On the Category Theoretic Implications on Induced Representation of Finite Groups

Part 2: Induced Representation Theory

Atonu Roy Chowdhury

BRAC University

atonuroychowdhury@gmail.com

September 21, 2024


Taz Chu 朱立福
@taz_chu
I've now discovered a lifehack for learning math, which is to sign up to give a talk about something I don't know super well
9:12 pm · 11 Jun 23 · 12.5K Views
7 Reposts 3 Quotes 240 Likes 9 Bookmarks

Groups are abstract objects, and hard to deal with.

Representations

Groups are abstract objects, and hard to deal with. What if group elements were matrices?

Representations

Groups are abstract objects, and hard to deal with. What if group elements were matrices?

Voila! We have Representation Theory!!!

Representations

Definition 1

A **representation** of a group G on a $\mathbb{K}$-vector space V is a homomorphism

$$\rho : G \to \mathrm{GL}(V).$$

Representations

Definition 1

A **representation** of a group G on a $\mathbb{K}$-vector space V is a homomorphism

$$\rho : G \to \mathrm{GL}(V).$$

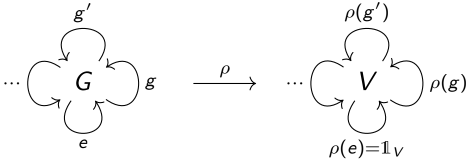
Sometimes we call V is the representation of G.

Representations

Categorically speaking, a representation of a group G is a functor $\mathcal{C}(G) \to \mathbf{Vect}_{\mathbb{K}}$.

$$\begin{tikzcd}[column sep=large] \cdots \; G \arrow[loop above, "g'"] \arrow[loop right, "g"] \arrow[loop below, "e"] \arrow[loop left] \arrow[r, "\rho"] & \cdots \; V \arrow[loop above, "\rho(g')"] \arrow[loop right, "\rho(g)"] \arrow[loop below, "\rho(e)=\mathbb{1}_V"] \arrow[loop left] \end{tikzcd}$$

Categorically speaking, a representation of a group G is a functor $\mathcal{C}(G) \to \mathbf{Vect}_{\mathbb{K}}$.



So we can form the functor category $\mathrm{Fun}\left(\mathcal{C}\left(G\right), \mathbf{Vect}_{\mathbb{K}}\right)$. This is the category of all $\mathbb{K}$-representations of the group G.

Representations

Categorically speaking, a representation of a group G is a functor $\mathcal{C}(G) \to \mathbf{Vect}_{\mathbb{K}}$.

$$\begin{tikzcd}
G \arrow[loop above, "g'"] \arrow[loop right, "g"] \arrow[loop below, "e"] \arrow[loop left, "\cdots"] \arrow[rr, "\rho"] & & V \arrow[loop above, "\rho(g')"] \arrow[loop right, "\rho(g)"] \arrow[loop below, "\rho(e)=\mathbb{1}_V"] \arrow[loop left, "\cdots"]
\end{tikzcd}$$

So we can form the functor category $\mathrm{Fun}\left(\mathcal{C}\left(G\right), \mathbf{Vect}_{\mathbb{K}}\right)$. This is the category of all $\mathbb{K}$-representations of the group G. We also call this category $\mathrm{Rep}_{\mathbb{K}}\left(G\right)$ or $\mathrm{Rep}(G)$.

Representations

What are the arrows in $\mathrm{Fun}\left(\mathcal{C}\left(G\right), \mathbf{Vect}_{\mathbb{K}}\right)$?

Representations

What are the arrows in $\mathrm{Fun}\left(\mathcal{C}\left(G\right), \mathbf{Vect}_{\mathbb{K}}\right)$?

Natural transformations $\eta : \rho \Rightarrow \sigma$!!!

Representations

What are the arrows in $\text{Fun}\left(\mathcal{C}\left(G\right), \mathbf{Vect}_{\mathbb{K}}\right)$?

Natural transformations $\eta : \rho \Rightarrow \sigma$!!!

Recall the definition of natural transformations: Let $F, G : \mathcal{C} \to \mathcal{D}$ be two functors. Then a **natural transformation** $\eta : F \Rightarrow G$ is a family of arrows

$$\left\{\eta_X : F\left(X\right) \to G\left(X\right)\right\}_{X \in \mathcal{C}_0}$$

in $\mathcal{D}$ such that for every arrow $f : X \to Y$ in $\mathcal{C}$, the following diagram commutes:

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\eta_X"'] & F(Y) \arrow[d, "\eta_Y"] \\ G(X) \arrow[r, "G(f)"'] & G(Y) \end{tikzcd}$$

In other words, $\eta_Y \circ F\left(f\right) = G\left(f\right) \circ \eta_X$.

Representations

In our case, $\mathcal{C} = \mathcal{C}(G)$, $\mathcal{D} = \mathbf{Vect}_{\mathbb{K}}$.

In our case, $\mathcal{C} = \mathcal{C}(G)$, $\mathcal{D} = \mathbf{Vect}_{\mathbb{K}}$. There is only one object in $\mathcal{C}$, i.e. G. So η has only one component. So we write $\eta_G : \rho(G) =: V \to \sigma(G) =: W$.

Representations

In our case, $\mathcal{C} = \mathcal{C}(G)$, $\mathcal{D} = \mathbf{Vect}_{\mathbb{K}}$. There is only one object in $\mathcal{C}$, i.e. G. So η has only one component. So we write $\eta_G : \rho(G) =: V \to \sigma(G) =: W$.

The arrows in the domain category are $g \in G$. We have to make the following diagram commute for every $g \in G$:

$$\begin{tikzcd} V \arrow[r, "\rho(g)"] \arrow[d, "\eta_G"'] & V \arrow[d, "\eta_G"] \\ W \arrow[r, "\sigma(g)"'] & W \end{tikzcd}$$

Representations

In our case, $\mathcal{C} = \mathcal{C}(G)$, $\mathcal{D} = \mathbf{Vect}_{\mathbb{K}}$. There is only one object in $\mathcal{C}$, i.e. G. So η has only one component. So we write $\eta_G : \rho(G) =: V \to \sigma(G) =: W$.

The arrows in the domain category are $g \in G$. We have to make the following diagram commute for every $g \in G$:

$$\begin{tikzcd} V \arrow[r, "\rho(g)"] \arrow[d, "\eta_G"'] & V \arrow[d, "\eta_G"] \\ W \arrow[r, "\sigma(g)"'] & W \end{tikzcd}$$

When a linear map $V \to W$ satisfies this commutativity, we call it a homomorphism of representations.

Representations

Definition 2

Let $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$ be two representations of a group G. A **homomorphism of representations** φ between two representations V and W of G is a linear map $\varphi : V \to W$ such that the following diagram commutes for every $g \in G$:

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & W \\ \rho(g)\big\downarrow & & \big\downarrow\sigma(g) \\ V & \xrightarrow[\varphi]{} & W \end{array}$$

In other words, $\varphi \circ \rho(g) = \sigma(g) \circ \varphi$.

Representations

Definition 2

Let $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$ be two representations of a group G. A **homomorphism of representations** φ between two representations V and W of G is a linear map $\varphi : V \to W$ such that the following diagram commutes for every $g \in G$:

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & W \\ \rho(g)\big\downarrow & & \big\downarrow\sigma(g) \\ V & \xrightarrow[\varphi]{} & W \end{array}$$

In other words, $\varphi \circ \rho(g) = \sigma(g) \circ \varphi$.

We also call it a G-linear map, or intertwining operator.

The category $\mathrm{Rep}(G)$

Let V and W be two representations of G. Then the space of all G-linear maps from V to W is

$$\mathrm{Hom}_{\mathrm{Rep}(G)}(V, W).$$

The category Rep(G)

Let V and W be two representations of G. Then the space of all G-linear maps from V to W is

$$\mathrm{Hom}_{\mathrm{Rep}(G)}(V, W).$$

We also write it as $\mathrm{Hom}_G(V, W)$.

The category Rep(G)

Let V and W be two representations of G. Then the space of all G-linear maps from V to W is

$$\mathrm{Hom}_{\mathrm{Rep}(G)}(V, W).$$

We also write it as $\mathrm{Hom}_G(V, W)$.

It has a vector space structure!!

The category Rep(G)

Suppose $\varphi, \psi \in \mathrm{Hom}_G(V, W)$ and $a \in \mathbb{K}$. Then we have the following commutative squares:

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & W \\ \rho(g)\downarrow & & \downarrow\sigma(g) \\ V & \xrightarrow[\varphi]{} & W \end{array} \qquad \begin{array}{ccc} V & \xrightarrow{\psi} & W \\ \rho(g)\downarrow & & \downarrow\sigma(g) \\ V & \xrightarrow[\psi]{} & W \end{array}$$

The category Rep(G)

Suppose $\varphi, \psi \in \mathrm{Hom}_G(V, W)$ and $a \in \mathbb{K}$. Then we have the following commutative squares:

$$
\begin{tikzcd}
V \arrow[r, "\varphi"] \arrow[d, "\rho(g)"'] & W \arrow[d, "\sigma(g)"] \\
V \arrow[r, "\varphi"'] & W
\end{tikzcd}
\qquad
\begin{tikzcd}
V \arrow[r, "\psi"] \arrow[d, "\rho(g)"'] & W \arrow[d, "\sigma(g)"] \\
V \arrow[r, "\psi"'] & W
\end{tikzcd}
$$

$$\sigma(g) \circ (a\varphi + \psi)(\mathbf{v}) = a\sigma(g)(\varphi(\mathbf{v})) + \sigma(g)(\psi(\mathbf{v}))$$

The category Rep(G)

Suppose $\varphi, \psi \in \mathrm{Hom}_G(V, W)$ and $a \in \mathbb{K}$. Then we have the following commutative squares:

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & W \\ {\scriptstyle \rho(g)}\downarrow & & \downarrow{\scriptstyle \sigma(g)} \\ V & \xrightarrow[\varphi]{} & W \end{array} \qquad \begin{array}{ccc} V & \xrightarrow{\psi} & W \\ {\scriptstyle \rho(g)}\downarrow & & \downarrow{\scriptstyle \sigma(g)} \\ V & \xrightarrow[\psi]{} & W \end{array}$$

$$\begin{aligned} \sigma(g) \circ (a\varphi + \psi)(\mathbf{v}) &= a\sigma(g)(\varphi(\mathbf{v})) + \sigma(g)(\psi(\mathbf{v})) \\ &= a\varphi(\rho(g)(\mathbf{v})) + \psi(\rho(g)(\mathbf{v})) \end{aligned}$$

The category Rep(G)

Suppose $\varphi, \psi \in \mathrm{Hom}_G(V, W)$ and $a \in \mathbb{K}$. Then we have the following commutative squares:

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "\rho(g)"'] & W \arrow[d, "\sigma(g)"] \\ V \arrow[r, "\varphi"'] & W \end{tikzcd} \qquad \begin{tikzcd} V \arrow[r, "\psi"] \arrow[d, "\rho(g)"'] & W \arrow[d, "\sigma(g)"] \\ V \arrow[r, "\psi"'] & W \end{tikzcd}$$

$$\begin{aligned} \sigma(g) \circ (a\varphi + \psi)(\mathbf{v}) &= a\sigma(g)(\varphi(\mathbf{v})) + \sigma(g)(\psi(\mathbf{v})) \\ &= a\varphi(\rho(g)(\mathbf{v})) + \psi(\rho(g)(\mathbf{v})) \\ &= (a\varphi + \psi)(\rho(g)\mathbf{v}). \end{aligned}$$

The category Rep(G)

Suppose $\varphi, \psi \in \mathrm{Hom}_G(V, W)$ and $a \in \mathbb{K}$. Then we have the following commutative squares:

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "\rho(g)"'] & W \arrow[d, "\sigma(g)"] \\ V \arrow[r, "\varphi"'] & W \end{tikzcd} \qquad \begin{tikzcd} V \arrow[r, "\psi"] \arrow[d, "\rho(g)"'] & W \arrow[d, "\sigma(g)"] \\ V \arrow[r, "\psi"'] & W \end{tikzcd}$$

$$\begin{aligned} \sigma(g) \circ (a\varphi + \psi)(\mathbf{v}) &= a\sigma(g)(\varphi(\mathbf{v})) + \sigma(g)(\psi(\mathbf{v})) \\ &= a\varphi(\rho(g)(\mathbf{v})) + \psi(\rho(g)(\mathbf{v})) \\ &= (a\varphi + \psi)(\rho(g)\mathbf{v}). \end{aligned}$$

This proves the commutativity of the following square:

$$\begin{tikzcd} V \arrow[r, "a\varphi+\psi"] \arrow[d, "\rho(g)"'] & W \arrow[d, "\sigma(g)"] \\ V \arrow[r, "a\varphi+\psi"'] & W \end{tikzcd}$$

Therefore, $z\varphi + \psi \in \mathrm{Hom}_G(V, W)$, i.e. $\mathrm{Hom}_G(V, W)$ is a $\mathbb{K}$-vector space.

New representations from old ones

Let $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$ be representations. Then there is a representation of G on the vector space $V \oplus W$.

$$\rho \oplus \sigma : G \to \mathrm{GL}\left(V \oplus W\right);$$

$$(\rho \oplus \sigma)(g)(\mathbf{v}, \mathbf{w}) = \left(\rho(g)\mathbf{v}, \sigma(g)\mathbf{w}\right), \tag{1}$$

for $g \in G$.

New representations from old ones

Let $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$ be representations. Then there is a representation of G on the tensor product vector space $V \otimes W$.

$$\rho \otimes \sigma : G \to \mathrm{GL}\left(V \otimes W\right);$$

$$(\rho \otimes \sigma)(g)(\mathbf{v} \otimes \mathbf{w}) = \rho(g)\mathbf{v} \otimes \sigma(g)\mathbf{w}, \tag{2}$$

for $g \in G$.

New representations from old ones

Let $\rho : G \to \mathrm{GL}(V)$ be a representation of G. Then there is a representation of G on the dual vector space V^*.

$$\rho^* : G \to \mathrm{GL}\left(V^*\right);$$

New representations from old ones

Let $\rho : G \to \mathrm{GL}(V)$ be a representation of G. Then there is a representation of G on the dual vector space V^*.

$$\rho^* : G \to \mathrm{GL}\left(V^*\right);$$

$$V \xrightarrow{\rho(g)} V$$

$$V^* \xleftarrow[\rho(g)^T]{} V^*$$

New representations from old ones

Let $\rho : G \to \mathrm{GL}(V)$ be a representation of G. Then there is a representation of G on the dual vector space V^*.

$$\rho^* : G \to \mathrm{GL}\left(V^*\right);$$

$$V \xrightarrow{\rho(g)} V$$

$$V^* \xleftarrow[\rho(g)^T]{} V^*$$

So we define

$$\rho^*\left(g\right) = \left[\rho\left(g^{-1}\right)\right]^T. \tag{3}$$

New representations from old ones

Let $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$ be representations. Then we can define a representation on the vector space $\mathrm{Hom}(V, W)$.

$$\gamma : G \to \mathrm{GL}(\mathrm{Hom}(V, W)).$$

New representations from old ones

Let $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$ be representations. Then we can define a representation on the vector space $\mathrm{Hom}\,(V, W)$.

$$\gamma : G \to \mathrm{GL}\,(\mathrm{Hom}\,(V, W))\,.$$

Given $f \in \mathrm{Hom}\,(V, W)$, $\gamma(g)f$ is supposed to be another element of $\mathrm{Hom}\,(V, W)$.

New representations from old ones

Let $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$ be representations. Then we can define a representation on the vector space $\mathrm{Hom}\,(V, W)$.

$$\gamma : G \to \mathrm{GL}\,(\mathrm{Hom}\,(V, W))\,.$$

Given $f \in \mathrm{Hom}\,(V, W)$, $\gamma(g)f$ is supposed to be another element of $\mathrm{Hom}\,(V, W)$. We have the following data

$$\begin{tikzcd}
V \arrow[r, "f"] \arrow[d, "\rho(g)"'] & W \arrow[d, "\sigma(g)"] \\
V & W
\end{tikzcd}$$

New representations from old ones

Let $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$ be representations. Then we can define a representation on the vector space $\mathrm{Hom}\,(V, W)$.

$$\gamma : G \to \mathrm{GL}\,(\mathrm{Hom}\,(V, W))\,.$$

Given $f \in \mathrm{Hom}\,(V, W)$, $\gamma(g)f$ is supposed to be another element of $\mathrm{Hom}\,(V, W)$. We have the following data

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\rho(g)"'] & W \arrow[d, "\sigma(g)"] \\ V & W \end{tikzcd}$$

So we can define $\gamma(g)f$ to be the missing arrow in the above diagram!

$$\gamma(g)f = \sigma\,(g) \circ f \circ \rho\,(g)^{-1}\,.$$

From now on, for the rest of the talk, all groups are finite groups. Also, all the vector spaces are finite dimensional.

Subgroup Representation

Let $\rho : G \to \mathrm{GL}(V)$ be a representation of G, and $H \subseteq G$ be a subgroup. Then ρ defines a representation of H as well!

$$\rho\big|_H : H \to \mathrm{GL}(V).$$

Subgroup Representation

Let $\rho : G \to \mathrm{GL}(V)$ be a representation of G, and $H \subseteq G$ be a subgroup. Then ρ defines a representation of H as well!

$$\rho\big|_H : H \to \mathrm{GL}(V).$$

Since we are restricting the domain of the representation $\rho : G \to \mathrm{GL}(V)$, we call this the **restriction** of the representation ρ of G.

This gives rise to a functor

$$\mathrm{Res}_H^G : \mathrm{Rep}(G) \to \mathrm{Rep}(H),$$

called the restriction functor. It takes a representation of G and restricts it to a representation of H.

Subgroup Representation

$\iota : H \to G$ is an inclusion, and it gives a functor between the category of representations, but in the reverse direction $\mathrm{Res}_H^G : \mathrm{Rep}(G) \to \mathrm{Rep}(H)$.

Subgroup Representation

$\iota : H \to G$ is an inclusion, and it gives a functor between the category of representations, but in the reverse direction $\text{Res}_H^G : \text{Rep}(G) \to \text{Rep}(H)$.

Can we do the same for any group homomorphism $f : G_1 \to G_2$? Does it give us a restriction functor

$$\text{Res} : \text{Rep}\,(G_2) \to \text{Rep}\,(G_1)?$$

Subgroup Representation

How did we define Res_H^G?

Subgroup Representation

How did we define Res_H^G? We took a representation $\rho : G \to \mathrm{GL}(V)$, and we restricted it to H. Now notice that

$$\rho\big|_H = \rho \circ \iota. \tag{4}$$

Subgroup Representation

How did we define Res_H^G? We took a representation $\rho : G \to \mathrm{GL}(V)$, and we restricted it to H. Now notice that

$$\rho\big|_H = \rho \circ \iota. \tag{4}$$

We can do the same for any group homomorphism $f : G_1 \to G_2$. If $\rho : G_2 \to \mathrm{GL}(V)$ is a representation of G_2, we get a representation of G_1:

$$\rho \circ f : G_1 \to \mathrm{GL}\left(V\right). \tag{5}$$

Induced Representation

Now suppose we have a representation W of a subgroup H. Can we generate a representation of G from this subgroup representation?

Induced Representation

Now suppose we have a representation W of a subgroup H. Can we generate a representation of G from this subgroup representation?

What do we want from this new representation, call it V?

Induced Representation

Now suppose we have a representation W of a subgroup H. Can we generate a representation of G from this subgroup representation?

What do we want from this new representation, call it V?

- When we restrict this new representation to H, we get W back. In other words, $\mathrm{Res}_H^G V = W$.

Induced Representation

Now suppose we have a representation W of a subgroup H. Can we generate a representation of G from this subgroup representation?

What do we want from this new representation, call it V?

- When we restrict this new representation to H, we get W back. In other words, $\mathrm{Res}_H^G V = W$.
- Also, we wish to construct the "most free" representation of G so as to not lose any information.

Induced Representation

Now suppose we have a representation W of a subgroup H. Can we generate a representation of G from this subgroup representation?

What do we want from this new representation, call it V?

- When we restrict this new representation to H, we get W back. In other words, $\mathrm{Res}_H^G V = W$.
- Also, we wish to construct the "most free" representation of G so as to not lose any information.

Information are encoded in Hom-sets, i.e. arrows (Yoneda lemma)!

Induced Representation

Definition 3

Let W be a representation of H and V be a representation of G. We say that $W \xrightarrow{\alpha} V$ is an **induction** if it satisfies the following universal property:

Definition 3

Let W be a representation of H and V be a representation of G. We say that $W \xrightarrow{\alpha} V$ is an **induction** if it satisfies the following universal property:

1. α is H-linear;

Induced Representation

Definition 3

Let W be a representation of H and V be a representation of G. We say that $W \xrightarrow{\alpha} V$ is an **induction** if it satisfies the following universal property:

1. α is H-linear;
2. if Z is another representation of G, and $\beta : W \to Z$ is a H-linear map, then there exists a unique G-linear map $\overline{\beta} : V \to Z$ such that the following diagram commutes:

$$\begin{tikzcd} W \arrow[r, "\alpha"] \arrow[dr, "\forall\, \beta"'] & V \arrow[d, dashed, "\exists!\ \overline{\beta}"] \\ & Z \end{tikzcd}$$

Induced Representation

How does this definition satisfy our requirements?

How does this definition satisfy our requirements?

All H-linear maps $\beta : W \to Z$ gets uniquely factored through $\alpha : W \to V$. Therefore, V preserves the "information" of $\mathrm{Hom}_H(W, -)$, where the black $-$ is replaced by a representation of G.

Induced Representation

Proposition 1

If induction of a H-representation W exists, then it's unique (up to isomorphism).

Induced Representation

Proposition 1

If induction of a H-representation W exists, then it's unique (up to isomorphism).

Proof.

Suppose $W \xrightarrow{\alpha_1} V_1$ and $W \xrightarrow{\alpha_2} V_2$ are two inductions of W.

Proposition 1

If induction of a H-representation W exists, then it's unique (up to isomorphism).

Proof.

Suppose $W \xrightarrow{\alpha_1} V_1$ and $W \xrightarrow{\alpha_2} V_2$ are two inductions of W.

$$\begin{tikzcd} W \arrow[r, "\alpha_1"] \arrow[dr, "\alpha_2"'] & V_1 \arrow[d, dashed, "\exists!\ f"] \\ & V_2 \end{tikzcd}$$

Proposition 1

If induction of a H-representation W exists, then it's unique (up to isomorphism).

Proof.

Suppose $W \xrightarrow{\alpha_1} V_1$ and $W \xrightarrow{\alpha_2} V_2$ are two inductions of W.

$$\begin{tikzcd} W \arrow[r, "\alpha_1"] \arrow[dr, "\alpha_2"'] & V_1 \arrow[d, dashed, "\exists!\ f"] \\ & V_2 \end{tikzcd} \qquad \begin{tikzcd} W \arrow[r, "\alpha_2"] \arrow[dr, "\alpha_1"'] & V_2 \arrow[d, dashed, "\exists!\ g"] \\ & V_1 \end{tikzcd}$$

Induced Representation

Proposition 1

If induction of a H-representation W exists, then it's unique (up to isomorphism).

Proof.

Suppose $W \xrightarrow{\alpha_1} V_1$ and $W \xrightarrow{\alpha_2} V_2$ are two inductions of W.

$$
\begin{tikzcd}
W \arrow[r, "\alpha_1"] \arrow[dr, "\alpha_2"'] & V_1 \arrow[d, dashed, "\exists!\ f"] \\
& V_2
\end{tikzcd}
\qquad
\begin{tikzcd}
W \arrow[r, "\alpha_2"] \arrow[dr, "\alpha_1"'] & V_2 \arrow[d, dashed, "\exists!\ g"] \\
& V_1
\end{tikzcd}
\implies
\begin{tikzcd}
& V_2 \arrow[d, "g"] \\
W \arrow[ur, "\alpha_2"] \arrow[r, "\alpha_1"] \arrow[dr, "\alpha_2"'] & V_1 \arrow[d, "f"] \\
& V_2
\end{tikzcd}
$$

Induced Representation

Proposition 1

If induction of a H-representation W exists, then it's unique (up to isomorphism).

Proof.

Suppose $W \xrightarrow{\alpha_1} V_1$ and $W \xrightarrow{\alpha_2} V_2$ are two inductions of W.

$$\begin{tikzcd} W \arrow[r, "\alpha_1"] \arrow[dr, "\alpha_2"'] & V_1 \arrow[d, dashed, "\exists!\ f"] \\ & V_2 \end{tikzcd} \qquad \begin{tikzcd} W \arrow[r, "\alpha_2"] \arrow[dr, "\alpha_1"'] & V_2 \arrow[d, dashed, "\exists!\ g"] \\ & V_1 \end{tikzcd} \implies \begin{tikzcd} & V_2 \arrow[d, "g"] \\ W \arrow[ur, "\alpha_2"] \arrow[r, "\alpha_1"] \arrow[dr, "\alpha_2"'] & V_1 \arrow[d, "f"] \\ & V_2 \end{tikzcd}$$

$$\begin{tikzcd} W \arrow[r, "\alpha_2"] \arrow[dr, "\alpha_2"'] & V_2 \arrow[d, "f \circ g"] \\ & V_2 \end{tikzcd}$$

Proposition 1

If induction of a H-representation W exists, then it's unique (up to isomorphism).

Proof.

Suppose $W \xrightarrow{\alpha_1} V_1$ and $W \xrightarrow{\alpha_2} V_2$ are two inductions of W.

$$\begin{tikzcd} W \arrow[r, "\alpha_1"] \arrow[dr, "\alpha_2"'] & V_1 \arrow[d, dashed, "\exists!\ f"] \\ & V_2 \end{tikzcd} \quad \begin{tikzcd} W \arrow[r, "\alpha_2"] \arrow[dr, "\alpha_1"'] & V_2 \arrow[d, dashed, "\exists!\ g"] \\ & V_1 \end{tikzcd} \implies \begin{tikzcd} & V_2 \arrow[d, "g"] \\ W \arrow[ur, "\alpha_2"] \arrow[r, "\alpha_1"] \arrow[dr, "\alpha_2"'] & V_1 \arrow[d, "f"] \\ & V_2 \end{tikzcd}$$

$$\begin{tikzcd} W \arrow[r, "\alpha_2"] \arrow[dr, "\alpha_2"'] & V_2 \arrow[d, "f \circ g"] \\ & V_2 \end{tikzcd} \implies f \circ g = \mathbb{1}_{V_2}.$$

Induced Representation

Proposition 1

If induction of a H-representation W exists, then it's unique (up to isomorphism).

Proof.

Suppose $W \xrightarrow{\alpha_1} V_1$ and $W \xrightarrow{\alpha_2} V_2$ are two inductions of W.

$$\begin{tikzcd} W \arrow[r, "\alpha_1"] \arrow[dr, "\alpha_2"'] & V_1 \arrow[d, dashed, "\exists!\ f"] \\ & V_2 \end{tikzcd} \quad \begin{tikzcd} W \arrow[r, "\alpha_2"] \arrow[dr, "\alpha_1"'] & V_2 \arrow[d, dashed, "\exists!\ g"] \\ & V_1 \end{tikzcd} \implies \begin{tikzcd} & V_2 \arrow[d, "g"] \\ W \arrow[ur, "\alpha_2"] \arrow[r, "\alpha_1"] \arrow[dr, "\alpha_2"'] & V_1 \arrow[d, "f"] \\ & V_2 \end{tikzcd}$$

$$\begin{tikzcd} W \arrow[r, "\alpha_2"] \arrow[dr, "\alpha_2"'] & V_2 \arrow[d, "f \circ g"] \\ & V_2 \end{tikzcd} \implies f \circ g = \mathbb{1}_{V_2}.$$ Similarly, $g \circ f = \mathbb{1}_{V_1}$.

■

But does it exist tho?

Induced Representation

But does it exist tho?

$$G = Hg_1 \cup Hg_2 \cup \cdots \cup Hg_n. \tag{6}$$

G is made out of n “copies” of H, so it’s expected that a representation of G that doesn’t lose any information will be made out of n copies of a representation of H.

But does it exist tho?

$$G = Hg_1 \cup Hg_2 \cup \cdots \cup Hg_n. \tag{6}$$

G is made out of n "copies" of H, so it's expected that a representation of G that doesn't lose any information will be made out of n copies of a representation of H. So we take

$$V = W \oplus W \oplus \cdots \oplus W = W^n. \tag{7}$$

But does it exist tho?

$$G = Hg_1 \cup Hg_2 \cup \cdots \cup Hg_n. \tag{6}$$

G is made out of n "copies" of H, so it's expected that a representation of G that doesn't lose any information will be made out of n copies of a representation of H. So we take

$$V = W \oplus W \oplus \cdots \oplus W = W^n. \tag{7}$$

We label them with the cosets, so

$$V = W_{Hg_1} \oplus W_{Hg_2} \oplus \cdots \oplus W_{Hg_n}. \tag{8}$$

Also, we take $g_1 = e$.

Induced Representation

Suppose $\rho_H : H \to \mathrm{GL}(W)$ be the representation. We define $\rho : G \to GL(V)$.

Induced Representation

Suppose $\rho_H : H \to \mathrm{GL}(W)$ be the representation. We define $\rho : G \to GL(V)$.

For $g \in G$, we are going to define $\rho(g)(\mathbf{v})$ for $\mathbf{v} \in W_{Hg_j}$.

Induced Representation

Suppose $\rho_H : H \to \mathrm{GL}(W)$ be the representation. We define $\rho : G \to GL(V)$.

For $g \in G$, we are going to define $\rho(g)(\mathbf{v})$ for $\mathbf{v} \in W_{Hg_j}$. Suppose $gg_j^{-1} = g_i^{-1}h$ for some $h \in H$. Then

$$\rho(g)\mathbf{v} := \text{ the copy of } \rho_H(h)\mathbf{v} \text{ in } W_{Hg_i}. \tag{9}$$

Induced Representation

Suppose $\rho_H : H \to \mathrm{GL}(W)$ be the representation. We define $\rho : G \to GL(V)$.

For $g \in G$, we are going to define $\rho(g)(\mathbf{v})$ for $\mathbf{v} \in W_{Hg_j}$. Suppose $gg_j^{-1} = g_i^{-1}h$ for some $h \in H$. Then

$$\rho(g)\mathbf{v} := \text{ the copy of } \rho_H(h)\mathbf{v} \text{ in } W_{Hg_i}. \tag{9}$$

More formally speaking, suppose $f_i : W = W_{He} \to W_{Hg_i}$ be the identification. Then for $\mathbf{v} \in W_{Hg_j}$ and $gg_j^{-1} = g_i^{-1}h$,

$$\rho(g)\mathbf{v} = \left[f_i \circ \rho_H(h) \circ f_j^{-1}\right]\mathbf{v}. \tag{10}$$

Induced Representation

Suppose $\rho_H : H \to \mathrm{GL}(W)$ be the representation. We define $\rho : G \to GL(V)$.

For $g \in G$, we are going to define $\rho(g)(\mathbf{v})$ for $\mathbf{v} \in W_{Hg_j}$. Suppose $gg_j^{-1} = g_i^{-1}h$ for some $h \in H$. Then

$$\rho(g)\mathbf{v} := \text{ the copy of } \rho_H(h)\mathbf{v} \text{ in } W_{Hg_i}. \tag{9}$$

More formally speaking, suppose $f_i : W = W_{He} \to W_{Hg_i}$ be the identification. Then for $\mathbf{v} \in W_{Hg_j}$ and $gg_j^{-1} = g_i^{-1}h$,

$$\rho(g)\mathbf{v} = \left[f_i \circ \rho_H(h) \circ f_j^{-1}\right]\mathbf{v}. \tag{10}$$

Exercise: Show that $\rho : G \to \mathrm{GL}(V)$ is a homomorphism.

Induced Representation

Now we are going to show that this construction satisfies the universal property of induction.

Suppose Z is another representation of G, and $\beta : W \to Z$ is a H-linear map.

$$
\begin{tikzcd}
W \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & V = \bigoplus_{i=1}^{n} W_{Hg_i} \arrow[d, dashed, "\overline{\beta}"] \\
& Z
\end{tikzcd}
$$

We need to show the existence and uniqueness of a G-linear map $\overline{\beta} : V \to Z$ such that the diagram commutes.

Suppose the corresponding group homomorphism of the representation Z is $\sigma : G \to \mathrm{GL}(Z)$.

First we show the uniqueness of $\overline{\beta}$.
Given $\mathbf{v} \in W_{Hg_j}$, $\rho(g_j)\mathbf{v} \in W_{He}$.

$$\begin{tikzcd} W \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & \bigoplus_{i=1}^{n} W_{Hg_i} \arrow[d, dashed, "\overline{\beta}"] \\ & Z \end{tikzcd}$$

First we show the uniqueness of $\overline{\beta}$.
Given $\mathbf{v} \in W_{Hg_j}$, $\rho(g_j)\mathbf{v} \in W_{He}$. Since α is the inclusion of W into W_{He},

$$\alpha(\rho(g_j)\mathbf{v}) = \rho(g_j)\mathbf{v}.$$

$$\begin{tikzcd} W \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & \bigoplus_{i=1}^{n} W_{Hg_i} \arrow[d, dashed, "\overline{\beta}"] \\ & Z \end{tikzcd}$$

Induced Representation

First we show the uniqueness of $\overline{\beta}$.
Given $\mathbf{v} \in W_{Hg_j}$, $\rho(g_j)\mathbf{v} \in W_{He}$. Since α is the inclusion of W into W_{He},

$$\alpha(\rho(g_j)\mathbf{v}) = \rho(g_j)\mathbf{v}.$$

$\overline{\beta} \circ \alpha = \beta$ gives us

$$\beta(\rho(g_j)\mathbf{v}) = \overline{\beta} \circ \alpha(\rho(g_j)\mathbf{v}) = \overline{\beta}(\rho(g_j)\mathbf{v}).$$

$$\begin{tikzcd} W \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & \bigoplus_{i=1}^{n} W_{Hg_i} \arrow[d, dashed, "\overline{\beta}"] \\ & Z \end{tikzcd}$$

First we show the uniqueness of $\overline{\beta}$.
Given $\mathbf{v} \in W_{Hg_j}$, $\rho(g_j)\mathbf{v} \in W_{He}$. Since α is the inclusion of W into W_{He},

$$\alpha(\rho(g_j)\mathbf{v}) = \rho(g_j)\mathbf{v}.$$

$\overline{\beta} \circ \alpha = \beta$ gives us

$$\beta(\rho(g_j)\mathbf{v}) = \overline{\beta} \circ \alpha(\rho(g_j)\mathbf{v}) = \overline{\beta}(\rho(g_j)\mathbf{v}).$$

$\overline{\beta}$ is G-linear, so

$$\beta(\rho(g_j)\mathbf{v}) = \sigma(g_j)\overline{\beta}(\mathbf{v}).$$

$$\begin{tikzcd} W \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & \bigoplus_{i=1}^{n} W_{Hg_i} \arrow[d, dashed, "\overline{\beta}"] \\ & Z \end{tikzcd}$$

Induced Representation

First we show the uniqueness of $\overline{\beta}$.
Given $\mathbf{v} \in W_{Hg_j}$, $\rho\left(g_j\right)\mathbf{v} \in W_{He}$. Since α is the inclusion of W into W_{He},

$$\alpha\left(\rho\left(g_j\right)\mathbf{v}\right) = \rho\left(g_j\right)\mathbf{v}.$$

$\overline{\beta} \circ \alpha = \beta$ gives us

$$\beta\left(\rho\left(g_j\right)\mathbf{v}\right) = \overline{\beta} \circ \alpha\left(\rho\left(g_j\right)\mathbf{v}\right) = \overline{\beta}\left(\rho\left(g_j\right)\mathbf{v}\right).$$

$\overline{\beta}$ is G-linear, so

$$\beta\left(\rho\left(g_j\right)\mathbf{v}\right) = \sigma\left(g_j\right)\overline{\beta}\left(\mathbf{v}\right).$$

$$\therefore \overline{\beta}\left(\mathbf{v}\right) = \sigma\left(g_j\right)^{-1}\beta\left(\rho\left(g_j\right)\mathbf{v}\right).$$

$$\begin{tikzcd} W \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & \bigoplus_{i=1}^{n} W_{Hg_i} \arrow[d, dashed, "\overline{\beta}"] \\ & Z \end{tikzcd}$$

Now we show the existence of $\overline{\beta}$, i.e. the formula we got earlier satisfies the required properties of $\overline{\beta}$.

$$\begin{tikzcd} W \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & \bigoplus_{i=1}^{n} W_{Hg_i} \arrow[d, dashed, "\overline{\beta}"] \\ & Z \end{tikzcd}$$

Induced Representation

Now we show the existence of $\overline{\beta}$, i.e. the formula we got earlier satisfies the required properties of $\overline{\beta}$.

It makes the diagram commute. We just need to check that $\overline{\beta}$ is G-linear.

$$\begin{tikzcd} W \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & \bigoplus_{i=1}^{n} W_{Hg_i} \arrow[d, dashed, "\overline{\beta}"] \\ & Z \end{tikzcd}$$

Induced Representation

Now we show the existence of $\overline{\beta}$, i.e. the formula we got earlier satisfies the required properties of $\overline{\beta}$.

It makes the diagram commute. We just need to check that $\overline{\beta}$ is G-linear.

$$\overline{\beta}\left(\mathbf{v}\right) = \sigma\left(g_j\right)^{-1}\beta\left(\rho\left(g_j\right)\mathbf{v}\right) \text{ for } \mathbf{v} \in W_{Hg_j}.$$

Let $gg_j^{-1} = g_i^{-1}h$ for some $h \in H$.

$$\begin{tikzcd} W \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & \bigoplus_{i=1}^{n} W_{Hg_i} \arrow[d, dashed, "\overline{\beta}"] \\ & Z \end{tikzcd}$$

Induced Representation

Now we show the existence of $\overline{\beta}$, i.e. the formula we got earlier satisfies the required properties of $\overline{\beta}$.

It makes the diagram commute. We just need to check that $\overline{\beta}$ is G-linear.

$$\overline{\beta}(\mathbf{v}) = \sigma(g_j)^{-1}\,\beta(\rho(g_j)\,\mathbf{v}) \text{ for } \mathbf{v} \in W_{Hg_j}.$$

$$\begin{tikzcd} W \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & \bigoplus_{i=1}^{n} W_{Hg_i} \arrow[d, dashed, "\overline{\beta}"] \\ & Z \end{tikzcd}$$

Let $gg_j^{-1} = g_i^{-1}h$ for some $h \in H$.

$$\overline{\beta}(\rho(g)\,\mathbf{v}) = \sigma(g_i)^{-1}\,\beta(\rho(g_i)\,\rho(g)\,\mathbf{v})$$

Now we show the existence of $\overline{\beta}$, i.e. the formula we got earlier satisfies the required properties of $\overline{\beta}$.

It makes the diagram commute. We just need to check that $\overline{\beta}$ is G-linear.

$$\overline{\beta}(\mathbf{v}) = \sigma(g_j)^{-1}\beta(\rho(g_j)\mathbf{v}) \text{ for } \mathbf{v} \in W_{Hg_j}.$$

Let $gg_j^{-1} = g_i^{-1}h$ for some $h \in H$.

$$\begin{aligned}\overline{\beta}(\rho(g)\mathbf{v}) &= \sigma(g_i)^{-1}\beta(\rho(g_i)\rho(g)\mathbf{v}) \\ &= \sigma(g_i)^{-1}\beta(\rho(h)\rho(g_j)\mathbf{v})\end{aligned}$$

$$\begin{tikzcd} W \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & \bigoplus_{i=1}^n W_{Hg_i} \arrow[d, dashed, "\overline{\beta}"] \\ & Z \end{tikzcd}$$

Induced Representation

Now we show the existence of $\overline{\beta}$, i.e. the formula we got earlier satisfies the required properties of $\overline{\beta}$.

It makes the diagram commute. We just need to check that $\overline{\beta}$ is G-linear.

$$\overline{\beta}\left(\mathbf{v}\right) = \sigma\left(g_j\right)^{-1}\beta\left(\rho\left(g_j\right)\mathbf{v}\right) \text{ for } \mathbf{v} \in W_{Hg_j}.$$

Let $gg_j^{-1} = g_i^{-1}h$ for some $h \in H$.

$$\begin{aligned}\overline{\beta}\left(\rho\left(g\right)\mathbf{v}\right) &= \sigma\left(g_i\right)^{-1}\beta\left(\rho\left(g_i\right)\rho\left(g\right)\mathbf{v}\right)\\ &= \sigma\left(g_i\right)^{-1}\beta\left(\rho\left(h\right)\rho\left(g_j\right)\mathbf{v}\right)\\ &= \sigma\left(g_i\right)^{-1}\sigma\left(h\right)\beta\left(\rho\left(g_j\right)\mathbf{v}\right)\end{aligned}$$

$$\begin{tikzcd}
W \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & \bigoplus_{i=1}^{n} W_{Hg_i} \arrow[d, dashed, "\overline{\beta}"] \\
& Z
\end{tikzcd}$$

Induced Representation

Now we show the existence of $\overline{\beta}$, i.e. the formula we got earlier satisfies the required properties of $\overline{\beta}$.

It makes the diagram commute. We just need to check that $\overline{\beta}$ is G-linear.

$$\overline{\beta}\left(\mathbf{v}\right) = \sigma\left(g_j\right)^{-1}\beta\left(\rho\left(g_j\right)\mathbf{v}\right) \text{ for } \mathbf{v} \in W_{Hg_j}.$$

Let $gg_j^{-1} = g_i^{-1}h$ for some $h \in H$.

$$\begin{aligned}
\overline{\beta}\left(\rho\left(g\right)\mathbf{v}\right) &= \sigma\left(g_i\right)^{-1}\beta\left(\rho\left(g_i\right)\rho\left(g\right)\mathbf{v}\right) \\
&= \sigma\left(g_i\right)^{-1}\beta\left(\rho\left(h\right)\rho\left(g_j\right)\mathbf{v}\right) \\
&= \sigma\left(g_i\right)^{-1}\sigma\left(h\right)\beta\left(\rho\left(g_j\right)\mathbf{v}\right) \\
&= \sigma\left(g\right)\sigma\left(g_j\right)^{-1}\beta\left(\rho\left(g_j\right)\mathbf{v}\right)
\end{aligned}$$

$$\begin{tikzcd}
W \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & \bigoplus_{i=1}^{n} W_{Hg_i} \arrow[d, dashed, "\overline{\beta}"] \\
& Z
\end{tikzcd}$$

Induced Representation

Now we show the existence of $\overline{\beta}$, i.e. the formula we got earlier satisfies the required properties of $\overline{\beta}$.

It makes the diagram commute. We just need to check that $\overline{\beta}$ is G-linear.

$$\overline{\beta}\left(\mathbf{v}\right) = \sigma\left(g_j\right)^{-1}\beta\left(\rho\left(g_j\right)\mathbf{v}\right) \text{ for } \mathbf{v} \in W_{Hg_j}.$$

Let $gg_j^{-1} = g_i^{-1}h$ for some $h \in H$.

$$\begin{aligned}
\overline{\beta}\left(\rho\left(g\right)\mathbf{v}\right) &= \sigma\left(g_i\right)^{-1}\beta\left(\rho\left(g_i\right)\rho\left(g\right)\mathbf{v}\right)\\
&= \sigma\left(g_i\right)^{-1}\beta\left(\rho\left(h\right)\rho\left(g_j\right)\mathbf{v}\right)\\
&= \sigma\left(g_i\right)^{-1}\sigma\left(h\right)\beta\left(\rho\left(g_j\right)\mathbf{v}\right)\\
&= \sigma\left(g\right)\sigma\left(g_j\right)^{-1}\beta\left(\rho\left(g_j\right)\mathbf{v}\right)\\
&= \sigma\left(g\right)\overline{\beta}\left(\mathbf{v}\right).
\end{aligned}$$

$$\begin{tikzcd}
W \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & \bigoplus_{i=1}^{n} W_{Hg_i} \arrow[d, dashed, "\overline{\beta}"] \\
& Z
\end{tikzcd}$$

So induced representation does exist, and hence it is unique up to isomorphism.

Induced Representation

So induced representation does exist, and hence it is unique up to isomorphism. If W is a representation of H, then the induced representation on G is denoted as $\mathrm{Ind}_H^G W$.

Induced Representation

So induced representation does exist, and hence it is unique up to isomorphism. If W is a representation of H, then the induced representation on G is denoted as $\operatorname{Ind}_H^G W$.

This gives us a functor

$$\operatorname{Ind}_H^G : \operatorname{Rep}(H) \to \operatorname{Rep}(G).$$

Induced Representation

So induced representation does exist, and hence it is unique up to isomorphism. If W is a representation of H, then the induced representation on G is denoted as $\mathrm{Ind}_H^G W$.

This gives us a functor
$$\mathrm{Ind}_H^G : \mathrm{Rep}(H) \to \mathrm{Rep}(G).$$

Does this work for group homomorphisms $f : G_1 \to G_2$?

Induced Representation

Let $f : G_1 \to G_2$ be a group homomorphism. If $\rho : G_1 \to \mathrm{GL}(W)$ is a representation of G_1, how can we get a representation of G_2 using f?

Induced Representation

Let $f : G_1 \to G_2$ be a group homomorphism. If $\rho : G_1 \to \mathrm{GL}(W)$ is a representation of G_1, how can we get a representation of G_2 using f?

We are going to use the **first isomorphism theorem**: $\operatorname{im} f \cong \frac{G_1}{\operatorname{Ker} f}$.

Induced Representation

Let $f : G_1 \to G_2$ be a group homomorphism. If $\rho : G_1 \to \mathrm{GL}(W)$ is a representation of G_1, how can we get a representation of G_2 using f?

We are going to use the **first isomorphism theorem**: $\operatorname{im} f \cong \frac{G_1}{\operatorname{Ker} f}$. We factorize f as follows:

$$G_1 \longrightarrow \frac{G_1}{\operatorname{Ker} f} \cong \operatorname{im} f \longrightarrow G_2$$

Now, given a representation W of G_1, we just have to find a representation of $\operatorname{im} f$. Then we can induce it to a representation of G_2 as earlier.

Let $\rho' : \operatorname{im} f \to \mathrm{GL}(W)$ be defined by

$$\rho'(f(g_1)) = \rho(g_1) \in \mathrm{GL}(W)\,. \tag{11}$$

Induced Representation

Let $\rho' : \operatorname{im} f \to \mathrm{GL}(W)$ be defined by

$$\rho'(f(g_1)) = \rho(g_1) \in \mathrm{GL}(W). \tag{11}$$

The issue is that for $f(g_1) = f(g_2)$, we don't necessarily have $\rho(g_1) = \rho(g_2)$.

Induced Representation

Let $\rho' : \operatorname{im} f \to \mathrm{GL}(W)$ be defined by

$$\rho'(f(g_1)) = \rho(g_1) \in \mathrm{GL}(W)\,. \tag{11}$$

The issue is that for $f(g_1) = f(g_2)$, we don't necessarily have $\rho(g_1) = \rho(g_2)$.

$$\rho(g_1)\,\mathbf{w} = \rho(g_2)\,\mathbf{w} \iff \rho\left(g_1 g_2^{-1}\right)\mathbf{w} = \mathbf{w}. \tag{12}$$

Let $\rho' : \operatorname{im} f \to \mathrm{GL}(W)$ be defined by

$$\rho'(f(g_1)) = \rho(g_1) \in \mathrm{GL}(W). \tag{11}$$

The issue is that for $f(g_1) = f(g_2)$, we don't necessarily have $\rho(g_1) = \rho(g_2)$.

$$\rho(g_1)\mathbf{w} = \rho(g_2)\mathbf{w} \iff \rho\left(g_1 g_2^{-1}\right)\mathbf{w} = \mathbf{w}. \tag{12}$$

But for $f(g_1) = f(g_2)$, we know $g_1 g_2^{-1} \in \operatorname{Ker} f$. This motivates us to define

$$W' = \frac{W}{\langle \mathbf{w} - \rho(k)\mathbf{w} \mid \mathbf{w} \in W, k \in \operatorname{Ker} f \rangle}. \tag{13}$$

Now W' is a representation of $\operatorname{im} f$, and using this, we can induce a representation of G_2.

Adjunction!

Recall the universal property of induced representation: $\alpha : W \to \operatorname{Ind} W$ is universal in the sense that if Z is another representation of G, and $\beta : W \to Z$ is a H-linear map, then there exists a unique G-linear map $\overline{\beta} : \operatorname{Ind} W \to Z$ such that the following diagram commutes:

$$\begin{tikzcd} W \arrow[r, "\alpha"] \arrow[dr, "\forall\, \beta"'] & \operatorname{Ind} W \arrow[d, dashed, "\exists!\ \overline{\beta}"] \\ & Z \end{tikzcd}$$

Adjunction!

Recall the universal property of induced representation: $\alpha : W \to \operatorname{Ind} W$ is universal in the sense that if Z is another representation of G, and $\beta : W \to Z$ is a H-linear map, then there exists a unique G-linear map $\overline{\beta} : \operatorname{Ind} W \to Z$ such that the following diagram commutes:

$$\begin{tikzcd} W \arrow[r, "\alpha"] \arrow[dr, "\forall\, \beta"'] & \operatorname{Ind} W \arrow[d, dashed, "\exists !\ \overline{\beta}"] \\ & Z \end{tikzcd}$$

So we have a 1-1 correspondence:

$$\{H\text{-linear maps } W \to Z\} \leftrightarrow \{G\text{-linear maps } \operatorname{Ind} W \to Z\}\,. \tag{14}$$

Adjunction!

Recall the universal property of induced representation: $\alpha : W \to \operatorname{Ind} W$ is universal in the sense that if Z is another representation of G, and $\beta : W \to Z$ is a H-linear map, then there exists a unique G-linear map $\overline{\beta} : \operatorname{Ind} W \to Z$ such that the following diagram commutes:

$$\begin{tikzcd} W \arrow[r, "\alpha"] \arrow[dr, "\forall\, \beta"'] & \operatorname{Ind} W \arrow[d, dashed, "\exists!\ \overline{\beta}"] \\ & Z \end{tikzcd}$$

So we have a 1-1 correspondence:

$$\{H\text{-linear maps } W \to Z\} \leftrightarrow \{G\text{-linear maps } \operatorname{Ind} W \to Z\}\,. \tag{14}$$

Or in other words,

$$\operatorname{Hom}_{\mathsf{Rep}(H)}(W, Z) \cong \operatorname{Hom}_{\mathsf{Rep}(G)}(\operatorname{Ind} W, Z)\,. \tag{15}$$

Adjunction!

So we have

$$\mathrm{Hom}_{\mathrm{Rep}(G)}\left(\mathrm{Ind}\, W, Z\right) \cong \mathrm{Hom}_{\mathrm{Rep}(H)}\left(W, \mathrm{Res}\, Z\right). \tag{16}$$

Adjunction!

So we have

$$\operatorname{Hom}_{\mathrm{Rep}(G)}\left(\operatorname{Ind} W, Z\right) \cong \operatorname{Hom}_{\mathrm{Rep}(H)}\left(W, \operatorname{Res} Z\right). \tag{16}$$

This also gives a natural isomorphism of functors

$$\begin{tikzcd}
\mathrm{Rep}(H)^{\mathrm{op}} \times \mathrm{Rep}(G) \arrow[rr, bend left, "\operatorname{Hom}_{\mathrm{Rep}(G)}(\operatorname{Ind}(-),-)"] \arrow[rr, bend right, "\operatorname{Hom}_{\mathrm{Rep}(H)}(-,\operatorname{Res}(-))"'] \arrow[rr, phantom, "\Downarrow\eta"] & & \textbf{Sets}
\end{tikzcd}$$

So we have

$$\mathrm{Hom}_{\mathrm{Rep}(G)}\left(\mathrm{Ind}\, W, Z\right) \cong \mathrm{Hom}_{\mathrm{Rep}(H)}\left(W, \mathrm{Res}\, Z\right). \tag{16}$$

This also gives a natural isomorphism of functors

$$\begin{tikzcd}
\mathrm{Rep}(H)^{\mathrm{op}} \times \mathrm{Rep}(G) \arrow[rr, bend left, "\mathrm{Hom}_{\mathrm{Rep}(G)}(\mathrm{Ind}(-), -)"] \arrow[rr, bend right, "\mathrm{Hom}_{\mathrm{Rep}(H)}(-, \mathrm{Res}(-))"'] \arrow[rr, phantom, "\Downarrow\eta"] & & \mathbf{Sets}
\end{tikzcd}$$

Therefore, Ind_H^G is the left-adjoint functor of Res_H^G.

Adjunction!

Proposition 2

Let $H \leq K \leq G$. Then $\operatorname{Ind}_H^G = \operatorname{Ind}_K^G \operatorname{Ind}_H^K$.

Adjunction!

Proposition 2

Let $H \leq K \leq G$. Then $\operatorname{Ind}_H^G = \operatorname{Ind}_K^G \operatorname{Ind}_H^K$.

Ind_H^G is the left adjoint of $\operatorname{Res}_H^G = \operatorname{Res}_H^K \operatorname{Res}_K^G$. Ind_K^G and Ind_H^K are left adjoints of Res_K^G and Res_H^K, respectively.

Adjunction!

Proposition 2

Let $H \leq K \leq G$. Then $\operatorname{Ind}_H^G = \operatorname{Ind}_K^G \operatorname{Ind}_H^K$.

Ind_H^G is the left adjoint of $\operatorname{Res}_H^G = \operatorname{Res}_H^K \operatorname{Res}_K^G$. Ind_K^G and Ind_H^K are left adjoints of Res_K^G and Res_H^K, respectively.

Theorem 3

Let $F_1 : \mathcal{C} \to \mathcal{D}$ and $F_2 : \mathcal{D} \to \mathcal{E}$ be left adjoints of the functors $G_1 : \mathcal{D} \to \mathcal{C}$ and $G_2 : \mathcal{E} \to \mathcal{D}$, respectively. Then $F_2 \circ F_1$ is the left adjoint of $G_1 \circ G_2$.

Adjunction!

Proposition 2

Let $H \leq K \leq G$. Then $\mathrm{Ind}_H^G = \mathrm{Ind}_K^G \mathrm{Ind}_H^K$.

Ind_H^G is the left adjoint of $\mathrm{Res}_H^G = \mathrm{Res}_H^K \mathrm{Res}_K^G$. Ind_K^G and Ind_H^K are left adjoints of Res_K^G and Res_H^K, respectively.

Theorem 3

Let $F_1 : \mathcal{C} \to \mathcal{D}$ and $F_2 : \mathcal{D} \to \mathcal{E}$ be left adjoints of the functors $G_1 : \mathcal{D} \to \mathcal{C}$ and $G_2 : \mathcal{E} \to \mathcal{D}$, respectively. Then $F_2 \circ F_1$ is the left adjoint of $G_1 \circ G_2$.

Proof.

$\blacksquare$

Adjunction!

Proposition 2

Let $H \leq K \leq G$. Then $\operatorname{Ind}_H^G = \operatorname{Ind}_K^G \operatorname{Ind}_H^K$.

Ind_H^G is the left adjoint of $\operatorname{Res}_H^G = \operatorname{Res}_H^K \operatorname{Res}_K^G$. Ind_K^G and Ind_H^K are left adjoints of Res_K^G and Res_H^K, respectively.

Theorem 3

Let $F_1 : \mathcal{C} \to \mathcal{D}$ and $F_2 : \mathcal{D} \to \mathcal{E}$ be left adjoints of the functors $G_1 : \mathcal{D} \to \mathcal{C}$ and $G_2 : \mathcal{E} \to \mathcal{D}$, respectively. Then $F_2 \circ F_1$ is the left adjoint of $G_1 \circ G_2$.

Proof.

$$\operatorname{Hom}_{\mathcal{E}}\left(F_2\left(F_1(-)\right), -\right) \cong \operatorname{Hom}_{\mathcal{D}}\left(F_1(-), G_2(-)\right)$$

■

Adjunction!

Proposition 2

Let $H \leq K \leq G$. Then $\operatorname{Ind}_H^G = \operatorname{Ind}_K^G \operatorname{Ind}_H^K$.

Ind_H^G is the left adjoint of $\operatorname{Res}_H^G = \operatorname{Res}_H^K \operatorname{Res}_K^G$. Ind_K^G and Ind_H^K are left adjoints of Res_K^G and Res_H^K, respectively.

Theorem 3

Let $F_1 : \mathcal{C} \to \mathcal{D}$ and $F_2 : \mathcal{D} \to \mathcal{E}$ be left adjoints of the functors $G_1 : \mathcal{D} \to \mathcal{C}$ and $G_2 : \mathcal{E} \to \mathcal{D}$, respectively. Then $F_2 \circ F_1$ is the left adjoint of $G_1 \circ G_2$.

Proof.

$$\begin{aligned}\operatorname{Hom}_{\mathcal{E}}\left(F_2\left(F_1(-)\right), -\right) &\cong \operatorname{Hom}_{\mathcal{D}}\left(F_1(-), G_2(-)\right) \\ &\cong \operatorname{Hom}_{\mathcal{C}}\left(-, G_1\left(G_2(-)\right)\right)\end{aligned}$$

∎

Theorem 4

Supposse $H \leq G$. Let U be a representation of G and W be a representation of H. Then

$$U \otimes \operatorname{Ind}_H^G W \cong \operatorname{Ind}_H^G \left(\operatorname{Res}_H^G U \otimes W \right).$$

Theorem 4

Suppose $H \leq G$. Let U be a representation of G and W be a representation of H. Then

$$U \otimes \mathrm{Ind}_H^G W \cong \mathrm{Ind}_H^G \left(\mathrm{Res}_H^G U \otimes W \right).$$

Proof.

Theorem 4

Suppose $H \leq G$. Let U be a representation of G and W be a representation of H. Then

$$U \otimes \mathrm{Ind}_H^G W \cong \mathrm{Ind}_H^G \left(\mathrm{Res}_H^G U \otimes W\right).$$

Proof.

$$\mathrm{Hom}_{\mathrm{Rep}(G)}(U \otimes \mathrm{Ind}\, W, -) \cong \mathrm{Hom}_{\mathrm{Rep}(G)}(\mathrm{Ind}\, W, \mathrm{Hom}(U, -)) \quad [\text{Hom-tensor adj}]$$

Theorem 4

Suppose $H \leq G$. Let U be a representation of G and W be a representation of H. Then

$$U \otimes \mathrm{Ind}_H^G W \cong \mathrm{Ind}_H^G \left(\mathrm{Res}_H^G U \otimes W \right).$$

Proof.

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Rep}(G)}(U \otimes \mathrm{Ind}\, W, -) &\cong \mathrm{Hom}_{\mathrm{Rep}(G)}(\mathrm{Ind}\, W, \mathrm{Hom}(U, -)) \quad \text{[Hom-tensor adj]} \\ &\cong \mathrm{Hom}_{\mathrm{Rep}(G)}(\mathrm{Ind}\, W, U^* \otimes -) \end{aligned}$$

Theorem 4

Supposse $H \leq G$. Let U be a representation of G and W be a representation of H. Then

$$U \otimes \operatorname{Ind}_H^G W \cong \operatorname{Ind}_H^G \left(\operatorname{Res}_H^G U \otimes W \right).$$

Proof.

$$\begin{aligned}
\operatorname{Hom}_{\operatorname{Rep}(G)}(U \otimes \operatorname{Ind} W, -) &\cong \operatorname{Hom}_{\operatorname{Rep}(G)}(\operatorname{Ind} W, \operatorname{Hom}(U, -)) && \text{[Hom-tensor adj]} \\
&\cong \operatorname{Hom}_{\operatorname{Rep}(G)}(\operatorname{Ind} W, U^* \otimes -) \\
&\cong \operatorname{Hom}_{\operatorname{Rep}(H)}(W, \operatorname{Res}(U^* \otimes -)) && \text{[Ind-Res adj]}
\end{aligned}$$

Theorem 4

Supposse $H \leq G$. Let U be a representation of G and W be a representation of H. Then

$$U \otimes \mathrm{Ind}_H^G W \cong \mathrm{Ind}_H^G \left(\mathrm{Res}_H^G U \otimes W\right).$$

Proof.

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Rep}(G)}(U \otimes \mathrm{Ind}\, W, -) &\cong \mathrm{Hom}_{\mathrm{Rep}(G)}(\mathrm{Ind}\, W, \mathrm{Hom}(U, -)) && \text{[Hom-tensor adj]} \\
&\cong \mathrm{Hom}_{\mathrm{Rep}(G)}(\mathrm{Ind}\, W, U^* \otimes -) \\
&\cong \mathrm{Hom}_{\mathrm{Rep}(H)}(W, \mathrm{Res}(U^* \otimes -)) && \text{[Ind-Res adj]} \\
&\cong \mathrm{Hom}_{\mathrm{Rep}(H)}\left(W, \mathrm{Res}(U)^* \otimes \mathrm{Res}(-)\right)
\end{aligned}$$

Theorem 4

Suppose $H \leq G$. Let U be a representation of G and W be a representation of H. Then

$$U \otimes \mathrm{Ind}_H^G W \cong \mathrm{Ind}_H^G \left(\mathrm{Res}_H^G U \otimes W\right).$$

Proof.

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Rep}(G)}(U \otimes \mathrm{Ind}\, W, -) &\cong \mathrm{Hom}_{\mathrm{Rep}(G)}(\mathrm{Ind}\, W, \mathrm{Hom}(U, -)) && \text{[Hom-tensor adj]}\\
&\cong \mathrm{Hom}_{\mathrm{Rep}(G)}(\mathrm{Ind}\, W, U^* \otimes -)\\
&\cong \mathrm{Hom}_{\mathrm{Rep}(H)}(W, \mathrm{Res}(U^* \otimes -)) && \text{[Ind-Res adj]}\\
&\cong \mathrm{Hom}_{\mathrm{Rep}(H)}\left(W, \mathrm{Res}(U)^* \otimes \mathrm{Res}(-)\right)\\
&\cong \mathrm{Hom}_{\mathrm{Rep}(H)}(\mathrm{Res}(U) \otimes W, \mathrm{Res}(-))
\end{aligned}$$

Theorem 4

Suppose $H \leq G$. Let U be a representation of G and W be a representation of H. Then

$$U \otimes \operatorname{Ind}_H^G W \cong \operatorname{Ind}_H^G \left(\operatorname{Res}_H^G U \otimes W \right).$$

Proof.

$$\begin{aligned}
\operatorname{Hom}_{\operatorname{Rep}(G)} (U \otimes \operatorname{Ind} W, -) &\cong \operatorname{Hom}_{\operatorname{Rep}(G)} (\operatorname{Ind} W, \operatorname{Hom}(U, -)) && \text{[Hom-tensor adj]} \\
&\cong \operatorname{Hom}_{\operatorname{Rep}(G)} (\operatorname{Ind} W, U^* \otimes -) \\
&\cong \operatorname{Hom}_{\operatorname{Rep}(H)} (W, \operatorname{Res}(U^* \otimes -)) && \text{[Ind-Res adj]} \\
&\cong \operatorname{Hom}_{\operatorname{Rep}(H)} \left(W, \operatorname{Res}(U)^* \otimes \operatorname{Res}(-) \right) \\
&\cong \operatorname{Hom}_{\operatorname{Rep}(H)} (\operatorname{Res}(U) \otimes W, \operatorname{Res}(-)) \\
&\cong \operatorname{Hom}_{\operatorname{Rep}(G)} (\operatorname{Ind}(\operatorname{Res}(U) \otimes W), -)
\end{aligned}$$

Theorem 4

Suppose $H \leq G$. Let U be a representation of G and W be a representation of H. Then

$$U \otimes \operatorname{Ind}_H^G W \cong \operatorname{Ind}_H^G \left(\operatorname{Res}_H^G U \otimes W \right).$$

Proof.

$$\begin{aligned}
\operatorname{Hom}_{\operatorname{Rep}(G)} (U \otimes \operatorname{Ind} W, -) &\cong \operatorname{Hom}_{\operatorname{Rep}(G)} (\operatorname{Ind} W, \operatorname{Hom}(U, -)) && \text{[Hom-tensor adj]} \\
&\cong \operatorname{Hom}_{\operatorname{Rep}(G)} (\operatorname{Ind} W, U^* \otimes -) \\
&\cong \operatorname{Hom}_{\operatorname{Rep}(H)} (W, \operatorname{Res}(U^* \otimes -)) && \text{[Ind-Res adj]} \\
&\cong \operatorname{Hom}_{\operatorname{Rep}(H)} \left(W, \operatorname{Res}(U)^* \otimes \operatorname{Res}(-)\right) \\
&\cong \operatorname{Hom}_{\operatorname{Rep}(H)} (\operatorname{Res}(U) \otimes W, \operatorname{Res}(-)) \\
&\cong \operatorname{Hom}_{\operatorname{Rep}(G)} (\operatorname{Ind}(\operatorname{Res}(U) \otimes W), -)
\end{aligned}$$

All of these isomorphisms are natural isomorphisms. Therefore, by Yoneda lemma, we are done! ∎

(Co)-induction?

Category theorists love to reverse the arrows and add the prefix "co". If we reverse the arrows in the universal property of induction, we get coinduction.

(Co)-induction?

Category theorists love to reverse the arrows and add the prefix "co". If we reverse the arrows in the universal property of induction, we get coinduction.

$$
\begin{tikzcd}
W \arrow[r, "\alpha"] \arrow[dr, "\forall\,\beta"'] & \operatorname{Ind} W \arrow[d, dashed, "\exists!\ \overline{\beta}"] \\
& Z
\end{tikzcd}
$$

Induction

$$
\begin{tikzcd}
W & \operatorname{Coind} W \arrow[l, "\alpha"'] \\
& Z \arrow[ul, "\forall\,\beta"] \arrow[u, dashed, "\exists!\ \overline{\beta}"']
\end{tikzcd}
$$

Coinduction

Coinduction

Definition 4

Let $H \subset G$. Given a representation W of H, and a representation V of G, we say that $V \xrightarrow{\alpha} W$ is a **coinduction** if it satisfies the following universal property:

Coinduction

Definition 4

Let $H \subset G$. Given a representation W of H, and a representation V of G, we say that $V \xrightarrow{\alpha} W$ is a **coinduction** if it satisfies the following universal property:

1. α is H-linear;

Coinduction

Definition 4

Let $H \subset G$. Given a representation W of H, and a representation V of G, we say that $V \xrightarrow{\alpha} W$ is a **coinduction** if it satisfies the following universal property:

1. α is H-linear;
2. if Z is another representation of G and $\beta : Z \to W$ is a H-linear map, then there exists a unique G-linear map $\overline{\beta} : Z \to V$ such that the following diagram commutes:

$$\begin{tikzcd}
W & W \arrow[l, "\alpha"'] \\
 & Z \arrow[ul, "\forall\, \beta"] \arrow[u, dashed, "\exists!\ \overline{\beta}"']
\end{tikzcd}$$

Coinduction

As earlier, one can prove that if coinduction exists, then it is unique up to isomorphism.

Coinduction

As earlier, one can prove that if coinduction exists, then it is unique up to isomorphism.

Furthermore, from the commutative diagram,

$$\begin{tikzcd} W & \operatorname{Coind} W \arrow[l, "\alpha"'] \\ & Z \arrow[ul, "\forall\, \beta"] \arrow[u, dashed, "\exists!\ \overline{\beta}"'] \end{tikzcd}$$

Coinduction

As earlier, one can prove that if coinduction exists, then it is unique up to isomorphism.

Furthermore, from the commutative diagram,

$$\begin{tikzcd} W & \operatorname{Coind} W \arrow[l, "\alpha"'] \\ & Z \arrow[ul, "\forall\, \beta"] \arrow[u, dashed, "\exists!\ \overline{\beta}"'] \end{tikzcd}$$

$$\operatorname{Hom}_{\operatorname{Rep}(H)}(Z, W) \cong \operatorname{Hom}_{\operatorname{Rep}(G)}(Z, \operatorname{Coind} W) \tag{17}$$

Coinduction

As earlier, one can prove that if coinduction exists, then it is unique up to isomorphism.

Furthermore, from the commutative diagram,

$$
\begin{tikzcd}
W & \operatorname{Coind} W \arrow[l, "\alpha"'] \\
& Z \arrow[ul, "\forall\, \beta"] \arrow[u, dashed, "\exists!\ \overline{\beta}"']
\end{tikzcd}
$$

$$\operatorname{Hom}_{\operatorname{Rep}(H)}(Z, W) \cong \operatorname{Hom}_{\operatorname{Rep}(G)}(Z, \operatorname{Coind} W) \tag{17}$$

In other words,

$$\operatorname{Hom}_{\operatorname{Rep}(H)}(\operatorname{Res}(Z), W) \cong \operatorname{Hom}_{\operatorname{Rep}(G)}(Z, \operatorname{Coind} W) \tag{18}$$

Coinduction

As earlier, one can prove that if coinduction exists, then it is unique up to isomorphism.

Furthermore, from the commutative diagram,

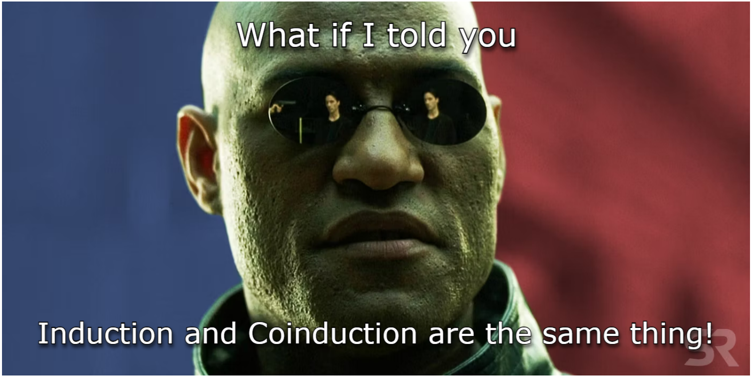
$$\begin{tikzcd} W & \mathrm{Coind}\, W \arrow[l, "\alpha"'] \\ & Z \arrow[ul, "\forall\, \beta"] \arrow[u, dashed, "\exists!\ \overline{\beta}"'] \end{tikzcd}$$

$$\mathrm{Hom}_{\mathrm{Rep}(H)}(Z, W) \cong \mathrm{Hom}_{\mathrm{Rep}(G)}(Z, \mathrm{Coind}\, W) \tag{17}$$

In other words,

$$\mathrm{Hom}_{\mathrm{Rep}(H)}(\mathrm{Res}(Z), W) \cong \mathrm{Hom}_{\mathrm{Rep}(G)}(Z, \mathrm{Coind}\, W) \tag{18}$$

So Coind is the **right adjoint** of Res!


What if I told you
Induction and Coinduction are the same thing!

Induction and Coinduction

Our construction of induction also satisfies the universal property of coinduction!

Our construction of induction also satisfies the universal property of coinduction!

Suppose V is another representation of G, and $\beta : V \to W$ is a H-linear map.

$$\begin{tikzcd}
W & \bigoplus_{i=1}^{n} W_{Hg_i} \arrow[l, "\alpha"'] \\
& V \arrow[ul, "\forall\, \beta"] \arrow[u, dashed, "\exists!\ \overline{\beta}"']
\end{tikzcd}$$

We need to show the existence and uniqueness of a G-linear map $\overline{\beta} : V \to \bigoplus_{i=1}^{n} W_{Hg_i}$ such that the diagram commutes.

Induction and Coinduction

Our construction of induction also satisfies the universal property of coinduction!

Suppose V is another representation of G, and $\beta : V \to W$ is a H-linear map.

$$\begin{tikzcd} W & \bigoplus_{i=1}^{n} W_{Hg_i} \arrow[l, "\alpha"'] \\ & V \arrow[ul, "\forall\, \beta"] \arrow[u, dashed, "\exists!\ \overline{\beta}"'] \end{tikzcd}$$

We need to show the existence and uniqueness of a G-linear map $\overline{\beta} : V \to \bigoplus_{i=1}^{n} W_{Hg_i}$ such that the diagram commutes.

Suppose the corresponding group homomorphism of the representation V is $\sigma : G \to \mathrm{GL}(V)$.

Induction and Coinduction

First we show the uniqueness of $\overline{\beta}$.

$$\begin{tikzcd} W & \bigoplus_{i=1}^{n} W_{Hg_i} \arrow[l, "\alpha"'] \\ & V \arrow[ul, "\beta"] \arrow[u, dashed, "\overline{\beta}"'] \end{tikzcd}$$

Induction and Coinduction

First we show the uniqueness of $\overline{\beta}$.
α is the projection onto the first component
$W_{Hg_1} = W$. Therefore, the first component of $\overline{\beta}(\mathbf{v})$ is
$\alpha \circ \overline{\beta}(\mathbf{v}) = \beta(\mathbf{v})$. In other words, $\overline{\beta}(\mathbf{v}) = (\beta(\mathbf{v}), \ldots)$.

$$
\begin{tikzcd}
W & \bigoplus_{i=1}^{n} W_{Hg_i} \arrow[l, "\alpha"'] \\
& V \arrow[ul, "\beta"] \arrow[u, dashed, "\overline{\beta}"']
\end{tikzcd}
$$

Induction and Coinduction

First we show the uniqueness of $\overline{\beta}$.
α is the projection onto the first component
$W_{Hg_1} = W$. Therefore, the first component of $\overline{\beta}(\mathbf{v})$ is
$\alpha \circ \overline{\beta}(\mathbf{v}) = \beta(\mathbf{v})$. In other words, $\overline{\beta}(\mathbf{v}) = (\beta(\mathbf{v}), \ldots)$.

$$\overline{\beta}(\sigma(g_j)\mathbf{v}) = \rho(g_j)\overline{\beta}(\mathbf{v}).$$

$$\begin{tikzcd}
W & \bigoplus_{i=1}^{n} W_{Hg_i} \arrow[l, "\alpha"'] \\
& V \arrow[ul, "\beta"] \arrow[u, dashed, "\overline{\beta}"']
\end{tikzcd}$$

Induction and Coinduction

First we show the uniqueness of $\overline{\beta}$.
α is the projection onto the first component $W_{\underline{Hg_1}} = W$. Therefore, the first component of $\overline{\beta}(\mathbf{v})$ is $\alpha \circ \overline{\beta}(\mathbf{v}) = \beta(\mathbf{v})$. In other words, $\overline{\beta}(\mathbf{v}) = (\beta(\mathbf{v}), \ldots)$.

$$\overline{\beta}(\sigma(g_j)\mathbf{v}) = \rho(g_j)\overline{\beta}(\mathbf{v}).$$

$\rho(g_j)$ will take the j-th component of $\overline{\beta}(\mathbf{v})$ into the first component. Therefore, the j-th component of $\overline{\beta}(\mathbf{v})$ will be

$$\begin{tikzcd} W & \bigoplus_{i=1}^{n} W_{Hg_i} \arrow[l, "\alpha"'] \\ & V \arrow[ul, "\beta"] \arrow[u, dashed, "\overline{\beta}"'] \end{tikzcd}$$

$$\alpha\left[\rho(g_j)\overline{\beta}(\mathbf{v})\right] = \alpha \circ \overline{\beta}(\sigma(g_j)\mathbf{v}) = \beta(\sigma(g_j)\mathbf{v})$$

.

First we show the uniqueness of $\overline{\beta}$.
α is the projection onto the first component $W_{H\underline{g_1}} = W$. Therefore, the first component of $\overline{\beta}\left(\mathbf{v}\right)$ is $\alpha \circ \overline{\beta}\left(\mathbf{v}\right) = \beta\left(\mathbf{v}\right)$. In other words, $\overline{\beta}\left(\mathbf{v}\right) = \left(\beta\left(\mathbf{v}\right), \ldots\right)$.

$$\overline{\beta}\left(\sigma\left(g_j\right)\mathbf{v}\right) = \rho\left(g_j\right)\overline{\beta}\left(\mathbf{v}\right).$$

$\rho\left(g_j\right)$ will take the j-th component of $\overline{\beta}\left(\mathbf{v}\right)$ into the first component. Therefore, the j-th component of $\overline{\beta}\left(\mathbf{v}\right)$ will be

$$\begin{tikzcd}
W & & \bigoplus_{i=1}^{n} W_{Hg_i} \arrow[ll, "\alpha"'] \\
& & V \arrow[ull, "\beta"] \arrow[u, dashed, "\overline{\beta}"']
\end{tikzcd}$$

$$\alpha\left[\rho\left(g_j\right)\overline{\beta}\left(\mathbf{v}\right)\right] = \alpha \circ \overline{\beta}\left(\sigma\left(g_j\right)\mathbf{v}\right) = \beta\left(\sigma\left(g_j\right)\mathbf{v}\right)$$

. So we get

$$\overline{\beta}\left(\mathbf{v}\right) = \left(\beta\left(\mathbf{v}\right), \beta\left(\sigma\left(g_2\right)\mathbf{v}\right), \cdots, \beta\left(\sigma\left(g_n\right)\mathbf{v}\right)\right).$$

Induction and Coinduction

Now we show the existence of $\overline{\beta}$.

$$\begin{tikzcd} W & \bigoplus_{i=1}^{n} W_{Hg_i} \arrow[l, "\alpha"'] \\ & V \arrow[ul, "\beta"] \arrow[u, dashed, "\overline{\beta}"'] \end{tikzcd}$$

Induction and Coinduction

Now we show the existence of $\overline{\beta}$. Clearly, $\overline{\beta} : V \to \bigoplus_{i=1}^{n} W_{Hg_i}$ defined by

$$\overline{\beta}\left(\mathbf{v}\right) = \left(\beta\left(\sigma\left(g_i\right)\mathbf{v}\right)\right)_{i=1}^{n}$$

makes the diagram commute! So we only need to show that it is G-linear.

$$\begin{tikzcd}
W & \bigoplus_{i=1}^{n} W_{Hg_i} \arrow[l, "\alpha"'] \\
& V \arrow[ul, "\beta"] \arrow[u, dashed, "\overline{\beta}"']
\end{tikzcd}$$

Induction and Coinduction

Now we show the existence of $\overline{\beta}$. Clearly,
$\overline{\beta} : V \to \bigoplus_{i=1}^{n} W_{Hg_i}$ defined by

$$\overline{\beta}\left(\mathbf{v}\right) = \left(\beta\left(\sigma\left(g_i\right)\mathbf{v}\right)\right)_{i=1}^{n}$$

makes the diagram commute! So we only need to show that it is G-linear. Given $g \in G$, let $g_i g = h_i g_{j(i)}$.

$$\begin{tikzcd} W & \bigoplus_{i=1}^{n} W_{Hg_i} \arrow[l, "\alpha"'] \\ & V \arrow[ul, "\beta"] \arrow[u, dashed, "\overline{\beta}"'] \end{tikzcd}$$

Now we show the existence of $\overline{\beta}$. Clearly, $\overline{\beta} : V \to \bigoplus_{i=1}^{n} W_{Hg_i}$ defined by

$$\overline{\beta}\left(\mathbf{v}\right) = \left(\beta\left(\sigma\left(g_i\right)\mathbf{v}\right)\right)_{i=1}^{n}$$

makes the diagram commute! So we only need to show that it is G-linear. Given $g \in G$, let $g_i g = h_i g_{j(i)}$.

$$\overline{\beta}\left(\sigma\left(g\right)\mathbf{v}\right) = \Big(\beta\left(\sigma\left(g_i\right)\sigma\left(g\right)\mathbf{v}\right)\Big)_{i=1}^{n}$$

$$\begin{tikzcd} W & \bigoplus_{i=1}^{n} W_{Hg_i} \arrow[l, "\alpha"'] \\ & V \arrow[ul, "\beta"] \arrow[u, dashed, "\overline{\beta}"'] \end{tikzcd}$$

Induction and Coinduction

Now we show the existence of $\overline{\beta}$. Clearly, $\overline{\beta} : V \to \bigoplus_{i=1}^{n} W_{Hg_i}$ defined by

$$\overline{\beta}\left(\mathbf{v}\right) = \left(\beta\left(\sigma\left(g_i\right)\mathbf{v}\right)\right)_{i=1}^{n}$$

makes the diagram commute! So we only need to show that it is G-linear. Given $g \in G$, let $g_i g = h_i g_{j(i)}$.

$$\begin{tikzcd}
W & \bigoplus_{i=1}^{n} W_{Hg_i} \arrow[l, "\alpha"'] \\
 & V \arrow[ul, "\beta"] \arrow[u, dashed, "\overline{\beta}"']
\end{tikzcd}$$

$$\begin{aligned}
\overline{\beta}\left(\sigma\left(g\right)\mathbf{v}\right) &= \left(\beta\left(\sigma\left(g_i\right)\sigma\left(g\right)\mathbf{v}\right)\right)_{i=1}^{n} \\
&= \left(\beta\left(\sigma\left(h_i\right)\sigma\left(g_{j(i)}\right)\mathbf{v}\right)\right)_{i=1}^{n}
\end{aligned}$$

Induction and Coinduction

Now we show the existence of $\overline{\beta}$. Clearly, $\overline{\beta} : V \to \bigoplus_{i=1}^{n} W_{Hg_i}$ defined by

$$\overline{\beta}\left(\mathbf{v}\right) = \left(\beta\left(\sigma\left(g_i\right)\mathbf{v}\right)\right)_{i=1}^{n}$$

makes the diagram commute! So we only need to show that it is G-linear. Given $g \in G$, let $g_i g = h_i g_{j(i)}$.

$$\begin{tikzcd} W & \bigoplus_{i=1}^{n} W_{Hg_i} \arrow[l, "\alpha"'] \\ & V \arrow[ul, "\beta"] \arrow[u, dashed, "\overline{\beta}"'] \end{tikzcd}$$

$$\begin{aligned} \overline{\beta}\left(\sigma\left(g\right)\mathbf{v}\right) &= \left(\beta\left(\sigma\left(g_i\right)\sigma\left(g\right)\mathbf{v}\right)\right)_{i=1}^{n} \\ &= \left(\beta\left(\sigma\left(h_i\right)\sigma\left(g_{j(i)}\right)\mathbf{v}\right)\right)_{i=1}^{n} \\ &= \left(\rho_W\left(h_i\right)\beta\left(\sigma\left(g_{j(i)}\right)\mathbf{v}\right)\right)_{i=1}^{n} \end{aligned}$$

$g_i g = h_i g_{j(i)}$ is equivalent to $gg_{j(i)}^{-1} = g_i^{-1} h_i$. Then $\rho(g)$ takes $W_{Hg_{j(i)}}$ on to W_{Hg_i}.

$$\begin{tikzcd}
W & \bigoplus_{i=1}^{n} W_{Hg_i} \arrow[l, "\alpha"'] \\
 & V \arrow[ul, "\beta"] \arrow[u, dashed, "\overline{\beta}"']
\end{tikzcd}$$

$g_i g = h_i g_{j(i)}$ is equivalent to $g g_{j(i)}^{-1} = g_i^{-1} h_i$. Then $\rho(g)$ takes $W_{Hg_{j(i)}}$ on to W_{Hg_i}. So the i-th component of $\rho(g)\,\beta(\mathbf{v})$ is

$$\rho(g)\left[\beta(\mathbf{v})\text{'s } j(i)\text{-th component}\right] = \rho_W(h_i)\,\beta\left(\sigma\left(g_{j(i)}\right)\mathbf{v}\right).$$

$$\begin{tikzcd} W & \bigoplus_{i=1}^{n} W_{Hg_i} \arrow[l, "\alpha"'] \\ & V \arrow[ul, "\beta"] \arrow[u, dashed, "\overline{\beta}"'] \end{tikzcd}$$

$g_i g = h_i g_{j(i)}$ is equivalent to $gg_{j(i)}^{-1} = g_i^{-1} h_i$. Then $\rho(g)$ takes $W_{Hg_{j(i)}}$ on to W_{Hg_i}. So the i-th component of $\rho(g)\,\beta(\mathbf{v})$ is

$$\rho(g)\left[\beta(\mathbf{v})\text{'s } j(i)\text{-th component}\right] = \rho_W(h_i)\,\beta\left(\sigma\left(g_{j(i)}\right)\mathbf{v}\right).$$

$$\begin{tikzcd} W & \bigoplus_{i=1}^{n} W_{Hg_i} \arrow[l, "\alpha"'] \\ & V \arrow[ul, "\beta"] \arrow[u, dashed, "\overline{\beta}"'] \end{tikzcd}$$

$$\overline{\beta}(\sigma(g)\,\mathbf{v}) = \left(\rho_W(h_i)\,\beta\left(\sigma\left(g_{j(i)}\right)\mathbf{v}\right)\right)_{i=1}^{n}$$

$g_i g = h_i g_{j(i)}$ is equivalent to $g g_{j(i)}^{-1} = g_i^{-1} h_i$. Then $\rho(g)$ takes $W_{Hg_{j(i)}}$ on to W_{Hg_i}. So the i-th component of $\rho(g)\beta(\mathbf{v})$ is

$$\rho(g)\left[\beta(\mathbf{v})\text{'s } j(i)\text{-th component}\right] = \rho_W(h_i)\,\beta\left(\sigma\left(g_{j(i)}\right)\mathbf{v}\right).$$

$$\begin{tikzcd}
W & \bigoplus_{i=1}^{n} W_{Hg_i} \arrow[l, "\alpha"'] \\
& V \arrow[ul, "\beta"] \arrow[u, dashed, "\overline{\beta}"']
\end{tikzcd}$$

$$\overline{\beta}(\sigma(g)\mathbf{v}) = \left(\rho_W(h_i)\,\beta\left(\sigma\left(g_{j(i)}\right)\mathbf{v}\right)\right)_{i=1}^{n}$$

So $\overline{\beta}(\sigma(g)\mathbf{v}) = \rho(g)\beta(\mathbf{v})$.

Induction and Coinduction

Does this always hold that induction and coinduction are the same? What if $|G| = \infty$?

Induction and Coinduction

Does this always hold that induction and coinduction are the same? What if $|G| = \infty$?

- The analysis we did holds verbatim when the index $[G : H] < \infty$.

Induction and Coinduction

Does this always hold that induction and coinduction are the same? What if $|G| = \infty$?

- The analysis we did holds verbatim when the index $[G : H] < \infty$.
- Induction and coinduction are, in general, different. The canonical definition of coinduction is

$$\text{Coind}\, W = \{f : G \to W \mid f(hg) = \rho(h) f(g) \;\; \forall\, h \in H\} \tag{19}$$

Induction and Coinduction

Does this always hold that induction and coinduction are the same? What if $|G| = \infty$?

- The analysis we did holds verbatim when the index $[G : H] < \infty$.
- Induction and coinduction are, in general, different. The canonical definition of coinduction is

$$\text{Coind}\, W = \{f : G \to W \mid f(hg) = \rho(h) f(g) \;\; \forall\, h \in H\} \tag{19}$$

This is not isomorhic to $\bigoplus W_{Hg_i}$ when the index of the subgroup is infinite.

References

1. *Representation Theory: A First Course*, by William Fulton and Joe Harris
2. *Representations of Finite and Compact Groups*, by Barry Simon
3. *Introduction to Representation Theory*, by Pavel Etingof
4. *Linear Representations of Finite Groups*, by Jean-Pierre Serre
5. `https://www.math.columbia.edu/~woit/LieGroups-2012/inducedreps.pdf`
6. `https://nms.kcl.ac.uk/james.newton/M3P12/induced.pdf`
7. `https://public.websites.umich.edu/~gadish/notes/rep/s5.pdf`
8. `https://duncan.math.sc.edu/s23/math742/notes/induction.pdf`

Thank you for joining!

The slides are available in my webpage
https://atonurc.github.io/assets/catrep_talk_2.pdf