{\#})_p - D^X_{p-1} \circ \partial^X_p \right] \\
&= \Big((F \circ j)_{\#} \Big)_p - \Big((F \circ i)_{\#} \Big)_p - (F_{\#})_p \circ D^X_{p-1} \circ \partial^X_p \\
&= (g_{\#})_p - (f_{\#})_p \, (F_{\#})_p \circ D^X_{p-1} \circ \partial^X_p,
\end{aligned} \tag{2.31}$$

which can be rearranged to obtain 2.30. The existence of the chain map $D^X : \mathcal{S}(X) \to \mathcal{S}(X \times I)$ is governed by the following lemma.

Lemma 2.6

There exists, for each space X, and each non-negative integer p, a homomorphism $D^X_p : S_p(X) \to S_{p+1}(X \times I)$ having the following properties:

(a) If $T : \Delta_p \to X$ is a singular p-simplex then

$$\partial^{X \times I}_{p+1} D^X_p T + D^X_{p-1} \partial^X_p T = (j_{\#})_p T - (i_{\#})_p T. \tag{2.32}$$

Here, the map $i : X \to X \times I$ carries x to $(x,0)$ and the map $j : X \to X \times I$ carries x to

$(x, 1)$.

(b) D_p^X is natural; i.e. given $f : X \to Y$ continuous, the following diagram commutes:

$$\begin{tikzcd} S_p(X) \arrow[r, "D_p^X"] \arrow[d, "(f_\#)_p"'] & S_{p+1}(X \times I) \arrow[d, "((f\times \mathrm{id}_I)_\#)_{p+1}"] \\ S_p(Y) \arrow[r, "D_p^Y"] & S_{p+1}(Y \times I) \end{tikzcd}$$

Note that continuous $f : X \to Y$ induces a continnuous map $f \times \mathrm{id}_I : X \times I \to Y \times I$ given by $(x, t) \mapsto (f(x), t)$. Hence there is a group homomorphism

$$\left((f \times \mathrm{id}_I)_\#\right)_p : S_p(X \times I) \to S_p(Y \times I)$$

for each non-negative integer p.

Proof of the lemma is omitted.

Theorem 2.7

If $f, g : (X, A) \to (Y, B)$ are homotopic, then $(f_*)_p = (g_*)_p$ for all p, with $(f_*)_p, (g_*)_p : H_p(X, A) \to H_p(Y, B)$ group homomorphisms. The same holds in the reduced homology if $A = B = \varnothing$.

Proof. Let $F : (X \times I, A \times I) \to (Y \times I, B \times I)$ be the homotopy between $f, g : (X, A) \to (Y, B)$. Let $i, j : (X, A) \to (X \times I, A \times I)$ be given by $i(x) = (x, 0)$ and $j(x) = (x, 1)$, for $x \in X$. Let $D_p^X : S_p(X) \to S_p(X \times I)$ be the group homomorphism associated with the chain homotopy $D^X : \mathcal{S}(X) \to \mathcal{S}(X \times I)$ constructed in Lemma 2.6. Naturality of D^X with respect to the inclusion map $\iota : A \hookrightarrow X$ dictates that the following diagram commutes:

$$\begin{tikzcd} S_p(A) \arrow[r, "D_p^A"] \arrow[d, "(\iota_\#)_p"'] & S_{p+1}(A \times I) \arrow[d, "((\iota\times \mathrm{id}_I)_\#)_{p+1}"] \\ S_p(X) \arrow[r, "D_p^X"] & S_{p+1}(X \times I) \end{tikzcd}$$

Consider $T \in S_{p+1}(A \times I)$ such that T is a $(p+1)$-singular simplex of $A \times I$, i.e. $T : \Delta_{p+1} \to A \times I$ is continuous. For a given $x \in \Delta_{p+1}$, let $T(x) = (a, t) \in A \times I$. Now,

$$\left((\iota \times \mathrm{id}_I)_\#\right)_{p+1} T(x) = (\iota \times \mathrm{id}_I) \circ T(x) = (\iota \times \mathrm{id}_I)(a, t) = (a, t) = T(x). \tag{2.33}$$

Hence, $\left((\iota \times \mathrm{id}_I)_\#\right)_{p+1} T = T$. So, we have

$$\left((\iota \times \mathrm{id}_I)_\#\right)_{p+1} \circ D_p^A = D_p^A. \tag{2.34}$$

Now, commutativity of the above diagram yields

$$\left((\iota \times \mathrm{id}_I)_\#\right)_{p+1} \circ D_p^A = D_p^X \circ (\iota_\#)_p = D_p^X\big|_{S_p(A)}. \tag{2.35}$$

Therefore, combining 2.34 and 2.35, we get

$$D_p^X\big|_{S_p(A)} = D_p^A. \tag{2.36}$$

In other words, $D_p^X : S_p(X) \to S_{p+1}(X \times I)$ carries $S_p(A)$ into $S_p(X \times I)$, and thus induces a chain homotopy on the relative level. The constituent group homomorphisms are given by

$$D_p^{(X,A)} : S_p(X, A) \to S_{p+1}(X \times I, A \times I). \tag{2.37}$$

Now, 2.32 indeed holds for $D_p^{(X,A)}$ as it is induced by D_p^X. Then we have

$$\left(F_\#\right)_{p+1} \circ D_p^{(X,A)} : S_p\left(X,A\right) \to S_{p+1}\left(Y,B\right),$$

where the homomorphism $\left(F_\#\right)_{p+1}$ associated with the chain map $F_\# : \mathcal{S}\left(X \times I, A \times I\right) \to \mathcal{S}\left(Y,B\right)$ is

$$\left(F_\#\right)_{p+1} : S_{p+1}\left(X \times I, A \times I\right) \to S_{p+1}\left(Y,B\right).$$

Then

$$\begin{aligned}
\partial_{p+1}^Y \circ \left(F_\#\right)_{p+1} \circ D_p^{(X,A)} &= \left(F_\#\right)_p \circ \partial_{p+1}^{X\times I} \circ D_p^{(X,A)} \\
&= \left(F_\#\right)_p \circ \left[\left(j_\#\right)_p - \left(i_\#\right)_p - D_{p-1}^{(X,A)} \circ \partial_p^X\right] \\
&= \left(\left(F \circ j\right)_\#\right)_p - \left(\left(F \circ i\right)_\#\right)_p - \left(F_\#\right)_p \circ D_{p-1}^{(X,A)} \circ \partial_p^X \\
&= \left(g_\#\right)_p - \left(f_\#\right)_p - \left(F_\#\right)_p \circ D_{p-1}^{(X,A)} \circ \partial_p^X. \qquad (2.38)
\end{aligned}$$

This proves that $F_\# \circ D^{(X,A)} : \mathcal{S}\left(X,A\right) \to \mathcal{S}\left(Y,B\right)$ is a chain homotopy between $f_\#, g_\# : \mathcal{S}\left(X,A\right) \to \mathcal{S}\left(Y,B\right)$. It now remains to prove that $\left(f_*\right)_p = \left(g_*\right)_p$ for all p.

Let $\alpha \in Z_p\left(X,A\right)$. It suffices to show that $\left(f_\#\right)_p\left(\alpha\right)$ and $\left(g_\#\right)_p\left(\alpha\right)$ differ by a boundary term. Given $\alpha \in Z_p\left(X,A\right)$, $\alpha = c_p + S_p\left(A\right)$ for some $c_p \in S_p\left(X\right)$ such that $\partial_p c_p$ is carried by A. By 2.38,

$$\begin{aligned}
\left(g_\#\right)_p\left(\alpha\right) - \left(f_\#\right)_p\left(\alpha\right) &= \partial_{p+1}^Y \circ \left(F_\#\right)_{p+1} \circ D_p^{(X,A)}\left(\alpha\right) + \left(F_\#\right)_p \circ D_{p-1}^{(X,A)} \circ \partial_p^X\left(\alpha\right) \\
&= \partial_{p+1}^Y \circ \left(F_\#\right)_{p+1} \circ D_p^{(X,A)}\left(\alpha\right), \qquad (2.39)
\end{aligned}$$

proving that $\left(f_\#\right)_p\left(\alpha\right)$ and $\left(g_\#\right)_p\left(\alpha\right)$ differ by a boundary term. Therefore, $\left(f_*\right)_p\left(\alpha + B_p\left(X,A\right)\right) = \left(f_*\right)_p\left(\alpha + B_p\left(X,A\right)\right)$.

The result in reduced homology is left as an exercise. ■

§2.4 Homotopy Equivalence

Definition 2.3 (Retraction)**.** Let $A \subset X$. A **retraction** of X onto A is a continuous map $r : X \to A$ such that $r\left(a\right) = a$ for every $a \in A$, i.e. $r\big|_A = \mathrm{id}_A$. If there is a retraction of X onto A, we say that A is a retract of X,

Definition 2.4 (Deformation retraction)**.** A **deformation retraction** of X onto A is a continuous map $F : X \times I \to X$ such that

$$F\left(x,0\right) = x\,,\;\; F\left(x,1\right) \in A\,,\;\; \text{and } F\left(a,t\right) = a \qquad (2.40)$$

for all $x \in X, a \in A, t \in I$.

If F is a deformation retraction of X onto A, then one can define

$$r\left(x\right) = F\left(x,1\right). \qquad (2.41)$$

Then 2.40 tells us that r is a map from X to A, and $r\left(a\right) = a$ for all $a \in A$. Hence, r is indeed a retraction of X onto A. Now, 2.40 also tells us that

$$F\left(x,0\right) = x = \mathrm{id}_X\left(x\right) \;\text{ and } F\left(x,1\right) = j \circ r\left(x\right), \qquad (2.42)$$

where $j : A \hookrightarrow X$ is the inclusion. Therefore, F is a homotopy between the identity map $\mathrm{id}_X : X \to X$ and $j \circ r : X \to X$.

Lemma 2.8

Deformation retract spaces have identical homology groups. In other words, if there exists a deformation retraction of X onto A, then $j : A \hookrightarrow X$ induces isomorphism in homology.

Proof. Suppose F is a deformation retraction of X onto A. If $j : A \hookrightarrow X$ is the inclusion map, and $r : X \to A$ is defined as $r(x) = F(x, 1)$, then $r \circ j = \mathrm{id}_A$. Therefore,

$$(r_*)_p \circ (j_*)_p = \mathrm{id}_{H_p(A)}. \tag{2.43}$$

Furthermore, F is a homotopy between $j \circ r$ and id_X. Therefore,

$$(j_*)_p \circ (r_*)_p = \mathrm{id}_{H_p(X)}. \tag{2.44}$$

Therefore, $(j_*)_p : H_p(A) \to H_p(X)$ is an isomorphism for all p. ■

Lemma 2.9

If the inclusion $j : A \hookrightarrow X$ induces homology isomorphism in all dimension, then $H_p(X, A) = 0$ for all p.

Proof. Consider the long exact homology sequence

$$\cdots \longrightarrow H_p(A) \xrightarrow{(j_*)_p} H_p(X) \xrightarrow{(\pi_*)_p} H_p(X, A) \xrightarrow{(\partial_*)_p} H_{p-1}(A) \xrightarrow{(j_*)_{p-1}} H_{p-1}(X) \longrightarrow \cdots$$

$\mathrm{Im}\,(\partial_*)_p = \mathrm{Ker}\,(j_*)_{p-1}$, and $(j_*)_{p-1}$ is an isomorphism. Therefore, $\mathrm{Im}\,(\partial_*)_p = 0$. So $\mathrm{Ker}\,(\partial_*)_p = H_p(X, A)$. By exactness, this is equal to $\mathrm{Im}\,(\pi_*)_p$. Hence, $(\pi_*)_p$ is a surjective map. Now, $\mathrm{Ker}\,(\pi_*)_p = \mathrm{Im}\,(j_*)_p = H_p(X)$. So $(\pi_*)_p$ is the zero map. Hence, $H_p(X, A) = 0$. ■

Combining Lemma 2.8 and Lemma 2.9 together, we get that if A is a deformation retract of X, then $H_p(X, A) = 0$ for all p.

Definition 2.5. Let $f : (X, A) \to (Y, B)$ be continuous. If there is a continuous map $g : (Y, B) \to (X, A)$ such that $g \circ f$ is homotopic to the identity map $\mathrm{id}_{(X,A)} : (X, A) \to (X, A)$ and $f \circ g$ is homotopic to the identity map $\mathrm{id}_{(Y,B)} : (Y, B) \to (Y, B)$, then we call f a **homotopy equivalence**, and we call g a **homotopy inverse** for f.

Theorem 2.10

Let $f : (X, A) \to (Y, B)$ be continuous.

(a) If f is a homotopy equivalence, then f_* is an isomorphism in relative homology.

(b) More generally, if $f : X \to Y$ and $f\big|_A : A \to B$ are homotopy equivalences, then f_* is an isomorphism in relative homology.

Proof. Let $f : (X, A) \to (Y, B)$ be a homotopy equivalence, and $g : (Y, B) \to (X, A)$ its homotopy inverse. Then $f \circ g \simeq \mathrm{id}_{(Y,B)}$ and $g \circ f \simeq \mathrm{id}_{(X,A)}$. Then by Theorem 2.7,

$$((f \circ g)_*)_p = \left(\left(\mathrm{id}_{(Y,B)}\right)_*\right)_p \text{ and } ((g \circ f)_*)_p = \left(\left(\mathrm{id}_{(X,A)}\right)_*\right)_p.$$

In other words,

$$(f_*)_p \circ (g_*)_p = \mathrm{id}_{H_p(Y,B)} \text{ and } (g_*)_p \circ (f_*)_p = \mathrm{id}_{H_p(X,A)}. \tag{2.45}$$

Therefore, $(f_*)_p : H_p(X, A) \to H_p(Y, B)$ is an isomorphism.

Now we shall prove (b). Consider the long exact sequence of the pairs (X, A) and (Y, B), separately with $(f_*)_p$ being the respective connecting homomorphisms.

$$\begin{tikzcd}
\cdots \arrow[r] & H_p(A) \arrow[r, "(i_*)_p"] \arrow[d, "\left(\left(f|_A\right)_*\right)_p"'] & H_p(X) \arrow[r, "(\pi_*)_p"] \arrow[d, "(f_*)_p"'] & H_p(X,A) \arrow[r, "(\partial_*)_p"] \arrow[d, "(f_*)_p"] & H_{p-1}(A) \arrow[r, "(i_*)_{p-1}"] \arrow[d, "\left(\left(f|_A\right)_*\right)_{p-1}"] & H_{p-1}(X) \arrow[r] \arrow[d, "(f_*)_{p-1}"] & \cdots \\
\cdots \arrow[r] & H_p(B) \arrow[r, "(i'_*)_p"'] & H_p(Y) \arrow[r, "(\pi'_*)_p"'] & H_p(Y,B) \arrow[r, "(\partial'_*)_p"'] & H_{p-1}(B) \arrow[r, "(i'_*)_{p-1}"'] & H_{p-1}(X) \arrow[r] & \cdots
\end{tikzcd}$$

By hypothesis, $f : (X, \varnothing) \to (Y, \varnothing)$ is a homotopy equivalence, and hence $(f_*)_p : H_p(X) \to H_p(Y)$ is an isomorphism. Similarly, by hypothesis, $f\big|_A : (A, \varnothing) \to (B, \varnothing)$ is a homotopy equivalence, and hence $\left(\left(f\big|_A\right)_*\right)_p : H_p(A) \to H_p(B)$ is an isomorphism. Now, applying Steenrod five lemma to the diagram above, one obtains that

$$(f_*)_p : H_p(X, A) \to H_p(Y, B)$$

is an isomorphism. ■

Remark 2.1. Let A and B be subspaces of X and Y, respectively. If X is homotopy equivalent to Y, and A is homotopy equivalent to B, then it is not, in general, true that (X, A) is homotopy equivalent to (Y, B). It is not either true that their relative homology groups will agree in general.

Note that in Theorem 2.10, the homotopy equivalences $f : X \to Y$ and $f\big|_A : A \to B$ are homotopy equivalences come from the **same** continuous map f. On the contrary, in our setup, we just know that X and Y are homotopy equivalent, and A and B are homotopy equivalent. There is a priori no connection between the two homotopy equivalences. Following is a counterexample that illustrates a case where even though X and Y are homotopy equivalent, and A and B are homotopy equivalent, the relative homology groups $H_p(X, A)$ and $H_p(Y, B)$ do not agre in all dimensions.

Example 2.1. We choose $X = Y = S^1 \times B^2$ as a subspace of $\mathbb{C}^2$; and $A = S^1 \times \{0\}$, $B = \{1\} \times S^1$. X and Y are the same space, so they are trivially homotopy equivalent. A and B are homeomorphic through the map $(z, 0) \mapsto (1, z)$; so they are also homotopy equivalent. However, the homology groups of the pair (X, A) are not isomorphic to the homology groups of the pair (X, B), as we shall now prove.

B^2 deformation retracts to 0. Indeed, the map $F : B^2 \times I \to B^2$ given by $F(z, t) = (1 - t) z$ takes $(z, 0)$ to z, $(z, 1)$ to 0, $(0, t)$ to 0. So F is a deformation retract of B^2 onto 0. Therefore, $S^1 \times B^2$ deformation retracts to $S^1 \times \{0\}$. To be precise, the deformation retraction is given by $G : S^1 \times B^2 \times I \to S^1 \times B^2$,

$$G(s, z, t) = (s, (1 - t) z). \tag{2.46}$$

So A is a deformation retraction of X. Therefore, by Lemma 2.8 and Lemma 2.9,

$$H_p(X, A) = 0, \tag{2.47}$$

for each p.

Now we shall prove that there exists p such that $H_p(X, B) \neq 0$. In fact we are going to show that $H_2(X, B) \neq 0$. Consider the inclusions

$$\begin{tikzcd}
\{1\} \times S^1 \arrow[r, hook] \arrow[rr, bend right, "k"'] & \{1\} \times B^2 \arrow[r, hook] & S^1 \times B^2.
\end{tikzcd} \tag{2.48}$$

$\{1\} \times B^2$ is homeomorphic to B^2, which is convex. So, at the homology level, we have

$$\begin{tikzcd}
H_p(B) \arrow[r] \arrow[rr, bend right, "(k_*)_p"'] & H_p(\{1\} \times B^2) = 0 \arrow[r] & H_p(X)
\end{tikzcd}, \quad \text{for } p \geq 1. \tag{2.49}$$

Therefore, if $k : B \hookrightarrow X$ is the inclusion map, $(k_*)_p = 0$ for $p \geq 1$. Now, we consider the long exact sequence of the pair (X, B).

$$\cdots \longrightarrow H_p(X,B) \xrightarrow{(\partial_*)_p} H_{p-1}(B) \xrightarrow{(k_*)_{p-1}} H_{p-1}(X) \longrightarrow \cdots$$

$H_1(B) = H_1\left(\{1\} \times S^1\right) \cong \mathbb{Z}$, and $(k_*)_1$ is the zero map. Therefore,

$$\operatorname{Im}(\partial_*)_2 = \operatorname{Ker}(k_*)_1 = H_1(B) \cong \mathbb{Z}. \tag{2.50}$$

So $\operatorname{Im}(\partial_*)_2$ is surjective. If $H_2(X,B) = 0$, $(\partial_*)_2$ could not have been surjective. Therefore, $H_2(X,B) \neq 0$. So $H_2(X,A) \ncong H_2(X,B)$.

Remark 2.2. If $f : (X, A) \to (Y, B)$ is a homotopy equivalence, then $f : X \to Y$ and $f\big|_A : A \to B$ are automaatically homotopy equivalences. However, the converse is not true. One counterexample is presented below.

Example 2.2. Consider the inclusion map $j : \left(B^n, S^{n-1}\right) \hookrightarrow (\mathbb{R}^n, \mathbb{R}^n \setminus \{\mathbf{0}\})$. $j : B^n \hookrightarrow \mathbb{R}^n$ has a homotopy inverse, so that B^n and $\mathbb{R}^n$ are homotopy equivalent. The homotopy inverse is given by $f : \mathbb{R}^n \to B^n$,

$$f(\mathbf{x}) = \begin{cases} \mathbf{x} & \text{if } \|\mathbf{x}\| \leq 1 \\ \frac{\mathbf{x}}{\|\mathbf{x}\|} & \text{if } \|\mathbf{x}\| > 1 \end{cases}. \tag{2.51}$$

Then $f(j(\mathbf{x})) = \mathbf{x}$, so $f \circ j = \mathrm{id}_{B^n}$. $j(f(\mathbf{x})) = f(\mathbf{x}) \in B^n$. So $F : \mathbb{R}^n \times I \to \mathbb{R}^n$ given by

$$F(\mathbf{x}, t) = (1-t)\,\mathbf{x} + tj \circ f(\mathbf{x}) \tag{2.52}$$

is a homotopy between $\mathrm{id}_{\mathbb{R}^n}$ and $j \circ f$. Therefore, f is the homotopy inverse of j.

In a similar manner, one can show that $j\big|_{S^{n-1}} : S^{n-1} \hookrightarrow \mathbb{R}^n \setminus \{\mathbf{0}\}$ also has a homotopy inverse. The homotopy inverse is $h : \mathbb{R}^n \setminus \{\mathbf{0}\} \to S^{n-1}$ given by

$$h(\mathbf{x}) = \frac{\mathbf{x}}{\|\mathbf{x}\|}. \tag{2.53}$$

Then $h \circ j\big|_{S^{n-1}} = \mathrm{id}_{S^{n-1}}$. Furthermore, $G : (\mathbb{R}^n \setminus \{\mathbf{0}\}) \times I \to \mathbb{R}^n \setminus \{\mathbf{0}\}$ given by

$$G(\mathbf{x}, t) = (1-t)\,\mathbf{x} + tj\big|_{S^{n-1}} \circ h(\mathbf{x}) = \left((1-t) + \frac{t}{\|\mathbf{x}\|}\right)\mathbf{x} \tag{2.54}$$

is a homotopy between $\mathrm{id}_{\mathbb{R}^n \setminus \{\mathbf{0}\}}$ and $j\big|_{S^{n-1}} \circ h$. Therefore, h is the homotopy inverse of j.

However, $j : \left(B^n, S^{n-1}\right) \hookrightarrow (\mathbb{R}^n, \mathbb{R}^n \setminus \{\mathbf{0}\})$ has no homotopy inverse although both $j : B^n \hookrightarrow \mathbb{R}^n$ and $j\big|_{S^{n-1}} : S^{n-1} \hookrightarrow \mathbb{R}^n \setminus \{\mathbf{0}\}$ have homotopy inverses. To show this, assume the contrary that $g : (\mathbb{R}^n, \mathbb{R}^n \setminus \{\mathbf{0}\}) \to \left(B^n, S^{n-1}\right)$ is a homotopy inverse of j. Then g is continuous, and it maps $\mathbb{R}^n \setminus \{\mathbf{0}\}$ into S^{n-1}. But $\mathbf{0}$ is a limit point of $\mathbb{R}^n \setminus \{\mathbf{0}\}$, and S^{n-1} is closed. Therefore, $g(\mathbf{0}) \in S^{n-1}$. In other words, g maps all of $\mathbb{R}^n$ into S^{n-1}. Hence, the composite

$$g \circ j : \left(B^n, S^{n-1}\right) \to \left(B^n, S^{n-1}\right) \tag{2.55}$$

maps all of B^n to S^{n-1}. If $T : \Delta_p \to B^n$ is a singular p-simplex, then for $T + S_p\left(S^{n-1}\right) \in S_p\left(B^n, S^{n-1}\right)$,

$$\left((g \circ j)_\#\right)_p \left(T + S_p\left(S^{n-1}\right)\right) = g \circ j \circ T + S_p\left(S^{n-1}\right). \tag{2.56}$$

But the image of $g \circ j \circ T$ lies entirely on S^{n-1}. So $\left((g \circ j)_\#\right)_p$ is the trivial chain map. Therefore, $((g \circ j)_*)_p : H_p\left(B^n, S^{n-1}\right) \to H_p\left(B^n, S^{n-1}\right)$ is the trivial map. However, since $g \circ j$ is homotopic with $\mathrm{id}_{(B^n, S^{n-1})}$, $((g \circ j)_*)_p$ is the identity homomorphism on $H_p\left(B^n, S^{n-1}\right)$. This can only be true if $H_p\left(B^n, S^{n-1}\right) = 0$. We shall soon see this is not true.

§2.5 Subdivision

Definition 2.6. Given a topological space X and a collection $\mathcal{A}$ of subsets of X whose interiors cover X, a singular simplex of X is said to be $\mathcal{A}$-small if its image set lies in an element of $\mathcal{A}$.

Given a singular chain of X, we show how to "chop it up" so that all its simplices are $\mathcal{A}$-small.

Definition 2.7 (Barycentric subdivision operator)**.** Let X be a topological space, we define a homomorphism $\mathrm{sd}_X : S_p(X) \to S_p(X)$ by induction. If $T : \Delta_0 \to X$ is a singular 0-simplex, we define

$$\mathrm{sd}_X T = T. \tag{2.57}$$

Now suppose sd_X is defined in dimensions less than p. We will first take $X : \Delta_p$ and choose the identity map $i_p : \Delta_p \to \Delta_p$, which is a singular p-simplex of Δ_p, i.e. $i_p \in S_p(\Delta_p)$. Let us denote by $\widehat{\Delta_p}$ the barycenter of Δ_p. Then we define $\mathrm{sd}_{\Delta_p} i_p$ as follows:

$$\mathrm{sd}_{\Delta_p} i_p = (-1)^p \left[\mathrm{sd}_{\Delta_p} \partial i_p, \widehat{\Delta_p}\right]. \tag{2.58}$$

Now, if $T : \Delta_p \to X$ is any singular p-simplex on X, then we define

$$\mathrm{sd}_X T = (T_\#)_p \left(\mathrm{sd}_{\Delta_p} i_p\right). \tag{2.59}$$

Observe that $\mathrm{sd}_{\Delta_p} i_p$ is expected to be in $S_p(\Delta_p)$. Since $\partial i_p \in S_{p-1}$ and sd_{Δ_p} is assumed to be defined in dimension less than p, $\mathrm{sd}_{\Delta_p} \partial i_p \in S_{p-1}(\Delta_p)$. The bracket operation on the RHS of 2.58, therefore, yields $\left[\mathrm{sd}_{\Delta_p} \partial i_p, \widehat{\Delta_p}\right] \in S_p(\Delta_p)$ so that indeed by 2.58, one obtains $\mathrm{sd}_{\Delta_p} i_p \in S_p(\Delta_p)$.

Lemma 2.11

The homomorphism sd_X is an augmentation preserving chain map. Furthermore, it is natural in the sense that for any continuou map $f : X \to Y$, one has $(f_\#)_p \circ \mathrm{sd}_X = \mathrm{sd}_Y \circ (f_\#)_p$. In other words, the following diagram commutes:

$$\begin{tikzcd} S_p(X) \arrow[r, "(f_\#)_p"] \arrow[d, "\mathrm{sd}_X"'] & S_p(Y) \arrow[d, "\mathrm{sd}_Y"] \\ S_p(X) \arrow[r, "(f_\#)_p"'] & S_p(Y). \end{tikzcd}$$

Proof. Recall that in dimension 0, for $T : \Delta_0 \to X$, one has $\mathrm{sd}_X T = T$. In other words, $\mathrm{sd}_X : S_0(X) \to S_0(X)$ is the identity map. Hence, in dimension 0, $\mathrm{sd}_X : S_0(X) \to S_0(X)$ is trivially augmentation preserving as the following diagram commutes:

$$\begin{tikzcd} S_0(X) \arrow[r, "\mathrm{sd}_X"] \arrow[d, "\epsilon"'] & S_0(X) \arrow[d, "\epsilon"] \\ \mathbb{Z} \arrow[r, "\mathrm{id}_{\mathbb{Z}}"] & \mathbb{Z}. \end{tikzcd}$$

Let us immediately find that the naturality of sd_X in dimension 0 holds. It follows trivially from the following commutative diagram.

$$\begin{tikzcd} S_0(X) \arrow[r, "(f_\#)_0"] \arrow[d, "\mathrm{sd}_X = \mathrm{id}"'] & S_0(Y) \arrow[d, "\mathrm{sd}_Y = \mathrm{id}"] \\ S_0(X) \arrow[r, "(f_\#)_0"'] & S_0(Y). \end{tikzcd}$$

Now, let's verify naturality in positive dimensions. Let $T : \Delta_p \to X$ be continuous. Then

$$(f_\#)_p (\mathrm{sd}_X T) = (f_\#)_p \left[(T_\#)_p \left(\mathrm{sd}_{\Delta_p} i_p \right) \right] = \left((f \circ T)_\# \right)_p \left(\mathrm{sd}_{\Delta_p} i_p \right). \tag{2.60}$$

Now, $f \circ T : \Delta_p \to Y$ is a singular p-simplex on Y. So we have

$$\mathrm{sd}_Y (f \circ T) = \left((f \circ T)_\# \right)_p \left(\mathrm{sd}_{\Delta_p} i_p \right). \tag{2.61}$$

Now, 2.60 and 2.61 together imply

$$(f_\#)_p (\mathrm{sd}_X T) = \mathrm{sd}_Y (f \circ T) = \mathrm{sd}_Y \left((f_\#)_p T \right). \tag{2.62}$$

Therefore, $(f_\#)_p \circ \mathrm{sd}_X = \mathrm{sd}_Y \circ (f_\#)_p$.

Finally, we shall prove that sd_X is a chain map by induction. We need to verify that sd commutes with the boundary operator. The fact that sd commutes with the boundary homomorphism in dimension 0 follows trivially from the following commutative diagram.

$$\begin{tikzcd}
S_0(X) \arrow[r, "\mathrm{sd}_X = \mathrm{id}_{S_0(X)}"] \arrow[d, "\partial_0"'] & S_0(X) \arrow[d, "\partial_0"] \\
0 \arrow[r, "\mathrm{id}"] & 0.
\end{tikzcd}$$

Now, assume that the result holds true in dimension less than p. Now,

$$\partial_p \left(\mathrm{sd}_{\Delta_p} i_p \right) = (-1)^p \partial_p \left[\mathrm{sd}_{\Delta_p} \partial i_p, \widehat{\Delta_p} \right], \tag{2.63}$$

where $i_p : \Delta_p \to \Delta_p$ is the identity map. Δ_p is star convex with respect to $\widehat{\Delta}_p$, and $\mathrm{sd}_{\Delta_p} \partial i_p$ is a $(p-1)$-chain of Δ_p. Then by Lemma 1.5,

$$\begin{aligned}
\partial_p \left[\mathrm{sd}_{\Delta_p} \partial i_p, \widehat{\Delta_p} \right] &= \begin{cases} \left[\partial_{p-1} \left(\mathrm{sd}_{\Delta_p} \partial_p i_p \right), \widehat{\Delta_p} \right] + (-1)^p \mathrm{sd}_{\Delta_p} \partial_p i_p & \text{if } p-1 > 0 \\ \epsilon \left(\mathrm{sd}_{\Delta_p} \partial_p i_p \right) T_0 - \mathrm{sd}_{\Delta_p} \partial i_p & \text{if } p-1 = 0 \end{cases} \\
&= \begin{cases} \left[\partial_{p-1} \left(\mathrm{sd}_{\Delta_p} \partial_p i_p \right), \widehat{\Delta_p} \right] + (-1)^p \mathrm{sd}_{\Delta_p} \partial_p i_p & \text{if } p > 1 \\ \epsilon \left(\mathrm{sd}_{\Delta_1} \partial_1 i_1 \right) T_0 - \mathrm{sd}_{\Delta_1} \partial_1 i_1 & \text{if } p = 1 \end{cases},
\end{aligned} \tag{2.64}$$

where T_0 is the singular 0-simplex whose image point is $\widehat{\Delta_1}$, the barycenter of Δ_1. If $p = 1$, since sd is augmentation preserving, the following diagram commutes:

$$\begin{tikzcd}
S_0(\Delta_1) \arrow[r, "\mathrm{sd}_{\Delta_1} = \mathrm{id}_{S_0(\Delta_1)}"] \arrow[d, "\epsilon"'] & S_0(\Delta_1) \arrow[d, "\epsilon"] \\
\mathbb{Z} \arrow[r, "\mathrm{id}_{\mathbb{Z}}"] & \mathbb{Z}.
\end{tikzcd}$$

So we get for $\partial_1 i_1 \in S_0(\Delta_1)$,

$$\epsilon \left(\mathrm{sd}_{\Delta_1} \partial_1 i_1 \right) = \epsilon \left(\partial_1 i_1 \right) = 0. \tag{2.65}$$

For $p > 1$, by the inductive hypothesis, the following diagram commutes:

$$\begin{tikzcd}
S_{p-1}(\Delta_p) \arrow[r, "\mathrm{sd}_{\Delta_p}"] \arrow[d, "\partial_{p-1}"'] & S_{p-1}(\Delta_p) \arrow[d, "\partial_{p-1}"] \\
S_{p-2}(\Delta_p) \arrow[r, "\mathrm{sd}_{\Delta_p}"] & S_{p-2}(\Delta_p).
\end{tikzcd}$$

Hence, for $\partial_p i_p \in S_{p-1}$,

$$\partial_{p-1}\left(\mathrm{sd}_{\Delta_p}\partial_p i_p\right) = \mathrm{sd}_{\Delta_p}\partial_{p-1}\partial_p i_p = 0. \tag{2.66}$$

Now, combining 2.65, 2.66 and plugging them into 2.64, we get

$$\partial_p\left[\mathrm{sd}_{\Delta_p}\partial i_p, \widehat{\Delta_p}\right] = (-1)^p\,\mathrm{sd}_{\Delta_p}\partial_p i_p \tag{2.67}$$

in both cases. Therefore, 2.63 gives us

$$\partial_p\left(\mathrm{sd}_{\Delta_p} i_p\right) = \mathrm{sd}_{\Delta_p}\partial_p i_p, \quad \forall\, p. \tag{2.68}$$

Now, in general, for $T : \Delta_p \to X$ continuous,

$$\partial_p\left(\mathrm{sd}_X T\right) = \partial_p\left[\left(T_\#\right)_p\left(\mathrm{sd}_{\Delta_p} i_p\right)\right] = \left(T_\#\right)_{p-1}\left[\partial_p\left(\mathrm{sd}_{\Delta_p} i_p\right)\right], \tag{2.69}$$

since $T_\#$ is a chain map and hence the following diagram commutes.

$$\begin{tikzcd}
S_p(\Delta_p) \arrow[r, "(T_\#)_p"] \arrow[d, "\partial_p"'] & S_p(X) \arrow[d, "\partial_p"] \\
S_{p-1}(\Delta_p) \arrow[r, "(T_\#)_{p-1}"'] & S_{p-1}(X)
\end{tikzcd}$$

So

$$\partial_p\left(\mathrm{sd}_X T\right) = \left(T_\#\right)_{p-1}\left[\partial_p\left(\mathrm{sd}_{\Delta_p} i_p\right)\right] = \left(T_\#\right)_{p-1}\left(\mathrm{sd}_{\Delta_p}\partial_p i_p\right) = \mathrm{sd}_X\left(T_\#\right)_{p-1}\left(\partial_p i_p\right), \tag{2.70}$$

using the naturality of sd. Hence,

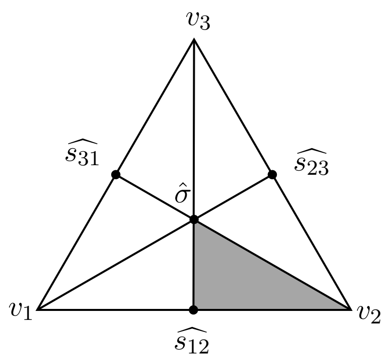
$$\partial_p\left(\mathrm{sd}_X T\right) = \mathrm{sd}_X\left(T_\#\right)_{p-1}\left(\partial_p i_p\right) = \mathrm{sd}_X\,\partial_p\left(sd_X\left(T_\#\right)_p i_p\right). \tag{2.71}$$

Now, $\left(T_\#\right)_p i_p = T \circ i_p = T$. Therefore,

$$\partial_p\left(\mathrm{sd}_X T\right) = \mathrm{sd}_X\,\partial_p T. \tag{2.72}$$

So sd_X indeed commutes with the boundary operator, and hence is a chain map. ■

Consider $\sigma = \Delta_2$ and its first barycentric subdivision.



Denote v_1v_2, v_2v_3 and v_3v_1 by s_{12}, s_{23} and s_{31}, respectively. Denote the barycenter of σ by $\widehat{\sigma}$, barycenter of s_{12} by $\widehat{s_1}2$ and so on. Observe that, for 0-simplices v_1, v_2, v_3, their barycenters are just themselves, i.e. $\widehat{v_i} = v_i$ for $i = 1, 2, 3$. Then we have a natural ordering. For example, $\sigma \succ s_{12} \succ v_2$, meaning s_{12} is a proper face of σ, v_2 is a proper face of s_{12}. Then we have a distinct 2-simplex $\widehat{\sigma}\widehat{s_{12}}\widehat{v_2}$ (colored gray in the above image) by joining the 3 barycenters $\widehat{\sigma}, \widehat{s_{12}}, \widehat{v_2}$. This 2-simplex belongs to the first barycentric subdivision of Δ_2, which we denote by $\mathrm{Sd}\,\Delta_2$ [1].

The first barycentric subdivision of Δ_2 contains also the following 2-simplices: $\widehat{\sigma}\widehat{s_{12}}\widehat{v_1}$, $\widehat{\sigma}\widehat{s_{23}}\widehat{v_2}$, $\widehat{\sigma}\widehat{s_{23}}\widehat{v_3}$, $\widehat{\sigma}\widehat{s_{31}}\widehat{v_1}$, $\widehat{\sigma}\widehat{s_{31}}\widehat{v_3}$. It contains the following 1-simplices: $\widehat{s_{12}}\widehat{v_1}$, $\widehat{s_{12}}\widehat{v_2}$, $\widehat{s_{23}}\widehat{v_2}$, $\widehat{s_{23}}\widehat{v_3}$, $\widehat{s_{31}}\widehat{v_1}$, $\widehat{s_{31}}\widehat{v_3}$ and the 0-simplices $\widehat{v_1}$, $\widehat{v_2}$, $\widehat{v_3}$, $\widehat{s_{12}}$, $\widehat{s_{23}}$, $\widehat{s_{31}}$, $\widehat{\sigma}$. We then have the following result:

[1] Note that, the subdivision operator $\mathrm{sd}_X : S_p(X) \to S_p(X)$ is written sd, and the barycentric subdivision of a simplicial complex (which we studied in AT2) is denoted by Sd, to avoid confusion.

Lemma 2.12

Let K be a simplicial complex. The complex $\operatorname{Sd} K$ equals the collection of all simplices of the form

$$\widehat{\sigma_1}\widehat{\sigma_2}\cdots\widehat{\sigma_n},$$

where $\widehat{\sigma_1} \succ \widehat{\sigma_2} \succ \cdots \succ \widehat{\sigma_n}$.

The proof of this lemma is omitted.

Lemma 2.13

Let $T : \Delta_p \to \sigma$ be a linear homeomorphism of Δ_p with the p-simplex σ. Then each term of $\operatorname{sd}_\sigma T$ is a linear homeomorphism of Δ_p with a simplex in the first barycentric subdivision of σ.

Proof. When $p = 0$, σ is a 0-simplex and the first barycentric subdivision of σ contains just the 0-simplex σ. And, given linear homeomorphism $T : \Delta_0 \to \sigma$, $\operatorname{sd}_\sigma T = T$ is the same linear homeomorphism of Δ_0 with the only simplex σ in the first barycentric subdivision of σ.

Now, suppose the lemma is true in dimension less than p. Consider the identity homeomorphism $i_p : \Delta_p \to \Delta_p$. Now,

$$\operatorname{sd}_{\Delta_p} i_p = (-1)^p \left[\operatorname{sd}_{\Delta_p} \partial i_p, \widehat{\Delta_p}\right].$$

Note that

$$\partial i_p = \sum_{j=0}^{p} (-1)^j \, i_p \circ l_{(\varepsilon_0,\ldots,\widehat{\varepsilon}_j,\ldots,\varepsilon_p)}$$

so that each term in this sum is a linear homeomorphism of Δ_{p-1} with a $(p-1)$-simplex in $\operatorname{Bd} \Delta_p$.

$$\operatorname{sd}_{\Delta_p} \partial i_p = \sum_{j=0}^{p} (-1)^j \operatorname{sd}_{\Delta_p} \left(i_p \circ l_{(\varepsilon_0,\ldots,\widehat{\varepsilon}_j,\ldots,\varepsilon_p)}\right).$$

By the inductive hypothesis, each term of $\operatorname{sd}_{\Delta_p} \left(i_p \circ l_{(\varepsilon_0,\ldots,\widehat{\varepsilon}_j,\ldots,\varepsilon_p)}\right)$ is a linear homeomorphism of Δ_{p-1} with a $(p-1)$-simplex $\widehat{s_1}\widehat{s_2}\cdots\widehat{s_p}$ in the first barycentric subdivision of $\operatorname{Bd} \Delta_p$.

$$\operatorname{sd}_{\Delta_p} \left(i_p \circ l_{(\varepsilon_0,\ldots,\widehat{\varepsilon}_j,\ldots,\varepsilon_p)}\right) = \sum_k \pm T_{jk}, \tag{2.73}$$

where T_{jk} is a linear homeomorphism of Δ_{p-1} with a $(p-1)$-simplex $\widehat{s_1}\widehat{s_2}\cdots\widehat{s_p}$ in the first barycentric subdivision of $\operatorname{Bd} \Delta_p$. So

$$\operatorname{sd}_{\Delta_p} \partial i_p = \sum_{j=0}^{p} \sum_k \pm T_{jk}. \tag{2.74}$$

Then $\left[T_{jk}, \widehat{\Delta_p}\right]$ is by definition a linear homeomorphism of Δ_p with the p-simplex $\widehat{\Delta_p}\widehat{s_1}\widehat{s_2}\cdots\widehat{s_p}$, which belongs to the first barycentric subdivision of Δ_p. Now,

$$\operatorname{sd}_{\Delta_p} i_p = \sum_{j=0}^{p} \sum_k \pm \left[T_{jk}, \widehat{\Delta_p}\right]. \tag{2.75}$$

Therefore, each term of $\operatorname{sd}_{\Delta_p} i_p$ is a linear homeomorphism of Δ_p with a p-simplex in the first barycentric subdivision of Δ_p.

Now consider a general linear homeomorphism $T : \Delta_p \to \sigma$. It's clear that T defines a linear homeomorphism between the first barycentric subdivision of Δ_p with that of σ, because T takes barycenter of Δ_p to the barycenter of σ (since T is linear).

$$\operatorname{sd}_\sigma T = (T_\#)_p \left(\operatorname{sd}_{\Delta_p} i_p\right),$$

with $T : \Delta_p \to \sigma$ being a linear homeomorphism. Using 2.75,

$$\operatorname{sd}_\sigma T = \sum_{j=0}^{p} \sum_k \pm T \circ \left[T_{jk}, \widehat{\Delta_p}\right]. \tag{2.76}$$

By construction, $\left[T_{jk}, \widehat{\Delta_p}\right] : \Delta_p \to \operatorname{Sd}(\Delta_p)$ is a linear homeomorphism onto its image, and $T : \Delta_p \to \sigma$ is a given linear homeomorphism. Hence, the composite $T \circ \left[T_{jk}, \widehat{\Delta_p}\right] : \Delta_p \to \sigma$ is a linear homeomorphism.

$\left[T_{jk}, \widehat{\Delta_p}\right]$ takes Δ_p linear homeomorphically to a p-simplex in the first barycentruc subdivision of Δ_p and we have seen that T is a linear homeomorphism between the first barycentric subdivision of Δ_p with that of σ. Hence, $T \circ \left[T_{jk}, \widehat{\Delta_p}\right]$ takes Δ_p linear homeomorphically to a p-simplex in the first barycentruc subdivision of σ. So the terms of $\operatorname{sd}_\sigma T$ are linear homeomorphisms of Δ_p with a p-somplex in the first barycentric subdivision of σ. ■

Theorem 2.14

Let $\mathcal{A}$ be a collection of subsets of X whose interiors cover X. Given $T : \Delta_p \to X$, there is an m such that each term of $\operatorname{sd}_X^m T$ is $\mathcal{A}$-small.

Proof. Apply Lemma 2.13 to each term of $\operatorname{sd}_\sigma L$, where $L : \Delta_p \to \sigma$ is a linear homeomorphism of Δ_p with a p-simplex σ. Each term of $\operatorname{sd}_\sigma L$ is a linear homeomorphism of Δ_p with a simplex in $\operatorname{Sd}\sigma$. Then each term of $\operatorname{sd}_\sigma^2 L$ is a linear homeomorphism of Δ_p with a simplex in $\operatorname{Sd}^2\sigma$. More generally, each term of $\operatorname{sd}_\sigma^m L$ is a linear homeomorphism of Δ_p with a simplex in the m-th barycentric subdivision of σ, i.e. $\operatorname{Sd}^m\sigma$.

Now, $\{\operatorname{Int} A \mid A \in \mathcal{A}\}$ covers X. Let us first cover Δ_p by open sets $T^{-1}(\operatorname{Int} A)$ with $A \in \mathcal{A}$. Δ_p is a compact metric space. Let λ be the Lebesgue number associated with this cover $\left\{T^{-1}(\operatorname{Int} A) \mid A \in \mathcal{A}\right\}$ of Δ_p. So every subset of Δ_p with diameter less than λ must be contained in $T^{-1}(\operatorname{Int} A)$ for some $A \in \mathcal{A}$.

Now, choose m large enough such that each simplex in the m-th barycentric subdivision has diameter less than λ. Now, in the opening paragraph of the proof, take $L = i_p : \Delta_p \to \Delta_p$, the identity map from Δ_p to itself. Then each term of $\operatorname{sd}_{\Delta_p}^m i_p$ is a linear homeomorphism of Δ_p with a p-simplex in the m-th barycentric subdivision of Δ_p, each of which has diameter smaller than λ.

Then by *Lebesgue number lemma*, the image of each term of $\operatorname{sd}_{\Delta_p}^m i_p$ is contained in $T^{-1}(\operatorname{Int} A)$ for some $A \in \mathcal{A}$. So, T composed with each term of $\operatorname{sd}_{\Delta_p}^m i_p$ is contained in $\operatorname{Int} A$ for some $A \in \mathcal{A}$. But T composed with each term of $\operatorname{sd}_{\Delta_p}^m i_p$ is nothing but each term of

$$(T_\#)_p\left(\operatorname{sd}_{\Delta_p}^m i_p\right) = \operatorname{sd}_X^m T. \tag{2.77}$$

Hence, each term of $\operatorname{sd}_X^m T$ has its image set contained in $\operatorname{Int} A$. In other words, each term of $\operatorname{sd}_X^m T$ is $\mathcal{A}$-small. ■

Remark 2.3. $\operatorname{sd}_X^m : S_p(X) \to S_p(X)$ is of course a map. In fact, it is a group homomorphism. But we can't talk about the image set of $\operatorname{sd}_X^m T$ even when $T : \Delta_p \to X$ is a singular p-simplex of X, as $\operatorname{sd}_X^m T$ is, in general, a p-chain, not a singular p-simplex.

Having shown how to chop up singular chains so that they are $\mathcal{A}$-small, we now show that these $\mathcal{A}$-small singular chains suffice to generate the homology of X. We first need a lemma.

Lemma 2.15

Let m be given. For each space X, there is a homomorphism $D_p^X : S_p(X) \to S_{p+1}(X)$ such that for each singular p-simplex T of X,

$$\partial_{p+1} D_p^X T + D_{p-1}^X \partial_p T = \mathrm{sd}_X^m T - \mathrm{id}_{S_p(X)} T. \tag{2.78}$$

Furthermore, D^X is natural; i.e., for continuous $f : X \to Y$, the following diagram commutes

$$\begin{tikzcd} S_p(X) \arrow[r, "(f_\#)_p"] \arrow[d, "D_p^X"'] & S_p(Y) \arrow[d, "D_p^Y"] \\ S_{p+1}(X) \arrow[r, "(f_\#)_{p+1}"'] & S_{p+1}(Y). \end{tikzcd}$$

In other words, $D_p^Y \circ (f_\#)_p = (f_\#)_{p+1} \circ D_p^X$.

Remark 2.4. The above lemma guarantees that there is a chain homotopy D^X between the chain maps $\mathrm{sd}_X^m, \mathrm{id}_{\mathcal{S}(X)} : \mathcal{S}(X) \to \mathcal{S}(X)$. Also, note that the naturality of sd_X^m and D^X shows that if A is a subspace of X, then sd_X^m and D^X carry $S_p(A)$ into $S_p(A)$ and $S_{p+1}(A)$, respectively. Thus they induce a chain map and a chain homotopy, respectively, on the relative chain complex $\mathcal{S}(X, A)$ as well.

§2.6 Excision

Definition 2.8. Let X be a topological space; let $\mathcal{A}$ be a covering of X. Let $S_p^{\mathcal{A}}(X)$ be the subgroup of $S_p(X)$ generated by singular p-simplices of X that are $\mathcal{A}$-small. Let $\mathcal{S}^{\mathcal{A}}(X)$ denote the chain complex whose chain groups are the groups $S_p^{\mathcal{A}}(X)$. $\mathcal{S}^{\mathcal{A}}(X)$ is a subchain complex of $\mathcal{S}(X)$, because if the singular p-simplex $T : \Delta_p \to X$ has its image set in $A \in \mathcal{A}$, then each term of $\partial_p T$ also has its image set contained in the same $A \in \mathcal{A}$.

Note that each singular 0-chain is automatically $\mathcal{A}$-small. Hence, $S_0^{\mathcal{A}}(X) = S_0(X)$, and consequently ϵ defines an augmentation for $\mathcal{S}^{\mathcal{A}}(X)$. Hence, by Remark 2.4, sd_X^m and D^X carry $\mathcal{S}^{\mathcal{A}}(X)$ into itself. In other words, if the image set of a singular p-simplex $T : \Delta_p \to X$ lies in $A \in \mathcal{A}$, then each term of $\mathrm{sd}_X^m T$ and $D_p^X T$ also has its image set lying in $A \in \mathcal{A}$.

Theorem 2.16

Let X be a topological space; let $\mathcal{A}$ be a collection of subsets of X whose interiors cover X. Then the inclusion map $\mathcal{S}^{\mathcal{A}}(X) \hookrightarrow \mathcal{S}(X)$ induces an isomorphism in homology, both ordinary and reduced.

Proof. Consider the short exact sequence of chain complexes

$$0 \longrightarrow \mathcal{S}^{\mathcal{A}}(X) \xrightarrow{i} \mathcal{S}(X) \longrightarrow \mathcal{S}(X)/\mathcal{S}^{\mathcal{A}}(X) \longrightarrow 0.$$

This, in fact, is a collection of short exact sequence of chain groups in each dimension p:

$$0 \longrightarrow S_p^{\mathcal{A}}(X) \xrightarrow{(i_\#)_p} S_p(X) \longrightarrow \mathcal{S}_p(X)/S_p^{\mathcal{A}}(X) \longrightarrow 0.$$

It gives rise to a long exact sequence in homology (either ordinary or reduced). Now, if we can prove that the homology groups of the chain complex $\left\{S_p(X)/S_p^{\mathcal{A}}(X), \partial_p^X\right\}$ vanish in every dimension p, then the long exact sequence in homology obtained from the short exact sequence above using Zig-Zag lemma will yield the following exact sequence:

$$0 \longrightarrow H_p^{\mathcal{A}}(X) \xrightarrow{(i_*)_p} H_p(X) \longrightarrow 0.$$

The exactness of this sequence will then dictate that $(i_*)_p : H_p^{\mathcal{A}}(X) \to H_p(X)$ is an isomorphism. Let us now prove that the homology groups of the chain complex $\left\{S_p(X)/S_p^{\mathcal{A}}(X), \partial_p^X\right\}$ vanish in every dimension p.

Let $c_p + S_p^{\mathcal{A}}(X) \in S_p(X)/S_p^{\mathcal{A}}(X)$, for $c_p \in S_p(X)$, such that it represents a cycle in $S_p(X)/S_p^{\mathcal{A}}(X)$. In other words, $\partial_p^X c_p$ belongs to $S_{p-1}^{\mathcal{A}}(X)$. We now want to show that this c_p necessarily represents a boundary, i.e. there exists some $d_{p+1} \in S_{p+1}(X)$ such that $c_p - \partial_{p+1}^X d_{p+1}$ belongs to $S_p^{\mathcal{A}}(X)$.

Note that c_p is a finite formal linear combination of singular p-simplices. In view of Theorem 2.14, we chan choose m large enough so that each singular p-simplex appearing in the expression for $\mathrm{sd}_X^m c_p$ is $\mathcal{A}$-small. Once m is chosen, let D^X be the chain homotopy of Lemma 2.15. $D_p^X : S_p(X) \to S_{p+1}(X)$. In fact, we shall show that $-D_p^X c_p$ is precisely the $d_{p+1} \in S_{p+1}(X)$ that we are looking for. In other words, we will show that $c_p + \partial_{p+1}^X D_p^X c_p$ belongs to $S_p^{\mathcal{A}}(X)$ and we are done!

By Lemma 2.15, we know that

$$\partial_{p+1}^X D_p^X c_p + D_{p-1}^X \partial_p^X c_p = \mathrm{sd}_X^m c_p - c_p \implies c_p + \partial_{p+1}^X D_p^X c_p = \mathrm{sd}_X^m c_p - D_{p-1}^X \partial_p^X c_p. \tag{2.79}$$

We have chosen m large enough so that $\mathrm{sd}_X^m c_p \in S_p^{\mathcal{A}}(X)$. Also, $\partial_p^X c_p \in S_{p-1}^{\mathcal{A}}(X)$, so that $D_{p-1}^X \partial_p^X c_p \in S_{p-1}^{\mathcal{A}}(X)$. Therefore, from 2.79, we can conclude that $c_p + \partial_{p+1}^X D_p^X c_p$. ∎

Corollary 2.17

Let X and $\mathcal{A}$ be as in the previous theorem. If $B \subseteq X$, let $S_p^{\mathcal{A}}(B)$ be generated by those singular p-simplices $T : \Delta_p \to B$ whose image sets lie in elements of $\mathcal{A}$. Obviously, $S_p^{\mathcal{A}}(B) \subseteq S_p^{\mathcal{A}}(X)$. Let us denote the quotient group by

$$S_p^{\mathcal{A}}(X, B) = S_p^{\mathcal{A}}(X) / S_p^{\mathcal{A}}(B).$$

Then the inclusion

$$i_p : S_p^{\mathcal{A}}(X, B) \hookrightarrow S_p(X, B)$$

induces a homology isomorphism.

Proof. Consider the following inclusion maps

$$\mathcal{S}^{\mathcal{A}}(B) \overset{i_B}{\hookrightarrow} \mathcal{S}(B),$$
$$\mathcal{S}^{\mathcal{A}}(X) \overset{i_X}{\hookrightarrow} \mathcal{S}(X),$$
$$\mathcal{S}^{\mathcal{A}}(X, B) \overset{i_{(X,B)}}{\hookrightarrow} \mathcal{S}(X, B),$$

and the 2 short exact sequences of chain complexes connected by the above 3 inclusions:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{S}^{\mathcal{A}}(B) \arrow[r] \arrow[d, "i_B"'] & \mathcal{S}^{\mathcal{A}}(X) \arrow[r] \arrow[d, "i_X"'] & \mathcal{S}^{\mathcal{A}}(X,B) \arrow[r] \arrow[d, "i_{(X,B)}"] & 0 \\
0 \arrow[r] & \mathcal{S}(B) \arrow[r] & \mathcal{S}(X) \arrow[r] & \mathcal{S}(X,B) \arrow[r] & 0
\end{tikzcd}$$

The above diagram commutes. To show that, it suffices to show that commutativity of the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & S_p^{\mathcal{A}}(B) \arrow[r] \arrow[d, "i_B"'] & S_p^{\mathcal{A}}(X) \arrow[r] \arrow[d, "i_X"'] & S_p^{\mathcal{A}}(X,B) \arrow[r] \arrow[d, "i_{(X,B)}"] & 0 \\
0 \arrow[r] & S_p(B) \arrow[r] & S_p(X) \arrow[r] & S_p(X,B) \arrow[r] & 0.
\end{tikzcd}$$

If we take $c \in S_p^{\mathcal{A}}(B)$, the inclusion maps take it to itself. So the left hand square commutes trivially. Now, we take $d \in S_p^{\mathcal{A}}(X)$. Then under the map $S_p^{\mathcal{A}}(X) \to S_p^{\mathcal{A}}(X, B)$, d goes to

$$d + S_p^{\mathcal{A}}(B).$$

Then under $i_{(X,B)}$, it goes to

$$d + S_p(B).$$

On the other hand, i_X takes d to itself. Then the map $S_p(X) \to S_p(X, B)$ takes it to

$$d + S_p(B).$$

Therefore, the right hand square commutes as well. Therefore, one obtains the following commutative diagram with the two corresponding long exact sequences connnected via induced group homomorphisms:

$$\begin{tikzcd}
\cdots \arrow[r] & H_p^{\mathcal{A}}(B) \arrow[r] \arrow[d, "((i_B)_*)_p"'] & H_p^{\mathcal{A}}(X) \arrow[r] \arrow[d, "((i_X)_*)_p"] & H_p^{\mathcal{A}}(X,B) \arrow[r] \arrow[d, "((i_{(X,B)})_*)_p"] & H_{p-1}^{\mathcal{A}}(B) \arrow[r] \arrow[d, "((i_B)_*)_{p-1}"] & H_{p-1}^{\mathcal{A}}(X) \arrow[r] \arrow[d, "((i_X)_*)_{p-1}"] & \cdots \\
\cdots \arrow[r] & H_p(B) \arrow[r] & H_p(X) \arrow[r] & H_p(X,B) \arrow[r] & H_{p-1}(B) \arrow[r] & H_{p-1}(X) \arrow[r] & \cdots
\end{tikzcd}$$

Now, $((i_B)_*)_p$, $((i_X)_*)_p$, $((i_B)_*)_{p-1}$, $((i_X)_*)_{p-1}$ are all isomorphisms by Theorem 2.16. Therefore, applying Steenrod five lemma, we conclude that $\left(\left(i_{(X,B)}\right)_*\right)_p : H_p^{\mathcal{A}}(X, B) \to H_p(X, B)$ is an isomorphism. ■

Theorem 2.18 (Excision for singular theory)
Let $A \subseteq X$. If U is a subset of X such that $\overline{U} \subseteq \operatorname{Int} A$, then the inclusion

$$j : (X \setminus U, A \setminus U) \hookrightarrow (X, A)$$

induces an isomorphism in singular homology.

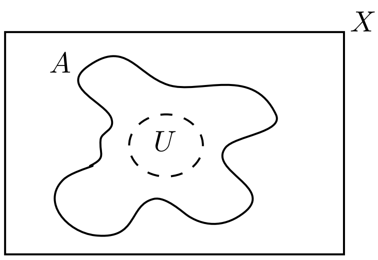
Proof. Let $\mathcal{A}$ denote the collection $\{X \setminus U, A\}$. Observe that the open set $X \setminus \overline{U}$ is precisely $\operatorname{Int}(X \setminus U)$. Also, since $\overline{U} \subseteq \operatorname{Int} A$,

$$X \setminus (\operatorname{Int} A) \subseteq X \setminus \overline{U} = \operatorname{Int}(X \setminus U).$$

Therefore,

$$X = [X \setminus (\operatorname{Int} A)] \cup (\operatorname{Int} A) \subseteq \operatorname{Int}(X \setminus U) \cup \operatorname{Int} A = \bigcup_{S \in \mathcal{A}} \operatorname{Int}(S).$$

Therefore, the interiors of sets in $\mathcal{A}$ cover X.



Now, consider the homomorphisms induced by inclusions

$$\frac{S_p(X \setminus U)}{S_p(A \setminus U)} \hookrightarrow \frac{S_p^{\mathcal{A}}(X)}{S_p^{\mathcal{A}}(A)} \text{ and } \frac{S_p^{\mathcal{A}}(X)}{S_p^{\mathcal{A}}(A)} \hookrightarrow \frac{S_p(X)}{S_p(A)}.$$

The first one is an inclusion since a p-chain in $X \setminus U$ is clearly in $S_p^{\mathcal{A}}(X)$ as $\mathcal{A} = \{X \setminus U, A\}$; and a p-chain in $A \setminus U$ is also clearly in $S_p^{\mathcal{A}}(A)$. The second inclusion is just $S_p^{\mathcal{A}}(X, A) \hookrightarrow S_p(X, A)$.

By Corollary 2.17, the latter homomorphism induces group isomorphism at the level of homology groups. We now intend to prove that

$$\frac{S_p\left(X \setminus U\right)}{S_p\left(A \setminus U\right)} \hookrightarrow \frac{S_p^{\mathcal{A}}\left(X\right)}{S_p^{\mathcal{A}}\left(A\right)}$$

is already an isomorphism at the chain level. Consider the map

$$\phi : S_p\left(X \setminus U\right) \to \frac{S_p^{\mathcal{A}}\left(X\right)}{S_p^{\mathcal{A}}\left(A\right)}, \qquad c_p \mapsto c_p + S_p^{\mathcal{A}}\left(A\right), \tag{2.80}$$

for $c_p \in S_p\left(X \setminus U\right)$. Note that ϕ is surjective. If c_p is a p-chain in $S_p^{\mathcal{A}}\left(X\right)$, then each term of c_p has image set lying in either $X \setminus U$ or in A. While forming the coset $c_p + S_p^{\mathcal{A}}\left(A\right)$, we can safely throw away the terms that have image sets in A. So every coset element in $\frac{S_p^{\mathcal{A}}(X)}{S_p^{\mathcal{A}}(A)}$ is of the form

$$d_p + S_p^{\mathcal{A}}\left(A\right)$$

for $d_p \in S_p\left(X \setminus U\right)$. Hence, ϕ is surjective. Now, $c_p \in \operatorname{Ker}\phi$ if $c_p \in S_p^{\mathcal{A}}\left(A\right)$. Since $\operatorname{Ker}\phi \subset S_p\left(X \setminus U\right)$, we have

$$c_p \in S_p\left(X \setminus U\right) \cap S_p^{\mathcal{A}}\left(A\right) = S_p\left(\left(X \setminus U\right) \cap A\right) = S_p\left(A \setminus U\right). \tag{2.81}$$

Therefore, $\operatorname{Ker}\phi = S_p\left(A \setminus U\right)$. Hence, by the first isomorphisim theorem,

$$\frac{S_p\left(X \setminus U\right)}{S_p\left(A \setminus U\right)} \cong \frac{S_p^{\mathcal{A}}\left(X\right)}{S_p^{\mathcal{A}}\left(A\right)}. \tag{2.82}$$

Therefore, $H_p\left(X \setminus U, A \setminus U\right) \cong H_p^{\mathcal{A}}\left(X, A\right)$. We already have $H_p^{\mathcal{A}}\left(X, A\right) \cong H_p\left(X, A\right)$ by Corollary 2.17. Therefore,

$$H_p\left(X \setminus U, A \setminus U\right) \cong H_p\left(X, A\right).$$

■

3 CW Complexes

§3.1 The Topology of CW Complexes

Definition 3.1. If X is a topological space and $\mathcal{C}$ is a collection of subspaces of X whose union is X, the topology of X is said to be **coherent** with the collection $\mathcal{C}$ provided a set $A \subseteq X$ is closed in X if and only if $A \cap C$ is closed in C for each $C \in \mathcal{C}$. It is equivalent to require that $U \subseteq X$ is open in X if and only if $U \cap C$ is open in C for each $C \in \mathcal{C}$.

Lemma 3.1

Let X be a set which is the union of topological space $\{X_\alpha\}$. If there is a topological space X_{T} having X as its underlying set, and each X_α is a subspace of X_{T}, then X has a topology (called the **coherent topology**), of which X_α are subspaces, that is coherent with the collection $\{X_\alpha\}$. This latter topology is, in general, finer than the topology of X_{T}.

Proof. Let us define a topological space X_{C} (whose underlying set is X) by declaring that $A \subseteq X$ is closed if and only if $A \cap X_\alpha$ is closed in X_α for each α. If A and B are closed in X_{C}, then both $A \cap X_\alpha$ and $B \cap X_\alpha$ are closed in X_α for each α. Therefore,

$$(A \cup B) \cap X_\alpha = (A \cap X_\alpha) \cup (B \cap X_\alpha) \tag{3.1}$$

is closed in X_α, proving that $A \cup B$ is closed. On the other hand, if $\{A_i\}_{i \in J}$ is an arbitrary collection of closed sets, each $A_i \cap X_\alpha$ is closed in X_α. Then

$$\left(\bigcap_{i \in J} A_i\right) \cap X_\alpha = \bigcap_{i \in J} (A_i \cap X_\alpha) \tag{3.2}$$

is closed in X_α. Therefore, $\bigcap_{i \in J} A_i$ is closed. Hence, X_{C} indeed defines a topology on X.

Now, if C is a closed set in X_{T}, then since X_α is a subspace of X_{T}, $C \cap X_\alpha$ must be closed in X_α for each α. Therefore, C is closed in X_{C}. Thus, the topology of X_{C} is finer than that of X_{T}.

Now we need to show that each X_α is a subspace of X_{C}. For this purpose, we show that the closed sets of X_α are of the form $C \cap X_\alpha$, where C is closed in X_{C}. First note that if C is closed in X_{C}, $C \cap X_\alpha$ is closed in X_α for each α. Conversely, if B is closed in X_α, since X_α is a subspace of X_{T}, $B = C \cap X_\alpha$ for some closed C in X_{T}. Now, since X_{C} is finer than X_{T}, C must also be closed in X_{C}. Thus $B = C \cap X_\alpha$ for some closed C in X_{C}, as desired. Therefore, each X_α is a subspace of X_{C}. So X_{C} is coherent with the collection $\{X_\alpha\}$. ■

Remark 3.1. We can always give a topology X_{T} to the underlying set $X = \bigcup_\alpha X_\alpha$, with each X_α being a topological space by its own right, so that X_α becomes a subspace of X_{T} (i.e. the topology of X_α that it had as an individual topological space from the beginning coincides with the subspace topology it inherits from X_{T}) with X_{T} not being coherent with its subspaces X_α. In such case, X_{C} will be strictly finer than X_{T}. When X_{T} is found to be coherent with its subspaces X_α, one has $X_{\mathsf{T}} = X_{\mathsf{C}}$.

Some useful terminologies

The m-dimensional ball B^m is the following subspace of $\mathbb{R}^m$

$$B^m = \{\mathbf{x} \in \mathbb{R}^m \mid \|\mathbf{x}\| \leq 1\}. \tag{3.3}$$

The open m-ball, denoted by $\operatorname{Int}(B^m)$, is the interior of B^m in $\mathbb{R}^m$.

$$\operatorname{Int} B^m = \{\mathbf{x} \in \mathbb{R}^m \mid \|\mathbf{x}\| < 1\}. \tag{3.4}$$

The boundary of B^m in $\mathbb{R}^m$ is the standard $(m-1)$-sphere.

$$S^{m-1} = \operatorname{Bd} B^m = \{\mathbf{x} \in \mathbb{R}^m \mid \|\mathbf{x}\| < 1\}. \tag{3.5}$$

We note that the 0-ball B^0 is equal to $\mathbb{R}^0 = \{0\}$. One has $\operatorname{Int} B^0 = B^0 = \{0\}$. Also, B^1 is the interval $[-1, 1]$ in $\mathbb{R}$, and $\operatorname{Int} B^1 = (-1, 1)$. So

$$S^0 = \operatorname{Bd} B^1 = \{-1, 1\}. \tag{3.6}$$

Cell decomposition and CW-complexes

Definition 3.2. An n-cell is a topological space homeomorphic to the open n-ball $\operatorname{Int} B^n$. A **cell** is a topological space which is an n-cell for some $n \geq 0$. Since $\operatorname{Int} B^n$ is homeomorphic to $\mathbb{R}^n$, we can talk about the dimension of an n-cell. An n-cell is rightly said to have dimension n.

Definition 3.3 (Cell decomposition)**.** A **cell decomposition** of a topological space X is a family $\mathcal{E} = \{e_\alpha \mid \alpha \in I\}$ of subspaces of X such that each e_α is a cell and

$$X = \bigsqcup_{\alpha \in I} e_\alpha. \tag{3.7}$$

The n-skeleton of X is the subspace

$$X^n = \bigsqcup_{\alpha \in I,\, \dim e_{\alpha \leq n}} e_\alpha. \tag{3.8}$$

Note that if $\mathcal{E}$ is a cell decomposition of X, then the cells of $\mathcal{E}$ can have many different dimensions. For example, consider a cell-decomposition of S^1 given by $\mathcal{E} = \{e_a, e_b\}$, where e_a is an arbitrary point $p \in S^1$ and $e_b = S^1 \setminus \{p\}$. Here, e_a is a 0-cell and e_b is a 1-cell. One can have uncountably many cells in a cell decomposition of a given topological space. A **finite cell decomposition** is a cell decomposition consisting of finitely many cells.

Definition 3.4 (CW complex)**.** A pair $(X, \mathcal{E})$ consisting of a Hausdorff space X and a cell decomposition $\mathcal{E}$ of X is called a **CW complex** if the following 3 axioms are satisfied:

Axiom 1 (Characteristic maps)**.** For each n-cell $e_\alpha \in \mathcal{E}$, there is a continuous map $f_\alpha : B^n \to X$ restricting to a homeomorphism

$$f_\alpha\big|_{\operatorname{Int} B^n} : \operatorname{Int} B^n \to e_\alpha$$

and taking $\operatorname{Bd} B^n = S^{n-1}$ into X^{n-1}.

Axiom 2 (Closure finiteness)**.** For any cell $e_\alpha \in \mathcal{E}$, the closure $\overline{e_\alpha}$ intersects only finitely many cells in $\mathcal{E}$.

Axiom 3 (Weak topology)**.** A subset $A \subseteq X$ is closed if and only if $A \cap \overline{e_\alpha}$ is closed in $\overline{e_\alpha}$ for each $e_\alpha \in \mathcal{E}$.

Remark 3.2. Here, the topology of the Hausdorff space $X = \bigcup_\alpha \overline{e_\alpha}$ is coherent with the subspaces $\{\overline{e_\alpha}\}_\alpha$, i.e. X is endowed with the finest topology with respect to which all these topological spaces $\overline{e_\alpha}$ become its subspaces. Axiom 3 basically demands this coherence.

Definition 3.5. The **dimension** of a CW complex (X, ε) is the largest dimension of a cell of $\mathcal{E}$, if such exists. Otherwise, it is said to be infinite.

Lemma 3.2

Let X be a Hausdorff space and $\mathcal{E} = \{e_\alpha\}_\alpha$ a cell decomposition of X. If $(X, \mathcal{E})$ satisfies Axiom 1 of CW complex, then we have $\overline{e_\alpha} = f_\alpha(B^n)$ for any n-cell e_α. In particular, $\overline{e_\alpha}$ is a compact subspace of X and the "cell boundary" $\dot{e}_\alpha := \overline{e_\alpha} \setminus e_\alpha = f_\alpha\left(S^{n-1}\right)$ lies in X^{n-1}.

Proof. Since $f_\alpha : B^n \to X$ is continuous associated with a given n-cell e_α, we have

$$\overline{e_\alpha} = \overline{f_\alpha(\operatorname{Int} B^n)} \supseteq f_\alpha\left(\overline{\operatorname{Int} B^n}\right) = f_\alpha(B^n). \tag{3.9}$$

So $f_\alpha(B^n) \subseteq \overline{e_\alpha}$. Since B^n is compact and f_α is continuous, $f_\alpha(B^n)$ is compact. Now, since X is Hausdorff, $f_\alpha(B^n)$ is closed. Since $e_\alpha = f_\alpha(\operatorname{Int} B^n)$,

$$f_\alpha(B^n) \supseteq e_\alpha \implies \overline{f_\alpha(B^n)} \supseteq \overline{e_\alpha} \implies f_\alpha(B^n) \supseteq \overline{e_\alpha}. \tag{3.10}$$

Therefore, $\overline{e_\alpha} = f_\alpha(B^n)$.

By Axiom 1, we have $f_\alpha(\operatorname{Int} B^n) = e_\alpha$ and $f_\alpha\left(S^{n-1}\right) \subseteq X^{n-1}$. So

$$f_\alpha\left(S^{n-1}\right) \cap e_\alpha = \varnothing. \tag{3.11}$$

But $f_\alpha\left(S^{n-1}\right) \subseteq f_\alpha(B^n) = \overline{e_\alpha}$. So we have

$$f_\alpha\left(S^{n-1}\right) \subseteq \overline{e_\alpha} \setminus e_\alpha. \tag{3.12}$$

Furthermore,

$$\overline{e_\alpha} \setminus e_\alpha = f_\alpha(B^n) \setminus f_\alpha(\operatorname{Int} B^n) \subseteq f_\alpha(B^n \setminus \operatorname{Int} B^n) = f_\alpha\left(S^{n-1}\right). \tag{3.13}$$

Therefore, $f_\alpha\left(S^{n-1}\right) = \overline{e_\alpha} \setminus e_\alpha =: \dot{e}_\alpha$. ∎

Subcomplexes

Lemma 3.3

Let $(X, \mathcal{E})$ be a CW complex, and $\mathcal{E}' = \{e_{\alpha'}\}_{\alpha'} \subseteq \mathcal{E}$ a collection of cells in it. Suppose $X' = \bigcup_{\alpha'} e_{\alpha'}$. Then the following are equivalent:

(a) The pair $(X', \mathcal{E}')$ is a CW complex.

(b) The subset X' is closed in X.

(c) $\overline{e_{\alpha'}} \subseteq X'$ for each $e_{\alpha'} \in \mathcal{E}'$, where $\overline{e_{\alpha'}}$ is the closure of $e_{\alpha'}$ in X.

Definition 3.6 (Subcomplex)**.** Let $(X, \mathcal{E})$ be a CW complex, and $(X', \mathcal{E}')$ be as above. Then $(X', \mathcal{E}')$ is called a **subcomplex** of $(X, \mathcal{E})$ if the 3 equivalent conditions stated in Lemma 3.3 are satisfied.

Corollary 3.4

Let $(X, \mathcal{E})$ be a CW complex. Then

(a) Let $\{A_i\}_{i \in I}$ be any family of subcomplexes of $(X, \mathcal{E})$. Then $\bigcup_{i \in I} A_i$ and $\bigcap_{i \in I} A_i$ are subcomplexes of $(X, \mathcal{E})$.

(b) The n-skeleton X^n is a subcomplex of $(X, \mathcal{E})$ for each $n \geq 0$.

(c) Let $\{e_i\}_{i \in I}$ be any arbitrary family of n-cells in $\mathcal{E}$. Then $X^{n-1} \cup \left(\bigcup_{i \in I} e_i\right)$ is a subcomplex.

Proof. We shall first prove (a). The others follow immediately from (a). Given the family of subcomplexes $\{A_i\}_{i\in I}$ of the CW complex $(X, \mathcal{E})$, each $A_i \subseteq X$ is a closed subspace of X. Then $\bigcap_{i\in I} A_i$ is closed in A. Therefore, by Lemma 3.3, $\bigcap_{i\in I} A_i$ is a subcomplex of $(X, \mathcal{E})$.

Now we shall prove that $\bigcup_{i\in I} A_i$ is a subcomplex. For this purpose, we shall use the characterization (c) of Lemma 3.3. Let $e \subseteq \bigcup_{i\in I} A_i$ be an n-cell. Then $e \subseteq A_j$ for some $j \in I$. By characterization (c), $\overline{e} \subseteq A_j$. Therefore, $\overline{e} \subseteq \bigcup_{i\in I} A_i$. So $\bigcup_{i\in I} A_i$ is a subcomplex.

Now, we shall prove (b). If e_α is a n-cell,

$$\overline{e_\alpha} = e_\alpha \cup \dot{e_\alpha} = e_\alpha \cup f_\alpha\left(S^{n-1}\right) \subseteq e_\alpha \cup X^{n-1}. \tag{3.14}$$

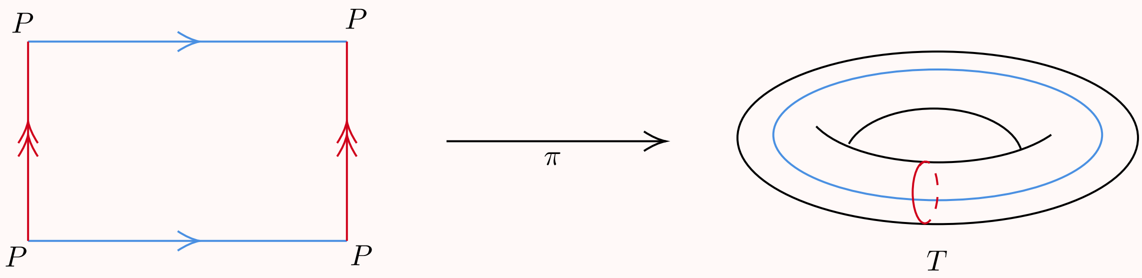
So $\overline{e_\alpha} \subseteq X^n$. If e_β is a k-cell for $k < n$, $\overline{e_\beta} \subseteq X^{n-1}$. Therefore, X^n is a subcomplex. For (c), a similar computation as 3.14 reveals that

$$\overline{e_i} \subseteq X^{n-1} \cup \left(\bigcup_{i\in I} e_i\right). \tag{3.15}$$

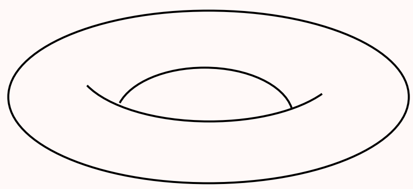
Therefore, $X^{n-1} \cup \left(\bigcup_{i\in I} e_i\right)$ is also a subcomplex. ∎

Example 3.1

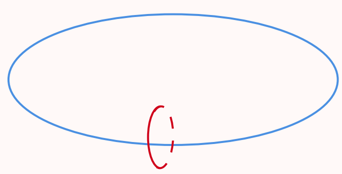
Consider the torus as a quotient space of a rectangle as usual (by identifying opposite sides of a rectangle).



We express T as a CW complex having a single 2-cell (the image under π of the interior of the rectangle), two 1-cells (the images of the 2 open edges of the rectangle under π), and one 0-cell (the image of the vertices of the rectangle under π). You should convince yourself that all the axioms in the definition of a CW complex are satisfied here.



2-skeleton of T

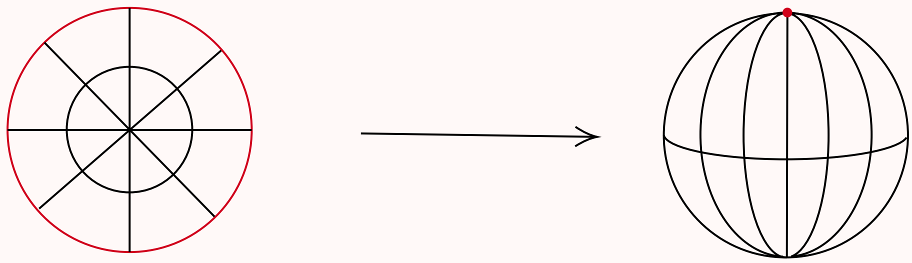


1-skeleton of T

0-skeleton of T

Example 3.2

The quotient space formed from B^n by collapsing $\operatorname{Bd} B^n$ to a point is homeomorphic to S^n. Hence, the Hausdorff topological space S^n can be expressed as a CW complex having one n-cell and a 0-cell, and no other cells at all.



§3.2 Adjunction Space

Definition 3.7 (Topological sum)**.** Let $\{X_\alpha\}_{\alpha\in J}$ be a family of topological spaces, not necessarily disjoint. Let E be the set that is the union of the disjoint topological spaces $E_\alpha = X_\alpha \times \{\alpha\}$. In other words,

$$E = \bigsqcup_{\alpha\in J} E_\alpha = \bigsqcup_{\alpha\in J} X_\alpha \times \{\alpha\}\,. \tag{3.16}$$

If we topologize E by declaring $U \subseteq E$ to be open if and only if $U \cap E_\alpha$ is open in E_α for each α, then E is called the **topological sum** of the topological spaces X_α.

One has a natural map $p : E \to \bigcup_\alpha X_\alpha$ which projects $X_\alpha \times \{\alpha\}$ onto X_α for each α. We now have the following important result.

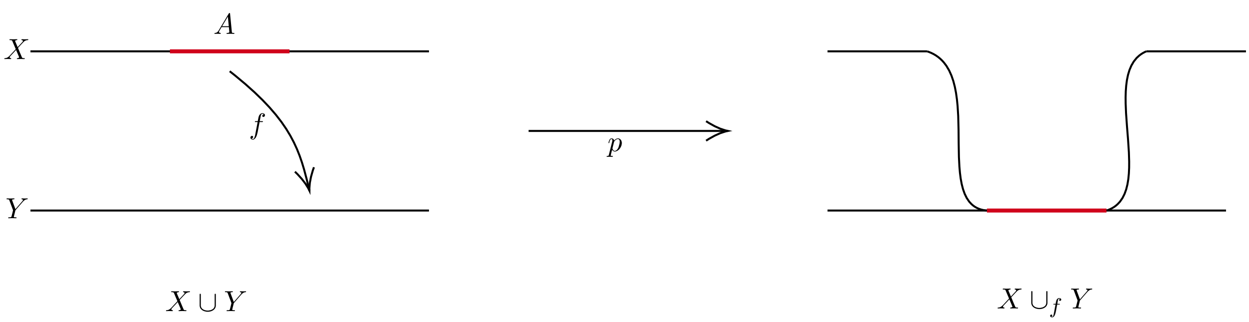
Lemma 3.5

Let X be a topological space which is the union of certain of its subspaces, i.e. $X = \bigcup_\alpha X_\alpha$. Let E be the topological sum of the subspaces X_α. Also, let $p : E \to \bigcup_\alpha X_\alpha$ be the natural projection. Then the topology of X is coherent with the subspaces if and only if p is a quotient map. In this situation, we often say that X is the **coherent union** of the spaces X_α.

Definition 3.8 (Adjunction space)**.** Let X and Y be disjoint topological spaces, and let A be a closed subspace of X. Let $f : A \to Y$ be a continuous map. We define a certain quotient space as follows: Topologize $X \cup Y$ as the topological sum, i.e. $U \subseteq X \cup Y$ is open if and only if both $U \cap X$ and $U \cap Y$ are open in X and Y, respectively. Form a quotient space by identifying each set

$$\{y\} \cup f^{-1}(y)\,,\ \ (\text{for } y \in Y)$$

to a point. That is, partition $X \cup Y$ into these sets, along with the singletons $\{x\}$ for $x \in X \setminus A$. We denote this quotient space by $X \cup_f Y$, and call it the **adjunction space** determined by f.



It is often useful to view a CW complex as a space built up from a collection of n-balls (possibly of different n) by forming appropriate quotient spaces.

Recall from point set topology that a topological space X is said to be **normal** if given any two disjoint closed sets E and F, there are open disjoint sets U and V such that $E \subseteq U$ and $F \subseteq V$.

Lemma 3.6

Let X be a space that is the countable union of certain closed subspaces X_n. Suppose the topology of X is coherent with those subspaces X_n. If each X is normal, so is X.

Theorem 3.7

If X and Y are normal, then so is the adjunction space $X \cup_f Y$.

Theorem 3.8

Suppose $(X, \mathcal{E})$ is a CW complex of dimension p. Then X is homeomorphic to an adjunction space formed from X^{p-1} and a topological sum $\bigsqcup_\alpha B^p_\alpha$ of p-balls B^p (here $B^p_\alpha = B^p \times \{\alpha\}$) by means of a continuous map $g : \bigsqcup_\alpha \operatorname{Bd} B^p_\alpha \to X^{p-1}$. It follows that X is normal.

Proof. Associated with each $e_\alpha \in \mathcal{E}$ of dimension p, there is a characteristic map $f_\alpha : B^p \to \overline{e_\alpha}$. Now, $B^p_\alpha = B^p \times \{\alpha\}$, and $\bigcup_\alpha B^p \times \{\alpha\} = \bigsqcup_\alpha B^p_\alpha$. Now, form the topological sum

$$E = X^{p-1} \cup \left(\bigsqcup_\alpha B^p_\alpha \right), \tag{3.17}$$

and define $\pi : E \to X$ by letting π equal inclusion on X^{p-1} and the composite

$$B^p_\alpha = B^p \times \{\alpha\} \to B^p \xrightarrow{f_\alpha} X \tag{3.18}$$

on B^p_α. We will now prove that π is a quotient map. This will prove that X is homeomorphic to the underlying quotient space $X^{p-1} \cup_g (\bigsqcup_\alpha B^p_\alpha)$, with g being the continuous map

$$g : \bigsqcup_\alpha \operatorname{Bd} B^p_\alpha \to X^{p-1}$$

induced from the characteristic maps f_α.

π is continuous on each of the disjoint components, so π is continuous. Furthermore, it is surjective. Indeed, for $x \in X$, x is either in X^{p-1} or in some p-cell e_α. In any case there is a pre-image of x, since f_α restricts to a homeomorphism of $\operatorname{Int} B^p$ with e_α.

Suppose $C \subseteq X$ and $\pi^{-1}(C)$ is closed in E. In order to show that π is a quotient map, we need to show that C is closed as well. X^{p-1} is a CW subcomplex of $(X, \mathcal{E})$ and hence X^{p-1} is a xlosed subspace of X. Therefore,

1. $\pi^{-1}(C) \cap X^{p-1}$ is closed in X^{p-1} in the subspace topology it inherits from E. But
$$\pi^{-1}(C) \cap X^{p-1} = \pi^{-1}\left(C \cap X^{p-1}\right) = C \cap X^{p-1}, \tag{3.19}$$
hence $C \cap X^{p-1}$ is closed in X^{p-1}. Since X^{p-1} is a CW complex in its own right, using the weak topology axiom, one obtains $C \cap \overline{e_\beta}$ is closed in $\overline{e_\beta}$ for $\dim e_\beta \leq p-1$.
2. Also, each B^p_α inherits subspace topology from $E = X^{p-1} \cup (\bigsqcup_\alpha B^p_\alpha)$. Since $\pi^{-1}(C)$ is closed in E, $\pi^{-1}(C) \cap B^p_\alpha$ is closed in B^p_α in subspace topology. Now, since each B^p_α is compact, and $\pi : E \to X$ is continuous,
$$\pi\left(\pi^{-1}(C) \cap B^p_\alpha\right) = C \cap f_\alpha\left(B^p_\alpha\right) = C \cap \overline{e_\alpha} \tag{3.20}$$
is compact (because closed subspace of compact set is compact, and so is continuous image of compact set). So we arrive at the fact that $C \cap \overline{e_\alpha}$ is compact. Since X is Hausdorff, $C \cap \overline{e_\alpha}$ is closed in X (compact subspace of Hausdorff is closed). $\overline{e_\alpha}$ is closed in X, $C \cap \overline{e_\alpha}$ is closed in $\overline{e_\alpha}$.

Therefore, we verified that $C \cap \overline{e_\alpha}$ is closed in $\overline{e_\alpha}$ with $\dim e_\alpha \leq p$. Since X is of dimension p, all the cells of $\mathcal{E}$ are of dimension at most p. Therefore, C is closed in X, proving that π is a quotient map.

We shall now prove that X is normal. We proceed inductively. X^0 is a discrete topological space, and hence normal. $\bigsqcup_\alpha B^1_\alpha$ is also normal. Therefore, the corresponding adjunction space $X^0 \cup_g \left(\bigsqcup_\alpha B^1_\alpha\right)$ (which is homeomorphic to X^1) is normal. In a similar manner, we can show that each X^i is normal. Therefore, $X^p = X$ is normal. ■

The converse of Theorem 3.8 can be stated as follows:

Theorem 3.9

Let $(Y, \mathcal{E})$ be a CW complex of dimension $p-1$. Let $\bigsqcup_\alpha B^p_\alpha$ be a topological sum of p-balls, and let $g : \bigsqcup_\alpha \operatorname{Bd} B^p_\alpha \to Y$ be a continuous map. Then the adjunction space

$$X = Y \cup_g \left(\bigsqcup_\alpha B^p_\alpha\right)$$

is the underlying topological space of a CW complex, and Y is its p-skeleton.

Sketch of proof. Use Theorem 3.7 to show that X is Hausdorff. Construct the quotient map

$$f : Y \cup \left(\bigsqcup_\alpha B^p_\alpha\right) \to X$$

by defining it as inclusion on Y, and by means of the given continuous map g on $\bigsqcup_\alpha \operatorname{Bd} B^p_\alpha$. f on $\operatorname{Int} B^p_\alpha$ is going to give the p cells $e_\alpha \in \mathcal{E}'$. This way form the p-cells in $\mathcal{E}'$. In particular, $e_\alpha = f\left(\operatorname{Int} B^p_\alpha\right)$. Some work needs to be done to show that e_α is a p-cell. The other cells in $\mathcal{E}'$ are the cells $e_\beta \in \mathcal{E}$ of dimension at most $p-1$. Now show that $(X, \mathcal{E}')$ thus constructed fulfills all 3 axioms of a CW complex. ■

Theorem 3.8 and Theorem 3.9 can be extended to construct infinite dimensional CW complexes. For that we need a lemma first.

Lemma 3.10

Let X be a set which is the union of topological space $\{X_\alpha\}$. If for each pair α, β of indices, the set $X_\alpha \cap X_\beta$ is closed in both X_α and X_β, and inherits the same subspace topology from each of them, then X has a topology coherent with the subspaces $\{X_\alpha\}$. Each X_α is closed in this topology.

We shall omit the proof of this lemma.

Theorem 3.11 (a) Let $(X, \mathcal{E})$ be a CW complex. Then X^p is a closed subspace of X^{p+1} for each p, and X is the coherent union of the spaces $X^0 \subseteq X^1 \subseteq X^2 \subseteq \cdots$. It follows that X is normal.

(b) Conversely, suppose $(X_p, \mathcal{E}_p)$ is a CW complex for each p, and X_p equals p-skeleton of X^{p+1} for each p. If X is the coherent union of the spaces X_p, then $(X, \mathcal{E})$ is a CW complex having X_p as its p-skeleton, where $\mathcal{E} = \bigcup_p \mathcal{E}_p$.

Proof. (a) By Lemma 3.3, both X^p and X^{p+1} are closed in X. Here, $X^p \subseteq X^{p+1} \subseteq X$. We prove that X^p is closed in X^{p+1}. It is equivalent to proving $X^{p+1} \setminus X^p$ is open in X^{p+1}.

$$X^{p+1} \setminus X^p = X^{p+1} \cap (X \setminus X^p), \tag{3.21}$$

and $X \setminus X^p$ is open in X. Therefore, $X^{p+1} \setminus X^p$ is open in X^{p+1} in subspace topology inherited from X. Hence, X^p is closed in X^{p+1}.

Now suppose $C \cap X^p$ is closed in X^p for each p. We need to prove that C is closed in X. Since $(X^p, \mathcal{E}_p)$ is a CW complex by its own right, by the weak topology axiom, $C \cap X^p \cap \overline{e_\alpha}$ is closed in $\overline{e_\alpha}$ for each $e_\alpha \in \mathcal{E}_p$. Since $\overline{e_\alpha} \subseteq X^p$, we have

$$C \cap X^p \cap \overline{e_\alpha} = C \cap \overline{e_\alpha}. \tag{3.22}$$

Therefore, $C \cap \overline{e_\alpha}$ is closed in $\overline{e_\alpha}$ for each $e_\alpha \in \mathcal{E}_p$. Since p is arbitrary, $C \cap \overline{e_\alpha}$ is closed in $\overline{e_\alpha}$ for each $e_\alpha \in \mathcal{E}$. Hence, C is closed in X.

Conversely, suppose C is closed in X. We know that $X^p \subseteq X$ is closed in X. Hence, $C \cap X^p$ is closed in X. Now,

$$X^p \setminus (C \cap X^p) = X^p \cap [X \setminus (C \cap X^p)]. \tag{3.23}$$

$X \setminus (C \cap X^p)$ is open in X. Therefore, $X^p \setminus (C \cap X^p)$ is open in X^p in the subspace topology it inherits from X. Therefore, $C \cap X^p$ is closed in X. We, therefore, conclude that C is closed in X if and only if $C \cap X^p$ is closed in X^p for each p. Therefore, X is the coherent union of the subspaces

$$X^0 \subseteq X^1 \subseteq X^2 \subseteq \cdots.$$

Normality of X follows by noting that $X = \bigcup_p X^p$, where $\{X^p\}_p$ is a countable collection of closed subspaces, and using Lemma 3.6.

(b) If $p < q$, then $X_p \cap X_q = X_p$ is a closed subspace of both X_p and X_q, since X_p is the p-skeleton of X_q. Therefore, by Lemma 3.10, there is a topology on X coherent with the subspaces $\{X_p\}_p$, and each X_p is closed in X. By Theorem 3.8, each X_p is normal. Using Lemma 3.6, X is normal as well (and in particular, Hausdorff). The closure-finiteness axiom follows trivially. Now we check the weak topology axiom.

Suppose $C \cap \overline{e_\alpha}$ is closed in $\overline{e_\alpha}$ for each cell e_α. Then $C \cap X_p$ is closed in X_p, since X_p is a CW complex. Then C is closed in X, because the topology of X is coherent with the spaces X_p.

Conversely, suppose C is closed in X. Then $C \cap X_p$ is closed in X_p for each p, because of the coherence. Since X_p is a CW complex, $C \cap X_p \cap \overline{e_\alpha}$ is closed in $\overline{e_\alpha}$ for each cell e_α with dimension at most p. But $C \cap X_p \cap \overline{e_\alpha} = C \cap \overline{e_\alpha}$. Therefore, $C \cap \overline{e_\alpha}$ is closed in $\overline{e_\alpha}$ for each cell e_α. This proves that X satisfies the weak topology axiom. Therefore, X is a CW complex.

■

§3.3 The Homology of CW Complexes

Let $(X, \mathcal{E})$ be a CW complex. Also, let $D_p(X) = H_p\left(X^p, X^{p-1}\right)$. Let $\partial_p : D_p(X) \to D_{p-1}(X)$ be defined to be the composite

$$\begin{tikzcd} H_p\left(X^p, X^{p-1}\right) \arrow[r, "(\partial_*)_p"] \arrow[rr, bend right, "(j_*)_{p-1} \circ (\partial_*)_p"'] & H_{p-1}\left(X^{p-1}\right) \arrow[r, "(j_*)_{p-1}"] & H_{p-1}\left(X^{p-1}, X^{p-2}\right) \end{tikzcd}$$

In other words,

$$\partial_p = (j_*)_{p-1} \circ (\partial_*)_p, \tag{3.24}$$

where $j : \left(X^{p-1}, \varnothing\right) \hookrightarrow \left(X^{p-1}, X^{p-2}\right)$ is the inclusion. One can verify that $\partial_{p-1} \circ \partial_p = 0$ by considering the long exact homology sequence of the pair $\left(X^{p-1}, X^{p-2}\right)$.

$$\cdots \longrightarrow H_{p-1}\left(X^{p-1}\right) \xrightarrow{(j_*)_{p-1}} H_{p-1}\left(X^{p-1}, X^{p-2}\right) \xrightarrow{(\partial_*)_{p-1}} H_{p-2}\left(X^{p-2}\right) \longrightarrow \cdots$$

Exactness of this sequence implies $(\partial_*)_{p-1} \circ (j_*)_{p-1} = 0$. Now,

$$\begin{aligned} \partial_{p-1} \circ \partial_p &= (j_*)_{p-2} \circ \left[(\partial_*)_{p-1} \circ (j_*)_{p-1}\right] \circ (\partial_*)_p \\ &= (j_*)_{p-2} \circ 0 \circ (\partial_*)_p = 0. \end{aligned} \tag{3.25}$$

The chain complex $\{D_p(X), \partial_p\}$ is called the **cellular chain complex** of $(X, \mathcal{E})$.

Lemma 3.12

Given a p-cell $e_\alpha \in \mathcal{E}$ of $(X, \mathcal{E})$, any characteristic map f_α associated with e_α,

$$f_\alpha : \left(B^p, S^{p-1}\right) \to \left(\overline{e_\alpha}, \dot{e_\alpha}\right)$$

induces an isomorphism in relative homology.

Proof. The result is trivial for $p = 0$. Let $p > 0$. The point $\mathbf{0}$ is the center of B^p. Let $\widehat{e_\alpha} = f_\alpha(\mathbf{0})$. One can form a continuous map between pairs of topological spaces given by the characteristic map $f_\alpha : B^p \to \overline{e_\alpha}$ associated with a given p-cell $e_\alpha \in \mathcal{E}$.

$$\left(B^p, S^{p-1}\right) \xrightarrow{f_\alpha} \left(\overline{e_\alpha}, \dot{e_\alpha}\right).$$

Here f_α is a quotient map, and nontrivial aspects of the quotient construction is happening at the boundary. Any open set containing $\operatorname{Bd} B^p$ is a saturated open set in B^p with respect to the quotient map $f_\alpha : B^p \to \overline{e_\alpha}$. In particular, $B^p \setminus \{\mathbf{0}\}$ is a saturated open set in B^p with respect to f_α. Hence, $f_\alpha\big|_{B^p\setminus\{\mathbf{0}\}}$ is a quotient map. We denote this restriction by f'_α.

$$f'_\alpha : B^p \setminus \{\mathbf{0}\} \to \overline{e_\alpha} \setminus \widehat{e_\alpha}$$

is a quotient map. Let $F : B^p \setminus \{\mathbf{0}\} \times I \to B^p \setminus \{\mathbf{0}\}$ be a deformation retract of $B^p \setminus \{\mathbf{0}\}$ onto S^{p-1}. Hence,

$$F(x,1) \in S^{p-1}\,,\ F(x,0) = x\ \forall\, x \in B^p \setminus \{\mathbf{0}\}\ \text{ and } F(a,t) = a\ \forall\, a \in S^{p-1},\ t \in I. \tag{3.26}$$

$$\begin{tikzcd}
(B^p \setminus \{\mathbf{0}\}) \times I \arrow[r, "F"] \arrow[d, "f'_\alpha \times \mathrm{id}_I"'] & B^p \setminus \{\mathbf{0}\} \arrow[d, "f'_\alpha"] \\
(\overline{e_\alpha} \setminus \widehat{e_\alpha}) \times I \arrow[r, "G"] & \overline{e_\alpha} \setminus \widehat{e_\alpha}
\end{tikzcd}$$

Here, G is the induced deformation retract from the quotient map f'_α.

$$\begin{tikzcd}
(B^p \setminus \{\mathbf{0}\}) \times I \arrow[d, "f'_\alpha \times \mathrm{id}_I"'] \arrow[dr, "f'_\alpha \circ F"] & \\
(\overline{e_\alpha} \setminus \widehat{e_\alpha}) \times I \arrow[r, "G"] & \overline{e_\alpha} \setminus \widehat{e_\alpha}
\end{tikzcd}$$

Now, $f'_\alpha \times \mathrm{id}_I$ is a quotient map, and $f'_\alpha \circ F$ is continuous. Furthermore, for each $(y,t) \in (\overline{e_\alpha} \setminus \widehat{e_\alpha}) \times I$, $f'_\alpha \circ F$ is constant on $\left(f'_\alpha \times \mathrm{id}_I\right)^{-1}(\{(y,t)\})$. Indeed, if $y \in e_\alpha$, then there is exactly one $x \in \operatorname{Int} B^p$ such that $f'_\alpha(x) = y$, so $f'_\alpha \circ F$ is constant on $\left(f'_\alpha \times \mathrm{id}_I\right)^{-1}(\{(y,t)\})$. Otherwise, if $y \in \dot{e}_\alpha$, let $a \in {f'_\alpha}^{-1}(\{y\})$. Then $a \in S^{p-1}$. The point (a,t) gets mapped to y under $f'_\alpha \circ F$. So $f'_\alpha \circ F$ is constant on $\left(f'_\alpha \times \mathrm{id}_I\right)^{-1}(\{(y,t)\})$. Therefore, there exists a unique continuous map $G : (\overline{e_\alpha} \setminus \widehat{e_\alpha}) \times I \to \overline{e_\alpha} \setminus \widehat{e_\alpha}$ such that the diagram above commutes.

Now, we want to show that G is a deformation retraction of $\overline{e}_\alpha \setminus \widehat{e}_\alpha$ onto $\dot{e}_\alpha$. For any $y \in \overline{e}_\alpha \setminus \widehat{e}_\alpha$, y is the preimage of some $x \in B^p \setminus \{\mathbf{0}\}$ under f'_α. Then $f'_\alpha \circ F$ maps $(x,1)$ to $f'_\alpha(z) \in \dot{e}_\alpha$ for some $z \in S^{p-1}$. Therefore, $G(y,1) \in \dot{e}_\alpha$. Furthermore, $f'_\alpha \circ F$ maps $(x,0)$ to $f'_\alpha(x) = y$. Hence, $G(y,0) = y$. Also, for $a \in \dot{e}_\alpha$, there exists some $b \in S^{p-1}$ such that $f'_\alpha(b) = a$. Then $f'_\alpha \circ F$ maps (b,t) to $f'_\alpha(b) = a$, proving that $G(a,t) = a$. Therefore, G is a deformation retract of $\overline{e}_\alpha \setminus \widehat{e}_\alpha$ onto $\dot{e}_\alpha$.

Now, consider the inclusion maps

$$i : \left(B^p, S^{p-1}\right) \hookrightarrow \left(B^p, B^p \setminus \{\mathbf{0}\}\right) \quad\text{and}\quad j : \left(\overline{e}_\alpha, \dot{e}_\alpha\right) \hookrightarrow \left(\overline{e}_\alpha, \overline{e}_\alpha \setminus \widehat{e}_\alpha\right).$$

By Theorem 2.10,

$$(i_*)_q : H_q\left(B^p, S^{p-1}\right) \to H_q\left(B^p, B^p \setminus \{\mathbf{0}\}\right) \quad\text{and}\quad (j_*)_q : H_q\left(\overline{e}_\alpha, \dot{e}_\alpha\right) \to H_q\left(\overline{e}_\alpha, \overline{e}_\alpha \setminus \widehat{e}_\alpha\right)$$

are isomorphisms in relative homology. Now, consider the following diagram:

$$\begin{tikzcd}
\left(B^p, S^{p-1}\right) \arrow[r, "i"] \arrow[d, "f_\alpha"'] & \left(B^p, B^p \setminus \{\mathbf{0}\}\right) \arrow[d, "\widetilde{f}_\alpha"'] & \left(\operatorname{Int} B^p, \operatorname{Int} B^p \setminus \{\mathbf{0}\}\right) \arrow[l, "k"'] \arrow[d, "g=\widetilde{f}_\alpha\big|_{(\operatorname{Int} B^p, \operatorname{Int} B^p\setminus\{\mathbf{0}\})}"] \\
\left(\overline{e}_\alpha, \dot{e}_\alpha\right) \arrow[r, "j"] & \left(\overline{e}_\alpha, \overline{e}_\alpha \setminus \widehat{e}_\alpha\right) & \left(e_\alpha, e_\alpha \setminus \widehat{e}_\alpha\right) \arrow[l, "l"']
\end{tikzcd}$$

$k : \left(B^p \setminus S^{p-1}, \left(B^p \setminus S^{p-1}\right) \setminus \{\mathbf{0}\}\right) \hookrightarrow \left(B^p, B^p \setminus \{\mathbf{0}\}\right)$ is the inclusion obtained from the pair $\left(B^p, B^p \setminus \{\mathbf{0}\}\right)$ by excising away S^{p-1}. Then by Theorem 2.18, $\left(k_*\right)_q$ is an isomorphism in singular homology. Similarly, $\left(l_*\right)_q : H_q\left(e_\alpha, e_\alpha \setminus \widehat{e}_\alpha\right) \to H_q\left(\overline{e}_\alpha, \overline{e}_\alpha \setminus \widehat{e}_\alpha\right)$ is also an isomorphism.

Furthermore, g is a homeomorphism (since f_α restricted to $\operatorname{Int} B^p$ is a homeomorphism). Hence, $\left(g_*\right)_q : H_q\left(\operatorname{Int} B^p, \operatorname{Int} B^p \setminus \{\mathbf{0}\}\right) \to H_q\left(e_\alpha, e_\alpha \setminus \widehat{e}_\alpha\right)$ is an isomorphism. Therefore,

$$\begin{aligned} H_q\left(B^p, S^{p-1}\right) &\cong H_q\left(B^p, B^p \setminus \{\mathbf{0}\}\right) \cong H_q\left(\operatorname{Int} B^p, \operatorname{Int} B^p \setminus \{\mathbf{0}\}\right) \\ &\cong H_q\left(e_\alpha, e_\alpha \setminus \widehat{e}_\alpha\right) \cong H_q\left(\overline{e}_\alpha, \overline{e}_\alpha \setminus \widehat{e}_\alpha\right) \cong H_q\left(\overline{e}_\alpha, \dot{e}_\alpha\right). \end{aligned} \tag{3.27}$$

Therefore, $H_q\left(B^p, S^{p-1}\right) \cong H_q\left(\overline{e}_\alpha, \dot{e}_\alpha\right)$. ■

Lemma 3.13

Let the map $f : X^{p-1} \cup \left(\bigsqcup_\alpha B^p_\alpha\right) \to X^p$ expresses X^p as the adjunction space obtained from X^{p-1} and a topological sum of p-balls $\bigsqcup_\alpha B^p_\alpha$ via a continuous map $g : \bigsqcup_\alpha \operatorname{Bd} B^p_\alpha \to X^{p-1}$, where $B^p_\alpha = B^p \times \{\alpha\}$. Then f induces an isomorphism in homology:

$$H_q\left(\bigsqcup_\alpha B^p_\alpha, \bigsqcup_\alpha \operatorname{Bd} B^p_\alpha\right) \cong H_q\left(X^p, X^{p-1}\right).$$

Proof. Let

$$f' = f\Big|_{X^{p-1}\cup\left(\bigsqcup_\alpha\left(B^p_\alpha\setminus\{\mathbf{0}_\alpha\}\right)\right)} : X^{p-1} \cup \left(\bigsqcup_\alpha \left(B^p_\alpha \setminus \{\mathbf{0}_\alpha\}\right)\right) \to X^p \setminus \bigcup_\alpha f\left(\mathbf{0}_\alpha\right),$$

where $\mathbf{0}_\alpha \in B^p_\alpha$ is the center of B^p_α. Observe that f' being the restriction of the quotient map f to the saturated open set $X^{p-1} \cup \left(\bigsqcup_\alpha \left(B^p_\alpha \setminus \{\mathbf{0}_\alpha\}\right)\right)$ is also a quotient map.

Suppose that $F : X^{p-1} \cup \left(\bigsqcup_\alpha \left(B^p_\alpha \setminus \{\mathbf{0}_\alpha\}\right)\right) \times I \to X^{p-1} \cup \left(\bigsqcup_\alpha \left(B^p_\alpha \setminus \{\mathbf{0}_\alpha\}\right)\right)$ is a deformation retraction of $X^{p-1} \cup \left(\bigsqcup_\alpha \left(B^p_\alpha \setminus \{\mathbf{0}_\alpha\}\right)\right)$ onto $X^{p-1} \cup \left(\bigsqcup_\alpha S^{p-1}_\alpha\right)$. Then there is a deformation retraction

$$G : X^p \setminus \bigcup_\alpha f\left(\mathbf{0}_\alpha\right) \times I \to X^p \setminus \bigcup_\alpha f\left(\mathbf{0}_\alpha\right)$$

of $X^p \setminus \bigcup_\alpha f\left(\mathbf{0}_\alpha\right)$ onto X^{p-1}.

$$\begin{tikzcd}
X^{p-1} \cup \left(\bigsqcup_\alpha \left(B^p_\alpha \setminus \{\mathbf{0}_\alpha\}\right)\right) \times I \arrow[d, "f' \times \mathrm{id}_I"'] \arrow[dr, "f' \circ F"] & \\
X^p \setminus \bigcup_\alpha f\left(\mathbf{0}_\alpha\right) \times I \arrow[r, "\exists! G"] & X^p \setminus \bigcup_\alpha f\left(\mathbf{0}_\alpha\right)
\end{tikzcd}$$

Similarly as in the previous lemma, one can show that G is a deformation retraction. Now, one has the following diagram:

$$\begin{tikzcd}
\left(\bigsqcup_\alpha B^p_\alpha, \bigsqcup_\alpha S^{p-1}_\alpha\right) \arrow[r, "i"] \arrow[d, "f"'] & \left(\bigsqcup_\alpha B^p_\alpha, \bigsqcup_\alpha \left(B^p_\alpha \setminus \{\mathbf{0}_\alpha\}\right)\right) \arrow[d, "\widetilde{f}"'] & \left(\bigsqcup_\alpha \operatorname{Int} B^p_\alpha, \bigsqcup_\alpha \left(\operatorname{Int} B^p_\alpha \setminus \{\mathbf{0}_\alpha\}\right)\right) \arrow[l, "k"'] \arrow[d, "g"] \\
\left(X^p, X^{p-1}\right) \arrow[r, "j"] & \left(X^p, X^p \setminus \bigcup_\alpha \widehat{e}_\alpha\right) & \left(\bigcup_\alpha e_\alpha, \bigcup_\alpha \left(e_\alpha \setminus \widehat{e}_\alpha\right)\right) \arrow[l, "l"']
\end{tikzcd}$$

By Theorem 2.10, $(i_*)_q$ and $(j_*)_q$ are isomorphisms in relative homology. Now, excising away S_α^{p-1} and $\dot{e}_\alpha$, one has the inclusions given by k and l, respectively. Therefore, by Theorem 2.18, $(k_*)_q$ and $(l_*)_q$ are isomorphisms in singular homology. Since g is a homeomorphism, $(g_*)_q$ is an isomorphism. Therefore, one concludes,

$$H_q\left(\bigsqcup_\alpha B_\alpha^p, \bigsqcup_\alpha S_\alpha^{p-1}\right) \cong H_q\left(X^p, X^{p-1}\right). \tag{3.28}$$

■

We shall now show that homology group of topological sum is isomorphic to the direct sum of homology groups of each components. In other words,

$$H_p\left(\bigsqcup_\alpha X_\alpha, \bigsqcup_\alpha Y_\alpha\right) \cong \bigoplus_\alpha H_p\left(X_\alpha, Y_\alpha\right). \tag{3.29}$$

In fact, we shall prove that this holds at chain level.

Theorem 3.14

Let $\{X_\alpha\}_{\alpha\in I}$ be a collection of topological spaces, and Y_α is a subspace of X_α, for each $\alpha \in I$. Let $X = \bigsqcup_{\alpha\in I} X_\alpha$ and $Y = \bigsqcup_{\alpha\in I} Y_\alpha$ denote the topological sum. Then

$$H_p\left(X, Y\right) \cong \bigoplus_{\alpha\in I} H_p\left(X_\alpha, Y_\alpha\right). \tag{3.30}$$

Proof. We shall first show that the chain group of topological sum of topological spaces is direct sum of chain groups of the topological spaces.

Claim — Let X and Y be as above. Then

$$S_p\left(X\right) = \bigoplus_{\alpha\in I} S_p\left(X_\alpha\right) \text{ and } S_p\left(Y\right) = \bigoplus_{\alpha\in I} S_p\left(Y_\alpha\right). \tag{3.31}$$

Proof. Let $c \in S_p(X)$. Then $c = \sum_i n_i f_i$, where $f_i : \Delta_p \to X$ is continuous. Since Δ_p is path connected, so is $f_i(\Delta_p)$. Therefore, $f_i(\Delta_p)$ lies in a path component of X.

Now, each X_α is an open subset of X, because $X_\alpha \cap X_\beta$ is either X_α (if $\alpha = \beta$), or $\varnothing$ (if $\alpha \neq \beta$). In any case, $X_\alpha \cap X_\beta$ is open in X_β, so X_α is open in X. Furthermore, as the X_α's are disjoint, X is not connected, let alone path connected.

If C is a connected component of X, each $C \cap X_\alpha$ is open in the subspace topology of C, they are all disjoint, and their union is C. Therefore, one of them is C and the other are empty. In other words, $C \subseteq X_\beta$ for some $\beta \in I$. Hence, $f_i(\Delta_p) \subseteq X_\alpha$ for some $\alpha \in I$. This means $f_i \in S_p(X_\alpha)$. So every element of $S_p(X)$ can be written as a sum of finitely elements from the $S_p(X_\alpha)$'s.

Since the X_α's are disjoint, so are their chain groups $S_p(X_\alpha)$'s. Therefore, every element of $S_p(X)$ can be written **uniquely** as a sum of finitely elements from the $S_p(X_\alpha)$'s. In other words,

$$S_p\left(X\right) = \bigoplus_{\alpha\in I} S_p\left(X_\alpha\right).$$

Similarly,

$$S_p\left(Y\right) = \bigoplus_{\alpha\in I} S_p\left(Y_\alpha\right).$$

□

Therefore,

$$S_p\left(X, Y\right) = \frac{S_p\left(X\right)}{S_p\left(Y\right)} = \frac{\bigoplus_{\alpha\in I} S_p\left(X_\alpha\right)}{\bigoplus_{\alpha\in I} S_p\left(Y_\alpha\right)} \cong \bigoplus_{\alpha\in I} \frac{S_p\left(X_\alpha\right)}{S_p\left(Y_\alpha\right)} = \bigoplus_{\alpha\in I} S_p\left(X_\alpha, Y_\alpha\right). \tag{3.32}$$

Since the isomorphism holds at the chain level, the isomorphism at the homology level follows. ■

Theorem 3.15

The group $H_i\left(X^p, X^{p-1}\right)$ vanishes for $i \neq p$, and is free abelian for $i = p$. If γ generates $H_p\left(B^p, S^{p-1}\right)$, then the elements $\left((f_\alpha)_*\right)_p \gamma$ form a basis for $H_p\left(X^p, X^{p-1}\right)$, where f_α ranges over the set of characteristic maps for the p-cells of X.

Proof. By Lemma 3.13 and Theorem 3.14,

$$H_i\left(X^p, X^{p-1}\right) \cong H_i\left(\bigsqcup_\alpha B^p_\alpha, \bigsqcup_\alpha S^{p-1}_\alpha\right) \cong \bigoplus_\alpha H_i\left(B^p_\alpha, S^{p-1}_\alpha\right). \tag{3.33}$$

Since $H_i\left(B^p_\alpha, S^{p-1}_\alpha\right) = 0$ for $i \neq p$, we have $H_i\left(X^p, X^{p-1}\right) = 0$ for $i \neq p$. Also, we have $H_p\left(B^p_\alpha, S^{p-1}_\alpha\right) \cong \mathbb{Z}$, so $H_p\left(X^p, X^{p-1}\right)$ is isomorphic to α-fold direct sum of $\mathbb{Z}$. Therefore, $H_p\left(X^p, X^{p-1}\right)$ is free abelian, and the elements $\left((f_\alpha)_*\right)_p \gamma$ form a basis for $H_p\left(X^p, X^{p-1}\right)$. ■

Homology sequence of a triple

Given a triple $B \subseteq A \subseteq X$ of topological spaces, one has the following inclusions

$$(A, B) \overset{i}{\hookrightarrow} (X, B) \overset{l}{\hookrightarrow} (X, A),$$

and a short exact sequence induced by these inclusions:

$$0 \longrightarrow S_p(A, B) \xrightarrow{(i_\#)_p} S_p(X, B) \xrightarrow{(l_\#)_p} S_p(X, A) \longrightarrow 0$$

By Zig-Zag lemma, one obtains the following long exact homology sequence

$$\cdots \longrightarrow H_p(A, B) \xrightarrow{(i_*)_p} H_p(X, B) \xrightarrow{(l_*)_p} H_p(X, A) \xrightarrow{(\widetilde{\partial}_*)_p} H_{p-1}(A, B) \longrightarrow \cdots.$$

This homology sequence is called the long exact homology sequence of a triple $B \subseteq A \subseteq X$.

Now, consider the case $(X, A, B) = \left(X^p, X^{p-1}, X^{p-2}\right)$. Then the boundary homorphism $\left(\widetilde{\partial}_*\right)_p$ in the above case coincides with the boundary homomorphism ∂_p of the cellular chain complex $\{D_p\left(X\right), \partial_p\}$. This follows from the commutativity of the following diagram:

$$\begin{tikzcd} H_p(X^p, X^{p-1}) \arrow[r, "(\partial_*)_p"] \arrow[rr, bend right, "(\widetilde{\partial}_*)_p"'] & H_{p-1}(X^{p-1}, \varnothing) \arrow[r, "(j_*)_{p-1}"] & H_{p-1}(X^{p-1}, X^{p-2}) \end{tikzcd}$$

The top line of the diagram comes from the long exact homology sequence of the pair $\left(X^p, X^{p-1}\right)$. Commutativity of the diagram above leads to

$$\left(\widetilde{\partial}_*\right)_p = (j_*)_{p-1} \circ (\partial_*)_p, \tag{3.34}$$

which is the same as the boundary homomorphism ∂_p of the cellular chain complex $\{D_p\left(X\right), \partial_p\}$. Using this fact, we will now prove that the cellular chain complex $\{D_p\left(X\right), \partial_p\}$ of the CW complex $(X, \mathcal{E})$ can be used to compute the homology of X.

For later purposes, we are going to prove this result in a bit more general form for filtered spaces. We shall assume that we have a space X that can be written as the union of a sequence of its subspaces:

$$X_0 \subset X_1 \subset X_2 \subset \cdots.$$

Then form the chain complex whose p-dimensional chain group is $H_p\left(X_p, X_{p-1}\right)$ and whose boundary operatoris the boundary homomorphism in the exact sequence of the triple (X_p, X_{p-1}, X_{p-2}). We shall show that under suitable hypotheses (which are satisfied in the case of a CW complex), this chain complex gives the homology of X.

Definition 3.9. If X is topological space, a filtration of X is a sequence $X_0 \subset X_1 \subset X_2 \subset \cdots$ of subspace of X whose union is X. A topological space together with a filtration of X is called a **filtered space**. If X and Y are filtered spaces, a continuous map $f : X \to Y$ is said to be **filtration preserving** if $f(X_p) \subseteq Y_p$, for all p.

Theorem 3.16

Let X be filtered by the subspaces $X_0 \subset X_1 \subset X_2 \subset \cdots$. Let $X_i = \varnothing$ for $i < 0$. Assume that $H_i(X_p, X_{p-1}) = 0$ for $i \neq p$. Suppose also that given any compact set $C \subseteq X$, there is an n such that $C \subseteq X_n$. Let $\mathcal{D}(X)$ be the chain complex defined by setting $D_p(X) = H_p(X_p, X_{p-1})$ and letting the boundary operator be the boundary homomorphism in the exact sequence of the triple (X_p, X_{p-1}, X_{p-2}). Then there is an isomorphism

$$\lambda : H_p(\mathcal{D}(X)) \to H_p(X).$$

It is natural with respect to the homomorphisms induced by filtration preserving continuous maps.

Proof. ***Step 1.*** We show that the homomorphism $(i_*)_p : H_p(X_{p+1}) \to H_p(X)$ induced by inclusion is an isomorphism. For this purpose, one first notes that

$$H_p(X_{p+1}) \to H_p(X_{p+2}) \to H_p(X_{p+3}) \to \cdots \tag{3.35}$$

induced by inclusions are isomorphisms. From the long exact sequence of the pair (X_{p+i+1}, X_{p+i}), we get the following exact sequence:

$$H_{p+1}(X_{p+i+1}, X_{p+i}) \longrightarrow H_p(X_{p+i}) \longrightarrow H_p(X_{p+i+1}) \longrightarrow H_p(X_{p+i+1}, X_{p+i})$$

Now that both end groups $H_{p+1}(X_{p+i+1}, X_{p+i})$ and $H_p(X_{p+i+1}, X_{p+i})$ vanish for $i \geq 1$, [since $H_q(X_p, X_{p-1}) = 0$ for $q \neq p$] so $H_p(X_{p+i}) \to H_p(X_{p+i+1})$ is an isomorphism for all $i \geq 1$.

Now, get back to the homomorphism $(i_*)_p : H_p(X_{p+1}) \to H_p(X)$. We first show that $(i_*)_p$ is surjective using the **compact support axiom**. Let $\beta \in H_p(X)$. Now choose a compact set $C \subseteq X$ such that β is in the image of the homomorphism $H_p(C) \to H_p(X)$ induced by the inclusion $C \hookrightarrow X$. Now, by hypothesis, $C \subseteq X_{p+k}$ for some $k \geq 1$ (The hypothesis only guarantees that $C \subseteq X_j$ for some j. But since $X_j \subseteq X_n$ for $n \geq j$, we can assume WLOG that $j > p + 1$.). Hence, β lies in the image of the homomorphism $H_p(X_{p+k}) \to H_p(X)$ induced by the inclusion $X_{p+k} \hookrightarrow X$. In other words, β is the image of an element in $H_p(X_{p+k})$ in the following diagram:

$$H_p(X_{p+1}) \to H_p(X_{p+k}) \to H_p(X).$$

Since $H_p(X_{p+1}) \to H_p(X_{p+k})$ is an isomorphism, β is the image of an element of $H_p(X_{p+1})$, establishing the surjectivity of $(i_*)_p : H_p(X_{p+1}) \to H_p(X)$.

Now let us show that $\operatorname{Ker}(i_*)_p = 0$. Suppose $(i_*)_p \beta = 0$ for some $\beta \in H_p(X_{p+1})$. One can choose compact $C' \subseteq X_{p+1}$ with $\beta \in H_p(C')$ so that

$$\left(\left(i|_{C'}\right)_*\right)_p \beta = 0 \in H_p(X).$$

By Theorem 2.5, there exists a compact set C with $C' \subseteq C \subseteq X$ and $l : C' \hookrightarrow C$ such that $(l_*)_p \beta = 0 \in H_p(C)$. But again, $C \subseteq X_{p+k}$ for some $k \geq 1$. This means that β lies in the kernel of $H_p(X_{p+1}) \to H_p(X_{p+k})$ induced by the inclusion map $X_{p+1} \hookrightarrow X_{p+k}$. But since $H_p(X_{p+1}) \to H_p(X_{p+k})$ is an isomorphism, $\beta = 0$. Hence, $\operatorname{Ker}(i_*)_p = 0$, completing the proof that $(i_*)_p$ is an isomorphism.

Step 2. We now show that the homomorphism

$$(j_*)_p : H_p(X_{p+1}) \to H_p(X_{p+1}, X_{p-2})$$

induced by inclusion is an isomorphism. This result will follow once one shows that the homomorphisms

$$H_p\left(X_{p+1}, \varnothing\right) \to H_p\left(X_{p+1}, X_0\right) \to H_p\left(X_{p+1}, X_1\right) \to \cdots \to H_p\left(X_{p+1}, X_{p-2}\right)$$

induced by inclusions are all isomorphisms. To prove this, consider the long exact sequence of the triple (X_{p+1}, X_i, X_{i-1}).

$$H_p(X_i, X_{i-1}) \longrightarrow H_p(X_{p+1}, X_{i-1}) \longrightarrow H_p(X_{p+1}, X_i) \longrightarrow H_{p-1}(X_i, X_{i-1})$$

If $i \leq p-2$, both end groups above vanish by hypothesis. Hence, the middle homomorphism $H_p(X_{p+1}, X_{i-1}) \to H_p(X_{p+1}, X_i)$ is an isomorphism. By plugging in $i = 0, 1, \ldots, p-2$, one obtains the following sequence of isomorphisms of homology groups:

$$H_p\left(X_{p+1}, \varnothing\right) \to H_p\left(X_{p+1}, X_0\right) \to H_p\left(X_{p+1}, X_1\right) \to \cdots \to H_p\left(X_{p+1}, X_{p-2}\right).$$

Step 3. We now prove the theorem. One has a quadruple: $X_{p-2} \subseteq X_{p-1} \subseteq X_p \subseteq X_{p+1}$, and 4 exact sequences of triples arranged in "overlapping sine curves" as follows:

$$\begin{tikzcd}
0 = H_p\left(X_{p-1}, X_{p-2}\right) \arrow[rr, bend left, "\alpha"] \arrow[dr] & & H_p\left(X_{p+1}, X_{p-2}\right) \arrow[rr, bend left, "\beta"] \arrow[dr] & & H_p\left(X_{p+1}, X_p\right) = 0 \\
& H_p\left(X_p, X_{p-2}\right) \arrow[ur, "(l_*)_p"] \arrow[dr, "(k_*)_p"'] & & H_p\left(X_{p+1}, X_{p-1}\right) \arrow[ur] \arrow[dr] & \\
H_{p+1}\left(X_{p+1}, X_p\right) \arrow[ur, "(\partial''_*)_{p+1}"] \arrow[rr, bend right, "(\partial_*)_{p+1}"'] & & H_p\left(X_p, X_{p-1}\right) \arrow[ur] \arrow[rr, bend right, "(\partial'_*)_p"'] & & H_{p-1}\left(X_{p-1}, X_{p-2}\right).
\end{tikzcd}$$

The homomorphisms of the exact sequences are colored with the same color in the above commutative diagram. It's easy to check that the diagram commutes, because the homomorphisms are either induced by inclusions, or induced by boundary operators, or a composition of them. This is an exact braid, so there exists an isomorphism

$$\Lambda : \frac{\operatorname{Ker}\left(\partial'_*\right)p}{\operatorname{im}\left(\partial_*\right)_{p+1}} \to \frac{\operatorname{Ker}\beta}{\operatorname{im}\alpha}. \tag{3.36}$$

$\frac{\operatorname{Ker}(\partial'_*)p}{\operatorname{im}(\partial_*)_{p+1}}$ is precisely $H_p\left(\mathcal{D}\left(X\right)\right)$. Since β is the zero map, its kernel is the whole $H_p\left(X_{p+1}, X_{p-2}\right)$. Furthermore, the domain of α is trivial, so $\operatorname{im}\alpha = 0$. Therefore,

$$H_p\left(\mathcal{D}\left(X\right)\right) \cong \frac{\operatorname{Ker}\beta}{\operatorname{im}\alpha} \cong H_p\left(X_{p+1}, X_{p-2}\right) \cong H_p\left(X_{p+1}\right) \cong H_p\left(X\right). \tag{3.37}$$

■

§3.4 The Homology of Triangulable CW Complexes

We now prove a version of Theorem 3.16 when the filtered space X is triangulable.

Theorem 3.17

Let X be filtered by the subspaces $X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots$. Suppose that X is the underlying space of a simplicial complex K, and each subspace X_p is the underlying space of a subcomplex of K of dimension at most p. Let H_i denote simplicial homology. Suppose $H_i\left(X_p, X_{p-1}\right) = 0$ for $i \neq p$.

Then $H_p(X_p, X_{p-1})$ is a subgroup of $C_p(K)$ consisting of all p-chains of K carried by X_p whose boundaries are carried by X_{p-1}. Furthermore, the isomorphism λ of Theorem 3.16 is induced by inclusion.

Proof. Any compact set in X lies in a finite subcomplex of K, so that it lies in X_i for some i. Therefore, the hypothesis of Theorem 3.16 are satisfied. Observe that

$$\frac{C_{p+1}(X_p)}{C_{p+1}(X_{p-1})} \xrightarrow{\widetilde{\partial}_{p+1}} \frac{C_p(X_p)}{C_p(X_{p-1})} \xrightarrow{\widetilde{\partial}_p} \frac{C_{p-1}(X_p)}{C_{p-1}(X_{p-1})}$$

$$\frac{\operatorname{Ker}\widetilde{\partial}_p}{\operatorname{im}\widetilde{\partial}_{p+1}} = H_p(X_p, X_{p-1}).$$

Since there are no $(p+1)$-chains either carried by X_p or X_{p-1}, both groups $C_{p+1}(X_p)$ and $C_{p+1}(X_{p-1})$ are trivial so that $\operatorname{im}\widetilde{\partial}_{p+1} = 0$ and $H_p(X_p, X_{p-1}) = \operatorname{Ker}\widetilde{\partial}_p$.

Also, there are no p-chains carried by X_{p-1} so that the group $C_p(X_{p-1})$ is trivial. Therefore, $\widetilde{\partial}_p$ is

$$C_p(X_p) \xrightarrow{\widetilde{\partial}_p} \frac{C_{p-1}(X_p)}{C_{p-1}(X_{p-1})}.$$

So $\operatorname{Ker}\widetilde{\partial}_p$ consists of simplicial p-chains in $C_p(X_p)$ whose boundaries are in $C_{p-1}(X_{p-1})$. In other words, boundaries are $(p-1)$-chains carried by X_{p-1}.

We must now check that the isomorphism λ of Theorem 3.16 is induced by inclusion. From the braid diagram in the step 3 of the proof, one extracts the following commutative diagram:

$$\begin{tikzcd}
H_p(X_p, X_{p-2}) \arrow[r, "(l_*)_p"] \arrow[d, "(k_*)_p"', "\cong"] & H_p(X_{p+1}, X_{p-2}) \arrow[r, "\cong", "(j_*)_p"'] & H_p(X_{p+1}) \arrow[r, "\cong", "(i_*)_p"'] & H_p(X) \\
\operatorname{Ker}(\partial'_*)_p & & &
\end{tikzcd}$$

Here $(k_*)_p$ is an injective homomorphism, and $(l_*)_p$ is a surjective homomorphism. The diagram above shows that

$$\phi := (i_*)_p \circ (j_*)_p \circ (l_*)_p \circ (k_*)_p^{-1} : \operatorname{Ker}\left(\partial'_*\right)_p \to H_p(X)$$

is a surjective group homomorphism induced by inclusion. Therefore, by the first isomorphism theorem,

$$\frac{\operatorname{Ker}(\partial'_*)_p}{\operatorname{Ker}\phi} \xrightarrow{\lambda} [\cong] H_p(X).$$

Now, $\operatorname{Ker}\phi = \operatorname{Ker}\left[(l_*)_p \circ (k_*)_p^{-1}\right]$. $(k_*)_p^{-1}$ is an isomorphism, so

$$\operatorname{Ker}\left[(l_*)_p \circ (k_*)_p^{-1}\right] = (k_*)_p\left(\operatorname{Ker}(l_*)_p\right) = (k_*)_p\left(\operatorname{im}\left(\partial''_*\right)_{p+1}\right) = \operatorname{im}(\partial_*)_{p+1}. \tag{3.38}$$

Therefore, λ is indeed the isomorphism of Theorem 3.16 from $\frac{\operatorname{Ker}(\partial'_*)_p}{\operatorname{im}(\partial_*)_{p+1}}$ to $H_p(X)$. For $a \in \operatorname{Ker}(\partial'_*)_p$,

$$\lambda\left(a + \operatorname{im}(\partial_*)_{p+1}\right) = \phi(a). \tag{3.39}$$

Since ϕ is induced by inclusion, so is λ. ■

Homology of torus and klein bottle using their CW complex structure

For each p-cell e_α of the CW complex $(X, \mathcal{E})$, the group $H_p(\overline{e}_\alpha, \dot{e}_\alpha)$ is infinite cyclic. So it has two choices for a generator, namely γ and γ^{-1}. These two generators will be called the two **orientations** of e_α. An oriented p-cell of X is a p-cell e_α together with an orientation of e_α.

The cellular chain group $D_p(X) = H_p\left(X^p, X^{p-1}\right)$ is a free abelian group. One obtains a basis for it by orienting each p cell e_α of X (say γ if not γ^{-1}) and passing it to the corresponding element of $H_p\left(X^p, X^{p-1}\right)$, i.e. by taking $(i_*)_p\gamma$ if not $(i_*)_p\gamma^{-1}$ with the homomorphism

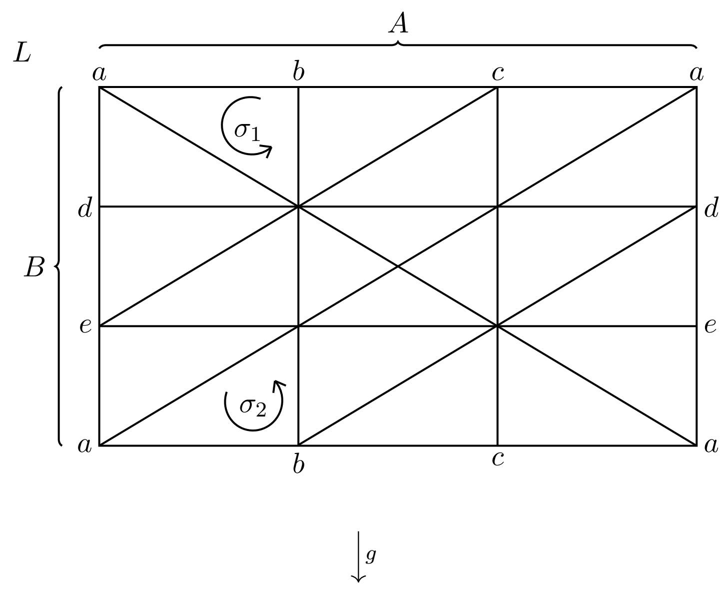
$$(i_*)_p : H_p(\overline{e}_\alpha, \dot{e}_\alpha) \to H_p\left(X^p, X^{p-1}\right)$$

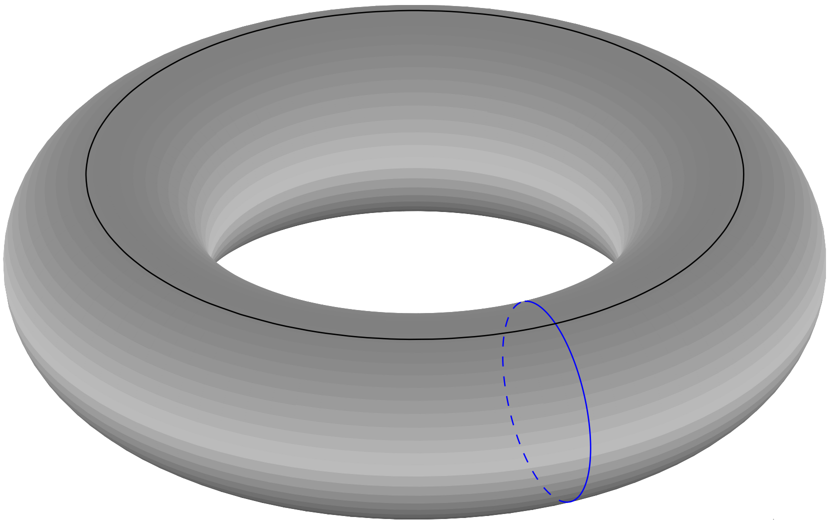
induced by the inclusion $i : (\overline{e}_\alpha, \dot{e}_\alpha) \hookrightarrow \left(X^p, X^{p-1}\right)$. The homology of the chain complex $\mathcal{D}(X)$ —to be more precise, cellular complex $\mathcal{D}(X)$ associated with the CW complex $(X, \mathcal{E})$— is isomorphic to the singular homology of X as proved in Theorem 3.16. In the special case when $(X, \mathcal{E})$ is a triangulable CW complex triangulated by a simplicial complex K, and H_p denotes simplicial homology, one observes the following fact: the fact that X^p and X^{p-1} are subcomplexes of K implies that each p-cell e_α is the union of open simplices of K of dimension at most p, so that $\overline{e}_\alpha$ is the polytope of a subcomplex of K. The group $H_p(\overline{e}_\alpha, \dot{e}_\alpha)$ equals the group of p-chains carried by $\overline{e}_\alpha$ whose boundaries are carried by $\dot{e}_\alpha$. The group $H_p(\overline{e}_\alpha, \dot{e}_\alpha)$ is infinite cyclic. Either generator of the group is called a **fundamental cycle** for $(\overline{e}_\alpha, \dot{e}_\alpha)$.

The cellular chain group $D_p(X)$ equals the group of all simplicial p-chains of X carried by X^p whose boundaries are carried by X^{p-1}. Any such p-chain can be written uniquely as a finite linear combination of fundamental cycles for those pairs $(\overline{e}_\alpha, \dot{e}_\alpha)$ for which $\dim e_\alpha = p$.

§3.4.i Torus

Let X denote the torus expressed as a quotient space of the rectangle L in the usual way. Then X is the underlying space of a triangulable CW complex.





If we consider the torus and denote the CW complex with the pais $(X, \mathcal{E})$, then $\mathcal{E}$ consists of a 2-cell e_2, two 1-cells e_1 and e_1' (which are the images of A and B under g, respectively) and one 0-cell e_0. Now $D_2(X) = H_2\left(X^2, X^1\right)$.

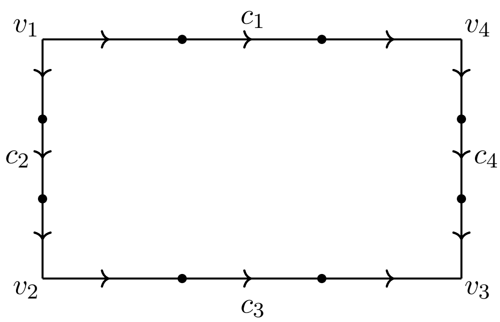
As we have done in AT2, all the 2-simplices are oriented counterclockwise, and the 1-simplices are oriented arbitrarily. Let the 2-chain d of L be the sum of the counterclockwise 2-simplices σ_i, i.e. $d = \sum_i \sigma_i$. As is done in AT2,

$$\partial_2^L d = 0. \tag{3.40}$$

Hence, d is a cycle of $(L, \operatorname{Bd} L)$, i.e. $d \in Z_2(L, \operatorname{Bd} L)$. Now, suppose $\alpha \in Z_2(L, \operatorname{Bd} L)$. In other words, $\partial_2^L \alpha$ is carried by $\operatorname{Bd} L$. Then by *Lemma 1.0.1(ii)* of AT2, $\alpha = pd$ for some $p \in \mathbb{Z}$. Hence, $Z_2(L, \operatorname{Bd} L)$ is generated by $d = \sum_i \sigma_i$. $B_2(L, \operatorname{Bd} L)$ is trivial since there are no 3-chains to consider. Hence, $H_2(L, \operatorname{Bd} L)$ is generated by d. Therefore, d is a **fundamental cycle** for $(L, \operatorname{Bd} L)$.

Since g is the underlying characteristic map associated with the 2-cell e_2, $\gamma = (g_*)_2\, d$ is a fundamental cycle for $(\bar{e}_2, \dot{e}_2)$ by Lemma 3.12. So we have

$$D_2(X) = H_2\left(X^2, X^1\right) \cong \mathbb{Z}. \tag{3.41}$$



Let c_1 be the sum of the 1-simplices along the top of L as indiced in the figure above. Let c_2, c_3, c_4 denote chains along the other edges of L. Any 1-cycle $c \in Z_1(\operatorname{Bd} L, \{v_1, v_2, v_3, v_4\})$ can be expressed as $c = m_1 c_1 + m_2 c_2$ for $m_i \in \mathbb{Z}$. Such 1-cycle c will bound if these exists a 2-chain d on $\operatorname{Bd} L$ such that $c - \partial_2^L d$ is carried by $\{v_1, v_2, v_3, v_4\}$. There is no non-trivial 2-chain on $\operatorname{Bd} L$. Hence, no nontrivial 1-chain bounds for $(\operatorname{Bd} L, \{v_1, v_2, v_3, v_4\})$. Therefore, $H_1(\operatorname{Bd} L, \{v_1, v_2, v_3, v_4\})$ is generated by c_1 and c_2, i.e. c_1 and c_2 are fundamental cycles for $(\operatorname{Bd} L, \{v_1, v_2, v_3, v_4\})$.

There are two 1-cells involved: $g\big|_A$ is the characteristic map associated with e_1 and $g\big|_B$ is the characteristic map associated with e_1', so that $w_1 = \left(\left(g\big|_{\operatorname{Bd} L}\right)_*\right)_1 c_1$ is a fundamental cycle for $(\bar{e}_1, \dot{e}_1)$, and $z_1 = \left(\left(g\big|_{\operatorname{Bd} L}\right)_*\right)_1 c_2$ is a fundamental cycle for $(\bar{e}_1', \dot{e}_1')$. One finds that

$$D_1(X) = H_1\left(X^1, X^0\right) \cong \mathbb{Z} \oplus \mathbb{Z}. \tag{3.42}$$

It's easy to see that $D_0(X) = H_0\left(X^0\right) \cong \mathbb{Z}$, since X^0 consists of just one 0-cell.

In terms of these basis elements, let us compute the boundary operator in the cellular chain complex $\mathcal{D}(X)$. We first compute ∂^L in the complex L as follows:

$$\partial_1^L c_1 = v_4 - v_1\,,\ \partial_1^L c_2 = v_2 - v_1. \tag{3.43}$$

$$\partial_2^L d = c_2 + c_3 - c_4 - c_1. \tag{3.44}$$

Let d_i be the i-th boundary operator on the cellular chain complex $\mathcal{D}(X)$. d_2 is the following composite:

$$\begin{tikzcd} H_2(X^2, X^1) \arrow[r, "(\partial_*)_2"] \arrow[rr, bend right, "d_2 = (j_*)_1 \circ (\partial_*)_2"'] & H_1(X^1) \arrow[r, "(j_*)_1"] & H_1(X^1, X^0) \end{tikzcd}$$

Here, $(\partial_*)_2$ is the homology boundary homomorphism in the long exact homology sequence of the pair $\left(X^2, X^1\right)$, and $(j_*)_1$ is induced by the inclusion $j : \left(X^1, \varnothing\right) \hookrightarrow \left(X^1, X^0\right)$. Now,

$$\begin{aligned}(\partial_*)_2\,\gamma &= (\partial_*)_2\,(g_*)_2\,d = (\partial_*)_2\left\{(g_\#)_2\,d\right\}\\ &= \left\{\partial_2^X\,(g_\#)_2\,d\right\} = \left\{(g_\#)_1\left(\partial_2^L d\right)\right\}\\ &= \left\{(g_\#)_2\,(0)\right\} = 0.\end{aligned} \tag{3.45}$$

Therefore, $(j_*)_1 \circ (\partial_*)_2\,\gamma = 0$. Since γ generates $D_2(X)$, the cellular chain map $d_2 = (j_*)_1 \circ (\partial_*)_2$ is trivial.

Now we shall see how the cellular boundary map $d_1 : H_1\left(X^1, X^0\right) \to H_0\left(X^0\right)$ works. d_1 is equal to the homology boundary homomorphism $(\partial'_*)_1$ of the pair $\left(X^1, X^0\right)$.

$$\begin{aligned}d_1(w_1) &= (\partial'_*)_1\left(\left(g|_{\mathrm{Bd}\,L}\right)_*\right)_1 c_1 = (\partial'_*)_1\left\{\left(\left(g|_{\mathrm{Bd}\,L}\right)_\#\right)_1 c_1\right\}\\ &= \left\{\partial_2^X\left(\left(\left(g|_{\mathrm{Bd}\,L}\right)_\#\right)_1 c_1\right)\right\} = \left\{\left(\left(g|_{\mathrm{Bd}\,L}\right)_\#\right)_0 \partial_1^L c_1\right\}\\ &= \left\{\left(\left(g|_{\mathrm{Bd}\,L}\right)_\#\right)_0 (v_4 - v_1)\right\} = 0,\end{aligned} \tag{3.46}$$

since $\left(\left(g|_{\mathrm{Bd}\,L}\right)_\#\right)_0 v_1 = \left(\left(g|_{\mathrm{Bd}\,L}\right)_\#\right)_0 v_4$. Also,

$$\begin{aligned}d_1(z_1) &= (\partial'_*)_1\left(\left(g|_{\mathrm{Bd}\,L}\right)_*\right)_1 c_2 = (\partial'_*)_1\left\{\left(\left(g|_{\mathrm{Bd}\,L}\right)_\#\right)_1 c_2\right\}\\ &= \left\{\partial_2^X\left(\left(\left(g|_{\mathrm{Bd}\,L}\right)_\#\right)_1 c_2\right)\right\} = \left\{\left(\left(g|_{\mathrm{Bd}\,L}\right)_\#\right)_0 \partial_1^L c_2\right\}\\ &= \left\{\left(\left(g|_{\mathrm{Bd}\,L}\right)_\#\right)_0 (v_2 - v_1)\right\} = 0,\end{aligned} \tag{3.47}$$

since $\left(\left(g|_{\mathrm{Bd}\,L}\right)_\#\right)_0 v_1 = \left(\left(g|_{\mathrm{Bd}\,L}\right)_\#\right)_0 v_2$. Since w_1 and z_1 generates $D_1(X) = H_1\left(X^1, X^0\right)$, and d_1 acting on both of them gives 0, so d_1 is also the 0 map.

Now, we have the cellular chain complex, with cellular chain groups $D_2(X), D_1(X), D_0(X)$, and cellular boundary maps $d_2 : D_2(X) \to D_1(X)$ and $d_1 : D_1(X) \to D_0(X)$. We have just seen that both d_2 and d_1 are 0 maps. Therefore, $Z_2(\mathcal{D}(X)) = D_2(X)$. Since $D_3(X)$ is trivial, so is $B_2(\mathcal{D}(X))$. Therefore,

$$H_2(\mathcal{D}(X)) = D_2(X) \cong \mathbb{Z}. \tag{3.48}$$

Since d_1 is the 0 map, $Z_1(\mathcal{D}(X)) = D_1(X)$. Image of d_2 is trivial, so $B_1(\mathcal{D}(X)) = 0$. Therefore,

$$H_1(\mathcal{D}(X)) = D_1(X) \cong \mathbb{Z} \oplus \mathbb{Z}. \tag{3.49}$$

$Z_0(\mathcal{D}(X)) = D_0(X)$. Image of d_1 is trivial, so $B_0(\mathcal{D}(X)) = 0$. Therefore,

$$H_0(\mathcal{D}(X)) = D_0(X) \cong \mathbb{Z}. \tag{3.50}$$

$D_n(X)$ is trivial for $n \geq 3$. Therefore, $H_n(\mathcal{D}(X))$ are also trivial for $n \geq 3$. Since $H_n(X) \cong H_n(\mathcal{D}(X))$, we have

$$H_n(X) \cong \begin{cases} \mathbb{Z} & \text{if } n = 0 \text{ or } n = 2, \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{3.51}$$

4 Cohomology Theory

§4.1 An Introductory Discussion on Hom Functor

Given two abelian groups A and G, there is a third abelian group $\mathrm{Hom}\,(A, G)$ consisting of all homomorphisms from A to G. We add two homomorphisms in the set $\mathrm{Hom}\,(A, G)$ by adding their values in G, i.e. given $a \in A$ and $\phi, \psi \in \mathrm{Hom}\,(A, G)$, we define

$$(\phi + \psi)(a) := \phi(a) + \psi(a)\,. \tag{4.1}$$

One can easily verify that $\phi + \psi \in \mathrm{Hom}\,(A, G)$. Indeed,

$$\begin{aligned}(\phi + \psi)(a + b) &= \phi(a + b) + \psi(a + b) = \phi(a) + \phi(b) + \psi(a) + \psi(b) \\ &= [\phi(a) + \psi(a)] + [\phi(b) + \psi(b)] \\ &= (\phi + \psi)(a) + (\phi + \psi)(b)\,.\end{aligned} \tag{4.2}$$

The identity element in $\mathrm{Hom}\,(A, G)$ is the homomorphism that maps all of A to 0_G, the identity of G.

Example 4.1

$\mathrm{Hom}\,(\mathbb{Z}, G)$ is isomorphic to G. The isomorphism assigns $\phi : \mathbb{Z} \to G$ to the element $\phi(1) \in G$. If one knows $\phi(1)$, one knows the homomorphism $\phi : \mathbb{Z} \to G$ completely as 1 generates $\mathbb{Z}$. Now we want to show that

$$i : \mathrm{Hom}\,(\mathbb{Z}, G) \to G\ , \qquad \phi \mapsto \phi(1)$$

is a group isomorphism. Indeed,

$$i(\phi + \psi) = (\phi + \psi)(1) = \phi(1) + \psi(1) = i(\phi) + i(\psi)\,.$$

So i is a group homomorphism. Given $g \in G$, we can define a homomorphism $f_g : \mathbb{Z} \to G$ by defining $f_g(1) = g$. So $i(f_g) = g$, proving that i is surjective. Now, take $f \in \mathrm{Ker}\, i$. So we have $f(1) = 0_G$. For $n > 0$,

$$f(n) = f(\underbrace{1 + 1 + \cdots + 1}_{n\text{-times}}) = f(1) + \cdots + f(1) = 0_G + \cdots + 0_G = 0_G.$$

Also, $f(-n) = -f(n) = -0_G = 0_G$. Therefore, f maps all of $\mathbb{Z}$ to 0_G, i.e. f is the identity of $\mathrm{Hom}\,(\mathbb{Z}, G)$. So $\mathrm{Ker}\, i$ is trivial, proving that i is injective. Therefore, i is an isomorphism.

Definition 4.1. A homomorphism $f : A \to B$ gives rise to a **dual homomorphism**

$$\mathrm{Hom}(A, G) \xleftarrow[\tilde{f}]{} \mathrm{Hom}(B, G)$$

going in the reverse direction. Given $\phi \in \mathrm{Hom}\,(B, G)$, $\tilde{f}$ is defined by

$$\tilde{f}(\phi) = \phi \circ f \in \mathrm{Hom}\,(A, G)\,. \tag{4.3}$$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "\phi\circ f"'] & B \arrow[r, "\phi"] & G \end{tikzcd}$$

Let us check quickly that $\widetilde{f}$ defined above is a group homomorphism: given $a \in A$ and $\phi, \psi \in \mathrm{Hom}\,(B, G)$,

$$\begin{aligned}\left[\widetilde{f}\,(\phi+\psi)\right](a) &= [(\phi+\psi)\circ f]\,(a) = (\phi+\psi)\,(f\,(a)) \\ &= (\phi\circ f)\,(a) + (\psi\circ f)\,(a) = \left[\widetilde{f}\,(\phi)\right](a) + \left[\widetilde{f}\,(\psi)\right](a) \\ &= \left[\widetilde{f}\,(\phi) + \widetilde{f}\,(\psi)\right](a)\,.\end{aligned}$$

Therefore, $\widetilde{f}\,(\phi+\psi) = \widetilde{f}\,(\phi) + \widetilde{f}\,(\psi)$.

Remark 4.1. Note that for a fixed abelian group G, the assignment $A \to \mathrm{Hom}\,(A, G)$ (at the level of objects) and $f \to \widetilde{f}$ (at the level of morphisms) defines a contravariant functor from the category of abelian groups and group homomorphisms to itself.

Category **Ab**

$$\begin{tikzcd} A \arrow[loop above, "\mathrm{id}_A"] \arrow[r, "f"] & B \arrow[r, "g"] & C \end{tikzcd}$$

$\big\downarrow$ Hom functor

Category **Ab**

$$\begin{tikzcd} \mathrm{Hom}(A,G) \arrow[loop above, "\widetilde{\mathrm{id}}_A"] & \mathrm{Hom}(B,G) \arrow[l, "\widetilde{f}"'] & \mathrm{Hom}(C,G) \arrow[l, "\widetilde{g}"'] \end{tikzcd}$$

Indeed, if $\mathrm{id}_A : A \to A$ is the identity group homomorphism, then $\widetilde{\mathrm{id}}_A : \mathrm{Hom}\,(A, G) \to \mathrm{Hom}\,(A, G)$ is

$$\widetilde{\mathrm{id}}_A\,(\phi) = \phi \circ \mathrm{id}_A = \phi.$$

Hence, $\widetilde{\mathrm{id}}_A = \mathrm{id}_{\mathrm{Hom}(A,G)}$. Furthermore, if $f : A \to B$ and $g : B \to C$ are homomorphisms, we need to show that $\widetilde{g \circ f} = \widetilde{f} \circ \widetilde{g}$. Indeed, for $\psi \in \mathrm{Hom}(C, G)$,

$$\widetilde{g\circ f}\,(\psi) = \psi \circ (g \circ f) = (\psi \circ g) \circ f = \widetilde{f}\,(\psi \circ g) = \widetilde{f}\,(\widetilde{g}\,(\psi))\,.$$

Therefore, the Hom functor is indeed a contravariant functor.

We have the following consequences of the above fact:

Theorem 4.1

Let f be a homomorphism, let $\widetilde{f}$ be the dual homomorphism.

(a) If f is an isomorphism, so is $\widetilde{f}$.

(b) If f is the zero homomorphism, so is $\widetilde{f}$.

(c) If f is surjective, then $\widetilde{f}$ is injective. That is, exactness of $B \xrightarrow{f} C \to 0$ implies the exactness of $\mathrm{Hom}\,(B, G) \xleftarrow{\widetilde{f}} \mathrm{Hom}\,(C, G) \leftarrow 0$.

Proof. (a) Let $f : A \to B$ be an isomorphism. First we shall show that $\widetilde{f} : \mathrm{Hom}\,(B, G) \to \mathrm{Hom}\,(A, G)$ is injective. Suppose $\widetilde{f}\,(\phi) = \widetilde{f}\,(\psi)$, for $\phi, \psi \in \mathrm{Hom}\,(B, G)$. Then $\phi \circ f = \psi \circ f$. Since f is surjective, for each $b \in B$, there exists $a \in A$ such that $f\,(a) = b$. Therefore,

$$(\phi \circ f)\,(a) = (\psi \circ f)\,(a) \implies \phi\,(b) = \psi\,(b)\,. \tag{4.4}$$

Therefore, $\phi(b) = \psi(b)$ for each $b \in B$, so that $\phi = \psi$ and $\widetilde{f}$ is injective.

Now we shall prove that $\widetilde{f}$ is surjective. For any $\psi \in \operatorname{Hom}(A, G)$, let $\phi = \psi \circ f^{-1} \in \operatorname{Hom}(B, G)$. Then we have

$$\widetilde{f}(\phi) = \phi \circ f = \left(\psi \circ f^{-1}\right) \circ f = \psi. \tag{4.5}$$

So $\widetilde{f}$ is surjective. Therefore, $\widetilde{f}$ is an isomorphism.

(b) Given $f : A \to B$ the zero homomorphism, $f(a) = 0_B$ for all $a \in A$. Then given $\phi \in \operatorname{Hom}(B, G)$,

$$\left[\widetilde{f}(\phi)\right](a) = (\phi \circ f)(a) = \phi(0_B) = 0_G. \tag{4.6}$$

Therefore, $\widetilde{f}(\phi)$ maps all of A to 0_G, so $\widetilde{f}(\phi)$ is the identity of $\operatorname{Hom}(A, G)$ for each $\phi \in \operatorname{Hom}(B, G)$. Hence, $\widetilde{f}$ is the zero homomorphism.

(c) This is exactly the same as the first part of (a).

■

Theorem 4.2

If the sequence

$$A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \tag{4.7}$$

is exact, then the dual sequence

$$\operatorname{Hom}(A, G) \xleftarrow{\widetilde{f}} \operatorname{Hom}(B, G) \xleftarrow{\widetilde{g}} \operatorname{Hom}(C, G) \longleftarrow 0 \tag{4.8}$$

is exact. Furthermore, if f is injective and the first sequence splits, then $\widetilde{f}$ is surjective and the second sequence splits.

Proof. Exactness of (4.7) implies that g is surjective. Then applying Theorem 4.1(c), $\widetilde{g}$ is injective. Hence, the sequence (4.8) is exact at $\operatorname{Hom}(C, G)$. Now we check exactness at $\operatorname{Hom}(B, G)$.

Exactness of (4.7) implies that $g \circ f = 0$. Therefore, by Theorem 4.1(b), $\widetilde{g \circ f} = \widetilde{f} \circ \widetilde{g} = 0$. So $\operatorname{im} \widetilde{g} \subseteq \operatorname{Ker} \widetilde{f}$. Now let us show the reverse inclusion $\operatorname{Ker} \widetilde{f} \subseteq \operatorname{im} \widetilde{g}$.

Suppose $\psi \in \operatorname{Ker} \widetilde{f}$, so that $\widetilde{f}(\psi) = 0_{\operatorname{Hom}(A,G)}$. We want to show that $\psi = \widetilde{g}(\phi)$ for some $\phi \in \operatorname{Hom}(C, G)$. Since $\widetilde{f}(\psi)$ is the 0-homomorphism, ψ vanishes on the subgroup $f(A) \subseteq B$. Since B is abelian, $f(A)$ is normal. Hence, $\psi : B \to G$ is a group homomorphism and $f(A) \subseteq \operatorname{Ker} \psi$. Then the homomorphism theorem tells us that there is an induced homomorphism

$$\psi' : \frac{B}{f(A)} \to G.$$

Now, exactness of (4.7) implies that $f(A) = \operatorname{Ker} g$. Besides, $g : B \to C$ is surjective by exactness at C. Hence, by first isomorphism theorem, g induces an isomorphism

$$g' : \frac{B}{f(A)} \to C.$$

So we have the following commutative diagram:

$$\begin{tikzcd}
G & B \arrow[l, "\psi"'] \arrow[r, "g"] \arrow[d, "\pi"'] & C \\
& \frac{B}{f(A)} \arrow[ul, "\psi'"] \arrow[ur, "g'"', "\cong"] &
\end{tikzcd}$$

The map $\phi = \psi' \circ (g')^{-1}$ is a homomorphism from C to G. Therefore,

$$\widetilde{g}(\phi) = \phi \circ g = \psi' \circ (g')^{-1} \circ g = \psi' \circ \pi = \psi. \tag{4.9}$$

Therefore, $\operatorname{Ker} \widetilde{f} \subseteq \operatorname{im} \widetilde{g}$, and (4.8) is exact.

Now, let us suppose that f is injective and the exact sequence (4.7) splits. Then the following short exact sequence splits:

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0.$$

Then from equivalent properties (*Theorem 4.1.1* of AT2) of a split short exact sequence, there is a homomorphism $\pi : B \to A$ such that $\pi \circ f = \mathrm{id}_A$.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] & B \arrow[l, bend left, "\pi"] \arrow[r, "g"] & C \arrow[r] & 0. \end{tikzcd}$$

From the functorial properties of the Hom functor, we have

$$\widetilde{f} \circ \widetilde{\pi} = \mathrm{id}_{\mathrm{Hom}(A,G)}\,. \tag{4.10}$$

Now, we want to show that $\widetilde{f}$ is surjective. For $\psi \in \mathrm{Hom}\,(A, G)$,

$$\widetilde{f}\,(\widetilde{\pi}\,(\psi)) = \mathrm{id}_{\mathrm{Hom}(A,G)}\,\psi = \psi. \tag{4.11}$$

Therefore, $\widetilde{f}$ is surjective. Therefore, (4.10) along with *Theorem 4.1.1* of AT2 implies that

$$\begin{tikzcd} 0 & \mathrm{Hom}(A,G) \arrow[l] \arrow[r, bend right, "\widetilde{\pi}"'] & \mathrm{Hom}(B,G) \arrow[l, "\widetilde{f}"'] & \mathrm{Hom}(C,G) \arrow[l, "\widetilde{g}"'] & 0 \arrow[l] \end{tikzcd}$$

is a split short exact sequence. ■

Remark 4.2. In general, exactness of a short exact sequence does not imply exactness of the dual sequence. To be more precise, exactness of (4.7) does not require f to be injective, in general. Only when f is injective, exactness of

$$A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

implies the existence of the following short exact sequence:

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0.$$

For instance, if $f : \mathbb{Z} \to \mathbb{Z}$ is defined by multiplication by 2, i.e. $f\,(n) = 2n$, then one has the following short exact sequence:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{f} \mathbb{Z} \xrightarrow{g} \mathbb{Z}/2 \longrightarrow 0,$$

with $\pi : \mathbb{Z} \to \mathbb{Z}/2$ being the canonical projection map. $\operatorname{Ker} \pi = \operatorname{im} f$ is the set of all even integers. But the dual sequence

$$0 \longleftarrow \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \xleftarrow{\widetilde{f}} \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \xleftarrow{\widetilde{\pi}} \mathrm{Hom}(\mathbb{Z}/2,\mathbb{Z}) \longleftarrow 0$$

is not exact. In particular, $\widetilde{f}$ is not surjective. Given $\phi \in \mathrm{Hom}\,(\mathbb{Z},\mathbb{Z})$, $\psi = \widetilde{f}\,(\phi)$ takes only even values, since

$$\psi\,(n) = \phi\,(f\,(n)) = \phi\,(2n) = 2\phi\,(n)\,.$$

Therefore, $\widetilde{f}$ is not surjective, and hence the dual sequence above is not exact.

Proposition 4.3
Let A, B, G be abelian groups. Then

$$\mathrm{Hom}\,(A \oplus B, G) \cong \mathrm{Hom}\,(A, G) \oplus \mathrm{Hom}\,(B, G)\,.$$

Proof. We define $F : \mathrm{Hom}\,(A \oplus B, G) \to \mathrm{Hom}\,(A, G) \oplus \mathrm{Hom}\,(B, G)$ as follows: for a homomorphism $\psi : A \oplus B \to G$, we define $F(\psi) = (\psi_1, \psi_2)$, where $\psi_1 : A \to G$ and $\psi_2 : B \to G$ are defined as follows:

$$\psi_1(a) = \psi(a, 0) \text{ and } \psi_2(b) = \psi(0, b)\,. \tag{4.12}$$

For $\phi, \psi \in \mathrm{Hom}\,(A \oplus B, G)$, $F(\phi + \psi) = ((\phi + \psi)_1, (\phi + \psi)_2)$. Now,

$$(\phi + \psi)_1(a) = (\phi + \psi)(a, 0) = \phi(a, 0) + \psi(a, 0) = \phi_1(a) + \psi_1(a) = (\phi_1 + \psi_1)(a)\,.$$

Therefore, $(\phi + \psi)_1 = \phi_1 + \psi_1$. Similarly, $(\phi + \psi)_2 = \phi_2 + \psi_2$. Hence,

$$F(\phi + \psi) = (\phi_1 + \psi_1, \phi_2 + \psi_2) = (\phi_1, \phi_2) + (\psi_1, \psi_2) = F(\phi) + F(\psi)\,. \tag{4.13}$$

So F is a group homomorphism. Now, let $\psi \in \mathrm{Ker}\, F$. So both ψ_1 and ψ_2 are zero maps. As a result,

$$\psi(a, b) = \psi(a, 0) + \psi(0, b) = \psi_1(a) + \psi_2(b) = 0.$$

So ψ is the zero map, and hence $\mathrm{Ker}\, F$ is trivial. Now, given any homomorphisms $\alpha : A \to G$ and $\beta : B \to G$, we can define $\gamma : A \oplus B \to G$ as follows:

$$\gamma(a, b) = \alpha(a) + \beta(b)\,.$$

Now,

$$\gamma_1(a) = \gamma(a, 0) = \alpha(a) \text{ and } \gamma_2(b) = \gamma(0, b) = \beta(b)\,,$$

so that $F(\gamma) = (\alpha, \beta)$, proving that F is surjective. Therefore, F is a bijective homomorphism, i.e. an isomorphism. ■

§4.2 Cohomology Theory

Let $\mathcal{C} = \{C_p, \partial_p\}$ be a chain complex.

$$\cdots \longrightarrow C_{p+1} \xrightarrow{\partial_{p+1}} C_p \xrightarrow{\partial_p} C_{p-1} \longrightarrow \cdots$$

Also, let G be an abelian group. We define the p-dimensional **cochain group** of $\mathcal{C}$, with coefficients in G by

$$C^p(\mathcal{C}; G) = \mathrm{Hom}\,(C_p, G)\,. \tag{4.14}$$

We define the **coboundary operator** δ_p to be the dual of the boundary homomrophism ∂_{p+1}, i.e. $\delta_p = \widetilde{\partial}_p$.

$$\begin{array}{c} \cdots \longrightarrow C_{p+1} \xrightarrow{\partial_{p+1}} C_p \xrightarrow{\partial_p} C_{p-1} \longrightarrow \cdots \\ \Big\downarrow \text{Hom functor} \\ \cdots \longleftarrow \mathrm{Hom}(C_{p+1}, G) \xleftarrow{\delta_{p+1}} \mathrm{Hom}(C_p, G) \xleftarrow{\delta_p} \mathrm{Hom}(C_{p-1}, G) \longleftarrow \cdots \end{array}$$

The boundary homomorphisms of the chain complex $\mathcal{C}$ satisfies $\partial_p \circ \partial_{p+1} = 0$. Therefore, using Theorem 4.1(b),

$$0 = \widetilde{\partial_p \circ \partial_{p+1}} = \widetilde{\partial_{p+1}} \circ \widetilde{\partial_p} = \delta_{p+1} \circ \delta_p. \tag{4.15}$$

Therefore, $\delta_{p+1} \circ \delta_p = 0$ for each p. The kernel of the group homorphism $\delta_{p+1} : C^p(\mathcal{C}; G) \to C^{p+1}(\mathcal{C}; G)$ being a subgroup of $C^p(\mathcal{C}; G)$ is called the **group of p-cocycles**, and is denoted by $Z^p(\mathcal{C}; G)$. The image of the homomorphism $\delta_p : C^{p-1}(\mathcal{C}; G) \to C^p(\mathcal{C}; G)$ is also a subgroup of $C^p(\mathcal{C}; G)$, which is called the **group of p-coboundaries** and is denoted by $B^p(\mathcal{C}; G)$. From $\delta_{p+1} \circ \delta_p = 0$, it is clear that $B^p(\mathcal{C}; G) \subseteq Z^p(\mathcal{C}; G)$. The resulting quotient group is called the **p-dimensional cohomology group** of $\mathcal{C}$, with coefficients in G:

$$H^p(\mathcal{C}; G) = \frac{Z^p(\mathcal{C}; G)}{B^p(\mathcal{C}; G)}. \tag{4.16}$$

If $\{\mathcal{C}, \epsilon\}$ is an augmented chain complex, then $\epsilon : C_0 \to \mathbb{Z}$ is a surjective group homomorphism satisfying $\epsilon \circ \partial_1 = 0$.

$$\begin{array}{ccc}
\cdots \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\epsilon} \mathbb{Z} \\
\Big\downarrow \text{Hom functor} \\
\cdots \longleftarrow \mathrm{Hom}(C_1, G) \xleftarrow{\delta_1} \mathrm{Hom}(C_0, G) \xleftarrow{\widetilde{\epsilon}} \mathrm{Hom}(\mathbb{Z}, G)
\end{array}$$

By Theorem 4.1(c), $\widetilde{\epsilon}$ is injective. One then defined the **reduced cohomology groups** of $\mathcal{C}$ by setting

$$\widetilde{H}^q(\mathcal{C}; G) = \begin{cases} H^q(\mathcal{C}; G) & \text{if } q > 0, \\ \frac{\mathrm{Ker}\,\delta_1}{\mathrm{im}\,\widetilde{\epsilon}} & \text{if } q = 0. \end{cases} \tag{4.17}$$

Now, if $\widetilde{H}_0(\mathcal{C})$ vanishes, then ϵ being surjective implies the following sequence is exact:

$$C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

Hence, by Theorem 4.2, the dual sequence

$$C^1(\mathcal{C}; G) \xleftarrow{\delta_1} C^0(\mathcal{C}; G) \xleftarrow{\widetilde{\epsilon}} \mathrm{Hom}(\mathbb{Z}, G) \longleftarrow 0$$

is also exact. Therefore, $\mathrm{Ker}\,\delta_1 = \mathrm{im}\,\widetilde{\epsilon}$, so that $\widetilde{H}^0(\mathcal{C}; G)$. Therefore, vanishing of reduced homology group in dimension 0 implies vanishing of reduced cohomology in dimension 0. In general, we have the following relationship between 0-dimensional ordinary and reduced cohomology groups:

Proposition 4.4
$H^0(\mathcal{C}; G) \cong \widetilde{H}^0(\mathcal{C}; G) \oplus G$.

Proof. For the unreduced cochain complex,

$$C^1(\mathcal{C}; G) \xleftarrow{\delta_1} C^0(\mathcal{C}; G) \longleftarrow 0$$

$H^0(\mathcal{C}; G) = \mathrm{Ker}\,\delta_1$; and for the reduced cochain complex,

$$C^1(\mathcal{C}; G) \xleftarrow{\delta_1} C^0(\mathcal{C}; G) \xleftarrow{\widetilde{\epsilon}} \mathrm{Hom}(\mathbb{Z}, G) \longleftarrow 0$$

$\widetilde{H}^0(\mathcal{C}; G) = \frac{\mathrm{Ker}\,\delta_1}{\mathrm{im}\,\widetilde{\epsilon}}$. We first want to show that $\mathrm{im}\,\widetilde{\epsilon}$ is a direct summand in $C^0(\mathcal{C}; G)$. Since ϵ is surjective, there exists some $c \in C_0$ such that $\epsilon(c) = 1$. We define $\psi : \mathbb{Z} \to C_0$ by $\psi(n) = nc$. Then

$$\epsilon(\psi(n)) = \epsilon(nc) = n\,\epsilon(c) = n. \tag{4.18}$$

So $\epsilon \circ \psi = \mathrm{id}_{\mathbb{Z}}$. After applying the Hom functor, we then get $\widetilde{\psi} \circ \widetilde{\epsilon} = \mathrm{id}_{\mathrm{Hom}(\mathbb{Z},G)}$.

$$C_0 \underset{\psi}{\overset{\epsilon}{\rightleftarrows}} \mathbb{Z} \xRightarrow{\text{Hom}} C^0(\mathcal{C};G) \underset{\widetilde{\psi}}{\overset{\widetilde{\epsilon}}{\leftrightarrows}} \text{Hom}(\mathbb{Z},G)$$

Now we claim that

$$C^0(\mathcal{C};G) = \operatorname{im}\widetilde{\epsilon} \oplus \operatorname{Ker}\widetilde{\psi}. \tag{4.19}$$

For any $f \in C^0(\mathcal{C};G)$, we can write f as

$$f = \widetilde{\epsilon}\left(\widetilde{\psi}(f)\right) + \left[f - \widetilde{\epsilon}\left(\widetilde{\psi}(f)\right)\right]. \tag{4.20}$$

Here, $\widetilde{\epsilon}\left(\widetilde{\psi}(f)\right) \in \operatorname{im}\widetilde{\epsilon}$, and

$$\widetilde{\psi}\left[f - \widetilde{\epsilon}\left(\widetilde{\psi}(f)\right)\right] = \widetilde{\psi}(f) - \widetilde{\psi}\left(\widetilde{\epsilon}\left(\widetilde{\psi}(f)\right)\right) = \widetilde{\psi}(f) - \widetilde{\psi}(f) = 0. \tag{4.21}$$

Therefore, $f - \widetilde{\epsilon}\left(\widetilde{\psi}(f)\right) \in \operatorname{Ker}\widetilde{\psi}$. Now we need to show that $\operatorname{im}\widetilde{\epsilon} \cap \operatorname{Ker}\widetilde{\psi}$ is trivial. If we take any $f \in \operatorname{im}\widetilde{\epsilon} \cap \operatorname{Ker}\widetilde{\psi}$, then $f = \widetilde{\epsilon}(g)$ for some $g \in \text{Hom}(\mathbb{Z},G)$, and $\widetilde{\psi}(f) = 0$. So

$$0 = \widetilde{\psi}(f) = \widetilde{\psi}(\widetilde{\epsilon}(g)) = g. \tag{4.22}$$

Therefore, $f = \widetilde{\epsilon}(g) = 0$. So $\operatorname{im}\widetilde{\epsilon} \cap \operatorname{Ker}\widetilde{\psi}$ is trivial, and hence $C^0(\mathcal{C};G) = \operatorname{im}\widetilde{\epsilon} \oplus \operatorname{Ker}\widetilde{\psi}$. Since $\operatorname{Ker}\delta_1$ is a subgroup of $C^0(\mathcal{C};G)$ containing $\operatorname{im}\widetilde{\epsilon}$, $\operatorname{im}\widetilde{\epsilon}$ is a direct summand in $\operatorname{Ker}\delta_1$ as well. Therefore,

$$\operatorname{Ker}\delta_1 = \operatorname{im}\widetilde{\epsilon} \oplus H, \tag{4.23}$$

where $H = \operatorname{Ker}\widetilde{\psi} \cap \operatorname{Ker}\delta_1$ is a subgroup of $\operatorname{Ker}\delta_1$.

It's easy to see that if $X = Y \oplus Z$, then Z is isomorphic to X/Y. For that purpose, we shall construct a homomorphism $\mu : X \to Z$ as follows: since $X = Y \oplus Z$, any $x \in X$ can be **uniquely** written as $x = y + z$, where $y \in Y$ and $z \in Z$. μ maps this $y + z$ to z. Then μ is a surjective homomorphism, with kernel Y. Therefore, by the first isomorphism theorem, $Z \cong X/Y$. Therefore,

$$\operatorname{Ker}\delta_1 = \operatorname{im}\widetilde{\epsilon} \oplus H \cong \operatorname{im}\widetilde{\epsilon} \oplus \frac{\operatorname{Ker}\delta_1}{\operatorname{im}\widetilde{\epsilon}}. \tag{4.24}$$

Since $\widetilde{\epsilon}$ is injective, its image is isomorphic to its domain $\text{Hom}(\mathbb{Z},G) \cong G$. Therefore,

$$\operatorname{Ker}\delta_1 \cong G \oplus \frac{\operatorname{Ker}\delta_1}{\operatorname{im}\widetilde{\epsilon}}. \tag{4.25}$$

$H^0(\mathcal{C};G) = \operatorname{Ker}\delta_1$ and $\widetilde{H}^0(\mathcal{C};G) = \frac{\operatorname{Ker}\delta_1}{\operatorname{im}\widetilde{\epsilon}}$. Hence,

$$H^0(\mathcal{C};G) \cong G \oplus \widetilde{H}^0(\mathcal{C};G). \tag{4.26}$$

■

§4.3 Cochain Maps and Cochain Homotopy

Definition 4.2 (Cochain map)**.** Suppose $\mathcal{C} = \{C_p, \partial_p\}$ and $\mathcal{C}' = \left\{C'_p, \partial'_p\right\}$ are chain complexes, and $\phi : \mathcal{C} \to \mathcal{C}'$ is a chain map, so that $\phi_{p-1} \circ \partial_p = \partial'_p \circ \phi_p$. Then applying the Hom functor, we get

$$\delta_p \circ \widetilde{\phi}_{p-1} = \widetilde{\phi}_p \circ \delta'_p. \tag{4.27}$$

The dual homomorphisms $C^p(\mathcal{C};G) \xleftarrow{\widetilde{\phi}_p} C^p(\mathcal{C}';G)$ form a family $\left\{\widetilde{\phi}_p\right\}$ of homomorphisms called a **cochain map**.

Since a cochain map commutes with the coboundary operator, it carries cocycles to cocycles and coboundaries to coboundaries. So it induced a homomorphism of cohomology groups

$$H^p(\mathcal{C};G) \xleftarrow{(\phi^*)_p} H^p(\mathcal{C}';G).$$

The assignment

$$\mathcal{C} \to H^p(\mathcal{C};G) \text{ and } \phi \to (\phi^*)_p$$

satisfies all the functorial properties.

If $\{\mathcal{C}, \epsilon\}$ and $\{\mathcal{C}', \epsilon'\}$ are augmented chain complexes and if $\phi : \mathcal{C} \to \mathcal{C}'$ is an augmentation preserving chain map, then the following diagram commutes:

$$\begin{tikzcd}
C_1 \arrow[r, "\partial_0"] \arrow[d, "\phi_1"] & C_0 \arrow[r, "\epsilon"] \arrow[d, "\phi_0"] & \mathbb{Z} \arrow[d, equal] \\
C'_1 \arrow[r, "\partial'_0"'] & C'_0 \arrow[r, "\epsilon'"'] & \mathbb{Z}
\end{tikzcd}$$

In other words, $\epsilon' \circ \phi_0 = \epsilon$. At the level of dual homomorphisms,

$$\widetilde{\epsilon} = \widetilde{\epsilon' \circ \phi_0} = \widetilde{\phi}_0 \circ \widetilde{\epsilon}'. \tag{4.28}$$

In this case, $\widetilde{\phi}$ induces homomorphisms of reduced as well as ordinary cohomology.

Suppose not that $\phi, \psi : \mathcal{C} \to \mathcal{C}'$ are chain maps and D is a chain homotopy between them, so that

$$\partial'_{p+1} D_p + D_{p-1}\partial_p = \phi_p - \psi_p. \tag{4.29}$$

$$\begin{tikzcd}
 & C'_{p+1} \arrow[d, "\partial'_{p+1}"] \\
C_p \arrow[ur, "D_p"] \arrow[r, shift left, "\phi_p"] \arrow[r, shift right, "\psi_p"'] \arrow[d, "\partial_p"'] & C'_p \\
C_{p-1} \arrow[ur, "D_{p-1}"'] &
\end{tikzcd}$$

Here $D_p : C_p \to C'_{p+1}$ is a group homomorphism, and $C^p(\mathcal{C};G) \xleftarrow{\widetilde{D}_p} C^{p+1}(\mathcal{C}';G)$ is the dual homomorphism satisfying

$$\widetilde{D}_p \delta'_{p+1} + \delta_p \widetilde{D}_{p-1} = \widetilde{\phi}_p - \widetilde{\psi}_p. \tag{4.30}$$

$$\begin{tikzcd}
 & C^{p+1}(\mathcal{C}';G) \arrow[dl, "\widetilde{D}_p"'] \\
C^p(\mathcal{C};G) & C^p(\mathcal{C}';G) \arrow[l, shift right, "\widetilde{\phi}_p"'] \arrow[l, shift left, "\widetilde{\psi}_p"] \arrow[u, "\delta'_{p+1}"'] \arrow[dl, "\widetilde{D}_{p-1}"] \\
C^{p-1}(\mathcal{C};G) \arrow[u, "\delta_p"] &
\end{tikzcd}$$

The family of group homomorphisms $\widetilde{D}_p : C^{p+1}(\mathcal{C}';G) \to C^p(\mathcal{C};G)$ is called a **cochain homotopy** between $\widetilde{\phi}$ and $\widetilde{\psi}$.

Given a p-cocycle $z^p \in Z^p(\mathcal{C}';G)$, one has

$$\begin{aligned}
& \widetilde{D}_p \delta'_{p+1} z^p + \delta_p \widetilde{D}_{p-1} z^p = \widetilde{\phi}_p(z^p) - \widetilde{\psi}_p(z^p) \\
\Longrightarrow \quad & \widetilde{\phi}_p(z^p) - \widetilde{\psi}_p(z^p) = \delta_p \widetilde{D}_{p-1} z^p.
\end{aligned} \tag{4.31}$$

Since $\delta_p \widetilde{D}_{p-1} z^p \in B^p(\mathcal{C};G)$, (4.31) tells us that $\widetilde{\phi}_p(z^p)$ and $\widetilde{\psi}_p(z^p)$ lie in the same cohomology class in $C^p(\mathcal{C};G)$. Therefore,

$$(\phi^*)_p\{z^p\} = (\psi^*)_p\{z^p\}. \tag{4.32}$$

It shows that $(\phi^*)_p, (\psi^*)_p : H^p(\mathcal{C}';G) \to H^p(\mathcal{C};G)$ are equal. This observation leads to the following theorem.

Theorem 4.5

Let $\mathcal{C}$ and $\mathcal{C}'$ be chain complexes; let $\phi : \mathcal{C} \to \mathcal{C}'$ be a chain equivalence. Then $(\phi_*)_p : H_p(\mathcal{C}) \to H_p(\mathcal{C}')$ and $(\phi^*)_p : H^p(\mathcal{C}';G) \to H_p(\mathcal{C};G)$ are isomorphisms of homology and cohomology, respectively. If $\mathcal{C}$ and $\mathcal{C}'$ are augmented chain complexes, and ϕ is an augmentation preserving chain equivalence, then $(\phi_*)_p$ and $(\phi^*)_p$ are isomorphisms of reduced homology and cohomology groups, respectively.

Proof. Since $\phi : \mathcal{C} \to \mathcal{C}'$ is a chain equivalence, there is a chain map $\psi : \mathcal{C}' \to \mathcal{C}$ such that $\phi \circ \psi$ is chain homotopic to $\mathrm{id}_{\mathcal{C}'}$ and $\psi \circ \phi$ is chain homotopic to $\mathrm{id}_{\mathcal{C}}$. Then $\widetilde{\phi \circ \psi} = \widetilde{\psi} \circ \widetilde{\phi}$ is cochain homotopic to the identity map on the cochain complex of $\mathcal{C}'$. Similarly, $\widetilde{\psi \circ \phi} = \widetilde{\phi} \circ \widetilde{\psi}$ is cochain homotopic to the identity map on the cochain complex of $\mathcal{C}$. Cochain homotopic maps induce **same** isomorphisms at the cohomology level. Therefore,

$$\begin{aligned}
(\phi^*)_p \circ (\psi^*)_p &= ((\psi \circ \phi)^*)_p = \mathrm{id}_{H^p(\mathcal{C};G)}, \\
(\psi^*)_p \circ (\phi^*)_p &= ((\phi \circ \psi)^*)_p = \mathrm{id}_{H^p(\mathcal{C}';G)}.
\end{aligned}$$

Therefore, $(\phi^*)_p : H^p(\mathcal{C}';G) \to (\mathcal{C};G)$ is an isomorphism. Isomorphism in homology follows from our earlier discussions. One can use similar arguments for reduced cohomology. ∎

Finally, suppose $0 \longrightarrow \mathcal{C} \xrightarrow{\phi} \mathcal{D} \xrightarrow{\psi} \mathcal{E} \longrightarrow 0$ is a short exact sequence of chain complexes that splits in each dimension, i.e.

$$0 \longrightarrow C_p \xrightarrow{\phi_p} D_p \xrightarrow{\psi_p} E_p \longrightarrow 0$$

is a split short exact sequence for each p. This occurs, for example, when E_p is free abelian (see *Corollary 4.1.2* of AT2). Then by Theorem 4.2, the following dual sequence is exact

$$0 \longleftarrow C^p(\mathcal{C};G) \xleftarrow{\widetilde{\phi}_p} C^p(\mathcal{D};G) \xleftarrow{\widetilde{\psi}_p} C^p(\mathcal{E};G) \longleftarrow 0.$$

Then there is a long exact cohomology sequence, by Zig-Zag lemma, as follows:

$$\begin{tikzcd}
 & \cdots & H^{p+1}(\mathcal{E};G) \arrow[l] \\
H^p(\mathcal{C};G) \arrow[urr, "(\delta^*)_{p+1}"] & H^p(\mathcal{D};G) \arrow[l, "(\phi^*)_p"'] & H^p(\mathcal{E};G) \arrow[l, "(\psi^*)_p"] \\
H^{p-1}(\mathcal{C};G) \arrow[urr, "(\delta^*)_p"'] & H^{p-1}(\mathcal{D};G) \arrow[l, "(\phi^*)_{p-1}"'] & \cdots \arrow[l]
\end{tikzcd}$$

where $(\delta^*)_p : H^p(\mathcal{E};G) \leftarrow H^{p-1}(\mathcal{C};G)$ is the coboundary cohomology homomorphism induced by the coboundary operator δ_p in the usual manner. This sequence is natural in the sense that if $f \equiv (f_1, f_2, f_3)$ is a homomorphism of short exact sequence of chain complexes,

$$\begin{tikzcd}
0 \arrow[r] & C_p \arrow[r, "\phi_p"] \arrow[d, "(f_1)_p"'] & D_p \arrow[r, "\psi_p"] \arrow[d, "(f_2)_p"] & E_p \arrow[r] \arrow[d, "(f_3)_p"] & 0 \\
0 \arrow[r] & C'_p \arrow[r, "\phi'_p"'] & D'_p \arrow[r, "\psi'_p"'] & E'_p \arrow[r] & 0,
\end{tikzcd}$$

(this diagram commutes for each p) then the following diagram commutes at the cohomology level:

$$\begin{tikzcd}
\cdots & H^p(\mathcal{C};G) \arrow[l] & H^p(\mathcal{D};G) \arrow[l, "(\phi^*)_p"'] & H^p(\mathcal{E};G) \arrow[l, "(\psi^*)_p"'] & H^{p-1}(\mathcal{C};G) \arrow[l, "(\delta^*)_p"'] & \cdots \arrow[l] \\
\cdots & H^p(\mathcal{C}';G) \arrow[l] \arrow[u, "(f_1^*)_p"] & H^p(\mathcal{D}';G) \arrow[l, "(\phi'^*)_p"] \arrow[u, "(f_2^*)_p"] & H^p(\mathcal{E}';G) \arrow[l, "(\psi'^*)_p"] \arrow[u, "(f_3^*)_p"] & H^{p-1}(\mathcal{C}';G) \arrow[l, "(\delta'^*)_p"] \arrow[u, "(f_1^*)_{p-1}"'] & \cdots \arrow[l]
\end{tikzcd}$$

§4.4 Eilenberg-Steenrod Axioms

Now we state the cohomology versions of the Eilenberg-Steenrod axioms. Given an admissible class $\mathcal{A}$ of pairs of spaces (X, A) and an abelian group G, a **cohomology theory** on $\mathcal{A}$ consists of the following:

1. A function defined for each integer p and each pair (X, A) in $\mathcal{A}$, whose values form an abelian group $H^p(X, A; G)$.
2. A function that assigns to each continuous map $h : (X, A) \to (Y, B)$ and each integer p, a homomorphism
$$H^p(X, A; G) \xleftarrow{(h^*)_p} H^p(Y, B; G).$$
3. A function that assigns to each pair $(X, A) \in \mathcal{A}$ and each integer p, a homomorphism
$$H^p(X, A; G) \xleftarrow{(\delta^*)_p} H^{p-1}(A; G).$$
($H^p(A; G)$ is $H^p(A, \varnothing; G)$.)

The following axioms are to be satisfied:

Axiom 1. If $(X, A) \xrightarrow{i} (X, A)$ is the identity map, then $H^p(X, A; G) \xleftarrow{(i^*)_p} H^p(X, A; G)$ is the identity map.

Axiom 2. Given $(X, A) \xrightarrow{h} (Y, B) \xrightarrow{k} (Z, C)$ continuous,

$$((k \circ h)^*)_p = (h^*)_p \circ (k^*)_p.$$

$$\begin{tikzcd}
H^p(X, A; G) & H^p(Y, B; G) \arrow[l, "(h^*)_p"'] & H^p(Z, C; G) \arrow[l, "(k^*)_p"'] \arrow[ll, bend left, "((k\circ h)^*)_p = (h^*)_p \circ (k^*)_p"]
\end{tikzcd}$$

Axiom 3. δ^* is a natural transformation of functors $\mathcal{F}$ and $\mathcal{G}$ (from the category of admissible pairs of topological spaces and continuous maps to the category of abelian groups and group homomorphisms) defined as follows:

$$(X, A) \overset{\mathcal{F}}{\longmapsto} H^q(X, A; G) \text{ and } (X, A) \overset{\mathcal{G}}{\longmapsto} H^{q-1}(A; G).$$

The morphisms transform as follows: given continuous $f : (X, A) \to (Y, B)$,

$$f \overset{\mathcal{F}}{\longmapsto} (f^*)_q \text{ and } f \overset{\mathcal{G}}{\longmapsto} ((f|_A)^*)_{q-1}.$$

If the components of the natural transformation δ^* at (X, A) and (Y, B) are denoted by $\left(\delta^*_{(X,A)}\right)_q$ and $\left(\delta^*_{(Y,B)}\right)_q$, respectively, in dimension q, the following diagram commutes from the naturality of δ^*:

$$\begin{tikzcd} H^q(X,A;G) & H^{q-1}(A;G) \arrow[l, "\left(\delta^*_{(X,A)}\right)_q"'] \\ H^q(Y,B;G) \arrow[u, "(f^*)_q"] & H^{q-1}(B;G) \arrow[l, "\left(\delta^*_{(Y,B)}\right)_q"] \arrow[u, "\left(\left(f|_A\right)^*\right)_{q-1}"'] \end{tikzcd}$$

Axiom 4. Given the inclusion of pairs $(A,\varnothing) \overset{i}{\hookrightarrow} (X,\varnothing)$ and $(X,\varnothing) \overset{j}{\hookrightarrow} (X,A)$, one has the following long exact sequence:

$$\cdots \longleftarrow H^p(A;G) \xleftarrow{(i^*)_p} H^p(X;G) \xleftarrow{(j^*)_p} H^p(X,A;G) \xleftarrow{(\delta^*)_p} H^{p-1}(A;G) \longleftarrow \cdots$$

Axiom 5. If $h,k:(X,A)\to(Y,B)$ are homotopic, then

$$(h^*)_p = (k^*)_p \ , \quad \forall\, p.$$

Axiom 6. If $U \subseteq X$ is open and $\overline{U} \subseteq \operatorname{Int} A$, and if $(X\setminus U, A\setminus U)$ is admissible, then the inclusion $(X\setminus U, A\setminus U) \overset{j}{\hookrightarrow} (X,A)$ induces a cohomology isomorphism

$$H^p(X\setminus U, A\setminus U;G) \xleftarrow{(j^*)_p} H^p(X,A;G).$$

Axiom 7. If P is a one-point space, then

$$H^q(P;G) \cong \begin{cases} 0 & \text{if } q\neq 0, \\ G & \text{if } q=0. \end{cases}$$

The axiom of compact support has no counterpart in cohomology theory.

§4.5 Singular Cohomology Theory

Now we consider singular cohomology theory and show it satisfies the axioms. The **singular cohomology groups** of a topological pair (X,A) with coefficients in the abelian group G are defined by

$$H^q(X,A;G) = H^q(\mathcal{S}(X,A);G), \tag{4.33}$$

where $\mathcal{S}(X,A)$ is the singular chain complex of (X,A). As usual, we delete A from the notation if $A=\varnothing$, and we delete G if it equals the group of integers. The **reduced singular cohomology groups** are defined by

$$\widetilde{H}^q(X;G) = \widetilde{H}^q(\mathcal{S}(X);G), \tag{4.34}$$

relative to the standard augmentation ϵ for the augmented singular chain complex $\{\mathcal{S}(X),\epsilon\}$.

Given a continuous map $h:(X,A)\to(Y,B)$, there is a chain map $(h_\#)_p : S_p(X,A)\to S_p(Y,B)$ (we defined it in 2.3). We customarily denote the dual cochain map by $\left(h^\#\right)_p : \operatorname{Hom}(S_p(X,A);G) \leftarrow \operatorname{Hom}(S_p(Y,B);G)$. It takes cocycles to cocycles and coboundaries to coboundaries, and hence it induces a homomorphism

$$H^p(X,A;G) \xleftarrow{(h^*)_p} H^p(Y,B;G).$$

The same holds in reduced cohomology if A and B are empty, since $(h_\#)_p$ is augmentation preserving. Axiom 1 and Axiom 2 (functorial properties) hold even at the cochain level.

Note that in the following short exact sequence,

$$0 \longrightarrow S_p(A) \longrightarrow S_p(X) \longrightarrow S_p(X,A) \longrightarrow 0,$$

$S_p(X,A)$ is free abelian, so the sequence splits. Therefore, by Theorem 4.2, the dual sequence

$$0 \longleftarrow \operatorname{Hom}\left(S_p\left(A\right), G\right) \longleftarrow \operatorname{Hom}\left(S_p\left(X\right), G\right) \longleftarrow \operatorname{Hom}\left(S_p\left(X,A\right), G\right) \longleftarrow 0,$$

is a short exact sequence. Therefore, by Zig-Zag lemma, one has the following long exact sequence:

$$\begin{tikzcd}
& \cdots & H^{p+1}(X,A;G) \arrow[l] \\
H^p(A;G) \arrow[urr, "(\delta^*)_{p+1}"] & H^p(X;G) \arrow[l] & H^p(X,A;G) \arrow[l] \\
H^{p-1}(A;G) \arrow[urr, "(\delta^*)_p"'] & H^{p-1}(X;G) \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

Now, Zig-Zag lemma assigns to a given short exact sequence of complexes, a long exact sequence of their cohomology groups. This assignment is "natural" as corroborated by *Theorem 5.1.1* of AT2. Given a continuous map $h:(X,A)\to(Y,B)$, one has induced homomorphism of long exact cohomology sequences:

$$\begin{tikzcd}
\cdots & H^p(A;G) \arrow[l] & H^p(X;G) \arrow[l] & H^p(X,A;G) \arrow[l] & H^{p-1}(A;G) \arrow[l, "\left(\delta^*_{(X,A)}\right)_p"'] & \cdots \arrow[l] \\
\cdots & H^p(B;G) \arrow[l] \arrow[u, "\left(\left(h\big|_A\right)^*\right)_q"] & H^p(Y;G) \arrow[l] \arrow[u, "(h^*)_p"] & H^p(Y,B;G) \arrow[l] \arrow[u, "(h^*)_p"] & H^{p-1}(B;G) \arrow[l, "\left(\delta^*_{(Y,B)}\right)_p"] \arrow[u, "\left(\left(h\big|_A\right)^*\right)_{q-1}"'] & \cdots \arrow[l]
\end{tikzcd}$$

This diagram commutes.

Let $h,k:(X,A)\to(Y,B)$ be homotopic. We have seen during the course of the proof of Theorem 2.7 that $h_\#$ and $k_\#$ are chain homotopic by constructing a chain homotopy between them. Therefore, $h^\#$ and $k^\#$ are cochain homotopic, and hence by (4.32),

$$\left(h^*\right)_p = \left(k^*\right)_p \;, \quad \forall\, p,$$

verifying Axiom 5.

To compute the cohomology of a one-point space P, recall that the singular chain complex has the following form: (Theorem 2.3)

$$\cdots \longrightarrow S_{2k}(P) \longrightarrow S_{2k-1}(P) \longrightarrow \cdots \longrightarrow S_1(P) \longrightarrow S_0(P) \longrightarrow 0$$

$$\cdots \xrightarrow{\bar{0}} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{\bar{0}} \cdots \longrightarrow \mathbb{Z} \xrightarrow{\bar{0}} \mathbb{Z} \longrightarrow 0$$

Here, $\bar{0}$ is the zero map that maps everything to $0\in\mathbb{Z}$. Using the fact that $\operatorname{Hom}(\mathbb{Z},G)\cong G$, we get the cochain complex:

$$\cdots \xleftarrow{\bar{0}} G \xleftarrow[j]{\cong} G \xleftarrow{\bar{0}} \cdots \longleftarrow G \xleftarrow{\bar{0}} G \xleftarrow{i} 0$$

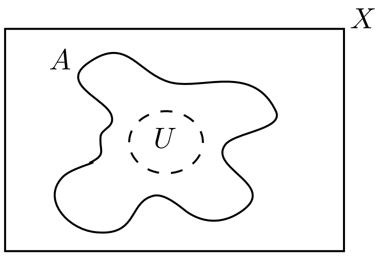
Here we used the fact from Theorem 4.1 that the dual of an isomorphism is also an isomorphism, and the dual of the zero map is also the zero map. One can now easily read off the cohomology groups of the one-point space P from the sequence.

$$H^0(P;G) \cong \frac{\operatorname{Ker}\bar{0}}{\operatorname{im} i} = \operatorname{Ker}\bar{0} = G,$$
$$H^{2k-1}(P;G) \cong \frac{\operatorname{Ker} j}{\operatorname{im}\bar{0}} = 0,$$
$$H^{2k}(P;G) \cong \frac{\operatorname{Ker}\bar{0}}{\operatorname{im} j} = \frac{G}{G} = 0.$$

Hence, Axiom 7 holds in singular cohomology.

Finally, we come to the excision property of singular cohomology. Let $U \subseteq X$ be subset such that $\overline{U} \subseteq \operatorname{Int} A$. The excision map, which is an inclusion map, is given by

$$j : (X \setminus U, A \setminus U) \hookrightarrow (X, A).$$



If we had showed that $j_\# : \mathcal{S}(X \setminus U, A \setminus U) \to \mathcal{S}(X, A)$ is a chain equivalence, then we would immediately obtain a corresponsing cochain equivalence $j^\#$. It would then follow by Theorem 4.5 that

$$(j^*)_p : H^p(\mathcal{S}(X \setminus U, A \setminus U); G) \leftarrow H^p(\mathcal{S}(X, A); G)$$

is an isomorphism. But instead of establishing a chain equivalence, we only proved a weaker ersult establishing only the isomorphism $(j_*)_p$ of the homology groups

$$(j_*)_p : H_p(X \setminus U, A \setminus U) \xrightarrow{\cong} H_p(X, A).$$

We have to use thiss isomorphism of homology groups to prove the isomorphism of the pertaining cohomology groups. For this we need a result that we will be stating now, and we shall prove it in the next section.

> Let $\mathcal{C}$ and $\mathcal{D}$ be free chain complexes (the abelian groups involved are all free); let $\phi : \mathcal{C} \to \mathcal{D}$be a chain map that induces homology isomorphisms in all dimensions. Then ϕ induces a cohomology isomorphism in all dimensions, for all coefficient groups G.

We apply the above result to the inclusion map $j : (X \setminus U, A \setminus U) \hookrightarrow (X, A)$, with $U \subseteq A \subseteq X$ and $\overline{U} \subseteq \operatorname{Int} A$. $j_\# : \mathcal{S}(X \setminus U, A \setminus U) \to \mathcal{S}(X, A)$ is the underlying chain map that induces an isomorphism in homology in all dimensions. Furthermore, the chain groups are all free. Therefore, j induces an isomorphism in cohomology

$$(j^*)_p : H^p(\mathcal{S}(X \setminus U, A \setminus U); G) \leftarrow H^p(\mathcal{S}(X, A); G).$$

Note that singular cohomology, like singular homology, satisfies an excision property slightly stronger than that stated in the axiom. One needs to have $\overline{U} \subseteq \operatorname{Int} A$, but one does not need U to be open, in order for excision to hold.

§4.6 The Cohomology of Free Chain Complexes

As promised earlier, we are going to prove a couple of theorems now. The first one says that for free chain complexes $\mathcal{C}$ and $\mathcal{D}$, any homomorphism $H_p(\mathcal{C}) \to H_p(\mathcal{D})$ of homology groups is induces by a chain map $\phi : \mathcal{C} \to \mathcal{D}$. The second one states that if a chain map $\phi : \mathcal{C} \to \mathcal{D}$ induces an isomorphism in homology, it induces an isomorphism in cohomology as well.

Definition 4.3. A short exact sequence of abelian groups $0 \to A \to B \to C \to 0$, where A and B are free, is called a **free resolution** of C. Any abelian group has a free resolution: take B to be the free abelian group generated by the elements of C, which we denote by $\widetilde{C}$, and the surjective map $B \to C$ is taken to be the canonical projection $\pi : \widetilde{C} \to C$ defined by

$$ng \mapsto \begin{cases} \underbrace{g * g * \cdots * g}_{n\text{-times}} & \text{if } n > 0, \\ \underbrace{g^{-1} * g^{-1} * \cdots * g^{-1}}_{-n\text{-times}} & \text{if } n < 0, \end{cases}$$

where $*$ is the group operation in C. For free resolution of C, we choose $A = \operatorname{Ker} \pi$. Then

$$0 \longrightarrow \operatorname{Ker} i \xhookrightarrow{i} \widetilde{C} \xrightarrow{\pi} C \longrightarrow 0$$

is a short exact sequence. It is called the **canonical free resolution** of C.

Free resolutions have the following useful property.

Proposition 4.6
In the following diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\phi"] & B \arrow[r, "\psi"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "\phi'"] & B' \arrow[r, "\psi'"] & C' \arrow[r] & 0 \end{tikzcd}$$

suppose the horizontal sequences are exact, and A and B are free. Then there exist homomorphism $\alpha : A \to A'$ and $\beta : B \to B'$ such that the following diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\phi"] \arrow[d, "\alpha"'] & B \arrow[r, "\psi"] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "\phi'"'] & B' \arrow[r, "\psi'"'] & C' \arrow[r] & 0 \end{tikzcd}$$

Proof. Let us define $\beta : B \to B'$ first. Choose a basis for B. If $b_i \in B$ is a basis element, we define $\beta(b_i)$ to be **any** element from the set $(\psi')^{-1}(\gamma(\psi(b_i)))$. This set is nonempty since ψ' is surjective. By this construction, $\psi'(\beta(b_i)) = \gamma(\psi(b_i))$ for basis elements b. Now, if we take any $b \in B$, b can be written as a finite $\mathbb{Z}$-linear combination of the basis elements $\{b_i\}_i$, i.e. $b = \sum_j n_j b_j$. Then

$$\begin{aligned} \psi'(\beta(b)) = \psi'\left(\beta\left(\sum_j n_j b_j\right)\right) &= \sum_j n_j \psi'(\beta(b_j)) \\ &= \sum_j n_j \gamma(\psi(b_j)) = \gamma\left(\psi\left(\sum_j n_j b_j\right)\right) \\ &= \gamma(\psi(b)) . \end{aligned} \tag{4.35}$$

Therefore, the right hand square of the following diagram commutes.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\phi"] \arrow[d, dashed, "\alpha"'] & B \arrow[r, "\psi"] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "\phi'"'] & B' \arrow[r, "\psi'"'] & C' \arrow[r] & 0 \end{tikzcd}$$

Now, we choose $b \in \operatorname{im} \phi = \operatorname{Ker} \psi$. So $\psi(b) = 0$. Then by (4.35),

$$\psi'(\beta(b)) = \gamma(\psi(b)) = 0. \tag{4.36}$$

So $\beta(b) \in \operatorname{Ker} \psi' = \operatorname{im} \phi'$. Hence, β takes $\operatorname{im} \phi$ to $\operatorname{im} \phi'$. Now, for a given $a \in A$, we define

$$\alpha(a) = \left(\phi'\right)^{-1}(\beta(\phi(a))). \tag{4.37}$$

This is well-defined since ϕ' is injective; and $\phi(a) \in \operatorname{im} \phi$, so $\beta(\phi(a)) \in \operatorname{im} \phi'$. After defining α this way, we have

$$\phi'(\alpha(a)) = \beta(\phi(a)). \tag{4.38}$$

Therefore, the left hand square in the diagram above commutes. ■

Theorem 4.7

Let $\mathcal{C}$ and $\mathcal{C}'$ be free chain complexes. If $\gamma_p : H_p(\mathcal{C}) \to H_p(\mathcal{C}')$ is a homomorphism defined for all p, then there is a chain map $\phi : \mathcal{C} \to \mathcal{C}'$ that induces γ. Indeed, if $\beta : Z_p \to Z'_p$ is any homomorphism between cycle groups inducing γ, then β extends to a chain map ϕ.

Proof. Let Z_p and B_p denote the group of p-cycles and p-boundaries, respectively, in the chain complex $\mathcal{C}$. Similarly, let Z'_p and B'_p denote the group of p-cycles and p-boundaries, respectively, in the chain complex $\mathcal{C}'$. We have the following diagram with horizontal sequences being exact:

$$\begin{tikzcd}
0 \arrow[r] & B_p \arrow[r, "i_p"] & Z_p \arrow[r, "\pi_p"] & H_p(\mathcal{C}) = \frac{Z_p}{B_p} \arrow[r] \arrow[d, "\gamma_p"] & 0 \\
0 \arrow[r] & B'_p \arrow[r, "i'_p"'] & Z'_p \arrow[r, "\pi'_p"'] & H_p(\mathcal{C}') = \frac{Z'_p}{B'_p} \arrow[r] & 0
\end{tikzcd}$$

Z_p and B_p are subgroups of a free abelian group C_p, so they are free abelian as well. Therefore, by Proposition 4.6, there exist group homomorphisms $\alpha_p : B_p \to B'_p$ and $\beta_p : Z_p \to Z'_p$ such that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & B_p \arrow[r, "i_p"] \arrow[d, "\alpha_p"'] & Z_p \arrow[r, "\pi_p"] \arrow[d, "\beta_p"'] & H_p(\mathcal{C}) = \frac{Z_p}{B_p} \arrow[r] \arrow[d, "\gamma_p"] & 0 \\
0 \arrow[r] & B'_p \arrow[r, "i_p"'] & Z'_p \arrow[r, "\pi_p"'] & H_p(\mathcal{C}') = \frac{Z'_p}{B'_p} \arrow[r] & 0
\end{tikzcd} \tag{4.39}$$

We seek to extend β_p to group homomorphisms $\phi_p : C_p \to C'_p$ between the respective chain groups. For this purpose, consider the following diagram with rows being short exact:

$$\begin{tikzcd}
0 \arrow[r] & Z_p \arrow[r, "j_p"] \arrow[d, "\beta_p"'] & C_p \arrow[r, "\widetilde{\partial}_p"] \arrow[d, dashed, "\phi_p"'] & B_{p-1} \arrow[r] \arrow[d, "\alpha_{p-1}"] & 0 \\
0 \arrow[r] & Z'_p \arrow[r, "j'_p"'] & C'_p \arrow[r, "\widetilde{\partial}'_p"'] & B'_{p-1} \arrow[r] & 0
\end{tikzcd}$$

Here $j_p : Z_p \to C_p$ is the inclusion, and $\widetilde{\partial}_p : C_p \to B_{p-1}$ is the surjective group homomorphism restricting the codomain of ∂_p to its range B_p.

$$\begin{tikzcd}
C_p \arrow[r, "\widetilde{\partial}_p"] & B_{p-1} \arrow[r, hook, "i_{p-1}"] & Z_{p-1} \arrow[r, hook, "j_{p-1}"] & C_{p-1}
\end{tikzcd}$$

$$\partial_p = j_{p-1} \circ i_{p-1} \circ \widetilde{\partial}_p. \tag{4.40}$$

Now, α_p and β_p have been constructed before. Since B_{p-1} and B'_{p-1} are free, the two horizontal sequences split. Choose subgroups U_p and U'_p of C_p and C'_p, respectively, such that

$$C_p = Z_p \oplus U_p \text{ and } C'_p = Z'_p \oplus U'_p. \tag{4.41}$$

Now we claim that $\widetilde{\partial}_p\big|_{U_p} : U_p \to B_{p-1}$ is an isomorphism. It's a restriction of a homomorphism to a subgroup of the domain, so it's a homomorphism. To see it's surjective, take any $b_{p-1} \in B_{p-1}$. Since $\widetilde{\partial}_p$ is surjective, there exists $c_p \in C_p$ such that $\widetilde{\partial}_p (c_p) = b_{p-1}$. Since $C_p = Z_p \oplus U_p$, c_p can be **uniquely** written as $z_p + u_p$ for some $z_p \in Z_p$ and $u_p \in U_p$. Now,

$$\widetilde{\partial}_p\big|_{U_p} (u_p) = \widetilde{\partial}_p (c_p - z_p) = \widetilde{\partial}_p (c_p) - \widetilde{\partial}_p (z_p) = b_{p-1} - 0 = b_{p-1}. \tag{4.42}$$

Therefore, $\widetilde{\partial}_p\big|_{U_p}$ is surjective. Now, for $u_p \in \operatorname{Ker} \widetilde{\partial}_p\big|_{U_p} \subseteq U_p$,

$$0 = \widetilde{\partial}_p\big|_{U_p} (u_p) = \widetilde{\partial}_p (u_p), \tag{4.43}$$

so $u_p \in \operatorname{Ker} \widetilde{\partial}_p = Z_p$. Since $C_p = Z_p \oplus U_p$, $Z_p \cap U_p$ is trivial. So $u_p = 0$, proving that $\operatorname{Ker} \widetilde{\partial}_p\big|_{U_p} = 0$. Hence, $\widetilde{\partial}_p\big|_{U_p} : U_p \to B_{p-1}$ is an isomorphism. Similarly, $\widetilde{\partial}'_p\big|_{U'_p} : U'_p \to B''_{p-1}$ is also an isomorphism.

Now we define $\phi_p : C_p \to C'_p$ as follows:

$$\phi_p\big|_{Z_p} = \beta_p \text{ and } \phi_p\big|_{U_p} = \left(\widetilde{\partial}'_p\big|_{U'_p}\right)^{-1} \circ \alpha_{p-1} \circ \widetilde{\partial}_p\big|_{U_p}. \tag{4.44}$$

In other words, for $c_p = z_p + u_p \in C_p$,

$$\phi_p (c_p) = \beta_p (z_p) + \left(\widetilde{\partial}'_p\big|_{U'_p}\right)^{-1} \left[\alpha_{p-1} \left(\widetilde{\partial}_p (u_p)\right)\right]. \tag{4.45}$$

Now, consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z_p & \xrightarrow{j_p} & C_p & \xrightarrow{\widetilde{\partial}_p} & B_{p-1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \beta_p} & & \downarrow{\scriptstyle \phi_p} & & \downarrow{\scriptstyle \alpha_{p-1}} & & \\ 0 & \longrightarrow & Z'_p & \xrightarrow[j'_p]{} & C'_p & \xrightarrow[\widetilde{\partial}'_p]{} & B'_{p-1} & \longrightarrow & 0 \end{array} \tag{4.46}$$

The first square commutes, because for $z_p \in Z_p$,

$$\phi_p (j_p (z_p)) = \phi_p (z_p) = \beta (z_p) = j'_p (\beta (z_p)),$$

so $\phi_p \circ j_p = j'_p \circ \beta_p$. The second square also commutes, because if we take $z_p + u_p \in C_p$,

$$z_p + u_p \overset{\widetilde{\partial}_p}{\longmapsto} \partial_p u_p \overset{\alpha_{p-1}}{\longmapsto} \alpha_{p-1} (\partial_p u_p),$$

$$z_p + u_p \overset{\phi_p}{\longmapsto} \beta_p (z_p) + \left(\widetilde{\partial}'_p\big|_{U'_p}\right)^{-1} \left[\alpha_{p-1} \left(\widetilde{\partial}_p (u_p)\right)\right] \overset{\widetilde{\partial}'_p}{\longmapsto} \alpha_{p-1} \left(\widetilde{\partial}_p (u_p)\right),$$

so $\alpha_{p-1} \circ \widetilde{\partial}_p = \widetilde{\partial}'_p \circ \phi_p$.

So, we have define $\phi_p : C_p \to C'_p$ such that the diagram above commutes. Now, consider the following diagram.

$$\begin{array}{ccccccc} C_p & \xrightarrow{\widetilde{\partial}_p} & B_{p-1} & \xrightarrow{i_{p-1}} & Z_{p-1} & \xrightarrow{j_{p-1}} & C_{p-1} \\ \downarrow{\scriptstyle \phi_p} & & \downarrow{\scriptstyle \alpha_{p-1}} & & \downarrow{\scriptstyle \beta_{p-1}} & & \downarrow{\scriptstyle \phi_{p-1}} \\ C'_p & \xrightarrow[\widetilde{\partial}'_p]{} & B'_{p-1} & \xrightarrow[i'_{p-1}]{} & Z'_{p-1} & \xrightarrow[j'_{p-1}]{} & C'_{p-1} \end{array}$$

The middle square commutes, as it is precisely the first square of (4.39) for $p-1$. The left hand square is the second square of (4.46), and the right hand square is the first square of (4.46) for $p-1$. Therefore, the diagram above commutes. Therefore, we have

$$\phi_{p-1} \circ j_{p-1} \circ i_{p-1} \circ \widetilde{\partial}_p = j'_{p-1} \circ i'_{p-1} \circ \widetilde{\partial}'_p \circ \phi_p.$$

In other words,

$$\phi_{p-1} \circ \partial_p = \partial'_p \circ \phi_p. \tag{4.47}$$

Hence, the family of maps $\{\phi_p\}$ denoted by $\phi : \mathcal{C} \to \mathcal{C}'$ is a chain map. Now, it only remains to show that γ is induced by ϕ, i.e. $\gamma_p = (\phi_*)_p$. Given a homology class $\{z_p\} = z_p + B_p \in H_p(\mathcal{C})$, with $z_p \in Z_p$,

$$\begin{aligned}(\phi_*)_p (z_p + B_p) &= \phi_p (z_p) + B'_p = \beta_p (z_p) + B'_p \\ &= \pi'_p (\beta_p (z_p)) = \gamma_p (\pi_p (z_p)) \\ &= \gamma_p (z_p + B_p), \end{aligned} \tag{4.48}$$

by the commutativity of the second square of (4.39). Therefore, $(\phi_*)_p = \gamma_p$, i.e. ϕ is our desired chain map that induces γ. ∎

Remark 4.3. Here we used the fact that any subgroup of a free abelian group is also free abelian. The proof can be found here. Since abelian groups are $\mathbb{Z}$-modules, we can rephrase the statement as follows:

> Any submodule of a free $\mathbb{Z}$-module is also a free $\mathbb{Z}$-module.

In fact, a general result is true. The result still holds if $\mathbb{Z}$ is replaced by a principal ideal domain R.

> Let R be a principal ideal domain. If M is a free R-module and N is a submodule of R, then N is also a free R-module.

Corollary 4.8

Suppose $\{\mathcal{C}, \epsilon\}$ and $\{\mathcal{C}', \epsilon'\}$ are free augmented chain complexes. If $\gamma_p : \widetilde{H}_p(\mathcal{C}) \to \widetilde{H}_p(\mathcal{C}')$ is a group homomorphism defined for all p, then γ_p is induced by an augmentation preserving chain map $\phi : \mathcal{C} \to \mathcal{C}'$.

We now prove our basic theorems. We begin by considering a special case.

Lemma 4.9

Let $0 \to \mathcal{C} \xrightarrow{\phi} \mathcal{D} \to \mathcal{E} \to 0$ be a short exact sequence of free chain complexes. If ϕ induces homology isomorphisms in all dimensions, i.e. $(\phi_*)_p : H_p(\mathcal{C}) \to H_p(\mathcal{D})$ are isomorphisms for all p, then it induces cohomology isomorphisms as well, denote by $(\phi^*)_p : H^p(\mathcal{C}; G) \leftarrow H^p(\mathcal{D}; G)$.

Proof. Let ψ be the chain map $\mathcal{D} \to \mathcal{E}$ in the given short exact sequence of free chain complexes. Consider the long exact sequence associated with the given short exact sequence of chain complexes:

$$\cdots \longrightarrow H_p(\mathcal{C}) \xrightarrow{(\phi_*)_p} H_p(\mathcal{D}) \xrightarrow{(\psi_*)_p} H_p(\mathcal{E}) \xrightarrow{(\partial_*)_p} H_{p-1}(\mathcal{C}) \xrightarrow{(\phi_*)_{p-1}} H_{p-1}(\mathcal{D}) \longrightarrow \cdots \tag{4.49}$$

Exactness at $H_{p-1}(\mathcal{C})$ tells us that

$$\operatorname{im}(\partial_*)_p = \operatorname{Ker}(\phi_*)_{p-1} = 0,$$

since $(\phi_*)_{p-1}$ is an isomorphism. Now, exactness at $H_p(\mathcal{D})$ demands that

$$\operatorname{Ker}(\psi_*)_p = \operatorname{im}(\phi_*)_p = H_p(\mathcal{D}),$$

since $(\phi_*)_p$ is an isomorphism. Therefore, the image of $(\psi_*)_p$ is only $0 \in H_p(\mathcal{E})$. Now, exactness at $H_p(\mathcal{E})$ tells us that

$$\operatorname{Ker}(\partial_*)_p = \operatorname{im}(\psi_*)_p = 0.$$

But we found earlier that $\operatorname{im}(\partial_*)_p = 0$. This is possible only when $H_p(\mathcal{E}) = 0$ for all p.

The dual sequence of (4.49) is

$$\cdots \longleftarrow H^{p+1}(\mathcal{E};G) \xleftarrow{(\delta^*)_{p+1}} H^p(\mathcal{C};G) \xleftarrow{(\phi^*)_p} H^p(\mathcal{D};G) \xleftarrow{(\psi^*)_p} H^p(\mathcal{E};G) \longleftarrow \cdots \tag{4.50}$$

We shall now prove that $H^p(\mathcal{E};G) = 0$ for all p. Let $B_p \subseteq Z_p \subseteq E_p$ denote the group of p-boundaries, the group of p-cycles and the group of p-chains of the complex $\mathcal{E}$, respectively. Consider the following short exact sequence:

$$0 \longrightarrow Z_p \xhookrightarrow{j_p} E_p \xrightarrow{\widetilde{\partial}_p} B_{p-1} \longrightarrow 0$$

This sequence splits, as B_{p-1} is free. Furthermore, since $H_p(\mathcal{E}) = 0$, $Z_p = B_p$ for all p. Therefore, we can write

$$E_p = Z_p \oplus U_p = B_p \oplus U_p, \tag{4.51}$$

for some subgroup U_p of E_p. So we have the following split short exact sequence:

$$0 \longrightarrow B_p \xhookrightarrow{j_p} B_p \oplus U_p \xrightarrow{\widetilde{\partial}_p} B_{p-1} \longrightarrow 0 \tag{4.52}$$

While proving Theorem 4.7, we proved that $\widetilde{\partial}_p\big|_{U_p} : U_p \to B_{p-1}$ is an isomorphism. Also $\widetilde{\partial}_p$ maps B_p to 0. By Proposition 4.3,

$$\operatorname{Hom}(E_p, G) \cong \operatorname{Hom}(B_p, G) \oplus \operatorname{Hom}(U_p, G).$$

Now, take the dual of (4.52) and denote the dual of $\widetilde{\partial}_p$ by $\widetilde{\delta}_p$.

$$0 \longleftarrow \operatorname{Hom}(B_p, G) \xleftarrow{\widetilde{j}_p} \operatorname{Hom}(B_p, G) \oplus \operatorname{Hom}(U_p, G) \xleftarrow{\widetilde{\delta}_p} \operatorname{Hom}(B_{p-1}, G) \longleftarrow 0 \tag{4.53}$$

$\widetilde{\delta}_p$ takes $\operatorname{Hom}(B_{p-1}, G)$ isomorphically to $\operatorname{Hom}(U_p, G)$, since $\widetilde{\delta}_p$ is the dual of $\widetilde{\partial}_p$ which takes U_p isomorphically to B_{p-1} and B_p to 0. (4.52) is split exact, so its dual (4.53) is also exact. A generic element of $\operatorname{im}\widetilde{\delta}_p$ is of the form $(0, f)$ with $f \in \operatorname{Hom}(U_p, G)$. By exactness, all such elements are mapped to the 0-homomorphism in $\operatorname{Hom}(B_p, G)$. In other words, $\widetilde{j}_p$ carries $\operatorname{Hom}(U_p, G)$ to $0 \in \operatorname{Hom}(B_p, G)$. Now consider the following diagram:

$$\begin{tikzcd}
\operatorname{Hom}(B_p, G) \arrow[d, "\widetilde{\delta}_{p+1}"'] & \operatorname{Hom}(B_p, G) \oplus \operatorname{Hom}(U_p, G) \arrow[l, "\widetilde{j}_p"'] & \operatorname{Hom}(B_{p-1}, G) \arrow[l, "\widetilde{\delta}_p"'] \\
\operatorname{Hom}(B_{p+1}, G) \oplus \operatorname{Hom}(U_{p+1}, G) & & \operatorname{Hom}(B_{p-1}, G) \oplus \operatorname{Hom}(U_{p-1}, G) \arrow[u, "\widetilde{j}_{p-1}"']
\end{tikzcd}$$

Now, $\delta_{p+1} = \widetilde{\delta}_{p+1} \circ \widetilde{j}_p$. Since $\widetilde{\delta}_{p+1}$ maps $\operatorname{Hom}(B_p, G)$ isomorphically to $\operatorname{Hom}(U_{p+1}, G)$, $\operatorname{Ker}\widetilde{\delta}_{p+1} = 0$. Therefore,

$$\operatorname{Ker}\delta_{p+1} = \widetilde{j}_p^{-1}\left(\operatorname{Ker}\widetilde{\delta}_{p+1}\right) = \widetilde{j}_p^{-1}(0) = \operatorname{Hom}(U_p, G). \tag{4.54}$$

Now, $\delta_p = \widetilde{\delta}_p \circ \widetilde{j}_{p-1}$. Since $\widetilde{j}_{p-1}$ is surjective, $\operatorname{im}\widetilde{j}_{p-1} = \operatorname{Hom}(B_p, G)$. Therefore,

$$\operatorname{im}\delta_p = \widetilde{\delta}_p\left(\operatorname{im}\widetilde{j}_{p-1}\right) = \widetilde{\delta}_p(\operatorname{Hom}(B_p, G)) = \operatorname{Hom}(U_p, G). \tag{4.55}$$

$\operatorname{Ker}\delta_{p+1} = \operatorname{im}\delta_p$. Therefore,

$$H^p(\mathcal{E};G) = \frac{\operatorname{Ker}\delta_{p+1}}{\operatorname{im}\delta_p} = 0. \tag{4.56}$$

Since $H^p(\mathcal{E};G) = 0$ for all p, from (4.50), we get the following exact sequence

$$0 \longleftarrow H^p(\mathcal{C};G) \xleftarrow{(\phi^*)_p} H^p(\mathcal{D};G) \longleftarrow 0$$

Exactness of this sequence implies that $H^p(\mathcal{C};G) \xleftarrow{(\phi^*)_p} H^p(\mathcal{D};G)$ is an isomorphism. ■

Lemma 4.10

Let $\mathcal{C}$ and $\mathcal{D}$ be free chain complexes; let $\phi : \mathcal{C} \to \mathcal{D}$ be a chain map. Then there is a free chain complex $\mathcal{D}'$ and injective chain maps $i : \mathcal{C} \to \mathcal{D}'$ and $j : \mathcal{D} \to \mathcal{D}'$ such that j induces homology isomorphisms $(j_*)_p : H_p(\mathcal{D}) \to H_p(\mathcal{D}')$ in all dimensions; and the following diagram commutes up to a chain homotopy:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\phi"] \arrow[dr, "i"'] & \mathcal{D} \arrow[d, "j"] \\
 & \mathcal{D}'
\end{tikzcd}$$

i.e. $j \circ \phi$ and i are chain homotopic.

Proof. Let ∂_p^C and ∂_p^D be boundary operators of the chain complexes $\mathcal{C}$ and $\mathcal{D}$, respectively. We define the chain complex $\mathcal{D}' = \left\{D'_p, \partial'_p\right\}$ as follows:

$$D'_p = C_p \oplus D_p \oplus C_{p-1}; \tag{4.57}$$
$$\partial'_p\left(c_p, 0, 0\right) = \left(\partial_p^C c_p, 0, 0\right), \tag{4.58}$$
$$\partial'_p\left(0, d_p, 0\right) = \left(0, \partial_p^D d_p, 0\right), \tag{4.59}$$
$$\partial'_p\left(0, 0, c_{p-1}\right) = \left(-c_{p-1}, \phi_{p-1}\left(c_{p-1}\right), -\partial_{p-1}^C c_{p-1}\right). \tag{4.60}$$

Let us now check that $\partial'_{p-1} \circ \partial'_p = 0$.

$$\begin{aligned}
\left(\partial'_{p-1} \circ \partial'_p\right)\left(c_p, 0, 0\right) &= \partial'_{p-1}\left(\partial_p^C c_p, 0, 0\right) = \left(\partial_{p-1}^C\left(\partial_p^C c_p\right), 0, 0\right) = (0,0,0). \\
\left(\partial'_{p-1} \circ \partial'_p\right)\left(0, d_p, 0\right) &= \partial'_{p-1}\left(0, \partial_p^D d_p, 0\right) = \left(0, \partial_{p-1}^D\left(\partial_p^D d_p\right), 0\right) = (0,0,0). \\
\left(\partial'_{p-1} \circ \partial'_p\right)\left(0, 0, c_{p-1}\right) &= \partial'_{p-1}\left(-c_{p-1}, \phi_p\left(c_{p-1}\right), -\partial_{p-1}^C c_{p-1}\right) \\
&= \partial'_{p-1}\left(-c_{p-1}, 0, 0\right) + \partial'_{p-1}\left(0, \phi_{p-1}\left(c_{p-1}\right), 0\right) + \partial'_{p-1}\left(0, 0, -\partial_{p-1}^C c_{p-1}\right) \\
&= \left(-\partial_{p-1}^C c_{p-1}, 0, 0\right) + \left(0, \partial_{p-1}^D \phi_{p-1}\left(c_{p-1}\right), 0\right) + \partial'_{p-1}\left(0, 0, -\partial_{p-1}^C c_{p-1}\right) \\
&= (0,0,0).
\end{aligned}$$

Now,
$$\partial'_{p-1}\left(0, 0, -\partial_{p-1}^C c_{p-1}\right) = \left(\partial_{p-1}^C c_{p-1}, \phi_{p-2}\left(-\partial_{p-1}^C c_{p-1}\right), \partial_{p-2}^C\left(\partial_{p-1}^C c_{p-1}\right)\right).$$
Since $\phi_{p-2}\left(-\partial_{p-1}^C c_{p-1}\right) = -\partial_{p-1}^D \phi_{p-1}\left(c_{p-1}\right)$, all the terms of $\left(\partial'_{p-1} \circ \partial'_p\right)\left(0, 0, c_{p-1}\right)$ gets cancelled. Hence,
$$\left(\partial'_{p-1} \circ \partial'_p\right)\left(0, 0, c_{p-1}\right) = (0,0,0).$$
Therefore, $\partial'_{p-1} \circ \partial'_p = 0$. So $\mathcal{D}' = \left\{D'_p, \partial'_p\right\}$ is a chain complex.

We define the injective chain maps $i : \mathcal{C} \to \mathcal{D}'$ and $j : \mathcal{D} \to \mathcal{D}'$ to be just inclusions:

$$i_p : C_p \hookrightarrow D'_p = C_p \oplus D_p \oplus C_{p-1}, \ c_p \mapsto \left(c_p, 0, 0\right),$$

$$j_p : D_p \hookrightarrow D'_p = C_p \oplus D_p \oplus C_{p-1}, \ d_p \mapsto \left(0, d_p, 0\right).$$

Now we shall prove that $j \circ \phi$ and i are chain homotopic. For that purpose, we shall construct a chain homotopy $F_p : C_p \to D'_{p+1} = C_{p+1} \oplus D_{p+1} \oplus C_p$ defined by

$$F_p\left(c_p\right) = \left(0, 0, c_p\right). \tag{4.61}$$

$$\begin{tikzcd}
& D'_{p+1} \arrow[d, "\partial'_{p+1}"] \\
C_p \arrow[ur, "F_p"] \arrow[r, shift left, "j_p\circ\phi_p"] \arrow[r, shift right, "i_p"'] \arrow[d, "\partial_p^C"'] & D'_p \\
C_{p-1} \arrow[ur, "F_{p-1}"'] &
\end{tikzcd}$$

For $c_p \in C_p$,

$$\begin{aligned}
\left(\partial'_{p+1} \circ F_p\right)(c_p) + \left(F_{p-1} \circ \partial_p^C\right)(c_p) &= \partial'_{p+1}(0,0,c_p) + F_{p-1}\left(\partial_p^C c_p\right) \\
&= \left(-c_p, \phi_p(c_p), -\partial_p^C c_p\right) + \left(0,0,\partial_p^C c_p\right) \\
&= (-c_p, 0, 0) + (0, \phi_p(c_p), 0) \\
&= -i_p(c_p) + j_p(\phi_p(c_p))
\end{aligned}$$

Therefore,

$$\partial'_{p+1} \circ F_p + F_{p-1} \circ \partial_p^C = j_p \circ \phi_p - i_p. \tag{4.62}$$

So F is indeed a chain homotopy between chain maps $j \circ \phi$ and i.

Now, it only remains to show that j induces homology isomorphisms in all dimensions. The following is a short exact sequence of chain complexes:

$$0 \longrightarrow \mathcal{D} \xrightarrow{\ j\ } \mathcal{D}' \longrightarrow \mathcal{D}'/\mathcal{D} \longrightarrow 0.$$

By Zig-Zag lemma, this short exact sequence gives rise to a long exact homology sequence:

$$\cdots \longrightarrow H_{p+1}(\mathcal{D}'/\mathcal{D}) \longrightarrow H_p(\mathcal{D}) \xrightarrow{(j_*)_p} H_p(\mathcal{D}') \longrightarrow H_p(\mathcal{D}'/\mathcal{D}) \longrightarrow \cdots \tag{4.63}$$

We shall now prove that $H_p(\mathcal{D}'/\mathcal{D})$ vanishes for all p. The p-th chain group of $\mathcal{D}'/\mathcal{D}$ is isomorphic to $C_p \oplus C_{p-1}$. Each element of the p-th chain group of $\mathcal{D}'/\mathcal{D}$ is of the form $(c_p, 0, c_{p-1}) + D_p$, which we can identify with (c_p, c_{p-1}) for $c_p \in C_p$ and $c_{p-1} \in C_{p-1}$.

Let ∂''_p be the boundary operator on $\mathcal{D}'/\mathcal{D}$. Then

$$\begin{aligned}
\partial''_p(c_p, 0) &= \partial''_p\left[(c_p,0,0) + D_p\right] = \partial'_p(c_p,0,0) + D_{p-1} \\
&= \left(\partial_p^C c_p, 0, 0\right) + D_{p-1} \\
&= \left(\partial_p^C c_p, 0\right). \\
\partial''_p(0, c_{p-1}) &= \partial''_p\left[(0,0,c_{p-1}) + D_p\right] = \partial'_p(0,0,c_{p-1}) + D_{p-1} \\
&= \left(-c_{p-1}, \phi_{p-1}(c_{p-1}), -\partial_{p-1}^C c_{p-1}\right) + D_{p-1} \\
&= \left(-c_{p-1}, 0, -\partial_{p-1}^C c_{p-1}\right) + D_{p-1} = \left(-c_{p-1}, -\partial_{p-1}^C c_{p-1}\right).
\end{aligned}$$

Therefore,

$$\partial''_p(c_p, c_{p-1}) = \left(\partial_p^C c_p - c_{p-1}, -\partial_{p-1}^C c_{p-1}\right). \tag{4.64}$$

Let $(c_p, c_{p-1}) \in Z^p(\mathcal{D}'/\mathcal{D})$ be a p-cycle. Then $\partial''_p(c_p, c_{p-1}) = (0,0)$ gives us $\partial_p^C c_p = c_{p-1}$. Now,

$$-\partial''_{p+1}(0, c_p) = -\left(-c_p, -\partial_p^C c_p\right) = \left(c_p, \partial_p^C c_p\right) = (c_p, c_{p-1}). \tag{4.65}$$

Therefore, $(c_p, c_{p-1}) \in B^p(\mathcal{D}'/\mathcal{D})$. Hence, every p-cycle of $\mathcal{D}'/\mathcal{D}$ bounds. As a result, $H_p(\mathcal{D}'/\mathcal{D}) = 0$.

Therefore, we get the following exact sequence from (4.63):

$$0 = H_{p+1}(\mathcal{D}'/\mathcal{D}) \longrightarrow H_p(\mathcal{D}) \xrightarrow{(j_*)_p} H_p(\mathcal{D}') \longrightarrow H_p(\mathcal{D}'/\mathcal{D}) = 0.$$

Exactness of this sequence then implies that $(j_*)_p$ is an isomorphism. ■

Theorem 4.11

Let $\mathcal{C}$ and $\mathcal{D}$ be free chain complexes; let $\phi : \mathcal{C} \to \mathcal{D}$ be a chain map that induces homology isomorphisms in all dimensions. Then ϕ induces a cohomology isomorphism in all dimensions.

Proof. By Lemma 4.10, there exists a free chain complex $\mathcal{D}$ and injective chain maps $i : \mathcal{C} \to \mathcal{D}'$, $j : \mathcal{D} \to \mathcal{D}'$ such that $j \circ \phi$ and i are chain homotopic. We then have the following exact sequence of free chain complexes:

$$0 \longrightarrow \mathcal{C} \xrightarrow{\ i\ } \mathcal{D}' \longrightarrow \mathcal{D}'/\mathcal{C} \longrightarrow 0 \tag{4.66}$$

$$0 \longrightarrow \mathcal{D} \xrightarrow{\ j\ } \mathcal{D}' \longrightarrow \mathcal{D}'/\mathcal{D} \longrightarrow 0 \tag{4.67}$$

Lemma 4.10 also says that $(j_*)_p : H_p(\mathcal{D}) \to H_p(\mathcal{D}')$ is a homology isomorphism for all p. By hypothesis, $(\phi_*)_p : H_p(\mathcal{C}) \to H_p(\mathcal{D})$ is a homology isomorphism for all p. Since $j \circ \phi$ and i are chain homotopic,

$$(i_*)_p = ((j \circ \phi)_*)_p = (j_*)_p \circ (\phi_*)_p \,.$$

So $(i_*)_p : H_p(\mathcal{C}) \to H_p(\mathcal{D}')$ is a homology isomorphism for all p. Now, if we apply Lemma 4.9 to the short exact sequence of free chain complexes given by (4.66) and (4.67), we get

$$(i^*)_p : H^p(\mathcal{C}; G) \leftarrow H^p\left(\mathcal{D}'; G\right) \text{ and } (j^*)_p : H^p(\mathcal{D}; G) \leftarrow H^p\left(\mathcal{D}'; G\right)$$

are cohomology isomorphisms for all p. Since i and $j \circ \phi$ are chain homotopic,

$$(i^*)_p = ((j \circ \phi)^*)_p = (\phi^*)_p \circ (j^*)_p \,. \tag{4.68}$$

Since $(i^*)_p$ and $(j^*)_p$ are both isomorphisms, it follows from (4.68) that $(\phi^*)_p : H^p(\mathcal{C}; G) \leftarrow H^p(\mathcal{D}; G)$ is also a cohomology isomorphism for all p. ■