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Differential Geometry II (MAT401)

Lecture Notes

Preface

This series of lecture notes has been prepared for aiding students who took the BRAC University course **Differential Geometry II (MAT401)** in Fall 2023 semester. These notes were typeset under the supervision of mathematician **Dr. Syed Hasibul Hassan Chowdhury**. The main goal of this typeset is to have an organized digital version of the notes, which is easier to share and handle. If you see any mistakes or typos, please send me an email at atonuroychowdhury@gmail.com

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References:

- *An Introduction to Manifolds*, by **Loring W. Tu**
- *An Introduction to Differentiable Manifolds and Riemannian Geometry*, by **William Boothby**
- *Introduction to Smooth Manifolds*, by **John M. Lee**
- *Lectures on Differential geometry*, by **S.S Chern, W.H. Chen and K.S. Lam**
- *Geometry of Differential Forms*, by **Shigeyuki Morita**
- *From Calculus to Cohomology: De Rham Cohomology and Characteristic Classes*, by **Ib Madsen and Jxrgen Tornehave**.

Contents

1 Review of Multilinear Algebra

§1.1 Dual Space

Let V and W be real vector spaces. We denote by $\mathrm{Hom}\left(V,W\right)$ the vector space of all linear maps $f:V\to W$. In particular, if we choose $W=\mathbb{R}$, we get the **dual space** V^*.

$$V^*=\mathrm{Hom}\left(V,\mathbb{R}\right).$$

The elements of V^* are called covectors on V. In the rest of the lecture, we will assume V to be a finite dimensional vector space. Let $\{\mathbf{e}_1,\mathbf{e}_2,\ldots,\mathbf{e}_n\}$ be a basis for V. Then every $\mathbf{v}\in V$ is a unique linear combination

$$\mathbf{v}=\sum_{i=1}^{n}v^i\mathbf{e}_i, \tag{1.1}$$

with $v^i\in\mathbb{R}$. v^i's are called the coordinates of $\mathbf{v}$ relative to the basis $\{\mathbf{e}_1,\mathbf{e}_2,\ldots,\mathbf{e}_n\}$. Let $\widehat{\alpha}^i$ be the linear function on V that picks up the i-th coordinate of the vector, i.e.

$$\widehat{\alpha}^i\left(\mathbf{v}\right)=\widehat{\alpha}^i\left(\sum_{i=1}^{n}v^i\mathbf{e}_i\right)=v^i. \tag{1.2}$$

When $\mathbf{v}$ is one of the basis vectors,

$$\widehat{\alpha}^i\left(\mathbf{e}_j\right)={\delta^i}_j=\begin{cases}1 & \text{if } i=j,\\ 0 & \text{if } i\neq j.\end{cases} \tag{1.3}$$

Proposition 1.1

The functions $\widehat{\alpha}^1,\ldots,\widehat{\alpha}^n$ form a basis for V^*.

Proof. Suppose $f\in V^*$. Then for any $\mathbf{v}=\sum_{i=1}^{n}v^i\mathbf{e}_i\in V$,

$$f\left(\mathbf{v}\right)=f\left(\sum_{i=1}^{n}v^i\mathbf{e}_i\right)=\sum_{i=1}^{n}v^if\left(\mathbf{e}_i\right)=\sum_{i=1}^{n}f\left(\mathbf{e}_i\right)\widehat{\alpha}^i\left(\mathbf{v}\right).$$

Since this holds for any $\mathbf{v}\in V$,

$$f=\sum_{i=1}^{n}f\left(\mathbf{e}_i\right)\widehat{\alpha}^i. \tag{1.4}$$

Therefore, $\widehat{\alpha}^1,\ldots,\widehat{\alpha}^n$ span V^*. As for linear independence, suppose

$$\sum_{i=1}^{n}c_i\widehat{\alpha}^i=\mathbf{0}, \tag{1.5}$$

where $\mathbf{0}$ is the function that takes all of V to $0\in\mathbb{R}$. If we evaluate (1.5) at $\mathbf{e}_j$, we get

$$0=\sum_{i=1}^{n}c_i\widehat{\alpha}^i\left(\mathbf{e}_j\right)=\sum_{i=1}^{n}c_i{\delta^i}_j=c_j. \tag{1.6}$$

So $c_j=0$, and this holds for each $j=1,2,\ldots,n$. Therefore, $\{\widehat{\alpha}^1,\ldots,\widehat{\alpha}^n\}$ is a linearly independent set that spans V^*, i.e. a basis. ■

Corollary 1.2

The dual space V^* of a finite dimensional vector space has the same dimension as V.

The basis $\{\widehat{\alpha}^1,\ldots,\widehat{\alpha}^n\}$ for V^* is said to be dual to the basis $\{\mathbf{e}_1,\mathbf{e}_2,\ldots,\mathbf{e}_n\}$ for V.

§1.2 Permutations

Fix a positive integer k. A permutation of the set $A = \{1, 2, \dots, k\}$ is a bijection $\sigma : A \to A$. The product of two permutations τ and σ is the composition $\tau \circ \sigma : A \to A$. The **cyclic permutation** $(a_1 \ a_2 \ \cdots \ a_r)$ is the permutation σ such that

$$\sigma(a_1) = a_2, \ \sigma(a_2) = a_3, \ \cdots, \ \sigma(a_{r-1}) = a_r, \ \text{and } \sigma(a_r) = 1,$$

leaving all other elements unchanged, i.e. $\sigma(j) = j$ if j is not one of the a_i's. A cyclic permutation $(a_1 \ a_2 \ \cdots \ a_r)$ is also called a **cycle** of length r or an r-cycle. A **transposition** is a permutation of the form $(a \ b)$ that interchanges a and b, leaving all other elements of A fixed.

A permutation $\sigma : A \to A$ can be described by

$$\begin{bmatrix} 1 & 2 & \cdots & k \\ \sigma(1) & \sigma(2) & \cdots & \sigma(k) \end{bmatrix}.$$

We also write it as

$$[1 \ 2 \ \cdots \ k] \overset{\sigma}{\mapsto} [\sigma(1) \ \sigma(2) \ \cdots \ \sigma(k)].$$

Example 1.1. Suppose $\sigma : \{1, 2, 3, 4, 5\} \to \{1, 2, 3, 4, 5\}$ is the permutation given by

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 5 & 1 & 3 \end{bmatrix}.$$

In other words, $\sigma(1) = 2$, $\sigma(2) = 4$, $\sigma(3) = 5$, $\sigma(4) = 1$, and $\sigma(5) = 3$.

$$[1 \ 2 \ 3 \ 4 \ 5] \overset{\sigma}{\mapsto} [2 \ 4 \ 5 \ 1 \ 3].$$

Observe that the cyclic permutation $\sigma' = (1 \ 2 \ 4)$ acts as $\sigma'(1) = 2$, $\sigma'(2) = 4$ and $\sigma'(4) = 1$, keeping 3 and 5 unchanged, i.e. $\sigma'(3) = 3$ and $\sigma'(5) = 5$.

$$[1 \ 2 \ 3 \ 4 \ 5] \xmapsto{(1 \ 2 \ 4)} [2 \ 4 \ 3 \ 1 \ 5].$$

Now the transposition $\sigma'' = (3 \ 5)$ acts as $\sigma''(3) = 5$ and $\sigma''(5) = 3$, keeping $1, 2, 4$ unchanged. Therefore,

$$\begin{tikzcd} {[1 \ 2 \ 3 \ 4 \ 5]} \arrow[r, mapsto, "(1 \ 2 \ 4)"] \arrow[rr, bend right, "(3 \ 5)(1 \ 2 \ 4)"'] & {[2 \ 4 \ 3 \ 1 \ 5]} \arrow[r, mapsto, "(3 \ 5)"] & {[2 \ 4 \ 5 \ 1 \ 3]} \end{tikzcd}$$

so that $\sigma = (3 \ 5)(1 \ 2 \ 4)$.

Let S_k be the group of permutations of the set $\{1, 2, \dots, k\}$. The order of this group is $k!$. A permutation is *even* or *odd* depending on whether it is the product of an even or an odd number of transpositions. The sign of a permutation σ is 1 if the permutation is even, and -1 otherwise. It is denoted by $\operatorname{sgn} \sigma$. For example, in Example 1.1, $\sigma = (3 \ 5)(1 \ 2 \ 4)$. Note that we can write $(1 \ 2 \ 4)$ as a product of two transpositions:

$$\begin{tikzcd} {[1 \ 2 \ 3 \ 4 \ 5]} \arrow[r, mapsto, "(1 \ 2)"] \arrow[rr, bend right, "(1 \ 4)(1 \ 2) = (1 \ 2 \ 4)"'] & {[2 \ 1 \ 3 \ 4 \ 5]} \arrow[r, mapsto, "(1 \ 4)"] & {[2 \ 4 \ 3 \ 1 \ 5]} \end{tikzcd}$$

In other words, $\sigma = (3 \ 5)(1 \ 4)(1 \ 2)$. Hence, $\operatorname{sgn} \sigma = -1$. One can easily check that

$$\operatorname{sgn}(\sigma\tau) = (\operatorname{sgn} \sigma)(\operatorname{sgn} \tau). \tag{1.7}$$

So $\operatorname{sgn} : S_k \to \{1, -1\}$ is a group homomorphism.

Example 1.2. Observe that the 5-cycle $(1\ 2\ 3\ 4\ 5)$ can be written as

$$(1\ 2\ 3\ 4\ 5) = (1\ 5)\,(1\ 4)\,(1\ 3)\,(1\ 2)\,.$$

Indeed,

$$\begin{tikzcd}
{[1\ 2\ 3\ 4\ 5]} \arrow[r, mapsto, "(1\ 2)"] \arrow[rrrr, bend right, "(1\ 2\ 3\ 4\ 5)=(1\ 5)(1\ 4)(1\ 3)(1\ 2)"'] & {[2\ 1\ 3\ 4\ 5]} \arrow[r, mapsto, "(1\ 3)"] & {[2\ 3\ 1\ 4\ 5]} \arrow[r, mapsto, "(1\ 4)"] & {[2\ 3\ 4\ 1\ 5]} \arrow[r, mapsto, "(1\ 5)"] & {[2\ 3\ 4\ 5\ 1]}
\end{tikzcd}$$

Therefore, $\operatorname{sgn}(1\ 2\ 3\ 4\ 5) = 1$.

An **inversion** in a permutation σ is an ordered pair $(\sigma(i), \sigma(j))$ such that $i < j$ but $\sigma(i) > \sigma(j)$. In Example 1.1, $\sigma(1) = 2$, $\sigma(2) = 4$, $\sigma(3) = 5$, $\sigma(4) = 1$, and $\sigma(5) = 3$. So, the inversions in this permutation are

$$(\sigma(1), \sigma(4))\,,\ (\sigma(2), \sigma(4))\,,\ (\sigma(2), \sigma(5))\,,\ (\sigma(3), \sigma(4))\,,\ (\sigma(3), \sigma(5))\,.$$

Hence, there are 5 inversions associated with the permutation σ. There is an efficient way of determining the sign of a permutation.

Proposition 1.3

A permutation is even if and only if it has an even number of inversions.

Proof. Let $\sigma \in S_k$ with n inversions. We shall prove that we can multiply σ by n transpositions and get the identity permutation. This will prove that $\operatorname{sgn}\sigma = (-1)^n$.

Suppose $\sigma(j_1) = 1$. Then for each $i < j_1$, $(\sigma(i), \sigma(j_1))$ is an inversion, and there are $j_1 - 1$ many of them. These are all the inversions with 1 in the second slot of the ordered pair of inversion. If we now multiply σ by the j_1-cycle

$$(\sigma(1)\ \ 1)\,(\sigma(2)\ \ 1) \cdots (\sigma(j_1 - 1)\ \ 1)$$

to the left of σ, the resulting permutation σ_1 would be

$$\begin{bmatrix} 1 & 2 & 3 & \cdots & j_1 & j_1 + 1 & \cdots & k \\ 1 & \sigma(1) & \sigma(2) & \cdots & \sigma(j_1 - 1) & \sigma(j_1 + 1) & \cdots & \sigma(k)\,. \end{bmatrix}$$

This permutation has no inversion with 1 in the second slot of the ordered pair of inversion. Suppose now that $\sigma(j_2) = 2$. Now observe that if $(\sigma_1(i), 2)$ is an inversion in σ_1, then either $(\sigma(i), 2)$ (if $i \geq j_1 + 1$) or $\sigma(i-1), 2$ (if $i \leq j_1 - 1$) is an inversion in σ. Therefore, the number of inversions in σ_1 ending in 2 is precisely the same as the number of inversions in σ ending in 2. So following a similar procedure as above, we can multiply σ_1 by i_2-many transpositions to the left (i_2 is the number of transpositions ending in 2) and get

$$\begin{bmatrix} 1 & 2 & 3 & \cdots & k \\ 1 & 2 & \sigma(1) & \cdots & \sigma(k)\,. \end{bmatrix}$$

We can continue these steps for each $j = 1, 2, \ldots, k$, and the number of transpositions required to move j to its natural position is the same as the number of inversions ending in j. In the end we achieve the identity permutation. Therefore, $\operatorname{sgn}\sigma = (-1)^n$, where n is the number of inversions. ■

§1.3 Multilinear Functions

Definition 1.1. Let V^k be the cartesian product of k-copies of a real vector space V.

$$V^k = \underbrace{V \times V \times \cdots \times V}_{k\text{-copies}}$$

A function $f : V^k \to \mathbb{R}$ is called k-linear if it is linear in each of its k arguments:

$$f(\ldots, a\mathbf{v} + b\mathbf{w}, \ldots) = a\,f(\ldots, \mathbf{v}, \ldots) + b\,f(\ldots, \mathbf{w}, \ldots), \tag{1.8}$$

for $a, b \in \mathbb{R}$ and $\mathbf{v}, \mathbf{w} \in V$.

Instead of 2-linear and 3-linear, it's customary to call "bilinear" and "trilinear", respectively. A k-linear function on V is called a k**-tensor** on V. We will denote the vector space of all k-tensors on V by $L_k(V)$. The vector addition and scalar multiplication of the real vector space $L_k(V)$ is the straightforward pointwise operation.

Example 1.3. The dot product $f(\mathbf{v}, \mathbf{w}) = \mathbf{v} \cdot \mathbf{w}$ on $\mathbb{R}^n$ is bilinear: if $\mathbf{v} = \sum_{i=1}^n v^i \mathbf{e}_i$ and $\mathbf{w} = \sum_{i=1}^n w^i \mathbf{e}_i$, then

$$f(\mathbf{v}, \mathbf{w}) = \mathbf{v} \cdot \mathbf{w} = \sum_{i=1}^n v^i w^i.$$

Example 1.4. The determinant

$$f(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n) = \det \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}$$

viewed as a function of the n column vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is n-linear.

Definition 1.2 (Symmetric and alternating function)**.** A k-linear function $f : V^k \to \mathbb{R}$ is **symmetric** if

$$f\left(\mathbf{v}_{\sigma(1)}, \mathbf{v}_{\sigma(2)}, \ldots, \mathbf{v}_{\sigma(k)}\right) = f(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k), \tag{1.9}$$

for all permutations $\sigma \in S_k$. It is **alternating** if

$$f\left(\mathbf{v}_{\sigma(1)}, \mathbf{v}_{\sigma(2)}, \ldots, \mathbf{v}_{\sigma(k)}\right) = (\operatorname{sgn}\sigma)\, f(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k), \tag{1.10}$$

for all permutations $\sigma \in S_k$.

The dor product function on $\mathbb{R}^n$ in Theorem 1.3 is symmetric, and the determinant function on $\mathbb{R}^n$ in Theorem 1.4 is alternating.

We are especially interested in the vector space $A_k(V)$ of all alternating k-linear functions on a vector space V, for $k > 0$. The elements of $A_k(V)$ are called alternating k-tensors (also known as k-covectors). We define $A_0(V)$ to be $\mathbb{R}$. The elements of $A_0(V)$ are simply constants, which we call 0-covectors. The elements of $A_1(V)$ are simply covectors, i.e. the elements of V^*.

Permutation action on k-linear functions

If $f \in L_k(V)$ and $\sigma \in S_k$, define $\sigma f \in L_k(V)$ as follows:

$$(\sigma f)(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k) = f\left(\mathbf{v}_{\sigma(1)}, \mathbf{v}_{\sigma(2)}, \ldots, \mathbf{v}_{\sigma(k)}\right). \tag{1.11}$$

Thus, f is symmetric if and only if $f = \sigma f$ for all $\sigma \in S_k$; and f is alternating if and only if $\sigma f = (\operatorname{sgn}\sigma) f$ for all $\sigma \in S_k$. When $k = 1$, S_k only has the identity permutation. In that case, a 1-linear function or simply linear function on V is both symmetric and alternating. In particular,

$$A_1(V) = L_1(V) = V^*.$$

Lemma 1.4

If $\sigma, \tau \in S_k$ and $f \in L_k(V)$, then $\tau(\sigma f) = (\tau\sigma) f$.

Proof. For any $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in V$,

$$\begin{aligned}
(\tau(\sigma f))(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k) &= (\sigma f)\left(\mathbf{v}_{\tau(1)}, \mathbf{v}_{\tau(2)}, \dots, \mathbf{v}_{\tau(k)}\right) \\
&= (\sigma f)(\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_k) && [\mathbf{w}_i = \mathbf{v}_{\tau(i)}] \\
&= f\left(\mathbf{w}_{\sigma(1)}, \mathbf{w}_{\sigma(2)}, \dots, \mathbf{w}_{\sigma(k)}\right) \\
&= f\left(\mathbf{v}_{\tau(\sigma(1))}, \mathbf{v}_{\tau(\sigma(2))}, \dots, \mathbf{v}_{\tau(\sigma(k))}\right) \\
&= ((\tau\sigma) f)(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k).
\end{aligned}$$

Therefore, $\tau(\sigma f) = (\tau\sigma) f$. ∎

Definition 1.3. If G is a group and X is a set, a map

$$\begin{aligned} G \times X &\to X \\ (g, x) &\mapsto g \cdot x \end{aligned}$$

is called a **left action** of G on X if

(i) $e \cdot x = x$, where e is the identity element in G and x is any element in X; and

(ii) $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$, for all $g_1, g_2 \in G$ and $x \in X$.

Similarly, a **right action** of G on X is a map

$$\begin{aligned} X \times G &\to X \\ (x, g) &\mapsto x \cdot g \end{aligned}$$

such that

(i) $x \cdot e = x$, for all $x \in X$; and

(ii) $(x \cdot g_1) \cdot g_2 = x \cdot (g_1 g_2)$, for all $g_1, g_2 \in G$ and $x \in X$.

Symmetrizing and alternating operators

Given $f \in L_k(V)$, there is a way to make it a symmetric k-linear function $\mathcal{S} f$ from it:

$$(\mathcal{S} f)(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k) = \sum_{\sigma \in S_k} f\left(\mathbf{v}_{\sigma(1)}, \mathbf{v}_{\sigma(2)}, \dots, \mathbf{v}_{\sigma(k)}\right). \tag{1.12}$$

In other words,

$$\mathcal{S} f = \sum_{\sigma \in S_k} \sigma f. \tag{1.13}$$

Similarly, there is a way to make an alternating k-linear function from f:

$$\mathcal{A} f = \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma)\, \sigma f. \tag{1.14}$$

Proposition 1.5 (i) The k-linear function $\mathcal{S} f$ is symmetric.

(ii) The k-linear function $\mathcal{A} f$ is alternating.

Proof. (i) Let $\tau \in S_k$. Then

$$\tau\left(\mathcal{S}f\right) = \tau\left(\sum_{\sigma\in S_k} \sigma f\right). \tag{1.15}$$

The group action of S_k on $L_k(V)$ is distributive over the vector space addition. Therefore,

$$\tau\left(\mathcal{S}f\right) = \sum_{\sigma\in S_k} \tau(\sigma f) = \sum_{\sigma\in S_k} (\tau\sigma) f. \tag{1.16}$$

As σ varies over S_k, $\tau\sigma$ also varies over S_k. Therefore, $\sum_{\sigma\in S_k}(\tau\sigma)f = \mathcal{S}f$. In other words,

$$\tau\left(\mathcal{S}f\right) = \mathcal{S}f, \tag{1.17}$$

i.e. $\mathcal{S}f$ is symmetric.

(ii) Let $\tau \in S_k$. Then

$$\tau\left(\mathcal{A}f\right) = \tau\left(\sum_{\sigma\in S_k} (\operatorname{sgn}\sigma)\,\sigma f\right) = \sum_{\sigma\in S_k} (\operatorname{sgn}\sigma)\,\tau(\sigma f) = \sum_{\sigma\in S_k} (\operatorname{sgn}\sigma)\,(\tau\sigma) f. \tag{1.18}$$

Since $(\operatorname{sgn}\tau)^2 = 1$,

$$\begin{aligned}\tau\left(\mathcal{A}f\right) &= \sum_{\sigma\in S_k} (\operatorname{sgn}\tau)^2 (\operatorname{sgn}\sigma)\,(\tau\sigma) f\\ &= (\operatorname{sgn}\tau) \sum_{\sigma\in S_k} (\operatorname{sgn}\tau)(\operatorname{sgn}\sigma)\,(\tau\sigma) f\\ &= (\operatorname{sgn}\tau) \sum_{\sigma\in S_k} (\operatorname{sgn}(\tau\sigma))\,(\tau\sigma) f. \end{aligned}\tag{1.19}$$

As σ varies over S_k, $\tau\sigma$ also varies over S_k. Therefore, $\sum_{\sigma\in S_k}(\operatorname{sgn}(\tau\sigma))\,(\tau\sigma)f = \mathcal{A}f$. In other words,

$$\tau\left(\mathcal{A}f\right) = \mathcal{A}f, \tag{1.20}$$

i.e. $\mathcal{A}f$ is alternating.

■

Lemma 1.6

If $f \in A_k(V)$, then $\mathcal{A}f = (k!)\,f$.

Proof. Since f is alternating,

$$\mathcal{A}f = \sum_{\sigma\in S_k} (\operatorname{sgn}\sigma)\,\sigma f = \sum_{\sigma\in S_k} (\operatorname{sgn}\sigma)^2 f = \sum_{\sigma\in S_k} f = (k!)\,f, \tag{1.21}$$

because the order of S_k is $k!$. ■

§1.4 Tensor Product and Wedge Product

Definition 1.4 (Tensor Product)**.** Let f be a k-linear function and g an l-linear function on a vector space V. Their tensor product $f\otimes g$ is the $(k+l)$-linear function defined by

$$(f\otimes g)(\mathbf{v}_1,\ldots,\mathbf{v}_k,\mathbf{v}_{k+1},\ldots,\mathbf{v}_{k+l}) = f(\mathbf{v}_1,\ldots,\mathbf{v}_k)\,g(\mathbf{v}_{k+1},\ldots,\mathbf{v}_{k+l}). \tag{1.22}$$

$(k+l)$-linearity of $f\otimes g$ follows from k-linearity of f and l-linearity of g.

Lemma 1.7 (Associativity of Tensor Product)
Let $f \in L_k(V)$, $g \in L_l(V)$ and $h \in L_m(V)$. Then

$$(f \otimes g) \otimes h = f \otimes (g \otimes h)\,.$$

Proof. For $\mathbf{v}_1, \ldots, \mathbf{v}_{k+l+m}$,

$$\begin{aligned}[(f \otimes g) \otimes h](\mathbf{v}_1, \ldots, \mathbf{v}_{k+l+m}) &= (f \otimes g)(\mathbf{v}_1, \ldots, \mathbf{v}_{k+l})\, h(\mathbf{v}_{k+l+1}, \ldots, \mathbf{v}_{k+l+m}) \\ &= f(\mathbf{v}_1, \ldots, \mathbf{v}_k)\, g(\mathbf{v}_{k+1}, \ldots, \mathbf{v}_{k+l})\, h(\mathbf{v}_{k+l+1}, \ldots, \mathbf{v}_{k+l+m})\,. \end{aligned} \tag{1.23}$$

$$\begin{aligned}[f \otimes (g \otimes h)](\mathbf{v}_1, \ldots, \mathbf{v}_{k+l+m}) &= f(\mathbf{v}_1, \ldots, \mathbf{v}_k)\,(g \otimes h)(\mathbf{v}_{k+1}, \ldots, \mathbf{v}_{k+l+m}) \\ &= f(\mathbf{v}_1, \ldots, \mathbf{v}_k)\, g(\mathbf{v}_{k+1}, \ldots, \mathbf{v}_{k+l})\, h(\mathbf{v}_{k+l+1}, \ldots, \mathbf{v}_{k+l+m})\,. \end{aligned} \tag{1.24}$$

Therefore, $(f \otimes g) \otimes h = f \otimes (g \otimes h)$, i.e. tensor product is associative. ■

Example 1.5. Let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be the standard basis for $\mathbb{R}^n$, and $\{\widehat{\alpha}^1, \ldots, \widehat{\alpha}^n\}$ its dual basis. The Euclidean inner product on $\mathbb{R}^n$ is the bilinear function

$$\langle\ ,\ \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$

defined by

$$\langle\ ,\ \rangle(\mathbf{v}, \mathbf{w}) = \langle \mathbf{v}, \mathbf{w} \rangle = \sum_{i=1}^{n} v^i w^i,$$

for $\mathbf{v} = \sum_{i=1}^n v^i \mathbf{e}_i$ and $\mathbf{w} = \sum_{i=1}^n w^i \mathbf{e}_i$. We can express $\langle\ ,\ \rangle$ in terms of tensor product as follows:

$$\langle\ ,\ \rangle(\mathbf{v}, \mathbf{w}) = \sum_{i=1}^{n} v^i w^i = \sum_{i=1}^{n} \widehat{\alpha}^i(\mathbf{v})\, \widehat{\alpha}^i(\mathbf{w}) = \sum_{i=1}^{n} \left(\widehat{\alpha}^i \otimes \widehat{\alpha}^i\right)(\mathbf{v}, \mathbf{w})\,.$$

Since $\mathbf{v}, \mathbf{w}$ are arbitrary,

$$\langle\ ,\ \rangle = \sum_{i=1}^{n} \left(\widehat{\alpha}^i \otimes \widehat{\alpha}^i\right). \tag{1.25}$$

If $f \in A_k(V)$ and $g \in A_l(V)$, then it's not true that $f \otimes g \in A_{k+l}(V)$, in general. We need to construct a product that is also alternating.

Definition 1.5 (Wedge Product). For $f \in A_k(V)$ and $g \in A_l(V)$, the wedge product of f and g is defined as follows:

$$f \wedge g = \frac{1}{k!l!} \mathcal{A}(f \otimes g)\,. \tag{1.26}$$

Explicitly,

$$\begin{aligned}(f \wedge g)(\mathbf{v}_1, \ldots, \mathbf{v}_{k+l}) &= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (\operatorname{sgn}\sigma)\, \sigma(f \otimes g)(\mathbf{v}_1, \ldots, \mathbf{v}_{k+l}) \\ &= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (\operatorname{sgn}\sigma)\,(f \otimes g)\left(\mathbf{v}_{\sigma(1)}, \ldots, \mathbf{v}_{\sigma(k+l)}\right) \\ &= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (\operatorname{sgn}\sigma)\, f\left(\mathbf{v}_{\sigma(1)}, \ldots, \mathbf{v}_{\sigma(k)}\right) g\left(\mathbf{v}_{\sigma(k+1)}, \ldots, \mathbf{v}_{\sigma(k+l)}\right). \end{aligned} \tag{1.27}$$

When $k = 0$, the element $f \in A_0(V)$ is simply a constant $c \in \mathbb{R}$ as discussed earlier. In this case, the wedge product $c \wedge g$ is just scalar multiplication as is evident from (1.27).

$$\begin{aligned}(c \wedge g)(\mathbf{v}_1, \ldots, \mathbf{v}_l) &= \frac{1}{l!} \sum_{\sigma \in S_l} (\operatorname{sgn} \sigma)\, c\, g\left(\mathbf{v}_{\sigma(1)}, \ldots, \mathbf{v}_{\sigma(l)}\right) \\ &= \frac{1}{l!} \sum_{\sigma \in S_l} (\operatorname{sgn} \sigma)\, c\, (\operatorname{sgn} \sigma)\, g(\mathbf{v}_1, \ldots, \mathbf{v}_l) \\ &= \frac{1}{l!} \sum_{\sigma \in S_l} c\, g(\mathbf{v}_1, \ldots, \mathbf{v}_l) \\ &= \frac{1}{l!} l! c\, g(\mathbf{v}_1, \ldots, \mathbf{v}_l) \\ &= c\, g(\mathbf{v}_1, \ldots, \mathbf{v}_l).\end{aligned}$$

Thus $c \wedge g = cg$, for $c \in \mathbb{R}$ and $g \in A_l(V)$.

Example 1.6. For $f \in A_2(V)$ and $g \in A_1(V)$,

$$\begin{aligned}\mathcal{A}(f \otimes g)(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = f(\mathbf{v}_1, \mathbf{v}_2)\, g(\mathbf{v}_3) - f(\mathbf{v}_1, \mathbf{v}_3)\, g(\mathbf{v}_2) - f(\mathbf{v}_2, \mathbf{v}_1)\, g(\mathbf{v}_3) \\ - f(\mathbf{v}_3, \mathbf{v}_2)\, g(\mathbf{v}_1) + f(\mathbf{v}_2, \mathbf{v}_3)\, g(\mathbf{v}_1) + f(\mathbf{v}_3, \mathbf{v}_1)\, g(\mathbf{v}_2).\end{aligned}$$

Among these 6 terms, there are 3 pairs of equal terms due to the alternating nature of f.

$$\begin{aligned}f(\mathbf{v}_1, \mathbf{v}_2)\, g(\mathbf{v}_3) &= -f(\mathbf{v}_2, \mathbf{v}_1)\, g(\mathbf{v}_3), \\ f(\mathbf{v}_3, \mathbf{v}_1)\, g(\mathbf{v}_2) &= -f(\mathbf{v}_1, \mathbf{v}_3)\, g(\mathbf{v}_2), \\ f(\mathbf{v}_2, \mathbf{v}_3)\, g(\mathbf{v}_1) &= -f(\mathbf{v}_3, \mathbf{v}_2)\, g(\mathbf{v}_1).\end{aligned}$$

Therefore,

$$\mathcal{A}(f \otimes g)(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = 2f(\mathbf{v}_1, \mathbf{v}_2)\, g(\mathbf{v}_3) + 2f(\mathbf{v}_3, \mathbf{v}_1)\, g(\mathbf{v}_2) + 2f(\mathbf{v}_2, \mathbf{v}_3)\, g(\mathbf{v}_1). \tag{1.28}$$

Hence,

$$\begin{aligned}(f \wedge g)(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) &= \frac{1}{2!1!} \mathcal{A}(f \otimes g)(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \\ &= f(\mathbf{v}_1, \mathbf{v}_2)\, g(\mathbf{v}_3) + f(\mathbf{v}_3, \mathbf{v}_1)\, g(\mathbf{v}_2) + f(\mathbf{v}_2, \mathbf{v}_3)\, g(\mathbf{v}_1). \qquad (1.29)\end{aligned}$$

Example 1.7 (Wedge product of 2 covectors)**.** If $f, g \in A_1(V)$, and $\mathbf{v}_1, \mathbf{v}_2 \in V$, then

$$(f \wedge g)(\mathbf{v}_1, \mathbf{v}_2) = \frac{1}{1!1!} \mathcal{A}(f \otimes g)(\mathbf{v}_1, \mathbf{v}_2).$$

S_2 has 2 elements: the identity element e and $(1\ 2)$. Therefore,

$$(f \wedge g)(\mathbf{v}_1, \mathbf{v}_2) = \mathcal{A}(f \otimes g)(\mathbf{v}_1, \mathbf{v}_2) = f(\mathbf{v}_1)\, g(\mathbf{v}_2) - f(\mathbf{v}_2)\, g(\mathbf{v}_1).$$

Proposition 1.8 (Anticommutativity of wedge product)

The wedge product is anticommutative: if $f \in A_k(V)$ and $g \in A_l(V)$, then

$$f \wedge g = (-1)^{kl}\, g \wedge f.$$

Proof. Define $\tau \in S_{k+l}$ to be the following permutation:

$$\begin{bmatrix} 1 & 2 & \cdots & l & l+1 & l+2 & \cdots & l+k \\ k+1 & k+2 & \cdots & k+l & 1 & 2 & \cdots & k \end{bmatrix}.$$

In other words,

$$\tau(i) = \begin{cases} k+i & \text{if } 1 \le i \le l, \\ i-l & \text{if } l+1 \le i \le l+k. \end{cases}$$

Then for any $\sigma \in S_{k+l}$,

$$\sigma(j) = \begin{cases} \sigma(\tau(l+j)) & \text{if } 1 \le j \le k, \\ \sigma(\tau(j-k)) & \text{if } k+1 \le j \le k+l. \end{cases} \tag{1.30}$$

Now, for any $\mathbf{v}_1, \ldots, \mathbf{v}_{k+l} \in V$,

$$\begin{aligned} \mathcal{A}(f \otimes g)(\mathbf{v}_1, \ldots, \mathbf{v}_{k+l}) &= \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma) f\left(\mathbf{v}_{\sigma(1)}, \ldots, \mathbf{v}_{\sigma(k)}\right) g\left(\mathbf{v}_{\sigma(k+1)}, \ldots, \mathbf{v}_{\sigma(k+l)}\right) \\ &= \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma) f\left(\mathbf{v}_{\sigma(\tau(l+1))}, \ldots, \mathbf{v}_{\sigma(\tau(l+k))}\right) g\left(\mathbf{v}_{\sigma(\tau(1))}, \ldots, \mathbf{v}_{\sigma(\tau(l))}\right) \\ &= (\operatorname{sgn} \tau) \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma)(\operatorname{sgn} \tau)\, g\left(\mathbf{v}_{\sigma(\tau(1))}, \ldots, \mathbf{v}_{\sigma(\tau(l))}\right) f\left(\mathbf{v}_{\sigma(\tau(l+1))}, \ldots, \mathbf{v}_{\sigma(\tau(l+k))}\right) \\ &= (\operatorname{sgn} \tau) \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma\tau)\, g\left(\mathbf{v}_{\sigma(\tau(1))}, \ldots, \mathbf{v}_{\sigma(\tau(l))}\right) f\left(\mathbf{v}_{\sigma(\tau(l+1))}, \ldots, \mathbf{v}_{\sigma(\tau(l+k))}\right). \end{aligned}$$

Again, as σ varies over S_{k+l}, $\sigma\tau$ also varies over S_{k+l}. Therefore,

$$\mathcal{A}(f \otimes g)(\mathbf{v}_1, \ldots, \mathbf{v}_{k+l}) = (\operatorname{sgn} \tau)\, \mathcal{A}(g \otimes f)(\mathbf{v}_1, \ldots, \mathbf{v}_{k+l}). \tag{1.31}$$

Now, let us evaluate the sign of the permutation τ. Let $(\tau(i), \tau(j))$ be an inversion of τ. Then it's not possible that $1 \le i < j \le l$, or $l+1 \le i < j \le l+k$; because if we have $1 \le i < j \le l$ or $l+1 \le i < j \le l+k$, then $\tau(i) < \tau(j)$. Therefore, i must be in between 1 and l (inclusive), and j must be in between $l+1$ and $l+k$ (inclusive). So there are l options for i, and k options for j. Therefore, τ has kl many inversions. So $\operatorname{sgn} \tau = (-1)^{kl}$. Using (1.31),

$$\mathcal{A}(f \otimes g) = (-1)^{kl}\, \mathcal{A}(g \otimes f). \tag{1.32}$$

Dividing by $k!l!$, we obtain

$$f \wedge g = (-1)^{kl}\, g \wedge f. \tag{1.33}$$

∎

Corollary 1.9

If f is a k-covector on V, i.e. $f \in A_k(V)$, and k is odd, then $f \wedge f = 0$.

Proof. By anticommutativity of wedge product,

$$f \wedge f = (-1)^{k^2} f \wedge f = -f \wedge f.$$

Therefore, $f \wedge f = 0$. ∎

If f is a k-covector and g is an l-covector, i.e. $f \in A_k(V)$ and $g \in A_l(V)$, then we have defined their wedge product to be the $(k+l)$-covector

$$f \wedge g = \frac{1}{k!l!}\mathcal{A}(f \otimes g). \tag{1.34}$$

We have the following lemmas associated with the alternating operator $\mathcal{A}$.

Lemma 1.10

Suppose $f \in L_k(V)$ and $g \in L_l(V)$. Then

(i) $\mathcal{A}(\mathcal{A}(f) \otimes g) = k!\mathcal{A}(f \otimes g)$.

(ii) $\mathcal{A}(f \otimes \mathcal{A}(g)) = l!\mathcal{A}(f \otimes g)$.

Proof. (i) By definition,

$$\begin{aligned}\mathcal{A}(\mathcal{A}(f) \otimes g) &= \sum_{\sigma \in S_{k+l}} (\operatorname{sgn}\sigma)\,\sigma(\mathcal{A}(f) \otimes g) \\ &= \sum_{\sigma \in S_{k+l}} (\operatorname{sgn}\sigma)\,\sigma\left[\sum_{\tau \in S_k} (\operatorname{sgn}\tau)(\tau f) \otimes g\right]. \qquad (1.35)\end{aligned}$$

We can view $\tau \in S_k$ as a permutation in the following way: define $\tau' \in S_{k+l}$ as follows

$$\tau'(i) = \begin{cases} \tau(i) & \text{if } i \leq k, \\ i & \text{if } i > k. \end{cases} \qquad (1.36)$$

Then for $\mathbf{v}_1, \ldots, \mathbf{v}_{k+l}$, we have

$$\begin{aligned}[(\tau f) \otimes g](\mathbf{v}_1, \ldots, \mathbf{v}_{k+l}) &= (\tau f)(\mathbf{v}_1, \ldots, \mathbf{v}_k)\, g(\mathbf{v}_{k+1}, \ldots, \mathbf{v}_{k+l}) \\ &= f\left(\mathbf{v}_{\tau(1)}, \ldots, \mathbf{v}_{\tau(k)}\right) g(\mathbf{v}_{k+1}, \ldots, \mathbf{v}_{k+l}) \\ &= f\left(\mathbf{v}_{\tau'(1)}, \ldots, \mathbf{v}_{\tau'(k)}\right) g\left(\mathbf{v}_{\tau'(k+1)}, \ldots, \mathbf{v}_{\tau'(k+l)}\right) \\ &= [\tau'(f \otimes g)](\mathbf{v}_1, \ldots, \mathbf{v}_{k+l}).\end{aligned}$$

Therefore, $(\tau f) \otimes g = \tau'(f \otimes g)$. Furthermore, $\operatorname{sgn}\tau = \operatorname{sgn}\tau'$ since the inversions $(\tau'(i), \tau'(j))$ occur only when $1 \leq i < j \leq k$, so that the τ and τ' has the same number of inversions.

Let us abuse notation a bit and denote by S_k the subgroup of permutations in S_{k+l} by keeping the last l arguments fixed. This subgroup of S_{k+l} is indeed isomorphic to S_k, so we will denote both these groups by S_k. Therefore, from (1.35),

$$\begin{aligned}\mathcal{A}(\mathcal{A}(f) \otimes g) &= \sum_{\sigma \in S_{k+l}} (\operatorname{sgn}\sigma)\,\sigma\left[\sum_{\tau' \in S_k \subseteq S_{k+l}} (\operatorname{sgn}\tau')\,\tau'(f \otimes g)\right] \\ &= \sum_{\sigma \in S_{k+l}} \sum_{\tau' \in S_k \subseteq S_{k+l}} (\operatorname{sgn}\sigma)(\operatorname{sgn}\tau')\,\sigma\tau'(f \otimes g) \\ &= \sum_{\tau' \in S_k \subseteq S_{k+l}} \sum_{\sigma \in S_{k+l}} (\operatorname{sgn}\sigma \operatorname{sgn}\tau')\left((\sigma\tau')(f \otimes g)\right).\end{aligned}$$

For a fixed τ', as σ varies over S_{k+l}, $\sigma\tau'$ also varies over S_{k+l}. Therefore,

$$\mathcal{A}(\mathcal{A}(f) \otimes g) = \sum_{\tau' \in S_k \subseteq S_{k+l}} \mathcal{A}(f \otimes g) = k!\mathcal{A}(f \otimes g). \qquad (1.37)$$

(ii) By (1.32),

$$\begin{aligned}\mathcal{A}(f \otimes \mathcal{A}(g)) &= \mathcal{A}\left((-1)^{kl}\mathcal{A}(g) \otimes f\right) \\ &= (-1)^{kl}\mathcal{A}(\mathcal{A}(g) \otimes f) \\ &= (-1)^{kl}\, l!\mathcal{A}(g \otimes f) \\ &= l!\mathcal{A}\left((-1)^{kl} g \otimes f\right) \\ &= l!\mathcal{A}(f \otimes g). \qquad (1.38)\end{aligned}$$

■

Proposition 1.11 (Associativity of wedge product)
Let V be a real vector space and f, g, h be alternating multilinear functions on V of degree k, l, m, respectively. Then

$$(f \wedge g) \wedge h = f \wedge (g \wedge h)\,.$$

Proof. Using the definition of wedge product,

$$\begin{aligned}
(f \wedge g) \wedge h &= \frac{1}{(k+l)!m!}\mathcal{A}\left[(f \wedge g) \otimes h\right] \\
&= \frac{1}{(k+l)!m!}\mathcal{A}\left[\frac{1}{k!l!}\mathcal{A}\left(f \otimes g\right) \otimes h\right] \\
&= \frac{1}{(k+l)!k!l!m!}\mathcal{A}\left[\mathcal{A}\left(f \otimes g\right) \otimes h\right] \\
&= \frac{(k+l)!}{(k+l)!k!l!m!}\mathcal{A}\left[(f \otimes g) \otimes h\right] \\
&= \frac{1}{k!l!m!}\mathcal{A}\left[(f \otimes g) \otimes h\right].
\end{aligned}$$

On the other hand,

$$\begin{aligned}
f \wedge (g \wedge h) &= \frac{1}{k!\,(l+m)!}\mathcal{A}\left[f \otimes (g \wedge h)\right] \\
&= \frac{1}{k!\,(l+m)!}\mathcal{A}\left[f \otimes \left(\frac{1}{l!m!}\mathcal{A}\left(g \otimes h\right)\right)\right] \\
&= \frac{1}{k!\,(l+m)!l!m!}\mathcal{A}\left[f \otimes \mathcal{A}\left(g \otimes h\right)\right] \\
&= \frac{(l+m)!}{k!\,(l+m)!l!m!}\mathcal{A}\left[f \otimes (g \otimes h)\right] \\
&= \frac{1}{k!l!m!}\mathcal{A}\left[f \otimes (g \otimes h)\right].
\end{aligned}$$

Since tensor product is associative (by Lemma 1.7), we conclude that

$$(f \wedge g) \wedge h = f \wedge (g \wedge h)\,. \tag{1.39}$$

■

By associativity, we can omit the parenthesis and write univocally $f \wedge g \wedge h$ instead of $(f \wedge g) \wedge h$ or $f \wedge (g \wedge h)$.

Corollary 1.12
Under the hypothesis of Proposition 1.11,

$$f \wedge g \wedge h = \frac{1}{k!l!m!}\mathcal{A}\left[f \otimes g \otimes h\right]. \tag{1.40}$$

This easily generalizes to an arbitrary number of factors: if $f_i \in A_{d_i}(V)$ for $i = 1, 2, \ldots, r$, i.e. f_i is an alternating d_i-linear function on V, then

$$f_1 \wedge \cdots \wedge f_r = \frac{1}{d_1! \cdots d_r!}\mathcal{A}\left(f_1 \otimes \cdots \otimes f_r\right). \tag{1.41}$$

Proposition 1.13

Let $\widehat{\alpha}^1, \widehat{\alpha}^2, \ldots, \widehat{\alpha}^k$ be linear functions on a real vector space V (i.e. $\widehat{\alpha}^i : V \to \mathbb{R}$) and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in V$. Then

$$\begin{aligned}\left(\widehat{\alpha}^1 \wedge \cdots \wedge \widehat{\alpha}^k\right)(\mathbf{v}_1, \ldots, \mathbf{v}_k) &= \det\left[\widehat{\alpha}^i(\mathbf{v}_j)\right] \\ &= \det\begin{bmatrix} \widehat{\alpha}^1(\mathbf{v}_1) & \widehat{\alpha}^1(\mathbf{v}_2) & \cdots & \widehat{\alpha}^1(\mathbf{v}_k) \\ \widehat{\alpha}^2(\mathbf{v}_1) & \widehat{\alpha}^2(\mathbf{v}_2) & \cdots & \widehat{\alpha}^2(\mathbf{v}_k) \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{\alpha}^k(\mathbf{v}_1) & \widehat{\alpha}^k(\mathbf{v}_2) & \cdots & \widehat{\alpha}^k(\mathbf{v}_k) \end{bmatrix}.\end{aligned}$$

Proof. By 1.41,

$$\left(\widehat{\alpha}^1 \wedge \cdots \wedge \widehat{\alpha}^k\right)(\mathbf{v}_1, \ldots, \mathbf{v}_k) = \mathcal{A}\left(\widehat{\alpha}^1 \otimes \cdots \otimes \widehat{\alpha}^k\right)(\mathbf{v}_1, \ldots, \mathbf{v}_k).$$

By the definition of the action of alternating operator,

$$\mathcal{A}\left(\widehat{\alpha}^1 \otimes \cdots \otimes \widehat{\alpha}^k\right)(\mathbf{v}_1, \ldots, \mathbf{v}_k) = \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma)\, \widehat{\alpha}^1\left(\mathbf{v}_{\sigma(1)}\right) \cdots \widehat{\alpha}^k\left(\mathbf{v}_{\sigma(k)}\right). \tag{1.42}$$

By the definition of determinant of a $k \times k$ matrix $A = [a_{ij}]$,

$$\det A = \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma)\, a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{k\sigma(k)}. \tag{1.43}$$

Using (1.43) in (1.42), we get

$$\mathcal{A}\left(\widehat{\alpha}^1 \otimes \cdots \otimes \widehat{\alpha}^k\right)(\mathbf{v}_1, \ldots, \mathbf{v}_k) = \det\left[\widehat{\alpha}^i(\mathbf{v}_j)\right]. \tag{1.44}$$

■

§1.5 A Basis for $A_k(V)$

Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be a basis for a real vector space V, and let $\{\widehat{\alpha}^1, \ldots, \widehat{\alpha}^n\}$ be the dual basis for V^*. Introduce the multi-index notation

$$I = (i_1, i_2, \ldots, i_k)$$

and write $\mathbf{e}_I$ for $(\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \ldots, \mathbf{e}_{i_k})$ and $\widehat{\alpha}^I$ for $\widehat{\alpha}^{i_1} \wedge \widehat{\alpha}^{i_2} \wedge \cdots \wedge \widehat{\alpha}^{i_k}$.

A k-linear function f on V is completely determined by its values on all k-tuples $(\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \ldots, \mathbf{e}_{i_k})$. If f is alternating, then f is completely determined by its values on all k-tuples $(\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \ldots, \mathbf{e}_{i_k})$ with

$$1 \leq i_1 < i_2 < \cdots < i_k \leq n.$$

In other words, it's sufficient to consider $\mathbf{e}_I$ with I in ascending order.

Lemma 1.14

Suppose I and J are ascending multi-indices of length k. Then

$$\widehat{\alpha}^I(\mathbf{e}_J) = \delta^I{}_J := \begin{cases} 1 & \text{if } I = J, \\ 0 & \text{if } I \neq J. \end{cases}$$

Proof. Suppose $I = (i_1, \ldots, i_k)$ and $J = (j_1, \ldots, j_k)$. Using (1.42), we get

$$\begin{aligned}\widehat{\alpha}^I(\mathbf{e}_J) &= \left(\widehat{\alpha}^{i_1} \wedge \widehat{\alpha}^{i_2} \wedge \cdots \wedge \widehat{\alpha}^{i_k}\right)(\mathbf{e}_{j_1}, \mathbf{e}_{j_2}, \ldots, \mathbf{e}_{j_k}) \\ &= \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma)\, \widehat{\alpha}^{i_1}\left(\mathbf{e}_{j_{\sigma(1)}}\right) \cdots \widehat{\alpha}^{i_k}\left(\mathbf{e}_{j_{\sigma(k)}}\right) \\ &= \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma)\, \delta^{i_1}{}_{j_{\sigma(1)}} \cdots \delta^{i_k}{}_{j_{\sigma(k)}}.\end{aligned} \tag{1.45}$$

The terms in the sum (1.45) contribute $\operatorname{sgn}\sigma$ if and only if

$$(i_1, i_2, \ldots, i_k) = \left(j_{\sigma(1)}, j_{\sigma(2)}, \ldots, j_{\sigma(k)}\right);$$

otherwise they contribute 0 to the sum. Both I and J are ascending multi-indices. Permuting the elements of J no longer gives an ascending multi-index (unless the permutation σ is the identity permutation). Therefore, in (1.45), all the summands corresponding to σ being a non-identity permutation contribute 0.

$$\widehat{\alpha}^I(\mathbf{e}_J) = \sum_{\sigma \in S_k} (\operatorname{sgn}\sigma)\, \delta^{i_1}{}_{j_{\sigma(1)}} \cdots \delta^{i_k}{}_{j_{\sigma(k)}} = \delta^{i_1}{}_{j_1} \cdots \delta^{i_k}{}_{j_k} = \begin{cases} 1 & \text{if } I = J, \\ 0 & \text{if } I \neq J. \end{cases} \tag{1.46}$$

■

Proposition 1.15

The alternating k-linear functions $\widehat{\alpha}^I$, $I = (i_1, \ldots, i_k)$, with $1 \leq i_1 < \cdots < i_k \leq n$ form a basis for the space $A_k(V)$ of alternating k-linear functions on V.

Proof. Let us first show linear independence. Suppose

$$\sum_I c_I \widehat{\alpha}^I = \mathbf{0}, \tag{1.47}$$

$c_I \in \mathbb{R}$ with I running over ascending multi-indices of length k. Applying $\mathbf{e}_J$ to both sides, we get

$$0 = \sum_I c_I \widehat{\alpha}^I(\mathbf{e}_J) = \sum_I c_I \delta^I{}_J = c_J. \tag{1.48}$$

Therefore, $\left\{\widehat{\alpha}^I \mid I \text{ is ascending multi-index of length } k\right\}$ is a linearly independent set. Now let us prove that this set spans $A_k(V)$. Let $f \in A_k(V)$. We claim that

$$f = \sum_I f(\mathbf{e}_I)\, \widehat{\alpha}^I. \tag{1.49}$$

Let $g = \sum_I f(\mathbf{e}_I)\, \widehat{\alpha}^I$. We need to prove that $f = g$. By k-linearity and alternating property, if two k-covectors agree on all $\mathbf{e}_J$ where J is an ascending multi-index, then they are equal. Now,

$$g(\mathbf{e}_J) = \sum_I f(\mathbf{e}_I)\, \widehat{\alpha}^I(\mathbf{e}_J) = \sum_I f(\mathbf{e}_I)\, \delta^I{}_J = f(\mathbf{e}_J). \tag{1.50}$$

Therefore, $f = g = \sum_I f(\mathbf{e}_I)\, \widehat{\alpha}^I$. ■

Corollary 1.16

If the vector space V has dimension n, then the vector space $A_k(V)$ of k-covectors on V has dimension $\binom{n}{k}$.

Proof. An ascending multi-index $I = (i_1, i_2, \ldots, i_k)$, $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ is obtained by choosing a k-element subset of $\{1, 2, \ldots, n\}$. This can be done in $\binom{n}{k}$ ways. ■

Corollary 1.17

If $k > \dim V$, then $A_k(V) = 0$.

Proof. If $k > \dim V = n$, then in the expression

$$\widehat{\alpha}^{i_1} \wedge \widehat{\alpha}^{i_2} \wedge \cdots \wedge \widehat{\alpha}^{i_k}$$

with each $i \in \{1, 2, \ldots, n\}$, there must be a repeated i_j's, say $\widehat{\alpha}^r$. Then $\widehat{\alpha}^r \wedge \widehat{\alpha}^r$ arises in the expression $\widehat{\alpha}^{i_1} \wedge \widehat{\alpha}^{i_2} \wedge \cdots \wedge \widehat{\alpha}^{i_k}$. But $\widehat{\alpha}^r \wedge \widehat{\alpha}^r = 0$ by Corollary 1.9. Hence, $\widehat{\alpha}^{i_1} \wedge \widehat{\alpha}^{i_2} \wedge \cdots \wedge \widehat{\alpha}^{i_k} = 0$. Therefore, the basis set of $A_k(V)$ is empty, meaning $A_k(V) = 0$. ■

2 Differential Forms on $\mathbb{R}^n$

Given an open set $U \subseteq \mathbb{R}^n$ and $p \in U$, T_pU is the set of tangent vectors at $p \in U$ is identified with the point derivations of C_p^∞ (germs of smooth functions at p), i.e. a tangent vector $X_p \in T_pU$ is a map $X_p : C_p^\infty \to \mathbb{R}$ such that X_p is $\mathbb{R}$-linear:

$$X_p\left(\alpha f + g\right) = \alpha\left(X_p f\right) + X_p g; \tag{2.1}$$

and satisfies the Leibniz condition:

$$X_p\left(fg\right) = \left(X_p f\right) g(p) + f(p)\left(X_p g\right).$$

In contrast to the notion of point derivation, there is this notion of derivation of aa algebra. If X is a C^∞ vector field on an open subset $U \subseteq \mathbb{R}^n$, i.e. $X \in \mathfrak{X}\left(U\right)$, and f is a C^∞ function on U, i.e. $f \in C^\infty(U)$, then $Xf \in C^\infty(U)$ defined by

$$\left(Xf\right)(p) = X_p f.$$

Remember that f in (2.1) and (2) is a representative of an equivalence class, the equivalence class of germs of C^∞ functions at $p \in U$. These equivalence classes constitute $C_p^\infty(U)$. It is of course an $\mathbb{R}$-algebra. While in (2), $f \in C^\infty(U)$, the algebra of C^∞ functions on U with no reference of p whatsoever.

From the discussion above, a C^∞ vector field X gives rise to an $\mathbb{R}$-linear map $C^\infty(U) \to C^\infty(U)$ by $f \mapsto Xf$ that additionally has to satisfy the following Leibniz condition:

$$X\left(fg\right) = \left(Xf\right) g + f\left(Xg\right). \tag{2.2}$$

Note that a derivation at p is not a derivation of the algebra C_p^∞. A derivation at p is a map from $C_p^\infty \to \mathbb{R}$ that satisfies (2), while a derivation of the algebra C_p^∞ is supposed to be a map from C_p^∞ to itself obeying Leibniz condition.

§2.1 1 form

From any C^∞ function $f : U \to \mathbb{R}$, one can construct a 1-form (dual notion of C^∞ vector field) $\mathrm{d}f$, the restriction of which to a given point $p \in U$ yields a covector $\left(\mathrm{d}f\right)_p \in T_p^*U$, the dual space of T_pU, in the following way:

$$\left(\mathrm{d}f\right)_p\left(X_p\right) = X_p f. \tag{2.3}$$

Proposition 2.1

If $x^1, x^2, \ldots, x^n$ are the standard coordinates on $\mathbb{R}^n$, then at each point $p \in \mathbb{R}^n$,

$$\left\{\left(\mathrm{d}x^1\right)_p, \left(\mathrm{d}x^2\right)_p, \ldots, \left(\mathrm{d}x^n\right)_p\right\}$$

is the basis for the cotangent space $T_p^*\mathbb{R}^n$ dual to the basis $\left\{\left.\frac{\partial}{\partial x^1}\right|_p, \ldots, \left.\frac{\partial}{\partial x^n}\right|_p\right\}$ for the tangent space $T_p\mathbb{R}^n$.

Proof. $\left(\mathrm{d}x^i\right)_p : T_p^*\mathbb{R}^n \to \mathbb{R}$ is a linear map for each i. Now,

$$\left(\mathrm{d}x^i\right)_p\left(\left.\frac{\partial}{\partial x^j}\right|_p\right) = \left.\frac{\partial}{\partial x^j}\right|_p\left(x^i\right) = \delta^i{}_j. \tag{2.4}$$

Therefore, $\left\{(\mathrm{d}x^1)_p, (\mathrm{d}x^2)_p, \ldots, (\mathrm{d}x^n)_p\right\}$ is the basis of $T_p^*\mathbb{R}^n$ dual to the basis $\left\{ \frac{\partial}{\partial x^1}\Big|_p, \ldots, \frac{\partial}{\partial x^n}\Big|_p \right\}$ for $T_p\mathbb{R}^n$. ■

If ω is a 1-form on an open subset $U \subseteq \mathbb{R}^n$, then by Proposition 2.1, there is a linear combination

$$\omega_p = \sum_{i=1}^{n} a_i(p) \left(\mathrm{d}x^i\right)_p, \tag{2.5}$$

for some $a_i(p) \in \mathbb{R}$. As p varies over U, the coefficients a_i become functions on U, and we may write

$$\omega = \sum_{i=1}^{n} a_i \mathrm{d}x^i. \tag{2.6}$$

The 1-form ω is said to be C^∞ on U if the coefficient functions a_i are all C^∞ functions on U.

Proposition 2.2 (The differential in terms of coordinates)
If $f : U \to \mathbb{R}$ is a C^∞ function on an open set $U \subseteq \mathbb{R}^n$, then

$$\mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} \mathrm{d}x^i.$$

Proof. By Proposition 2.1, at each point $p \in U$,

$$(\mathrm{d}f)_p = \sum_{i=1}^{n} a_i(p) \left(\mathrm{d}x^i\right)_p, \tag{2.7}$$

for some constants $a_i(p)$ depending on p. Thus

$$\mathrm{d}f = \sum_{i=1}^{n} a_i \ \mathrm{d}x^i, \tag{2.8}$$

for some functions a_i on U. To evaluate a_j, apply both sides of (2.8) to the coordinate vector field $\frac{\partial}{\partial x^j}$:

$$\mathrm{d}f\left(\frac{\partial}{\partial x^j}\right) = \sum_{i=1}^{n} a_i \ \mathrm{d}x^i \left(\frac{\partial}{\partial x^j}\right) = \sum_{i=1}^{n} a_i \delta^i{}_j = a_j. \tag{2.9}$$

On the other hand, using $(\mathrm{d}f)_p (X_p) = X_p f = (Xf)(p)$, we get $(\mathrm{d}f)(X) = Xf$. So

$$\mathrm{d}f\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial f}{\partial x^j}. \tag{2.10}$$

Therefore, $a_j = \frac{\partial f}{\partial x^j}$. Hence,

$$\mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} \ \mathrm{d}x^i. \tag{2.11}$$

■

(2.11) tells us that $\mathrm{d}f$ will be a C^∞ 1-form if $\frac{\partial f}{\partial x^i}$ is C^∞ on U. Hence, it is sufficient to have f as a C^∞ function on U in order to have $\mathrm{d}f$ as a C^∞ 1-form.

§2.2 Differential k-forms

A differential form ω of degree k (or a k-form) on an open subset $U \subseteq \mathbb{R}^n$ is a map that assigns to each point $p \in U$, an alternating k-linear function on the tangent space $T_p\mathbb{R}^n$, i.e.

$$\omega_p \in A_k\left(T_p\mathbb{R}^n\right).$$

By Proposition 1.15, a basis for $A_k\left(T_p\mathbb{R}^n\right)$ is

$$\left(\mathrm{d}x^I\right)_p = \left(\mathrm{d}x^{i_1}\right)_p \wedge \left(\mathrm{d}x^{i_2}\right)_p \wedge \cdots \wedge \left(\mathrm{d}x^{i_k}\right)_p,$$

where $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. Therefore, at each point $p \in U$, ω_p is a linear combination

$$\omega_p = \sum_I a_I(p)\left(\mathrm{d}x^I\right)_p, \tag{2.12}$$

and a k-form ω on U is a linear combination

$$\omega = \sum_I a_I \, \mathrm{d}x^I, \tag{2.13}$$

with function coefficients $a_I : U \to \mathbb{R}$. We say that a k-form ω is **smooth** on U if all the coeffieicnts a_I are C^∞ functions on U.

Denote by $\Omega^k(U)$ the vector space of C^∞ k-forms on U. A 0-form on U assigns to each point $p \in U$ an element of $A_0\left(T_p\mathbb{R}^n\right) = \mathbb{R}$. Thus a 0-form on U is simply a real-valued function on U, and $\Omega^0(U) = C^\infty(U)$.

Since one can multiply a C^∞ k-form by a C^∞ function on U from the left, the set $\Omega^k(U)$ of C^∞ k-forms on U is both a real vector space and a $C^\infty(U)$-module. With the wedge product as multiplication, the direct sum

$$\Omega^*(U) = \bigoplus_{k=0}^{n} \Omega^k(U)$$

becomes an algebra over $\mathbb{R}$ as well as a module over $C^\infty(U)$. As an algebra, it is anticommutative and associative.

Example 2.1. Let x, y, z be the coordinates on $\mathbb{R}^3$. The C^∞ 1-forms on $\mathbb{R}^3$ are

$$P(x,y,z)\,\mathrm{d}x + Q(x,y,z)\,\mathrm{d}y + R(x,y,z)\,\mathrm{d}z,$$

where P, Q, R range over all C^∞ functions on $\mathbb{R}^3$. The C^∞ 2-forms are

$$A(x,y,z)\,\mathrm{d}y \wedge \mathrm{d}z + B(x,y,z)\,\mathrm{d}x \wedge \mathrm{d}z + C(x,y,z)\,\mathrm{d}x \wedge \mathrm{d}y;$$

and the C^∞ 1-forms are

$$a(x,y,z)\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z.$$

Example 2.2 (A basis for 3-covectors)**.** Let x^1, x^2, x^3, x^4 be the standard coordinates on $\mathbb{R}^4$, and $p \in \mathbb{R}^4$. A basis for $A_3\left(T_p\mathbb{R}^4\right)$ is

$$\left\{\left(\mathrm{d}x^1\right)_p \wedge \left(\mathrm{d}x^2\right)_p \wedge \left(\mathrm{d}x^3\right)_p, \left(\mathrm{d}x^1\right)_p \wedge \left(\mathrm{d}x^2\right)_p \wedge \left(\mathrm{d}x^4\right)_p, \right.$$
$$\left. \left(\mathrm{d}x^1\right)_p \wedge \left(\mathrm{d}x^3\right)_p \wedge \left(\mathrm{d}x^4\right)_p, \left(\mathrm{d}x^2\right)_p \wedge \left(\mathrm{d}x^3\right)_p \wedge \left(\mathrm{d}x^4\right)_p\right\}.$$

So $\dim\left(A_3\left(T_p\mathbb{R}^n\right)\right) = 4$.

§2.3 Exterior Derivative

Before defining exterior derivative of a C^∞ k-form on an open subset $U \subseteq \mathbb{R}^n$, we first define it on 0-forms. The exterior derivative of a C^∞ function $f \in C^\infty(U)$ is its differential:

$$\mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} \mathrm{d}x^i \in \Omega^1(U).$$

Definition 2.1 (Exterior Derivative)**.** If $\omega = \sum_I a_I \, \mathrm{d}x^I \in \omega^K(U)$, then its exterior derivative is defined as follows:

$$\mathrm{d}\omega = \sum_I \mathrm{d}a_I \wedge \mathrm{d}x^I = \sum_I \left(\sum_{j=1}^{n} \frac{\partial a_I}{\partial x^j} \, \mathrm{d}x^j \right) \wedge \mathrm{d}x^I \in \Omega^{k+1}(U). \tag{2.14}$$

Example 2.3. Let ω be the 1 form $f\mathrm{d}x + g\mathrm{d}y$ on $\mathbb{R}^2$, where f and g are C^∞ functions on $\mathbb{R}^2$. Let us write $f_x = \frac{\partial f}{\partial x}$ and $f_y = \frac{\partial f}{\partial y}$. Then

$$\begin{aligned}
\mathrm{d}\omega &= \mathrm{d}f \wedge \mathrm{d}x + \mathrm{d}g \wedge \mathrm{d}y \\
&= (f_x \mathrm{d}x + f_y \mathrm{d}y) \wedge \mathrm{d}x + (g_x \mathrm{d}x + g_y \mathrm{d}y) \wedge \mathrm{d}y \\
&= -f_y \, \mathrm{d}x \wedge \mathrm{d}y + g_x \, \mathrm{d}x \wedge \mathrm{d}y \\
&= (g_x - f_y) \, \mathrm{d}x \wedge \mathrm{d}y.
\end{aligned}$$

Definition 2.2 (Graded Algebra)**.** An algebra A over a field $\mathbb{K}$ is said to be **graded** if it can be written as a direct sum

$$A = \bigoplus_{k=0}^{\infty} A^k$$

of vector spaces over $\mathbb{K}$ so that the multiplication map sends $A^k \times A^l$ to A^{k+l}.

The notation $A = \bigoplus_{k=0}^{\infty} A^k$ means that each element of A is uniquely a **finite sum**

$$a = a_{i_1} + a_{i_2} + \cdots + a_{i_m},$$

where $a_{i_j} \in A^{i_j}$.

Example 2.4. The polynomial algebra

$$\mathbb{R}[x, y] = \bigoplus_{k=0}^{\infty} A^k$$

with A^k being the vector space of homogenous polynomials of degree k in x and y. Observe that the 0 polynomial is trivially homogenous of any degree, and hence belongs to A^k for all $k \geq 0$. Multiplication of degree k homogenous polynomial with a degree l homogenous polynomial in x and y will result in a homogenous polynomial of degree $k + l$ in x and y.

Example 2.5. The algebra $\Omega^*(U)$ of C^∞ differential forms on U is also graded by the degree of differential forms. Each $\Omega^k(U)$ is a vector space. Multiplication of differential forms is defined by wedge product between them. The wedge product of a degree k differential form on U with a degree l differential form results in a degree $k + l$ differential form.

Definition 2.3 (Anti-derivation)**.** Let $A = \bigoplus_{k=0}^{\infty} A^k$ be a graded algebra over a field $\mathbb{K}$. An **anti-derivation** of the graded algebra A is a $\mathbb{K}$-linear map $D : A \to A$ such that for $\omega \in A^k$ and $\tau \in A^l$, one has

$$D(\omega\tau) = (D\omega)\,\tau + (-1)^k\,\omega\,(D\tau)\,. \tag{2.15}$$

If the antiderivation D sends $\omega \in A^k$ to $D\omega \in A^{k+m}$, we say that it is an antiderivation of degree m.

Proposition 2.3 (i) The exterior derivative $\mathrm{d} : \Omega^*(U) \to \Omega^*(U)$ is an antiderivation of degree 1:

$$\mathrm{d}(\omega \wedge \tau) = (\mathrm{d}\omega) \wedge \tau + (-1)^{\deg\omega}\,\omega \wedge \mathrm{d}\tau. \tag{2.16}$$

(ii) $\mathrm{d}^2 = 0$.

(iii) If $f \in \Omega^0(U) = C^\infty(U)$ and $X \in \mathfrak{X}(U)$ (the space of C^∞ vector fields), then $(\mathrm{d}f)(X) = Xf$.

Proof. (i) Since the exterior derivative operator $\mathrm{d} : \Omega^k(U) \to \Omega^{k+1}(U)$ is linear, it suffices to check the equality (2.16) for $\omega = f\,\mathrm{d}x^I$ and $\tau = g\,\mathrm{d}x^J$ with $I = (i_1, \ldots, i_k)$ and $J = (j_1, \ldots, j_l)$ being strictly ascending multi-indices.

$$\begin{aligned}
\mathrm{d}(\omega \wedge \tau) &= \mathrm{d}\left(fg\mathrm{d}x^I \wedge \mathrm{d}x^J\right)\\
&= \sum_{i=1}^{n} \frac{\partial (fg)}{\partial x^i}\mathrm{d}x^i \wedge \mathrm{d}x^I \wedge \mathrm{d}x^J\\
&= \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} \cdot g\mathrm{d}x^i \wedge \mathrm{d}x^I \wedge \mathrm{d}x^J + \sum_{i=1}^{n} f \cdot \frac{\partial g}{\partial x^i}\mathrm{d}x^i \wedge \mathrm{d}x^I \wedge \mathrm{d}x^J\\
&= \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}\mathrm{d}x^i \wedge \mathrm{d}x^I \wedge g\mathrm{d}x^J + \sum_{i=1}^{n} \frac{\partial g}{\partial x^i}\mathrm{d}x^i \wedge f\,\mathrm{d}x^I \wedge \mathrm{d}x^J.
\end{aligned} \tag{2.17}$$

Now, in the second sum in (2.17), one has to push $\frac{\partial g}{\partial x^i}\mathrm{d}x^i$ through the k-fold wedge product $\mathrm{d}x^I$ and hence in the process picks out a sign $(-1)^k$. Therefore,

$$\sum_{i=1}^{n} \frac{\partial g}{\partial x^i}\mathrm{d}x^i \wedge f\,\mathrm{d}x^I \wedge \mathrm{d}x^J = (-1)^k f\,\mathrm{d}x^I \wedge \frac{\partial g}{\partial x^i}\mathrm{d}x^i \wedge \mathrm{d}x^J. \tag{2.18}$$

Now, observe that

$$\mathrm{d}\omega = \mathrm{d}\left(f\,\mathrm{d}x^I\right) = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}\mathrm{d}x^i \wedge \mathrm{d}x^I, \text{ and} \tag{2.19}$$

$$\mathrm{d}\tau = \mathrm{d}\left(g\,\mathrm{d}x^J\right) \sum_{i=1}^{n} \frac{\partial g}{\partial x^i}\mathrm{d}x^i \wedge \mathrm{d}x^J. \tag{2.20}$$

Therefore,

$$\mathrm{d}(\omega \wedge \tau) = \mathrm{d}\omega \wedge \tau + (-1)^k\,\omega \wedge \mathrm{d}\tau. \tag{2.21}$$

(ii) Again, by $\mathbb{R}$-linearity of d, it suffices to show that $\mathrm{d}^2\omega = 0$ for $\omega = f\mathrm{d}x^I$.

$$\begin{aligned}
\mathrm{d}^2\left(f\mathrm{d}x^I\right) &= \mathrm{d}\left(\sum_{i=1}^{n} \frac{\partial f}{\partial x^i}\mathrm{d}x^i \wedge \mathrm{d}x^I\right)\\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x^j \partial x^i}\mathrm{d}x^j \wedge \mathrm{d}x^i \wedge \mathrm{d}x^I.
\end{aligned} \tag{2.22}$$

If $i = j$, then $\mathrm{d}x^j \wedge \mathrm{d}x^i = 0$. If $i \neq j$, then $\frac{\partial^2 f}{\partial x^j \partial x^i}$ is symmetric in i and j, but $\mathrm{d}x^j \wedge \mathrm{d}x^i$ is alternating in i and j. Therefore, the terms with $i \neq j$ pair up and cancel out.

(iii) Let $X = \sum_{i=1}^n a^i \frac{\partial}{\partial x^i}$. Then

$$
\begin{aligned}
(\mathrm{d}f)(X) &= \left(\sum_{j=1}^n \frac{\partial f}{\partial x^j}\mathrm{d}x^j\right)\left(\sum_{i=1}^n a^i \frac{\partial}{\partial x^i}\right) \\
&= \sum_{j=1}^n \sum_{i=1}^n a^i \frac{\partial f}{\partial x^j}\mathrm{d}x^j\left(\frac{\partial}{\partial x^i}\right) \\
&= \sum_{j=1}^n \sum_{i=1}^n a^i \frac{\partial f}{\partial x^j}\delta^j{}_i \\
&= \sum_{i=1}^n a^i \frac{\partial f}{\partial x^i} = Xf.
\end{aligned} \tag{2.23}
$$

■

Proposition 2.4 (Characterization of exterior derivative)

The 3 properties of Proposition 2.3 uniquely characterize exterior derivative on an open set $U \subseteq \mathbb{R}^n$. In other words, if $D : \Omega^*(U) \to \Omega^*(U)$ is an antiderivation of degree 1 such that $D^2 = 0$ and for $f \in C^\infty(U)$ and $X \in \mathfrak{X}(U)$, $(Df)(X) = Xf$, then $D = \mathrm{d}$.

Proof. Since every k-form on U is a sum of terms such as $f\mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2} \wedge \cdots \wedge \mathrm{d}x^{i_k}$, by linearity of d, it suffices to show that $D = \mathrm{d}$ on a k-form of this type. Applying property (iii) for $f = x^i$, one has

$$Dx^i(X) = X\left(x^i\right).$$

Writing $X = \sum_{j=1}^n a^j \frac{\partial}{\partial x^j}$, we get

$$Dx^i\left(\sum_{j=1}^n a^j \frac{\partial}{\partial x^j}\right) = \sum_{j=1}^n a^j \frac{\partial}{\partial x^j}\left(x^i\right) = a^i = \mathrm{d}x^i\left(\sum_{j=1}^n a^j \frac{\partial}{\partial x^j}\right).$$

Therefore,

$$Dx^i = \mathrm{d}x^i. \tag{2.24}$$

Now,

$$
\begin{aligned}
D\left(f\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}\right) &= D\left(fDx^{i_1} \wedge \cdots \wedge Dx^{i_k}\right) \\
&= Df \wedge \left(Dx^{i_1} \wedge \cdots \wedge Dx^{i_k}\right) + (-1)^0 fD\left(Dx^{i_1} \wedge \cdots \wedge Dx^{i_k}\right).
\end{aligned} \tag{2.25}
$$

Now, since $\mathrm{d}f(X) = Xf = Df(X)$ for any $X \in \mathfrak{X}(U)$, $\mathrm{d}f = Df$. Furthermore, $D\left(Dx^{i_1}\right) = 0$, and

$$
\begin{aligned}
D\left(Dx^{i_1} \wedge \cdots \wedge Dx^{i_k}\right) &= D^2x^{i_1} \wedge Dx^{i_2} \wedge \cdots \wedge Dx^{i_k} - Dx^{i_1} \wedge D\left(Dx^{i_2} \wedge \cdots \wedge Dx^{i_k}\right) \\
&= -Dx^{i_1} \wedge D\left(Dx^{i_2} \wedge \cdots \wedge Dx^{i_k}\right).
\end{aligned} \tag{2.26}
$$

Therefore, by induction on k,

$$D\left(Dx^{i_1} \wedge \cdots \wedge Dx^{i_k}\right) = 0. \tag{2.27}$$

Hence, from (2.25),

$$
\begin{aligned}
D\left(f\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}\right) &= Df \wedge \left(Dx^{i_1} \wedge \cdots \wedge Dx^{i_k}\right) \\
&= \mathrm{d}f \wedge \left(\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}\right) \\
&= \mathrm{d}\left(f\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}\right).
\end{aligned} \tag{2.28}
$$

So $D = \mathrm{d}$ on $\Omega^*(U)$. ■