Homology Groups of Torus as a CW Complex

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In this note, we shall compute the singular homology groups of a 2-torus with the help of its CW-complex structure.

§1 Generator of $H_p\left(B^p, S^{p-1}\right)$

We know that $H_p\left(B^p, S^{p-1}\right)$ is an infinite cyclic group, and hence it is isomorphic to $\mathbb{Z}$, the additive group of integers. Therefore, it has two choices for a generator. In this section, we shall explicitly compute the generator.

Let Δ_p be the standard p-simplex. Since there is a homeomorphism of Δ_p onto B^p that takes $\operatorname{Bd}\Delta_p$ onto S^{p-1}, $H_p\left(B^p, S^{p-1}\right) \cong H_p\left(\Delta_p, \operatorname{Bd}\Delta_p\right)$. So it suffices to compute a generator for $H_p\left(\Delta_p, \operatorname{Bd}\Delta_p\right)$. If f is the said isomorphism, and α is a generator of $H_p\left(\Delta_p, \operatorname{Bd}\Delta_p\right)$, then $\left(f_*\right)_p \alpha$ will be a generator of $H_p\left(B^p, S^{p-1}\right)$.

Lemma 1. Let $i : \Delta_p \to \Delta_p$ be the identity map. Then $\{i\}$ is a generator of $H_p\left(\Delta_p, \operatorname{Bd}\Delta_p\right)$.

Proof. Clearly, $\partial_p i$ is carried by $\operatorname{Bd}\Delta_p$, so $i \in Z_p\left(\Delta_p, \operatorname{Bd}\Delta_p\right)$ and $\{i\} \in H_p\left(\Delta_p, \operatorname{Bd}\Delta_p\right)$. Let $\{g\}$ be a generator of $H_p\left(\Delta_p, \operatorname{Bd}\Delta_p\right)$, where $g : \Delta_p \to \Delta_p$ is continuous with $\partial_p g$ carried by $\operatorname{Bd}\Delta_p$. So

$$\{i\} = n\{g\}\ , \quad \text{for some } n \in \mathbb{Z}. \tag{1}$$

Now, notice that

$$\left(g_*\right)_p \{i\} = \left\{\left(g_\#\right)_p i\right\} = \{g \circ i\} = \{g\}\,. \tag{2}$$

So $\operatorname{im}\left(g_*\right)_p$ contains $\{g\}$. Therefore, $\operatorname{im}\left(g_*\right)_p$ is the whole $H_p\left(\Delta_p, \operatorname{Bd}\Delta_p\right)$. In other words, $\left(g_*\right)_p$ is surjective. The only surjective map from an infinite cyclic group to itself is either the identity or negative identity. So

$$\left(g_*\right)_p = \{g\} \ \text{ or } \ -\{g\}\,. \tag{3}$$

Now,

$$\{g\} = \left(g_*\right)_p \{i\} = n\left(g_*\right)_p \{g\} = n\{g\} \ \text{ or } \ -n\{g\}\,. \tag{4}$$

Therefore, n must be either 1 or -1. As a result, either $\{g\} = \{i\}$, or $\{g\} = -\{i\}$, proving that $\{i\}$ is a generator. ∎

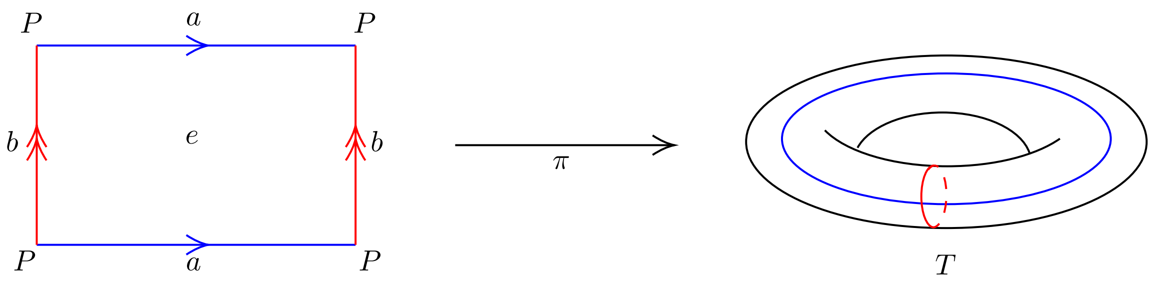
Suppose $f : \left(\Delta_p, \operatorname{Bd}\Delta_p\right) \to \left(B^p, S^{p-1}\right)$ is a homeomorphism. Since $\{i\}$ is a generator of $H_p\left(\Delta_p, \operatorname{Bd}\Delta_p\right)$, a generator of $H_p\left(B^p, S^{p-1}\right)$ would be

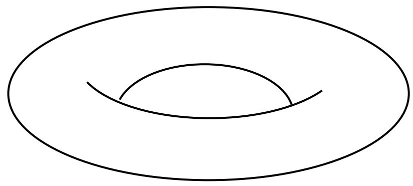
$$\left(f_*\right)_p \{i\} = \left\{\left(f_\#\right)_p i\right\} = \{f \circ i\} = \{f\}\,. \tag{5}$$

Therefore, $\{f\}$ is a generator of $H_p\left(B^p, S^{p-1}\right)$.

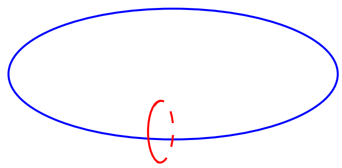
§2 CW Structure of Torus

We consider the torus as a quotient space of a rectangle as usual (by identifying opposite sides of a rectangle).





2-skeleton of T



1-skeleton of T

0-skeleton of T

We express T as a CW complex having a single 2-cell e (the image under π of the interior of the rectangle), two 1-cells a and b (the images of the 2 open edges of the rectangle under π), and one 0-cell P (the image of the vertices of the rectangle under π).

Now, we form the cellular complex of T. Let X^2, X^1 and X^0 denote the 2-skeleton, 1-skeleton and 0-skeleton of T, respectively. Since we have only one 2-cell e,

$$H_2\left(X^2, X^1\right) = H_2\left(\bar{e}, \dot{e}\right) \cong \mathbb{Z}. \tag{6}$$

Also, we have two 1-cells a and b, so

$$H_1\left(X^1, X^0\right) = H_1\left(\bar{a}, \dot{a}\right) \oplus H_1\left(\bar{b}, \dot{b}\right) \cong \mathbb{Z} \oplus \mathbb{Z}. \tag{7}$$

There is only one 1-cell, hence

$$H_0\left(X^0, X^{-1}\right) = H_0\left(X^0, \varnothing\right) = H_0\left(P, \varnothing\right) \cong \mathbb{Z}. \tag{8}$$

Since there are no cells in dimension higher than 2, $H_n\left(X^n, X^{n-1}\right) = 0$ for $n \geq 2$. So we have the following cellular complex $\mathcal{D}(T)$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_2(X^2, X^1) & \xrightarrow{d_2} & H_1(X^1, X^0) & \xrightarrow{d_1} & H_0(X^0, \varnothing) & \longrightarrow & 0 \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & 0
\end{array}$$

d_2 and d_1 are the cellular boundary maps.

§3 Computation of Homology Groups

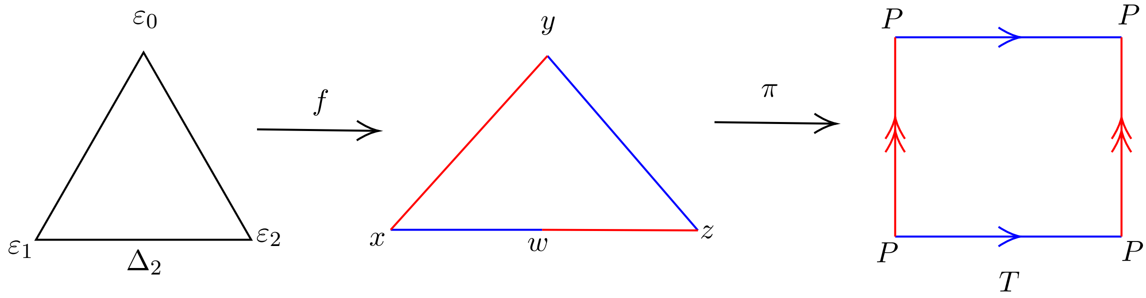
We know that

$$H_n\left(\mathcal{D}\left(T\right)\right) \cong H_n\left(T\right). \tag{9}$$

So, we only need to compute the homology groups of the cellular complex $\mathcal{D}\left(T\right)$. For that purpose, we first compute how the cellular boundary map d_2 works. d_2 is the composite

$$\begin{tikzcd} H_2(X^2, X^1) \arrow[r, "(\partial_*)_2"] \arrow[rr, bend right, "d_2 = (j_*)_1 \circ (\partial_*)_2"'] & H_1(X^1) \arrow[r, "(j_*)_1"] & H_1(X^1, X^0) \end{tikzcd}$$

For the 2-cell e, the following composite $g = \pi \circ f$ is the characteristic map (we replaced B^2 by Δ_2 since they are homeomorphic).



Now, let $i : \Delta_2 \to \Delta_2$ be the identity map. $\{i\}$ generates $H_2\left(\Delta_2, \operatorname{Bd}\Delta_2\right)$. So $\left(g_*\right)_2\{i\}$ generates $H_2\left(X^2, X^1\right)$. We want to see d_2's action on $\left(g_*\right)_2\{i\}$.

$$\left(\partial_*\right)_2\left(g_*\right)_2\{i\} = \left(\partial_*\right)_2\{g\} = \{\partial_2 g\} \in H_1\left(X^1\right). \tag{10}$$

Now,

$$\partial_2 g = \partial_2\left(\pi \circ f\right) = \partial_2\left(\left(\pi_\#\right)_2 f\right) = \left(\pi_\#\right)_1\left(\partial_2 f\right). \tag{11}$$

Now,

$$\partial_2 f = f \circ l_{(\varepsilon_1,\varepsilon_2)} - f \circ l_{(\varepsilon_0,\varepsilon_2)} + f \circ l_{(\varepsilon_0,\varepsilon_1)} = l_{(x,z)} - l_{(y,z)} + l_{(y,x)}. \tag{12}$$

Now we claim that $l_{(x,z)}$ is homologous to $l_{(x,w)} - l_{(z,w)}$. Indeed,

$$\partial_2 l_{(x,z,w)} = l_{(z,w)} - l_{(x,w)} + l_{(x,z)}. \tag{13}$$

Since $l_{(x,z)}$ and $l_{(x,w)} - l_{(z,w)}$ differ by a boundary term $\partial_2 l_{(x,z,w)}$, they are homologous. Therefore, $\partial_2 f$ is homologous to

$$l_{(x,w)} - l_{(z,w)} - l_{(y,z)} + l_{(y,x)} \tag{14}$$

Now, after a composition with π, $l_{(x,w)}$ and $l_{(y,z)}$ are the same. Furthermore, $l_{(z,w)}$ and $l_{(y,x)}$ are also the same after a composition with π. Therefore,

$$\{\partial_2 g\} = \left\{\left(\pi_\#\right)_1\left(\partial_2 f\right)\right\} = \left\{\pi \circ l_{(x,w)} - \pi \circ l_{(z,w)} - \pi \circ l_{(y,z)} + \pi \circ l_{(y,x)}\right\} = 0. \tag{15}$$

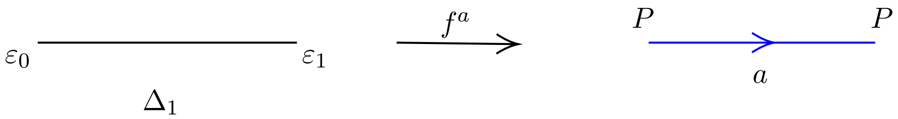
Therefore, $\left(\partial_*\right)_2\left(g_*\right)_2\{i\} = 0$. This proves that d_2 is the zero map.

Now we shall see how the cellular boundary map $d_1 : H_1\left(X^1, X^0\right) \to H_0\left(X^0\right)$ works. d_1 is equal to the homology boundary homomorphism $\left(\partial_*\right)_1$ of the pair $\left(X^1, X^0\right)$.

We have two 1 cells in X, namely a and b. Therefore,

$$H_1\left(X^1, X^0\right) = H_1\left(\overline{a}, \dot{a}\right) \oplus H_1\left(\overline{b}, \dot{b}\right). \tag{16}$$

Let's consider a first. As shown in the figure in previous page, a is the image of the blue line. Its characteristic map is f^a in the following image:



Since $H_1\left(\Delta_1, \operatorname{Bd}\Delta_1\right)$ is generated by $\{j\}$, where j is the identity $j : \Delta_1 \to \Delta_1$, $H_1\left(\overline{a}, \dot{a}\right)$ is generated by $\left(f^a_*\right)_1\{j\}$. We shall now see d_1's action on $\left(f^a_*\right)_1\{j\}$.

$$\left(\partial_*\right)_1\left(f^a_*\right)_1\{j\} = \left(\partial_*\right)_1\{f^a\} = \{\partial_1 f^a\}. \tag{17}$$

Now,

$$\partial_1 f^a = f^a \circ l_{\left(\widehat{\varepsilon_0}, \varepsilon_1\right)} - f^a \circ l_{\left(\varepsilon_0, \widehat{\varepsilon_1}\right)}. \tag{18}$$

Observe that

$$f^a \circ l_{\left(\widehat{\varepsilon_0}, \varepsilon_1\right)}(1, 0, \dots) = f^a\left(\varepsilon_0\right) = P, \tag{19}$$
$$f^a \circ l_{\left(\varepsilon_0, \widehat{\varepsilon_1}\right)}(1, 0, \dots) = f^a\left(\varepsilon_1\right) = P. \tag{20}$$

Therefore, $\partial_1 f^a = 0$. This proves that $\left(\partial_*\right)_1\left(f^a_*\right)_1\{j\} = 0$. So d_1's action on the generator of $H_1\left(\overline{a}, \dot{a}\right)$ gives 0. Similarly, d_1's action on the generator of $H_1\left(\overline{b}, \dot{b}\right)$ is also 0. Therefore, d_1 is the 0 map.

Now, we are ready to compute the homology groups of $\mathcal{D}(T)$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_2(X^2, X^1) & \xrightarrow{d_2} & H_1(X^1, X^0) & \xrightarrow{d_1} & H_0(X^0, \varnothing) & \longrightarrow & 0 \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & 0
\end{array}$$

$$H_0(\mathcal{D}(T)) = \frac{\operatorname{Ker} d_0}{\operatorname{im} d_1} = \frac{H_0\left(X^0\right)}{\{0\}} = H_0\left(X^0\right) \cong \mathbb{Z}, \tag{21}$$
$$H_1(\mathcal{D}(T)) = \frac{\operatorname{Ker} d_1}{\operatorname{im} d_2} = \frac{H_1\left(X^1, X^0\right)}{\{0\}} = H_1\left(X^1, X^0\right) \cong \mathbb{Z} \oplus \mathbb{Z}, \tag{22}$$
$$H_2(\mathcal{D}(T)) = \frac{\operatorname{Ker} d_2}{\operatorname{im} d_3} = \frac{H_2\left(X^2, X^1\right)}{\{0\}} = H_2\left(X^2, X^1\right) \cong \mathbb{Z}, \tag{23}$$
$$H_n(\mathcal{D}(T)) = 0 \text{ for } n \geq 3. \tag{24}$$

Therefore, the homology groups of torus are

$$H_n(T) \cong \begin{cases} \mathbb{Z} & \text{if } n = 0 \text{ or } n = 2, \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{25}$$