Algebriac Topology III (MAT484)

Lecture Notes

Contents

1 Singular Homology Groups

Let $\mathbb{R}^\infty$ denote the generalized Euclidean space $\mathbb{E}^J$, with J being the set of positive integers. An element of the vector space $\mathbb{R}^\infty$ is an infinite sequence of real numbers (functions from $\mathbb{N}$ to $\mathbb{R}$) with finitely many nonzero entries. Let Δ_p denote the p-simplex in $\mathbb{R}^\infty$ having vertices

$$\begin{aligned}
\varepsilon_0 &= (1,0,0,\dots,0,\dots)\ ,\\
\varepsilon_1 &= (0,1,0,\dots,0,\dots)\ ,\\
&\dots\\
\varepsilon_p &= (0,0,0,\dots,\underbrace{1}_{(p+1)\text{-th entry}},\dots)\,.
\end{aligned}$$

We call Δ_p the **standard p-simplex**. In this notation, Δ_{p-1} is a face of Δ_p.

> **Definition 1.1** (Singular p-simplex)**.** Let X be a topological space. We define a **singular p-simplex** of X to be a continuous map $T:\Delta_p\to X$. The free abelian group generated by singular p-simplices of X is denoted by $S_p(X)$, and is called the **singular chain group** of X in dimension p. We shall denote an element of $S_p(X)$ by a $\mathbb{Z}$-linear combination of singular p-simplices of X.

Singular means that T could be a "bad" map, i.e. it may not be an imbedding. All we want that T is just continuous. Now, recall that

$$\Delta_p=\left\{(x_0,x_1,\dots,x_p,0,\dots)\in\mathbb{R}^\infty|0\le x_i\le 1 \text{ and } \sum_{i=0}^p x_i=1\right\}. \tag{1.1}$$

Given $a_0,a_1,\dots,a_p\in\mathbb{R}^\infty$, there is a unique affine map $l_{(a_0,\dots,a_p)}:\Delta_p\to\mathbb{R}^\infty$ that maps ε_i to a_i. It is defined by

$$\begin{aligned}
l_{(a_0,\dots,a_p)}(x_0,x_1,\dots,x_p,0,\dots) &= \sum_{i=0}^p x_ia_i=\sum_{i=0}^p x_ia_i+a_0-\sum_{i=0}^p x_ia_0\\
&= a_0+\sum_{i=0}^p x_i(a_i-a_0)\,.
\end{aligned}\tag{1.2}$$

We call this map the **linear singular simplex** determined by $a_0,a_1,\dots,a_p\in\mathbb{R}^\infty$. Now, what is $l_{(\varepsilon_0,\dots,\varepsilon_p)}$? Observe that

$$l_{(\varepsilon_0,\dots,\varepsilon_p)}\varepsilon_i=l_{(\varepsilon_0,\dots,\varepsilon_p)}(0,\dots,0,\underbrace{1}_{(i+1)\text{-th entry}},0,\dots)=\varepsilon_i. \tag{1.3}$$

Therefore, $l_{(\varepsilon_0,\dots,\varepsilon_p)}$ maps ε_i to itself, for every $i=0,1,\dots,p$. Also,

$$l_{(\varepsilon_0,\dots,\varepsilon_p)}(x_0,x_1,\dots,x_p,0,\dots)=\sum_{i=0}^p x_i\varepsilon_i=(x_0,x_1,\dots,x_p,0,\dots)\,. \tag{1.4}$$

Therefore, $l_{(\varepsilon_0,\dots,\varepsilon_p)}$ is just the inclusion map of Δ_p into $\mathbb{R}^\infty$. Now, suppose $(x_0,x_1,\dots,x_{p-1},0,\dots)\in\Delta_{p-1}$, so that $\sum_{i=0}^{p-1}x_i=1$. Then

$$\begin{aligned}
l_{(\varepsilon_0,\dots,\widehat{\varepsilon_i},\dots,\varepsilon_p)}(x_0,x_1,\dots,x_{p-1},0,\dots) &= x_0\varepsilon_0+\dots+x_{i-1}\varepsilon_{i-1}+0\cdot\varepsilon_i+x_{i+1}\varepsilon_{i+1}+\dots+x_{p-1}\varepsilon_p\\
&=(x_0,\dots,x_{i-1},0,x_{i+1},\dots,x_{p-1},0,\dots)\,,
\end{aligned}\tag{1.5}$$

which is a point on the face of Δ_p opposite to the vertex ε_i. In fact, $l_{(\varepsilon_0,\dots,\widehat{\varepsilon_i},\dots,\varepsilon_p)}$ is a linear homomorphism of Δ_{p-1} into the face of Δ_p that is opposite to the vertex ε_i. In other words,

$$l_{(\varepsilon_0,\dots,\widehat{\varepsilon_i},\dots,\varepsilon_p)} : \Delta_{p-1} \to \Delta_p$$

maps Δ_{p-1} to the face of Δ_p opposite to the vertex ε_i. Therefore, given a singular p-simplex $T : \Delta_p \to X$, one can form the composite

$$T \circ l_{(\varepsilon_0,\dots,\widehat{\varepsilon_i},\dots,\varepsilon_p)} : \Delta_{p-1} \to X,$$

which is a singular $(p-1)$-simplex. We think of it as the i-th face of the singular p-simplex T.

> **Definition 1.2** (Boundary homomorphism). We define $\partial : S_p(X) \to S_{p-1}(X)$ as follows. If $T : \Delta_p \to X$ is a singular p-simplex, we define ∂T to be
>
> $$\partial T = \sum_{i=0}^{p} (-1)^i \, T \circ l_{(\varepsilon_0,\dots,\widehat{\varepsilon_i},\dots,\varepsilon_p)}. \tag{1.6}$$
>
> In other words, ∂T is a formal sum of singular simplices of dimension $p-1$, which are the faces of T.

If $f : X \to Y$ is a continuous map, we define a group homomorphism $f_\# : S_p(X) \to S_p(Y)$ by defining it on singular p-simplices by the equation

$$f_\#(T) = f \circ T \tag{1.7}$$

for a singular p-simplex T.

$$\begin{tikzcd} \Delta_p \arrow[r, "T"] \arrow[rr, bend right, "f\circ T"'] & X \arrow[r, "f"] & Y \end{tikzcd}$$

> **Theorem 1.1**
>
> The homomorphism $f_\#$ commutes with ∂. Furthermore, $\partial^2 = 0$.

Proof. Given a singular p-simplex T,

$$\partial f_\#(T) = \partial(f \circ T) = \sum_{i=0}^{p} (-1)^i (f \circ T) \circ l_{(\varepsilon_0,\dots,\widehat{\varepsilon_i},\dots,\varepsilon_p)}. \tag{1.8}$$

$$f_\#(\partial T) = f_\#\left(\sum_{i=0}^{p} (-1)^i \, T \circ l_{(\varepsilon_0,\dots,\widehat{\varepsilon_i},\dots,\varepsilon_p)}\right) = \sum_{i=0}^{p} (-1)^i \, f \circ T \circ l_{(\varepsilon_0,\dots,\widehat{\varepsilon_i},\dots,\varepsilon_p)}. \tag{1.9}$$

Therefore, $\partial f_\#(T) = f_\#(\partial T)$. Now, to prove $\partial^2 = 0$, we first compute ∂ for linear singular simplices $l_{(a_0,\dots,a_p)}$.

$$\partial l_{(a_0,\dots,a_p)} = \sum_{i=0}^{p} (-1)^i \, l_{(a_0,\dots,a_p)} \circ l_{(\varepsilon_0,\dots,\widehat{\varepsilon_i},\dots,\varepsilon_p)}. \tag{1.10}$$

Observe that

$$\begin{aligned} l_{(a_0,\dots,a_p)} \circ l_{(\varepsilon_0,\dots,\widehat{\varepsilon_i},\dots,\varepsilon_p)}(x_0,\dots,x_{p-1},0,\dots) &= l_{(a_0,\dots,a_p)}(x_0,\dots,x_{i-1},0,x_i x_{p-1},0,) \\ &= x_0 a_0 + \dots + x_{i-1}a_{i-1} + 0 \cdot a_i + x_i a_{i+1} + \cdots x_{p-1}a_p \\ &= l_{(a_0,\dots,\widehat{a_i},\dots a_p)}(x_0,\dots,x_{p-1},0,\dots). \end{aligned} \tag{1.11}$$

Hence,

$$l_{(a_0,\dots,a_p)} \circ l_{(\varepsilon_0,\dots,\widehat{\varepsilon_i},\dots,\varepsilon_p)} = l_{(a_0,\dots,\widehat{a_i},\dots a_p)}. \tag{1.12}$$

Therefore, from 1.10, it follows that

$$\partial l_{(a_0,\ldots,a_p)} = \sum_{i=0}^{p} (-1)^i \, l_{(a_0,\ldots,\widehat{a_i},\ldots a_p)}. \tag{1.13}$$

Let's now evaluate $\partial\partial l_{(a_0,\ldots,a_p)}$.

$$\begin{aligned}
\partial\partial l_{(a_0,\ldots,a_p)} &= \sum_{i=0}^{p} (-1)^i \, \partial l_{(a_0,\ldots,\widehat{a_i},\ldots a_p)} \\
&= \sum_{i=0}^{p} (-1)^i \sum_{j<i} (-1)^j \, l_{(a_0,\ldots,\widehat{a_j},\ldots \widehat{a_i},\ldots a_p)} + \sum_{i=0}^{p} (-1)^i \sum_{j>i} (-1)^{j-1} \, l_{(a_0,\ldots,\widehat{a_i},\ldots \widehat{a_j},\ldots a_p)} \\
&= \sum_{i=0}^{p} \sum_{j<i} (-1)^{i+j} \, l_{(a_0,\ldots,\widehat{a_j},\ldots \widehat{a_i},\ldots a_p)} - \sum_{i=0}^{p} \sum_{j>i} (-1)^{i+j} \, l_{(a_0,\ldots,\widehat{a_i},\ldots \widehat{a_j},\ldots a_p)}.
\end{aligned} \tag{1.14}$$

Now fix $0 \leq j_0 < i_0 \leq p$. In the first summand of 1.14, the contribution of $i = i_0, j = j_0$ is

$$(-1)^{i_0+j_0} \, l_{\left(a_0,\ldots,\widehat{a_{j_0}},\ldots \widehat{a_{i_0}},\ldots a_p\right)}. \tag{1.15}$$

On the other hand, in the second summand of 1.14, the contribution of $i = j_0, j = i_0$ is also

$$(-1)^{i_0+j_0} \, l_{\left(a_0,\ldots,\widehat{a_{j_0}},\ldots \widehat{a_{i_0}},\ldots a_p\right)}. \tag{1.16}$$

These two contributions cancel each other. This way, one finds that the RHS of 1.14 vanishes. Hence,

$$\partial\partial l_{(a_0,\ldots,a_p)} = 0. \tag{1.17}$$

In particular,

$$\partial\partial l_{(\varepsilon_0,\ldots,\varepsilon_p)} = 0. \tag{1.18}$$

Now, $l_{(\varepsilon_0,\ldots,\varepsilon_p)} : \Delta_p \to \Delta_p$ is continuous, so $l_{(\varepsilon_0,\ldots,\varepsilon_p)} \in S_p(\Delta_p)$. Furthermore, it is the identity map as we have seen in 1.4. Since $T : \Delta_p \to X$ is continuous, we can form $T_\# : S_p(\Delta_p) \to S_p(X)$.

$$T_\# \left(l_{(\varepsilon_0,\ldots,\varepsilon_p)} \right) = T \circ l_{(\varepsilon_0,\ldots,\varepsilon_p)} = T \circ \mathrm{id}_{\Delta_p} = T. \tag{1.19}$$

Therefore, using the fact that $T_\#$ commutes with ∂, we obtain

$$\partial\partial T = \partial\partial T_\# \left(l_{(\varepsilon_0,\ldots,\varepsilon_p)} \right) = T_\# \left(\partial\partial l_{(\varepsilon_0,\ldots,\varepsilon_p)} \right) = 0. \tag{1.20}$$

Hence, $\partial^2 T = 0$. ■

Definition 1.3 (Singular homology groups)**.** Th family of groups $S_p(X)$ and homomorphisms $\partial_p : S_p(X) \to S_{p-1}(X)$ is called **singular chain complex** of X, and is denoted by $\mathcal{S}(X)$. We will be attaching the index p with the homomorphism while dealing with singular chain complex:

$$\cdots \longrightarrow S_{p+1}(X) \xrightarrow{\partial_{p+1}} S_p(X) \xrightarrow{\partial_p} S_{p-1}(X) \longrightarrow \cdots$$

The homology groups of this chain complex are called the **singular homology groups** of X, and are denoted by $H_p(X)$.

Definition 1.4 (Augmentation map)**.** The chain complex $\mathcal{S}(X)$ is augmented by the homomorphism $\epsilon : S_0(X) \to \mathbb{Z}$ defined by setting $\epsilon(T) = 1$ for each singular 0-simplex $T : \Delta_0 \to X$. (A generic singular 0-chain is a $\mathbb{Z}$-linear combination of singular 0-simplices.)

It's immediate that if T is a singular 1-simplex, then $\epsilon(\partial T) = 0$. Indeed,

$$\epsilon(\partial T) = \epsilon\left(T \circ l_{(\widehat{\varepsilon_0},\varepsilon_1)}\right) - \epsilon\left(T \circ l_{(\varepsilon_0,\widehat{\varepsilon_1})}\right) = 0. \tag{1.21}$$

Definition 1.5 (Reduced homology groups)**.** The homology groups of $\{\mathcal{S}(X), \epsilon\}$ are called the **reduced singular homology groups** of X, and are denoted by $\widetilde{H}_p(X)$.

Now, given continuous map $f : X \to Y$ and $T : \Delta_0 \to X$ a singular 0-simplex on X, then $f_{\#}(T) = f \circ T : \Delta_0 \to Y$.

$$\begin{tikzcd} \Delta_0 \arrow[r, "T"] \arrow[rr, bend right, "f\circ T"'] & X \arrow[r, "f"] & Y \end{tikzcd}$$

Now, consider the augmented singular chain complexes $\{\mathcal{S}(X), \epsilon^X\}$ and $\{\mathcal{S}(Y), \epsilon^Y\}$. Noting continuous $T : \Delta_0 \to X$ and $f_{\#}(T) : \Delta_0 \to Y$, one obtains $\epsilon^X(T) = 1$ and $\epsilon^Y(f_{\#}(T)) = 1$. In other words, the following diagram commutes

$$\begin{tikzcd} S_0(X) \arrow[r, "\epsilon^X"] \arrow[d, "(f_\#)_0"'] & \mathbb{Z} \arrow[d, "\mathrm{id}"] \\ S_0(Y) \arrow[r, "\epsilon^Y"] & \mathbb{Z} \end{tikzcd}$$

Therefore, $f_{\#} : S_p(X) \to S_p(Y)$ is an **augmentation preserving chain map** between $\{\mathcal{S}(X), \epsilon^X\}$ and $\{\mathcal{S}(Y), \epsilon^Y\}$. Thus, $f_{\#}$ induces a homomorphism f_* in both ordinary and reduced singular homology.

In Theorem 1.1, we saw that the chain map $f_{\#}$ commutes with the boundary operator ∂. In other words, $(f_{\#})_p : S_p(X) \to S_p(Y)$ takes cycles to cycles and boundaries to boundaries. Suppose $c_p \in Z_p(X) = \operatorname{Ker} \partial_p^X$, so that $\partial_p^X c_p = 0$. Now,

$$\partial_p^Y \left((f_{\#})_p c_p \right) = (f_{\#})_{p-1} \left(\partial_p^X c_p \right) = 0. \tag{1.22}$$

Hence, $(f_{\#})_p c_p \in Z_p(Y)$. On the other hand, let $b_p \in B_p(X) = \operatorname{im} \partial_{p+1}^X$. Then $b_p = \partial_{p+1}^X d_{p+1}$ for some $d_{p+1} \in S_{p+1}(X)$. Then

$$(f_{\#})_p b_p = (f_{\#})_p \left(\partial_{p+1}^X d_{p+1} \right) = \partial_{p+1}^Y \left((f_{\#})_{p+1} d_{p+1} \right). \tag{1.23}$$

In other words, $(f_{\#})_p b_p \in B_p(Y)$. This reflects the fact that $(f_{\#})_p : S_p(X) \to S_p(Y)$ induces a homomorphism between the singular homology groups $(f_*)_p : H_p(X) \to H_p(Y)$. $(f_*)_p$ is given by

$$(f_*)_p (c_p + B_p(X)) = (f_{\#})_p c_p + B_p(Y). \tag{1.24}$$

If the reduced homology groups of X vanishes in all dimensions, we say that X is **acyclic** (in singular homology).

Theorem 1.2

If $i : X \to X$ is the identity, then so is $(i_*)_p : H_p(X) \to H_p(X)$. If $f : X \to Y$ and $g : Y \to Z$ are continuous, then $((g \circ f)_*)_p = (g_*)_p \circ (f_*)_p$.

Proof. It is sufficient to show that the equations hold at the chain level. We know from the definition of $(f_{\#})_p : S_p(X) \to S_p(Y)$ that it maps $T \in S_p(X)$ to $f \circ T \in S_p(Y)$. Since $i : X \to X$ is the identity map,

$$(i_{\#})_p (T) = i \circ T = T. \tag{1.25}$$

So $(i_{\#})_p : S_p(X) \to S_p(X)$ is the identity homomorphism. As a result,

$$(i_*)_p (c_p + B_p(X)) = (i_{\#})_p c_p + B_p(X) = c_p + B_p(X). \tag{1.26}$$

Therefore, $(i_*)_p = \mathrm{id}_{H_p(X)}$.

Given continuous $f : X \to Y$ and $g : Y \to Z$, $\left((g \circ f)_{\#}\right)_p : S_p(X) \to S_p(Z)$ is defined by

$$\left((g \circ f)_{\#}\right)_p T = (g \circ f) \circ T = g \circ (f \circ T) = (g_{\#})_p \left((f_{\#})_p T\right). \tag{1.27}$$

Therefore, $\left((g \circ f)_{\#}\right)_p = (g_{\#})_p \circ (f_{\#})_p$. Now, at the homology level, for $c_p + B_p(X) \in H_p(X) = Z_p(X) / B_p(X)$

$$((g \circ f)_*)_p (c_p + B_p(X)) = \left((g \circ f)_{\#}\right)_p c_p + B_p(Z) = (g_{\#})_p \left((f_{\#})_p c_p\right) + B_p(Z). \tag{1.28}$$

Also,

$$(g_*)_p \circ (f_*)_p (c_p + B_p(X)) = (g_*)_p \left((f_{\#})_p c_p + B_p(Y)\right) = (g_{\#})_p \left((f_{\#})_p c_p\right) + B_p(Z). \tag{1.29}$$

From 1.28 and 1.29, we can deduce that $((g \circ f)_*)_p = (g_*)_p \circ (f_*)_p$. ■

Corollary 1.3

If $h : X \to Y$ is a homeomorphims, then $(h_*)_p : H_p(X) \to H_p(Y)$ is an isomorphism.

Proof. Both $h : X \to Y$ and $h^{-1} : Y \to X$ are continuous, and $h \circ h^{-1} = \mathrm{id}_Y$. Therefore,

$$(h_*)_p \circ \left((h^{-1})_*\right)_p = \left((h \circ h^{-1})_*\right)_p = ((\mathrm{id}_Y)_*)_p = \mathrm{id}_{H_p(Y)}. \tag{1.30}$$

Similarly, starting with $h^{-1} \circ h = \mathrm{id}_X$, we will get $\left((h^{-1})_*\right)_p \circ (h_*)_p = \mathrm{id}_{H_p(X)}$. Therefore, $\left((h^{-1})_*\right)_p$ is the inverse of $(h_*)_p$. In other words, $(h_*)_p$ is an invertible homomorphism, i.e. an isomorphism. ■

Theorem 1.4

Let X be a topological space. Then $H_0(X)$ is free abelian. If $\{X_\alpha\}$ is the collection of path components of X, and if T_α is a singular 0-simplex with image in X_α for each α, then the homology classes of the chains T_α form a basis for $H_0(X)$. The group $\widetilde{H}_0(X)$ is also free abelian; it vanishes if X is path connected. Otherwise, let α_0 be a fixed index, then the homology classes of the chains $T_\alpha - T_{\alpha_0}$ for $\alpha \neq \alpha_0$ form a basis for $\widetilde{H}_0(X)$.

Proof. Let $x_\alpha = T_\alpha(\Delta_0) \in X_\alpha$, with $T_\alpha : \Delta_0 \to X$ being a singular 0-simplex. Here, Δ_0 consists of the point $\varepsilon_0 = (1, 0, 0, \ldots) \in \mathbb{R}^\infty$. Also, let $T : \Delta_0 \to X$ be any singular 0-simplex such that $T(\Delta_0) \in X_\alpha$. Since X_α is path connected, there is a path connecting $T(\Delta_0)$ and $T_\alpha(\Delta_0)$. In other words, there is a singular 1-simplex $f : \Delta_1 \to X$ such that

$$f(1, 0, 0 \ldots) = T(\Delta_0) \text{ and } f(0, 1, 0 \ldots) = T_\alpha(\Delta_0). \tag{1.31}$$

Then we have

$$\partial_1 f = f \circ l_{(\widehat{\varepsilon_0}, \varepsilon_1)} - f \circ l_{(\varepsilon_0, \widehat{\varepsilon_1})}. \tag{1.32}$$

Now,

$$f \circ l_{(\varepsilon_0, \widehat{\varepsilon_1})}(1, 0, 0, \ldots) = f(1, 0, 0, \ldots) = T(\Delta_0) = T(1, 0, 0, \ldots), \tag{1.33}$$
$$f \circ l_{(\widehat{\varepsilon_0}, \varepsilon_1)}(1, 0, 0, \ldots) = f(0, 1, 0, \ldots) = T_\alpha(\Delta_0) = T_\alpha(1, 0, 0, \ldots). \tag{1.34}$$

Therefore, $\partial_1 f = T_\alpha - T$.

An arbitrary singular 0-chain is a $\mathbb{Z}$-linear combination of singular 0-simplices. Let's take $c \in S_0(X)$. Then $c = \sum_\beta m_\beta T'_\beta$, with $m_\beta \in \mathbb{Z}$ and T'_β being singular 0-simplices. Each $T'_\beta(\Delta_0)$ belongs to some

X_α, and hence homologous to T_α. Therefore, c is homologous to some $\mathbb{Z}$-linear combination $\sum_\alpha n_\alpha T_\alpha$ of the T_α's. We will now show that no such nontrivial 0-chain $\sum_\alpha n_\alpha T_\alpha$ bounds.

Assume the contrary that $\sum_\alpha n_\alpha T_\alpha = \partial_1 d$ for some $d \in S_1(X)$. Now, the singular 1-chain d is a formal linear combination of singular 1-simplices with path connected image, i.e. the image lies in one of the path components X_α. Thus we can write $d = \sum_\alpha d_\alpha$, where d_α consists of the terms whose images are in X_α. Therefore,

$$\sum_\alpha n_\alpha T_\alpha = \partial_1 d = \sum_\alpha \partial_1 d_\alpha. \tag{1.35}$$

Hence, we get

$$n_\alpha T_\alpha = \partial_1 d_\alpha \tag{1.36}$$

for each α. Applying ϵ to both sides of 1.36, we get

$$\epsilon\left(n_\alpha T_\alpha\right) = \epsilon\left(\partial_1 d_\alpha\right) \implies n_\alpha = 0. \tag{1.37}$$

Therefore, no non-trivial 0-chain $\sum_\alpha n_\alpha T_\alpha$ bounds. Since every 0-chain is automatically a 0-cycle, an element of $H_0(X)$ is homologous to a 0-chain of the form $\sum_\alpha n_\alpha T_\alpha$. Hence, the homology classes of the singular 0-simplices $\{T_\alpha\}$ form a basis for the free abelian group $H_0(X)$.

$$S_1(X) \xrightarrow{\partial_1} S_0(X) \xrightarrow{\epsilon} \mathbb{Z}$$

$\widetilde{H}_0(X)$ is defined as $\widetilde{H}_0(X) = \operatorname{Ker}\epsilon / \operatorname{im}\partial_1$. Given a singular 0-chain $T \in S_0(X)$, we've seen that T is homologous to a 0-chain of the form $T' = \sum_\alpha n_\alpha T_\alpha$; and T' bounds iff $T' = 0$, i.e. $n_\alpha = 0$ for every α. If further $T \in \operatorname{Ker}\epsilon$, then $\epsilon(T) = 0$. Since T and T' are homologous, $T = T' + \partial_1 d$ for some $d \in S_1(X)$. Therefore,

$$0 = \epsilon(T) = \epsilon\left(T'\right) + \epsilon\left(\partial_1 d\right) = \epsilon\left(\sum_\alpha n_\alpha T_\alpha\right) = \sum_\alpha n_\alpha. \tag{1.38}$$

If X is path connected, there is only one component, and hence there is only one n_α involved. Thus $n_\alpha = 0$ from 1.38. This gives us $T' = 0$, leading to the fact that every $T \in \operatorname{Ker}\epsilon$ is homologous to 0, i.e. $T = 0 + \partial_1 d$ for some $d \in S_1(X)$. So $\operatorname{Ker}\epsilon = \operatorname{im}\partial_1$. Therefore, $\widetilde{H}_0(X) = 0$, when X is path connected.

Now, suppose X has more than one path components. Fix α_0. Then from 1.38, we get

$$0 = \sum_\alpha n_\alpha = n_{\alpha_0} + \sum_{\alpha \neq \alpha_0} n_\alpha \implies n_{\alpha_0} = -\sum_{\alpha \neq \alpha_0} n_\alpha. \tag{1.39}$$

Then T' is

$$T' = \sum_\alpha n_\alpha T_\alpha = \sum_{\alpha \neq \alpha_0} n_\alpha T_\alpha + n_{\alpha_0} T_{\alpha_0} = \sum_{\alpha \neq \alpha_0} n_\alpha T_\alpha - \sum_{\alpha \neq \alpha_0} n_\alpha T_{\alpha_0} = \sum_{\alpha \neq \alpha_0} n_\alpha \left(T_\alpha - T_{\alpha_0}\right). \tag{1.40}$$

1.40 suggests that T' is a linear combination of the singular 0-chains $\{T_\alpha - T_{\alpha_0}\}_{\alpha \neq \alpha_0}$. And T' bounds iff it is trivial, as shown earlier. Therefore, the homology classes of 0-chains $\{T_\alpha - T_{\alpha_0}\}_{\alpha \neq \alpha_0}$ form a basis for $\widetilde{H}_0(X)$. ■

Theorem 1.4 illustrates the following result:

$$H_p(X) = \begin{cases} \widetilde{H}_p(X) & \text{if } p > 0 \\ \widetilde{H}_0(X) \oplus \mathbb{Z} & \text{if } p = 0 \end{cases}. \tag{1.41}$$