Algebriac Topology III (MAT484)

Lecture Notes

Preface

This series of lecture notes has been prepared for aiding students who took the BRAC University course **Algebraic Topology III (MAT484)** in Spring 2023 semester. These notes were typeset under the supervision of mathematician **Dr. Syed Hasibul Hassan Chowdhury**. The main goal of this typeset is to have an organized digital version of the notes, which is easier to share and handle. Lecture notes of the previous Algebraic Topology courses can be found in the following links.

- **Algebraic Topology I (MAT431)**: https://atonurc.github.io/assets/MAT431_AT1.pdf
- **Algebraic Topology II (MAT432)**: https://atonurc.github.io/assets/MAT432_AT2.pdf

If you see any mistakes or typos, please send me an email at atonuroychowdhury@gmail.com

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References:

- *Elements of Algebraic Topology*, by James R. Munkres
- *Topology*, by Klaus Jänich, translated by Silvio Levy.
- Note on CW Complexes, by Soren Hansen. Link: https://www.math.ksu.edu/~hansen/CWcomplexes.pdf

Contents

1 Singular Homology Theory

§1.1 Singular Homology Groups

Let $\mathbb{R}^\infty$ denote the generalized Euclidean space $\mathbb{E}^J$, with J being the set of positive integers. An element of the vector space $\mathbb{R}^\infty$ is an infinite sequence of real numbers (functions from $\mathbb{N}$ to $\mathbb{R}$) with finitely many nonzero entries. Let Δ_p denote the p-simplex in $\mathbb{R}^\infty$ having vertices

$$\begin{aligned}
\varepsilon_0 &= (1, 0, 0, \dots, 0, \dots) \,, \\
\varepsilon_1 &= (0, 1, 0, \dots, 0, \dots) \,, \\
&\cdots \\
\varepsilon_p &= (0, 0, 0, \dots, \underbrace{1}_{(p+1)\text{-th entry}}, \dots) \,.
\end{aligned}$$

We call Δ_p the **standard p-simplex**. In this notation, Δ_{p-1} is a face of Δ_p.

> **Definition 1.1** (Singular p-simplex)**.** Let X be a topological space. We define a **singular p-simplex** of X to be a continuous map $T : \Delta_p \to X$. The free abelian group generated by singular p-simplices of X is denoted by $S_p(X)$, and is called the **singular chain group** of X in dimension p. We shall denote an element of $S_p(X)$ by a $\mathbb{Z}$-linear combination of singular p-simplices of X.

Singular means that T could be a "bad" map, i.e. it may not be an imbedding. All we want that T is just continuous. Now, recall that

$$\Delta_p = \left\{ (x_0, x_1, \dots, x_p, 0, \dots) \in \mathbb{R}^\infty | 0 \le x_i \le 1 \text{ and } \sum_{i=0}^{p} x_i = 1 \right\}. \tag{1.1}$$

Given $a_0, a_1, \dots, a_p \in \mathbb{R}^\infty$, there is a unique affine map $l_{(a_0,\dots,a_p)} : \Delta_p \to \mathbb{R}^\infty$ that maps ε_i to a_i. It is defined by

$$\begin{aligned}
l_{(a_0,\dots,a_p)}(x_0, x_1, \dots, x_p, 0, \dots) &= \sum_{i=0}^{p} x_i a_i = \sum_{i=0}^{p} x_i a_i + a_0 - \sum_{i=0}^{p} x_i a_0 \\
&= a_0 + \sum_{i=0}^{p} x_i (a_i - a_0) \,.
\end{aligned} \tag{1.2}$$

We call this map the **linear singular simplex** determined by $a_0, a_1, \dots, a_p \in \mathbb{R}^\infty$. Now, what is $l_{(\varepsilon_0,\dots,\varepsilon_p)}$? Observe that

$$l_{(\varepsilon_0,\dots,\varepsilon_p)}\varepsilon_i = l_{(\varepsilon_0,\dots,\varepsilon_p)}(0, \dots, 0, \underbrace{1}_{(i+1)\text{-th entry}}, 0, \dots) = \varepsilon_i. \tag{1.3}$$

Therefore, $l_{(\varepsilon_0,\dots,\varepsilon_p)}$ maps ε_i to itself, for every $i = 0, 1, \dots, p$. Also,

$$l_{(\varepsilon_0,\dots,\varepsilon_p)}(x_0, x_1, \dots, x_p, 0, \dots) = \sum_{i=0}^{p} x_i \varepsilon_i = (x_0, x_1, \dots, x_p, 0, \dots) \,. \tag{1.4}$$

Therefore, $l_{(\varepsilon_0,\dots,\varepsilon_p)}$ is just the inclusion map of Δ_p into $\mathbb{R}^\infty$. Now, suppose $(x_0, x_1, \dots, x_{p-1}, 0, \dots) \in \Delta_{p-1}$, so that $\sum_{i=0}^{p-1} x_i = 1$. Then

$$\begin{aligned}
l_{(\varepsilon_0,\dots,\widehat{\varepsilon_i},\dots,\varepsilon_p)}(x_0, x_1, \dots, x_{p-1}, 0, \dots) &= x_0\varepsilon_0 + \cdots + x_{i-1}\varepsilon_{i-1} + 0 \cdot \varepsilon_i + x_{i+1}\varepsilon_{i+1} + \cdots + x_{p-1}\varepsilon_p \\
&= (x_0, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_{p-1}, 0, \dots) \,,
\end{aligned} \tag{1.5}$$

which is a point on the face of Δ_p opposite to the vertex ε_i. In fact, $l_{(\varepsilon_0,\ldots,\widehat{\varepsilon_i},\ldots,\varepsilon_p)}$ is a linear homomorphism of Δ_{p-1} into the face of Δ_p that is opposite to the vertex ε_i. In other words,

$$l_{(\varepsilon_0,\ldots,\widehat{\varepsilon_i},\ldots,\varepsilon_p)} : \Delta_{p-1} \to \Delta_p$$

maps Δ_{p-1} to the face of Δ_p opposite to the vertex ε_i. Therefore, given a singular p-simplex $T : \Delta_p \to X$, one can form the composite

$$T \circ l_{(\varepsilon_0,\ldots,\widehat{\varepsilon_i},\ldots,\varepsilon_p)} : \Delta_{p-1} \to X,$$

which is a singular $(p-1)$-simplex. We think of it as the i-th face of the singular p-simplex T.

Definition 1.2 (Boundary homomorphism)**.** We define $\partial : S_p(X) \to S_{p-1}(X)$ as follows. If $T : \Delta_p \to X$ is a singular p-simplex, we define ∂T to be

$$\partial T = \sum_{i=0}^{p} (-1)^i \, T \circ l_{(\varepsilon_0,\ldots,\widehat{\varepsilon_i},\ldots,\varepsilon_p)}. \tag{1.6}$$

In other words, ∂T is a formal sum of singular simplices of dimension $p-1$, which are the faces of T.

Remark 1.1 (IMPORTANT!)**.** Note that only the singular p-simplices are maps, not the singular p-chains. The p-chains are just formal sum of continuous maps from Δ_p to X. If T_1 and T_2 are two singular p-simplices, i.e. continuous maps $\Delta_p \to X$, then $T_1 + T_2$ is **NOT** a map. The sum present here is nothing but a formal notation. So one cannot act $T_1 + T_2$ on a point of Δ_p. For the same reason, ∂T_1 is not a map. It is merely a formal linear combination of the continuous maps $T \circ l_{(\varepsilon_0,\ldots,\widehat{\varepsilon_i},\ldots,\varepsilon_p)}$.

If $f : X \to Y$ is a continuous map, we define a group homomorphism $f_\# : S_p(X) \to S_p(Y)$ by defining it on singular p-simplices by the equation

$$f_\#(T) = f \circ T \tag{1.7}$$

for a singular p-simplex T.

$$\begin{tikzcd} \Delta_p \arrow[r, "T"] \arrow[rr, bend right, "f\circ T"'] & X \arrow[r, "f"] & Y \end{tikzcd}$$

Theorem 1.1

The homomorphism $f_\#$ commutes with ∂. Furthermore, $\partial^2 = 0$.

Proof. Given a singular p-simplex T,

$$\partial f_\#(T) = \partial(f \circ T) = \sum_{i=0}^{p} (-1)^i (f \circ T) \circ l_{(\varepsilon_0,\ldots,\widehat{\varepsilon_i},\ldots,\varepsilon_p)}. \tag{1.8}$$

$$f_\#(\partial T) = f_\# \left(\sum_{i=0}^{p} (-1)^i \, T \circ l_{(\varepsilon_0,\ldots,\widehat{\varepsilon_i},\ldots,\varepsilon_p)} \right) = \sum_{i=0}^{p} (-1)^i f \circ T \circ l_{(\varepsilon_0,\ldots,\widehat{\varepsilon_i},\ldots,\varepsilon_p)}. \tag{1.9}$$

Therefore, $\partial f_\#(T) = f_\#(\partial T)$. Now, to prove $\partial^2 = 0$, we first compute ∂ for linear singular simplices $l_{(a_0,\ldots,a_p)}$.

$$\partial l_{(a_0,\ldots,a_p)} = \sum_{i=0}^{p} (-1)^i \, l_{(a_0,\ldots,a_p)} \circ l_{(\varepsilon_0,\ldots,\widehat{\varepsilon_i},\ldots,\varepsilon_p)}. \tag{1.10}$$

Observe that

$$\begin{aligned}l_{(a_0,\ldots,a_p)} \circ l_{(\varepsilon_0,\ldots,\widehat{\varepsilon_i},\ldots,\varepsilon_p)} (x_0,\ldots,x_{p-1},0,\ldots) &= l_{(a_0,\ldots,a_p)} (x_0,\ldots,x_{i-1},0,x_i x_{p-1},0,) \\ &= x_0 a_0 + \cdots + x_{i-1}a_{i-1} + 0\cdot a_i + x_i a_{i+1} + \cdots x_{p-1}a_p \\ &= l_{(a_0,\ldots,\widehat{a_i},\ldots a_p)} (x_0,\ldots,x_{p-1},0,\ldots). \end{aligned} \tag{1.11}$$

Hence,

$$l_{(a_0,\ldots,a_p)} \circ l_{(\varepsilon_0,\ldots,\widehat{\varepsilon_i},\ldots,\varepsilon_p)} = l_{(a_0,\ldots,\widehat{a_i},\ldots a_p)}. \tag{1.12}$$

Therefore, from 1.10, it follows that

$$\partial l_{(a_0,\ldots,a_p)} = \sum_{i=0}^{p} (-1)^i \, l_{(a_0,\ldots,\widehat{a_i},\ldots a_p)}. \tag{1.13}$$

Let's now evaluate $\partial\partial l_{(a_0,\ldots,a_p)}$.

$$\begin{aligned}\partial\partial l_{(a_0,\ldots,a_p)} &= \sum_{i=0}^{p} (-1)^i \, \partial l_{(a_0,\ldots,\widehat{a_i},\ldots a_p)} \\ &= \sum_{i=0}^{p} (-1)^i \sum_{j<i} (-1)^j \, l_{(a_0,\ldots,\widehat{a_j},\ldots\widehat{a_i},\ldots a_p)} + \sum_{i=0}^{p} (-1)^i \sum_{j>i} (-1)^{j-1} \, l_{(a_0,\ldots,\widehat{a_i},\ldots\widehat{a_j},\ldots a_p)} \\ &= \sum_{i=0}^{p} \sum_{j<i} (-1)^{i+j} \, l_{(a_0,\ldots,\widehat{a_j},\ldots\widehat{a_i},\ldots a_p)} - \sum_{i=0}^{p} \sum_{j>i} (-1)^{i+j} \, l_{(a_0,\ldots,\widehat{a_i},\ldots\widehat{a_j},\ldots a_p)}. \end{aligned} \tag{1.14}$$

Now fix $0 \le j_0 < i_0 \le p$. In the first summand of 1.14, the contribution of $i = i_0, j = j_0$ is

$$(-1)^{i_0+j_0} \, l_{\left(a_0,\ldots,\widehat{a_{j_0}},\ldots\widehat{a_{i_0}},\ldots a_p\right)}. \tag{1.15}$$

On the other hand, in the second summand of 1.14, the contribution of $i = j_0, j = i_0$ is also

$$(-1)^{i_0+j_0} \, l_{\left(a_0,\ldots,\widehat{a_{j_0}},\ldots\widehat{a_{i_0}},\ldots a_p\right)}. \tag{1.16}$$

These two contributions cancel each other. This way, one finds that the RHS of 1.14 vanishes. Hence,

$$\partial\partial l_{(a_0,\ldots,a_p)} = 0. \tag{1.17}$$

In particular,

$$\partial\partial l_{(\varepsilon_0,\ldots,\varepsilon_p)} = 0. \tag{1.18}$$

Now, $l_{(\varepsilon_0,\ldots,\varepsilon_p)} : \Delta_p \to \Delta_p$ is continuous, so $l_{(\varepsilon_0,\ldots,\varepsilon_p)} \in S_p(\Delta_p)$. Furthermore, it is the identity map as we have seen in 1.4. Since $T : \Delta_p \to X$ is continuous, we can form $T_\# : S_p(\Delta_p) \to S_p(X)$.

$$T_\# \left(l_{(\varepsilon_0,\ldots,\varepsilon_p)}\right) = T \circ l_{(\varepsilon_0,\ldots,\varepsilon_p)} = T \circ \mathrm{id}_{\Delta_p} = T. \tag{1.19}$$

Therefore, using the fact that $T_\#$ commutes with ∂, we obtain

$$\partial\partial T = \partial\partial T_\# \left(l_{(\varepsilon_0,\ldots,\varepsilon_p)}\right) = T_\# \left(\partial\partial l_{(\varepsilon_0,\ldots,\varepsilon_p)}\right) = 0. \tag{1.20}$$

Hence, $\partial^2 T = 0$. ■

Definition 1.3 (Singular homology groups)**.** Th family of groups $S_p(X)$ and homomorphisms $\partial_p : S_p(X) \to S_{p-1}(X)$ is called **singular chain complex** of X, and is denoted by $\mathcal{S}(X)$.

$$\cdots \longrightarrow S_{p+1}(X) \xrightarrow{\partial_{p+1}} S_p(X) \xrightarrow{\partial_p} S_{p-1}(X) \longrightarrow \cdots$$

The homology groups of this chain complex are called the **singular homology groups** of X, and are denoted by $H_p(X)$.

Definition 1.4 (Augmentation map)**.** The chain complex $\mathcal{S}(X)$ is augmented by the homomorphism $\epsilon : S_0(X) \to \mathbb{Z}$ defined by setting $\epsilon(T) = 1$ for each singular 0-simplex $T : \Delta_0 \to X$. (A generic singular 0-chain is a $\mathbb{Z}$-linear combination of singular 0-simplices.)

It's immediate that if T is a singular 1-simplex, then $\epsilon(\partial T) = 0$. Indeed,

$$\epsilon(\partial T) = \epsilon\left(T \circ l_{(\widehat{\varepsilon_0}, \varepsilon_1)}\right) - \epsilon\left(T \circ l_{(\varepsilon_0, \widehat{\varepsilon_1})}\right) = 0. \tag{1.21}$$

Definition 1.5 (Reduced homology groups)**.** The homology groups of $\{\mathcal{S}(X), \epsilon\}$ are called the **reduced singular homology groups** of X, and are denoted by $\widetilde{H}_p(X)$.

Now, given continuous map $f : X \to Y$ and $T : \Delta_0 \to X$ a singular 0-simplex on X, then $f_\#(T) = f \circ T : \Delta_0 \to Y$.

$$\begin{tikzcd} \Delta_0 \arrow[r, "T"] \arrow[rr, bend right, "f\circ T"'] & X \arrow[r, "f"] & Y \end{tikzcd}$$

Now, consider the augmented singular chain complexes $\left\{\mathcal{S}(X), \epsilon^X\right\}$ and $\left\{\mathcal{S}(Y), \epsilon^Y\right\}$. Noting continuous $T : \Delta_0 \to X$ and $f_\#(T) : \Delta_0 \to Y$, one obtains $\epsilon^X(T) = 1$ and $\epsilon^Y(f_\#(T)) = 1$. In other words, the following diagram commutes

$$\begin{tikzcd} S_0(X) \arrow[r, "\epsilon^X"] \arrow[d, "(f_\#)_0"'] & \mathbb{Z} \arrow[d, "\mathrm{id}"] \\ S_0(Y) \arrow[r, "\epsilon^Y"] & \mathbb{Z} \end{tikzcd}$$

Therefore, $f_\# : S_p(X) \to S_p(Y)$ is an **augmentation preserving chain map** between $\left\{\mathcal{S}(X), \epsilon^X\right\}$ and $\left\{\mathcal{S}(Y), \epsilon^Y\right\}$. Thus, $f_\#$ induces a homomorphism f_* in both ordinary and reduced singular homology.

In Theorem 1.1, we saw that the chain map $f_\#$ commutes with the boundary operator ∂. In other words, $(f_\#)_p : S_p(X) \to S_p(Y)$ takes cycles to cycles and boundaries to boundaries. Suppose $c_p \in Z_p(X) = \operatorname{Ker} \partial_p^X$, so that $\partial_p^X c_p = 0$. Now,

$$\partial_p^Y\left((f_\#)_p c_p\right) = (f_\#)_{p-1}\left(\partial_p^X c_p\right) = 0. \tag{1.22}$$

Hence, $(f_\#)_p c_p \in Z_p(Y)$. On the other hand, let $b_p \in B_p(X) = \operatorname{Im} \partial_{p+1}^X$. Then $b_p = \partial_{p+1}^X d_{p+1}$ for some $d_{p+1} \in S_{p+1}(X)$. Then

$$(f_\#)_p b_p = (f_\#)_p\left(\partial_{p+1}^X d_{p+1}\right) = \partial_{p+1}^Y\left((f_\#)_{p+1} d_{p+1}\right). \tag{1.23}$$

In other words, $(f_\#)_p b_p \in B_p(Y)$. This reflects the fact that $(f_\#)_p : S_p(X) \to S_p(Y)$ induces a homomorphism between the singular homology groups $(f_*)_p : H_p(X) \to H_p(Y)$. $(f_*)_p$ is given by

$$(f_*)_p(c_p + B_p(X)) = (f_\#)_p c_p + B_p(Y). \tag{1.24}$$

If the reduced homology groups of X vanishes in all dimensions, we say that X is **acyclic** (in singular homology).

Theorem 1.2

If $i : X \to X$ is the identity, then so is $(i_*)_p : H_p(X) \to H_p(X)$. If $f : X \to Y$ and $g : Y \to Z$ are continuous, then $((g \circ f)_*)_p = (g_*)_p \circ (f_*)_p$.

Proof. It is sufficient to show that the equations hold at the chain level. We know from the definition of $(f_\#)_p : S_p(X) \to S_p(Y)$ that it maps $T \in S_p(X)$ to $f \circ T \in S_p(Y)$. Since $i : X \to X$ is the identity map,

$$(i_\#)_p(T) = i \circ T = T. \tag{1.25}$$

So $(i_\#)_p : S_p(X) \to S_p(X)$ is the identity homomorphism. As a result,

$$(i_*)_p(c_p + B_p(X)) = (i_\#)_p c_p + B_p(X) = c_p + B_p(X). \tag{1.26}$$

Therefore, $(i_*)_p = \mathrm{id}_{H_p(X)}$.

Given continuous $f : X \to Y$ and $g : Y \to Z$, $\left((g \circ f)_\#\right)_p : S_p(X) \to S_p(Z)$ is defined by

$$\left((g \circ f)_\#\right)_p T = (g \circ f) \circ T = g \circ (f \circ T) = (g_\#)_p \left((f_\#)_p T\right). \tag{1.27}$$

Therefore, $\left((g \circ f)_\#\right)_p = (g_\#)_p \circ (f_\#)_p$. Now, at the homology level, for $c_p + B_p(X) \in H_p(X) = Z_p(X)/B_p(X)$

$$((g \circ f)_*)_p(c_p + B_p(X)) = \left((g \circ f)_\#\right)_p c_p + B_p(Z) = (g_\#)_p\left((f_\#)_p c_p\right) + B_p(Z). \tag{1.28}$$

Also,

$$(g_*)_p \circ (f_*)_p(c_p + B_p(X)) = (g_*)_p\left((f_\#)_p c_p + B_p(Y)\right) = (g_\#)_p\left((f_\#)_p c_p\right) + B_p(Z). \tag{1.29}$$

From 1.28 and 1.29, we can deduce that $((g \circ f)_*)_p = (g_*)_p \circ (f_*)_p$. ∎

Corollary 1.3

If $h : X \to Y$ is a homeomorphims, then $(h_*)_p : H_p(X) \to H_p(Y)$ is an isomorphism.

Proof. Both $h : X \to Y$ and $h^{-1} : Y \to X$ are continuous, and $h \circ h^{-1} = \mathrm{id}_Y$. Therefore,

$$(h_*)_p \circ \left(\left(h^{-1}\right)_*\right)_p = \left(\left(h \circ h^{-1}\right)_*\right)_p = ((\mathrm{id}_Y)_*)_p = \mathrm{id}_{H_p(Y)}. \tag{1.30}$$

Similarly, starting with $h^{-1} \circ h = \mathrm{id}_X$, we will get $\left(\left(h^{-1}\right)_*\right)_p \circ (h_*)_p = \mathrm{id}_{H_p(X)}$. Therefore, $\left(\left(h^{-1}\right)_*\right)_p$ is the inverse of $(h_*)_p$. In other words, $(h_*)_p$ is an invertible homomorphism, i.e. an isomorphism. ∎

Theorem 1.4

Let X be a topological space. Then $H_0(X)$ is free abelian. If $\{X_\alpha\}$ is the collection of path components of X, and if T_α is a singular 0-simplex with image in X_α for each α, then the homology classes of the chains T_α form a basis for $H_0(X)$. The group $\widetilde{H}_0(X)$ is also free abelian; it vanishes if X is path connected. Otherwise, let α_0 be a fixed index, then the homology classes of the chains $T_\alpha - T_{\alpha_0}$ for $\alpha \neq \alpha_0$ form a basis for $\widetilde{H}_0(X)$.

Proof. Let $x_\alpha = T_\alpha(\Delta_0) \in X_\alpha$, with $T_\alpha : \Delta_0 \to X$ being a singular 0-simplex. Here, Δ_0 consists of the point $\varepsilon_0 = (1, 0, 0, \ldots) \in \mathbb{R}^\infty$. Also, let $T : \Delta_0 \to X$ be any singular 0-simplex such that $T(\Delta_0) \in X_\alpha$. Since X_α is path connected, there is a path connecting $T(\Delta_0)$ and $T_\alpha(\Delta_0)$. In other words, there is a singular 1-simplex $f : \Delta_1 \to X$ such that

$$f(1,0,0\ldots) = T(\Delta_0) \text{ and } f(0,1,0\ldots) = T_\alpha(\Delta_0). \tag{1.31}$$

Then we have

$$\partial_1 f = f \circ l_{(\widehat{\varepsilon_0},\varepsilon_1)} - f \circ l_{(\varepsilon_0,\widehat{\varepsilon_1})}. \tag{1.32}$$

Now,

$$f \circ l_{(\varepsilon_0,\widehat{\varepsilon_1})}(1,0,0,\ldots) = f(1,0,0,\ldots) = T(\Delta_0) = T(1,0,0,\ldots), \tag{1.33}$$

$$f \circ l_{(\widehat{\varepsilon_0},\varepsilon_1)}(1,0,0,\ldots) = f(0,1,0,\ldots) = T_\alpha(\Delta_0) = T_\alpha(1,0,0,\ldots). \tag{1.34}$$

Therefore, $\partial_1 f = T_\alpha - T$.

An arbitrary singular 0-chain is a $\mathbb{Z}$-linear combination of singular 0-simplices. Let's take $c \in S_0(X)$. Then $c = \sum_\beta m_\beta T'_\beta$, with $m_\beta \in \mathbb{Z}$ and T'_β being singular 0-simplices. Each $T'_\beta(\Delta_0)$ belongs to some X_α, and hence homologous to T_α. Therefore, c is homologous to some $\mathbb{Z}$-linear combination $\sum_\alpha n_\alpha T_\alpha$ of the T_α's. We will now show that no such nontrivial 0-chain $\sum_\alpha n_\alpha T_\alpha$ bounds.

Assume the contrary that $\sum_\alpha n_\alpha T_\alpha = \partial_1 d$ for some $d \in S_1(X)$. Now, the singular 1-chain d is a formal linear combination of singular 1-simplices with path connected image, i.e. the image lies in one of the path components X_α. Thus we can write $d = \sum_\alpha d_\alpha$, where d_α consists of the terms whose images are in X_α. Therefore,

$$\sum_\alpha n_\alpha T_\alpha = \partial_1 d = \sum_\alpha \partial_1 d_\alpha. \tag{1.35}$$

Hence, we get

$$n_\alpha T_\alpha = \partial_1 d_\alpha \tag{1.36}$$

for each α. Applying ϵ to both sides of 1.36, we get

$$\epsilon(n_\alpha T_\alpha) = \epsilon(\partial_1 d_\alpha) \implies n_\alpha = 0. \tag{1.37}$$

Therefore, no non-trivial 0-chain $\sum_\alpha n_\alpha T_\alpha$ bounds. Since every 0-chain is automatically a 0-cycle, an element of $H_0(X)$ is homologous to a 0-chain of the form $\sum_\alpha n_\alpha T_\alpha$. Hence, the homology classes of the singular 0-simplices $\{T_\alpha\}$ form a basis for the free abelian group $H_0(X)$.

$$S_1(X) \xrightarrow{\partial_1} S_0(X) \xrightarrow{\epsilon} \mathbb{Z}$$

$\widetilde{H}_0(X)$ is defined as $\widetilde{H}_0(X) = \operatorname{Ker}\epsilon / \operatorname{Im}\partial_1$. Given a singular 0-chain $T \in S_0(X)$, we've seen that T is homologous to a 0-chain of the form $T' = \sum_\alpha n_\alpha T_\alpha$; and T' bounds iff $T' = 0$, i.e. $n_\alpha = 0$ for every α. If further $T \in \operatorname{Ker}\epsilon$, then $\epsilon(T) = 0$. Since T and T' are homologous, $T = T' + \partial_1 d$ for some $d \in S_1(X)$. Therefore,

$$0 = \epsilon(T) = \epsilon\left(T'\right) + \epsilon(\partial_1 d) = \epsilon\left(\sum_\alpha n_\alpha T_\alpha\right) = \sum_\alpha n_\alpha. \tag{1.38}$$

If X is path connected, there is only one component, and hence there is only one n_α involved. Thus $n_\alpha = 0$ from 1.38. This gives us $T' = 0$, leading to the fact that every $T \in \operatorname{Ker}\epsilon$ is homologous to 0, i.e. $T = 0 + \partial_1 d$ for some $d \in S_1(X)$. So $\operatorname{Ker}\epsilon = \operatorname{Im}\partial_1$. Therefore, $\widetilde{H}_0(X) = 0$, when X is path connected.

Now, suppose X has more than one path components. Fix α_0. Then from 1.38, we get

$$0 = \sum_\alpha n_\alpha = n_{\alpha_0} + \sum_{\alpha \neq \alpha_0} n_\alpha \implies n_{\alpha_0} = -\sum_{\alpha \neq \alpha_0} n_\alpha. \tag{1.39}$$

Then T' is

$$T' = \sum_{\alpha} n_\alpha T_\alpha = \sum_{\alpha\neq\alpha_0} n_\alpha T_\alpha + n_{\alpha_0} T_{\alpha_0} = \sum_{\alpha\neq\alpha_0} n_\alpha T_\alpha - \sum_{\alpha\neq\alpha_0} n_\alpha T_{\alpha_0} = \sum_{\alpha\neq\alpha_0} n_\alpha \left(T_\alpha - T_{\alpha_0}\right). \tag{1.40}$$

1.40 suggests that T' is a linear combination of the singular 0-chains $\{T_\alpha - T_{\alpha_0}\}_{\alpha\neq\alpha_0}$. And T' bounds iff it is trivial, as shown earlier. Therefore, the homology classes of 0-chains $\{T_\alpha - T_{\alpha_0}\}_{\alpha\neq\alpha_0}$ form a basis for $\widetilde{H}_0(X)$. ■

Theorem 1.4 illustrates the following result:

$$H_p(X) = \begin{cases} \widetilde{H}_p(X) & \text{if } p > 0 \\ \widetilde{H}_0(X) \oplus \mathbb{Z} & \text{if } p = 0 \end{cases}. \tag{1.41}$$

§1.2 Bracket Operation

Definition 1.6 (Star convex set)**.** A set $X \subseteq \mathbb{E}^J$ is said to be **star convex** relative to the point $w \in X$, if for each $x \in X$, the line segment from x to w lies in X.

Definition 1.7 (Bracket operation)**.** Suppose $X \in \mathbb{E}^J$ is star convex relative to w. We define bracket operation on singular chains of X. Let us first define it for singular p-simplices. Let $T : \Delta_p \to X$ be a singular p-simplex of X. Define a singular $(p+1)$-simplex

$$[T, w] : \Delta_{p+1} \to X$$

by letting $[T, w]$ carry the line segment from x to ε_{p+1}, for $x \in\in \Delta_p$ (the collection of all such line segments as x varies in Δ_p constitutes Δ_{p+1}), linearly onto the line segment $T(x)$ to w in X. In other words,

$$[T, w]\left(t\varepsilon_{p+1} + (1-t)x\right) = tw + (1-t)T(x), \tag{1.42}$$

for $t \in [0, 1]$. Now, extend the definition of bracket operation to arbitrary p-chains as follows: if $c = \sum n_i T_i$ is a singular p-chain of X with each T_i being a singular p-simplex, then we define

$$[c, w] = \sum n_i [T_i, w]. \tag{1.43}$$

In other words, $[\cdot\,, w] : S_p(X) \to S_{p+1}(X)$, $c \mapsto [c, w]$ is a homomorphism.

From Figure 1.1, it's immediate that the restriction of $[T, w]$ to the face Δ_p of Δ_{p+1} is just the map T. Now, consider the case when T is the linear singular simplex $l_{(a_0,\ldots,a_p)}$ for $a_0, \ldots, a_p \in \mathbb{R}^\infty$. We want to calculate what $\left[l_{(a_0,\ldots,a_p)}, w\right]$ is.

Recall that $l_{(a_0,\ldots,a_p)} : \Delta_p \to \mathbb{R}^\infty$ is defined as

$$l_{(a_0,\ldots,a_p)}(x_0, \ldots, x_p) = \sum_{i=0}^{p} x_i a_i. \tag{1.44}$$

Consider a point $(x_0, \ldots, x_p, x_{p+1}, 0, \ldots) \in \Delta_{p+1}$. We want to see where $\left[l_{(a_0,\ldots,a_p)}, w\right]$ takes this point to. Since $(x_0, \ldots, x_p, x_{p+1}, 0, \ldots) \in \Delta_{p+1}$, each x_i is nonnegative with $\sum_{i=0}^{p+1} x_i = 1$. Now,

$$\sum_{i=0}^{p} \frac{x_i}{1 - x_{p+1}} = 1, \tag{1.45}$$

so $\left(\frac{x_0}{1-x_{p+1}}, \frac{x_1}{1-x_{p+1}}, \ldots, \frac{x_p}{1-x_{p+1}}, 0, \ldots\right) \in \Delta_p$. Therefore,

$$(x_0, \ldots, x_p, x_{p+1}, 0, \ldots) = (1 - x_{p+1})\left(\frac{x_0}{1-x_{p+1}}, \frac{x_1}{1-x_{p+1}}, \ldots, \frac{x_p}{1-x_{p+1}}, 0, \ldots\right) + x_{p+1}\varepsilon_{p+1}. \tag{1.46}$$

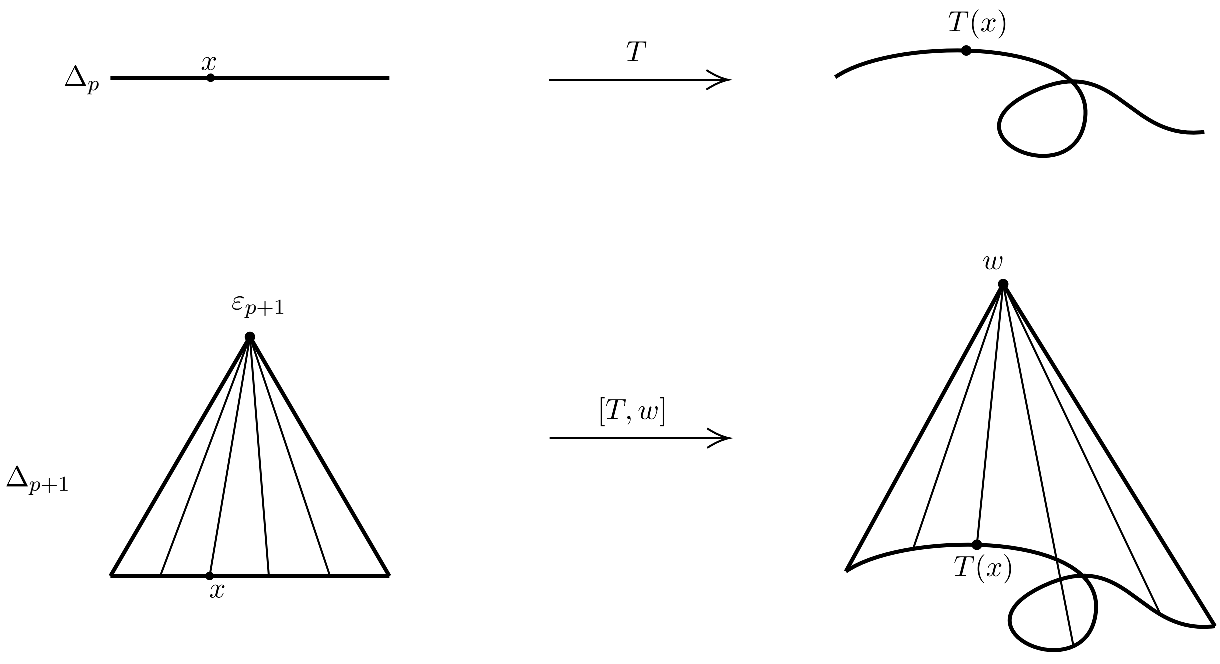


Figure 1.1

By the definition of bracket operation,

$$\begin{aligned}
& \left[l_{(a_0,\ldots,a_p)}, w\right](x_0,\ldots,x_p,x_{p+1},0,\ldots) \\
= \; & (1-x_{p+1})\, l_{(a_0,\ldots,a_p)}\left(\frac{x_0}{1-x_{p+1}}, \frac{x_1}{1-x_{p+1}}, \ldots, \frac{x_p}{1-x_{p+1}}, 0, \ldots\right) + x_{p+1}w \\
= \; & (1-x_{p+1}) \sum_{i=0}^{p} \frac{x_i}{1-x_{p+1}} a_i + x_{p+1}w \\
= \; & \sum_{i=0}^{p} x_i a_i + x_{p+1}w. \qquad (1.47)
\end{aligned}$$

Furthermore,

$$l_{(a_0,\ldots,a_p,w)}(x_0,\ldots,x_p,x_{p+1},0,\ldots) = x_0a_0 + \cdots + x_pa_p + x_{p+1}w = \sum_{i=0}^{p} x_ia_i + x_{p+1}w. \qquad (1.48)$$

Equating 1.47 and 1.48, we get

$$\left[l_{(a_0,\ldots,a_p)}, w\right] = l_{(a_0,\ldots,a_p,w)}. \qquad (1.49)$$

Now we will showo that $[T,w] : \Delta_{p+1} \to X$ is continuous. We have seen earlier that given $x \in \Delta_p$, a point in Δ_{p+1} is expressed as $t\varepsilon_{p+1} + (1-t)\,x$, with $0 \le t \le 1$. Hence, we are concerned with the following quotient map $\pi : \Delta_p \times [0,1] \to \Delta_{p+1}$ defined by

$$\pi(x,t) = t\varepsilon_{p+1} + (1-t)\,x. \qquad (1.50)$$

If $x = (x_0,\ldots,x_p,0,\ldots) \in \Delta_p$, then 1.50 takes the familiar form

$$\pi((x_0,\ldots,x_p,0,\ldots),t) = ((1-t)\,x_0,\ldots,(1-t)\,x_p,t,0,\ldots). \qquad (1.51)$$

Observe that $\pi\big|_{\Delta_p\times[0,1)} : \Delta_p \times [0,1) \to \Delta_{p+1}$ is 1-1, and $\pi(\Delta_p \times \{1\}) = \{\varepsilon_{p+1}\}$, showing that π collapses $\Delta_p \times \{1\}$ to the $(p+1)$-th vertex ε_{p+1} of Δ_{p+1}. Now, the continuous map $f : \Delta_p \times [0,1] \to X$ defined by

$$f(x,t) = tw + (1-t)\,T(x) \qquad (1.52)$$

is constant on $\Delta_p \times \{1\}$. In fact, $f(\Delta_p \times \{1\}) = \{w\}$. Since π is $1-1$ for other points, f is seen to be constant for $\pi^{-1}(y)$ with $y \in \Delta_{p+1} \setminus \{\varepsilon_{p+1}\}$. In other words, $f : \Delta_p \times [0,1] \to X$ is constant for each $\pi^{-1}(y)$ with $y \in \Delta_{p+1}$. Therefore, f induces a unique continuous map $\widetilde{f} : \Delta_{p+1} \to X$ such that the following diagram commutes

$$\begin{tikzcd}
\Delta_p \times [0,1] \arrow[d, "\pi"'] \arrow[dr, "f"] & \\
\Delta_{p+1} \arrow[r, "\widetilde{f}"] & X
\end{tikzcd}$$

This unique map $\widetilde{f}$ is precisely $[T,w]$, since

$$([T,w] \circ \pi)(x,t) = [T,w](t\varepsilon_{p+1} + (1-t)x) = tw + (1-t)T(x) = f(x,t). \tag{1.53}$$

Therefore, $\widetilde{f} = [T,w]$, and hence it is continuous. So $[T,w]$ is indeed a singular $(p+1)$-simplex.

Lemma 1.5

Let X be a star convex set with respect to w; let c be a singular p-chain of X. Then

$$\partial [c,w] = \begin{cases} [\partial c, w] + (-1)^{p+1} c & \text{if } p > 0 \\ \epsilon(c) T_w - c & \text{if } p = 0 \end{cases}, \tag{1.54}$$

where T_w is the singular 0-simplex mappting Δ_0 to w.

Proof. If T is a singular 0-simplex, $[T,w]$ is a singular 1-simplex. Then

$$\partial [T,w] = [T,w] \circ l_{(\widehat{\varepsilon}_0, \varepsilon_1)} - [T,w] \circ l_{(\varepsilon_0, \widehat{\varepsilon}_1)}. \tag{1.55}$$

Now, recall $[T,w] : \Delta_1 \to X$ maps the line joining ε_1 to ε_0 to the line joining w to $T(\varepsilon_0)$. So

$$[T,w](1-t, t, 0, \ldots) = tw + (1-t)T(\varepsilon_0). \tag{1.56}$$

Now,

$$\left([T,w] \circ l_{(\widehat{\varepsilon}_0, \varepsilon_1)}\right)(1,0,\ldots) = [T,w](0,1,0,\ldots) = w = T_w(1,0,\ldots). \tag{1.57}$$

Therefore, $\left([T,w] \circ l_{(\widehat{\varepsilon}_0, \varepsilon_1)}\right) = T_w$.

$$\left([T,w] \circ l_{(\varepsilon_0, \widehat{\varepsilon}_1)}\right)(1,0,\ldots) = [T,w](1,0,\ldots) = T(\varepsilon_0) = T(1,0,\ldots), \tag{1.58}$$

so $[T,w] \circ l_{(\varepsilon_0, \widehat{\varepsilon}_1)} = T$. By 1.55, we get

$$\partial [T,w] = T_w - T. \tag{1.59}$$

Now, let $c = \sum_i n_i T_i$ be a singular 0-chain with T_i being singular 0-simplices. Then

$$\partial \left[\sum_i n_i T_i, w\right] = \sum_i n_i \partial [T_i, w] = \sum_i n_i (T_w - T_i) = \left(\sum_i n_i\right) T_w - \sum_i n_i T_i. \tag{1.60}$$

Now, applying the augmentation map to c, we get

$$\epsilon(c) = \epsilon\left(\sum_i n_i T_i\right) = \sum_i n_i \epsilon(T_i) = \sum_i n_i. \tag{1.61}$$

Therefore, 1.60 gives us

$$\partial [c,w] = \epsilon(c) T_w - c. \tag{1.62}$$

Now we shall consider the case when T is a singular p-simplex, and we shall prove that $\partial[T,w] = [\partial T, w] + (-1)^{p+1} T$.

$$\begin{aligned}\partial[T,w] &= \sum_{i=0}^{p+1} (-1)^i [T,w] \circ l_{(\varepsilon_0,\dots\widehat{\varepsilon}_i,\dots,\varepsilon_{p+1})} \\ &= \sum_{i=0}^{p} (-1)^i [T,w] \circ l_{(\varepsilon_0,\dots\widehat{\varepsilon}_i,\dots,\varepsilon_{p+1})} + (-1)^{p+1} [T,w] \circ l_{(\varepsilon_0,\dots,\varepsilon_p,\widehat{\varepsilon}_{p+1})}. \end{aligned} \tag{1.63}$$

$l_{(\varepsilon_0,\dots,\varepsilon_p,\widehat{\varepsilon}_{p+1})}$ is the inclusion map of Δ_p into Δ_{p+1}. So $[T,w] \circ l_{(\varepsilon_0,\dots,\varepsilon_p,\widehat{\varepsilon}_{p+1})}$ is nothing but the restriction of $[T,w]$ to Δ_p, which is the same as T. Now we want to show that

$$[T,w] \circ l_{(\varepsilon_0,\dots\widehat{\varepsilon}_i,\dots,\varepsilon_{p+1})} = \left[T \circ l_{(\varepsilon_0,\dots\widehat{\varepsilon}_i,\dots,\varepsilon_p)}, w\right]. \tag{1.64}$$

Both sides of 1.64 are maps from Δ_p to X. Let $(x_0,\dots,x_p,0,\dots) \in \Delta_p$. Then

$$\left([T,w] \circ l_{(\varepsilon_0,\dots\widehat{\varepsilon}_i,\dots,\varepsilon_{p+1})}\right)(x_0,\dots,x_p,0,\dots) = [T,w](x_0,\dots,x_{i-1},0,x_i,\dots,x_{p-1},x_p,0,\dots). \tag{1.65}$$

Now, $(x_0,\dots,x_{i-1},0,x_i,\dots,x_{p-1},x_p,0,\dots)$ is a point in Δ_{p+1}. We can write it as

$$(x_0,\dots,x_{i-1},0,x_i,\dots,x_{p-1},x_p,0,\dots) = (1-x_p)\left(\frac{x_0}{1-x_p},\dots,\frac{x_{i-1}}{1-x_p},0,\frac{x_i}{1-x_p},\dots,\frac{x_{p-1}}{1-x_p},0,\dots\right) + x_p\varepsilon_{p+1}. \tag{1.66}$$

Now, $\left(\frac{x_0}{1-x_p},\dots,\frac{x_{i-1}}{1-x_p},0,\frac{x_i}{1-x_p},\dots,\frac{x_{p-1}}{1-x_p},0,\dots\right)$ is a point in Δ_p since its nonzero components are all non-negative and they add to 1. Therefore,

$$\begin{aligned}&[T,w](x_0,\dots,x_{i-1},0,x_i,\dots,x_{p-1},x_p,0,\dots) \\ =\ &(1-x_p)\,T\left(\frac{x_0}{1-x_p},\dots,\frac{x_{i-1}}{1-x_p},0,\frac{x_i}{1-x_p},\dots,\frac{x_{p-1}}{1-x_p},0,\dots\right) + x_p w. \end{aligned} \tag{1.67}$$

On the other hand, we can write $(x_0,\dots,x_p,0,\dots)$ as

$$(x_0,\dots,x_p,0,\dots) = (1-x_p)\left(\frac{x_0}{1-x_p},\dots,\frac{x_{p-1}}{1-x_p},0,\dots\right) + x_p\varepsilon_p, \tag{1.68}$$

where $\left(\frac{x_0}{1-x_p},\dots,\frac{x_{p-1}}{1-x_p},0,\dots\right) \in \Delta_{p-1}$. So

$$\begin{aligned}&\left[T \circ l_{(\varepsilon_0,\dots\widehat{\varepsilon}_i,\dots,\varepsilon_p)}, w\right](x_0,\dots,x_p,0,\dots) \\ =\ &x_p w + (1-x_p)\left(T \circ l_{(\varepsilon_0,\dots\widehat{\varepsilon}_i,\dots,\varepsilon_p)}\right)\left(\frac{x_0}{1-x_p},\dots,\frac{x_{p-1}}{1-x_p},0,\dots\right) \\ =\ &x_p w + (1-x_p)\,T\left(\frac{x_0}{1-x_p},\dots,\frac{x_{i-1}}{1-x_p},0,\frac{x_i}{1-x_p},\dots,\frac{x_{p-1}}{1-x_p},0,\dots\right). \end{aligned} \tag{1.69}$$

Combining 1.65, 1.67 and 1.69, we get that 1.64 indeed holds, i.e.

$$[T,w] \circ l_{(\varepsilon_0,\dots\widehat{\varepsilon}_i,\dots,\varepsilon_{p+1})} = \left[T \circ l_{(\varepsilon_0,\dots\widehat{\varepsilon}_i,\dots,\varepsilon_p)}, w\right].$$

Now, from 1.63, we then get

$$\begin{aligned}\partial[T,w] &= \sum_{i=0}^{p} (-1)^i \left[T \circ l_{(\varepsilon_0,\dots\widehat{\varepsilon}_i,\dots,\varepsilon_p)}, w\right] + (-1)^{p+1} T \\ &= \left[\sum_{i=0}^{p} (-1)^i T \circ l_{(\varepsilon_0,\dots\widehat{\varepsilon}_i,\dots,\varepsilon_p)}, w\right] + (-1)^{p+1} T \\ &= [\partial T, w] + (-1)^{p+1} T. \end{aligned} \tag{1.70}$$

Now, if $c = \sum_i n_i T_i$ is a singular p-chain with T_i being singular 0-simplices, then

$$\partial[c,w] = \sum_i n_i \partial[T_i,w] = \sum_i n_i [\partial T_i, w] + (-1)^{p+1} \sum_i n_i T_i = [\partial c, w] + (-1)^{p+1} c. \tag{1.71}$$

■

Theorem 1.6

Let $X \subseteq \mathbb{E}^J$ be star convex with respect to w. Then X is acyclic in singular homology.

Proof. To show that $\widetilde{H}_0\left(X\right) = 0$, let $c \in \operatorname{Ker} \epsilon$.

$$S_1(X) \xrightarrow{\partial_1} S_0(X) \xrightarrow{\epsilon} \mathbb{Z}$$

So $\epsilon\left(c\right) = 0$. Now, by Lemma 1.5,

$$\partial_1 \left[c, w\right] = \epsilon\left(c\right) T_w - c = -c. \tag{1.72}$$

Hence, $c \in \operatorname{Im} \partial_1$ leading to $\operatorname{Ker} \epsilon \subseteq \operatorname{Im} \partial_1$. We already know Hence, $\operatorname{Im} \partial_1 \subseteq \operatorname{Ker} \epsilon$. Therefore, $\widetilde{H}_0\left(X\right) = 0$.

Now we shall show that $H_p\left(X\right) = 0$ for $p > 0$. Let $z \in \operatorname{Ker} \partial_p$. Then $\partial_p z = 0$. By Lemma 1.5 again,

$$\partial_{p+1} \left[z, w\right] = \left[\partial_p z, w\right] + \left(-1\right)^{p+1} z = \left(-1\right)^{p+1} z. \tag{1.73}$$

Hence, $z \in \operatorname{Im} \partial_{p+1}$. Therefore, $H_p\left(X\right) = 0$. In other words, $\widetilde{H}_p\left(X\right) = 0$ for all p, i.e. X is acyclic. ■

Corollary 1.7

Any simplex is acyclic in singular homology.

2 Axioms of Singular Homology

In this chapter, we shall verify that singular homology does, in fact, satisfy the Eilenberg-Steenrod axioms. The axioms can be found in chapter 6 of [AT2 lecture notes].

§2.1 Relative Homology Groups

If X is a space and A is a subspace of X, there is a natural inclusion $S_p(A) \hookrightarrow S_p(X)$. The group of **relative singular chains** is defined by

$$S_p(X, A) = S_p(X) / S_p(A). \tag{2.1}$$

The boundary operator $\partial_p^X : S_p(X) \to S_{p-1}(X)$ restricts to the boundary operator on $S_p(A)$, i.e. $\partial_p^X|_{S_p(A)} : S_p(A) \to S_{p-1}(A)$. It, therefore, induces a boundary operator at the relative singular chain level:

$$\begin{aligned} \partial_p^{(X,A)} &: S_p(X, A) \to S_{p-1}(X, A), \\ T + S_p(A) &\mapsto \partial_p^X T + S_{p-1}(A), \end{aligned} \tag{2.2}$$

with $T = \sum_\alpha n_\alpha T_\alpha$ being a singular p-chain, where $n_\alpha \in \mathbb{Z}$ and T_α singular p-simplices. If any of the T_α's are such that $T_\alpha(\Delta_p) \subseteq A$, then $T_\alpha \in S_p(A)$. So, we can assume $T_\alpha(\Delta_p) \setminus A \neq \varnothing$. Such T_α's generate the group $S_p(X, A)$, and so $S_p(X, A)$ is a free abelian group.

The family of groups $S_p(X, A)$ and homomorphisms $\partial_p^{(X,A)}$ is called **the singular chain complex** of the pair (X, A), and is denoted by $\mathcal{S}(X, A)$. The homology groups of the chain complex $\mathcal{S}(X, A)$ of the pair (X, A) are called the **singular homology groups** of the pair (X, A), and are denoted by $H_p(X, A)$.

The chain complex $\mathcal{S}(X, A)$ is free, i.e. $S_p(X, A)$ is free for each p. The group $S_p(X, A)$ has as basis all the cosets of the form $T + S_p(A)$, where T is a singular p-simplex with $T(\Delta_p) \setminus A \neq \varnothing$.

If $f : (X, A) \to (Y, B)$ is a continuous map (recall that by the continuity of f between pairs (X, A) and (Y, B), we actually mean that $f : X \to Y$ is continuous, with $f(A) \subseteq B$), then homomorphisms $(f_\#)_p : S_p(X) \to S_p(Y)$ carries singular p-chains of A into singular p-chains of B. So it induces a homomorphism (also denoted by $(f_\#)_p$) at the level of relative singular p-chains:

$$\begin{aligned} (f_\#)_p &: S_p(X, A) \to S_p(Y, B), \\ T + S_p(A) &\mapsto (f_\#)_p T + S_p(B) = f \circ T + S_p(B). \end{aligned} \tag{2.3}$$

where T is a singular p-simplex with $T(\Delta_p) \setminus A \neq \varnothing$. This map can be seen to commute with the boundary oeprator at the relative singular chain level. To be precise,

$$(f_\#)_{p-1} \circ \partial_p^{(X,A)} = \partial_p^{(Y,B)} \circ (f_\#)_p. \tag{2.4}$$

In other words, the following diagram commutes.

$$\begin{array}{ccc} S_p(X,A) & \xrightarrow{\partial_p^{(X,A)}} & S_{p-1}(X,A) \\ {\scriptstyle (f_\#)_p}\downarrow & & \downarrow{\scriptstyle (f_\#)_{p-1}} \\ S_p(Y,B) & \xrightarrow[\partial_p^{(Y,B)}]{} & S_{p-1}(Y,B) \end{array}$$

Therefore, $f_\#$ induces a homomorphism

$$\begin{aligned} (f_*)_p &: H_p(X, A) \to H_p(Y, B), \\ c + \operatorname{Im} \partial_{p+1}^{(X,A)} &\mapsto (f_\#)_p c + \operatorname{Im} \partial_{p+1}^{(Y,B)}. \end{aligned} \tag{2.5}$$

Theorem 2.1

If $i : (X, A) \to (X, A)$ is the identity, then so is $(i_*)_p : H_p(X, A) \to H_p(X, A)$. If $h : (X, A) \to (Y, B)$ and $k : (Y, B) \to (Z, C)$ are continuous, then $((k \circ h)_*)_p = (k_*)_p \circ (h_*)_p$.

Proof. Since $(i_\#)_p : S_p(X) \to S_p(X)$ is the identity map (as proven while proving Theorem 1.2), so is $(i_\#)_p : S_p(X, A) \to S_p(X, A)$. Then from 2.5, we get that $(i_*)_p : H_p(X, A) \to H_p(X, A)$ is the identity, i.e. $(i_*)_p = \mathrm{id}_{H_p(X,A)}$.

Now, let us prove $\left((k \circ h)_\#\right)_p = (k_\#)_p \circ (h_\#)_p$. The equality at the homology level will then follow from 2.5.

$$(h_\#)_p : S_p(X, A) \to S_p(Y, B)\ ,\ \ (k_\#)_p : S_p(Y, B) \to S_p(Z, C)\,.$$

We choose a singular p-simplex T such that $T(\Delta_p) \setminus A \neq \varnothing$. Then the cosets of the form $T + S_p(A)$ form a basis of $S_p(X, A)$.

$$\begin{tikzcd} \Delta_p \arrow[r, "T"] & X \arrow[r, "h"] \arrow[rr, bend right, "k\circ h"'] & Y \arrow[r, "k"] & Z \end{tikzcd}$$

Using 2.3, we get

$$(h_\#)_p(T + S_p(A)) = h \circ T + S_p(B)\,, \tag{2.6}$$

$$(k_\#)_p\left((h_\#)_p(T + S_p(A))\right) = (k_\#)_p(h \circ T + S_p(B)) = k \circ h \circ T + S_p(C)\,, \tag{2.7}$$

$$\left((k \circ h)_\#\right)_p(T + S_p(A)) = k \circ h \circ T + S_p(C)\,. \tag{2.8}$$

Therefore, we can conclude that $\left((k \circ h)_\#\right)_p = (k_\#)_p \circ (h_\#)_p$. ■

Theorem 2.2

There is a homomorphism $(\partial_*)_p : H_p(X, A) \to H_{p-1}(A)$, defined for $A \subset X$ and all p, such that the sequence

$$\cdots \longrightarrow H_p(A) \xrightarrow{(i_*)_p} H_p(X) \xrightarrow{(\pi_*)_p} H_p(X, A) \xrightarrow{(\partial_*)_p} H_{p-1}(A) \longrightarrow \cdots$$

is exact, where i and π are the inclusions

$$(A, \varnothing) \xhookrightarrow{i} (X, \varnothing) \xhookrightarrow{\pi} (X, A).$$

The same holds if reduced homology is used for X and A, provided $A \neq \varnothing$.

A continuous map $f : (X, A) \to (Y, B)$ induces a homomorphism of the corresponding exact sequences in singular homology, either ordinary or reduced.

Proof. Let us recall the Zig-Zag lemma (Lemma 4.4.1 in the lecture note of AT2). Given a short exact sequence of chain complexes $\mathcal{C} = \left\{C_p, \partial_p^C\right\}$, $\mathcal{D} = \left\{D_p, \partial_p^D\right\}$ and $\mathcal{E} = \left\{E_p, \partial_p^E\right\}$, i.e.

$$0 \longrightarrow \mathcal{C} \xrightarrow{\phi} \mathcal{D} \xrightarrow{\psi} \mathcal{E} \longrightarrow 0$$

with ϕ and ψ being chain maps, i.e. family of homomorphisms $\{\phi_p\}$ and $\{\psi_p\}$ such that

$$0 \longrightarrow C_p \xrightarrow{\phi_p} D_p \xrightarrow{\psi_p} E_p \longrightarrow 0$$

is exact for each p, then there is a long exact homology sequence

$$\begin{tikzcd}
& \cdots & \cdots \arrow[dll] \\
H_p(\mathcal{C}) \arrow[r, "(\phi_p)_*"'] & H_p(\mathcal{D}) \arrow[r, "(\psi_p)_*"] & H_p(\mathcal{E}) \arrow[dll, "(\partial_*)_p"] \\
H_{p-1}(\mathcal{C}) \arrow[r, "(\phi_{p-1})_*"'] & H_{p-1}(\mathcal{D}) \arrow[r] & \cdots
\end{tikzcd}$$

We shall use Zig-Zag lemma with $C_p = S_p(A)$, $D_p = S_p(X)$ and $E_p = S_p(X, A)$, with chain maps given as follows:

$$0 \longrightarrow S_p(A) \xrightarrow{(i_\#)_p} S_p(X) \xrightarrow{(\pi_\#)_p} S_p(X, A) \longrightarrow 0.$$

Then the above sequence is exact, since $S_p(X, A) = S_p(X)/S_p(A)$. Now, Zig-Zag lemma guarantees the existence of the homomorphism $(\partial_*)_p : H_p(X, A) \to H_{p-1}(A)$ and the following long-exact sequence

$$\cdots \longrightarrow H_p(A) \xrightarrow{(i_*)_p} H_p(X) \xrightarrow{(\pi_*)_p} H_p(X, A) \xrightarrow{(\partial_*)_p} H_{p-1}(A) \qquad \cdots .$$

Now, given a continuous map $f : (X, A) \to (Y, B)$, we shall verify that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & S_p(A) \arrow[r, "(i_\#)_p"] \arrow[d, "\left(\left(f\big|_A\right)_\#\right)_p"'] & S_p(X) \arrow[r, "(\pi_\#)_p"] \arrow[d, "\left(\left(f\big|_X\right)_\#\right)_p"'] & S_p(X, A) \arrow[r] \arrow[d, "(f_\#)_p"] & 0 \\
0 \arrow[r] & S_p(B) \arrow[r, "(i'_\#)_p"'] & S_p(Y) \arrow[r, "(\pi'_\#)_p"'] & S_p(Y, B) \arrow[r] & 0
\end{tikzcd}$$

Here, by $f\big|_X$, we mean the map $f : X \to Y$. First, let's show the commutativity of the left hand square. Let's take a singular p-simplex T of A, i.e. $T : \Delta_p \to A$ is continuous. Then

$$(i_\#)_p T = i \circ T = T\,,\ (f_\#)_p \Big((i_\#)_p T\Big) = f \circ T\,. \tag{2.9}$$

$$\Big(\left(f\big|_A\right)_\#\Big)_p T = f\big|_A \circ T = f \circ T\,,\ (i'_\#)_p \left(\Big(\left(f\big|_A\right)_\#\Big)_p T\right) = i' \circ f \circ T = f \circ T. \tag{2.10}$$

$f\big|_A \circ T = f \circ T$ because the image of T lies entirely in A. Therefore, the left hand square commutes. Now we shall show that the right hand square commutes as well. Let's take a singular p-simplex T of X, i.e. $T : \Delta_p \to X$ is continuous.

$$(\pi_\#)_p T = T + S_p(A)\ ,\ (f_\#)_p \Big((\pi_\#)_p T\Big) = (f_\#)_p T + S_p(B) = (\pi'_\#)_p \Big((f_\#)_p T\Big)\,. \tag{2.11}$$

Therefore, the right hand square commutes. So the diagram is commutative. Now, applying Theorem 5.1.1 from the lecture note of AT2, we obtain that the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r] & H_p(A) \arrow[r, "(i_*)_p"] \arrow[d, "\left(\left(f\big|_A\right)_*\right)_p"'] & H_p(X) \arrow[r, "(\pi_*)_p"] \arrow[d, "(f_*)_p"'] & H_p(X, A) \arrow[r, "(\partial_*)_p"] \arrow[d, "(f_*)_p"] & H_{p-1}(A) \arrow[r] \arrow[d, "\left(\left(f\big|_A\right)_*\right)_{p-1}"] & \cdots \\
\cdots \arrow[r] & H_p(B) \arrow[r, "(i'_*)_p"'] & H_p(Y) \arrow[r, "(\pi'_*)_p"'] & H_p(Y, B) \arrow[r, "(\partial'_*)_p"'] & H_{p-1}(B) \arrow[r] & \cdots
\end{tikzcd}$$

This establishes the induced homomorphisms between the respective long exact sequences of the singular homology. Following the same procedure, one can show that the same result holds in reduced homology. ■

Theorem 2.3

If P is a one-point space, then $H_p(P) = 0$ for $p \neq 0$, and $H_0(P) \cong \mathbb{Z}$.

Proof. We provide a direct proof here. We first compute the chain complex $\mathcal{S}(P)$. Observe that there is exactly one singular p-simplex in each non-negative dimention $p \geq 0$: $T_p : \Delta_p \to P$, because P is a singleton. Therefore, the group of p-chains $S_p(P) \cong \mathbb{Z}$, which is infinite cyclic. Each of the "faces" of $T_p : \Delta_p \to P$ is given

$$T_p \circ l_{(\varepsilon_0,\dots,\widehat{\varepsilon}_i,\dots,\varepsilon_p)} : \Delta_p \to P$$

and is precisely T_{p-1}. All $(p+1)$ faces of T_p are just T_{p-1}. Therefore, if p is even, then the singular p-simplex $(p+1)$ faces, which is an odd number. Hence, in the formula

$$\partial_p T_p = \sum_{i=0}^{p} (-1)^i \, T_p \circ l_{(\varepsilon_0,\dots,\widehat{\varepsilon}_i,\dots,\varepsilon_p)}, \tag{2.12}$$

only one term will survive, the others will cancel in pairs. Hence, we find that $\partial_p T_p = T_{p-1}$, when p is even.

On the other hand, when p is odd, T_p will have an even number of faces, and all the terms in 2.12 will cancel in pairs. Therefore, $\partial_p T_p = 0$, when p is odd. The chain complex $\mathcal{S}(P)$ is, thus, of the following form:

$$\cdots \longrightarrow S_{2k}(P) \longrightarrow S_{2k-1}(P) \longrightarrow \cdots \longrightarrow S_1(P) \longrightarrow S_0(P) \longrightarrow 0$$

$$\cdots \longrightarrow \mathbb{Z} \xrightarrow{\ \cong\ } \mathbb{Z} \xrightarrow{\ \bar{0}\ } \cdots \longrightarrow \mathbb{Z} \xrightarrow{\ \bar{0}\ } \mathbb{Z} \longrightarrow 0$$

Here, $\bar{0}$ maps everything to 0. In dimension $(2k-1)$, every $(2k-1)$-chain is a cycle, and every $(2k-1)$-chain can be seen to be a boundary of a $2k$-chain. Hence, there is no nontrivial $(2k-1)$-cycle that is not a $(2k-1)$-boundary. Therefore, $H_{2k-1}(P) = 0$.

In dimension $2k$, for $k > 0$, there is no nontrivial chain that is a cycle. Hence, $H_{2k} = 0$. In dimension 0, every chain is a cycle, and no nontrivial 0-chain is a bounday. Therefore, $H_0(P) \cong \mathbb{Z}$. ■

§2.2 Compact Support Axiom

In this section, we shall verify that singular homology theory satisfies the compact support axiom.

Definition 2.1 (Minimal carrier). If $T : \Delta_p \to X$ is a singular p-simplex of X, then the **minimal carrier** of T is defined to be the image set $T(\Delta_p)$. If $c = \sum n_i T_i$ is a singular p-chain, with T_i being singular p-simplices and each n_i nonzero, then the minimal carrier of c is defined to be the union of the minimal carriers of the singular p-simplices T_i.

A singular p-simplex T is a continuous map from Δ_p to X. Since Δ_p is compact, so is $T(\Delta_p)$ since continuous map takes compact sets to compact sets. Now, a finite union of compact sets is also compact. Therefore, the minimal carrier of a singular p-chain is compact.

Theorem 2.4

Given $\alpha \in H_p(X, A)$, there is a compact pair $(X_0, A_0) \subseteq (X, A)$, with $\iota : (X_0, A_0) \hookrightarrow (X, A)$ such that $(\iota_*)_p(\beta) = \alpha$ for some $\beta \in H_p(X_0, A_0)$, where $(\iota_*)_p : H_p(X_0, A_0) \to H_p(X, A)$ is the homomorphism induced by the inclusion ι.

Proof. Given $\alpha \in H_p(X, A) = Z_p(X, A) / B_p(X, A)$, α is of the form $C + B_p(X, A)$, with $C \in Z_p(X, A) \subset S_p(X, A) = S_p(X) / S_p(A)$. Therefore,

$$\alpha = (c_p + S_p(A)) + B_p(X, A), \tag{2.13}$$

where $c_p \in S_p(X)$ such that $\partial_p c_p$ is carried by A. The minimal carrier of $\partial_p c_p$ is a compact set contained in A. Let us denote this compact set by A_0. On the other hand, c_p is minimally carried by a compact set X_0 contained in X. Now, we define

$$D = c_p + S_p(A_0) \in S_p(X_0, A_0). \tag{2.14}$$

Since $\partial_p c_p$ is carried by A_0, $D \in Z_p(X_0, A_0)$. Now, we claim that

$$\beta = D + B_p(X_0, A_0) = (c_p + S_p(A_0)) + B_p(X_0, A_0) \in H_p(X_0, A_0) \tag{2.15}$$

is the required element of $H_p(X_0, A_0)$ whose image under $(\iota_*)_p$ is α. Now,

$$(\iota_*)_p(\beta) = (\iota_*)_p((c_p + S_p(A_0)) + B_p(X_0, A_0)) = \Big((\iota_\#)_p c_p + S_p(A)\Big) + B_p(X, A). \tag{2.16}$$

If $c_p = \sum n_i T_i$, with T_i being singular p-simplices, then

$$(\iota_\#)_p c_p = \sum n_i (\iota_\#)_p (T_i) = \sum n_i (\iota \circ T_i) = \sum n_i T_i = c_p. \tag{2.17}$$

Therefore,

$$(\iota_*)_p(\beta) = (c_p + S_p(A)) + B_p(X, A) = \alpha. \tag{2.18}$$

■

Theorem 2.5

Let $i : (X_0, A_0) \hookrightarrow (X, A)$ be inclusion, where (X_0, A_0) is a compact pair. If $\alpha \in H_p(X_0, A_0)$ with $(i_*)_p(\alpha) = 0$, then there are a compact pais (X_1, A_1) and inclusions

$$(X_0, A_0) \xhookrightarrow{j} (X_1, A_1) \xhookrightarrow{k} (X, A)$$

such that $(j_*)_p(\alpha) = 0$.

Proof. Let $\alpha = (c_p + S_p(A_0)) + B_p(X_0, A_0) \in H_p(X_0, A_0)$, where $c_p \in S_p(X_0)$ and $\partial_p c_p$ is carried by A_0. Now, $(i_*)_p : H_p(X_0, A_0) \to H_p(X, A)$, so $(i_*)_p(\alpha) = 0 + B_p(X, A)$.

$$0 + B_p(X, A) = (i_*)_p(\alpha) = \Big((i_\#)_p c_p + S_p(A)\Big) + B_p(X, A). \tag{2.19}$$

Using a similar method as in 2.17, one can show that $(i_\#)_p c_p = c_p$. So 2.19 reads

$$0 + B_p(X, A) = (c_p + S_p(A)) + B_p(X, A). \tag{2.20}$$

Therefore, $c_p + S_p(A) \in B_p(X, A)$. In other words, there exists a $(p+1)$-chain d_{p+1} such that $c_p - \partial_{p+1} d_{p+1}$ is carried by A. Now, d_{p+1} is carried by

$$X_1 = X_0 \cup (\text{minimal carrier of } d_{p+1}),$$

and $c_p - \partial_{p+1} d_{p+1}$ is carried by

$$A_1 = A_0 \cup (\text{minimal carrier of } c_p - \partial_{p+1} d_{p+1}).$$

Consider the inclusion maps

$$\begin{tikzcd} (X_0, A_0) \arrow[r, hook, "j"] \arrow[rr, bend right, hook, "i = k \circ j"'] & (X_1, A_1) \arrow[r, hook, "k"] & (X, A). \end{tikzcd}$$

Then $(j_*)_p(\alpha)$ is

$$(j_*)_p(\alpha) = (j_*)_p((c_p + S_p(A_0)) + B_p(X_0, A_0)) = \Big((j_\#)_p c_p + S_p(A_1)\Big) + B_p(X_1, A_1). \tag{2.21}$$

Again, similarly as in 2.17, one can show that $(j_\#)_p c_p = c_p$.

$$(j_*)_p(\alpha) = (c_p + S_p(A_1)) + B_p(X_1, A_1). \tag{2.22}$$

$c_p - \partial_{p+1} d_{p+1}$ is carried by A_1, so $c_p - \partial_{p+1} d_{p+1} \in S_p(A_1)$. Therefore,

$$\begin{aligned} c_p + S_p(A_1) &= c_p - (c_p - \partial_{p+1} d_{p+1}) + S_p(A_1) = \partial_{p+1} d_{p+1} + S_p(A_1) \\ &= \partial_{p+1}(d_{p+1} + S_{p+1}(A_1)) \in B_p(X_1, A_1). \end{aligned} \tag{2.23}$$

Combining 2.22 and 2.23, we get

$$(j_*)_p(\alpha) = \partial_{p+1}(d_{p+1} + S_{p+1}(A_1)) + B_p(X_1, A_1) = 0 + B_p(X_1, A_1). \tag{2.24}$$

■

§2.3 Chain Homotopy

Definition 2.2. Given chain complexes $\mathcal{C} = \{C_p, \partial_p\}$ and $\mathcal{C}' = \{C'_p, \partial'_p\}$ and chain maps $\phi, \psi : \mathcal{C} \to \mathcal{C}'$, a **chain homotopy** of ϕ to ψ is a family of homomorphisms $D_p : C_p \to C'_{p+1}$ such that the following holds

$$\partial'_{p+1} D_p + D_{p-1} \partial_p = \psi_p - \phi_p. \tag{2.25}$$

The following diagram might be useful for to understand the above formula in 2.25. Note that this is **NOT** a commutative diagram.

$$\begin{tikzcd} & C'_{p+1} \arrow[d, "\partial'_{p+1}"] \\ C_p \arrow[ur, "D_p"] \arrow[r, shift left, "\phi_p"] \arrow[r, shift right, "\psi_p"'] \arrow[d, "\partial_p"'] & C'_p \\ C_{p-1} \arrow[ur, "D_{p-1}"'] & \end{tikzcd}$$

Now, consider the inclusions $i, j : X \to X \times I$ (I is the unit interval $[0,1]$) given by

$$i(x) = (x, 0) \text{ and } j(x) = (x, 1). \tag{2.26}$$

The corresponding chain maps are denoted by $(i_\#)_p, (j_\#)_p : S_p(X) \to S_p(X \times I)$. Construct a chain homotopy D^X between the chain map $i_\#$ and $j_\#$ as follows:

$$\begin{aligned} D^X &: \mathcal{S}(X) \to \mathcal{S}(X \times I), \\ D^X_p &: S_p(X) \to S_p(X \times I). \end{aligned} \tag{2.27}$$

For D^X to be a chain homotopy, the following equation must hold:

$$\partial^{X \times I}_{p+1} \circ D^X_p + D^X_{p-1} \circ \partial^X_p = (j_\#)_p - (i_\#)_p. \tag{2.28}$$

$$\begin{tikzcd} & S_{p+1}(X \times I) \arrow[d, "\partial^{X\times I}_{p+1}"] \\ S_p(X) \arrow[ur, "D^X_p"] \arrow[r, shift left, "(i_\#)_p"] \arrow[r, shift right, "(j_\#)_p"'] \arrow[d, "\partial^X_p"'] & S_p(X \times I) \\ S_{p-1}(X) \arrow[ur, "D^X_{p-1}"'] & \end{tikzcd}$$

One can now construct the following diagram to find that $F_{\#} \circ D^X$ is a chain homotopy between the chain maps $f_{\#}, g_{\#} : \mathcal{S}(X) \to \mathcal{S}(Y)$, where X and Y are topological spaces and F is a homotopy between the maps $f, g : X \to Y$, i.e. $F : X \times I \to Y$ is a continuous map such that

$$F(x,0) = f(x) \text{ and } F(x,1) = g(x).$$

Using 2.26, we then have

$$F \circ i = f \text{ and } F \circ j = g. \tag{2.29}$$

$F_{\#} : \mathcal{S}(X \times I) \to \mathcal{S}(Y)$. In order to show that $F_{\#} \circ D^X$ is a chain homotopy between $f_{\#}$ and $g_{\#}$, one needs to prove that

$$\partial_{p+1}^Y \circ (F_{\#})_{p+1} \circ D_p^X + (F_{\#})_p \circ D_{p-1}^X \circ \partial_p^X = (g_{\#})_p - (f_{\#})_p. \tag{2.30}$$

$$\begin{tikzcd}
 & & S_{p+1}(Y) \arrow[dd, "\partial_{p+1}^Y"] \\
 & S_{p+1}(X \times I) \arrow[ur, "(F_{\#})_{p+1}"] & \\
S_p(X) \arrow[ur, "D_p^X"] \arrow[rr, shift left, "(f_{\#})_p"] \arrow[rr, shift right, "(g_{\#})_p"'] \arrow[dd, "\partial_{p+1}^X"'] & & S_p(Y) \\
 & S_p(X \times I) \arrow[ur, "(F_{\#})_p"'] & \\
S_{p-1}(X) \arrow[ur, "D_{p-1}^X"'] & &
\end{tikzcd}$$

Let us quickly see how 2.30 comes from 2.28. Since chain maps commute with the boundary operator, we have the following commutative diagram:

$$\begin{tikzcd}
S_{p+1}(X \times I) \arrow[r, "(F_{\#})_{p+1}"] \arrow[d, "\partial_{p+1}^{X \times I}"'] & S_{p+1}(Y) \arrow[d, "\partial_{p+1}^Y"] \\
S_p(X \times I) \arrow[r, "(F_{\#})_p"'] & S_p(Y)
\end{tikzcd}$$

i.e. $\partial_{p+1}^Y \circ (F_{\#})_{p+1} = (F_{\#})_p \circ \partial_{p+1}^{X \times I}$. Therefore, one obtains

$$\begin{aligned}
\partial_{p+1}^Y \circ (F_{\#})_{p+1} \circ D_p^X &= (F_{\#})_p \circ \partial_{p+1}^{X \times I} \circ D_p^X \\
&= (F_{\#})_p \circ \left[(j_{\#})_p - (i_{\#})_p - D_{p-1}^X \circ \partial_p^X \right] \\
&= \Big((F \circ j)_{\#} \Big)_p - \Big((F \circ i)_{\#} \Big)_p - (F_{\#})_p \circ D_{p-1}^X \circ \partial_p^X \\
&= (g_{\#})_p - (f_{\#})_p \, (F_{\#})_p \circ D_{p-1}^X \circ \partial_p^X,
\end{aligned} \tag{2.31}$$

which can be rearranged to obtain 2.30. The existence of the chain map $D^X : \mathcal{S}(X) \to \mathcal{S}(X \times I)$ is governed by the following lemma.

Lemma 2.6

There exists, for each space X, and each non-negative integer p, a homomorphism $D_p^X : S_p(X) \to S_{p+1}(X \times I)$ having the following properties:

(a) If $T : \Delta_p \to X$ is a singular p-simplex then

$$\partial_{p+1}^{X \times I} D_p^X T + D_{p-1}^X \partial_p^X T = (j_{\#})_p T - (i_{\#})_p T. \tag{2.32}$$

Here, the map $i : X \to X \times I$ carries x to $(x, 0)$ and the map $j : X \to X \times I$ carries x to

$(x, 1)$.

(b) D_p^X is natural; i.e. given $f : X \to Y$ continuous, the following diagram commutes:

$$\begin{tikzcd} S_p(X) \arrow[r, "D_p^X"] \arrow[d, "(f_\#)_p"'] & S_{p+1}(X \times I) \arrow[d, "((f\times \mathrm{id}_I)_\#)_{p+1}"] \\ S_p(Y) \arrow[r, "D_p^Y"] & S_{p+1}(Y \times I) \end{tikzcd}$$

Note that continuous $f : X \to Y$ induces a continnuous map $f \times \mathrm{id}_I : X \times I \to Y \times I$ given by $(x, t) \mapsto (f(x), t)$. Hence there is a group homomorphism

$$\left((f \times \mathrm{id}_I)_\#\right)_p : S_p(X \times I) \to S_p(Y \times I)$$

for each non-negative integer p.

Proof of the lemma is omitted.

Theorem 2.7

If $f, g : (X, A) \to (Y, B)$ are homotopic, then $(f_*)_p = (g_*)_p$ for all p, with $(f_*)_p, (g_*)_p : H_p(X, A) \to H_p(Y, B)$ group homomorphisms. The same holds in the reduced homology if $A = B = \varnothing$.

Proof. Let $F : (X \times I, A \times I) \to (Y \times I, B \times I)$ be the homotopy between $f, g : (X, A) \to (Y, B)$. Let $i, j : (X, A) \to (X \times I, A \times I)$ be given by $i(x) = (x, 0)$ and $j(x) = (x, 1)$, for $x \in X$. Let $D_p^X : S_p(X) \to S_p(X \times I)$ be the group homomorphism associated with the chain homotopy $D^X : \mathcal{S}(X) \to \mathcal{S}(X \times I)$ constructed in Lemma 2.6. Naturality of D^X with respect to the inclusion map $\iota : A \hookrightarrow X$ dictates that the following diagram commutes:

$$\begin{tikzcd} S_p(A) \arrow[r, "D_p^A"] \arrow[d, "(\iota_\#)_p"'] & S_{p+1}(A \times I) \arrow[d, "((\iota\times \mathrm{id}_I)_\#)_{p+1}"] \\ S_p(X) \arrow[r, "D_p^X"] & S_{p+1}(X \times I) \end{tikzcd}$$

Consider $T \in S_{p+1}(A \times I)$ such that T is a $(p+1)$-singular simplex of $A \times I$, i.e. $T : \Delta_{p+1} \to A \times I$ is continuous. For a given $x \in \Delta_{p+1}$, let $T(x) = (a, t) \in A \times I$. Now,

$$\left((\iota \times \mathrm{id}_I)_\#\right)_{p+1} T(x) = (\iota \times \mathrm{id}_I) \circ T(x) = (\iota \times \mathrm{id}_I)(a, t) = (a, t) = T(x). \tag{2.33}$$

Hence, $\left((\iota \times \mathrm{id}_I)_\#\right)_{p+1} T = T$. So, we have

$$\left((\iota \times \mathrm{id}_I)_\#\right)_{p+1} \circ D_p^A = D_p^A. \tag{2.34}$$

Now, commutativity of the above diagram yields

$$\left((\iota \times \mathrm{id}_I)_\#\right)_{p+1} \circ D_p^A = D_p^X \circ (\iota_\#)_p = D_p^X\big|_{S_p(A)}. \tag{2.35}$$

Therefore, combining 2.34 and 2.35, we get

$$D_p^X\big|_{S_p(A)} = D_p^A. \tag{2.36}$$

In other words, $D_p^X : S_p(X) \to S_{p+1}(X \times I)$ carries $S_p(A)$ into $S_p(X \times I)$, and thus induces a chain homotopy on the relative level. The constituent group homomorphisms are given by

$$D_p^{(X,A)} : S_p(X, A) \to S_{p+1}(X \times I, A \times I). \tag{2.37}$$

Now, 2.32 indeed holds for $D_p^{(X,A)}$ as it is induced by D_p^X. Then we have

$$\left(F_\#\right)_{p+1} \circ D_p^{(X,A)} : S_p\left(X,A\right) \to S_{p+1}\left(Y,B\right),$$

where the homomorphism $\left(F_\#\right)_{p+1}$ associated with the chain map $F_\# : \mathcal{S}\left(X \times I, A \times I\right) \to \mathcal{S}\left(Y,B\right)$ is

$$\left(F_\#\right)_{p+1} : S_{p+1}\left(X \times I, A \times I\right) \to S_{p+1}\left(Y,B\right).$$

Then

$$\begin{aligned}
\partial_{p+1}^Y \circ \left(F_\#\right)_{p+1} \circ D_p^{(X,A)} &= \left(F_\#\right)_p \circ \partial_{p+1}^{X\times I} \circ D_p^{(X,A)} \\
&= \left(F_\#\right)_p \circ \left[\left(j_\#\right)_p - \left(i_\#\right)_p - D_{p-1}^{(X,A)} \circ \partial_p^X\right] \\
&= \left(\left(F \circ j\right)_\#\right)_p - \left(\left(F \circ i\right)_\#\right)_p - \left(F_\#\right)_p \circ D_{p-1}^{(X,A)} \circ \partial_p^X \\
&= \left(g_\#\right)_p - \left(f_\#\right)_p - \left(F_\#\right)_p \circ D_{p-1}^{(X,A)} \circ \partial_p^X. \qquad (2.38)
\end{aligned}$$

This proves that $F_\# \circ D^{(X,A)} : \mathcal{S}\left(X,A\right) \to \mathcal{S}\left(Y,B\right)$ is a chain homotopy between $f_\#, g_\# : \mathcal{S}\left(X,A\right) \to \mathcal{S}\left(Y,B\right)$. It now remains to prove that $\left(f_*\right)_p = \left(g_*\right)_p$ for all p.

Let $\alpha \in Z_p\left(X,A\right)$. It suffices to show that $\left(f_\#\right)_p\left(\alpha\right)$ and $\left(g_\#\right)_p\left(\alpha\right)$ differ by a boundary term. Given $\alpha \in Z_p\left(X,A\right)$, $\alpha = c_p + S_p\left(A\right)$ for some $c_p \in S_p\left(X\right)$ such that $\partial_p c_p$ is carried by A. By 2.38,

$$\begin{aligned}
\left(g_\#\right)_p\left(\alpha\right) - \left(f_\#\right)_p\left(\alpha\right) &= \partial_{p+1}^Y \circ \left(F_\#\right)_{p+1} \circ D_p^{(X,A)}\left(\alpha\right) + \left(F_\#\right)_p \circ D_{p-1}^{(X,A)} \circ \partial_p^X\left(\alpha\right) \\
&= \partial_{p+1}^Y \circ \left(F_\#\right)_{p+1} \circ D_p^{(X,A)}\left(\alpha\right), \qquad (2.39)
\end{aligned}$$

proving that $\left(f_\#\right)_p\left(\alpha\right)$ and $\left(g_\#\right)_p\left(\alpha\right)$ differ by a boundary term. Therefore, $\left(f_*\right)_p\left(\alpha + B_p\left(X,A\right)\right) = \left(f_*\right)_p\left(\alpha + B_p\left(X,A\right)\right)$.

The result in reduced homology is left as an exercise. ■

§2.4 Homotopy Equivalence

Definition 2.3 (Retraction). Let $A \subset X$. A **retraction** of X onto A is a continuous map $r : X \to A$ such that $r\left(a\right) = a$ for every $a \in A$, i.e. $r\big|_A = \mathrm{id}_A$. If there is a retraction of X onto A, we say that A is a retract of X,

Definition 2.4 (Deformation retraction). A **deformation retraction** of X onto A is a continuous map $F : X \times I \to X$ such that

$$F\left(x,0\right) = x\,, \;\; F\left(x,1\right) \in A\,, \;\; \text{and } F\left(a,t\right) = a \qquad (2.40)$$

for all $x \in X, a \in A, t \in I$.

If F is a deformation retraction of X onto A, then one can define

$$r\left(x\right) = F\left(x,1\right). \qquad (2.41)$$

Then 2.40 tells us that r is a map from X to A, and $r\left(a\right) = a$ for all $a \in A$. Hence, r is indeed a retraction of X onto A. Now, 2.40 also tells us that

$$F\left(x,0\right) = x = \mathrm{id}_X\left(x\right) \;\text{ and } F\left(x,1\right) = j \circ r\left(x\right), \qquad (2.42)$$

where $j : A \hookrightarrow X$ is the inclusion. Therefore, F is a homotopy between the identity map $\mathrm{id}_X : X \to X$ and $j \circ r : X \to X$.

Definition 2.5. Let $f:(X,A)\to(Y,B)$ be continuous. If there is a continuous map $g:(Y,B)\to(X,A)$ such that $g\circ f$ is homotopic to the identity map $\mathrm{id}_{(X,A)}:(X,A)\to(X,A)$ and $f\circ g$ is homotopic to the identity map $\mathrm{id}_{(Y,B)}:(Y,B)\to(Y,B)$, then we call f a **homotopy equivalence**, and we call g a **homotopy inverse** for f.

Theorem 2.8

Let $f:(X,A)\to(Y,B)$ be continuous.

(a) If f is a homotopy equivalence, then f_* is an isomorphism in relative homology.

(b) More generally, if $f:X\to Y$ and $f\big|_A:A\to B$ are homotopy equivalences, then f_* is an isomorphism in relative homology.

Proof. Let $f:(X,A)\to(Y,B)$ be a homotopy equivalence, and $g:(Y,B)\to(X,A)$ its homotopy inverse. Then $f\circ g\simeq\mathrm{id}_{(Y,B)}$ and $g\circ f\simeq\mathrm{id}_{(X,A)}$. Then by Theorem 2.7,

$$((f\circ g)_*)_p=\left(\left(\mathrm{id}_{(Y,B)}\right)_*\right)_p \text{ and } ((g\circ f)_*)_p=\left(\left(\mathrm{id}_{(X,A)}\right)_*\right)_p.$$

In other words,

$$(f_*)_p\circ(g_*)_p=\mathrm{id}_{H_p(Y,B)} \text{ and } (g_*)_p\circ(f_*)_p=\mathrm{id}_{H_p(X,A)}. \tag{2.43}$$

Therefore, $(f_*)_p:H_p(X,A)\to H_p(Y,B)$ is an isomorphism.

Now we shall prove (b). Consider the long exact sequence of the pairs (X,A) and (Y,B), separately with $(f_*)_p$ being the respective connecting homomorphisms.

$$\begin{tikzcd}
\cdots \arrow[r] & H_p(A) \arrow[r, "(i_*)_p"] \arrow[d, "\left(\left(f|_A\right)_*\right)_p"'] & H_p(X) \arrow[r, "(\pi_*)_p"] \arrow[d, "(f_*)_p"'] & H_p(X,A) \arrow[r, "(\partial_*)_p"] \arrow[d, "(f_*)_p"] & H_{p-1}(A) \arrow[r, "(i_*)_{p-1}"] \arrow[d, "\left(\left(f|_A\right)_*\right)_{p-1}"] & H_{p-1}(X) \arrow[r] \arrow[d, "(f_*)_{p-1}"] & \cdots \\
\cdots \arrow[r] & H_p(B) \arrow[r, "(i'_*)_p"'] & H_p(Y) \arrow[r, "(\pi'_*)_p"'] & H_p(Y,B) \arrow[r, "(\partial'_*)_p"'] & H_{p-1}(B) \arrow[r, "(i'_*)_{p-1}"'] & H_{p-1}(X) \arrow[r] & \cdots
\end{tikzcd}$$

By hypothesis, $f:(X,\varnothing)\to(Y,\varnothing)$ is a homotopy equivalence, and hence $(f_*)_p:H_p(X)\to H_p(Y)$ is an isomorphism. Similarly, by hypothesis, $f\big|_A:(A,\varnothing)\to(B,\varnothing)$ is a homotopy equivalence, and hence $\left(\left(f\big|_A\right)_*\right)_p:H_p(A)\to H_p(B)$ is an isomorphism. Now, applying Steenrod five lemma to the diagram above, one obtains that

$$(f_*)_p:H_p(X,A)\to H_p(Y,B)$$

is an isomorphism. ∎

Remark 2.1. If $f:(X,A)\to(Y,B)$ is a homotopy equivalence, then $f:X\to Y$ and $f\big|_A:A\to B$ are automaatically homotopy equivalences. However, the converse is not true. One counterexample is presented below.

Example 2.1

Consider the inclusion map $j:\left(B^n,S^{n-1}\right)\hookrightarrow(\mathbb{R}^n,\mathbb{R}^n\setminus\{\mathbf{0}\})$. $j:B^n\hookrightarrow\mathbb{R}^n$ has a homotopy inverse, so that B^n and $\mathbb{R}^n$ are homotopy equivalent. The homotopy inverse is given by $f:\mathbb{R}^n\to B^n$,

$$f(\mathbf{x})=\begin{cases}\mathbf{x} & \text{if } \|\mathbf{x}\|\le 1\\ \frac{\mathbf{x}}{\|\mathbf{x}\|} & \text{if } \|\mathbf{x}\|>1\end{cases}. \tag{2.44}$$

Then $f(j(\mathbf{x})) = \mathbf{x}$, so $f \circ j = \mathrm{id}_{B^n}$. $j(f(\mathbf{x})) = f(\mathbf{x}) \in B^n$. So $F : \mathbb{R}^n \times I \to \mathbb{R}^n$ given by

$$F(\mathbf{x}, t) = (1-t)\mathbf{x} + tj \circ f(\mathbf{x}) \tag{2.45}$$

is a homotopy between $\mathrm{id}_{\mathbb{R}^n}$ and $j \circ f$. Therefore, f is the homotopy inverse of j.

In a similar manner, one can show that $j\big|_{S^{n-1}} : S^{n-1} \hookrightarrow \mathbb{R}^n \setminus \{\mathbf{0}\}$ also has a homotopy inverse. The homotopy inverse is $h : \mathbb{R}^n \setminus \{\mathbf{0}\} \to S^{n-1}$ given by

$$h(\mathbf{x}) = \frac{\mathbf{x}}{\|\mathbf{x}\|}. \tag{2.46}$$

Then $h \circ j\big|_{S^{n-1}} = \mathrm{id}_{S^{n-1}}$. Furthermore, $G : (\mathbb{R}^n \setminus \{\mathbf{0}\}) \times I \to \mathbb{R}^n \setminus \{\mathbf{0}\}$ given by

$$G(\mathbf{x}, t) = (1-t)\mathbf{x} + tj\big|_{S^{n-1}} \circ h(\mathbf{x}) = \left((1-t) + \frac{t}{\|\mathbf{x}\|}\right)\mathbf{x} \tag{2.47}$$

is a homotopy between $\mathrm{id}_{\mathbb{R}^n \setminus \{\mathbf{0}\}}$ and $j\big|_{S^{n-1}} \circ h$. Therefore, h is the homotopy inverse of j.

However, $j : (B^n, S^{n-1}) \hookrightarrow (\mathbb{R}^n, \mathbb{R}^n \setminus \{\mathbf{0}\})$ has no homotopy inverse although both $j : B^n \hookrightarrow \mathbb{R}^n$ and $j\big|_{S^{n-1}} : S^{n-1} \hookrightarrow \mathbb{R}^n \setminus \{\mathbf{0}\}$ have homotopy inverses. To show this, assume the contrary that $g : (\mathbb{R}^n, \mathbb{R}^n \setminus \{\mathbf{0}\}) \to (B^n, S^{n-1})$ is a homotopy inverse of j. Then g is continuous, and it maps $\mathbb{R}^n \setminus \{\mathbf{0}\}$ into S^{n-1}. But $\mathbf{0}$ is a limit point of $\mathbb{R}^n \setminus \{\mathbf{0}\}$, and S^{n-1} is closed. Therefore, $g(\mathbf{0}) \in S^{n-1}$. In other words, g maps all of $\mathbb{R}^n$ into S^{n-1}. Hence, the composite

$$g \circ j : (B^n, S^{n-1}) \to (B^n, S^{n-1}) \tag{2.48}$$

maps all of B^n to S^{n-1}. If $T : \Delta_p \to B^n$ is a singular p-simplex, then for $T + S_p(S^{n-1}) \in S_p(B^n, S^{n-1})$,

$$\left((g \circ j)_\#\right)_p \left(T + S_p(S^{n-1})\right) = g \circ j \circ T + S_p(S^{n-1}). \tag{2.49}$$

But the image of $g \circ j \circ T$ lies entirely on S^{n-1}. So $\left((g \circ j)_\#\right)_p$ is the trivial chain map. Therefore, $((g \circ j)_*)_p : H_p(B^n, S^{n-1}) \to H_p(B^n, S^{n-1})$ is the trivial map. However, since $g \circ j$ is homotopic with $\mathrm{id}_{(B^n, S^{n-1})}$, $((g \circ j)_*)_p$ is the identity homomorphism on $H_p(B^n, S^{n-1})$. This can only be true if $H_p(B^n, S^{n-1}) = 0$. We shall soon see this is not true.

§2.5 Subdivision

Definition 2.6. Given a topological space X and a collection $\mathcal{A}$ of subsets of X whose interiors cover X, a singular simplex of X is said to be $\mathcal{A}$-small if its image set lies in an element of $\mathcal{A}$.

Given a singular chain of X, we show how to “chop it up” so that all its simplices are $\mathcal{A}$-small.

Definition 2.7 (Barycentric subdivision operator)**.** Let X be a topological space, we define a homomorphism $\mathrm{sd}_X : S_p(X) \to S_p(X)$ by induction. If $T : \Delta_0 \to X$ is a singular 0-simplex, we define

$$\mathrm{sd}_X T = T. \tag{2.50}$$

Now suppose sd_X is defined in dimensions less than p. We will first take $X : \Delta_p$ and choose the identity map $i_p : \Delta_p \to \Delta_p$, which is a singular p-simplex of Δ_p, i.e. $i_p \in S_p(\Delta_p)$. Let us denote by $\widehat{\Delta_p}$ the barycenter of Δ_p. Then we define $\mathrm{sd}_{\Delta_p} i_p$ as follows:

$$\mathrm{sd}_{\Delta_p} i_p = (-1)^p \left[\mathrm{sd}_{\Delta_p} \partial i_p, \widehat{\Delta_p}\right]. \tag{2.51}$$

Now, if $T : \Delta_p \to X$ is any singular p-simplex on X, then we define

$$\mathrm{sd}_X T = (T_\#)_p \left(\mathrm{sd}_{\Delta_p} i_p\right). \tag{2.52}$$

Observe that $\mathrm{sd}_{\Delta_p} i_p$ is expected to be in $S_p(\Delta_p)$. Since $\partial i_p \in S_{p-1}$ and sd_{Δ_p} is assumed to be defined in dimension less than p, $\mathrm{sd}_{\Delta_p} \partial i_p \in S_{p-1}(\Delta_p)$. The bracket operation on the RHS of 2.51, therefore, yields $\left[\mathrm{sd}_{\Delta_p} \partial i_p, \widehat{\Delta_p}\right] \in S_p(\Delta_p)$ so that indeed by 2.51, one obtains $\mathrm{sd}_{\Delta_p} i_p \in S_p(\Delta_p)$.

Lemma 2.9

The homomorphism sd_X is an augmentation preserving chain map. Furthermore, it is natural in the sense that for any continuou map $f : X \to Y$, one has $(f_\#)_p \circ \mathrm{sd}_X = \mathrm{sd}_Y \circ (f_\#)_p$. In other words, the following diagram commutes:

$$\begin{tikzcd} S_p(X) \arrow[r, "(f_\#)_p"] \arrow[d, "\mathrm{sd}_X"'] & S_p(Y) \arrow[d, "\mathrm{sd}_Y"] \\ S_p(X) \arrow[r, "(f_\#)_p"'] & S_p(Y). \end{tikzcd}$$

Proof. Recall that in dimension 0, for $T : \Delta_0 \to X$, one has $\mathrm{sd}_X T = T$. In other words, $\mathrm{sd}_X : S_0(X) \to S_0(X)$ is the identity map. Hence, in dimension 0, $\mathrm{sd}_X : S_0(X) \to S_0(X)$ is trivially augmentation preserving as the following diagram commutes:

$$\begin{tikzcd} S_0(X) \arrow[r, "\mathrm{sd}_X"] \arrow[d, "\epsilon"'] & S_0(X) \arrow[d, "\epsilon"] \\ \mathbb{Z} \arrow[r, "\mathrm{id}_{\mathbb{Z}}"] & \mathbb{Z}. \end{tikzcd}$$

Let us immediately find that the naturality of sd_X in dimension 0 holds. It follows trivially from the following commutative diagram.

$$\begin{tikzcd} S_0(X) \arrow[r, "(f_\#)_0"] \arrow[d, "\mathrm{sd}_X=\mathrm{id}"'] & S_0(Y) \arrow[d, "\mathrm{sd}_Y=\mathrm{id}"] \\ S_0(X) \arrow[r, "(f_\#)_0"'] & S_0(Y). \end{tikzcd}$$

Now, let's verify naturality in positive dimensions. Let $T : \Delta_p \to X$ be continuous. Then

$$(f_\#)_p (\mathrm{sd}_X T) = (f_\#)_p \left[(T_\#)_p \left(\mathrm{sd}_{\Delta_p} i_p\right)\right] = \left((f \circ T)_\#\right)_p \left(\mathrm{sd}_{\Delta_p} i_p\right). \tag{2.53}$$

Now, $f \circ T : \Delta_p \to Y$ is a singular p-simplex on Y. So we have

$$\mathrm{sd}_Y (f \circ T) = \left((f \circ T)_\#\right)_p \left(\mathrm{sd}_{\Delta_p} i_p\right). \tag{2.54}$$

Now, 2.53 and 2.54 together imply

$$(f_\#)_p (\mathrm{sd}_X T) = \mathrm{sd}_Y (f \circ T) = \mathrm{sd}_Y \left((f_\#)_p T\right). \tag{2.55}$$

Therefore, $(f_\#)_p \circ \mathrm{sd}_X = \mathrm{sd}_Y \circ (f_\#)_p$.

Finally, we shall prove that sd_X is a chain map by induction. We need to verify that sd commutes with the boundary operator. The fact that sd commutes with the boundary homomorphism in dimension 0 follows trivially from the following commutative diagram.

$$\begin{tikzcd} S_0(X) \arrow[r, "\mathrm{sd}_X = \mathrm{id}_{S_0(X)}"] \arrow[d, "\partial_0"'] & S_0(X) \arrow[d, "\partial_0"] \\ 0 \arrow[r, "\mathrm{id}"] & 0. \end{tikzcd}$$

Now, assume that the result holds true in dimension less than p. Now,

$$\partial_p\left(\mathrm{sd}_{\Delta_p} i_p\right) = (-1)^p\, \partial_p\left[\mathrm{sd}_{\Delta_p} \partial i_p, \widehat{\Delta_p}\right], \tag{2.56}$$

where $i_p : \Delta_p \to \Delta_p$ is the identity map. Δ_p is star convex with respect to $\widehat{\Delta_p}$, and $\mathrm{sd}_{\Delta_p} \partial i_p$ is a $(p-1)$-chain of Δ_p. Then by Lemma 1.5,

$$\begin{aligned}
\partial_p\left[\mathrm{sd}_{\Delta_p} \partial i_p, \widehat{\Delta_p}\right] &= \begin{cases} \left[\partial_{p-1}\left(\mathrm{sd}_{\Delta_p} \partial_p i_p\right), \widehat{\Delta_p}\right] + (-1)^p\, \mathrm{sd}_{\Delta_p} \partial_p i_p & \text{if } p-1>0 \\ \epsilon\left(\mathrm{sd}_{\Delta_p} \partial_p i_p\right) T_0 - \mathrm{sd}_{\Delta_p} \partial i_p & \text{if } p-1=0 \end{cases} \\
&= \begin{cases} \left[\partial_{p-1}\left(\mathrm{sd}_{\Delta_p} \partial_p i_p\right), \widehat{\Delta_p}\right] + (-1)^p\, \mathrm{sd}_{\Delta_p} \partial_p i_p & \text{if } p>1 \\ \epsilon\left(\mathrm{sd}_{\Delta_1} \partial_1 i_1\right) T_0 - \mathrm{sd}_{\Delta_1} \partial_1 i_1 & \text{if } p=1 \end{cases},
\end{aligned} \tag{2.57}$$

where T_0 is the singular 0-simplex whose image point is $\widehat{\Delta_1}$, the barycenter of Δ_1. If $p=1$, since sd is augmentation preserving, the following diagram commutes:

$$\begin{tikzcd} S_0(\Delta_1) \arrow[r, "\mathrm{sd}_{\Delta_1} = \mathrm{id}_{S_0(\Delta_1)}"] \arrow[d, "\epsilon"'] & S_0(\Delta_1) \arrow[d, "\epsilon"] \\ \mathbb{Z} \arrow[r, "\mathrm{id}_{\mathbb{Z}}"] & \mathbb{Z}. \end{tikzcd}$$

So we get for $\partial_1 i_1 \in S_0(\Delta_1)$,

$$\epsilon\left(\mathrm{sd}_{\Delta_1} \partial_1 i_1\right) = \epsilon\left(\partial_1 i_1\right) = 0. \tag{2.58}$$

For $p>1$, by the inductive hypothesis, the following diagram commutes:

$$\begin{tikzcd} S_{p-1}(\Delta_p) \arrow[r, "\mathrm{sd}_{\Delta_p}"] \arrow[d, "\partial_{p-1}"'] & S_{p-1}(\Delta_p) \arrow[d, "\partial_{p-1}"] \\ S_{p-2}(\Delta_p) \arrow[r, "\mathrm{sd}_{\Delta_p}"] & S_{p-2}(\Delta_p). \end{tikzcd}$$

Hence, for $\partial_p i_p \in S_{p-1}$,

$$\partial_{p-1}\left(\mathrm{sd}_{\Delta_p} \partial_p i_p\right) = \mathrm{sd}_{\Delta_p} \partial_{p-1}\partial_p i_p = 0. \tag{2.59}$$

Now, combining 2.58, 2.59 and plugging them into 2.57, we get

$$\partial_p\left[\mathrm{sd}_{\Delta_p} \partial i_p, \widehat{\Delta_p}\right] = (-1)^p\, \mathrm{sd}_{\Delta_p} \partial_p i_p \tag{2.60}$$

in both cases. Therefore, 2.56 gives us

$$\partial_p\left(\mathrm{sd}_{\Delta_p} i_p\right) = \mathrm{sd}_{\Delta_p} \partial_p i_p, \quad \forall\, p. \tag{2.61}$$

Now, in general, for $T : \Delta_p \to X$ continuous,

$$\partial_p\left(\mathrm{sd}_X T\right) = \partial_p\left[\left(T_\#\right)_p \left(\mathrm{sd}_{\Delta_p} i_p\right)\right] = \left(T_\#\right)_{p-1}\left[\partial_p\left(\mathrm{sd}_{\Delta_p} i_p\right)\right], \tag{2.62}$$

since $T_\#$ is a chain map and hence the following diagram commutes.

$$\begin{tikzcd} S_p(\Delta_p) \arrow[r, "(T_\#)_p"] \arrow[d, "\partial_p"'] & S_p(X) \arrow[d, "\partial_p"] \\ S_{p-1}(\Delta_p) \arrow[r, "(T_\#)_{p-1}"'] & S_{p-1}(X) \end{tikzcd}$$

So

$$\partial_p\left(\mathrm{sd}_X T\right) = \left(T_\#\right)_{p-1}\left[\partial_p\left(\mathrm{sd}_{\Delta_p} i_p\right)\right] = \left(T_\#\right)_{p-1}\left(\mathrm{sd}_{\Delta_p}\partial_p i_p\right) = \mathrm{sd}_X\left(T_\#\right)_{p-1}\left(\partial_p i_p\right), \tag{2.63}$$

using the naturality of sd. Hence,

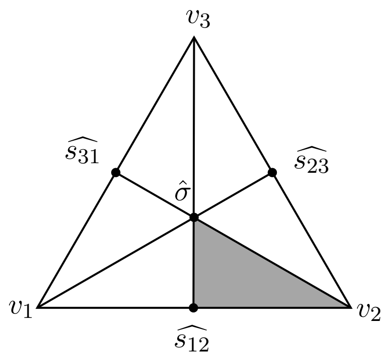
$$\partial_p\left(\mathrm{sd}_X T\right) = \mathrm{sd}_X\left(T_\#\right)_{p-1}\left(\partial_p i_p\right) = \mathrm{sd}_X\,\partial_p\left(sd_X\left(T_\#\right)_p i_p\right). \tag{2.64}$$

Now, $\left(T_\#\right)_p i_p = T \circ i_p = T$. Therefore,

$$\partial_p\left(\mathrm{sd}_X T\right) = \mathrm{sd}_X\,\partial_p T. \tag{2.65}$$

So sd_X indeed commutes with the boundary operator, and hence is a chain map. ■

Consider $\sigma = \Delta_2$ and its first barycentric subdivision.



Denote v_1v_2, v_2v_3 and v_3v_1 by s_{12}, s_{23} and s_{31}, respectively. Denote the barycenter of σ by $\widehat{\sigma}$, barycenter of s_{12} by $\widehat{s_1 2}$ and so on. Observe that, for 0-simplices v_1, v_2, v_3, their barycenters are just themselves, i.e. $\widehat{v_i} = v_i$ for $i = 1, 2, 3$. Then we have a natural ordering. For example, $\sigma \succ s_{12} \succ v_2$, meaning s_{12} is a proper face of σ, v_2 is a proper face of s_{12}. Then we have a distinct 2-simplex $\widehat{\sigma}\widehat{s_{12}}\widehat{v_2}$ (colored gray in the above image) by joining the 3 barycenters $\widehat{\sigma}, \widehat{s_{12}}, \widehat{v_2}$. This 2-simplex belongs to the first barycentric subdivision of Δ_2, which we denote by $\mathrm{Sd}\,\Delta_2$ [1].

The first barycentric subdivision of Δ_2 contains also the following 2-simplices: $\widehat{\sigma}\widehat{s_{12}}\widehat{v_1}$, $\widehat{\sigma}\widehat{s_{23}}\widehat{v_2}$, $\widehat{\sigma}\widehat{s_{23}}\widehat{v_3}$, $\widehat{\sigma}\widehat{s_{31}}\widehat{v_1}$, $\widehat{\sigma}\widehat{s_{31}}\widehat{v_3}$. It contains the following 1-simplices: $\widehat{s_{12}}\widehat{v_1}$, $\widehat{s_{12}}\widehat{v_2}$, $\widehat{s_{23}}\widehat{v_2}$, $\widehat{s_{23}}\widehat{v_3}$, $\widehat{s_{31}}\widehat{v_1}$, $\widehat{s_{31}}\widehat{v_3}$ and the 0-simplices $\widehat{v_1}$, $\widehat{v_2}$, $\widehat{v_3}$, $\widehat{s_{12}}$, $\widehat{s_{23}}$, $\widehat{s_{31}}$, $\widehat{\sigma}$. We then have the following result:

Lemma 2.10

Let K be a simplicial complex. The complex $\mathrm{Sd}\,K$ equals the collection of all simplices of the form

$$\widehat{\sigma_1}\widehat{\sigma_2}\cdots\widehat{\sigma_n},$$

where $\widehat{\sigma_1} \succ \widehat{\sigma_2} \succ \cdots \succ \widehat{\sigma_n}$.

The proof of this lemma is omitted.

Lemma 2.11

Let $T : \Delta_p \to \sigma$ be a linear homeomorphism of Δ_p with the p-simplex σ. Then each term of $\mathrm{sd}_\sigma T$ is a linear homeomorphism of Δ_p with a simplex in the first barycentric subdivision of σ.

Proof. When $p = 0$, σ is a 0-simplex and the first barycentric subdivision of σ contains just the 0-simplex σ. And, given linear homeomorphism $T : \Delta_0 \to \sigma$, $\mathrm{sd}_\sigma T = T$ is the same linear homeomorphism of Δ_0 with the only simplex σ in the first barycentric subdivision of σ.

[1]Note that, the subdivision operator $\mathrm{sd}_X : S_p(X) \to S_p(X)$ is written sd, and the barycentric subdivision of a simplicial complex (which we studied in AT2) is denoted by Sd, to avoid confusion.

Now, suppose the lemma is true in dimension less than p. Consider the identity homeomorphism $i_p : \Delta_p \to \Delta_p$. Now,

$$\operatorname{sd}_{\Delta_p} i_p = (-1)^p \left[\operatorname{sd}_{\Delta_p} \partial i_p, \widehat{\Delta_p}\right].$$

Note that

$$\partial i_p = \sum_{j=0}^{p} (-1)^j \, i_p \circ l_{(\varepsilon_0,\ldots,\widehat{\varepsilon}_j,\ldots,\varepsilon_p)}$$

so that each term in this sum is a linear homeomorphism of Δ_{p-1} with a $(p-1)$-simplex in $\operatorname{Bd} \Delta_p$.

$$\operatorname{sd}_{\Delta_p} \partial i_p = \sum_{j=0}^{p} (-1)^j \operatorname{sd}_{\Delta_p} \left(i_p \circ l_{(\varepsilon_0,\ldots,\widehat{\varepsilon}_j,\ldots,\varepsilon_p)}\right).$$

By the inductive hypothesis, each term of $\operatorname{sd}_{\Delta_p} \left(i_p \circ l_{(\varepsilon_0,\ldots,\widehat{\varepsilon}_j,\ldots,\varepsilon_p)}\right)$ is a linear homeomorphism of Δ_{p-1} with a $(p-1)$-simplex $\widehat{s}_1\widehat{s}_2\cdots\widehat{s}_p$ in the first barycentric subdivision of $\operatorname{Bd} \Delta_p$.

$$\operatorname{sd}_{\Delta_p} \left(i_p \circ l_{(\varepsilon_0,\ldots,\widehat{\varepsilon}_j,\ldots,\varepsilon_p)}\right) = \sum_k \pm T_{jk}, \tag{2.66}$$

where T_{jk} is a linear homeomorphism of Δ_{p-1} with a $(p-1)$-simplex $\widehat{s}_1\widehat{s}_2\cdots\widehat{s}_p$ in the first barycentric subdivision of $\operatorname{Bd} \Delta_p$. So

$$\operatorname{sd}_{\Delta_p} \partial i_p = \sum_{j=0}^{p} \sum_k \pm T_{jk}. \tag{2.67}$$

Then $\left[T_{jk}, \widehat{\Delta_p}\right]$ is by definition a linear homeomorphism of Δ_p with the p-simplex $\widehat{\Delta_p}\widehat{s}_1\widehat{s}_2\cdots\widehat{s}_p$, which belongs to the first barycentric subdivision of Δ_p. Now,

$$\operatorname{sd}_{\Delta_p} i_p = \sum_{j=0}^{p} \sum_k \pm \left[T_{jk}, \widehat{\Delta_p}\right]. \tag{2.68}$$

Therefore, each term of $\operatorname{sd}_{\Delta_p} i_p$ is a linear homeomorphism of Δ_p with a p-simplex in the first barycentric subdivision of Δ_p.

Now consider a general linear homeomorphism $T : \Delta_p \to \sigma$. It's clear that T defines a linear homeomorphism between the first barycentric subdivision of Δ_p with that of σ, because T takes barycenter of Δ_p to the barycenter of σ (since T is linear).

$$\operatorname{sd}_\sigma T = (T_\#)_p \left(\operatorname{sd}_{\Delta_p} i_p\right),$$

with $T : \Delta_p \to \sigma$ being a linear homeomorphism. Using 2.68,

$$\operatorname{sd}_\sigma T = \sum_{j=0}^{p} \sum_k \pm T \circ \left[T_{jk}, \widehat{\Delta_p}\right]. \tag{2.69}$$

By construction, $\left[T_{jk}, \widehat{\Delta_p}\right] : \Delta_p \to \operatorname{Sd}(\Delta_p)$ is a linear homeomorphism onto its image, and $T : \Delta_p \to \sigma$ is a given linear homeomorphism. Hence, the composite $T \circ \left[T_{jk}, \widehat{\Delta_p}\right] : \Delta_p \to \sigma$ is a linear homeomorphism.

$\left[T_{jk}, \widehat{\Delta_p}\right]$ takes Δ_p linear homeomorphically to a p-simplex in the first barycentruc subdivision of Δ_p and we have seen that T is a linear homeomorphism between the first barycentric subdivision of Δ_p with that of σ. Hence, $T \circ \left[T_{jk}, \widehat{\Delta_p}\right]$ takes Δ_p linear homeomorphically to a p-simplex in the first barycentruc subdivision of σ. So the terms of $\operatorname{sd}_\sigma T$ are linear homeomorphisms of Δ_p with a p-somplex in the first barycentric subdivision of σ. ■

Theorem 2.12

Let $\mathcal{A}$ be a collection of subsets of X whose interiors cover X. Given $T : \Delta_p \to X$, there is an m such that each term of $\operatorname{sd}_X^m T$ is $\mathcal{A}$-small.

Proof. Apply Lemma 2.11 to each term of $\operatorname{sd}_\sigma L$, where $L : \Delta_p \to \sigma$ is a linear homeomorphism of Δ_p with a p-simplex σ. Each term of $\operatorname{sd}_\sigma L$ is a linear homeomorphism of Δ_p with a simplex in $\operatorname{Sd}\sigma$. Then each term of $\operatorname{sd}_\sigma^2 L$ is a linear homeomorphism of Δ_p with a simplex in $\operatorname{Sd}^2\sigma$. More generally, each term of $\operatorname{sd}_\sigma^m L$ is a linear homeomorphism of Δ_p with a simplex in the m-th barycentric subdivision of σ, i.e. $\operatorname{Sd}^m \sigma$.

Now, $\{\operatorname{Int} A \mid A \in \mathcal{A}\}$ covers X. Let us first cover Δ_p by open sets $T^{-1}(\operatorname{Int} A)$ with $A \in \mathcal{A}$. Δ_p is a compact metric space. Let λ be the Lebesgue number associated with this cover $\left\{T^{-1}(\operatorname{Int} A) \mid A \in \mathcal{A}\right\}$ of Δ_p. So every subset of Δ_p with diameter less than λ must be contained in $T^{-1}(\operatorname{Int} A)$ for some $A \in \mathcal{A}$.

Now, choose m large enough such that each simplex in the m-th barycentric subdivision has diameter less than λ. Now, in the opening paragraph of the proof, take $L = i_p : \Delta_p \to \Delta_p$, the identity map from Δ_p to itself. Then each term of $\operatorname{sd}_{\Delta_p}^m i_p$ is a linear homeomorphism of Δ_p with a p-simplex in the m-th barycentric subdivision of Δ_p, each of which has diameter smaller than λ.

Then by *Lebesgue number lemma*, the image of each term of $\operatorname{sd}_{\Delta_p}^m i_p$ is contained in $T^{-1}(\operatorname{Int} A)$ for some $A \in \mathcal{A}$. So, T composed with each term of $\operatorname{sd}_{\Delta_p}^m i_p$ is contained in $\operatorname{Int} A$ for some $A \in \mathcal{A}$. But T composed with each term of $\operatorname{sd}_{\Delta_p}^m i_p$ is nothing but each term of

$$(T_\#)_p\left(\operatorname{sd}_{\Delta_p}^m i_p\right) = \operatorname{sd}_X^m T. \tag{2.70}$$

Hence, each term of $\operatorname{sd}_X^m T$ has its image set contained in $\operatorname{Int} A$. In other words, each term of $\operatorname{sd}_X^m T$ is $\mathcal{A}$-small. ∎

Remark 2.2. $\operatorname{sd}_X^m : S_p(X) \to S_p(X)$ is of course a map. In fact, it is a group homomorphism. But we can't talk about the image set of $\operatorname{sd}_X^m T$ even when $T : \Delta_p \to X$ is a singular p-simplex of X, as $\operatorname{sd}_X^m T$ is, in general, a p-chain, not a singular p-simplex.

Having shown how to chop up singular chains so that they are $\mathcal{A}$-small, we now show that these $\mathcal{A}$-small singular chains suffice to generate the homology of X. We first need a lemma.

Lemma 2.13

Let m be given. For each space X, there is a homomorphism $D_p^X : S_p(X) \to S_{p+1}(X)$ such that for each singular p-simplex T of X,

$$\partial_{p+1} D_p^X T + D_{p-1}^X \partial_p T = \operatorname{sd}_X^m T - \operatorname{id}_{S_p(X)} T. \tag{2.71}$$

Furthermore, D^X is natural; i.e., for continuous $f : X \to Y$, the following diagram commutes

$$\begin{tikzcd} S_p(X) \arrow[r, "(f_\#)_p"] \arrow[d, "D_p^X"'] & S_p(Y) \arrow[d, "D_p^Y"] \\ S_{p+1}(X) \arrow[r, "(f_\#)_{p+1}"'] & S_{p+1}(Y). \end{tikzcd}$$

In other words, $D_p^Y \circ (f_\#)_p = (f_\#)_{p+1} \circ D_p^X$.

Remark 2.3. The above lemma guarantees that there is a chain homotopy D^X between the chain maps $\mathrm{sd}_X^m, \mathrm{id}_{\mathcal{S}(X)} : \mathcal{S}(X) \to \mathcal{S}(X)$. Also, note that the naturality of sd_X^m and D^X shows that if A is a subspace of X, then sd_X^m and D^X carry $S_p(A)$ into $S_p(A)$ and $S_{p+1}(A)$, respectively. Thus they induce a chain map and a chain homotopy, respectively, on the relative chain complex $\mathcal{S}(X, A)$ as well.

§2.6 Excision

Definition 2.8. Let X be a topological space; let $\mathcal{A}$ be a covering of X. Let $S_p^{\mathcal{A}}(X)$ be the subgroup of $S_p(X)$ generated by singular p-simplices of X that are $\mathcal{A}$-small. Let $\mathcal{S}^{\mathcal{A}}(X)$ denote the chain complex whose chain groups are the groups $S_p^{\mathcal{A}}(X)$. $\mathcal{S}^{\mathcal{A}}(X)$ is a subchain complex of $\mathcal{S}(X)$, because if the singular p-simplex $T : \Delta_p \to X$ has its image set in $A \in \mathcal{A}$, then each term of $\partial_p T$ also has its image set contained in the same $A \in \mathcal{A}$.

Note that each singular 0-chain is automatically $\mathcal{A}$-small. Hence, $S_0^{\mathcal{A}}(X) = S_0(X)$, and consequently ϵ defines an augmentation for $\mathcal{S}^{\mathcal{A}}(X)$. Hence, by Remark 2.3, sd_X^m and D^X carry $\mathcal{S}^{\mathcal{A}}(X)$ into itself. In other words, if the image set of a singular p-simplex $T : \Delta_p \to X$ lies in $A \in \mathcal{A}$, then each term of $\mathrm{sd}_X^m T$ and $D_p^X T$ also has its image set lying in $A \in \mathcal{A}$.

Theorem 2.14

Let X be a topological space; let $\mathcal{A}$ be a collection of subsets of X whose interiors cover X. Then the inclusion map $\mathcal{S}^{\mathcal{A}}(X) \hookrightarrow \mathcal{S}(X)$ induces an isomorphism in homology, both ordinary and reduced.

Proof. Consider the short exact sequence of chain complexes

$$0 \longrightarrow \mathcal{S}^{\mathcal{A}}(X) \xrightarrow{i} \mathcal{S}(X) \longrightarrow \mathcal{S}(X)/\mathcal{S}^{\mathcal{A}}(X) \longrightarrow 0.$$

This, in fact, is a collection of short exact sequence of chain groups in each dimension p:

$$0 \longrightarrow S_p^{\mathcal{A}}(X) \xrightarrow{(i_\#)_p} S_p(X) \longrightarrow \mathcal{S}_p(X)/S_p^{\mathcal{A}}(X) \longrightarrow 0.$$

It gives rise to a long exact sequence in homology (either ordinary or reduced). Now, if we can prove that the homology groups of the chain complex $\left\{S_p(X)/S_p^{\mathcal{A}}(X), \partial_p^X\right\}$ vanish in every dimension p, then the long exact sequence in homology obtained from the short exact sequence above using Zig-Zag lemma will yield the following exact sequence:

$$0 \longrightarrow H_p^{\mathcal{A}}(X) \xrightarrow{(i_*)_p} H_p(X) \longrightarrow 0.$$

The exactness of this sequence will then dictate that $(i_*)_p : H_p^{\mathcal{A}}(X) \to H_p(X)$ is an isomorphism. Let us now prove that the homology groups of the chain complex $\left\{S_p(X)/S_p^{\mathcal{A}}(X), \partial_p^X\right\}$ vanish in every dimension p.

Let $c_p + S_p^{\mathcal{A}}(X) \in S_p(X)/S_p^{\mathcal{A}}(X)$, for $c_p \in S_p(X)$, such that it represents a cycle in $S_p(X)/S_p^{\mathcal{A}}(X)$. In other words, $\partial_p^X c_p$ belongs to $S_{p-1}^{\mathcal{A}}(X)$. We now want to show that this c_p necessarily represents a boundary, i.e. there exists some $d_{p+1} \in S_{p+1}(X)$ such that $c_p - \partial_{p+1}^X d_{p+1}$ belongs to $S_p^{\mathcal{A}}(X)$.

Note that c_p is a finite formal linear combination of singular p-simplices. In view of Theorem 2.12, we chan choose m large enough so that each singular p-simplex appearing in the expression for $\mathrm{sd}_X^m c_p$ is $\mathcal{A}$-small. Once m is chosen, let D^X be the chain homotopy of Lemma 2.13. $D_p^X : S_p(X) \to S_{p+1}(X)$. In fact, we shall show that $-D_p^X c_p$ is precisely the $d_{p+1} \in S_{p+1}(X)$ that we are looking for. In other words, we will show that $c_p + \partial_{p+1}^X D_p^X c_p$ belongs to $S_p^{\mathcal{A}}(X)$ and we are done!

By Lemma 2.13, we know that

$$\partial_{p+1}^{X} D_p^{X} c_p + D_{p-1}^{X} \partial_p^{X} c_p = \mathrm{sd}_X^m c_p - c_p \implies c_p + \partial_{p+1}^{X} D_p^{X} c_p = \mathrm{sd}_X^m c_p - D_{p-1}^{X} \partial_p^{X} c_p. \tag{2.72}$$

We have chosen m large enough so that $\mathrm{sd}_X^m c_p \in S_p^{\mathcal{A}}(X)$. Also, $\partial_p^X c_p \in S_{p-1}^{\mathcal{A}}(X)$, so that $D_{p-1}^X \partial_p^X c_p \in S_{p-1}^{\mathcal{A}}(X)$. Therefore, from 2.72, we can conclude that $c_p + \partial_{p+1}^X D_p^X c_p$. ■

Corollary 2.15

Let X and $\mathcal{A}$ be as in the previous theorem. If $B \subseteq X$, let $S_p^{\mathcal{A}}(B)$ be generated by those singular p-simplices $T : \Delta_p \to B$ whose image sets lie in elements of $\mathcal{A}$. Obviously, $S_p^{\mathcal{A}}(B) \subseteq S_p^{\mathcal{A}}(X)$. Let us denote the quotient group by

$$S_p^{\mathcal{A}}(X, B) = S_p^{\mathcal{A}}(X) / S_p^{\mathcal{A}}(B).$$

Then the inclusion

$$i_p : S_p^{\mathcal{A}}(X, B) \hookrightarrow S_p(X, B)$$

induces a homology isomorphism.

Proof. Consider the following inclusion maps

$$\mathcal{S}^{\mathcal{A}}(B) \xhookrightarrow{i_B} \mathcal{S}(B),$$
$$\mathcal{S}^{\mathcal{A}}(X) \xhookrightarrow{i_X} \mathcal{S}(X),$$
$$\mathcal{S}^{\mathcal{A}}(X, B) \xhookrightarrow{i_{(X,B)}} \mathcal{S}(X, B),$$

and the 2 short exact sequences of chain complexes connected by the above 3 inclusions:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{S}^{\mathcal{A}}(B) \arrow[r] \arrow[d, "i_B"'] & \mathcal{S}^{\mathcal{A}}(X) \arrow[r] \arrow[d, "i_X"'] & \mathcal{S}^{\mathcal{A}}(X,B) \arrow[r] \arrow[d, "i_{(X,B)}"] & 0 \\
0 \arrow[r] & \mathcal{S}(B) \arrow[r] & \mathcal{S}(X) \arrow[r] & \mathcal{S}(X,B) \arrow[r] & 0
\end{tikzcd}$$

The above diagram commutes. To show that, it suffices to show that commutativity of the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & S_p^{\mathcal{A}}(B) \arrow[r] \arrow[d, "i_B"'] & S_p^{\mathcal{A}}(X) \arrow[r] \arrow[d, "i_X"'] & S_p^{\mathcal{A}}(X,B) \arrow[r] \arrow[d, "i_{(X,B)}"] & 0 \\
0 \arrow[r] & S_p(B) \arrow[r] & S_p(X) \arrow[r] & S_p(X,B) \arrow[r] & 0.
\end{tikzcd}$$

If we take $c \in S_p^{\mathcal{A}}(B)$, the inclusion maps take it to itself. So the left hand square commutes trivially. Now, we take $d \in S_p^{\mathcal{A}}(X)$. Then under the map $S_p^{\mathcal{A}}(X) \to S_p^{\mathcal{A}}(X, B)$, d goes to

$$d + S_p^{\mathcal{A}}(B).$$

Then under $i_{(X,B)}$, it goes to

$$d + S_p(B).$$

On the other hand, i_X takes d to itself. Then the map $S_p(X) \to S_p(X, B)$ takes it to

$$d + S_p(B).$$

Therefore, the right hand square commutes as well. Therefore, one obtains the following commutative diagram with the two corresponding long exact sequences connnected via induced group homomorphisms:

$$
\begin{tikzcd}
\cdots \arrow[r] & H_p^{\mathcal{A}}(B) \arrow[r] \arrow[d, "((i_B)_*)_p"'] & H_p^{\mathcal{A}}(X) \arrow[r] \arrow[d, "((i_X)_*)_p"] & H_p^{\mathcal{A}}(X,B) \arrow[r] \arrow[d, "\left(\left(i_{(X,B)}\right)_*\right)_p"] & H_{p-1}^{\mathcal{A}}(B) \arrow[r] \arrow[d, "((i_B)_*)_{p-1}"] & H_{p-1}^{\mathcal{A}}(X) \arrow[r] \arrow[d, "((i_X)_*)_{p-1}"] & \cdots \\
\cdots \arrow[r] & H_p(B) \arrow[r] & H_p(X) \arrow[r] & H_p(X,B) \arrow[r] & H_{p-1}(B) \arrow[r] & H_{p-1}(X) \arrow[r] & \cdots
\end{tikzcd}
$$

Now, $((i_B)_*)_p$, $((i_X)_*)_p$, $((i_B)_*)_{p-1}$, $((i_X)_*)_{p-1}$ are all isomorphisms by Theorem 2.14. Therefore, applying Steenrod five lemma, we conclude that $\left(\left(i_{(X,B)}\right)_*\right)_p : H_p^{\mathcal{A}}(X,B) \to H_p(X,B)$ is an isomorphism. ∎

Theorem 2.16 (Excision for singular theory)
Let $A \subseteq X$. If U is a subset of X such that $\overline{U} \subseteq \operatorname{Int} A$, then the inclusion

$$j : (X \setminus U, A \setminus U) \hookrightarrow (X, A)$$

induces an isomorphism in singular homology.

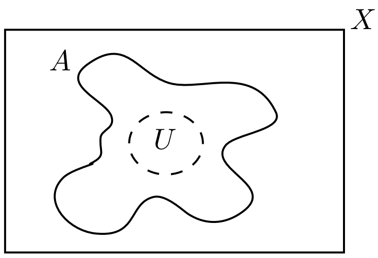
Proof. Let $\mathcal{A}$ denote the collection $\{X \setminus U, A\}$. Observe that the open set $X \setminus \overline{U}$ is precisely $\operatorname{Int}(X \setminus U)$. Also, since $\overline{U} \subseteq \operatorname{Int} A$,

$$X \setminus (\operatorname{Int} A) \subseteq X \setminus \overline{U} = \operatorname{Int}(X \setminus U).$$

Therefore,

$$X = [X \setminus (\operatorname{Int} A)] \cup (\operatorname{Int} A) \subseteq \operatorname{Int}(X \setminus U) \cup \operatorname{Int} A = \bigcup_{S \in \mathcal{A}} \operatorname{Int}(S).$$

Therefore, the interiors of sets in $\mathcal{A}$ cover X.



Now, consider the homomorphisms induced by inclusions

$$\frac{S_p(X \setminus U)}{S_p(A \setminus U)} \hookrightarrow \frac{S_p^{\mathcal{A}}(X)}{S_p^{\mathcal{A}}(A)} \text{ and } \frac{S_p^{\mathcal{A}}(X)}{S_p^{\mathcal{A}}(A)} \hookrightarrow \frac{S_p(X)}{S_p(A)}.$$

The first one is an inclusion since a p-chain in $X \setminus U$ is clearly in $S_p^{\mathcal{A}}(X)$ as $\mathcal{A} = \{X \setminus U, A\}$; and a p-chain in $A \setminus U$ is also clearly in $S_p^{\mathcal{A}}(A)$. The second inclusion is just $S_p^{\mathcal{A}}(X,A) \hookrightarrow S_p(X,A)$.

By Corollary 2.15, the latter homomorphism induces group isomorphism at the level of homology groups. We now intend to prove that

$$\frac{S_p(X \setminus U)}{S_p(A \setminus U)} \hookrightarrow \frac{S_p^{\mathcal{A}}(X)}{S_p^{\mathcal{A}}(A)}$$

is already an isomorphism at the chain level. Consider the map

$$\phi : S_p(X \setminus U) \to \frac{S_p^{\mathcal{A}}(X)}{S_p^{\mathcal{A}}(A)}, \quad c_p \mapsto c_p + S_p^{\mathcal{A}}(A), \tag{2.73}$$

for $c_p \in S_p(X \setminus U)$. Note that ϕ is surjective. If c_p is a p-chain in $S_p^{\mathcal{A}}(X)$, then each term of c_p has image set lying in either $X \setminus U$ or in A. While forming the coset $c_p + S_p^{\mathcal{A}}(A)$, we can safely throw away the terms that have image sets in A. So every coset element in $\frac{S_p^{\mathcal{A}}(X)}{S_p^{\mathcal{A}}(A)}$ is of the form

$$d_p + S_p^{\mathcal{A}}(A)$$

for $d_p \in S_p(X \setminus U)$. Hence, ϕ is surjective. Now, $c_p \in \operatorname{Ker}\phi$ if $c_p \in S_p^{\mathcal{A}}(A)$. Since $\operatorname{Ker}\phi \subset S_p(X \setminus U)$, we have

$$c_p \in S_p(X \setminus U) \cap S_p^{\mathcal{A}}(A) = S_p((X \setminus U) \cap A) = S_p(A \setminus U). \tag{2.74}$$

Therefore, $\operatorname{Ker}\phi = S_p(A \setminus U)$. Hence, by the first isomorphisim theorem,

$$\frac{S_p(X \setminus U)}{S_p(A \setminus U)} \cong \frac{S_p^{\mathcal{A}}(X)}{S_p^{\mathcal{A}}(A)}. \tag{2.75}$$

Therefore, $H_p(X \setminus U, A \setminus U) \cong H_p^{\mathcal{A}}(X, A)$. We already have $H_p^{\mathcal{A}}(X, A) \cong H_p(X, A)$ by Corollary 2.15. Therefore,

$$H_p(X \setminus U, A \setminus U) \cong H_p(X, A).$$

■

3 CW Complexes

§3.1 The Topology of CW Complexes

Definition 3.1. If X is a topological space and $\mathcal{C}$ is a collection of subspaces of X whose union is X, the topology of X is said to be **coherent** with the collection $\mathcal{C}$ provided a set $A \subseteq X$ is closed in X if and only if $A \cap C$ is closed in C for each $C \in \mathcal{C}$. It is equivalent to require that $U \subseteq X$ is open in X if and only if $U \cap C$ is open in C for each $C \in \mathcal{C}$.

Lemma 3.1

Let X be a set which is the union of topological space $\{X_\alpha\}$. If there is a topological space X_T having X as its underlying set, and each X_α is a subspace of X_T, then X has a topology (called the **coherent topology**), of which X_α are subspaces, that is coherent with the collection $\{X_\alpha\}$. This latter topology is, in general, finer than the topology of X_T.

Proof. Let us define a topological space X_C (whose underlying set is X) by declaring that $A \subseteq X$ is closed if and only if $A \cap X_\alpha$ is closed in X_α for each α. If A and B are closed in X_C, then both $A \cap X_\alpha$ and $B \cap X_\alpha$ are closed in X_α for each α. Therefore,

$$(A \cup B) \cap X_\alpha = (A \cap X_\alpha) \cup (B \cap X_\alpha) \tag{3.1}$$

is closed in X_α, proving that $A \cup B$ is closed. On the other hand, if $\{A_i\}_{i\in J}$ is an arbitrary collection of closed sets, each $A_i \cap X_\alpha$ is closed in X_α. Then

$$\left(\bigcap_{i\in J} A_i\right) \cap X_\alpha = \bigcap_{i\in J} (A_i \cap X_\alpha) \tag{3.2}$$

is closed in X_α. Therefore, $\bigcap_{i\in J} A_i$ is closed. Hence, X_C indeed defines a topology on X.

Now, if C is a closed set in X_T, then since X_α is a subspace of X_T, $C \cap X_\alpha$ must be closed in X_α for each α. Therefore, C is closed in X_C. Thus, the topology of X_C is finer than that of X_T.

Now we need to show that each X_α is a subspace of X_C. For this purpose, we show that the closed sets of X_α are of the form $C \cap X_\alpha$, where C is closed in X_C. First note that if C is closed in X_C, $C \cap X_\alpha$ is closed in X_α for each α. Conversely, if B is closed in X_α, since X_α is a subspace of X_T, $B = C \cap X_\alpha$ for some closed C in X_T. Now, since X_C is finer than X_T, C must also be closed in X_C. Thus $B = C \cap X_\alpha$ for some closed C in X_C, as desired. Therefore, each X_α is a subspace of X_C. So X_C is coherent with the collection $\{X_\alpha\}$. ■

Remark 3.1. We can always give a topology X_T to the underlying set $X = \bigcup_\alpha X_\alpha$, with each X_α being a topological space by its own right, so that X_α becomes a subspace of X_T (i.e. the topology of X_α that it had as an individual topological space from the beginning coincides with the subspace topology it inherits from X_T) with X_T not being coherent with its subspaces X_α. In such case, X_C will be strictly finer than X_T. When X_T is found to be coherent with its subspaces X_α, one has $X_\mathsf{T} = X_\mathsf{C}$.

Some useful terminologies

The m-dimensional ball B^m is the following subspace of $\mathbb{R}^m$

$$B^m = \{\mathbf{x} \in \mathbb{R}^m \mid \|\mathbf{x}\| \leq 1\}. \tag{3.3}$$

The open m-ball, denoted by $\operatorname{Int}(B^m)$, is the interior of B^m in $\mathbb{R}^m$.

$$\operatorname{Int} B^m = \{\mathbf{x} \in \mathbb{R}^m \mid \|\mathbf{x}\| < 1\}. \tag{3.4}$$

The boundary of B^m in $\mathbb{R}^m$ is the standard $(m-1)$-sphere.

$$S^{m-1} = \operatorname{Bd} B^m = \{\mathbf{x} \in \mathbb{R}^m \mid \|\mathbf{x}\| < 1\}. \tag{3.5}$$

We note that the 0-ball B^0 is equal to $\mathbb{R}^0 = \{0\}$. One has $\operatorname{Int} B^0 = B^0 = \{0\}$. Also, B^1 is the interval $[-1,1]$ in $\mathbb{R}$, and $\operatorname{Int} B^1 = (-1,1)$. So

$$S^0 = \operatorname{Bd} B^1 = \{-1,1\}. \tag{3.6}$$

Cell decomposition and CW-complexes

Definition 3.2. An n-cell is a topological space homeomorphic to the open n-ball $\operatorname{Int} B^n$. A **cell** is a topological space which is an n-cell for some $n \geq 0$. Since $\operatorname{Int} B^n$ is homeomorphic to $\mathbb{R}^n$, we can talk about the dimension of an n-cell. An n-cell is rightly said to have dimension n.

Definition 3.3 (Cell decomposition)**.** A **cell decomposition** of a topological space X is a family $\mathcal{E} = \left\{e_{\alpha \mid \alpha \in I}\right\}$ of subspaces of X such that each e_α is a cell and

$$X = \bigsqcup_{\alpha \in I} e_\alpha. \tag{3.7}$$

The n-skeleton of X is the subspace

$$X^n = \bigsqcup_{\alpha \in I,\, \dim e_{\alpha \leq n}} e_\alpha. \tag{3.8}$$

Note that if $\mathcal{E}$ is a cell decomposition of X, then the cells of $\mathcal{E}$ can have many different dimensions. For example, consider a cell-decomposition of S^1 given by $\mathcal{E} = \{e_a, e_b\}$, where e_a is an arbitrary point $p \in S^1$ and $e_b = S^1 \setminus \{p\}$. Here, e_a is a 0-cell and e_b is a 1-cell. One can have uncountably many cells in a cell decomposition of a given topological space. A **finite cell decomposition** is a cell decomposition consisting of finitely many cells.

Definition 3.4 (CW complex)**.** A pair $(X, \mathcal{E})$ consisting of a Hausdorff space X and a cell decomposition $\mathcal{E}$ of X is called a **CW complex** if the following 3 axioms are satisfied:

Axiom 1. (Characteristic maps) For each n-cell $e_\alpha \in \mathcal{E}$, there is a continuous map $f_\alpha : B^n \to X$ restricting to a homeomorphism

$$f_\alpha\big|_{\operatorname{Int} B^n} : \operatorname{Int} B^n \to e_\alpha$$

and taking $\operatorname{Bd} B^n = S^{n-1}$ into X^{n-1}.

Axiom 2 (Closure finiteness)**.** For any cell $e_\alpha \in \mathcal{E}$, the closure $\overline{e_\alpha}$ intersects only finitely many cells in $\mathcal{E}$.

Axiom 3 (Weak topology)**.** A subset $A \subseteq X$ is closed if and only if $A \cap \overline{e_\alpha}$ is closed in $\overline{e_\alpha}$ for each $e_\alpha \in \mathcal{E}$.

Remark 3.2. Here, the topology of the Hausdorff space $X = \bigcup_\alpha \overline{e_\alpha}$ is coherent with the subspaces $\{\overline{e_\alpha}\}_\alpha$, i.e. X is endowed with the finest topology with respect to which all these topological spaces $\overline{e_\alpha}$ become its subspaces. Axiom 3 basically demands this coherence.

Definition 3.5. The **dimension** of a CW complex (X, ε) is the largest dimension of a cell of $\mathcal{E}$, if such exists. Otherwise, it is said to be infinite.

Lemma 3.2

Let X be a Hausdorff space and $\mathcal{E} = \{\overline{e_\alpha}\}_\alpha$ a cell decomposition of X. If $(X, \mathcal{E})$ satisfies Axiom 1 of CW complex, then we have $\overline{e_\alpha} = f_\alpha(B^n)$ for any n-cell e_α. In particular, $\overline{e_\alpha}$ is a compact subspace of X and the "cell boundary" $\dot{e}_\alpha := \overline{e_\alpha} \setminus e_\alpha = f_\alpha(S^{n-1})$ lies in X^{n-1}.

Proof. Since $f_\alpha : B^n \to X$ is continuous associated with a given n-cell e_α, we have

$$\overline{e_\alpha} = \overline{f_\alpha(\operatorname{Int} B^n)} \supseteq f_\alpha\left(\overline{\operatorname{Int} B^n}\right) = f_\alpha(B^n). \tag{3.9}$$

So $f_\alpha(B^n) \subseteq \overline{e_\alpha}$. Since B^n is compact and f_α is continuous, $f_\alpha(B^n)$ is compact. Now, since X is Hausdorff, $f_\alpha(B^n)$ is closed. Since $e_\alpha = f_\alpha(\operatorname{Int} B^n)$,

$$f_\alpha(B^n) \supseteq e_\alpha \implies \overline{f_\alpha(B^n)} \supseteq \overline{e_\alpha} \implies f_\alpha(B^n) \supseteq \overline{e_\alpha}. \tag{3.10}$$

Therefore, $\overline{e_\alpha} = f_\alpha(B^n)$.

By Axiom 1, we have $f_\alpha(\operatorname{Int} B^n) = e_\alpha$ and $f_\alpha\left(S^{n-1}\right) \subseteq X^{n-1}$. So

$$f_\alpha\left(S^{n-1}\right) \cap e_\alpha = \varnothing. \tag{3.11}$$

But $f_\alpha\left(S^{n-1}\right) \subseteq f_\alpha(B^n) = \overline{e_\alpha}$. So we have

$$f_\alpha\left(S^{n-1}\right) \subseteq \overline{e_\alpha} \setminus e_\alpha. \tag{3.12}$$

Furthermore,

$$\overline{e_\alpha} \setminus e_\alpha = f_\alpha(B^n) \setminus f_\alpha(\operatorname{Int} B^n) \subseteq f_\alpha(B^n \setminus \operatorname{Int} B^n) = f_\alpha\left(S^{n-1}\right). \tag{3.13}$$

Therefore, $f_\alpha\left(S^{n-1}\right) = \overline{e_\alpha} \setminus e_\alpha =: \dot{e}_\alpha$. ∎

Subcomplexes

Lemma 3.3

Let $(X, \mathcal{E})$ be a CW complex, and $\mathcal{E}' = \{e_{\alpha'}\}_{\alpha'} \subseteq \mathcal{E}$ a collection of cells in it. Suppose $X' = \bigcup_{\alpha'} e_{\alpha'}$. Then the following are equivalent:

(a) The pair $(X', \mathcal{E}')$ is a CW complex.

(b) The subset X' is closed in X.

(c) $\overline{e_{\alpha'}} \subseteq X'$ for each $e_{\alpha'} \in \mathcal{E}'$, where $\overline{e_{\alpha'}}$ is the closure of $e_{\alpha'}$ in X.

Definition 3.6 (Subcomplex)**.** Let $(X, \mathcal{E})$ be a CW complex, and $(X', \mathcal{E}')$ be as above. Then $(X', \mathcal{E}')$ is called a **subcomplex** of $(X, \mathcal{E})$ if the 3 equivalent conditions stated in Lemma 3.3 are satisfied.

Corollary 3.4

Let $(X, \mathcal{E})$ be a CW complex. Then

(a) Let $\{A_i\}_{i \in I}$ be any family of subcomplexes of $(X, \mathcal{E})$. Then $\bigcup_{i \in I} A_i$ and $\bigcap_{i \in I} A_i$ are subcomplexes of $(X, \mathcal{E})$.

(b) The n-skeleton X^n is a subcomplex of $(X, \mathcal{E})$ for each $n \geq 0$.

(c) Let $\{e_i\}_{i \in I}$ be any arbitrary family of n-cells in $\mathcal{E}$. Then $X^{n-1} \cup \left(\bigcup_{i \in I} e_i\right)$ is a subcomplex.

Proof. We shall first prove (a). The others follow immediately from (a). Given the family of subcomplexes $\{A_i\}_{i\in I}$ of the CW complex $(X,\mathcal{E})$, each $A_i \subseteq X$ is a closed subspace of X. Then $\bigcap_{i\in I} A_i$ is closed in A. Therefore, by Lemma 3.3, $\bigcap_{i\in I} A_i$ is a subcomplex of $(X,\mathcal{E})$.

Now we shall prove that $\bigcup_{i\in I} A_i$ is a subcomplex. For this purpose, we shall use the characterization (c) of Lemma 3.3. Let $e \subseteq \bigcup_{i\in I} A_i$ be an n-cell. Then $e \subseteq A_j$ for some $j \in I$. By characterization (c), $\overline{e} \subseteq A_j$. Therefore, $\overline{e} \subseteq \bigcup_{i\in I} A_i$. So $\bigcup_{i\in I} A_i$ is a subcomplex.

Now, we shall prove (b). If e_α is a n-cell,

$$\overline{e_\alpha} = e_\alpha \cup \dot{e_\alpha} = e_\alpha \cup f_\alpha\left(S^{n-1}\right) \subseteq e_\alpha \cup X^{n-1}. \tag{3.14}$$

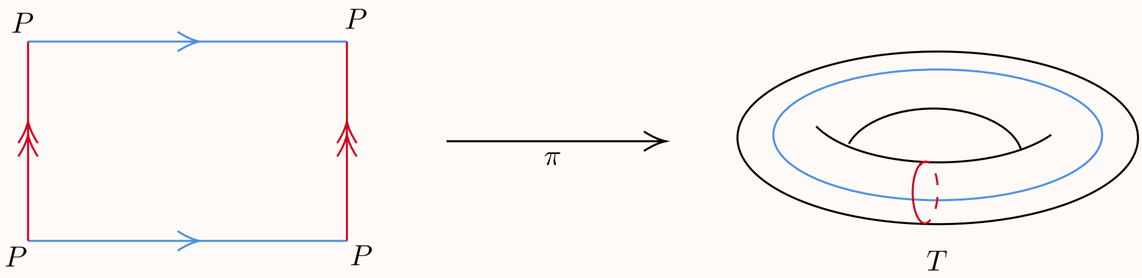
So $\overline{e_\alpha} \subseteq X^n$. If e_β is a k-cell for $k < n$, $\overline{e_\beta} \subseteq X^{n-1}$. Therefore, X^n is a subcomplex. For (c), a similar computation as 3.14 reveals that

$$\overline{e_i} \subseteq X^{n-1} \cup \left(\bigcup_{i\in I} e_i\right). \tag{3.15}$$

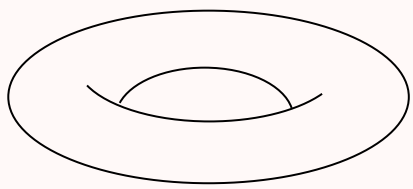
Therefore, $X^{n-1} \cup \left(\bigcup_{i\in I} e_i\right)$ is also a subcomplex. ■

Example 3.1

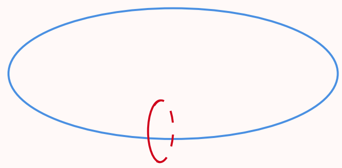
Consider the torus as a quotient space of a rectangle as usual (by identifying opposite sides of a rectangle).



We express T as a CW complex having a single 2-cell (the image under π of the interior of the rectangle), two 1-cells (the images of the 2 open edges of the rectangle under π), and one 0-cell (the image of the vertices of the rectangle under π). You should convince yourself that all the axioms in the definition of a CW complex are satisfied here.



2-skeleton of T

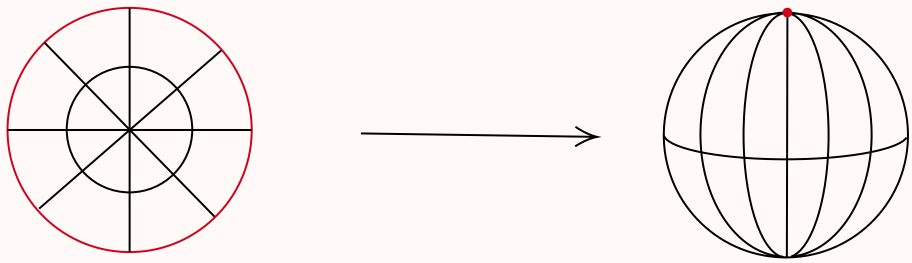


1-skeleton of T

0-skeleton of T

Example 3.2

The quotient space formed from B^n by collapsing $\operatorname{Bd} B^n$ to a point is homeomorphic to S^n. Hence, the Hausdorff topological space S^n can be expressed as a CW complex having one n-cell and a 0-cell, and no other cells at all.



§3.2 Adjunction Space

Definition 3.7 (Topological sum)**.** Let $\{X_\alpha\}_{\alpha\in J}$ be a family of topological spaces, not necessarily disjoint. Let E be the set that is the union of the disjoint topological spaces $E_\alpha = X_\alpha \times \{\alpha\}$. In other words,

$$E = \bigsqcup_{\alpha\in J} E_\alpha = \bigsqcup_{\alpha\in J} X_\alpha \times \{\alpha\}\,. \tag{3.16}$$

If we topologize E by declaring $U \subseteq E$ to be open if and only if $U \cap E_\alpha$ is open in E_α for each α, then E is called the **topological sum** of the topological spaces X_α.

One has a natural map $p : E \to \bigcup_\alpha X_\alpha$ which projects $X_\alpha \times \{\alpha\}$ onto X_α for each α. We now have the following important result.

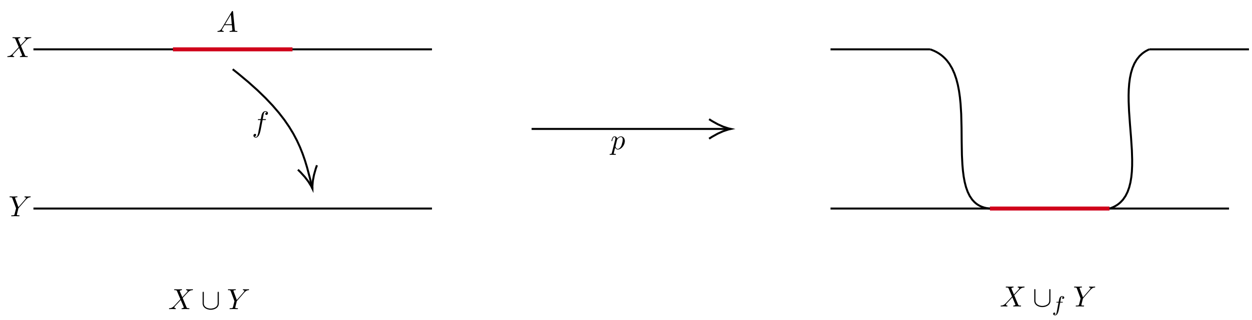
Lemma 3.5

Let X be a topological space which is the union of certain of its subspaces, i.e. $X = \bigcup_\alpha X_\alpha$. Let E be the topological sum of the subspaces X_α. Also, let $p : E \to \bigcup_\alpha X_\alpha$ be the natural projection. Then the topology of X is coherent with the subspaces if and only if p is a quotient map. In this situation, we often say that X is the **coherent union** of the spaces X_α.

Definition 3.8 (Adjection space)**.** Let X and Y be disjoint topological spaces, and let A be a closed subspace of X. Let $f : A \to Y$ be a continuous map. We define a certain quotient space as follows: Topologize $X \cup Y$ as the topological sum, i.e. $U \subseteq X \cup Y$ is open if and only if both $U \cap X$ and $U \cap Y$ are open in X and Y, respectively. Form a quotient space by identifying each set

$$\{y\} \cup f^{-1}(Y)\,, \ \text{(for } y \in Y)$$

to a point. That is, partition $X \cup Y$ into these sets, along with the singletons $\{x\}$ for $x \in X \setminus A$. We denote this quotient space by $X \cup_f Y$, and call it the **adjunction space** determined by f.



It is often useful to view a CW complex as a space built up from a collection of n-balls (possibly of different n) by forming appropriate quotient spaces.

Recall from point set topology that a topological space X is said to be **normal** if given any two disjoint closed sets E and F, there are open disjoint sets U and V such that $E \subseteq U$ and $F \subseteq V$.

Lemma 3.6

Let X be a space that is the countable union of certain closed subspaces X_n. Suppose the topology of X is coherent with those subspaces X_n. If each X is normal, so is X.

Theorem 3.7

If X and Y are normal, then so is the adjunction space $X \cup_f Y$.

Theorem 3.8

Suppose $(X, \mathcal{E})$ is a CW complex of dimension p. Then X is homeomorphic to an adjunction space formed from X^{p-1} and a topological sum $\bigsqcup_\alpha B_\alpha^p$ of p-balls B^p (here $B_\alpha^p = B^p \times \{\alpha\}$) by means of a continuous map $g : \bigsqcup_\alpha \operatorname{Bd} B_\alpha^p \to X^{p-1}$. It follows that X is normal.

Proof. Associated with each $e_\alpha \in \mathcal{E}$ of dimension p, there is a characteristic map $f_\alpha : B^p \to \overline{e_\alpha}$. Now, $B_\alpha^p = B^p \times \{\alpha\}$, and $\bigcup_\alpha B^p \times \{\alpha\} = \bigsqcup_\alpha B_\alpha^p$. Now, form the topological sum

$$E = X^{p-1} \cup \left(\bigsqcup_\alpha B_\alpha^p \right), \tag{3.17}$$

and define $\pi : E \to X$ by letting π equal inclusion on X^{p-1} and the composite

$$B_\alpha^p = B^p \times \{\alpha\} \to B^p \xrightarrow{f_\alpha} X \tag{3.18}$$

on B_α^p. We will now prove that π is a quotient map. This will prove that X is homeomorphic to the underlying quotient space $X^{p-1} \cup_g (\bigsqcup_\alpha B_\alpha^p)$, with g being the continuous map

$$g : \bigsqcup_\alpha \operatorname{Bd} B_\alpha^p \to X^{p-1}$$

induced from the characteristic maps f_α.

π is continuous on each of the disjoint components, so π is continuous. Furthermore, it is surjective. Indeed, for $x \in X$, x is either in X^{p-1} or in some p-cell e_α. In any case there is a pre-image of x, since f_α restricts to a homeomorphism of $\operatorname{Int} B^p$ with e_α.

Suppose $C \subseteq X$ and $\pi^{-1}(C)$ is closed in E. In order to show that π is a quotient map, we need to show that C is closed as well. X^{p-1} is a CW subcomplex of $(X, \mathcal{E})$ and hence X^{p-1} is a xlosed subspace of X. Therefore,

1. $\pi^{-1}(C) \cap X^{p-1}$ is closed in X^{p-1} in the subspace topology it inherits from E. But

$$\pi^{-1}(C) \cap X^{p-1} = \pi^{-1}\left(C \cap X^{p-1}\right) = C \cap X^{p-1}, \tag{3.19}$$

hence $C \cap X^{p-1}$ is closed in X^{p-1}. Since X^{p-1} is a CW complex in its own right, using the weak topology axiom, one obtains $C \cap \overline{e_\beta}$ is closed in $\overline{e_\beta}$ for $\dim e_\beta \leq p-1$.

2. Also, each B_α^p inherits subspace topology from $E = X^{p-1} \cup (\bigsqcup_\alpha B_\alpha^p)$. Since $\pi^{-1}(C)$ is closed in E, $\pi^{-1}(C) \cap B_\alpha^p$ is closed in B_α^p in subspace topology. Now, since each B_α^p is compact, and $\pi : E \to X$ is continuous,

$$\pi\left(\pi^{-1}(C) \cap B_\alpha^p\right) = C \cap f_\alpha\left(B_\alpha^p\right) = C \cap \overline{e_\alpha} \tag{3.20}$$

is compact (because closed subspace of compact set is compact, and so is continuous image of compact set). So we arrive at the fact that $C \cap \overline{e_\alpha}$ is compact. Since X is Hausdorff, $C \cap \overline{e_\alpha}$ is closed in X (compact subspace of Hausdorff is closed). $\overline{e_\alpha}$ is closed in X, $C \cap \overline{e_\alpha}$ is closed in $\overline{e_\alpha}$.

Therefore, we verified that $C \cap \overline{e_\alpha}$ is closed in $\overline{e_\alpha}$ with $\dim e_\alpha \leq p$. Since X is of dimension p, all the cells of $\mathcal{E}$ are of dimension at most p. Therefore, C is closed in X, proving that π is a quotient map.

We shall now prove that X is normal. We proceed inductively. X^0 is a discrete topological space, and hence normal. $\bigsqcup_\alpha B^1_\alpha$ is also normal. Therefore, the corresponding adjunction space $X^0 \cup_g \left(\bigsqcup_\alpha B^1_\alpha\right)$ (which is homeomorphic to X^1) is normal. In a similar manner, we can show that each X^i is normal. Therefore, $X^p = X$ is normal. ∎

The converse of Theorem 3.8 can be stated as follows:

Theorem 3.9

Let $(Y, \mathcal{E})$ be a CW complex of dimension $p-1$. Let $\bigsqcup_\alpha B^p_\alpha$ be a topological sum of p-balls, and let $g : \bigsqcup_\alpha \operatorname{Bd} B^p_\alpha \to Y$ be a continuous map. Then the adjunction space

$$X = Y \cup_g \left(\bigsqcup_\alpha B^p_\alpha \right)$$

is the underlying topological space of a CW compleex, and Y is its p-skeleton.

Sketch of proof. Use Theorem 3.7 to show that X is Hausdorff. Construct the quotient map

$$f : Y \cup \left(\bigsqcup_\alpha B^p_\alpha \right) \to X$$

by defining it as inclusion on Y, and by means of the given continuous map g on $\bigsqcup_\alpha \operatorname{Bd} B^p_\alpha$. f on $\operatorname{Int} B^p_\alpha$ is going to give the p cells $e_\alpha \in \mathcal{E}'$. This way form the p-cells in $\mathcal{E}'$. In particular, $e_\alpha = f\left(\operatorname{Int} B^p_\alpha\right)$. Some work needs to be done to show that e_α is a p-cell. The other cells in $\mathcal{E}'$ are the cells $e_\beta \in \mathcal{E}$ of dimension at most $p-1$. Now show that $(X, \mathcal{E}')$ thus constructed fulfills all 3 axioms of a CW complex. ∎

Theorem 3.8 and Theorem 3.9 can be extended to construct infinite dimensional CW complexes. For that we need a lemma first.

Lemma 3.10

Let X be a set which is the union of topological space $\{X_\alpha\}$. If for each pair α, β of indices, the set $X_\alpha \cap X_\beta$ is closed in both X_α and X_β, and inherits the same subspace topology from each of them, then X has a topology coherent with the subspaces $\{X_\alpha\}$. Each X_α is closed in this topology.

We shall omit the proof of this lemma.

Theorem 3.11 (a) Let $(X, \mathcal{E})$ be a CW complex. Then X^p is a closed subspace of X^{p+1} for each p, and X is the coherent union of the spaces $X^0 \subseteq X^1 \subseteq X^2 \subseteq \cdots$. It follows that X is normal.

(b) Conversely, suppose $(X_p, \mathcal{E}_p)$ is a CW complex for each p, and X_p equals p-skeleton of X^{p+1} for each p. If X is the coherent union of the spaces X_p, then $(X, \mathcal{E})$ is a CW complex having X_p as its p-skeleton, where $\mathcal{E} = \bigcup_p \mathcal{E}_p$.

Proof. (a) By Lemma 3.3, both X^p and X^{p+1} are closed in X. Here, $X^p \subseteq X^{p+1} \subseteq X$. We prove that X^p is closed in X^{p+1}. It is equivalent to proving $X^{p+1} \subseteq X^p$ is open in X^{p+1}.

$$X^{p+1} \setminus X^p = X^{p+1} \cap (X \setminus X^p)\,, \tag{3.21}$$

and $X \setminus X^p$ is open in X. Therefore, $X^{p+1} \setminus X^p$ is open in X^{p+1} in subspace topology inherited from X. Hence, X^p is closed in X^{p+1}.

Now suppose $C \cap X^p$ is closed in X^p for each p. We need to prove that C is closed in X. Since $(X^p, \mathcal{E}_p)$ is a CW complex by its own right, by the weak topology axiom, $C \cap X^p \cap \overline{e_\alpha}$ is closed in $\overline{e_\alpha}$ for each $e_\alpha \in \mathcal{E}_p$. Since $\overline{e_\alpha} \subseteq X^p$, we have

$$C \cap X^p \cap \overline{e_\alpha} = C \cap \overline{e_\alpha}. \tag{3.22}$$

Therefore, $C \cap \overline{e_\alpha}$ is closed in $\overline{e_\alpha}$ for each $e_\alpha \in \mathcal{E}_p$. Since p is arbitrary, $C \cap \overline{e_\alpha}$ is closed in $\overline{e_\alpha}$ for each $e_\alpha \in \mathcal{E}$. Hence, C is closed in X.

Conversely, suppose C is closed in X. We know that $X^p \subseteq X$ is closed in X. Hence, $C \cap X^p$ is closed in X. Now,

$$X^p \setminus (C \cap X^p) = X^p \cap [X \setminus (C \cap X^p)]. \tag{3.23}$$

$X \setminus (C \cap X^p)$ is open in X. Therefore, $X^p \setminus (C \cap X^p)$ is open in X^p in the subspace topology it inherits from X. Therefore, $C \cap X^p$ is closed in X. We, therefore, conclude that C is closed in X if and only if $C \cap X^p$ is closed in X^p for each p. Therefore, X is the coherent union of the subspaces

$$X^0 \subseteq X^1 \subseteq X^2 \subseteq \cdots.$$

Normality of X follows by noting that $X = \bigcup_p X^p$, where $\{X^p\}_p$ is a countable collection of closed subspaces, and using Lemma 3.6.

(b) If $p < q$, then $X_p \cap X_q = X_p$ is a closed subspace of both X_p and X_q, since X_p is the p-skeleton of X_q. Therefore, by Lemma 3.10, there is a topology on X coherent with the subspaces $\{X_p\}_p$, and each X_p is closed in X. By Theorem 3.8, each X_p is normal. Using Lemma 3.6, X is normal as well (and in particular, Hausdorff). The closure-finiteness axiom follows trivially. Now we check the weak topology axiom.

Suppose $C \cap \overline{e_\alpha}$ is closed in $\overline{e_\alpha}$ for each cell e_α. Then $C \cap X_p$ is closed in X_p, since X_p is a CW complex. Then C is closed in X, because the topology of X is coherent with the spaces X_p.

Conversely, suppose C is closed in X. Then $C \cap X_p$ is closed in X_p for each p, because of the coherence. Since X_p is a CW complex, $C \cap X_p \cap \overline{e_\alpha}$ is closed in $\overline{e_\alpha}$ for each cell e_α with dimension at most p. But $C \cap X_p \cap \overline{e_\alpha} = C \cap \overline{e_\alpha}$. Therefore, $C \cap \overline{e_\alpha}$ is closed in $\overline{e_\alpha}$ for each cell e_α. This proves that X satisfies the weak topology axiom. Therefore, X is a CW complex.

■

§3.3 The Homology of CW Complexes

Let $(X, \mathcal{E})$ be a CW complex. Also, let $D_p(X) = H_p\left(X^p, X^{p-1}\right)$. Let $\partial_p : D_p(X) \to D_{p-1}(X)$ be defined to be the composite

$$\begin{tikzcd} H_p\left(X^p, X^{p-1}\right) \arrow[r, "(\partial_*)_p"] \arrow[rr, bend right, "(j_*)_{p-1} \circ (\partial_*)_p"'] & H_{p-1}\left(X^{p-1}\right) \arrow[r, "(j_*)_{p-1}"] & H_{p-1}\left(X^{p-1}, X^{p-2}\right) \end{tikzcd}$$

In other words,

$$\partial_p = (j_*)_{p-1} \circ (\partial_*)_p, \tag{3.24}$$

where $j : \left(X^{p-1}, \varnothing\right) \hookrightarrow \left(X^{p-1}, X^{p-2}\right)$ is the inclusion. One can verify that $\partial_{p-1} \circ \partial_p = 0$ by considering the long exact homology sequence of the pair $\left(X^{p-1}, X^{p-2}\right)$.

$$\cdots \longrightarrow H_{p-1}\left(X^{p-1}\right) \xrightarrow{(j_*)_{p-1}} H_{p-1}\left(X^{p-1}, X^{p-2}\right) \xrightarrow{(\partial_*)_{p-1}} H_{p-2}\left(X^{p-2}\right) \longrightarrow \cdots$$

Exactness of this sequence implies $(\partial_*)_{p-1} \circ (j_*)_{p-1} = 0$. Now,

$$\begin{aligned} \partial_{p-1} \circ \partial_p &= (j_*)_{p-2} \circ \left[(\partial_*)_{p-1} \circ (j_*)_{p-1}\right] \circ (\partial_*)_p \\ &= (j_*)_{p-2} \circ 0 \circ (\partial_*)_p = 0. \end{aligned} \tag{3.25}$$