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Topology (MAT411)

Lecture Notes

Preface

This series of lecture notes has been prepared for aiding students who took the BRAC University course **Topology (MAT411)** in Fall 2023 semester. These notes were typeset under the supervision of mathematician **Dr. Syed Hasibul Hassan Chowdhury**. The main goal of this typeset is to have an organized digital version of the notes, which is easier to share and handle. If you see any mistakes or typos, please send me an email at atonuroychowdhury@gmail.com

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References:

- *Topology*, by **James R. Munkres**

Contents

1 Topological Spaces

§1.1 Basic Definitions

Definition 1.1. Let X be a set. A **topology** on X is a collection $\mathcal{T}$ of subsets of X having the following properties:

1. $\varnothing$ and X are in $\mathcal{T}$.
2. For any subcollection $\{U_\alpha\}_{\alpha \in J}$ of $\mathcal{T}$, the union $\bigcup_{\alpha \in J} U_\alpha$ is in $\mathcal{T}$.
3. For any finite subcoluction $\{U_1, \ldots, U_n\}$ of $\mathcal{T}$, the intersection $\bigcap_{i=1}^n U_i$ is in $\mathcal{T}$.

A **topological space** $(X, \mathcal{T})$ is a set X with a given topology $\mathcal{T}$. A subset $U \subset X$ with $U \in \mathcal{T}$ is said to be an open set.

Example 1.1 (Two extreme examples)**.** Let X be a set. Following are 2 examples of topologies on X :

1. (Discrete topology) The discrete Topology on X, denoted by $\mathcal{T}_{\text{disc}}$ is the topology where all subsets $U \subset X$ are defined to be open. Hence, $\mathcal{T}_{\text{disc}} = \mathscr{P}(X)$, the power set of X. One can easily check that $\mathcal{T}_{\text{disc}}$ is indeed a topology.
2. (Indiscrete topology) The indiscrete topology on X, denoted by $\mathcal{T}_{\text{indis}}$ is the topology where only the subsets X and $\varnothing$ are defined to be open sets. In other words, $\mathcal{T}_{\text{indis}} = \{\varnothing, X\}$.

Definition 1.2 (Finite topological space)**.** If X is a finite set and $\mathcal{T}$ is a topology on X, we call $(X, \mathcal{T})$ a **finite topological space**.

Example 1.2. Let X be a 3-element set, $X = \{1, 2, 3\}$. Verify that the following are examples of finite topological spaces:

1. $\mathcal{T} = \{\varnothing, \{1, 2, 3\}\}$.
2. $\mathcal{T} = \{\varnothing, \{1, 2, 3\}, \{1, 2\}, \{2, 3\}, \{2\}\}$.
3. $\mathcal{T} = \{\varnothing, \{1, 2, 3\}, \{1\}, \{1, 2\}\}$.

Non-example: The collection $\mathcal{T} = \{\varnothing, \{1, 2, 3\}, \{1\}, \{2\}\}$ is not a topology on $X = \{1, 2, 3\}$, since it is not closed under union.

Definition 1.3. Let $\mathcal{T}$ and $\mathcal{T}'$ be 2 topologies on the same set X. If $\mathcal{T}' \supseteq \mathcal{T}$, we soy that $\mathcal{T}'$ is **finer** than $\mathcal{T}$, or $\mathcal{T}$ is **coarser** than $\mathcal{T}'$. If the containment above is proper, we say that $\mathcal{T}'$ is strictly finer than $\mathcal{T}$, or $\mathcal{T}$ is strictly coarser than $\mathcal{T}'$.

Example 1.3. In the context of Example 1.2, for the 3-element set $X = \{1, 2, 3\}$, consider the following 4 topologies:

1. $\mathcal{T} = \{\{1, 2, 3\}, \varnothing, \{1, 2\}, \{2\}, \{2, 3\}\}$.
2. $\mathcal{T}_1 = \{\{1, 2, 3\}, \varnothing\}$.
3. $\mathcal{T}_2 = \{\{1, 2, 3\}, \varnothing, \{2\}\}$
4. $\mathcal{T}_3 = \{\{1, 2, 3\}, \varnothing, \{1, 2\}\}$

Observe that $\mathcal{T}$ is strictly finer than all 3 of $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$. Also, one has $\mathcal{T}_1 \subset \mathcal{T}_3$, and $\mathcal{T}_1 \subset \mathcal{T}_2$, i.e. $\mathcal{T}_3$ is strictly finer than $\mathcal{T}_1$, and $\mathcal{T}_2$ is strict finer than $\mathcal{T}_1$.

§1.2 Review of Metric Space

Definition 1.4. A **metric** on a set X is a function $d : X \times X \to \mathbb{R}$ such that:

1. (Non-negativity) $d(x, y) \geq 0$ for any $x, y \in X$, and $d(x, y) = 0$ if and only if $x = y$.
2. (Symmetry) $d(x, y) = d(y, x)$, for any $x, y \in X$.
3. (Triangle inequality) $d(x, z) \leq d(x, y) + d(y, z)$ for any $x, y, z \in X$.

A **metric space** (X, d) is a set X equipped with a metric d.

Example 1.4. The real line $\mathbb{R}$ is a metric space, with distance function $d_{\text{Euc}}(x, y) = |y - x|$. More generally, in $\mathbb{R}^n$, one can define the Euclidean distance

$$d_{\text{Euc}}(\mathbf{x}, \mathbf{y}) = \|\mathbf{y} - \mathbf{x}\| = \sqrt{(y_1 - x_1)^2 + \cdots + (y_n - x_n)^2}, \tag{1.1}$$

for $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$. We call $(\mathbb{R}^n, d_{\text{Euc}})$ the Euclidean n-space.

Definition 1.5. Let (X, d) be a metric space. For each point $x \in X$ and each $\varepsilon < 0$, let

$$B_d(x, \varepsilon) = \{y \in X \mid d(x, y) < \varepsilon\}. \tag{1.2}$$

Then the set $B_d(x, \varepsilon)$ is called ε-ball around x in (X, d).

Definition 1.6 (Metric topology)**.** Let (X, d) be a metric space. The metric topology $\mathcal{T}_d$ on X is the collection of subsets $U \subset X$ such that for each $x \in U$, there exists $\varepsilon > 0$ such that $B_d(x, \varepsilon) \subset U$.

Lemma 1.1

The collection $\mathcal{T}_d$ is a topology on X.

Proof. Observe that $\varnothing$ is vacuously open in metric topology, i.e. $\varnothing \in \mathcal{T}_d$ since there is no element in $\varnothing$ to open the argument with. Also, the whole set $X \in \mathcal{T}_d$, i.e. the whole set X itself is open in the metric topology. This is so because for any $x \in X$, one can choose $B_d(x, 1) = \{y \in X \mid d(x, y) < 1\} \subseteq X$ proving that X is open in the metric topology.

Next, let $\{U_\alpha\}_{\alpha \in J}$ be a subcollection of $\mathcal{T}_d$. Let $W = \bigcup_\alpha U_\alpha$. Consider $x \in W = \bigcup_\alpha U_\alpha$. Hence, there is some $\alpha_0 \in J$ such that $x \in U_{\alpha_0}$. Since $U_{\alpha_0} \in \mathcal{T}_d$, there exists $\varepsilon > 0$ such that

$$B_d(x, \varepsilon) \subset U_{\alpha_0} \subset \bigcup_{\alpha \in J} U_\alpha = W. \tag{1.3}$$

Hence, $\bigcup_{\alpha \in J} U_\alpha \in \mathcal{T}_d$.

Now, let $\{U_1, \ldots, U_n\}$ be a finite subcolution of $\mathcal{T}_d$. Let $V = U_1 \cap \cdots \cap U_n$ and consider $x \in V$. Hence, $x \in U_i$ for each $i \in \{1, \ldots, n\}$. Since, each $U_i \in \mathcal{T}_d$, there exists $\varepsilon_i > 0$, such that $B_d(x, \varepsilon_i) \subset U_i$, for each $i \in \{1, \ldots, n\}$. Choose $\varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_n\} > 0$. Then one has $B_d(x, \varepsilon) \subset B_d(x, \varepsilon_i) \subset U_i$, for any i. Therefore,

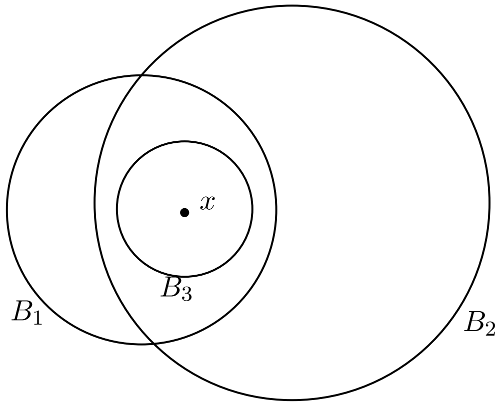
$$B_d(x, \varepsilon) \subset \bigcap_{i=1}^{n} U_i, \tag{1.4}$$

proving that $V = \bigcap_{i=1}^n U_i \in \mathcal{T}_d$. ■

§1.3 Basis for a Topology

Definition 1.7 (Basis)**.** Let X be a set. A **basis** for a topology on X is a collection $\mathscr{B}$ of subsets of X (called *basis elements*) such that

1. for each $x \in X$, there exists $B \in \mathscr{B}$ such that $x \in B \subset X$;
2. if $x \in B_1 \cap B_2$ with $B_1, B_2 \in \mathscr{B}$, then there exists $B_3 \in \mathscr{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.



Definition 1.8 (Topology generated by a basis)**.** Let $\mathscr{B}$ be a basis for a topology topology on a given set X. The topology $\mathcal{T}$ generated by $\mathscr{B}$ is the collection of subsets $U \subset X$ such that for each $x \in U$, there exists $B \in \mathscr{B}$ with $x \in B \subset U$. In other words, a subset $U \subset X$ is defined to be open in this topology if for each $x \in U$, there exists a basis element $B \subset U$ with $x \in B$.

Lemma 1.2

The collection $\mathcal{T}$ generated by a basis $\mathcal{B}$ as defined above is a topology on X.

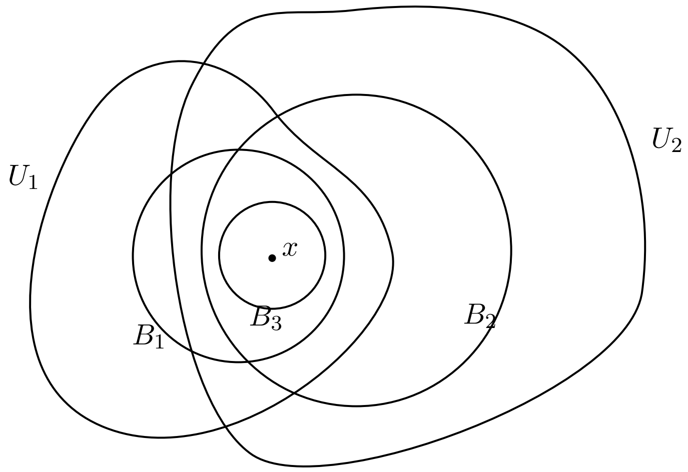
Proof. $\varnothing \in \mathcal{T}$ since there is no element in $\varnothing$ to verify the conditions, and hence $\varnothing$ is vacuously open. By the first condition of basis, for each $x \in X$, there exists $B \in \mathscr{B}$ such that $x \in B \subset X$. Therefore, from the definition of the topology generated by a basis, X is open, i.e. $X \in \mathcal{T}$.

Now, let $\{U_\alpha\}_{\alpha \in J}$ be a subcollection of $\mathcal{T}$. Also, let $\bigcup_{\alpha \in J} U_\alpha = W$. We need to show that $W \in \mathcal{T}$. Consider $x \in W = \bigcup_\alpha U_\alpha$. Hence, there is some $\alpha_0 \in J$ such that $x \in U_{\alpha_0}$. Since $U_{\alpha_0} \in \mathcal{T}$, there exists $B \in \mathscr{B}$ for which $x \in B \subset U_{\alpha_0}$ holds. In other words,

$$x \in B \subset U_{\alpha_0} \subset \bigcup_{\alpha \in J} U_\alpha = W. \tag{1.5}$$

Therefore, $W \in \mathcal{T}$.

Now, let $U_1, U_2 \in \mathcal{T}$. Given $x \in U_1 \cap U_2$, x is in both U_1 and U_2. Since $U_1, U_2 \in \mathcal{T}$, by the definition of topology generated by a basis, there exist basis elements $B_1, B_2 \in \mathscr{B}$ such that $x \in B_1 \subset U_1$ and $x \in B_2 \subset U_2$. Then we have $x \in B_1 \cap B_2$.



By the second condition for a basis, there is some $B_3 \in \mathscr{B}$ such that $x \in B_3 \subset B_1 \cap B_2$. Therefore,

$$x \in B_3 \subset B_1 \cap B_2 \subset U_1 \cap U_2. \tag{1.6}$$

So $U_1 \cap U_2 \in \mathcal{T}$. Now we use induction to prove that $V = \bigcap_{i=1}^{n} U_i \in \mathcal{T}$, where each $U_i \in \mathcal{T}$. The base case $n = 1$ is trivial. Now suppose that this is true for $n-1$, i.e. $\bigcap_{i=1}^{n-1} U_i \in \mathcal{T}$. We also have $U_n \in \mathcal{T}$. We have just proved that the intersection of two elements of $\mathcal{T}$ also belongs to $\mathcal{T}$. Therefore,

$$\left(\bigcap_{i=1}^{n-1} U_i\right) \cap U_n = \bigcap_{i=1}^{n} U_i \in \mathcal{T}. \tag{1.7}$$

Therefore, $\mathcal{T}$ is a topology on X. ■

Lemma 1.3

In any metric space (X, d), the collection of ε-balls

$$\mathscr{B} = \{B_d(x, \varepsilon) \mid x \in X, \varepsilon > 0\}$$

is a basis.

Proof. 1. For each $x \in X$, the 1-ball $B_d(x, 1) \in \mathscr{B}$.

2. Given $B_1 = B_d(x_1, \varepsilon_1)$ and $B_2 = B_d(x_2, \varepsilon_2)$, consider $x \in B_1 \cap B_2$. It is evident that

$$\varepsilon_1 - d(x, x_1) > 0 \text{ and } \varepsilon_2 - d(x, x_2) > 0. \tag{1.8}$$

Let $\varepsilon = \min\{\varepsilon_1 - d(x, x_1), \varepsilon_2 - d(x, x_2)\}$. Then $\varepsilon > 0$. Now we claim that $x \in B_d(x, \varepsilon) =: B_3 \subset B_1 \cap B_2$. Let $y \in B_3 = B_d(x, \varepsilon)$, so that $d(x, y) < \varepsilon$. Then

$$d(x, y) < \varepsilon \leq \varepsilon_1 - d(x, x_1).$$

By the triangle inequality,

$$d(x_1, y) \leq d(x, x_1) + d(x, y) < \varepsilon_1, \tag{1.9}$$

which implies that $y \in B_1 = B_d(x_1, \varepsilon_1)$. So $B_3 \subset B_1$. Similarly, $B_3 \subset B_2$. Therefore, $B_3 = B_d(x, \varepsilon) \subset B_1 \cap B_2$, as required. ■

Proposition 1.4

The metric topology $\mathcal{T}_d$ defined earlier on the metric space coincides with the topology $\mathcal{T}_d$ on (X, d) generated by the basis of ε-balls as in Lemma 1.3.

Proof. Suppose $U \in \mathcal{T}_d$. Hence, from the definition of metric topology, for each $y \in U$, there exists $\delta > 0$ such that $B_d(y,\delta) \subset U$. Since $B_d(y,\delta) \in \mathscr{B}$, and $y \in B_d(y,\delta) \subset U$, $U \in \mathcal{T}$, the topology on (X,d) generated by the basis $\mathscr{B}$. In other words, $\mathcal{T}_d \subset \mathcal{T}$.

Now conversely, suppose $U \in \mathcal{T}$. Hence, given $y \in U$, there is a basis element $B_d(x,\varepsilon) \in \mathscr{B}$ such that $y \in B_d(x,\varepsilon) \subset U$. Hence, $d(x,y) < \varepsilon$. Define $\delta = \varepsilon - d(x,y) > 0$. Then one immediately finds $B_d(y,\delta) \subset B_d(x,\varepsilon)$. Indeed, if $z \in B_d(y,\delta)$, then $d(y,z) < \delta = \varepsilon - d(x,y)$. By the triangle inequality,

$$d(x,z) \leq d(x,y) + d(y,z) < \varepsilon. \tag{1.10}$$

Therefore, $z \in B_d(x,\varepsilon)$, proving that $y \in B_d(y,\delta) \subset B_d(x,\varepsilon) \subset U$. So we have proved that given $y \in U$, there exists $\delta > 0$ such that $B_d(y,\delta) \subset U$. In other words, $U \in \mathcal{T}_d$, so that $\mathcal{T} \subset \mathcal{T}_d$. Hence, $\mathcal{T} = \mathcal{T}_d$. ∎

Example 1.5. Let $X = \mathbb{R}^2$, and $\mathscr{B}$ be the collection of all circular regions (interior of circles) in the plane. This is the collection of all ε-balls

$$B_{\text{Euc}}(\mathbf{x},\varepsilon) = \left\{\mathbf{y} \in \mathbb{R}^2 \mid d(\mathbf{x},\mathbf{y}) < \varepsilon\right\}$$

with respect to the Euclidean metric $d_{\text{Euc}}(\mathbf{x},\mathbf{y}) = \|\mathbf{x}-\mathbf{y}\| = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$, with $\mathbf{x} = (x_1,x_2), \mathbf{y} = (y_1,y_2) \in \mathbb{R}^2$. Indeed, $(\mathbb{R}^2, d_{\text{Euc}})$ is a metric space, and by means of Lemma 1.3 and Proposition 1.4, the collection

$$\mathscr{B} = \left\{B_{\text{Euc}}(\mathbf{x},\varepsilon) \mid \mathbf{x} \in \mathbb{R}^2,\ \varepsilon > 0\right\}$$

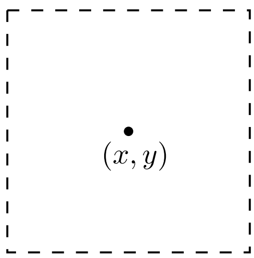
is a basis for the metric topology with respect to the Euclidean metric on $\mathbb{R}^2$.

Example 1.6. Let $X = \mathbb{R}^2$, but in contrast to Example 1.5, here choose $\mathscr{B}'$ to be the collection of all rectangular regions (interior of rectangles) in the plane $\mathbb{R}^2$. This is the collection of all sets of the form

$$(a,b) \times (c,d) \in \mathbb{R} \times \mathbb{R},$$

with $a < b$ and $c < d$. This is the open rectangular area bounded by the vertical lines $x = a$ and $x = b$, and horizontal lines $y = c$ and $y = d$. Let us verify that such a collection, indeed, satisfies the two conditions for a basis:

1. For each $(x,y) \in \mathbb{R}^2$, $(x,y) \in (x-1,x+1) \times (y-1,y+1)$, with $(x-1,x+1) \times (y-1,y+1) \in \mathscr{B}'$.



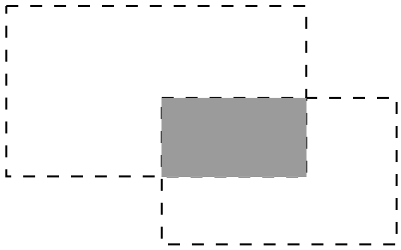
$(x-1,x+1) \times (y-1,y+1)$

2. Consider $B_1 = (a_1,b_1) \times (c_1,d_1)$ and $B_2 = (a_2,b_2) \times (c_2,d_2)$ to be two elements in $\mathscr{B}'$. Take $(x_0,y_0) \in B_1 \cap B_2$. Since $a_1 < x_0 < b_1$ and $a_2 < x_0 < b_2$, one has

$$a := \max\{a_1,a_2\} < x_0 < \min\{b_1,b_2\} =: b,$$

Similarly,

$$c := \max\{c_1,c_2\} < y_0 < \min\{d_1,d_2\} =: d.$$



Then $(x_0, y_0) \in (a, b) \times (c, d) =: B_3 = B_1 \cap B_2$, the shaded open rectangle in the diagram above. The diagram above is the case when $B_1 \cap B_2 \neq \varnothing$. The condition for this to happen is $a < b$ and $c < d$. Otherwise, the intersection is empty, and the second condition for basis is vacuously satisfied.

Proposition 1.5

Let $\mathscr{B}$ be a basis for a topology $\mathcal{T}$ on X, i.e. $\mathcal{T}$ is the topology on X generated by the basis $\mathscr{B}$. Then $\mathcal{T}$ equals the collection of all unions of elements of $\mathscr{B}$.

Proof. Let us first prove that $\mathcal{T}$ is contained in the collection of all unions of elements of $\mathscr{B}$. Let $U \in \mathcal{T}$. For each $x \in U$, there exists $B_x \in \mathscr{B}$ with $x \in B_x \subset U$. Then one easily has $U = \bigcup_{x \in U} B_x$. Indeed, since $x \in B_x \subset U$, taking union over all $x \in U$ gives us

$$\bigcup_{x \in U} \{x\} \subset \bigcup_{x \in U} B_x \subset U.$$

In other word,

$$U \subset \bigcup_{x \in U} B_x \subset U. \tag{1.11}$$

So $U = \bigcup_{x \in U} B_x$. Therefore, any open set on X in the topology $\mathcal{T}$ generated by a basis $\mathscr{B}$ is a union of basis elements from $\mathscr{B}$.

To prove the converse, i.e. any union of basis elements from $\mathscr{B}$ belongs to $\mathcal{T}$, note that every basis element B of $\mathscr{B}$ is open, i.e. it belongs to $\mathcal{T}$. This is because for each $x \in B$, there is a basis element, namely B itself, such that $x \in B \subset B$, proving that $B \in \mathcal{T}$, the topology generated by the basis $\mathscr{B}$. From the definition of topology, it follows that arbitrary union of basis elements from $\mathscr{B}$ will be in $\mathcal{T}$ as well. ∎

Example 1.7. If X is any set, the collection of all one-point subsets of X is a basis for the discrete topology $\mathcal{T}_{\text{dis}}$ on X. For example, if $X = \{a, b, c\}$, then

$$\mathcal{T}_{\text{dis}} = \{\{a, b, c\}, \varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}\} = \mathscr{P}(X);$$

and $\mathscr{B} = \{\{a\}, \{b\}, \{c\}\}$. Indeed, $\mathcal{T}_{\text{dis}}$ can be obtained from $\mathscr{B}$ by taking all possible unions. $\varnothing$ is understoor as the union of no basis elements at all.

Lemma 1.6 (Comparing topologies using bases)

Let $\mathscr{B}$ and $\mathscr{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$ on X, respectively. Then the following are equivalent:

1. $\mathcal{T}'$ is finer than $\mathcal{T}$.
2. For each $x \in X$ and any basis element $B \in \mathscr{B}$ containing x, there is a basis element $B' \in \mathscr{B}'$ such that $x \in B' \subset B$.

Proof. **(1⇒2)** Let $x \in X$ and $B \in \mathscr{B}$ with $x \in B$. We have seen in the proof of Proposition 1.5 that, $B \in \mathcal{T}$. By hypothesis, $\mathcal{T} \subset \mathcal{T}'$. Hence, $B \in \mathcal{T}'$. Since $\mathcal{T}'$ is the topology generated by $\mathscr{B}'$, there exists $B' \in \mathscr{B}'$ with $x \in B' \subset B$.

(2⇒1) Let $U \in \mathcal{T}$. Since $\mathcal{T}$ is generated by $\mathscr{B}$, for each $x \in U$, there exists some $B \in \mathscr{B}$ with $x \in B \subset U$. By hypothesis, there exists a $B' \in \mathscr{B}'$ with $x \in B' \subset B$. Therefore, $B' \in U$. We, therefore, have shown that for each $x \in U$, there exists $B' \in \mathscr{B}'$ with $x \in B' \subset U$. Hence, $U \in \mathcal{T}'$, the topology generated by $\mathscr{B}'$. Therefore, $\mathcal{T} \subset \mathcal{T}'$. ■

Corollary 1.7

Two bases $\mathscr{B}$ and $\mathscr{B}'$ for topologies on X generate the same topology if and only if

1. for each $x \in B \in \mathscr{B}$, there is a basis element $B' \in \mathscr{B}'$ such that $x \in B' \subset B$; and furthermore,
2. for each $x \in B' \in \mathscr{B}'$, there is a basis element $B \in \mathscr{B}$ such that $x \in B \subset B'$.

Proof. Let $\mathscr{B}$ and $\mathscr{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$ on X, respectively. By Lemma 1.6, $\mathcal{T} \subseteq \mathcal{T}'$ is equivalent to (1). By Lemma 1.6, $\mathcal{T}' \subseteq \mathcal{T}$ is equivalent to (2). ■

Example 1.8. The basis $\mathscr{B}$ of open circular regions in the plane $\mathbb{R}^2$ and the basis $\mathscr{B}'$ of open rectanglular regions generate the same topology on $\mathbb{R}^2$, namely the metric topology.