Representation Theory (MAT440)

Lecture Notes

In memory of the unsung heroes of the July revolution, whose courage and sacrifice lit the path of justice and change.

Preface

This series of lecture notes has been prepared for aiding students who took the BRAC University course **Representation Theory (MAT440)** in Summer 2024 semester. These notes were typeset under the supervision of mathematician **Dr. Syed Hasibul Hassan Chowdhury**. The main goal of this typeset is to have an organized digital version of the notes, which is easier to share and handle. If you see any mistakes or typos, please send me an email at atonuroychowdhury@gmail.com

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1 Representation of Finite Groups

§1.1 Definitions

Definition 1.1 (Representation)**.** A **representation** of a finite group G on a finite dimensional complex vector space V is a homomorphism $\rho : G \to \mathrm{GL}(V)$ of G to the group of invertible linear transformations on V. We often say that such a homomorphism gives V the structure of a G-module. The dimension of V is sometimes called the **degree** of the representation ρ. We also sometimes call V itself a representation of G.

Definition 1.2. A **map** φ between two representations V and W of G is a linear map $\varphi : V \to W$ such that the following diagram commutes for every $g \in G$:

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "\rho(g)"'] & W \arrow[d, "\sigma(g)"] \\ V \arrow[r, "\varphi"'] & W \end{tikzcd}$$

In other words, $\varphi \circ \rho(g) = \sigma(g) \circ \varphi$. Here, $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$ are two group homomorphisms in question. We distinguish such a linear map $\varphi : V \to W$ between two representations of G from an ordinary linear map between vector spaces by calling it a **G-linear map**.

One can then define G-module structure on $\operatorname{Ker}\varphi$ and $\operatorname{im}\varphi$ by restricting the group homomorphisms $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$, namely,

$$\rho_1 : G \to \mathrm{GL}\left(\operatorname{Ker}\varphi\right) \text{ and } \sigma_1 : G \to \mathrm{GL}\left(\operatorname{im}\varphi\right).$$

Suppose $\mathbf{v} \in \operatorname{Ker}\varphi$. Then $\rho\left(g\right)\left(\mathbf{v}\right) \in \operatorname{Ker}\varphi$, because

$$\varphi\left(\rho\left(g\right)\left(\mathbf{v}\right)\right) = \sigma\left(g\right)\left(\varphi\left(\mathbf{v}\right)\right) = \sigma\left(g\right)\left(\mathbf{0}\right) = \mathbf{0}. \tag{1.1}$$

Also, let $\mathbf{w} \in \operatorname{im}\varphi$. Then $\mathbf{w} = \varphi(\mathbf{v})$ for some $\mathbf{v} \in V$. Then $\sigma\left(g\right)\left(\mathbf{w}\right) \in \operatorname{im}\varphi$, because

$$\sigma\left(g\right)\varphi(\mathbf{v}) = \varphi\left(\rho\left(g\right)\left(\mathbf{v}\right)\right) \in \operatorname{im}\varphi. \tag{1.2}$$

One can also give the quotient vector space $W/\operatorname{im}\varphi = \operatorname{Coker}\varphi$ a G-module structure by introducing the group homomorphism $\sigma_2 : G \to \mathrm{GL}(\operatorname{Coker}\varphi)$. Given $\mathbf{w} + \operatorname{im}\varphi \in \operatorname{Coker}\varphi$ and $g \in G$, one defines

$$\sigma_2\left(g\right)\left(\mathbf{w} + \operatorname{im}\varphi\right) = \sigma(g)\left(\mathbf{w}\right) + \operatorname{im}\varphi \in \operatorname{Coker}\varphi. \tag{1.3}$$

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The space of all G-linear maps from V to W is denoted $\mathrm{Hom}_G(V, W)$. It has a vector space structure. Suppose $\varphi, \psi \in \mathrm{Hom}_G(V, W)$ and $z \in \mathbb{C}$. Then we have the following commutative squares:

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "\rho(g)"'] & W \arrow[d, "\sigma(g)"] \\ V \arrow[r, "\varphi"'] & W \end{tikzcd} \qquad \begin{tikzcd} V \arrow[r, "\psi"] \arrow[d, "\rho(g)"'] & W \arrow[d, "\sigma(g)"] \\ V \arrow[r, "\psi"'] & W \end{tikzcd}$$

Then one can show that $z\varphi + \psi$ is also a G-linear map. Indeed,

$$\begin{aligned}\sigma(g)\circ(z\varphi+\psi)(\mathbf{v}) &= z\sigma(g)(\varphi(\mathbf{v})) + \sigma(g)(\psi(\mathbf{v}))\\ &= z\varphi(\rho(g)(\mathbf{v})) + \psi(\rho(g)(\mathbf{v}))\\ &= (z\varphi+\psi)(\rho(g)\mathbf{v}).\end{aligned}$$

This proves the commutativity of the following square:

$$\begin{tikzcd} V \arrow[r, "z\varphi+\psi"] \arrow[d, "\rho(g)"'] & W \arrow[d, "\sigma(g)"] \\ V \arrow[r, "z\varphi+\psi"'] & W \end{tikzcd}$$

Therefore, $z\varphi + \psi \in \mathrm{Hom}_G(V, W)$, i.e. $\mathrm{Hom}_G(V, W)$ is a complex vector space.

Definition 1.3 (Subrepresentation)**.** Suppose one is given a representation V of G with the help of the group homomorphism $\rho : G \to \mathrm{GL}(V)$ and $W \subset V$ be a vector subspace. One calls W **invariant** under the action of G if for all $g \in G$ and all $\mathbf{w} \in W$, one has $\rho(g)\mathbf{w} \in W$.

A **subrepresentation** of a representation V of G is a vector subspace W of V that is invariant under the action of G. A representation V of G is called **irreducible** if there is no proper nonzero invariant subspace W of V, i.e., there is no invariant subsapce $W \subset V$ such that $W \neq \{\mathbf{0}\}$ and $W \neq V$.

§1.2 Linear algebra revisited

Definition 1.4 (Tensor product)**.** The **tensor product** of two complex vector spaces V and W is another complex vector space $V \otimes W$ equipped with a bilinear map $\theta : V \times W \to V \otimes W$ that is *universal*: for any bilinear map $\beta : V \times W \to U$ to a complex vector space U, there exists a unique linear map $\alpha : V \otimes W \to U$ such that the following diagram commutes:

$$\begin{tikzcd} V \times W \arrow[r, "\theta"] \arrow[d, "\beta"'] & V \otimes W \arrow[dl, dashed, "\exists!\,\alpha"] \\ U & \end{tikzcd}$$

In other words, $\beta = \alpha \circ \theta$.

If we want the ground field $\mathbb{C}$ to be mentioned, we write the tensor product by $V \otimes_{\mathbb{C}} W$. If $\{\mathbf{e}_i\}$ and $\{\mathbf{f}_j\}$ are bases of V and W, respectively, $\{\mathbf{e}_i \otimes \mathbf{f}_j\}$ form a basis for $V \otimes W$. Similarly, one can form the tensor product $V_1 \otimes \cdots \otimes V_n$ of n vector spaces, with the universal (in the above sense) multilinear map

$$\begin{aligned}\theta : V_1 \times \cdots \times V_n &\to V_1 \otimes \cdots \otimes V_n\\ (\mathbf{v}_1, \ldots, \mathbf{v}_n) &\mapsto \mathbf{v}_1 \otimes \cdots \otimes \mathbf{v}_n.\end{aligned} \tag{1.4}$$

In particular, one can construct

$$V^{\otimes n} = \underbrace{V \otimes \cdots \otimes V}_{n\text{-copies}},$$

for a fixed complex vector space V. If $\{\mathbf{e}_i \mid i = 1, 2, \ldots, m\}$ is a basis for V, then the set

$$\{\mathbf{e}_{i_1} \otimes \mathbf{e}_{i_2} \otimes \cdots \otimes \mathbf{e}_{i_n} \mid i_1, \ldots, i_n \in \{1, 2, \ldots, m\}\} \tag{1.5}$$

is a basis for $V^{\otimes n}$. It follows that $\dim V^{\otimes n} = m^n$.

Let $\mathfrak{S}_n$ be the symmetric group on the set $\{1, 2, \ldots, n\}$. It is a finite group of order $n!$ that consists of all the permutations (i.e. bijections) on the set $\{1, 2, \ldots, n\}$. An alternating multilinear map $\beta : V \times \cdots \times V \to U$ satisfies

$$\beta\left(\mathbf{v}_{\sigma(1)}, \ldots, \mathbf{v}_{\sigma(n)}\right) = \operatorname{sgn} \sigma \beta\left(\mathbf{v}_1, \ldots, \mathbf{v}_n\right), \tag{1.6}$$

for every $\sigma \in \mathfrak{S}_n$.

Definition 1.5 (Exterior power). The **exterior power** of a complex vector spaces V is another complex vector space $\Lambda^n V$ equipped with an alternating multilinear map

$$\begin{aligned} \kappa : V \times \cdots \times V &\to \Lambda^n V \\ (\mathbf{v}_1, \ldots, \mathbf{v}_n) &\mapsto \mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_n, \end{aligned}$$

that is *universal*: for any alternating multilinear map $\beta : V \times \cdots \times V \to U$ to a complex vector space U, there exists a unique linear map $\alpha : \Lambda^n V \to U$ such that the following diagram commutes:

$$\begin{tikzcd} V \times \cdots \times V \arrow[r, "\kappa"] \arrow[d, "\beta"'] & \Lambda^n V \arrow[dl, dashed, "\exists!\,\alpha"] \\ U & \end{tikzcd}$$

In other words, $\beta = \alpha \circ \kappa$.

If $\{\mathbf{e}_i \mid i = 1, 2, \ldots, m\}$ is a basis for V, then the set

$$\{\mathbf{e}_{i_1} \wedge \mathbf{e}_{i_2} \wedge \cdots \wedge \mathbf{e}_{i_n} \mid 1 \leq i_1 < i_2 < \cdots < i_n \leq m\} \tag{1.7}$$

is a basis for $\Lambda^n V$. It follows that $\dim \Lambda^n V = \binom{m}{n}$.

A symmetric multilinear map $\beta : V \times \cdots \times V \to U$ satisfies

$$\beta\left(\mathbf{v}_{\sigma(1)}, \ldots, \mathbf{v}_{\sigma(n)}\right) = \beta\left(\mathbf{v}_1, \ldots, \mathbf{v}_n\right), \tag{1.8}$$

for every $\sigma \in \mathfrak{S}_n$.

Definition 1.6 (Symmetric power). The **symmetric power** of a complex vector spaces V is another complex vector space $\operatorname{Sym}^n V$ equipped with an symmetric multilinear map

$$\begin{aligned} \delta : V \times \cdots \times V &\to \operatorname{Sym}^n V \\ (\mathbf{v}_1, \ldots, \mathbf{v}_n) &\mapsto \mathbf{v}_1 \odot \cdots \odot \mathbf{v}_n, \end{aligned}$$

that is *universal*: for any symmetric multilinear map $\beta : V \times \cdots \times V \to U$ to a complex vector space U, there exists a unique linear map $\alpha : \operatorname{Sym}^n V \to U$ such that the following diagram commutes:

$$\begin{tikzcd} V \times \cdots \times V \arrow[r, "\delta"] \arrow[d, "\beta"'] & \operatorname{Sym}^n V \arrow[dl, dashed, "\exists!\,\alpha"] \\ U & \end{tikzcd}$$

In other words, $\beta = \alpha \circ \delta$.

If $\{\mathbf{e}_i \mid i = 1, 2, \ldots, m\}$ is a basis for V, then the set

$$\{\mathbf{e}_{i_1} \odot \mathbf{e}_{i_2} \odot \cdots \odot \mathbf{e}_{i_n} \mid 1 \leq i_1 \leq i_2 \leq \cdots \leq i_n \leq m\} \tag{1.9}$$

is a basis for $\operatorname{Sym}^n V$. It follows that $\dim \operatorname{Sym}^n V = \binom{m+n-1}{n}$.

§1.3 New representations from old ones

If V and W are representations of G, then so are the direct sum $V \oplus W$ and the tensor product $V \otimes W$. More explicitly, suppose $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$ are the relevant group homomorphisms. Then, one defines $\rho \oplus \sigma : G \to \mathrm{GL}(V \oplus W)$ by

$$(\rho \oplus \sigma)(g)(\mathbf{v} \oplus \mathbf{w}) = \rho(g)\mathbf{v} \oplus \sigma(g)\mathbf{w}, \tag{1.10}$$

for $g \in G$. Similarly, one can define the group homomorphism $\rho \otimes \sigma : G \to \mathrm{GL}(V \otimes W)$ by

$$(\rho \otimes \sigma)(g)(\mathbf{v} \otimes \mathbf{w}) = \rho(g)\mathbf{v} \otimes \sigma(g)\mathbf{w} \tag{1.11}$$

for $g \in G$.

For a representation V of G, the nth tensor power $V^{\otimes n}$ is again a representation of G:

$$(\rho \otimes \rho \otimes \cdots \otimes \rho)(g)(\mathbf{v}_1 \otimes \mathbf{v}_2 \otimes \cdots \otimes \mathbf{v}_n) = \rho(g)\mathbf{v}_1 \otimes \rho(g)\mathbf{v}_2 \otimes \cdots \otimes \rho(g)\mathbf{v}_n, \tag{1.12}$$

for $g \in G$. The exterior power $\Lambda^n(V)$ and the symmetric power $\operatorname{Sym}^n(V)$ are subrepresentations of $V^{\otimes n}$. Given the group homomorphism $\rho : G \to \mathrm{GL}(V)$, we defined the nth tensor power representation $\rho^{\otimes n} : G \to \mathrm{GL}\left(V^{\otimes n}\right)$ by (1.12). Now, the exterior power representation $\Lambda^n \rho : G \to \mathrm{GL}(\Lambda^n V)$, being a subrepresentation of $V^{\otimes n}$, can be defined as follows:

$$(\Lambda^n \rho)(g)(\mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \cdots \wedge \mathbf{v}_n) = \rho(g)\mathbf{v}_1 \wedge \rho(g)\mathbf{v}_2 \wedge \cdots \wedge \rho(g)\mathbf{v}_n. \tag{1.13}$$

One can now write down the group homomorphism $\operatorname{Sym}^n \rho : G \to \mathrm{GL}(\operatorname{Sym}^n V)$ associated with the subrepresentation $\operatorname{Sym}^n V$ of the representation $V^{\otimes n}$ of G:

$$(\operatorname{Sym}^n \rho)(g)(\mathbf{v}_1 \odot \mathbf{v}_2 \odot \cdots \odot \mathbf{v}_n) = \rho(g)\mathbf{v}_1 \odot \rho(g)\mathbf{v}_2 \odot \cdots \odot \rho(g)\mathbf{v}_n. \tag{1.14}$$

Now, let us define $\rho^* : G \to \mathrm{GL}(V^*)$, given $\rho : G \to \mathrm{GL}(V)$. Suppose $\{\mathbf{e}_i\}_{i=1}^m$ and $\{\widehat{\alpha}^i\}_{i=1}^m$ are bases of V and V^*, respectively. Here, $V^* = \mathrm{Hom}(V, \mathbb{C})$, the dual vector space of linear functionals on V. Any linear functional $\widehat{\omega} \in V^*$ can be written as

$$\widehat{\omega} = \sum_{i=1}^m \omega_i \widehat{\alpha}^i \tag{1.15}$$

Also, any vector $\mathbf{v} \in V$ can be written as

$$\mathbf{v} = \sum_{i=1}^m v^i \mathbf{e}_i. \tag{1.16}$$

In a given basis $\{\mathbf{e}_i\}_{i=1}^m$ of V and its dual basis $\{\widehat{\alpha}^i\}_{i=1}^m$ of V^*, $\omega \in V^*$ can be coordinated as a column vector $\begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_m \end{bmatrix}$, whereas a vector $\mathbf{v} \in V$ can be coordinated as $\begin{bmatrix} v^1 \\ v^2 \\ \vdots \\ v^m \end{bmatrix}$. We will simply denote the column vector $\begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_m \end{bmatrix}$ by $\widehat{\omega}$, and the column vector $\begin{bmatrix} v^1 \\ v^2 \\ \vdots \\ v^m \end{bmatrix}$ by $\mathbf{v}$. We then write the dual pairing

$$\langle \widehat{\omega}, \mathbf{v} \rangle = \widehat{\omega}(\mathbf{v}) = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_m \end{bmatrix}^T \begin{bmatrix} v^1 \\ v^2 \\ \vdots \\ v^m \end{bmatrix} = \widehat{\omega}^T \mathbf{v}. \tag{1.17}$$

Now, we want the dual representation V^* of V to satisfy

$$\langle \rho^* g(\widehat{\omega}), \rho(g)\mathbf{v} \rangle = \langle \widehat{\omega}, \mathbf{v} \rangle \tag{1.18}$$

for $g \in G$, $\mathbf{v} \in V$ and $\widehat{\omega} \in V^*$. Now, we claim that $\rho^* : V^* \to V^*$ defined by

$$\rho^*(g)(\widehat{\omega}) = \left[\rho\left(g^{-1}\right)\right]^T \widehat{\omega} \tag{1.19}$$

satisfies (1.18). Indeed,

$$\begin{aligned} \langle \rho^* g(\widehat{\omega}), \rho(g)\mathbf{v} \rangle &= \rho^* g(\widehat{\omega}) \left(\rho(g)\mathbf{v}\right) \\ &= \left[\rho\left(g^{-1}\right)\right]^T \widehat{\omega} \left[\rho(g)\mathbf{v}\right] \\ &= \widehat{\omega}\left(\rho\left(g^{-1}\right)\rho(g)\mathbf{v}\right) \\ &= \widehat{\omega}(\mathbf{v}) = \langle \widehat{\omega}, \mathbf{v} \rangle . \end{aligned}$$

Here we used the following definition of transpose: given a linear map $f : V \to W$, its transpose map $f^T : W^* \to V^*$ is defined as $f^T(\widehat{\omega})(\mathbf{v}) = \widehat{\omega}(f(\mathbf{v}))$. In light of this, we can also write (1.19) as

$$\rho^*(g)(\widehat{\omega})(\mathbf{v}) = \left[\rho\left(g^{-1}\right)\right]^T \widehat{\omega}(\mathbf{v}) = \widehat{\omega}\left(\rho\left(g^{-1}\right)\mathbf{v}\right). \tag{1.20}$$

Now, if V and W are representations of G, then so is $\mathrm{Hom}(V, W)$. In order to see this, we shall use the fact that

$$\mathrm{Hom}(V, W) \cong V^* \otimes W. \tag{1.21}$$

Note here that both V and W are finite dimensional complex vector spaces. Consider the group homomorphisms $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$. Now, the group homomorphism associated with dual representation on V^* of G is given by $\rho^* : G \to GL(V^*)$. Note that for $\widehat{\omega} \in V^*$, one has $\widehat{\omega}(\mathbf{e}_i) = \omega_i$, and for $\mathbf{v} \in V$, $\widehat{\alpha}^i(\mathbf{v}) = v^i$, where $\{\mathbf{e}_i\}_{i=1}^m$ is a basis for V and $\{\widehat{\alpha}^i\}_{i=1}^m$ is the dual basis for V^*. Note that $\widehat{\alpha}^i(\mathbf{e}_j) = \delta^i{}_j$.

Given $\varphi \in \mathrm{Hom}(V, W)$, define $\widetilde{g} : \mathrm{Hom}(V, W) \to V^* \otimes W$ by

$$\widetilde{g}(\varphi) = \sum_{i=1}^m \widehat{\alpha}^i \otimes \varphi(\mathbf{e}_i). \tag{1.22}$$

On the other hand, define $\widetilde{f} : V^* \otimes W \to \mathrm{Hom}(V, W)$ by

$$\widetilde{f}(\widehat{\kappa} \otimes \mathbf{w})(\mathbf{v}) = \widehat{\kappa}(\mathbf{v})\,\mathbf{w}, \tag{1.23}$$

where $\widehat{\kappa} \in V^*$, $\mathbf{v} \in V$, $\mathbf{w} \in W$. Then observe that $\widetilde{f}$ and $\widetilde{g}$ are inverses of each other. In fact,

$$\begin{aligned}
\widetilde{f}\left(\widetilde{g}\left(\varphi\right)\right)\left(\mathbf{v}\right) &= \widetilde{f}\left(\sum_{i=1}^{m} \widehat{\alpha}^i \otimes \varphi\left(\mathbf{e}_i\right)\right)\left(\mathbf{v}\right) \\
&= \sum_{i=1}^{m} \widetilde{f}\left(\widehat{\alpha}^i \otimes \varphi\left(\mathbf{e}_i\right)\right)\left(\mathbf{v}\right) \\
&= \sum_{i=1}^{m} \widehat{\alpha}^i\left(\mathbf{v}\right)\varphi\left(\mathbf{e}_i\right) \\
&= \sum_{i=1}^{m} v^i \varphi\left(\mathbf{e}_i\right) \\
&= \varphi\left(\sum_{i=1}^{m} v^i \mathbf{e}_i\right) \\
&= \varphi\left(\mathbf{v}\right).
\end{aligned}$$

Therefore,

$$\widetilde{f} \circ \widetilde{g} = \mathbb{1}_{\mathrm{Hom}(V,W)}. \tag{1.24}$$

Now, for a given $\widehat{\kappa} \otimes \mathbf{w} \in V^* \otimes W$,

$$\begin{aligned}
\widetilde{g}\left(\widetilde{f}\left(\widehat{\kappa} \otimes \mathbf{w}\right)\right) &= \sum_{i=1}^{m} \widehat{\alpha}^i \otimes \widetilde{f}\left(\widehat{\kappa} \otimes \mathbf{w}\right)\left(\mathbf{e}_i\right) \\
&= \sum_{i=1}^{m} \widehat{\alpha}^i \otimes \widehat{\kappa}\left(\mathbf{e}_i\right)\mathbf{w} \\
&= \sum_{i=1}^{m} \widehat{\kappa}\left(\mathbf{e}_i\right)\widehat{\alpha}^i \otimes \mathbf{w} \\
&= \sum_{i=1}^{m} \kappa_i \widehat{\alpha}^i \otimes \mathbf{w} \\
&= \widehat{\kappa} \otimes \mathbf{w}.
\end{aligned}$$

Therefore,

$$\widetilde{g} \circ \widetilde{f} = \mathbb{1}_{V^* \otimes W}. \tag{1.25}$$

(1.24) and (1.25) together imply that $\mathrm{Hom}\left(V, W\right) \cong V^* \otimes W$. We now define the representation of G on $\mathrm{Hom}(V, W)$ via the representation of G on $V^* \otimes W$. In fact, G acts on $V^* \otimes W$ via the map $\rho^* \otimes \sigma : G \to \mathrm{GL}\left(V^* \otimes W\right)$, so that $\left(\rho^* \otimes \sigma\right) g\left(\widehat{\kappa} \otimes \mathbf{w}\right) \in V^* \otimes W$. Then via $\widetilde{f} : V^* \otimes W \to \mathrm{Hom}(V, W)$, one has $\widetilde{f}\left(\left(\rho^* \otimes \sigma\right) g\left(\widehat{\kappa} \otimes \mathbf{w}\right)\right) \in \mathrm{Hom}(V, W)$. This is, by definition, the representation of G on $\mathrm{Hom}(V, W)$. In other words, $\gamma : G \to \mathrm{GL}\left(\mathrm{Hom}(V, W)\right)$ is defined by

$$\begin{aligned}
\gamma\left(g\right)\left(\widetilde{f}\left(\widehat{\kappa} \otimes \mathbf{w}\right)\right)\left(\mathbf{v}\right) &= \widetilde{f}\left(\left(\rho^* \otimes \sigma\right) g\left(\widehat{\kappa} \otimes \mathbf{w}\right)\right)\left(\mathbf{v}\right) \\
&= \widetilde{f}\left(\rho^*\left(g\right)\widehat{\kappa} \otimes \sigma(g)\mathbf{w}\right)\left(\mathbf{v}\right) \\
&= \left(\rho^*\left(g\right)\widehat{\kappa}\right)\left(\mathbf{v}\right)\sigma(g)\mathbf{w} \\
&= \widehat{\kappa}\left(\rho\left(g^{-1}\right)\mathbf{v}\right)\sigma(g)\mathbf{w} \\
&= \sigma\left(g\right)\left(\widehat{\kappa}\left(\rho\left(g^{-1}\right)\mathbf{v}\right)\mathbf{w}\right).
\end{aligned} \tag{1.26}$$

Now, let us write $\widetilde{f}\left(\widehat{\kappa} \otimes \mathbf{w}\right) = \varphi \in \mathrm{Hom}(V, W)$. So we have

$$\varphi\left(\mathbf{v}\right) = \widetilde{f}\left(\widehat{\kappa} \otimes \mathbf{w}\right)\left(\mathbf{v}\right) = \widehat{\kappa}\left(\mathbf{v}\right)\mathbf{w}. \tag{1.27}$$

As a result,

$$\varphi\left(\rho\left(g^{-1}\right)\mathbf{v}\right) = \widehat{\kappa}\left(\rho\left(g^{-1}\right)\mathbf{v}\right)\mathbf{w}. \tag{1.28}$$

(1.26) and (1.28) together imply that

$$(\gamma(g)\varphi)(\mathbf{v}) = \sigma(g)\left(\varphi\left(\rho\left(g^{-1}\right)\mathbf{v}\right)\right). \tag{1.29}$$

(1.29) can be expressed by means of the commutativity of the following diagram:

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "\rho(g)"'] & W \arrow[d, "\sigma(g)"] \\ V \arrow[r, "\gamma(g)\varphi"'] & W \end{tikzcd}$$

Proposition 1.1

Given representations $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$ of a finite group G, $\widetilde{f} : V^* \otimes W \to \mathrm{Hom}(V, W)$ is an isomorphism of representations.

Proof. We have already shown that $\widetilde{f} : V^* \otimes W \to \mathrm{Hom}(V, W)$ is an isomorphism of vector spaces. We now need to show that $\widetilde{f}$ is a map between the representations $\rho^* \otimes \sigma$ and γ. For that purpose, we need to show the commutativity of the following square:

$$\begin{tikzcd} V^* \otimes W \arrow[r, "\widetilde{f}"] \arrow[d, "(\rho^*\otimes\sigma)(g)"'] & \mathrm{Hom}(V,W) \arrow[d, "\gamma(g)"] \\ V^* \otimes W \arrow[r, "\widetilde{f}"'] & \mathrm{Hom}(V,W) \end{tikzcd} \tag{1.30}$$

Given any $\widehat{\kappa} \in V^*$ and $\mathbf{w} \in W$, we need to show that

$$\gamma(g) \circ \widetilde{f}(\widehat{\kappa} \otimes \mathbf{w}) = \widetilde{f} \circ (\rho^* \otimes \sigma)(g)(\widehat{\kappa} \otimes \mathbf{w}). \tag{1.31}$$

Both sides of (1.31) are in $\mathrm{Hom}(V, W)$. In order to show their equality, we need to show the equality of them evaluated at an arbitrary $\mathbf{v} \in V$. So, we are going to show that

$$\left[\gamma(g) \circ \widetilde{f}(\widehat{\kappa} \otimes \mathbf{w})\right](\mathbf{v}) = \left[\widetilde{f} \circ (\rho^* \otimes \sigma)(g)(\widehat{\kappa} \otimes \mathbf{w})\right](\mathbf{v}). \tag{1.32}$$

The RHS of (1.32) is

$$\begin{aligned} \text{RHS} &= \left[\widetilde{f} \circ (\rho^* \otimes \sigma)(g)(\widehat{\kappa} \otimes \mathbf{w})\right](\mathbf{v}) \\ &= \left[\widetilde{f}(\rho^*(g)\widehat{\kappa} \otimes \sigma(g)\mathbf{w})\right](\mathbf{v}) \\ &= (\rho^*(g)\widehat{\kappa})(\mathbf{v}) \cdot \sigma(g)(\mathbf{w}) && [\,\cdot \text{ is the scalar multiplication in } W] \\ &= \widehat{\kappa}\left(\rho(g)^{-1}\mathbf{v}\right) \cdot \sigma(g)(\mathbf{w}) \end{aligned}$$

Before computing the LHS of (1.32), let us quickly recall the definition of γ. $\gamma : G \to \mathrm{GL}(\mathrm{Hom}(V, W))$ is defined so that the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "\rho(g)"'] & W \arrow[d, "\sigma(g)"] \\ V \arrow[r, "\gamma(g)\varphi"'] & W \end{tikzcd}$$

In other words,

$$\gamma(g)(\varphi) = \sigma(g) \circ \varphi \circ \rho(g)^{-1}. \tag{1.33}$$

Now, the LHS of (1.32) is

$$\begin{aligned}
\text{LHS} &= \left[\gamma(g) \circ \widetilde{f}(\widehat{\kappa} \otimes \mathbf{w})\right](\mathbf{v}) \\
&= \left[\sigma(g) \circ \left(\widetilde{f}(\widehat{\kappa} \otimes \mathbf{w})\right) \circ \rho(g)^{-1}\right](\mathbf{v}) \\
&= \sigma(g)\left(\widetilde{f}(\widehat{\kappa} \otimes \mathbf{w})\left(\rho(g)^{-1}\mathbf{v}\right)\right) \\
&= \sigma(g)\left(\widehat{\kappa}\left(\rho(g)^{-1}\mathbf{v}\right) \cdot \mathbf{w}\right) && [\,\cdot \text{ is the scalar multiplication in } W] \\
&= \widehat{\kappa}\left(\rho(g)^{-1}\mathbf{v}\right) \cdot \sigma(g)(\mathbf{w}).
\end{aligned}$$

Therefore, LHS = RHS, so (1.32) holds. As a result, (1.30) commutes, and hence, $\widetilde{f}$ is a G-linear map, as required. ■

§1.4 Complete reducibility

Definition 1.7 (Hermitian inner product)**.** If V is a complex vector space, then a **Hermitian inner product** is a positive definite sesquilinear map $H : V \times V \to \mathbb{C}$ that satisfies the following:

(i) $H(a\mathbf{u} + b\mathbf{v}, \mathbf{w}) = \bar{a}H(\mathbf{u}, \mathbf{w}) + \bar{b}H(\mathbf{v}, \mathbf{w})$ and $H(\mathbf{w}, a\mathbf{u} + b\mathbf{v}) = aH(\mathbf{w}, \mathbf{u}) + bH(\mathbf{w}, \mathbf{v})$ for all $a, b \in \mathbb{C}$, $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$.

(ii) $H(\mathbf{u}, \mathbf{v}) = \overline{H(\mathbf{v}, \mathbf{u})}$, for all $\mathbf{u}, \mathbf{v} \in V$.

(iii) $H(\mathbf{u}, \mathbf{u}) > 0$, for every $\mathbf{u} \in V \setminus \{\mathbf{0}\}$ (positive definite).

If $W \subset V$ is a vector subspace of a complex vector space with a Hermitian inner product, we define the following subspace:

$$W^{\perp} = \{\mathbf{v} \in V \mid H(\mathbf{v}, \mathbf{w}) = 0, \text{ for all } \mathbf{w} \in W\}. \tag{1.34}$$

If V is a finite dimensional complex vector space, then we can write $V = W \oplus W^{\perp}$, i.e. $W^{\perp}$ is the the orthogonal complement of W. We also say that $W^{\perp}$ is the complementary subspace of V.

Definition 1.8. A Hermitian inner product H on a finite dimensional representation V of a finite group G ($\rho : G \to \mathrm{GL}(V)$) is said to be **preserved under group action** if

$$H(\rho(g)\mathbf{u}, \rho(g)\mathbf{w}) = H(\mathbf{u}, \mathbf{w}) \tag{1.35}$$

for all $g \in G$ and $\mathbf{u}, \mathbf{w} \in V$. H is then called a ***G*-invariant** Hermitian inner product.

If H is a G-invariant Hermitian inner product on a finite dimensional representation V of a finite group G, then we have

$$\begin{aligned}
H(\rho(g)\mathbf{v}, \mathbf{w}) &= H\left(\rho(g)\mathbf{v}, \rho(g)\rho\left(g^{-1}\right)\mathbf{w}\right) \\
&= H\left(\mathbf{v}, \rho\left(g^{-1}\right)\mathbf{w}\right).
\end{aligned} \tag{1.36}$$

Lemma 1.2

If $H : V \times V \to \mathbb{C}$ is a G-invariant Hermitian inner product on a finite dimensional representation V of a finite group G and $W \subset V$ is a subrepresentation, then $W^{\perp}$ is a G-invariant complement to W.

Proof. Since we are dealing with finite dimensional complex vector spaces, $W^{\perp}$ is a complement to W. It, therefore, suffices to show that $W^{\perp}$ is G-invariant.

Suppose $g \in G$, $\mathbf{u} \in W^{\perp}$, and $\mathbf{w} \in W$. Let us denote the group homomorphism associated with the finite dimensional complex representation by $\rho : G \to \mathrm{GL}(V)$. Since the Hermitian inner product $H : V \times V \to \mathbb{C}$ is G-invariant, one has

$$H(\rho(g)\mathbf{u}, \mathbf{w}) = H(\mathbf{u}, \rho(g^{-1})\mathbf{w}). \tag{1.37}$$

Since W is a subrepresentation of V, one must have $\rho(g^{-1})\mathbf{w} \in W$ for any $g \in G$ and $\mathbf{w} \in W$. Hence, $H(\mathbf{u}, \rho(g^{-1})\mathbf{w}) = 0$ in (1.37) leads to

$$H(\rho(g)\mathbf{u}, \mathbf{w}) = 0 \tag{1.38}$$

This is true for all $\mathbf{w} \in W$. Therefore, from the definition of $W^{\perp}$, one then must have $\rho(g)\mathbf{u} \in W^{\perp}$ for any $g \in G$, which then implies that the subspace $W^{\perp}$ is G-invariant. ■

Proposition 1.3

If V is a complex representation of a finite group G, then there is a G-invariant Hermitian inner product on V.

Proof. Pick a Hermitian inner product $H_0 : V \times V \to \mathbb{C}$ on the finite dimensional complex vector space V with respect to which a given basis of V is orthonormal, i.e., choose a basis $\{\mathbf{e}_i\}_{i=1}^{m}$ of V and define $H_0(\mathbf{e}_i, \mathbf{e}_j) = \delta_{ij}$ and extend H_0 to all of $V \times V$ sesquilinearly. Given $\mathbf{v} = \sum_{i=1}^{m} v^i \mathbf{e}_i$ and $\mathbf{w} = \sum_{j=1}^{m} w^j \mathbf{e}_j$, we then have

$$H_0(\mathbf{v}, \mathbf{w}) = H_0\left(\sum_{i=1}^{m} v^i \mathbf{e}_i, \sum_{j=1}^{m} w^j \mathbf{e}_j\right) = \sum_{i=1}^{m} \overline{v^i} w^i. \tag{1.39}$$

Then define a new Hermitian inner product $H_1 : V \times V \to \mathbb{C}$ by averaging over all of G via representation $\rho : G \to \mathrm{GL}(V)$:

$$H_1(\mathbf{v}, \mathbf{w}) = \frac{1}{|G|} \sum_{g \in G} H_0(\rho(g)\mathbf{v}, \rho(g)\mathbf{w}). \tag{1.40}$$

Using the Hermitian inner product properties of H_0, one can verify that H_1 is also a Hermitian inner product on V. Additionally,

$$\begin{aligned} H_1(\rho(h)\mathbf{v}, \rho(h)\mathbf{w}) &= \frac{1}{|G|} \sum_{g \in G} H_0(\rho(g)\rho(h)\mathbf{v}, \rho(g)\rho(h)\mathbf{w}) \\ &= \frac{1}{|G|} \sum_{g \in G} H_0(\rho(gh)\mathbf{v}, \rho(gh)\mathbf{w}) \\ &= \frac{1}{|G|} \sum_{g' \in G} H_0(\rho(g')\mathbf{v}, \rho(g')\mathbf{w}) \quad (\text{where } g' = gh) \\ &= H_1(\mathbf{v}, \mathbf{w}). \end{aligned} \tag{1.41}$$

Then (1.41) implies that the Hermitian inner product $H_1 : V \times V \to \mathbb{C}$ defined by (1.40) on V is G-invariant. ■

Corollary 1.4

If W is a subrepresentation of a finite dimensional complex representation V of a finite group G, then there exists a complementary invariant subspace $W^{\perp}$ of V so that $V = W \oplus W^{\perp}$.

Proof. Given that V is a complex representation of a finite group G, there is a G-invariant Hermitian inner product on V by Proposition 1.3. Now, if W is a subrepresentation of V, then by Lemma 1.2, the complementary subspace $W^{\perp}$ is G-invariant, i.e., $V = W \oplus W^{\perp}$. ■

Corollary 1.5 (Maschke's theorem)
Any complex representation of a finite group can be expressed as a direct sum of irreducible representations.

Remark 1.1. The property of a representation being expressed as a direct sum of irreducibles is called complete reducibility (semisimplicity). Maschke's theorem tells us that any complex representation of a finite group is semisimple. The additive group $\mathbb{R}$, being an infinite group, doesn't have this property; for example, the representation

$$a \mapsto \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$$

is not semisimple.

The extent to which the decomposition of an arbitrary complex representation into a direct sum of irreducibles is unique is one of the consequences of the following.

Lemma 1.6 (Schur's lemma)
Recall that $\mathrm{Hom}_G(V, W)$ is the vector space of G-linear maps between two finite dimensional complex representations V and W of the finite group G. Suppose V and W are irreducible complex representations of G. Then

(a) Every element of $\mathrm{Hom}_G(V, W)$ is either 0 or an isomorphism.

(b) $\dim_{\mathbb{C}} \mathrm{Hom}_G(V, W) = 0$ or 1.

Proof. (a) Let $\varphi : V \to W$ be a non-zero G-linear map. We have verified in (1.1) that $\mathrm{Ker}\,\varphi \subseteq V$ is a G-invariant subspace of V. Since V is irreducible, by hypothesis, one has

$$\mathrm{Ker}\,\varphi = \{\mathbf{0}\}\,, \tag{1.42}$$

because $\mathrm{Ker}\,\varphi \neq V$, as φ is chosen to be nonzero.

We also know from (1.2) that $\mathrm{im}\,\varphi \subseteq W$ is a G-invariant subspace of W, i.e., $\mathrm{Im}\varphi$ is a subrepresentation of W. Since W is also irreducible, by hypothesis, one must have

$$\mathrm{im}\,\varphi = W, \tag{1.43}$$

because $\mathrm{im}\,\varphi \neq \{\mathbf{0}\}$ as φ is chosen to be nonzero.

Now, $\mathrm{Ker}\,\varphi = \{\mathbf{0}\}$ and $\mathrm{im}\,\varphi = W$ together imply that $\varphi : V \to W$ is a bijective linear map from V to W, i.e., φ is an isomorphism between vector spaces.

(b) Suppose $\varphi_1, \varphi_2 \in \mathrm{Hom}_G(V, W)$ with both being nonzero. Then by (a), φ_1 and φ_2 are both isomorphisms. Since $\varphi_1^{-1} : W \to V$ and $\varphi_2 : V \to W$, one can compose them to obtain $\varphi = \varphi_1^{-1} \circ \varphi_2 \in \mathrm{Hom}_G(V, V)$.

Now, $\varphi : V \to V$ is a linear operator on the finite dimensional complex vector space V. Also, since $\mathbb{C}$ is algebraically closed, $\det(\varphi - \lambda\,\mathbb{1}_V) = 0$ has a solution (here $\varphi - \lambda\,\mathbb{1}_V$ is considered a square matrix) which implies that $\mathrm{Ker}(\varphi - \lambda\,\mathbb{1}_V) \neq \{\mathbf{0}\}$, i.e., $\varphi - \lambda\,\mathbb{1}_V$ is not an isomorphism belonging to the vector space $\mathrm{Hom}_G(V, V)$. Then, by (a), one concludes that $\varphi - \lambda\,\mathbb{1}_V$ must be the 0-map in $\mathrm{Hom}_G(V, V)$, i.e.,

$$\varphi = \varphi_1^{-1} \circ \varphi_2 = \lambda\,\mathbb{1}_V\,.$$

In other words, $\varphi_2 = \lambda\varphi_1$. Since this is true for any pair of G-linear maps $\varphi_1, \varphi_2 \in \mathrm{Hom}_G(V, W)$, we have $\dim_{\mathbb{C}} \mathrm{Hom}_G(V, W) = 1$.

■

Lemma 1.7

Suppose V_1, V_2, W are finite dimensional complex representation of the finite group G. Then one has the following vector space isomorphisms:

$$\begin{aligned}\mathrm{Hom}_G\left(V_1 \oplus V_2, W\right) &\cong \mathrm{Hom}_G\left(V_1, W\right) \oplus \mathrm{Hom}_G\left(V_2, W\right),\\ \mathrm{Hom}_G\left(W, V_1 \oplus V_2\right) &\cong \mathrm{Hom}_G\left(W, V_1\right) \oplus \mathrm{Hom}_G\left(W, V_2\right).\end{aligned}$$

Proof. Following are the required linear maps that can easily be verified to be isomorphisms:

$$\begin{aligned}s : \mathrm{Hom}_G\left(V_1, W\right) \oplus \mathrm{Hom}_G\left(V_2, W\right) &\to \mathrm{Hom}_G\left(V_1 \oplus V_2, W\right),\\ s\left(\varphi_1, \varphi_2\right)\left(\mathbf{v}_1, \mathbf{v}_2\right) &= \varphi_1\left(\mathbf{v}_1\right) + \varphi_2\left(\mathbf{v}_2\right).\end{aligned} \tag{1.44}$$

$$\begin{aligned}u : \mathrm{Hom}_G\left(V_1 \oplus V_2, W\right) &\to \mathrm{Hom}_G\left(V_1, W\right) \oplus \mathrm{Hom}_G\left(V_2, W\right)\\ u\left(\varphi\right) &= \left(\varphi \circ i_1, \varphi \circ i_2\right),\end{aligned} \tag{1.45}$$

where $i_1 : V_1 \to V_1 \oplus V_2$ and $i_2 : V_2 \to V_1 \oplus V_2$ are the canoncial inclusions defined by

$$i_1\left(\mathbf{v}_1\right) = \left(\mathbf{v}_1, \mathbf{0}_{V_2}\right) \quad \text{and} \quad i_2\left(\mathbf{v}_2\right) = \left(\mathbf{0}_{V_1}, \mathbf{v}_2\right).$$

Now, one can check that $u \circ s = \mathbb{1}_{\mathrm{Hom}_G(V_1,W)\oplus\mathrm{Hom}_G(V_2,W)}$ and $s \circ u = \mathbb{1}_{\mathrm{Hom}_G(V_1\oplus V_2,W)}$. Indeed,

$$\begin{aligned}\left(u \circ s\right)\left(\varphi_1, \varphi_2\right) &= u\left(s\left(\varphi_1, \varphi_2\right)\right)\\ &= \left(s\left(\varphi_1, \varphi_2\right) \circ i_1, s\left(\varphi_1, \varphi_2\right) \circ i_2\right).\end{aligned}$$

Now,

$$\begin{aligned}\left(s\left(\varphi_1, \varphi_2\right) \circ i_1\right)\left(\mathbf{v}_1\right) &= s\left(\varphi_1, \varphi_2\right)\left(i_1\left(\mathbf{v}_1\right)\right)\\ &= s\left(\varphi_1, \varphi_2\right)\left(\mathbf{v}_1, \mathbf{0}_{V_2}\right)\\ &= \varphi_1\left(\mathbf{v}_1\right) + \varphi_2\left(\mathbf{0}_{V_2}\right)\\ &= \varphi_1\left(\mathbf{v}_1\right).\end{aligned}$$

Therefore, $s\left(\varphi_1, \varphi_2\right) \circ i_1 = \varphi_1$. Similarly, $s\left(\varphi_1, \varphi_2\right) \circ i_2 = \varphi_2$. Hence,

$$\left(u \circ s\right)\left(\varphi_1, \varphi_2\right) = \left(s\left(\varphi_1, \varphi_2\right) \circ i_1, s\left(\varphi_1, \varphi_2\right) \circ i_2\right) = \left(\varphi_1, \varphi_2\right).$$

So we have

$$u \circ s = \mathbb{1}_{\mathrm{Hom}_G(V_1,W)\oplus\mathrm{Hom}_G(V_2,W)}. \tag{1.46}$$

On the other hand, given $\varphi \in \mathrm{Hom}_G\left(V_1 \oplus V_2, W\right)$,

$$\begin{aligned}\left[\left(s \circ u\right)\left(\varphi\right)\right]\left(\mathbf{v}_1, \mathbf{v}_2\right) &= \left[s\left(\varphi \circ i_1, \varphi \circ i_2\right)\right]\left(\mathbf{v}_1, \mathbf{v}_2\right)\\ &= \left(\varphi \circ i_1\right)\left(\mathbf{v}_1\right) + \left(\varphi \circ i_2\right)\left(\mathbf{v}_2\right)\\ &= \varphi\left(\mathbf{v}_1, \mathbf{0}_{V_2}\right) + \varphi\left(\mathbf{0}_{V_1}, \mathbf{v}_2\right)\\ &= \varphi\left(\mathbf{v}_1, \mathbf{v}_2\right).\end{aligned}$$

Therefore,

$$s \circ u = \mathbb{1}_{\mathrm{Hom}_G(V_1\oplus V_2,W)}. \tag{1.47}$$

So $s : \mathrm{Hom}_G\left(V_1, W\right) \oplus \mathrm{Hom}_G\left(V_2, W\right) \to \mathrm{Hom}_G\left(V_1 \oplus V_2, W\right)$ is an isomorphism.

Now consider the following linear maps

$$\begin{aligned}t : \mathrm{Hom}_G\left(W, V_1\right) \oplus \mathrm{Hom}_G\left(W, V_2\right) &\to \mathrm{Hom}_G\left(W, V_1 \oplus V_2\right)\\ t\left(\varphi_1, \varphi_2\right)\left(\mathbf{w}\right) &= \left(\varphi_1\left(\mathbf{w}\right), \varphi_2\left(\mathbf{w}\right)\right).\end{aligned} \tag{1.48}$$

$$\begin{aligned} v : \mathrm{Hom}_G(W, V_1 \oplus V_2) &\to \mathrm{Hom}_G(W, V_1) \oplus \mathrm{Hom}_G(W, V_2) \\ v(\varphi) &= (q_1 \circ \varphi, q_2 \circ \varphi), \end{aligned} \tag{1.49}$$

where $q_1 : V_1 \oplus V_2 \to V_1$ and $q_2 : V_1 \oplus V_2 \to V_2$ are the canonical projections, defined by

$$q_1(\mathbf{v}_1, \mathbf{v}_2) = \mathbf{v}_1 \quad \text{and} \quad q_2(\mathbf{v}_1, \mathbf{v}_2) = \mathbf{v}_2.$$

Now, one can check that $v \circ t = \mathbb{1}_{\mathrm{Hom}_G(W,V_1)\oplus \mathrm{Hom}_G(W,V_2)}$ and $t \circ v = \mathbb{1}_{\mathrm{Hom}_G(W,V_1\oplus V_2)}$. Indeed,

$$\begin{aligned} (v \circ t)(\varphi_1, \varphi_2) &= v(t(\varphi_1, \varphi_2)) \\ &= (q_1 \circ t(\varphi_1, \varphi_2), q_2 \circ t(\varphi_1, \varphi_2)). \end{aligned}$$

Now,

$$\begin{aligned} (q_1 \circ t(\varphi_1, \varphi_2))(\mathbf{w}) &= q_1 [t(\varphi_1, \varphi_2)\mathbf{w}] \\ &= q_1(\varphi_1(\mathbf{w}), \varphi_2(\mathbf{w})) \\ &= \varphi_1(\mathbf{w}). \end{aligned}$$

Therefore, $q_1 \circ t(\varphi_1, \varphi_2) = \varphi_1$. Similarly, $q_2 \circ t(\varphi_1, \varphi_2) = \varphi_2$. Hence,

$$(v \circ t)(\varphi_1, \varphi_2) = (q_1 \circ t(\varphi_1, \varphi_2), q_2 \circ t(\varphi_1, \varphi_2)) = (\varphi_1, \varphi_2).$$

So we have

$$v \circ t = \mathbb{1}_{\mathrm{Hom}_G(W,V_1)\oplus \mathrm{Hom}_G(W,V_2)}. \tag{1.50}$$

On the other hand, given $\varphi \in \mathrm{Hom}_G(W, V_1 \oplus V_2)$, let $\varphi(\mathbf{w}) = (\mathbf{v}_1, \mathbf{v}_2)$. Then

$$\begin{aligned} [(t \circ v)(\varphi)](\mathbf{w}) &= t(q_1 \circ \varphi, q_2 \circ \varphi)(\mathbf{w}) \\ &= ((q_1 \circ \varphi)(\mathbf{w}), (q_2 \circ \varphi)(\mathbf{w})) \\ &= (\mathbf{v}_1, \mathbf{v}_2) = \varphi(\mathbf{w}). \end{aligned}$$

Therefore,

$$t \circ v = \mathbb{1}_{\mathrm{Hom}_G(W,V_1\oplus V_2)}. \tag{1.51}$$

So $t : \mathrm{Hom}_G(W, V_1) \oplus \mathrm{Hom}_G(W, V_2) \to \mathrm{Hom}_G(W, V_1 \oplus V_2)$ is an isomorphism. ■

Now, let G be a finite group and V be a finite dimensional complex representation of G. Since V is a direct sum of irreducible representations by Maschke's theorem, up to isomorphism we can group together the isomorphic representations and say that

$$V \cong V_1^{r_1} \oplus \cdots \oplus V_m^{r_m} \tag{1.52}$$

Here $V_i^{r_i}$ is the shorthand for r_i fold direct sum of V_i with itself.

$$V_i^{r_i} = \underbrace{V_i \oplus V_i \oplus \cdots \oplus V_i}_{r_i\text{-fold direct sum}}. \tag{1.53}$$

Here, for distinct i and j, V_i and V_j are non-isomorphic, and the integers $r_i \geq 1$.

Remark 1.2. While grouping together in (1.52), we are grouping isomorphic representations together, NOT isomorphic vector spaces. V_1 and V_2 may be isomorphic as vector spaces, but we don't group them together unless they are isomorphic representations. In other words, if $\rho : G \to \mathrm{GL}(V)$ is the said representation of G into V, we group two irreducible subrepresentations W_1 and W_2 together while writing (1.52) if there exists a vector space isomorphism $\psi : W_1 \to W_2$ such that the following diagram commutes for every $g \in G$:

$$\begin{tikzcd} W_1 \arrow[r, "\psi"] \arrow[d, "\rho(g)\big|_{W_1}"'] & W_2 \arrow[d, "\rho(g)\big|_{W_2}"] \\ W_1 \arrow[r, "\psi"'] & W_2 \end{tikzcd}$$

When we say V_i and V_j are not isomorphic for $i \neq j$ in (1.52), we mean that they are not isomorphic as representations, i.e. there is no isomorphism in $\mathrm{Hom}_G(V_i, V_j)$. In principle, they can be isomorphic as vector spaces, but that's not our concern here.

Proposition 1.8

In (1.52), $r_i = \dim_{\mathbb{C}} \mathrm{Hom}_G(V_i, V) = \dim_{\mathbb{C}} \mathrm{Hom}_G(V, V_i)$.

Proof. By Lemma 1.7,

$$\mathrm{Hom}_G(V_i, V) \cong \mathrm{Hom}_G\left(V_i, \bigoplus_{j=1}^{m} V_j^{r_j}\right) \cong \bigoplus_{j=1}^{m} \mathrm{Hom}_G\left(V_i, V_j^{r_j}\right). \tag{1.54}$$

But $\mathrm{Hom}_G\left(V_i, V_j^{r_j}\right)$ is

$$\mathrm{Hom}_G\left(V_i, V_j^{r_j}\right) = \mathrm{Hom}_G\left(V_i, \underbrace{V_j \oplus \cdots \oplus V_j}_{r_j\text{-fold direct sum}}\right) \cong \underbrace{\mathrm{Hom}_G(V_i, V_j) \oplus \cdots \oplus \mathrm{Hom}_G(V_i, V_j)}_{r_j\text{-fold direct sum}}. \tag{1.55}$$

Since V_i's are pairwise non-isomorphic for $j \neq i$, we have $\mathrm{Hom}_G(V_i, V_j) = \{\mathbf{0}\}$, so that

$$\dim_{\mathbb{C}} \mathrm{Hom}_G(V_i, V_j) = 0 \quad \text{and} \quad \dim_{\mathbb{C}} \mathrm{Hom}_G(V_i, V_j) = 1. \tag{1.56}$$

So we have

$$\begin{aligned} \dim_{\mathbb{C}} \mathrm{Hom}_G(V_i, V) &= \dim_{\mathbb{C}}\left(\bigoplus_{j=1}^{m} \mathrm{Hom}_G\left(V_i, V_j^{r_j}\right)\right) \\ &= \sum_{j=1}^{m} \dim_{\mathbb{C}} \mathrm{Hom}_G\left(V_i, V_j^{r_j}\right) \\ &= \dim_{\mathbb{C}} \mathrm{Hom}_G\left(V_i, V_i^{r_i}\right) \\ &= \dim_{\mathbb{C}}\left(\underbrace{\mathrm{Hom}_G(V_i, V_i) \oplus \cdots \oplus \mathrm{Hom}_G(V_i, V_i)}_{r_i\text{-fold direct sum}}\right) \\ &= \underbrace{1 + 1 + \cdots + 1}_{r_i\text{-fold sum}} \\ &= r_i. \end{aligned} \tag{1.57}$$

Similarly, $\dim_{\mathbb{C}} \mathrm{Hom}_G(V, V_i) = r_i$. ∎

Proposition 1.9

The decomposition (1.52) is unique up to replacement of each V_i by an isomorphic representation.

Proof. Suppose

$$V \cong V_1^{r_1} \oplus \cdots \oplus V_m^{r_m} \cong W_1^{s_1} \oplus \cdots \oplus W_n^{s_n} \tag{1.58}$$

are two decompositions into non-isomorphic irreducible representations of G. By Proposition 1.8, for $i_0 \in \{1, 2, \dots, m\}$,

$$\begin{aligned} r_{i_0} &= \dim_{\mathbb{C}} \operatorname{Hom}_G (V_{i_0}, V) \\ &= \dim_{\mathbb{C}} \operatorname{Hom}_G \left(V_{i_0}, \bigoplus_{j=1}^{n} W_j^{s_j} \right) \\ &= \dim_{\mathbb{C}} \left(\bigoplus_{j=1}^{n} \operatorname{Hom}_G \left(V_{i_0}, W_j^{s_j} \right) \right) \\ &= \sum_{j=1}^{n} s_j \dim \operatorname{Hom}_G (V_{i_0}, W_j). \end{aligned} \tag{1.59}$$

Since $r_{i_0} > 0$, there must exist some $j_0 \in \{1, 2, \dots, n\}$ such that $\operatorname{Hom}_G (V_{i_0}, W_{j_0}) \neq \{\mathbf{0}\}$, i.e. it is nontrivial. Then by Schur's lemma, $W_{j_0} \cong V_{i_0}$. The j_0 must also be unique because W_j's are pairwise non-isomorphic. In other words, the only nonvanishing contribution in the sum (1.59) is due to the unique value $j = j_0$, for which

$$\dim_{\mathbb{C}} \operatorname{Hom}_G (V_{i_0}, W_{j_0}) = 1 \quad \text{and} \quad \dim_{\mathbb{C}} \operatorname{Hom}_G (V_{i_0}, W_j) = 0 \text{ for } j \neq j_0. \tag{1.60}$$

Hence, by (1.59) and (1.60), $r_{i_0} = s_{j_0}$. Thus we have an injection $\sigma : \{1, 2, \dots, m\} \to \{1, 2, \dots, n\}$ such that $V_{i_0} \cong W_{j_0} = W_{\sigma(i_0)}$ and $r_{i_0} = s_{j_0} = s_{\sigma(i_0)}$ for each i_0.

In a similar manner, interchanging V_i and W_j throughout above, we have an injection $\tau : \{1, 2, \dots, n\} \to \{1, 2, \dots, m\}$ such that $W_{j_0} \cong V_{\tau(j_0)}$ and $s_{j_0} = r_{\tau(i_0)}$ for each j_0. The first injection σ implies that $m \leq n$. The latter injection τ gives $n \leq m$. Therefore, $m = n$, and σ and τ are permutations, i.e. $\sigma \in \mathfrak{S}_n$. Hence, (1.52) is unique up to replacement of each V_{i_0} by an isomorphic representation W_{j_0}. ■

Corollary 1.10

The irreducible complex representations of a finite abelian group G are all 1-dimensional.

Proof. Let V be a complex irreducible representation of a finite group G and $\rho : G \to \mathrm{GL}(V)$ be the underlying group homomorphism. Then, for each $g \in G$, the map $\rho(g) : V \to V$ is G-linear:

$$\begin{tikzcd} V \arrow[r, "\rho(g)"] \arrow[d, "\rho(h)"'] & V \arrow[d, "\rho(h)"] \\ V \arrow[r, "\rho(g)"'] & V \end{tikzcd}$$

The diagram above is commutative for all $h \in G$ for a given $g \in G$. Indeed,

$$\rho(g)\rho(h) = \rho(gh) = \rho(hg) = \rho(h)\rho(g).$$

We, therefore, have $\rho(g) \in \operatorname{Hom}_G(V, V)$. By Schur's lemma, $\dim_{\mathbb{C}} \operatorname{Hom}_G(V, V) = 1$, so $\rho(g) = \lambda_g \mathbb{1}_V$ for some $\lambda_g \in \mathbb{C}$.

Now, choose a non-zero vector $\mathbf{v} \in V$ and consider the 1-dimensional subspace

$$\langle \mathbf{v} \rangle = \mathbb{C}\mathbf{v} \subset V,$$

by taking all complex multiples of the nonzero vector $\mathbf{v}$. Observe that $\langle \mathbf{v} \rangle$ is G-invariant. Indeed,

$$\rho(g)\mathbf{v} = \lambda_g \mathbb{1}_V \mathbf{v} = \lambda_g \mathbf{v} \in \langle \mathbf{v} \rangle,$$

i.e. $\langle \mathbf{v} \rangle$ is a G-invariant subspace of V, i.e. a subrepresentation. But V is irreducible by hypothesis. Hence, $\langle \mathbf{v} \rangle = V$. In other words, V is 1-dimensional. ■

Definition 1.9 (Faithful representation). A complex representation V of a finite group G is called **faithful** if the homomorphism $\rho : G \to \mathrm{GL}(V)$ is injective.

Corollary 1.11

If G has a faithful complex irreducible representation, then $Z(G)$ is cyclic.

Proof. Let $\rho : G \to \mathrm{GL}(V)$ be the injective group homomorphism associated with a faithful irreducible complex representation V of a finite group G. Now, let $z \in Z(G)$ so that $zg = gz$ for all $g \in G$. Now consider the map $\rho(z) : V \to V$. Since z commutes with all $g \in G$, the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "\rho(z)"] \arrow[d, "\rho(g)"'] & V \arrow[d, "\rho(g)"] \\ V \arrow[r, "\rho(z)"'] & V \end{tikzcd}$$

Hence, $\rho(z) \in \mathrm{Hom}_G(V, V)$. By Schur's lemma, $\dim_{\mathbb{C}} \mathrm{Hom}_G(V, V) = 1$, so $\rho(z) = \lambda_z \mathbb{1}_V$ for some $\lambda_z \in \mathbb{C}^\times := \mathbb{C} \setminus \{0\}$.

Now, the map $Z(G) \to \mathbb{C}^\times = \mathrm{GL}(\mathbb{C})$ given by $z \mapsto \lambda_z$ is a representation of the subgroup $Z(G)$ of G. Moreover, this representation is faithful, because

$$\begin{aligned} \lambda_z = \lambda_{z'} &\implies \lambda_z \mathbb{1}_V = \lambda_{z'} \mathbb{1}_V \\ &\implies \rho(z) = \rho(z') \\ &\implies z = z', \end{aligned}$$

since ρ is injective. Therefore, the map $Z(G) \to \mathbb{C}^\times = \mathrm{GL}(\mathbb{C})$ given by $z \mapsto \lambda_z$ is injective. So $Z(G)$ is isomorphic to a finite subgroup of $\mathbb{C}^\times$. Finite subgroups of the multiplicative group of a field is a cyclic group. Hence, $Z(G)$ is cyclic. ∎

One also knows from elementary group theory that every finite abelian group is isomorphic to a direct product of cyclic groups. In other words, if G is a finite abelian group, then we can write G as

$$G = C_{n_1} \times \cdots \times C_{n_r}, \tag{1.61}$$

where each C_{n_i} is a cyclic group of order n_i.

Proposition 1.12

A finite abelian group G has precisely $|G|$-many irreducible complex representations.

Proof. We write G as a direct product of cyclic groups as follows:

$$G = \langle x_1 \rangle \times \cdots \times \langle x_r \rangle, \tag{1.62}$$

where $|\langle x_j \rangle| = n_j$, and x_j generates the cyclic group $\langle x_j \rangle$. Suppose $\rho : G \to \mathbb{C}^\times$ is an irreducible representation of the finite abelian group G (which is 1-dimensional by Corollary 1.10). Let

$$\rho(e_1, \ldots, e_{j-1}, x_j, e_{j+1}, \ldots, e_r) = \lambda_j \in \mathbb{C}^\times, \tag{1.63}$$

where e_k's are the identity elements of the cyclic group $C_{n_k} = \langle x_k \rangle$. Since $x_j^{n_j} = e_j$, and since $\rho : G \to \mathbb{C}^\times$ is a group homomorphism, one must have

$$1 = \rho(e_1, \ldots, e_r) = \rho\left(e_1, \ldots, e_{j-1}, x_j^{n_j}, e_{j+1}, \ldots, e_r\right) = \lambda_j^{n_j}. \tag{1.64}$$

Then $\lambda_j^{n_j} = 1$ gives us that λ_j is a n_j-th root of unity. Also, observe that

$$\rho\left(x_1^{j_1}, \ldots, x_r^{j_r}\right) = \lambda_1^{j_1} \cdots \lambda_r^{j_r}, \tag{1.65}$$

for $1 \leq j_k \leq n_k$ for each k. Thus, the r-tuple $(\lambda_1, \ldots, \lambda_r)$ completely determines the homomorphism $\rho : G \to \mathbb{C}^\times$. There are n_j many n_j-th root of unity, so there are n_j many choices for λ_j. Hence, there are total $n_1 \cdots n_r$ many choices for the r-tuple $(\lambda_1, \ldots, \lambda_r)$. Therefore, there are $n_1 \cdots n_r$ many irreducible representations $\rho : G \to \mathbb{C}^\times$. But

$$|G| = |\langle x_1 \rangle \times \cdots \times \langle x_r \rangle| = \prod_{j=1}^{r} |\langle x_j \rangle| = \prod_{j=1}^{r} n_j. \tag{1.66}$$

Hence, there are $|G|$ many irreducible complex representation of the finite abelian group G. ■

Example 1.1 (Example of finite abelian group representations). (i) Consider the finite abelian group $G = C_2 \times C_2 = \langle x_1 \rangle \times \langle x_2 \rangle$, with $x_1^2 = e_1$ and $x_2^2 = e_2$.[1]

We are concerned with the 2nd roots of unity, namely 1 and -1. There are 4 possible choices for (λ_1, λ_2), they are $(1,1), (1,-1), (-1,1), (-1,-1)$. Corresponding to these 4 choices, there are 4 irreducible representations $\rho_1, \rho_2, \rho_3, \rho_4$. The way these 4 irreducible representations map is illustrates in the following table:

(λ_1, λ_2)	(e_1, e_2)	(x_1, e_2)	(e_1, x_2)	(x_1, x_2)
$\rho_1 \equiv (1,1)$	1	1	1	1
$\rho_2 \equiv (1,-1)$	1	1	-1	-1
$\rho_3 \equiv (-1,1)$	1	-1	1	-1
$\rho_4 \equiv (-1,-1)$	1	-1	-1	1

From this table, we can see that there is no irreducible faithful representation of G.

(ii) Now consider the cyclic group $G = C_4 = \langle x \rangle$. This group has 4 elements: e, x, x^2, x^3, and $x^4 = e$. There are 4 roots of unity, namely $1, -1, i, -i$. Corresponding to these 4 roots of unity, there are 4 irreducible representations $\rho_1, \rho_2, \rho_3, \rho_4$. The way these 4 irreducible representations map is illustrates in the following table:

λ	e	x	x^2	x^3
$\rho_1 \equiv 1$	1	1	1	1
$\rho_2 \equiv -1$	1	-1	1	-1
$\rho_3 \equiv i$	1	i	-1	$-i$
$\rho_4 \equiv -i$	1	$-i$	-1	i

From the table, we can see that ρ_3 and ρ_4 are faithful.

[1]This is the Klein four-group. Geometrically, it represents the group of all symmetries of a non-square rectangle.

2 Character Theory

§2.1 Characters

Definition 2.1. Let V be a finite dimensional complex representation of a finite group G and $\rho : G : \mathrm{GL}(V)$ be the corresponding group homomorphism. Then the **character** χ_ρ of the representation V is the function $\chi_\rho : G \to \mathbb{C}$ defined by

$$\chi_\rho(g) = \operatorname{Tr} \rho(g). \tag{2.1}$$

The right side of (2.1) is unambiguous. In fact, $\rho(g) \in \mathrm{GL}(V)$ is a invertible linear transformation on the finite dimensional vector space V. In different bases of V, $\rho(g)$ can be represented by different $n \times n$ complex matrices if the dimension of V is n. But $\operatorname{Tr} \rho(g)$ will be the same for all these matrices following from the invariance of trace under conjugation: denote the $n \times n$ complex matrix $[\rho(g)]_{\mathcal{B}}$ representing the invertible linear transformation $\rho(g) \in \mathrm{GL}(V)$ in the basis $\mathcal{B}$ of the finite dimensional complex vector space V. Also, let $[\rho(g)]_{\mathcal{B}'}$ be the matrix representation of $\rho(g) \in \mathrm{GL}(V)$ with respect to the basis $\mathcal{B}'$ of V. We know from basic linear algebra that there exists an invertible $n \times n$ complex matrix T such that

$$[\rho(g)]_{\mathcal{B}'} = T^{-1}[\rho(g)]_{\mathcal{B}}T. \tag{2.2}$$

The cyclicity of trace (i.e. $\operatorname{Tr}(ABC) = \operatorname{Tr}(CAB) = \operatorname{Tr}(BCA)$) then guarantees

$$\operatorname{Tr}[\rho(g)]_{\mathcal{B}'} = \operatorname{Tr}[\rho(g)]_{\mathcal{B}}. \tag{2.3}$$

The basis independent complex number given by (2.3) is precisely the right side of (2.1), namely $\operatorname{Tr} \rho(g)$.

Remark 2.1. In general, not every invertible linear map has an eigenbasis, i.e. not every linear map is diagonalizable. But the situation is much simpler when we are dealing with representations of finite groups. Since $|G|$ is finite, $g^{|G|} = e$, for every $g \in G$. Therefore,

$$\rho(g)^{|G|} = \rho(e) = \mathbb{1}_V, \tag{2.4}$$

i.e. $\rho(g)$ is of finite order. Linear maps that are of finite order are diagonalizable, because of the following theorem from linear algebra:

> A linear map is diagonalizable if and only if its minimal polynomial doesn't have repeated roots.

Since $\rho(g)$ satisfies $\rho(g)^{|G|} - \mathbb{1}_V = 0$, it is the zero of the polynomial $x^{|G|} - 1$. Therefore, the minimal polynomial of $\rho(g)$ divides $x^{|G|} - 1$. But the roots of $x^{|G|} - 1$ are the $|G|$-th roots of unity. In particular, the roots of $x^{|G|} - 1$ are all distinct. Therfore, the minimal polynomial of $\rho(g)$ can't have repeated roots. As a result, we can pick a basis of V using eigenvectors of $\rho(g)$. In this basis, the trace of $\rho(g)$ is the sum of its eigenvalues. So we can write

$$\chi_\rho(g) = \sum_{\lambda \text{ eigenvalues of } \rho(g)} \lambda. \tag{2.5}$$

Furthermore, the roots of the minimal polynomial of $\rho(g)$ are also roots of $x^{|G|} - 1$, which are the $|G|$-th roots of unity. So the eigenvalues of $\rho(g)$ have modulus 1.

Remark 2.2. Note that ther character $\chi_\rho : G \to \mathbb{C}$ of the representation $\rho : G \to \mathrm{GL}(V)$ is constant on the conjugacy classes of G. In other words,

$$\chi_\rho\left(h^{-1}gh\right) = \chi_\rho(g) \tag{2.6}$$

for every $h \in G$. Also,

$$\chi_\rho(e) = \operatorname{Tr}\rho(e) = \operatorname{Tr}\mathbb{1}_V = \dim V, \tag{2.7}$$

where $e \in G$ is the identity element.

Proposition 2.1

Let V and W be representations of G with $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$ being the respective group homomorphisms. Then

(a) $\chi_{\rho\oplus\sigma} = \chi_\rho + \chi_\sigma$;

(b) $\chi_{\rho\otimes\sigma} = \chi_\rho \cdot \chi_\sigma$;

(c) $\chi_{\rho^*}(g) = \overline{\chi_\rho(g)}$ for every $g \in G$;

(d) $\chi_{\Lambda^2\rho}(g) = \frac{1}{2}\left[(\chi_\rho(g))^2 - \chi_\rho\left(g^2\right)\right]$ for every $g \in G$.

Proof. (a) Suppose $n = \dim V$ and $m = \dim W$. Recall that $\rho\oplus\sigma : G \to \mathrm{GL}\left(V \oplus W\right)$ is defined as $\left(\rho\oplus\sigma\right)g\left(\mathbf{v},\mathbf{w}\right) = \left(\rho(g)\mathbf{v}, \sigma(g)\mathbf{w}\right)$, for $\mathbf{v} \in V$ and $\mathbf{w} \in W$. Let $\mathcal{B}_1$ be a basis for V and $\mathcal{B}_2$ be a basis for W so that $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ is a basis for $V \oplus W$.

Now, $\rho(g) \in \mathrm{GL}(V)$ can be represented by the $n\times n$ complex matrix $[\rho(g)]_{\mathcal{B}_1}$, and $\sigma(g) \in \mathrm{GL}(W)$ can be represented by the $m \times m$ complex matrix $[\sigma(g)]_{\mathcal{B}_2}$. Then $\left(\rho\oplus\sigma\right)g \in \mathrm{GL}\left(V\oplus W\right)$ can be represented by an $(m+n)\times(m+n)$ complex matrix

$$\left[\left(\rho\oplus\sigma\right)g\right]_{\mathcal{B}} = \begin{bmatrix} [\rho(g)]_{\mathcal{B}_1} & 0_{n\times m} \\ 0_{m\times n} & [\sigma(g)]_{\mathcal{B}_2} \end{bmatrix}. \tag{2.8}$$

From (2.8), it follows that

$$\chi_{\rho\oplus\sigma}(g) = \operatorname{Tr}\left[\left(\rho\oplus\sigma\right)g\right]_{\mathcal{B}} = \operatorname{Tr}\left[\rho(g)\right]_{\mathcal{B}_1} + \operatorname{Tr}\left[\sigma(g)\right]_{\mathcal{B}_2} = \chi_\rho(g) + \chi_\sigma(g). \tag{2.9}$$

(b) Recall that $\left(\rho\otimes\sigma\right)g\left(\mathbf{v}\otimes\mathbf{w}\right) = \rho(g)\mathbf{v}\otimes\sigma(g)\mathbf{w}$, for $\mathbf{v} \in V$ and $\mathbf{w} \in W$. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be an eigenbasis of V with respect to $\rho(g) \in \mathrm{GL}(V)$ and $\{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ be an eigenbasis of W with respect to $\sigma(g) \in \mathrm{GL}(W)$. Then

$$\rho(g)\mathbf{v}_i = \lambda_i\mathbf{v}_i \quad\text{and}\quad \sigma(g)\mathbf{w}_j = \mu_j\mathbf{w}_j, \tag{2.10}$$

for $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Then

$$\left(\rho\otimes\sigma\right)g\left(\mathbf{v}_i\otimes\mathbf{w}_j\right) = \rho(g)\mathbf{v}_i\otimes\sigma(g)\mathbf{w}_j = \lambda_i\mathbf{v}_i\otimes\mu_j\mathbf{w}_j = \lambda_i\mu_j\,\mathbf{v}_i\otimes\mathbf{w}_j. \tag{2.11}$$

Therefore, $\mathbf{v}_i\otimes\mathbf{w}_j$ is an eigenvector of $\left(\rho\otimes\sigma\right)g$ with the eigenvalue $\lambda_i\mu_j$. We, therefore, see that $\{\mathbf{v}_i\otimes\mathbf{w}_j \mid i = 1, \ldots, n; j = 1, \ldots, m\}$ forms an eigenbasis of $V\otimes W$. Therefore,

$$\begin{aligned} \chi_{\rho\otimes\sigma}(g) &= \sum_{\lambda \text{ eigenvalues of } (\rho\otimes\sigma)g} \lambda \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m}\lambda_i\mu_j \\ &= \sum_{i=1}^{n}\lambda_i\sum_{j=1}^{m}\mu_j \\ &= \chi_\rho\left(g\right)\cdot\chi_\sigma\left(g\right). \end{aligned} \tag{2.12}$$

(c) Recall that $\rho^* : G \to \mathrm{GL}(V^*)$ is defined by $(\rho^*(g)\,\widehat{\omega})(\mathbf{v}) = \widehat{\omega}\left(\rho\left(g^{-1}\right)\mathbf{v}\right)$, for $\widehat{\omega} \in V^*$ and $\mathbf{v} \in V$. The relevant eigenvalue equations are $\rho(g)\mathbf{v}_i = \lambda_i \mathbf{v}_i$.

Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be an eigenbasis of V with respect to $\rho(g) \in \mathrm{GL}(V)$, and let $\{\widehat{\alpha}^1, \ldots, \widehat{\alpha}^n\}$ be the associated dual basis of V^*. Then

$$\begin{aligned}\left(\rho^*(g)\,\widehat{\alpha}^j\right)(\mathbf{v}_i) &= \widehat{\alpha}^j\left(\rho\left(g^{-1}\right)\mathbf{v}_i\right)\\ &= \widehat{\alpha}^j\left(\frac{1}{\lambda_i}\mathbf{v}_i\right)\\ &= \frac{1}{\lambda_j}\,\widehat{\alpha}^j(\mathbf{v}_i),\end{aligned}$$

since $\widehat{\alpha}^j(\mathbf{v}_i) = \delta^j{}_i$. In other words,

$$\rho^*(g)\,\widehat{\alpha}^j = \frac{1}{\lambda_j}\,\widehat{\alpha}^j. \tag{2.13}$$

So $\{\widehat{\alpha}^1, \ldots, \widehat{\alpha}^n\}$ is an eigenbasis of V^* with respect to $\rho^*(g) \in \mathrm{GL}(V^*)$. The eigenvalues are $\frac{1}{\lambda_j}$. By Remark 2.1, $|\lambda_j| = 1$, so $\frac{1}{\lambda_j} = \overline{\lambda_j}$. So we have

$$\chi_{\rho^*}(g) = \sum_{j=1}^{n} \overline{\lambda_j} = \overline{\sum_{j=1}^{n} \lambda_j} = \overline{\chi_\rho(g)}. \tag{2.14}$$

(d) Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be an eigenbasis of V with respect to $\rho(g) \in \mathrm{GL}(V)$. The relevant eigenvalue equations are $\rho(g)\mathbf{v}_i = \lambda_i \mathbf{v}_i$, for $i = 1, \ldots, n$. Then for $1 \leq i < j \leq n$,

$$\Lambda^2\rho(g)(\mathbf{v}_i \wedge \mathbf{v}_j) = \rho(g)\mathbf{v}_i \wedge \rho(g)\mathbf{v}_j = \lambda_i\mathbf{v}_i \wedge \lambda_i\mathbf{v}_j = \lambda_i\lambda_j\,\mathbf{v}_i \wedge \mathbf{v}_j. \tag{2.15}$$

So $\{\mathbf{v}_i \wedge \mathbf{v}_j\}_{1 \leq i < j \leq n}$ forms an eigenbasis of $\Lambda^2 V$ with respect to $\Lambda^2\rho(g)$. Therefore,

$$\chi_{\Lambda^2\rho}(g) = \sum_{1 \leq i < j \leq n} \lambda_i\lambda_j. \tag{2.16}$$

Now, the eigenvalues of $\rho\left(g^2\right)$ are λ_i^2.

$$(\chi_\rho(g))^2 - \chi_\rho\left(g^2\right) = \left(\sum_{i=1}^{n} \lambda_i\right)^2 - \sum_{i=1}^{n} \lambda_i^2 = 2\sum_{1 \leq i < j \leq n} \lambda_i\lambda_j. \tag{2.17}$$

Therefore,

$$\chi_{\Lambda^2\rho}(g) = \frac{1}{2}\left[(\chi_\rho(g))^2 - \chi_\rho\left(g^2\right)\right]. \tag{2.18}$$

■

Remark 2.3. One can similarly compute the character of the second symmetric power of a given representation, namely

$$\chi_{\mathrm{Sym}^2\rho}(g) = \frac{1}{2}\left[(\chi_\rho(g))^2 + \chi_\rho\left(g^2\right)\right]. \tag{2.19}$$

Indeed, $V^{\otimes 2} \cong \Lambda^2 V \oplus \mathrm{Sym}^2 V$, and $\rho \otimes \rho = \Lambda^2\rho \oplus \mathrm{Sym}^2\rho$ so that we have

$$\chi_{\rho\otimes\rho} = \chi_{\Lambda^2\rho} + \chi_{\mathrm{Sym}^2\rho}. \tag{2.20}$$

For any $g \in G$, we then compute

$$\begin{aligned}\chi_{\mathrm{Sym}^2\rho}(g) &= \chi_{\rho\otimes\rho}(g) - \chi_{\Lambda^2\rho}(g)\\ &= \chi_\rho(g)\,\chi_\rho(g) - \chi_{\Lambda^2\rho}(g)\\ &= \chi_\rho(g)^2 - \frac{1}{2}\left[(\chi_\rho(g))^2 - \chi_\rho\left(g^2\right)\right]\\ &= \frac{1}{2}\left[(\chi_\rho(g))^2 + \chi_\rho\left(g^2\right)\right].\end{aligned} \tag{2.21}$$

§2.2 Permutation representation and regular representation

Let X be a finite set and $\sigma : G \to \mathrm{Aut}(X)$ is a group homomorphism from the finite group G to the premutation group of X. That is, given $g \in G$ and $x \in X$, $\sigma(g) : X \to X$ is a bijection, so that $\sigma(g)x \in X$. In other words, $\sigma(g)$ permutes the elements of X.

Now, construct the $|X|$-dimensional complex vector space V as follows: V is the vector space with basis $\{e_x \mid x \in X\}$. Now, define the representation $\rho : G \to \mathrm{GL}(V)$ by

$$\rho(g)\left(\sum_{x\in X} a_x e_x\right) = \sum_{x\in X} a_x e_{\sigma(g)x}, \tag{2.22}$$

with $a_x \in \mathbb{C}$. The representation of G on the complex vector space V constructed above is called the **permutation representation**.

Lemma 2.2

If V is the permutation representation associated with the action of a group G on a finite set X, where $\rho : G \to \mathrm{GL}(V)$ is the corresponding group homomorphism, then $\chi_\rho(g)$ is the number of elements of X fixed by g.

Proof. We need to show that $\chi_\rho(g)$ is the number of elements of X fixed by $\sigma(g)$. Suppose we have enumerated the elements of X:

$$X = \{x_1, x_2, \dots, x_n\}\,. \tag{2.23}$$

Then the n-dimensional vector space V has an ordered basis:

$$\mathcal{B} = \{e_{x_1}, e_{x_2}, \dots, e_{x_n}\}\,. \tag{2.24}$$

Now let's consider the $n \times n$ matrix representation of $\rho(g)$ in the basis $\mathcal{B}$. Suppose $[\rho(g)]_{\mathcal{B}} = [A_{ij}]_{i,j=1}^n$. We claim that

$$A_{ii} = \begin{cases} 1 & \text{if } \sigma\left(x_i\right) = x_i, \\ 0 & \text{otherwise.} \end{cases} \tag{2.25}$$

The i-th column of $[A_{ij}]_{i,j=1}^n$ looks like $\begin{bmatrix} A_{1i} \\ A_{2i} \\ \vdots \\ A_{ni} \end{bmatrix}$. It signifies that the coordinate of $\rho\left(g\right)\left(e_{x_i}\right)$ in the aforementioned basis is $\begin{bmatrix} A_{1i} \\ A_{2i} \\ \vdots \\ A_{ni} \end{bmatrix}$. In other words,

$$\rho\left(g\right)\left(e_{x_i}\right) = \sum_{j=1}^n A_{ji} e_{x_j}. \tag{2.26}$$

But $\rho\left(g\right)\left(e_{x_i}\right) = e_{\sigma(g)(x_i)}$. So we have

$$\rho\left(g\right)\left(e_{x_i}\right) = \sum_{j=1}^n A_{ji} e_{x_j} = e_{\sigma(g)(x_i)}. \tag{2.27}$$

Since every vector in a vector space can be **uniquely** written as a linear combination of the basis vecctors, we can conclude from (2.27) that

$$A_{ji} = \begin{cases} 1 & \text{if } x_j = \sigma(x_i), \\ 0 & \text{otherwise.} \end{cases} \tag{2.28}$$

Hence,

$$A_{ii} = \begin{cases} 1 & \text{if } \sigma(x_i) = x_i, \\ 0 & \text{otherwise;} \end{cases} \tag{2.29}$$

and our claim is proved. Therefore,

$$\chi_\rho(g) = \operatorname{Tr} \rho(g) = \operatorname{Tr} [\rho(g)]_{\mathcal{B}} = \sum_{i=1}^{n} A_{ii}. \tag{2.30}$$

We have shown that $A_{ii} = 1$ if and only if $\sigma(g)$ fixes x_i, and $A_{ii} = 0$ otherwise. Therefore, $\sum_{i=1}^{n} A_{ii}$ is equal to the number of x_i's such that $\sigma(g)$ fixes x_i. So

$$\chi_\rho(g) = \sum_{i=1}^{n} A_{ii} = |\{x \in X \mid \sigma(g) \text{ fixes } x\}|. \tag{2.31}$$

■

There is another important representation called the **regular representation** of a given finite group G, which is actually a special case of permutation representation. In this case, $X = G_{\mathbf{Set}}$, the underlying set of the finite group G, and $\sigma : G \to \operatorname{Aut}(G_{\mathbf{Set}}) \cong \mathfrak{S}_n$, where $n = |G|$. Here $\operatorname{Aut}(G_{\mathbf{Set}})$ is the group of all bijections from the set $G_{\mathbf{Set}}$ to itself. Since $|G| = n$, there is a bijection from G to $\{1, 2, \ldots, n\}$. So we can actually identify $\operatorname{Aut}(G_{\mathbf{Set}})$ to $\mathfrak{S}_n$.

Take $V = \mathbb{C}[G]$, the group algebra corresponding to the finite group G. An element $x \in \mathbb{C}[G]$ is a complex valued function on the finite set G. $\mathbb{C}[G]$ is easily seen to be a complex vector space with basis $\{\delta_g \mid g \in G\}$, where $\delta_g : G \to \mathbb{C}$ is defined by

$$\delta_g(h) = \begin{cases} 1 & \text{if } h = g, \\ 0 & \text{if } h \neq g. \end{cases} \tag{2.32}$$

A generic element $f \in \mathbb{C}[G]$ can be represented as

$$\alpha = \sum_{g \in G} a_g \delta_g, \tag{2.33}$$

with $a_g \in \mathbb{C}$ is the value α takes at $g \in G$, i.e. $a_g = \alpha(g)$. We don't talk about the algebra structure of $\mathbb{C}[G]$ at the moment. All we need here is the vector space structure of $\mathbb{C}[G]$. With these given data, the regular representation of the finite group G is the associated permutation representation. If $\rho : G \to \operatorname{GL}(\mathbb{C}[G])$ is the representation, then for a given $h \in G$, $\rho(h) : \mathbb{C}[G] \to \mathbb{C}[G]$ is a linear map, and $\rho(h)\left(\sum_{g \in G} a_g \delta_g\right)$ is a function from G to $\mathbb{C}$. This is defined as follows: given $k \in G$,

$$\begin{aligned} \rho(h)\left(\sum_{g \in G} a_g \delta_g\right)(k) &= \sum_{g \in G} a_g \delta_{\sigma(h)g}(k) \\ &= a_g \qquad \text{such that } \sigma(h)g = k \\ &= a_{\sigma(h^{-1})k} \\ &= \sum_{g \in G} a_g \delta_g\left(\sigma\left(h^{-1}\right)k\right). \end{aligned} \tag{2.34}$$

If we denote $\sum_{g \in G} a_g \delta_g$ by α, then we can rewrite (2.34) as

$$(\rho(h)\alpha)(k) = \alpha\left(\sigma\left(h^{-1}\right)k\right). \tag{2.35}$$

For the **left-regular representation**, we define the homomorphism $\sigma : G \to \operatorname{Aut}(G)$ as

$$\sigma(g)(h) = gh. \tag{2.36}$$

In this case, (2.35) reads

$$(\rho(h)\,\alpha)(k) = \alpha\left(h^{-1}k\right). \tag{2.37}$$

In a similar manner, we can also define the **right-regular representation**, where $\sigma : G \to \mathrm{Aut}(G)$ is defined as

$$\sigma(g)(h) = hg^{-1}. \tag{2.38}$$

In this case, (2.35) reads

$$(\rho(h)\,\alpha)(k) = \alpha(kh). \tag{2.39}$$

§2.3 An example of $\mathfrak{S}_3$

Consider $G = \mathfrak{S}_3$. It has 6 elements, $1, (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)$. There are 3 conjugacy classes:

$$\{1\}, \quad \{(1\ 2), (1\ 3), (2\ 3)\}, \quad \{(1\ 2\ 3), (1\ 3\ 2)\}. \tag{2.40}$$

Here, $G = \mathrm{Aut}(X)$ with $X = \{1,2,3\}$. Consider V to be the vector space of all complex valued functions on X. It is isomorphic to $\mathbb{C}^3$, and the basis we choose for V is $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

Here $\mathbf{e}_x$ can be seen as a complex valued function on $X = \{1,2,3\}$, i.e. $\mathbf{e}_x : X \to \mathbb{C}$, defined by

$$\mathbf{e}_x(y) = \begin{cases} 1 & \text{if } y = x, \\ 0 & \text{if } y \neq x. \end{cases} \tag{2.41}$$

So the linear combination $\sum_{x\in X} a_x \mathbf{e}_x$ is also seen as a complex valued function on X. Now, (2.22) reads

$$\rho(g)\left(\sum_{x\in X} a_x \mathbf{e}_x\right) = \sum_{x\in X} a_x \mathbf{e}_{\sigma(g)x}, \tag{2.42}$$

so that for $y \in X$,

$$\begin{aligned} \rho(g)\left(\sum_{x\in X} a_x \mathbf{e}_x\right)(y) &= \sum_{x\in X} a_x \mathbf{e}_{\sigma(g)x}(y) \\ &= a_x \qquad \text{such that } \sigma(g)x = y \\ &= a_{\sigma(g^{-1})y} \\ &= \left(\sum_{x\in X} a_{\sigma(g^{-1})x} \mathbf{e}_x\right)(y). \end{aligned}$$

Therefore,

$$\rho(g)\left(\sum_{x\in X} a_x \mathbf{e}_x\right) = \sum_{x\in X} a_{\sigma(g^{-1})x} \mathbf{e}_x. \tag{2.43}$$

We can identify a complex valued function on $X = \{1,2,3\}$ by the column vector $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \mathbb{C}^3$, and the action of $g \in \mathfrak{S}_3$ on this triple is realized as

$$\rho(g)\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_{\sigma(g^{-1})1} \\ a_{\sigma(g^{-1})2} \\ a_{\sigma(g^{-1})3} \end{bmatrix}. \tag{2.44}$$

For $g = (1\ 2\ 3)$, $g^{-1} = (1\ 3\ 2)$.

$$\rho((1\ 2\ 3))\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_{\sigma((1\ 3\ 2))1} \\ a_{\sigma((1\ 3\ 2))2} \\ a_{\sigma((1\ 3\ 2))3} \end{bmatrix} = \begin{bmatrix} a_3 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}. \tag{2.45}$$

Therefore, $\chi_\rho((1\ 2\ 3)) = 0$. Similarly, for $g = (1\ 2)$, $g^{-1} = (1\ 2)$.

$$\rho((1\ 2)) \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_{\sigma((1\ 2))1} \\ a_{\sigma((1\ 2))2} \\ a_{\sigma((1\ 2))3} \end{bmatrix} = \begin{bmatrix} a_2 \\ a_1 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}. \tag{2.46}$$

So $\chi_\rho((1\ 2)) = 1$. Finally,

$$\rho(1) \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \tag{2.47}$$

So $\chi_\rho(1) = 3$.

The permutation representation $\mathbb{C}^3$ associated with the group homomorphism $\rho : \mathfrak{S}_3 \to \mathrm{GL}(\mathbb{C}^3)$ that we studied above is not irreducible. If we take the subspace

$$\left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \mathbb{C}^3 \mid a_1 = a_2 = a_3 \right\},$$

which is a 1-dimensional subspace of $\mathbb{C}^3$, it is invariant under the action of the permutation group as all the coefficients a_1, a_2, a_3 are the same. This 1-dimensional subspace of $\mathbb{C}^3$ is spanned by $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. T1he complementary subspace of this one-dimensional subspace is given by the set

$$V = \left\{ \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \in \mathbb{C}^3 \mid z_1 + z_2 + z_3 = 0 \right\}.$$

This is a 2-dimensional vector subspace of $\mathbb{C}^3$ that is also left invariant under the action of the permutation group by Corollary 1.4. One can verify that the subrepresentations mentioned above are irreducible representations of $\mathfrak{S}_3$. The 2-dimensional irreducible representation of $\mathfrak{S}_3$ is called the **standard representation** of $\mathfrak{S}_3$.

Let us denote the group homomorphism associated with the standard representation V of $\mathfrak{S}_3$ by ρ_V. Observe that $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ form a basis $\mathcal{B}_V$ for the 2-dimensional subspace V of $\mathbb{C}^3$. Since

$$\rho_V(1\ 2\ 3) \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \rho_V(1\ 2\ 3) \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \tag{2.48}$$

one has the matrix representation of $\rho_V(1\ 2\ 3)$ in the above basis as

$$[\rho_V((1\ 2\ 3))]_{\mathcal{B}_V} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}. \tag{2.49}$$

Similarly,

$$[\rho_V((1\ 2))]_{\mathcal{B}_V} = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}, \text{ and } [\rho_V(1)]_{\mathcal{B}_V} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{2.50}$$

so that

$$\chi_{\rho_V}((1\ 2\ 3)) = -1,\ \chi_{\rho_V}((1\ 2)) = 0,\ \chi_{\rho_V}(1) = 2. \tag{2.51}$$

Recall that an element of $\mathfrak{S}_3$ is even or odd if it can be written as a product of an even or odd number of transpositions. The sign of an element of $\mathfrak{S}_3$ is 1 if it is even and is -1 if it is odd. For example,

$\mathrm{sgn}((1\ 2\ 3)) = 1$ as $(1\ 2\ 3) = (1\ 2)(1\ 3)$. Now, the alternating representation $\sigma' : \mathfrak{S}_3 \to \mathrm{GL}(\mathbb{C}) \cong \mathbb{C}^\times$ is given by

$$\sigma'(g)v = \mathrm{sgn}(g)v, \tag{2.52}$$

for $g \in \mathfrak{S}_3$ and $v \in \mathbb{C}$. This is indeed a representation as

$$\sigma'\left(g'\right)\left(\sigma'\left(g\right)v\right) = \sigma'\left(g'\right)\left(\mathrm{sgn}(g)v\right) = \mathrm{sgn}(g')\mathrm{sgn}(g)v = \mathrm{sgn}(g'g)v = \sigma'(g'g)v.$$

Explicitly, considering $\mathrm{GL}(\mathbb{C}) \cong \mathbb{C}^\times$,

$$\sigma'(1) = 1, \quad \sigma'((1\ 2)) = \sigma'((1\ 3)) = \sigma'((2\ 3)) = -1, \quad \sigma'((1\ 2\ 3)) = \sigma'((1\ 3\ 2)) = 1.$$

And the character of the alternating representation is given by

$$\chi_{\sigma'}(1) = 1, \quad \chi_{\sigma'}((1\ 2)) = -1, \quad \chi_{\sigma'}((1\ 2\ 3)) = 1. \tag{2.53}$$

The alternating representation is a 1-dimensional (irreducible) representation of $\mathfrak{S}_3$. And there is this trivial 1-dimensional representation of $\mathfrak{S}_3$, $\sigma : \mathfrak{S}_3 \to \mathrm{GL}(\mathbb{C}) \cong \mathbb{C}^\times$ given by

$$\sigma(g) = 1, \quad \forall\, g \in \mathfrak{S}_3. \tag{2.54}$$

Then the character is given by

$$\chi_\sigma(1) = 1, \quad \chi_\sigma((1\ 2)) = 1, \quad \chi_\sigma((1\ 2\ 3)) = 1. \tag{2.55}$$

Now, take an arbitrary representation W of $\mathfrak{S}_3$ whose associated homomorphism is given by $\rho_W : \mathfrak{S}_3 \to \mathrm{GL}(W)$. Now, $\mathfrak{S}_3$ has an abelian subgroup of order 3, that is generated by a 3-cycle, say $(1\ 2\ 3)$. This finite abelian group is isomorphic to $\mathbb{Z}_3$. Let us denote this finite abelian subgroup by $\mathfrak{A}_3$. Let us denote by g_1 one of the two 3-cycles that generate $\mathfrak{A}_3$, i.e. $\mathfrak{A}_3 = \langle g_1 \rangle$. Then W is also a representation of $\mathfrak{A}_3$.

The complex vector space W has an eigenbasis with respect to $\rho(g_1) \in \mathrm{GL}(V)$. By Remark 2.1, the eiganvalues are cubic roots of unity, namely $1, \omega, \omega^2$. Then we write the respective eigenvalue equations as

$$\rho\left(g_1\right)\mathbf{v}_i = \omega^{\alpha_i}\mathbf{v}_i, \tag{2.56}$$

with $\{\mathbf{v}_i\}_{i=1}^n$ being the eigenbasis. Thus the representation W of $\mathfrak{A}_3$ is decomoposed into one dimensional complex vector spaces:

$$W = \bigoplus_{i=1}^n V_i,$$

where $V_i = \mathbb{C}\mathbf{v}_i$. This decomposition only refers to 3 elements: $g_1 = (1\ 2\ 3), g_1^2 = (1\ 3\ 2), g_1^3 = e$ of $\mathfrak{S}_3$. How does the decomposition (2.3) respond to when the rest of the elements of $\mathfrak{S}_3$ are considered? Choose a transposition, say $(1\ 2)$ of $\mathfrak{S}_3$ and denote it by g_2. Observe that $g_2 = (1\ 2)$ and $g_1 = (1\ 2\ 3)$ generate the whole of $\mathfrak{S}_3$. Indeed, one has $g_2 g_1 g_2 = g_1^2$, i.e. $g_1 g_2 = g_2 g_1^2$, since $g_2 = g_2^{-1}$. We are trying to find proper $\mathfrak{S}_3$-invariant subspace that can't be further decomposed. Now, for $\mathbf{v}_i \in W$ satisfying (2.56), one has

$$\begin{aligned}
\rho_W(g_1)\left(\rho_W(g_2)\mathbf{v}_i\right) &= \rho_W\left(g_1 g_2\right)\mathbf{v}_i \\
&= \rho_W\left(g_2 g_1^2\right)\mathbf{v}_i \\
&= \rho_W\left(g_2\right)\rho_W\left(g_1^2\right)\mathbf{v}_i \\
&= \rho_W(g_2)\left(\omega^{2\alpha_i}\mathbf{v}_i\right) \\
&= \omega^{2\alpha_i}\left(\rho_W(g_2)\mathbf{v}_i\right).
\end{aligned} \tag{2.57}$$

So $\rho_W(g_2)\mathbf{v}_i$ is an eigenvector of $\rho_W\left(g_1\right)$ with eigenvalue $\omega^{2\alpha_i}$. To check $\mathfrak{S}_3$-invariance of a proper subspace of the complex vector space W, it is sufficient to verify the invariance of the subspace in question under the action of $\rho_W(g_1)$ and $\rho_W(g_2)$, as g_1 and g_2 generate $\mathfrak{S}_3$. Also, let

$$\mathbf{s} = \begin{bmatrix} \omega \\ 1 \\ \omega^2 \end{bmatrix} \quad \text{and} \quad \mathbf{t} = \begin{bmatrix} 1 \\ \omega \\ \omega^2 \end{bmatrix}$$

with $\mathbf{s}, \mathbf{t}$ being a basis for the 2-dimensional vector subspace V of $\mathbb{C}^3$ that is known as the standard representation of $\mathfrak{S}_3$. Recall that the permutation representation $\rho : \mathfrak{S}_3 \to \mathrm{GL}(\mathbb{C}^3)$ satisfies

$$\rho(g_1)\begin{bmatrix}\omega\\1\\\omega^2\end{bmatrix} = \begin{bmatrix}\omega^2\\\omega\\1\end{bmatrix} = \omega\begin{bmatrix}\omega\\1\\\omega^2\end{bmatrix};$$

$$\rho(g_1)\begin{bmatrix}1\\\omega\\\omega^2\end{bmatrix} = \begin{bmatrix}\omega^2\\1\\\omega\end{bmatrix} = \omega^2\begin{bmatrix}1\\\omega\\\omega^2\end{bmatrix};$$

$$\rho(g_2)\begin{bmatrix}\omega\\1\\\omega^2\end{bmatrix} = \begin{bmatrix}1\\\omega\\\omega^2\end{bmatrix};$$

$$\rho(g_1)\begin{bmatrix}1\\\omega\\\omega^2\end{bmatrix} = \begin{bmatrix}\omega\\1\\\omega^2\end{bmatrix}.$$

Altogether, one has the following:

$$\rho(g_1)\mathbf{s} = \omega\mathbf{s}, \quad \rho(g_1)\mathbf{t} = \omega^2\mathbf{t}, \quad \rho(g_2)\mathbf{s} = \mathbf{t}, \quad \rho(g_2)\mathbf{t} = \mathbf{s}. \tag{2.58}$$

Suppose that we start with an eigenvector $\mathbf{v}$ of $\rho_W(g_1)$. Then we have the following possibilities:

1. The eigenvalue of $\rho_W(g_1)$ corresponding to the eigenvactor $\mathbf{v}$ is ω^i, where $\omega^i \neq 1$. Then $\omega^{2i} \neq \omega^i$. In terms of the eigenvalue equations, one has

$$\rho_W(g_1)\mathbf{v} = \omega^i\mathbf{v} \text{ and } \rho_W(g_1)\rho_W(g_2)\mathbf{v} = \omega^{2i}\rho_W(g_2)\mathbf{v}. \tag{2.59}$$

Since $\mathbf{v}$ and $\rho_W(g_2)\mathbf{v}$ are eigenvectors of two different eigenvalues, they are linearly independent. In other words, $\operatorname{span}\{\mathbf{v}, \rho_W(g_2)\mathbf{v}\} =: V'$ is a 2-dimensional vector subspace of W that is invariant under the action of $\mathfrak{S}_3$ (as g_1 and g_2 generate $\mathfrak{S}_3$).

Furthermore, this 2-dimensional representation V' is isomorphic to the standard representation V of $\mathfrak{S}_3$. In order to show this isomorphism, we need to prove the commutativity of the following square for each $g \in \mathfrak{S}_3$:

$$\begin{tikzcd} V \arrow[r, "j"] \arrow[d, "\rho(g)"'] & V' \arrow[d, "\rho_W(g)"] \\ V \arrow[r, "j"'] & V', \end{tikzcd}$$

where $j : V \to V'$ is a vector space isomorphism. It suffices to verify the commutativity for g_1 and g_2 as they generate $\mathfrak{S}_3$.

Here $V = \operatorname{span}\{\mathbf{s}, \mathbf{t}\}$ and $V' = \operatorname{span}\{\mathbf{v}, \rho_W(g_2)\mathbf{v}\}$. Consider the case $i = 1$ first. Then we define $j(\mathbf{s}) = \mathbf{v}$ and $j(\mathbf{t}) = \rho_W(g_2)\mathbf{v}$. This is an isomorphism of vector spaces.

$$\begin{aligned}(\rho_W(g_2) \circ j)(c_1\mathbf{s} + c_2\mathbf{t}) &= c_1\rho_W(g_2)(j(\mathbf{s})) + c_2\rho_W(g_2)(j(\mathbf{t}))\\ &= c_1\rho_W(g_2)\mathbf{v} + c_2\rho_W(g_2)\rho_W(g_2)\mathbf{v}\\ &= c_1\rho_W(g_2)\mathbf{v} + c_2\mathbf{v}\\ (j \circ \rho(g_2))(c_1\mathbf{s} + c_2\mathbf{t}) &= c_1 j(\rho(g_2)\mathbf{s}) + c_2 j(\rho(g_2)\mathbf{t})\\ &= c_1 j(\mathbf{t}) + c_2 j(\mathbf{s})\\ &= c_1\rho_W(g_2)\mathbf{v} + c_2\mathbf{v}.\end{aligned}$$

$$\begin{aligned}
\left(\rho_W\left(g_1\right) \circ j\right)\left(c_1 \mathbf{s}+c_2 \mathbf{t}\right) &= c_1 \rho_W\left(g_1\right)(j(\mathbf{s}))+c_2 \rho_W\left(g_1\right)(j(\mathbf{t})) \\
&= c_1 \rho_W\left(g_1\right) \mathbf{v}+c_2 \rho_W\left(g_1\right) \rho_W\left(g_2\right) \mathbf{v} \\
&= c_1 \omega \mathbf{v}+c_2 \omega^2 \rho_W\left(g_2\right) \mathbf{v} \\
\left(j \circ \rho\left(g_1\right)\right)\left(c_1 \mathbf{s}+c_2 \mathbf{t}\right) &= c_1 j\left(\rho\left(g_1\right) \mathbf{s}\right)+c_2 j\left(\rho\left(g_1\right) \mathbf{t}\right) \\
&= c_1 j(\omega \mathbf{s})+c_2 j\left(\omega^2 \mathbf{t}\right) \\
&= c_1 \omega \mathbf{v}+c_2 \omega^2 \rho_W\left(g_2\right) \mathbf{v}.
\end{aligned}$$

Therefore, the following diagrams commute

$$\begin{tikzcd}
V \arrow[r, "j"] \arrow[d, "\rho(g_2)"'] & V' \arrow[d, "\rho_W(g_2)"] \\
V \arrow[r, "j"'] & V',
\end{tikzcd}
\qquad
\begin{tikzcd}
V \arrow[r, "j"] \arrow[d, "\rho(g_1)"'] & V' \arrow[d, "\rho_W(g_1)"] \\
V \arrow[r, "j"'] & V'.
\end{tikzcd}$$

So $j : V \to V'$ is an isomorphism of representations.

Now we are left with the case $i = 2$. We define $j(\mathbf{s}) = \rho_W(g_2)\mathbf{v}$ and $j(\mathbf{t}) = \mathbf{v}$. This is an isomorphism of vector spaces.

$$\begin{aligned}
\left(\rho_W\left(g_2\right) \circ j\right)\left(c_1 \mathbf{t}+c_2 \mathbf{s}\right) &= c_1 \rho_W\left(g_2\right)(j(\mathbf{t}))+c_2 \rho_W\left(g_2\right)(j(\mathbf{s})) \\
&= c_1 \rho_W\left(g_2\right) \mathbf{v}+c_2 \rho_W\left(g_2\right) \rho_W\left(g_2\right) \mathbf{v} \\
&= c_1 \rho_W\left(g_2\right) \mathbf{v}+c_2 \mathbf{v} \\
\left(j \circ \rho\left(g_2\right)\right)\left(c_t \mathbf{s}+c_2 \mathbf{s}\right) &= c_1 j\left(\rho\left(g_2\right) \mathbf{t}\right)+c_2 j\left(\rho\left(g_2\right) \mathbf{s}\right) \\
&= c_1 j(\mathbf{s})+c_2 j(\mathbf{t}) \\
&= c_1 \rho_W\left(g_2\right) \mathbf{v}+c_2 \mathbf{v}.
\end{aligned}$$

$$\begin{aligned}
\left(\rho_W\left(g_1\right) \circ j\right)\left(c_1 \mathbf{t}+c_2 \mathbf{s}\right) &= c_1 \rho_W\left(g_1\right)(j(\mathbf{t}))+c_2 \rho_W\left(g_1\right)(j(\mathbf{s})) \\
&= c_1 \rho_W\left(g_1\right) \mathbf{v}+c_2 \rho_W\left(g_1\right) \rho_W\left(g_2\right) \mathbf{v} \\
&= c_1 \omega^2 \mathbf{v}+c_2 \omega^4 \rho_W\left(g_2\right) \mathbf{v} \\
\left(j \circ \rho\left(g_1\right)\right)\left(c_1 \mathbf{t}+c_2 \mathbf{s}\right) &= c_1 j\left(\rho\left(g_1\right) \mathbf{t}\right)+c_2 j\left(\rho\left(g_1\right) \mathbf{s}\right) \\
&= c_1 j\left(\omega^2 \mathbf{t}\right)+c_2 j(\omega \mathbf{s}) \\
&= c_1 \omega^2 \mathbf{v}+c_2 \omega \rho_W\left(g_2\right) \mathbf{v}.
\end{aligned}$$

Therefore, the following diagrams commute

$$\begin{tikzcd}
V \arrow[r, "j"] \arrow[d, "\rho(g_2)"'] & V' \arrow[d, "\rho_W(g_2)"] \\
V \arrow[r, "j"'] & V',
\end{tikzcd}
\qquad
\begin{tikzcd}
V \arrow[r, "j"] \arrow[d, "\rho(g_1)"'] & V' \arrow[d, "\rho_W(g_1)"] \\
V \arrow[r, "j"'] & V'.
\end{tikzcd}$$

So $j : V \to V'$ is an isomorphism of representations in $i = 2$ case as well.

Therefore, the 2-dimensional representation V' is isomorphic to V for both cases. Since V is irreducible, so is V'.

2. Now suppose the eigenvalue of $\rho_W(g_1)$ corresponding to the eigenvector $\mathbf{v}$ is 1. By (2.57),

$$\rho_W\left(g_1\right)\left(\rho_W\left(g_2\right) \mathbf{v}\right)=\rho_W\left(g_2\right) \mathbf{v}. \tag{2.60}$$

In other words, $\rho_W(g_2)\mathbf{v}$ is an eigenvector of $\rho_W(g_1)$ with eigenvalue 1. But $\mathbf{v}$ is also an eigenvector of $\rho_W(g_1)$ with eigenvalue 1.

Case 2(i): If $\mathbf{v}$ and $\rho_W(g_2)\mathbf{v}$ are linearly dependent, then $\mathbf{v}$ is an eigenvector of $\rho_W(g_2)$. Since $g_2^2 = e$, the eigenvalue of $\rho_W(g_2)$ corresponding to the eigenvector $\mathbf{v}$ will be 1 or -1.

If the eigenvalue is 1, then $\rho_W(g_1)\mathbf{v} = \mathbf{v}$ and $\rho_W(g_2)\mathbf{v} = \mathbf{v}$. Since g_1 and g_2 generate $\mathfrak{S}_3$, $\rho_W(g)\mathbf{v} = \mathbf{v}$ for every $g \in \mathfrak{S}_3$. So $\mathbb{C}\mathbf{v}$ is a 1-dimensional representation of $\mathfrak{S}_3$ isomorphic to the trivial representation.

If the eigenvalue is -1, then $\rho_W(g_1)\mathbf{v} = \mathbf{v}$ and $\rho_W(g_2)\mathbf{v} = -\mathbf{v}$. Then the equation $\rho_W(g)\mathbf{v} = (\operatorname{sgn} g)\mathbf{v}$ holds for $g = g_1, g_2$. Since g_1, g_2 generate $\mathfrak{S}_3$, this holds for all $g \in \mathfrak{S}_3$. Therefore, $\mathbb{C}\mathbf{v}$ is a 1-dimensional representation of $\mathfrak{S}_3$ isomorphic to the alternating representation.

Case 2(ii): Now suppose $\mathbf{v}$ and $\rho_W(g_2)\mathbf{v}$ are linearly independent. Then $\mathbf{v} + \rho_W(g_2)\mathbf{v}$ span a 1-dimensional representation of $\mathfrak{S}_3$ iromosphic to the trivial representation of $\mathfrak{S}_3$. Indeed,

$$\rho_W(g_1)(\mathbf{v} + \rho_W(g_2)\mathbf{v}) = \mathbf{v} + \rho_W(g_2)\mathbf{v}, \quad \rho_W(g_2)(\mathbf{v} + \rho_W(g_2)\mathbf{v}) = \rho_W(g_2)\mathbf{v} + \mathbf{v}. \tag{2.61}$$

Since g_1 and g_2 generate $\mathfrak{S}_3$, $\rho_W(g)(\rho_W(g_2)\mathbf{v} + \mathbf{v}) = \rho_W(g_2)\mathbf{v} + \mathbf{v}$ for every $g \in \mathfrak{S}_3$. Therefore, $\operatorname{span}\{\rho_W(g_2)\mathbf{v} + \mathbf{v}\}$ is a 1-dimensional representation of $\mathfrak{S}_3$ isomosphic to the trivial representation of $\mathfrak{S}_3$. On the other hand,

$$\rho_W(g_1)(\mathbf{v} - \rho_W(g_2)\mathbf{v}) = \mathbf{v} - \rho_W(g_2)\mathbf{v}, \quad \rho_W(g_2)(\mathbf{v} - \rho_W(g_2)\mathbf{v}) = \rho_W(g_2)\mathbf{v} - \mathbf{v}. \tag{2.62}$$

The equation $\rho_W(g)(\mathbf{v} - \rho_W(g_2)\mathbf{v}) = (\operatorname{sgn} g)(\mathbf{v} - \rho_W(g_2)\mathbf{v})$ holds for $g = g_1, g_2$. Since g_1, g_2 generate $\mathfrak{S}_3$, this holds for all $g \in \mathfrak{S}_3$. Therefore, $\operatorname{span}\{\mathbf{v} - \rho_W(g_2)\mathbf{v}\}$ is a 1-dimensional representation of $\mathfrak{S}_3$ isomorphic to the alternating representation of $\mathfrak{S}_3$.

In conclusion, there are 3 irreducible subrepresentations of W of $\mathfrak{S}_3$: the 2-dimensional irreducible representation isomorphic to the standard representation; the 1-dimensional irreducible representation isomorphic to the trivial representation; the 1-dimensional irreducible representation isomorphic to the alternating representation. By Maschke's theorem, W can be expressed as a direct sum of the 3 irreducible representations stated above:

$$\rho_W \cong \sigma^{\otimes a} \oplus (\sigma')^{\otimes b} \oplus \rho_V^{\otimes c}. \tag{2.63}$$

Here, $\sigma^{\otimes a}$ stands for the a-fold direct sum of the trivial representation $\sigma : \mathfrak{S}_3 \to \mathrm{GL}(\mathbb{C})$ with itself; $(\sigma')^{\otimes b}$ stands for the b-fold direct sum of the alternating representation $\sigma' : \mathfrak{S}_3 \to \mathrm{GL}(\mathbb{C})$ with itself; $\rho_V^{\otimes c}$ stands for the c-fold direct sum of the standard representation $\rho_V : \mathfrak{S}_3 \to \mathrm{GL}(V)$ with itself. Now, how do we determine the multiplicities a, b, c?

Suppose $\mathbf{v} \in V$ is an eigenvector of $\rho_V(g_1) \in \mathrm{GL}(V)$ with eigenvalue ω. Then $\rho_V(g_1)\mathbf{v} = \omega\mathbf{v}$. Take $\rho_V^{\otimes c}(g_1) \in \mathrm{GL}(V^c)$. There is a $\mathbf{v}$ in each copy of V in V^c. There are c-many linearly independent eigenvectors in W of $\rho_W(g_1)$ with eigenvalue ω, namely

$$\begin{bmatrix} \mathbf{0}_{a\times 1} \\ \mathbf{0}_{b\times 1} \\ \mathbf{v} \\ \mathbf{0}_{2\times 1} \\ \cdots \\ \mathbf{0}_{2\times 1} \end{bmatrix}, \begin{bmatrix} \mathbf{0}_{a\times 1} \\ \mathbf{0}_{b\times 1} \\ \mathbf{0}_{2\times 1} \\ \mathbf{v} \\ \cdots \\ \mathbf{0}_{2\times 1} \end{bmatrix}, \dots, \begin{bmatrix} \mathbf{0}_{a\times 1} \\ \mathbf{0}_{b\times 1} \\ \mathbf{0}_{2\times 1} \\ \cdots \\ \mathbf{0}_{2\times 1} \\ \mathbf{v} \end{bmatrix}.$$

Therefore, the number of linearly independent eigenvectors in W of $\rho_W(g_1)$ with eigenvalue ω is equal to c. Now, $\rho_W(g_2)$ has eigenvalues 1 or -1. It has $a + c$ eigenvectors of eigenvalue 1, namely

$$\begin{bmatrix} 1 \\ 0 \\ \cdots \\ 0 \\ \mathbf{0}_{b\times 1} \\ \mathbf{0}_{2c\times 1} \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ 0 \\ \cdots \\ 1 \\ \mathbf{0}_{b\times 1} \\ \mathbf{0}_{2c\times 1} \end{bmatrix}, \begin{bmatrix} \mathbf{0}_{a\times 1} \\ \mathbf{0}_{b\times 1} \\ \mathbf{v} + \rho_V(g_2)\mathbf{v} \\ \mathbf{0}_{2\times 1} \\ \cdots \\ \mathbf{0}_{2\times 1} \end{bmatrix}, \dots, \begin{bmatrix} \mathbf{0}_{a\times 1} \\ \mathbf{0}_{b\times 1} \\ \mathbf{0}_{2\times 1} \\ \cdots \\ \mathbf{0}_{2\times 1} \\ \mathbf{v} + \rho_V(g_2)\mathbf{v} \end{bmatrix}.$$

Finally, $\rho_W(g_2)$ has $b+c$ eigenvectors of eigenvalue -1, namely

$$\begin{bmatrix} \mathbf{0}_{a\times 1} \\ 1 \\ 0 \\ \cdots \\ 0 \\ \mathbf{0}_{2c\times 1} \end{bmatrix}, \ldots, \begin{bmatrix} \mathbf{0}_{a\times 1} \\ 0 \\ 0 \\ \cdots \\ 1 \\ \mathbf{0}_{2c\times 1} \end{bmatrix}, \begin{bmatrix} \mathbf{0}_{a\times 1} \\ \mathbf{0}_{b\times 1} \\ \mathbf{v}-\rho_V(g_2)\mathbf{v} \\ \mathbf{0}_{2\times 1} \\ \cdots \\ \mathbf{0}_{2\times 1} \end{bmatrix}, \ldots, \begin{bmatrix} \mathbf{0}_{a\times 1} \\ \mathbf{0}_{b\times 1} \\ \mathbf{0}_{2\times 1} \\ \cdots \\ \mathbf{0}_{2\times 1} \\ \mathbf{v}-\rho_V(g_2)\mathbf{v} \end{bmatrix}.$$

Hence, the nonnegative integer $a+c$ in (2.63) is the multiplicity of the eigenvalue 1 of $\rho_W(g_2)$; and $b+c$ is the multiplicity of the eigenvalue -1 of $\rho_W(g_2)$.

§2.4 Projection formulae

Recall that if V and W are two finite dimensional complex representations of a finite group G, then $\mathrm{Hom}_G(V,W)$ is the vector space of all G-linear maps (sometimes called G-module homomorphisms) from the finite dimensional complex representation V to the finite dimensional complex representation W of the finite group G. Now, given any representation $\rho : G \to \mathrm{GL}(V)$, we define

$$V^G = \{\in V \mid \rho(g)\mathbf{v} = \mathbf{v} \text{ for every } g \in G\}. \tag{2.64}$$

Observe that for a given $g_0 \in G$, the automorphism $\rho(g_0) : V \to V$ is not necessarily a G-module homomorphism as $\rho(g) \circ \rho(g_0)$ and $\rho(g_0) \circ \rho(g)$ are not necessarily equal for every $g \in G$. If we, instead, take the average of all the automorphisms $\rho(g) \in \mathrm{GL}(V)$, for all $g \in G$, and denote it by φ, i.e.

$$\varphi = \frac{1}{|G|}\sum_{g\in G}\rho(g), \tag{2.65}$$

then φ is a G-module homomorphism. Indeed, for any $g' \in G$,

$$\rho(g') \circ \varphi = \frac{1}{|G|}\rho(g')\sum_{g\in G}\rho(g) = \frac{1}{|G|}\sum_{g\in G}\rho(g')\rho(g) = \frac{1}{|G|}\sum_{g\in G}\rho(g'g) = \frac{1}{|G|}\sum_{g\in G}\rho(g).$$

$$\varphi \circ \rho(g') = \frac{1}{|G|}\left(\sum_{g\in G}\rho(g)\right)\rho(g') = \frac{1}{|G|}\sum_{g\in G}\rho(g)\rho(g') = \frac{1}{|G|}\sum_{g\in G}\rho(g') = \frac{1}{|G|}\sum_{g\in G}\rho(g).$$

Therefore,

$$\rho(g') \circ \varphi = \varphi \circ \rho(g') = \varphi \tag{2.66}$$

for every $g' \in G$, i.e. the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "\rho(g')"'] & V \arrow[d, "\rho(g')"] \\ V \arrow[r, "\varphi"'] & V \end{tikzcd}$$

Proposition 2.3
The map $\varphi : V \to V^G$ is a projection of V onto V^G.

Proof. Let us first show that $\operatorname{im}\varphi = V^G$. Suppose $\mathbf{v} = \varphi(\mathbf{w})$. Then for any $h \in G$,

$$\rho(h)\mathbf{v} = [\rho(h)\circ\varphi](\mathbf{w}) = \varphi(\mathbf{w}) = \mathbf{v}, \tag{2.67}$$

since we proved $\rho(h) \circ \varphi = \varphi$ in (2.66). So we have $\rho(h)\mathbf{v} = \mathbf{v}$ for any $h \in G$. Therefore, $\mathbf{v} \in V^G$, i.e. $\operatorname{im}\varphi \subseteq V^G$.

Conversely, suppose $\mathbf{v} \in V^G$. Then $\rho(g)\,\mathbf{v} = \mathbf{v}$ for any $g \in G$. So

$$\varphi(\mathbf{v}) = \frac{1}{|G|}\sum_{g\in G}\rho(g)\mathbf{v} = \frac{1}{|G|}\sum_{g\in G}\mathbf{v} = \mathbf{v}. \tag{2.68}$$

So $\mathbf{v} = \varphi(\mathbf{v}) \in \operatorname{im}\varphi$, i.e. $V^G \subseteq \operatorname{im}\varphi$. Hence, $\operatorname{im}\varphi = V^G$.

Now, for $\mathbf{v} \in V$,

$$(\varphi\circ\varphi)(\mathbf{v}) = \varphi(\varphi(\mathbf{v})) = \varphi(\mathbf{v}), \tag{2.69}$$

since $\varphi(\mathbf{v}) \in V^G$ and we showed earlier that $\varphi(\mathbf{w}) = \mathbf{w}$ for $\mathbf{w} \in V^G$. Therefore, $\varphi : V \to V^G$ is a surjective map satisfying $\varphi\circ\varphi = \varphi$. So it is a projection map of V onto V^G. ■

Given a finite dimensional complex representation V of the finite group G, we want to calculate the dimension of the vector space V^G. We refer back to the projection map $\varphi : V \to V^G$. One can decompose V as $V = V^G \oplus \operatorname{Ker}\varphi$. Now, one can form a basis $\mathcal{B}$ of V by taking the union of a basis of V^G and a basis of $\operatorname{Ker}\varphi$. In this chosen basis of V, φ can be expressed as the following block-diagonal matrix:

$$[\varphi]_{\mathcal{B}} = \begin{bmatrix} \mathbb{1}_{V^G} & \\ & \mathbf{0}_{k\times k}\end{bmatrix},$$

where $k = \dim\operatorname{Ker}\varphi$. From this block-diagonal form, one obtains

$$\dim V^G = \operatorname{Tr}\varphi = \frac{1}{G}\sum_{g\in G}\operatorname{Tr}(\rho(g)) = \frac{1}{G}\sum_{g\in G}\chi_\rho(g). \tag{2.70}$$

If one denotes $\dim_{\mathbb{C}} V^G = m$, then one immediately finds that the nonnegative integer m is precisely the number of times the trivial (1-dimensional) representation of G appears in the direct sum decomposition of V. In particular, if V is an irreducible representation other than the trivial representation of G, then since there is no ptoper G-invariant subspace of V, one must have $\dim V^G = 0$. In other words, if $\rho : G \to \mathrm{GL}(V)$ is an irreducible representation (other than the trivial representation), then

$$\sum_{g\in G}\chi_\rho(g) = 0. \tag{2.71}$$

Now, given two finite dimensional representations V and W with associated group homomorphisms $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$, $\operatorname{Hom}(V, W)$ is also a representation with group homomorphism $\gamma : G \to \mathrm{GL}(\operatorname{Hom}(V, W))$ defined by

$$\gamma(g)\,\psi = \sigma(g)\circ\psi\circ\rho\left(g^{-1}\right). \tag{2.72}$$

In other words, the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "\psi"] & W \arrow[d, "\sigma(g)"] \\ V \arrow[u, "\rho(g^{-1})"] \arrow[r, "\gamma(g)\psi"'] & W \end{tikzcd}$$

Now, using the definition (2.64),

$$\operatorname{Hom}(V, W)^G = \{\psi \in \operatorname{Hom}(V, W) \mid \gamma(g)\,\psi = \psi \text{ for every } g \in G\} \tag{2.73}$$

Proposition 2.4

$\operatorname{Hom}_G(V, W) = \operatorname{Hom}(V, W)^G$.

Proof.

$$\begin{aligned}
\psi \in \mathrm{Hom}\,(V,W)^G &\iff \gamma\,(g)\,\psi = \psi \text{ for every } g \in G\\
&\iff \sigma\,(g)\circ\psi\circ\rho\left(g^{-1}\right) = \psi \text{ for every } g\in G\\
&\iff \sigma\,(g)\circ\psi = \psi\circ\rho\,(g) \text{ for every } g \in G\\
&\iff \psi \in \mathrm{Hom}_G\,(V,W)\,,
\end{aligned}$$

because $\sigma\,(g)\circ\psi = \psi\circ\rho\,(g)$ is equivalent to the commutativity of the following diagram:

$$\begin{tikzcd}
V \arrow[r, "\psi"] \arrow[d, "\rho(g)"'] & W \arrow[d, "\sigma(g)"] \\
V \arrow[r, "\psi"'] & W
\end{tikzcd}$$

Therefore, $\mathrm{Hom}\,(V,W)^G = \mathrm{Hom}_G\,(V,W)$. ■

Remark 2.4. Note that in Proposition 2.4, on the right side, $\mathrm{Hom}\,(V,W)$ is the representation of G given by the group homomorphism $\gamma : G \to \mathrm{GL}\,(\mathrm{Hom}\,(V,W))$ defined by (2.72). On the left hand side of Proposition 2.4, $\mathrm{Hom}_G\,(V,W)$ is the vector space of all G-module homomorphisms from the finite dimensional complex representation V to the finite dimensional complex representation W.

If V is irreducible and W is reducible with the multiplicity of V in the decomposition of W being m, i.e. $W = V^m \oplus \cdots$, then by Proposition 1.8,

$$m = \dim_{\mathbb{C}} \mathrm{Hom}_G\,(V,W) = \dim_{\mathbb{C}} \mathrm{Hom}\,(V,W)^G\,. \tag{2.74}$$

Similarly, if W is irreducible and V is reducible with the multiplicity of W in the decomposition of V being n, i.e. $V = W^n \oplus \cdots$, then by Proposition 1.8,

$$n = \dim_{\mathbb{C}} \mathrm{Hom}_G\,(V,W) = \dim_{\mathbb{C}} \mathrm{Hom}\,(V,W)^G\,. \tag{2.75}$$

When both the representations V and W of the finite group G are irreducibles, then

$$\dim_{\mathbb{C}} \mathrm{Hom}\,(V,W)^G = \begin{cases} 1 & \text{if } V \cong W \text{ as representations;}\\ 0 & \text{if } V \ncong W \text{ as representations.}\end{cases} \tag{2.76}$$

In Proposition 1.1, we showed that $\mathrm{Hom}\,(V,W)$ and $V^* \otimes W$ are isomorphic as representations, i.e. $\gamma \cong \rho^* \otimes \sigma$. Now, using Proposition 2.1, we get

$$\chi_\gamma\,(g) = \overline{\chi_\rho\,(g)}\,\chi_\sigma(g). \tag{2.77}$$

Now, using (2.70),

$$\dim_{\mathbb{C}} \mathrm{Hom}\,(V,W)^G = \frac{1}{|G|}\sum_{g\in G}\chi_\gamma\,(g) = \frac{1}{|G|}\sum_{g\in G}\overline{\chi_\rho\,(g)}\,\chi_\sigma(g). \tag{2.78}$$

In the case when both V and W are irreducible representations, with the respective group homomorphisms $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$, then

$$\frac{1}{|G|}\sum_{g\in G}\overline{\chi_\rho\,(g)}\,\chi_\sigma(g) = \begin{cases} 1 & \text{if } V \cong W,\\ 0 & \text{if } V \ncong W.\end{cases} \tag{2.79}$$

(Here, the isomorphism is isomorphism of representations.)

Definition 2.2 (Class functions). A **class function** on G is a complex valued function $f : G \to \mathbb{C}$ that is constant on the conjugacy classes of G. We will denote the space of all class functions on a finite group G by $\mathbb{C}_{\text{class}}[G]$.

Character associated with a finite dimensional representation is an example of a class function. Now we define a Hermitian inner product on $\mathbb{C}_{\text{class}}[G]$ by

$$(\alpha, \beta) = \frac{1}{|G|} \sum_{g \in G} \overline{\alpha(g)}\, \beta(g). \tag{2.80}$$

Then (2.79) translates into the following theorem.

Theorem 2.5

In terms of the inner product (2.80), the characters of the irreducible representations of G are orthonormal.

$\mathbb{C}_{\text{class}}[G]$ is, in fact, a complex inner product space endowed with the hermitial inner product given by (2.80). The dimension of $\mathbb{C}_{\text{class}}[G]$ is the number of conjugacy classes of G. Theorem 2.5 tells us that the irreducible characters are linearly independent, so that the number of irreducible representations is less than or equal to the number of conjugacy classes. We will soon prove that these two are, indeed, the same.

Corollary 2.6

Any representation is determined by its character. In other words, if $\sigma_1 : G \to \mathrm{GL}(V), \sigma_2 : G \to \mathrm{GL}(W)$ are two representations of G such that $\chi_{\sigma_1} = \chi_{\sigma_2}$, then σ_1 and σ_2 are isomorphic representations.

Proof. Suppose $\rho_i : G \to \mathrm{GL}(V_i)$ are all the irreducible representations, for $i = 1, \ldots, k$. We express V and W as direct sum of irreducible representations:

$$V = \bigoplus_{i=1}^{k} V_i^{a_i} \text{ and } W = \bigoplus_{i=1}^{k} V_i^{b_i}, \tag{2.81}$$

for $a_i, b_i \in \mathbb{Z}_{\geq 0}$.

$$\chi_{\sigma_1} = \sum_{i=1}^{k} a_i \chi_{\rho_i} \text{ and } \chi_{\sigma_2} = \sum_{i=1}^{k} b_i \chi_{\rho_i}. \tag{2.82}$$

Since $\chi_{\sigma_1} = \chi_{\sigma_2}$, we have

$$\sum_{i=1}^{k} (a_i - b_i)\, \chi_{\rho_i} = 0. \tag{2.83}$$

Since $\{\chi_{\rho_i}\}_{i=1}^{k}$ is a linearly independent set in the space of all class functions, we must have $a_i - b_i = 0$ for each i. Therefore, $a_i = b_i$ for each i, and hence, σ_1 and σ_2 are isomorphic representations. ∎

Corollary 2.7

A representation $\rho : G \to \mathrm{GL}(V)$ is irreducible if and only if $(\chi_\rho, \chi_\rho) = 1$.

Proof. We have already proved one direction: if $\rho : G \to \mathrm{GL}(V)$ is irreducible, then $(\chi_\rho, \chi_\rho) = 1$, by Theorem 2.5. Conversely, suppose $(\chi_\rho, \chi_\rho) = 1$. Suppose $\rho_i : G \to \mathrm{GL}(V_i)$ are all the irreducible representations, for $i = 1, \ldots, k$. We express V and W as direct sum of irreducible representations:

$$V = \bigoplus_{i=1}^{k} V_i^{a_i}, \tag{2.84}$$

for $a_i \in \mathbb{Z}_{\geq 0}$. Then

$$\chi_\rho = \sum_{i=1}^{k} a_i \chi_{\rho_i}. \tag{2.85}$$

Now, the sesqui-linearity of inner product along with the orthonormality of irreducible character gives us

$$\begin{aligned} 1 &= (\chi_\rho, \chi_\rho) \\ &= \left(\sum_{i=1}^{k} a_i \chi_{\rho_i}, \sum_{j=1}^{k} a_j \chi_{\rho_j} \right) \\ &= \sum_{i,j=1}^{k} \overline{a_i}\, a_j \left(\chi_{\rho_i}, \chi_{\rho_j}\right) \\ &= \sum_{i,j=1}^{k} \overline{a_i}\, a_j\ \delta_{ij} \\ &= \sum_{i=1}^{k} |a_i|^2 . \end{aligned} \tag{2.86}$$

a_i are each non-negative integers, and their square-sum is 1. This is only possible when $a_{i_0} = 1$ for some i_0, and $a_i = 0$ for other $i \neq i_0$. Therefore, $\rho = \rho_{i_0}$, and hence ρ is irreducible. ∎

Corollary 2.8
Let $\rho_i : G \to \mathrm{GL}(V_i)$ be an irreducible representation, and $\rho : G \to \mathrm{GL}(V)$ be any other representation. Then the multiplicity a_i of V_i in V is given by

$$a_i = (\chi_\rho, \chi_{\rho_i}) = (\chi_{\rho_i}, \chi_\rho) . \tag{2.87}$$

Proof. Follows trivially from (2.74), (2.75), (2.78). ∎

Corollary 2.9
Any irreducible representation V_i appears in the regular representation with multiplicity $\dim V_i$.

Proof. Let $R = \mathbb{C}[G]$ be the vector space on which the regular representation acts on, and $\rho : G \to \mathrm{GL}\left(\mathbb{C}\left[G\right]\right)$ be the associated group homomorphism. As we know that regular representation is a special case of permutation representation, with the set X being $G_{\mathbf{Set}}$.

$$\rho\left(h\right) \left(\sum_{g \in G} a_g \delta_g \right) = \sum_{g \in G} a_g \delta_{\sigma(h) g}, \tag{2.88}$$

where $\sigma(h) : G_{\mathbf{Set}} \to G_{\mathbf{Set}}$ is a bijection (i.e. permutation), which we define as $\sigma(h)g = hg$. Therefore, the character $\chi_\rho(h)$ of the regular representation indicates the number of elements of $G_{\mathbf{Set}}$ fixed by $\sigma\left(h\right)$ (Lemma 2.2).

$$\sigma(h)g = g \iff hg = g \iff h = e. \tag{2.89}$$

If $h = e$, then all the elements of $G_{\mathbf{Set}}$ are fixed by $\sigma(e)$. Otherwise, none of the elements are fixed. So

$$\chi_\rho(h) = \begin{cases} |G| & \text{if } h = e, \\ 0 & \text{otherwise.} \end{cases} \tag{2.90}$$

Let $\rho_i : G \to \mathrm{GL}(V_i)$ be an irreducible representation. Then the number of times V_i appears in the regular representation R is given by

$$(\chi_\rho, \chi_{\rho_i}) = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_\rho(g)}\, \chi_{\rho_i}(g) = \frac{1}{|G|} \overline{\chi_\rho(e)}\, \chi_{\rho_i}(e) = \frac{1}{|G|}\, |G| \dim V_i = \dim V_i. \tag{2.91}$$

Therefore, V_i appears in R with multiplicity $\dim V_i$. ■

Corollary 2.10

Let $\rho_i : G \to \mathrm{GL}(V_i)$, $i = 1, 2, \ldots, m$ be the irreducible representations of G. Then

$$\sum_{i=1}^{m} (\dim V_i)^2 = |G|\,. \tag{2.92}$$

Proof. By Corollary 2.9,

$$\mathbb{C}[G] = V_1^{\dim V_1} \oplus V_2^{\dim V_2} \oplus \cdots \oplus V_m^{\dim V_m} = \bigoplus_{i=1}^{m} V_i^{\dim V_i}. \tag{2.93}$$

The dimension of $\mathbb{C}[G]$ is $|G|$. So equating the dimensions in (2.93), we have

$$|G| = \sum_{i=1}^{m} \dim \left(V_i^{\dim V_i} \right). \tag{2.94}$$

$\dim \left(V^k \right) = k \dim V$, so

$$|G| = \sum_{i=1}^{m} \dim \left(V_i^{\dim V_i} \right) = \sum_{i=1}^{m} (\dim V_i)\,(\dim V_i) = \sum_{i=1}^{m} (\dim V_i)^2\,. \tag{2.95}$$

■

Proposition 2.11

Let $\alpha : G \to \mathbb{C}$ be any function on G, and V be a complex representation of G with the group homomorphism $\rho : G \to \mathrm{GL}(V)$. Let

$$\phi_{\alpha,V} = \sum_{g \in G} \alpha(g)\rho(g) : V \to V$$

be a linear map. Then $\phi_{\alpha,V} \in \mathrm{Hom}_G(V, V)$ for all V if and only if α is a class function.

Proof. Suppose α is a class function. To prove that $\phi_{\alpha,V} \in \mathrm{Hom}_G(V, V)$, we need to show that for all $h \in G$, $\phi_{\alpha,V} \circ \rho(h) = \rho(h) \circ \phi_{\alpha,V}$.

$$\phi_{\alpha,V} \circ \rho(h) = \sum_{g \in G} \alpha(g)\rho(g)\rho(h)\,. \tag{2.96}$$

Write $h^{-1}gh = g'$ so that $g = hg'h^{-1}$. Since h is fixed, as g varies in G, g' also varies in G. Hence,

$$\phi_{\alpha,V} \circ \rho(h) = \sum_{g' \in G} \alpha\left(hg'h^{-1}\right) \rho\left(hg'h^{-1}\right) \rho(h)\,. \tag{2.97}$$

Since α is a class function, $\alpha(hg'h^{-1}) = \alpha(g')$. So

$$\phi_{\alpha,V} \circ \rho(h) = \sum_{g' \in G} \alpha(g')\, \rho(hg') = \rho(h) \sum_{g' \in G} \alpha(g')\, \rho(g') = \rho(h) \circ \phi_{\alpha,V}. \tag{2.98}$$

So $\phi_{\alpha,V} \in \mathrm{Hom}_G(V,V)$.

Conversely, assume α is not a class function. Then we shall prove that $\phi_{\alpha,V}$ is not a G-linear map, for $V = \mathbb{C}[G]$, the regular representation. Since α is not a class function, there exists $h, k \in G$ such that $\alpha(h^{-1}k) \neq \alpha(kh^{-1})$.

Assume for the sake of contradiction that $\phi_{\alpha,V}$ is a G-linear map. Then, $\phi_{\alpha,V} \circ \rho(h) = \rho(h) \circ \phi_{\alpha,V}$. In other words,

$$\left[\sum_{g\in G} \alpha(g)\rho(g)\right] \circ \rho h = \rho(h) \circ \left[\sum_{g\in G} \alpha(g)\rho(g)\right]. \tag{2.99}$$

We can rewrite it as follows:

$$\sum_{g\in G} \alpha(g)\rho(gh) = \sum_{g\in G} \alpha(g)\rho(hg). \tag{2.100}$$

With the change of variable $gh \to g'$ on LHS and $hg \to g'$ on RHS, we have

$$\sum_{g'\in G} \alpha\left(g'h^{-1}\right)\rho(g') = \sum_{g\in G} \alpha\left(h^{-1}g'\right)\rho(g'). \tag{2.101}$$

Since these two are equal, they'll yield the same value when acted on $\delta_e \in \mathbb{C}[G]$. Hence,

$$\begin{aligned} &\sum_{g'\in G} \alpha\left(g'h^{-1}\right)\rho(g')(\delta_e) = \sum_{g\in G} \alpha\left(h^{-1}g'\right)\rho(g')(\delta_e) \\ \implies &\sum_{g'\in G} \alpha\left(g'h^{-1}\right)\delta_{g'} = \sum_{g'\in G} \alpha\left(h^{-1}g'\right)\delta_{g'}. \end{aligned} \tag{2.102}$$

Since $\{\delta_{g'}\}_{g'\in G}$ is a basis for $\mathbb{C}[G]$, (2.102) gives us that $\alpha(g'h^{-1}) = \alpha(h^{-1}g')$ for every $g' \in G$. But we know that there exists $k \in G$ with $\alpha(h^{-1}k) \neq \alpha(kh^{-1})$. Thus we arrive at a contradiction! Therefore, $\phi_{\alpha,V}$ is not a G-linear map, for $V = \mathbb{C}[G]$, if α is not a class function. ■

Lemma 2.12

A complex representation $\rho : G \to \mathrm{GL}(V)$ is irreducible if and only if its dual representation $\rho^* : G \to \mathrm{GL}(V^*)$ is irreducible.

Proof.

$$\begin{aligned} \rho \text{ is irreducible} &\iff (\chi_\rho, \chi_\rho) = 1 \\ &\iff \frac{1}{|G|}\sum_{g\in G} \overline{\chi_\rho(g)}\,\chi_\rho(g) = 1 \\ &\iff \frac{1}{|G|}\sum_{g\in G} \chi_{\rho^*}(g)\,\overline{\chi_{\rho^*}(g)} = 1 \\ &\iff \frac{1}{|G|}\sum_{g\in G} (\chi_{\rho^*}, \chi_{\rho^*}) = 1 \\ &\iff \rho^* \text{ is irreducible.} \end{aligned}$$

■

Definition 2.3 (Irreducible Characters). The characters of the irreducible representations are called **irreducible characters**.

Theorem 2.13

The set of irreducible characters forms an orthonormal basis of $\mathbb{C}_{\text{class}}[G]$.

Proof. Let $\alpha \in \mathbb{C}_{\text{class}}[G]$ and $(\alpha, \chi_\rho) = 0$ for every irreducible representation $\rho : G \to \mathrm{GL}(V)$. We need to show that $\alpha = 0$. That would prove that $\{\chi_\rho\}_\rho$ spans $\mathbb{C}_{\text{class}}[G]$.

Consider $\phi_{\alpha,V} = \sum_{g\in G} \alpha(g)\rho(g) : V \to V$. By Proposition 2.11, $\phi_{\alpha,V} \in \mathrm{Hom}_G(V,V)$. Since V is an irreducible representation, by Schur's lemma, $\dim \mathrm{Hom}_G(V,V) = 1$. Since $\mathbb{1}_V \in \mathrm{Hom}_G(V,V)$, one must have $\phi_{\alpha,V} = \lambda \mathbb{1}_V$ for some $\lambda \in \mathbb{C}$. Let $n = \dim V$. Taking trace on both sides of $\phi_{\alpha,V} = \lambda \mathbb{1}_V$, we have

$$\begin{aligned}
\operatorname{Tr} \phi_{\alpha,V} = \lambda \operatorname{Tr} \mathbb{1}_V \implies & \operatorname{Tr}\left[\sum_{g\in G} \alpha(g)\rho(g)\right] = \lambda n \\
\implies & \sum_{g\in G} \alpha(g) \operatorname{Tr} \rho(g) = \lambda n \\
\implies & \sum_{g\in G} \alpha(g)\chi_\rho(g) = \lambda n \\
\implies & \sum_{g\in G} \overline{\alpha(g)}\, \overline{\chi_\rho(g)} = \overline{\lambda n} = \lambda n \\
\implies & \frac{1}{|G|} \sum_{g\in G} \overline{\alpha(g)}\chi_{\rho^*}(g) = \frac{n}{|G|}\lambda \\
\implies & (\alpha, \chi_{\rho^*}) = \frac{n}{|G|}\lambda
\end{aligned}$$

Since ρ is irreducible, so is ρ^*. By hypothesis, $(\alpha, \chi_\rho) = 0$ for every irreducible representation ρ. Therefore, $(\alpha, \chi_{\rho^*}) = 0$. $\frac{n}{|G|} \neq 0$, so $\lambda = 0$. This gives us $\phi_{\alpha,V} = 0$ for every irreducible representation $\rho : G \to \mathrm{GL}(V)$, i.e.

$$\sum_{g\in G} \alpha(g)\rho(g) = 0. \tag{2.103}$$

One can, therefore, conclude that for any representation $W = \bigoplus_{i=1}^k V_i^{r_i}$ of G, associated with the group homomorphism $\sigma : G \to \mathrm{GL}(W) = \mathrm{GL}\left(\bigoplus_{i=1}^k V_i^{r_i}\right)$,

$$\phi_{\alpha,W} = \sum_{g\in G} \alpha(g)\,\sigma(g) = 0, \tag{2.104}$$

i.e. the endomorophism $\phi_{\alpha,W}$ is the zero map. In particular, (2.104) holds for the left-regular representation $\mathbb{C}[G]$ of G. The group homomorphism associated with the left-regular representation is $\sigma : G \to \mathrm{GL}(\mathbb{C}[G])$. Here, $\{\delta_g \mid g \in G\}$ is a basis for $\mathbb{C}[G]$. Since $\phi_{\alpha,\mathbb{C}[G]} = 0$, it will give out 0 if acted upon δ_e. Hence,

$$0 = \left[\sum_{g\in G} \alpha(g)\,\sigma(g)\right](\delta_e) = \sum_{g\in G} \alpha(g)\,\delta_g. \tag{2.105}$$

$\{\delta_g \mid g \in G\}$ is a basis for $\mathbb{C}[G]$. Therefore, $\sum_{g\in G} \alpha(g)\,\delta_g = 0$ implies $\alpha(g) = 0$ for every $g \in G$, i.e. $\alpha : G \to \mathbb{C}$ has to be the 0-function. ∎

Note that $\mathbb{C}_{\text{class}}[G]$ has a basis of complex valued functions which are 1 on a given conjugacy class and 0 otherwise (characteristic functions on conjugacy classes of the group). The number of such characteristic functions is precisely the total number of conjugacy classes of the group. Hence, the dimension of the complex vector space $\mathbb{C}_{\text{class}}[G]$ is the number of conjugacy classes of the group G. By Theorem 2.13, on the other hand, the number of irreducible characters and hence the number of irreducible representation of G is also equal to the dimension of $\mathbb{C}_{\text{class}}[G]$.

Corollary 2.14

The number of irreducible representations of G is equal to the number of conjugacy class of G.

Representation ring of G

Take the isomorphism classes of representations of G. Suppose the group homomorphism $\rho : G \to \mathrm{GL}(V)$ defines a representation of G on the finite dimensional vector space V. By the class $[\rho]$, one denotes the isomorphism classes of all such group homomorphism. We call $[\rho]$ an isomorphism class of representations of G. Then form the free abelian group generated by these isomorphism classes. Elements of the resulting abelian group are like $m\,[\rho] + n\,[\sigma] + p\,[\tau]$, where $m, n, p \in \mathbb{Z}$ and $[\rho]\,, [\sigma]\,, [\tau]$ are isomorphism classes of representations of the finite group G.

Now take the quotient group $R(G)$ of the above free abelian group by modding out the subgroup generated by elements of the form $[\rho] + [\sigma] - [\rho \oplus \sigma]$. For example, in this quotient group $R(G)$ of the free abelian group, $[\rho] + 2\,[\sigma]$ is the same as $[\rho \oplus \sigma \oplus \sigma]$. Now, a ring structure can be imposed on $R(G)$ as follows:

$$[\rho] \cdot [\sigma] := [\rho \otimes \sigma]\,. \tag{2.106}$$

One can then extend the product on the whole of $R(G)$ by linearity. For instance, the product of $[\rho] + 2\,[\sigma]$ and $[\rho] - [\sigma]$ in $R(G)$ reads as

$$([\rho] + 2\,[\sigma]) \cdot ([\rho] - [\sigma]) = [\rho \otimes \rho] - [\rho \otimes \sigma] + 2\,[\sigma \otimes \rho] - 2\,[\sigma \otimes \sigma]\,. \tag{2.107}$$

Now let us revisit the terms that we are familiar with using representation ring of a finite group G. The character defines a map

$$\chi : R\,(G) \to \mathbb{C}_{\text{class}}[G]$$

by $\chi\,([\rho]) = \chi_\rho$. We have seen that $\mathbb{C}_{\text{class}}[G]$ is a complex inner product space. The set of class functions $\mathbb{C}_{\text{class}}[G]$ also comes equipped with certain algebraic structures, namely those of a ring: it is a commutative ring under pointwise addition and multiplication:

$$(f_1 + f_2)\,(g) = f_1\,(g) + f_2\,(g) \;\text{ and }\; (f_1 \cdot f_2)\,(g) = f_1\,(g)\,f_2\,(g)\,. \tag{2.108}$$

The additive identity is the constant function with value 0; and the multiplicative ideneity is the constant function with value 1. By Proposition 2.1, $\chi : R\,(G) \to \mathbb{C}_{\text{class}}[G]$ is a ring homomorphism.

The multiplicative identity of $R(G)$ is the trivial representation $\mathbf{1}_\mathbb{C}$, which is the 1-dimensional trivial representation of the group G. Indeed, $\mathbf{1}_\mathbb{C} \otimes \rho$ and ρ belong to the same isomorphism class in $R(G)$, so that we have

$$[\mathbf{1}_C] \cdot [\rho] = [\rho] \tag{2.109}$$

in $R(G)$. Since $\mathbf{1}_\mathbb{C}$ is the one-dimensional trivial representation of G, $\chi_{\mathbf{1}_\mathbb{C}}$ is the constant function that maps all the group elements to the constant $1 \in \mathbb{C}$, which is precisely the multiplicative identity of $\mathbb{C}_{\text{class}}[G]$. One also has

$$\begin{aligned}\chi\,([\rho] + [\sigma]) = \chi\,([\rho \oplus \sigma]) = \chi_{\rho\oplus\sigma} = \chi_\rho + \chi_\sigma = \chi\,([\rho]) + \chi\,([\sigma])\,;\\ \chi\,([\rho] \cdot [\sigma]) = \chi\,([\rho \otimes \sigma]) = \chi_{\rho\otimes\sigma} = \chi_\rho \cdot \chi_\sigma = \chi\,([\rho]) \cdot \chi\,([\sigma])\,.\end{aligned}$$

Therefore, $\chi : R\,(G) \to \mathbb{C}_{\text{class}}[G]$ is, indeed, a ring homomorphism. However, it is not an isomorphism. It's injective, as a representation is uniquely determined by its character (Corollary 2.6). There are too many elements in the codomain $\mathbb{C}_{\text{class}}[G]$. $\mathbb{C}_{\text{class}}[G]$ is a complex vector space, while $R(G)$ is a $\mathbb{Z}$-module. We can form the tensor product $R(G) \otimes \mathbb{C}$ which will then be a free $\mathbb{C}$-vector space of all isomorphism classes of representations of G modulo the $\mathbb{C}$-subspace spanned by elements of the form $[\rho] + [\sigma] - [\rho \oplus \sigma]$. One then has an isomorphism

$$\chi_\mathbb{C} : R(G) \otimes \mathbb{C} \to \mathbb{C}_{\text{class}}[G].$$

Proposition 2.15

Let V be a finite dimensional complex representation of a finite group G, and $\rho : G \to \mathrm{GL}(V)$ be the associated group homomorphism. Let $\rho = \bigoplus_{i=1}^m \rho_i^{\oplus a_i}$ be the canonical decomposition of ρ into irreducibles $\rho_i : G \to \mathrm{GL}\,(V_i)$, for $i = 1, 2, \ldots, m$, so that the representation space decomposes as

$V = \bigoplus_{i=1}^m V_i^{a_i}$. Then

$$\pi_i = \dim V_i \frac{1}{|G|} \sum_{g \in G} \overline{\chi_{\rho_i}(g)}\, \rho(g) \in \mathrm{End}(V) \tag{2.110}$$

is the projection of V onto $V_i^{a_i}$.

Proof. Let us first prove that

$$p_i^j = \frac{\dim V_i}{|G|} \sum_{g \in G} \overline{\chi_{\rho_i}(g)}\, \rho_j(g) : V_j \to V_j$$

satisfies $p_i^j = \delta_i^j\, \mathbb{1}_{V_j}$. First observe that $p_i^j \in \mathrm{Hom}_G(V_j, V_j)$, since χ_{ρ_i} is a class function (by Proposition 2.11). Now, since $\mathbb{1}_{V_j} \in \mathrm{Hom}_G(V_j, V_j)$, and V_j is irreducible, by Schur's lemma, $\dim \mathrm{Hom}_G(V_j, V_j) = 1$, so that $p_i^j = \lambda\, \mathbb{1}_{V_j}$ for some $\lambda \in \mathbb{C}$. Taking trace, we have

$$\begin{aligned}
\lambda \dim V_j = \mathrm{Tr}\, p_i^j &= \frac{\dim V_i}{|G|} \sum_{g \in G} \overline{\chi_{\rho_i}(g)}\, \mathrm{Tr}\, \rho_j(g) \\
&= \frac{\dim V_i}{|G|} \sum_{g \in G} \overline{\chi_{\rho_i}(g)}\, \chi_{\rho_j}(g) \\
&= \dim V_i \left(\chi_{\rho_i}, \chi_{\rho_j} \right) \\
&= \dim V_i \delta_i^j .
\end{aligned} \tag{2.111}$$

Therefore, $\lambda = \frac{\dim V_i}{\dim V_j} \delta_i^j = \delta_i^j$. As a result,

$$p_i^j = \delta_i^j\, \mathbb{1}_{V_j} . \tag{2.112}$$

Now, write an element $\mathbf{v} \in V = \bigoplus_{i=1}^m V_i^{a_i}$ as $\begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_m \end{bmatrix}$, with $\mathbf{v}_i \in V_i^{a_i}$. Then

$$\begin{aligned}
\pi_i(\mathbf{v}) &= \dim V_i \frac{1}{|G|} \sum_{g \in G} \overline{\chi_{\rho_i}(g)} \left(\bigoplus_{i=1}^m \rho_i^{\oplus a_i} \right)(g) \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_m \end{bmatrix} \\
&= \begin{bmatrix} \dim V_i \frac{1}{|G|} \sum_{g \in G} \overline{\chi_{\rho_i}(g)} \rho_1^{\oplus a_1}(g)\, \mathbf{v}_1 \\ \vdots \\ \dim V_i \frac{1}{|G|} \sum_{g \in G} \overline{\chi_{\rho_i}(g)} \rho_i^{\oplus a_i}(g)\, \mathbf{v}_i \\ \vdots \\ \dim V_i \frac{1}{|G|} \sum_{g \in G} \overline{\chi_{\rho_i}(g)} \rho_m^{\oplus a_m}(g)\, \mathbf{v}_m \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ \vdots \\ \mathbf{v}_i \\ \vdots \\ 0 \end{bmatrix} .
\end{aligned}$$

Therefore, $\pi_i(\mathbf{v}) = \mathbf{v}_i$, so π_i is the projection onto $V_i^{a_i}$. ∎

3 Character Table

The character of a representation of a group G is actually a function on the set of conjugacy classes of G. In the character table, we list the character values on the conjugacy classes in different rows. We write the number of elements in each conjugacy class just above the class. By Corollary 2.14, the number of irreducible representations of G is equal to the number of conjugacy class of G. Therefore, a character table will have the same number of rows and columns. A typical character table looks as follows:

# G	g_1	g_2	$\cdots$	g_m
ρ_1				
ρ_2				
$\cdots$				
ρ_m				

Here, $g_1, \ldots, g_m$ are representatives of the conjugacy classes. Above these group elements, we write the size of the conjugacy classes. Then we fill out the table by writing out the values of the character of irreducible representations $\rho_1, \ldots, \rho_m$ on each conjugacy classes.

Example 3.1. We have already calculated the irreducible representations of $\mathfrak{S}_3$. There are 3 conjugacy classes of $\mathfrak{S}_3$:

$$\{1\}, \quad \{(1\ 2), (1\ 3), (2\ 3)\}, \quad \{(1\ 2\ 3), (1\ 3\ 2)\}.$$

Likewise, there are 3 irreducible representations: the trivial representation $\sigma : \mathfrak{S}_3 \to \mathbb{C}^\times$ that maps $g \in \mathfrak{S}_3$ to 1; the sign representation $\sigma' : \mathfrak{S}_3 \to \mathbb{C}^\times$ that maps $g \in \mathfrak{S}_3$ to $\operatorname{sgn} g$; the standard representation $\rho_{\text{std}} : \mathfrak{S}_3 \to \mathrm{GL}(V)$, where $V \subseteq \mathbb{C}^3$ is the subapace

$$V = \left\{ \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \mid z_1 + z_2 + z_3 = 0 \right\}.$$

We have calculated the characters in (2.51).

$$\chi_{\rho_{\text{std}}}((1\ 2\ 3)) = -1, \quad \chi_{\rho_{\text{std}}}((1\ 2)) = 0, \quad \chi_{\rho_{\text{std}}}(1) = 2. \tag{3.1}$$

So, the character table of $\mathfrak{S}_3$ is

# $\mathfrak{S}_3$	1 1	2 $(1\ 2\ 3)$	3 $(1\ 2)$
σ	1	1	1
σ'	1	1	-1
ρ_{std}	2	-1	0

§3.1 Conjugacy classes of symmetric group $\mathfrak{S}_n$

Two group elements $x_1, x_2 \in G$ are conjugate if and only if there exists another group element $y \in G$ such that $x_1 = yx_2y^{-1}$. The group can be divided into classes of conjugate group elements. Indeed, conjugacy is an equivalence relation, and the equivalence classes (i.e. the classes of conjugate elements) form a partition of the group.

For example, take $x_1 = (1\ 5\ 3\ 6\ 7\ 4\ 2)\,(8\ 10) \in \mathfrak{S}_n$. Let us represent $y \in \mathfrak{S}_n$ by the follwing array:

$$y = \begin{pmatrix} 1 & 2 & 3 & \cdots & 10 & \cdots \\ i_1 & i_2 & i_3 & \cdots & i_{10} & \cdots \end{pmatrix}.$$

Let $x_2 = yx_1y^{-1}$. Then

$$x_2\,(i_j) = yx_1y^{-1}\,(i_j) = yx_1\,(j) = i_{x_1(j)}. \tag{3.2}$$

Since x_1 does not change values larger than 10, x_2 will not change i_j for $j > 10$. So we can express x_2 as the following array:

$$x_2 = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 & i_6 & i_7 & i_8 & i_9 & i_{10} & \cdots \\ i_5 & i_1 & i_6 & i_2 & i_3 & i_7 & i_4 & i_{10} & i_9 & i_8 & \cdots \end{pmatrix}.$$

Therefore,

$$x_2 = (i_1\ i_5\ i_3\ i_6\ i_7\ i_4\ i_2)\,(i_8\ i_{10})\,.$$

It has the same cycle structure as x_1 (it is comprised of a 7-cycle and a 2-cycle). Thus, one verifies that elements in the same conjugacy class of $\mathfrak{S}_n$ have the same cycle structure. We, then, have the following result on the symmetric group $\mathfrak{S}_n$:

Theorem 3.1 (a) Every permutation, i.e. an element of $\mathfrak{S}_n$ can be represented by a product of disjoint cycles. This decomposition is unique up to an ordering of factors– the product of disjoint cycles is commutative.

(b) Every permutation may be represented by a product of transpositions. The number of transpositions in any decomposition of a given $g \in \mathfrak{S}_n$ is invariant mod 2.

Since the cycle lengths themselves are characterized by partition of n and all the elements in the same conjugacy classes have the same cycle structure, the number of conjugacy classes is precisely the number of distinct partitions of n. For instance $\mathfrak{S}_3$ has 3 conjugacy classes, as there are 3 partitions of 3: 3, $2+1$, $1+1+1$. Also, $\mathfrak{S}_4$ has 5 conjugacy classes, since there are 5 partitions of 4: 4, $3+1$, $2+2$, $2+1+1$, $1+1+1+1$.

Because the disjoint cycles commute, we can order them from large to small. The partitions may be characterized by the set of non-negative integers $\alpha = (\alpha_1, \ldots, \alpha_n)$ such that

$$n = \alpha_1 + 2\alpha_2 + 3\alpha_3 + \cdots + n\alpha_n. \tag{3.3}$$

Example 3.2. Elements in the same conjugacy class of $\mathfrak{S}_n$ have the same cycle structure. If an element of $\mathfrak{S}_n$ is given by the cycle structure $\alpha = (\alpha_1, \ldots, \alpha_n)$ such that $n = \alpha_1 + 2\alpha_2 + 3\alpha_3 + \cdots + n\alpha_n$, then the element is written as a product of α_i i-cycles, for $i = 1, 2, \ldots, n$. Then the number of elements in the conjugacy class of $\mathfrak{S}_n$ containing this element is

$$h_\alpha = \frac{n!}{\prod_{j=1}^n \alpha_j! j^{\alpha_j}}. \tag{3.4}$$

Indeed, there are $\frac{n!}{\prod_{j=1}^n j!^{\alpha_j}}$ many ways to divide n numbers into α_1-many subsets of size 1, α_2-many subsets of size 2, …, α_n-many subsets of size n. The ordering of the subsets of same size doesn't really matter, so we need to divide by $\prod_{j=1}^n \alpha_j!$. Then each of the subsets of size j gives us $(j-1)!$ many different j-cycles. So we need to further multiply it by $\prod_{j=1}^n (j-1)!^{\alpha_j}$. Finally, the result we get is

$$\frac{n!}{\prod_{j=1}^n j!^{\alpha_j}} \frac{1}{\prod_{j=1}^n \alpha_j!} \prod_{j=1}^n (j-1)!^{\alpha_j} = \frac{n!}{\prod_{j=1}^n \alpha_j! j^{\alpha_j}}.$$

Consider $\mathfrak{S}_4$. For $n = 4$, then the possible candidates for the quadruple $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ are

$$(4,0,0,0)\,,(2,1,0,0)\,,(1,0,1,0)\,,(0,2,0,0)\,,(0,0,0,1)\,.$$

Cycle Structure $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$	Relevant partition of $n = 4$	number of elements in conjugacy class
$(4,0,0,0)$	$1+1+1+1$	$\frac{4!}{4!\cdot 1^4} = 1$
$(1,0,1,0)$	$3+1$	$\frac{4!}{1!\cdot 1^1 \cdot 1!\cdot 3} = 8$
$(2,1,0,0)$	$2+1+1$	$\frac{4!}{2!\cdot 1^2 \cdot 1!\cdot 2^1} = 6$
$(0,2,0,0)$	$2+2$	$\frac{4!}{2!\cdot 2^2} = 3$
$(0,0,0,1)$	4	$\frac{4!}{1!\cdot 4^1} = 6$

Theorem 3.2
The order of a conjugacy class divides the order of the group G.

Proof. We define a subgroup U_x, called the centralizer of $x \in G$:

$$U_x = \left\{ y \in G \mid yxy^{-1} = x \right\}. \tag{3.5}$$

Now observe that, two elements uxu^{-1} and vxv^{-1} are identical if and only if u and v belong to the same left coset of U_x. Indeed,

$$\begin{aligned} uxu^{-1} = vxv^{-1} &\iff x = \left(u^{-1}v\right) x \left(v^{-1}u\right) = \left(u^{-1}v\right) x \left(u^{-1}v\right)^{-1} \\ &\iff u^{-1}v \in U_x \iff v \in uU_x. \end{aligned}$$

Hence, if two elements uxu^{-1} and vxv^{-1} that are conjugate to x are distinct, then u and v must belong to distinct cosets of U_x and vice versa. Therefore, the number of distinct elements that are conjugate to x is precisely the number of left cosets of the subgroup U_x. This is the index of the subgroup U_x, which is $\frac{|G|}{|U_x|}$. Clearly, this number divides $|G|$. ■

§3.2 Character table properties

Before we compute the character table of some interesting groups, we need some results about the character table. Theorem 2.5 says that the rows of the character table are orthonormal. Similarly, the columns are also orthogonal, which is illustrated in the following result.

Theorem 3.3
If $g, h \in G$, then

$$\sum_{i=1}^{m} \overline{\chi_i(g)}\, \chi_i(h) = \begin{cases} \frac{|G|}{c(g)} & \text{if } g \text{ is conjugate to } h, \\ 0 & \text{otherwise;} \end{cases} \tag{3.6}$$

where m is the number of irreducible representations of G, and $c(g)$ is the number of group elements that belong to the conjugacy class containing g.

Proof. By Corollary 2.14, the number of irreducible representations is the same as the number of conjugacy classes of the group G. This means that the character table $T = [\chi_i(c_j)]$ is a square matrix. Hence, there will be m conjugacy classes in this case. We denote a representative of the j-th conjugacy class by g_j. The size of the j-th conjugacy class is then $c(g_j)$. The row orthonormality condition (Theorem 2.5) gives us that

$$\begin{aligned} (\chi_i, \chi_j) = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_i(g)}\, \chi_j(g) = \delta_{ij} \\ \implies \frac{1}{|G|} \sum_{g \in G} c(g_k)\, \overline{\chi_i(g_k)}\, \chi_j(g_k) = \delta_{ij}. \end{aligned}$$

In the matrix form, this equation translates to

$$T\begin{bmatrix} \frac{c(g_1)}{|G|} & 0 & \cdots & 0 \\ 0 & \frac{c(g_2)}{|G|} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{c(g_m)}{|G|} \end{bmatrix} T^\dagger = I_{m\times m}. \tag{3.7}$$

In other words, $TDT^\dagger = I$, where D is the diagonal matrix in the middle. Therefore, $T^\dagger T = D^{-1}$. In the component form, this gives us

$$\begin{aligned} \left(T^\dagger T\right)_{ij} &= \left(D^{-1}\right)_{ij} \\ \implies \sum_{k=1}^{m} T^\dagger_{ik} T_{kj} &= \left(D^{-1}\right)_{ij} \\ \implies \sum_{k=1}^{m} \overline{T}_{ki} T_{kj} &= \left(D^{-1}\right)_{ij} \\ \implies \sum_{k=1}^{m} \overline{\chi_k\left(g_i\right)}\, \chi_k\left(g_j\right) &= \begin{cases} \frac{|G|}{c(g_j)} & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Therefore, we have the desired result

$$\sum_{i=1}^{m} \overline{\chi_i\left(g\right)}\, \chi_i\left(h\right) = \begin{cases} \frac{|G|}{c(g)} & \text{if } g \text{ is conjugate to } h, \\ 0 & \text{otherwise.} \end{cases} \tag{3.8}$$

■

Remark 3.1. By Theorem 3.2, the index of U_g (the centralizer of g) in G is the order $c\left(g\right)$ of the conjugacy class containing g, which is $\frac{|G|}{|U_g|}$. Therefore,

$$|U_g| = \frac{|G|}{c\left(g\right)}. \tag{3.9}$$

Therefore, when g, h are conjugate to each other,

$$\sum_{i=1}^{m} \overline{\chi_i\left(g\right)}\, \chi_i\left(h\right) = |U_g|\,. \tag{3.10}$$

Remark 3.2. Theorem 2.5 and Theorem 3.3 will help us fill in the missing rows or columns of character table without explicitly computing the representations.

Lemma 3.4

Let $\sigma : G \to \mathrm{GL}(V)$ be an irreducible representation of the finite group G. If $\rho : G \to \mathbb{C}^\times$ is a 1-dimensional representation of G, then $\sigma \otimes \rho$ is also an irreducible representation of G.

Proof. Since ρ is a 1-dimensional representation, $\rho\left(g\right) \in \mathbb{C}^\times$ is some root of unity. So $\left|\chi_\rho\left(g\right)\right|^2 = 1$.

$$\begin{aligned} \left(\chi_{\sigma\otimes\rho}, \chi_{\sigma\otimes\rho}\right) &= \frac{1}{|G|}\sum_{g\in G} \overline{\chi_{\sigma\otimes\rho}\left(g\right)}\, \chi_{\sigma\otimes\rho}\left(g\right) \\ &= \frac{1}{|G|}\sum_{g\in G} \overline{\chi_\sigma\left(g\right)\chi_\rho(g)}\, \chi_\sigma\left(g\right)\chi_\rho(g) \\ &= \frac{1}{|G|}\sum_{g\in G} \overline{\chi_\sigma\left(g\right)}\, \chi_\sigma\left(g\right) \overline{\chi_\rho(g)}\, \chi_\rho(g) \\ &= \frac{1}{|G|}\sum_{g\in G} \overline{\chi_\sigma\left(g\right)}\, \chi_\sigma\left(g\right) = 1. \end{aligned}$$

So $\sigma \otimes \rho$ is also an irreducible representation by Corollary 2.7. ■

Lemma 3.5

Let $\rho : G \to \mathrm{GL}(V)$ be a representation of G. If $\chi_\rho(g) = \chi_\rho(e)$, then $\rho(g) = \rho(e)$.

Proof. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $\rho(g)$. Since $\rho(g)^{|G|} = \rho\left(g^{|G|}\right) = \rho(1) = \mathbb{1}_V$, we have $\lambda_j^{|G|} = 1$ for each j, i.e. each λ_j is some root of unity. In particular, $|\lambda_j| = 1$. Now,

$$n = |\chi_\rho(g)| = \left|\sum_{j=1}^n \lambda_j\right| \le \sum_{j=1}^n |\lambda_j| = n. \tag{3.11}$$

The equality case of triangle inequality occurs when all the summands have the same argument. Since all λ_j's have the same modulus as well, all λ_j's are equal; i.e. $\lambda_1 = \lambda_2 = \cdots = \lambda_n =: \lambda$.

$$n = \chi_\rho(g) = \mathrm{Tr}\left[\rho(g)\right] = \sum_{j=1}^n \lambda_j = n\lambda. \tag{3.12}$$

So $\lambda = 1$. Now, $\rho(g)$ is diagonalizable, with all the eigenvalues being 1. Therefore, $\rho(g) = \mathbb{1}_V = \rho(e)$. ∎

Theorem 3.6

If V is a faithful representation of G, i.e., $\rho : G \to \mathrm{GL}(V)$ is injective, then any irreducible representation of G is contained in some tensor power $V^{\otimes n}$ of V.

Proof. If $\chi_\rho(g) = \chi_\rho(e)$, then $\rho(g) = \rho(e)$ (by Lemma 3.5). But in this case, ρ is a faithful representation. So, there does not exist any $g \in G$ such that $\chi_\rho(g) = \chi_\rho(e)$.

Let $\sigma : G \to \mathrm{GL}(W)$ be an irreducible representation of G.

$$W \text{ is contained in } V^{\otimes n} \text{ as a direct summand} \iff (\chi_\sigma, \chi_{\rho^{\otimes n}}) \in \mathbb{Z}_{>0}.$$

But $(\chi_\sigma, \chi_{\rho^{\otimes n}})$ is a non-negative integer, so it suffices to prove that this inner product is nonzero. Consider

$$a_n = (\chi_\sigma, \chi_{\rho^{\otimes n}}) = \left(\chi_\sigma, \chi_\rho^n\right) = \frac{1}{|G|}\sum_{g\in G} \overline{\chi_\sigma(g)}\,\chi_\rho(g)^n. \tag{3.13}$$

Assume for the sake of contradiction that $a_n = 0$ for every n. Then the formal power series $\sum_{n=0}^\infty a_n t^n$ is identically 0.

$$\begin{aligned} 0 &= \sum_{n=0}^\infty a_n t^n \\ &= \frac{1}{|G|}\sum_{n=0}^\infty \sum_{g\in G} \overline{\chi_\sigma(g)}\,\chi_\rho(g)^n t^n \\ &= \frac{1}{|G|}\sum_{n=0}^\infty \sum_{C} |C|\,\overline{\chi_\sigma(g_C)}\,\chi_\rho(g_C)^n t^n, \end{aligned} \tag{3.14}$$

where the sum runs over all the conjugacy classes C of G, g_C is a representative of C. Since the C-sum is a finite sum, it commutes with the n-sum. Therefore,

$$\begin{aligned} 0 &= \frac{1}{|G|}\sum_{C} |C|\,\overline{\chi_\sigma(g_C)} \sum_{n=0}^\infty \chi_\rho(g_C)^n t^n \\ &= \frac{1}{|G|}\sum_{C} |C|\,\overline{\chi_\sigma(g_C)}\,\frac{1}{1-\chi_\rho(g_C)\,t}. \end{aligned} \tag{3.15}$$

So we have

$$\sum_C \frac{|C|\,\overline{\chi_\sigma(g_C)}}{1-\chi_\rho(g_C)\,t} = 0. \tag{3.16}$$

For $C \neq \{e\}$, $1 - \chi_\rho(g_C)\,t \neq 1 - \chi_\rho(e)t$. So, the sum is of the following form:

$$\frac{c_1}{1-b_1t} + \frac{c_2}{1-b_2t} + \cdots + \frac{c_k}{1-b_kt} = 0, \tag{3.17}$$

where b_i's are pairwise disjoint. WLOG, $b_1 = \chi_\rho(e) = \dim V \neq 0$. So $c_1 = \overline{\chi_\sigma(e)} = \dim W$. The other b_i's are $\chi_\rho(g_C)$. If there exists C, C' with $\chi_\rho(g_C) = \chi_\rho(g_{C'})$, we combine them into one term $\frac{a_i}{1-b_it}$.

Multiplying (3.17) by $\prod_{i=1}^k (1-b_it)$, we get

$$c_1 \prod_{i\neq 1}(1-b_it) + c_2 \prod_{i\neq 2}(1-b_it) + \cdots + c_k \prod_{i\neq k}(1-b_it) = 0. \tag{3.18}$$

This holds for all values of t. Plugging in $t = \frac{1}{b_1}$, we have $1 - b_1t = 0$. As a result, (3.18) becomes

$$c_1 \prod_{i\neq 1}\left(1 - \frac{b_i}{b_1}\right) = 0. \tag{3.19}$$

So we have $c_1 = 0$. But $c_1 = \dim W \neq 0$, since irreducible representations are by definition nonzero. So we have a contradiction. Therefore, there must exist some n such that $a_n \neq 0$, and we are done! ■

§3.3 Character table of $\mathfrak{S}_4$

Let us now compute the character table of $\mathfrak{S}_4$. As we have seen earlier, there are 5 conjugacy classes, and hence there are 5 irreducible representations. There is a trivial representation $\sigma : \mathfrak{S}_4 \to \mathrm{GL}(\mathbb{C}) \equiv \mathbb{C}^\times$ with $\sigma(g) = 1$ for every $g \in \mathfrak{S}_4$. There is an alternating representation as well: $\sigma' : \mathfrak{S}_4 \to \mathrm{GL}(\mathbb{C}) \equiv \mathbb{C}^\times$ given by $\sigma'(g) = \operatorname{sgn} g$ for every $g \in \mathfrak{S}_4$.

Cycle Structure $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$	**Representative of conjugacy class**	**number of elements in conjugacy class**	**sign of permutation**
$(4,0,0,0)$	1	1	1
$(1,0,1,0)$	$(1\ 2\ 3) = (1\ 2)\,(2\ 3)$	8	1
$(2,1,0,0)$	$(1\ 2)$	6	-1
$(0,2,0,0)$	$(1\ 2)\,(3\ 4)$	3	1
$(0,0,0,1)$	$(1\ 2\ 3\ 4) = (1\ 3)\,(3\ 4)\,(1\ 2)$	6	-1

Let ρ be the permutation representation of $\mathfrak{S}_4$ on the set $\{1,2,3,4\}$. By Lemma 2.2, $\chi_\rho(g)$ is the number of elements of $X = \{1,2,3,4\}$ fixed by g. Therefore,

$$\chi_\rho(1) = 4, \quad \chi_\rho((1\ 2)) = 2, \quad \chi_\rho((1\ 2\ 3)) = 1, \quad \chi_\rho((1\ 2\ 3\ 4)) = 0, \quad \chi_\rho((1\ 2)\,(3\ 4)) = 0. \tag{3.20}$$

Now,

$$\begin{aligned}(\chi_\rho, \chi_\rho) &= \frac{1}{24}\Big[\overline{\chi_\rho(1)}\,\chi_\rho(1) + 8\,\overline{\chi_\rho((1\ 2\ 3))}\,\chi_\rho((1\ 2\ 3)) + 6\,\overline{\chi_\rho((1\ 2))}\,\chi_\rho((1\ 2)) \\ &\quad + 3\,\overline{\chi_\rho((1\ 2)\,(3\ 4))}\,\chi_\rho((1\ 2)\,(3\ 4)) + 6\,\overline{\chi_\rho((1\ 2\ 3\ 4))}\,\chi_\rho((1\ 2\ 3\ 4))\Big] \\ &= \frac{1}{24}\,[16 + 8 + 24 + 0 + 0] = 2 \neq 1.\end{aligned}$$

Therefore, the permutation representation ρ is not irreducible. By (2.86), $(\chi_\rho, \chi_\rho) = \sum_i |a_i|^2$, where a_i are the multiplicities of the irreducible representations. Here, $\sum_i |a_i|^2 = 2$, so there are two i such that $a_i = 1$, and the rest a_i's are all 0. In other words, the permutation representation is the direct sum of two irreducibles. Now,

$$\begin{aligned}(\chi_\sigma, \chi_\rho) = \frac{1}{24}\sum_{g\in\mathfrak{S}_4}\chi_\rho(g) &= \frac{1}{24}\Big[\chi_\rho(1) + 8\,\chi_\rho((1\ 2\ 3)) + 6\,\chi_\rho((1\ 2)) \\ &\quad + 3\,\chi_\rho((1\ 2)\,(3\ 4)) + 6\,\chi_\rho((1\ 2\ 3\ 4))\Big] \\ &= \frac{1}{24}\,[4 + 8 + 12 + 0 + 0] = 1.\end{aligned}$$

Therefore, the trivial representation σ is a direct summand of the permutation representation ρ. The other direct summand is called the standard representation, which we denote as ρ_{std}. Therefore,

$$\rho \cong \sigma \oplus \rho_{\text{std}}, \tag{3.21}$$

so that

$$\chi_{\rho_{\text{std}}} = \chi_\rho - \chi_\sigma = \chi_\rho - 1. \tag{3.22}$$

Using this, we can fill out the first 3 rows of the character table.

#	1	8	6	3	6
$\mathfrak{S}_4$	1	(1 2 3)	(1 2)	(1 2) (3 4)	(1 2 3 4)
σ	1	1	1	1	1
σ'	1	1	-1	1	-1
ρ_{std}	3	0	1	-1	-1

By Lemma 3.4, $\rho_{\text{std}} \otimes \sigma'$ is also an irreducible representation. Its character is given by $\chi_{\rho_{\text{std}}}\chi_{\sigma'}$.

#	1	8	6	3	6
$\mathfrak{S}_4$	1	(1 2 3)	(1 2)	(1 2) (3 4)	(1 2 3 4)
σ	1	1	1	1	1
σ'	1	1	-1	1	-1
ρ_{std}	3	0	1	-1	-1
$\rho_{\text{std}} \otimes \sigma'$	3	0	-1	-1	1
τ	x	y	z	t	p

By Corollary 2.10, $1^2 + 1^2 + 3^2 + 3^2 + x^2 = 24$. Therefore, $x = 2$. So τ is a 2-dimensional irreducible representation. By Lemma 3.4, $\tau \otimes \sigma'$ is also an irreducible representation. But there are no more 2-dimensional irreducible representations. Therefore, $\tau \cong \tau \otimes \sigma'$. As a result, $z = p = 0$, because otherwise, $\tau \otimes \sigma'$ will have character values $-z$ and $-p$ for the conjugacy classes of (1 2) and (1 2 3 4), respectively, which is different from that of τ.

Now, we are only left with the value of y and p. Orthogonality of the first and second column gives us that

$$1 \cdot 1 + 1 \cdot 1 + 3 \cdot 0 + 3 \cdot 0 + xy = 0 \implies y = -1, \tag{3.23}$$

since $x = 2$. $(\chi_\sigma, \chi_\tau) = 0$ gives us that $\sum_{g \in G} \chi_\tau(g) = 0$. Hence,

$$2 + 8 \cdot (-1) + 6 \cdot 0 + 3t + 6 \cdot 0 = 0 \implies t = 2. \tag{3.24}$$

Therefore, the complete character table of $\mathfrak{S}_4$ is

#	1	8	6	3	6
$\mathfrak{S}_4$	1	(1 2 3)	(1 2)	(1 2) (3 4)	(1 2 3 4)
σ	1	1	1	1	1
σ'	1	1	-1	1	-1
ρ_{std}	3	0	1	-1	-1
$\rho_{\text{std}} \otimes \sigma'$	3	0	-1	-1	1
τ	2	-1	0	2	0

§3.4 Character table of $\mathfrak{S}_5$

There are 7 partitions of $n = 5$, corresponding to 7 conjugacy classes.

Cycle Structure $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$	**Relevant partition of** $n = 5$	**Representative of conjugacy class**	**number of elements in conjugacy class**
$(5, 0, 0, 0)$	$1 + 1 + 1 + 1 + 1$	1	$\frac{5!}{5! \cdot 1^5} = 1$
$(1, 0, 0, 1, 0)$	$4 + 1$	$(1\ 2\ 3\ 4)$	$\frac{5!}{1! \cdot 1^1 \cdot 1! \cdot 4^1} = 30$
$(0, 1, 1, 0, 0)$	$3 + 2$	$(1\ 2\ 3)\,(4\ 5)$	$\frac{5!}{1! \cdot 3^1 \cdot 1! \cdot 2^1} = 20$
$(2, 0, 1, 0, 0)$	$3 + 1 + 1$	$(1\ 2\ 3)$	$\frac{5!}{1! \cdot 3^1 \cdot 2! \cdot 1^2} = 20$
$(3, 1, 0, 0, 0)$	$2 + 1 + 1 + 1$	$(1\ 2)$	$\frac{5!}{3! \cdot 1^3 \cdot 1! \cdot 2^1} = 10$
$(0, 0, 0, 0, 1)$	5	$(1\ 2\ 3\ 4\ 5)$	$\frac{5!}{5^1 \cdot 1!} = 24$
$(1, 2, 0, 0, 0)$	$2 + 2 + 1$	$(1\ 2)\,(3\ 4)$	$\frac{5!}{2! \cdot 2^2 \cdot 1! \cdot 1^1} = 15$

So, there are 7 irreducible representations of $\mathfrak{S}_5$. We get 4 irreducible representations for free, using our knowledge of the representation theory of $\mathfrak{S}_4$: the trivial representation σ, the alternating representation σ', the standard representation ρ_{std}, tensor product of the standard representation and the alternating representation $\rho_{\text{std}} \otimes \sigma'$.

Let ρ be the permutation representation of $\mathfrak{S}_5$ on the set $\{1, 2, 3, 4, 5\}$. By Lemma 2.2, $\chi_\rho(g)$ is the number of elements of $X = \{1, 2, 3, 4, 5\}$ fixed by g. Therefore,

$$\begin{aligned} \chi_\rho(1) = 5, \quad \chi_\rho((1\ 2)) = 3, \quad \chi_\rho((1\ 2\ 3)) = 2, \quad \chi_\rho((1\ 2\ 3\ 4)) = 1, \\ \chi_\rho((1\ 2)\,(3\ 4)) = 1, \quad \chi_\rho((1\ 2\ 3)\,(4\ 5)) = 0, \quad \chi_\rho((1\ 2\ 3\ 4\ 5)) = 0. \end{aligned} \tag{3.25}$$

Now,

$$\begin{aligned} (\chi_\rho, \chi_\rho) = \frac{1}{120} \Big[& \overline{\chi_\rho(1)}\,\chi_\rho(1) + 30\,\overline{\chi_\rho((1\ 2\ 3\ 4))}\,\chi_\rho((1\ 2\ 3\ 4)) + 20\,\overline{\chi_\rho((1\ 2\ 3)\,(4\ 5))}\,\chi_\rho((1\ 2\ 3)\,(4\ 5)) \\ & + 20\,\overline{\chi_\rho((1\ 2\ 3))}\,\chi_\rho((1\ 2\ 3)) + 10\,\overline{\chi_\rho((1\ 2))}\,\chi_\rho((1\ 2)) \\ & + 24\,\overline{\chi_\rho((1\ 2\ 3\ 4\ 5))}\,\chi_\rho((1\ 2\ 3\ 4\ 5)) + 15\,\overline{\chi_\rho((1\ 2)\,(3\ 4))}\,\chi_\rho((1\ 2)\,(3\ 4)) \Big] \\ = \frac{1}{120} & [25 + 30 + 0 + 80 + 90 + 0 + 15] = 2 \neq 1. \end{aligned}$$

Therefore, the permutation representation ρ is not irreducible. By (2.86), $(\chi_\rho, \chi_\rho) = \sum_i |a_i|^2$, where a_i are the multiplicities of the irreducible representations. Here, $\sum_i |a_i|^2 = 2$, so there are two i such that $a_i = 1$, and the rest a_i's are all 0. In other words, the permutation representation is the direct sum of two irreducibles. Now,

$$\begin{aligned} (\chi_\sigma, \chi_\rho) = \frac{1}{120} \sum_{g \in \mathfrak{S}_5} \chi_\rho(g) = \frac{1}{120} \Big[& \chi_\rho(1) + 30\,\chi_\rho((1\ 2\ 3\ 4)) + 20\,\chi_\rho((1\ 2\ 3)\,(4\ 5)) \\ & + 20\,\chi_\rho((1\ 2\ 3)) + 10\,\chi_\rho((1\ 2)) \\ & + 24\,\chi_\rho((1\ 2\ 3\ 4\ 5)) + 15\,\chi_\rho((1\ 2)\,(3\ 4)) \Big] \\ = \frac{1}{120} & [5 + 30 + 0 + 40 + 30 + 0 + 15] = 1. \end{aligned}$$

Therefore, the trivial representation σ is a direct summand of the permutation representation ρ. The other direct summand is called the standard representation, which we denote as ρ_{std}. Therefore,

$$\rho \cong \sigma \oplus \rho_{\text{std}}, \tag{3.26}$$

so that

$$\chi_{\rho_{\text{std}}} = \chi_\rho - \chi_\sigma = \chi_\rho - 1. \tag{3.27}$$

Using this, we can fill out the first 3 rows of the character table.

#	1	30	20	20	10	24	15
$\mathfrak{S}_5$	1	(1 2 3 4)	(1 2 3) (4 5)	(1 2 3)	(1 2)	(1 2 3 4 5)	(1 2) (3 4)
σ	1	1	1	1	1	1	1
σ'	1	-1	-1	1	-1	1	1
ρ_{std}	4	0	-1	1	2	-1	0

By Lemma 3.4, $\rho_{\text{std}} \otimes \sigma'$ is also an irreducible representation. Its character is given by $\chi_{\rho_{\text{std}}}\chi_{\sigma'}$.

#	1	30	20	20	10	24	15
$\mathfrak{S}_5$	1	(1 2 3 4)	(1 2 3) (4 5)	(1 2 3)	(1 2)	(1 2 3 4 5)	(1 2) (3 4)
σ	1	1	1	1	1	1	1
σ'	1	-1	-1	1	-1	1	1
ρ_{std}	4	0	-1	1	2	-1	0
$\rho_{\text{std}} \otimes \sigma'$	4	0	1	1	-2	-1	0

Now, ρ_{std} is a faithful representation. Indeed, if $g \in \operatorname{Ker} \rho_{\text{std}}$, then $\rho_{\text{std}}(g) = \rho_{\text{std}}(1)$. By taking trace, we have $\chi_{\rho_{\text{std}}}(g) = \chi_{\rho_{\text{std}}}(1) = 4$. From the character table, we can see that no element other than the identity element $1 \in \mathfrak{S}_5$ satisfies $\chi_{\rho_{\text{std}}}(g) = 4$. Therefore, $\operatorname{Ker} \rho_{\text{std}} = \{1\}$, and hence, ρ_{std} is a faithful irreducible representation.

Then by Theorem 3.6, every irreducible representation of $\mathfrak{S}_5$ is contained in some tensor power ${\rho_{\text{std}}}^{\otimes n}$ as a direct summand. So it's natural to look inside $\rho_{\text{std}} \otimes \rho_{\text{std}}$. We know that

$$\rho_{\text{std}} \otimes \rho_{\text{std}} \cong \Lambda^2 \rho_{\text{std}} \oplus \operatorname{Sym}^2 \rho_{\text{std}}. \tag{3.28}$$

Let's now find the characters $\chi_{\Lambda^2\rho_{\text{std}}}$ and $\chi_{\operatorname{Sym}^2\rho_{\text{std}}}$. By Proposition 2.1(d),

$$\chi_{\Lambda^2\rho_{\text{std}}}(g) = \frac{1}{2}\left[(\chi_{\rho_{\text{std}}}(g))^2 - \chi_{\rho_{\text{std}}}\left(g^2\right)\right]. \tag{3.29}$$

$$\begin{aligned}
\chi_{\Lambda^2\rho_{\text{std}}}(1) &= \frac{1}{2}\left[(\chi_{\rho_{\text{std}}}(1))^2 - \chi_{\rho_{\text{std}}}\left(1^2\right)\right] = \frac{1}{2}\left[4^2 - 4\right] = 6. \\
\chi_{\Lambda^2\rho_{\text{std}}}((1\ 2\ 3\ 4)) &= \frac{1}{2}\left[(\chi_{\rho_{\text{std}}}((1\ 2\ 3\ 4)))^2 - \chi_{\rho_{\text{std}}}\left((1\ 2\ 3\ 4)^2\right)\right] \\
&= \frac{1}{2}\left[(\chi_{\rho_{\text{std}}}((1\ 2\ 3\ 4)))^2 - \chi_{\rho_{\text{std}}}((1\ 3)\,(2\ 4))\right] = \frac{1}{2}\left[0^2 - 0\right] = 0. \\
\chi_{\Lambda^2\rho_{\text{std}}}((1\ 2\ 3)\,(4\ 5)) &= \frac{1}{2}\left[(\chi_{\rho_{\text{std}}}((1\ 2\ 3)\,(4\ 5)))^2 - \chi_{\rho_{\text{std}}}\left(((1\ 2\ 3)\,(4\ 5))^2\right)\right] \\
&= \frac{1}{2}\left[(-1)^2 - \chi_{\rho_{\text{std}}}((1\ 3\ 2))\right] = \frac{1}{2}[1 - 1] = 0. \\
\chi_{\Lambda^2\rho_{\text{std}}}((1\ 2\ 3)) &= \frac{1}{2}\left[(\chi_{\rho_{\text{std}}}((1\ 2\ 3)))^2 - \chi_{\rho_{\text{std}}}\left((1\ 2\ 3)^2\right)\right] \\
&= \frac{1}{2}\left[(1)^2 - \chi_{\rho_{\text{std}}}((1\ 3\ 2))\right] = \frac{1}{2}[1 - 1] = 0.
\end{aligned}$$

$$\begin{aligned}
\chi_{\Lambda^2\rho_{\mathrm{std}}}((1\ 2)) &= \frac{1}{2}\left[(\chi_{\rho_{\mathrm{std}}}((1\ 2)))^2 - \chi_{\rho_{\mathrm{std}}}\left((1\ 2)^2\right)\right] \\
&= \frac{1}{2}\left[(2)^2 - \chi_{\rho_{\mathrm{std}}}(1)\right] = \frac{1}{2}[4-4] = 0. \\
\chi_{\Lambda^2\rho_{\mathrm{std}}}((1\ 2\ 3\ 4\ 5)) &= \frac{1}{2}\left[(\chi_{\rho_{\mathrm{std}}}((1\ 2\ 3\ 4\ 5)))^2 - \chi_{\rho_{\mathrm{std}}}\left((1\ 2\ 3\ 4\ 5)^2\right)\right] \\
&= \frac{1}{2}\left[(-1)^2 - \chi_{\rho_{\mathrm{std}}}((1\ 3\ 5\ 2\ 4))\right] = \frac{1}{2}[1-(-1)] = 1. \\
\chi_{\Lambda^2\rho_{\mathrm{std}}}((1\ 2)\,(3\ 4)) &= \frac{1}{2}\left[(\chi_{\rho_{\mathrm{std}}}((1\ 2)\,(3\ 4)))^2 - \chi_{\rho_{\mathrm{std}}}\left(((1\ 2)\,(3\ 4))^2\right)\right] \\
&= \frac{1}{2}\left[(0)^2 - \chi_{\rho_{\mathrm{std}}}(1)\right] = \frac{1}{2}[-4] = -2.
\end{aligned}$$

Then its norm is

$$\begin{aligned}
\left(\chi_{\Lambda^2\rho_{\mathrm{std}}}, \chi_{\Lambda^2\rho_{\mathrm{std}}}\right) = \frac{1}{120}\Big[&\overline{\chi_{\Lambda^2\rho_{\mathrm{std}}}(1)}\,\chi_{\Lambda^2\rho_{\mathrm{std}}}(1) + 30\,\overline{\chi_{\Lambda^2\rho_{\mathrm{std}}}((1\ 2\ 3\ 4))}\,\chi_{\Lambda^2\rho_{\mathrm{std}}}((1\ 2\ 3\ 4)) \\
&+ 20\,\overline{\chi_{\Lambda^2\rho_{\mathrm{std}}}((1\ 2\ 3)\,(4\ 5))}\,\chi_{\Lambda^2\rho_{\mathrm{std}}}((1\ 2\ 3)\,(4\ 5)) \\
&+ 20\,\overline{\chi_{\Lambda^2\rho_{\mathrm{std}}}((1\ 2\ 3))}\,\chi_{\Lambda^2\rho_{\mathrm{std}}}((1\ 2\ 3)) + 10\,\overline{\chi_{\Lambda^2\rho_{\mathrm{std}}}((1\ 2))}\,\chi_{\Lambda^2\rho_{\mathrm{std}}}((1\ 2)) \\
&+ 24\,\overline{\chi_{\Lambda^2\rho_{\mathrm{std}}}((1\ 2\ 3\ 4\ 5))}\,\chi_{\Lambda^2\rho_{\mathrm{std}}}((1\ 2\ 3\ 4\ 5)) \\
&+ 15\,\overline{\chi_{\Lambda^2\rho_{\mathrm{std}}}((1\ 2)\,(3\ 4))}\,\chi_{\Lambda^2\rho_{\mathrm{std}}}((1\ 2)\,(3\ 4))\Big] \\
= \frac{1}{120}&[36+0+0+0+0+24+60] = 1.
\end{aligned}$$

Therefore, $\Lambda^2\rho_{\mathrm{std}}$ is an irreducible representation of $\mathfrak{S}_5$. So we can update the character table:

#	1	30	20	20	10	24	15
$\mathfrak{S}_5$	1	(1 2 3 4)	(1 2 3) (4 5)	(1 2 3)	(1 2)	(1 2 3 4 5)	(1 2) (3 4)
σ	1	1	1	1	1	1	1
σ'	1	-1	-1	1	-1	1	1
ρ_{std}	4	0	-1	1	2	-1	0
$\rho_{\mathrm{std}}\otimes\sigma'$	4	0	1	1	-2	-1	0
$\Lambda^2\rho_{\mathrm{std}}$	6	0	0	0	0	1	-2

Tensoring it with σ' doesn't give us a new representation. Let's now calculate the character of $\mathrm{Sym}^2\rho_{\mathrm{std}}$. By Remark 2.3,

$$\chi_{\mathrm{Sym}^2\rho_{\mathrm{std}}}(g) = \frac{1}{2}\left[(\chi_{\rho_{\mathrm{std}}}(g))^2 + \chi_{\rho_{\mathrm{std}}}\left(g^2\right)\right]. \tag{3.30}$$

$$\begin{aligned}
\chi_{\mathrm{Sym}^2\rho_{\mathrm{std}}}(1) &= \frac{1}{2}\left[(\chi_{\rho_{\mathrm{std}}}(1))^2 + \chi_{\rho_{\mathrm{std}}}\left(1^2\right)\right] = \frac{1}{2}\left[4^2+4\right] = 10. \\
\chi_{\mathrm{Sym}^2\rho_{\mathrm{std}}}((1\ 2\ 3\ 4)) &= \frac{1}{2}\left[(\chi_{\rho_{\mathrm{std}}}((1\ 2\ 3\ 4)))^2 + \chi_{\rho_{\mathrm{std}}}\left((1\ 2\ 3\ 4)^2\right)\right] \\
&= \frac{1}{2}\left[(\chi_{\rho_{\mathrm{std}}}((1\ 2\ 3\ 4)))^2 + \chi_{\rho_{\mathrm{std}}}((1\ 3)\,(2\ 4))\right] = \frac{1}{2}\left[0^2+0\right] = 0. \\
\chi_{\mathrm{Sym}^2\rho_{\mathrm{std}}}((1\ 2\ 3)\,(4\ 5)) &= \frac{1}{2}\left[(\chi_{\rho_{\mathrm{std}}}((1\ 2\ 3)\,(4\ 5)))^2 + \chi_{\rho_{\mathrm{std}}}\left(((1\ 2\ 3)\,(4\ 5))^2\right)\right] \\
&= \frac{1}{2}\left[(-1)^2 + \chi_{\rho_{\mathrm{std}}}((1\ 3\ 2))\right] = \frac{1}{2}[1+1] = 1. \\
\chi_{\mathrm{Sym}^2\rho_{\mathrm{std}}}((1\ 2\ 3)) &= \frac{1}{2}\left[(\chi_{\rho_{\mathrm{std}}}((1\ 2\ 3)))^2 + \chi_{\rho_{\mathrm{std}}}\left((1\ 2\ 3)^2\right)\right] \\
&= \frac{1}{2}\left[(1)^2 + \chi_{\rho_{\mathrm{std}}}((1\ 3\ 2))\right] = \frac{1}{2}[1+1] = 1.
\end{aligned}$$

$$\begin{aligned}
\chi_{\mathrm{Sym}^2\rho_{\mathrm{std}}}((1\ 2)) &= \frac{1}{2}\left[(\chi_{\rho_{\mathrm{std}}}((1\ 2)))^2 + \chi_{\rho_{\mathrm{std}}}\left((1\ 2)^2\right)\right] \\
&= \frac{1}{2}\left[(2)^2 + \chi_{\rho_{\mathrm{std}}}(1)\right] = \frac{1}{2}[4+4] = 4. \\
\chi_{\mathrm{Sym}^2\rho_{\mathrm{std}}}((1\ 2\ 3\ 4\ 5)) &= \frac{1}{2}\left[(\chi_{\rho_{\mathrm{std}}}((1\ 2\ 3\ 4\ 5)))^2 + \chi_{\rho_{\mathrm{std}}}\left((1\ 2\ 3\ 4\ 5)^2\right)\right] \\
&= \frac{1}{2}\left[(-1)^2 + \chi_{\rho_{\mathrm{std}}}((1\ 3\ 5\ 2\ 4))\right] = \frac{1}{2}[1+(-1)] = 0. \\
\chi_{\mathrm{Sym}^2\rho_{\mathrm{std}}}((1\ 2)\,(3\ 4)) &= \frac{1}{2}\left[(\chi_{\rho_{\mathrm{std}}}((1\ 2)\,(3\ 4)))^2 + \chi_{\rho_{\mathrm{std}}}\left(((1\ 2)\,(3\ 4))^2\right)\right] \\
&= \frac{1}{2}\left[(0)^2 + \chi_{\rho_{\mathrm{std}}}(1)\right] = \frac{1}{2}[0+4] = 2.
\end{aligned}$$

Let's now calculate the norm of $\chi_{\mathrm{Sym}^2\rho_{\mathrm{std}}}$.

$$\begin{aligned}
\left(\chi_{\mathrm{Sym}^2\rho_{\mathrm{std}}}, \chi_{\mathrm{Sym}^2\rho_{\mathrm{std}}}\right) = \frac{1}{120}\Big[&\overline{\chi_{\mathrm{Sym}^2\rho_{\mathrm{std}}}(1)}\,\chi_{\mathrm{Sym}^2\rho_{\mathrm{std}}}(1) + 30\,\overline{\chi_{\mathrm{Sym}^2\rho_{\mathrm{std}}}((1\ 2\ 3\ 4))}\,\chi_{\mathrm{Sym}^2\rho_{\mathrm{std}}}((1\ 2\ 3\ 4)) \\
&+ 20\,\overline{\chi_{\mathrm{Sym}^2\rho_{\mathrm{std}}}((1\ 2\ 3)\,(4\ 5))}\,\chi_{\mathrm{Sym}^2\rho_{\mathrm{std}}}((1\ 2\ 3)\,(4\ 5)) \\
&+ 20\,\overline{\chi_{\mathrm{Sym}^2\rho_{\mathrm{std}}}((1\ 2\ 3))}\,\chi_{\mathrm{Sym}^2\rho_{\mathrm{std}}}((1\ 2\ 3)) \\
&+ 10\,\overline{\chi_{\mathrm{Sym}^2\rho_{\mathrm{std}}}((1\ 2))}\,\chi_{\mathrm{Sym}^2\rho_{\mathrm{std}}}((1\ 2)) \\
&+ 24\,\overline{\chi_{\mathrm{Sym}^2\rho_{\mathrm{std}}}((1\ 2\ 3\ 4\ 5))}\,\chi_{\mathrm{Sym}^2\rho_{\mathrm{std}}}((1\ 2\ 3\ 4\ 5)) \\
&+ 15\,\overline{\chi_{\mathrm{Sym}^2\rho_{\mathrm{std}}}((1\ 2)\,(3\ 4))}\,\chi_{\mathrm{Sym}^2\rho_{\mathrm{std}}}((1\ 2)\,(3\ 4))\Big] \\
= \frac{1}{120}&[100+0+20+20+160+0+60] = 3.
\end{aligned}$$

Therefore, the permutation representation $\mathrm{Sym}^2\rho_{\mathrm{std}}$ is not irreducible. By (2.86), $\left(\chi_{\mathrm{Sym}^2\rho_{\mathrm{std}}}, \chi_{\mathrm{Sym}^2\rho_{\mathrm{std}}}\right) = \sum_i |a_i|^2$, where a_i are the multiplicities of the irreducible representations. Here, $\sum_i |a_i|^2 = 3$, so there are three i such that $a_i = 1$, and the rest a_i's are all 0. In other words, the permutation representation is the direct sum of three irreducibles. Now,

$$\begin{aligned}
\left(\chi_\sigma, \chi_{\mathrm{Sym}^2\rho_{\mathrm{std}}}\right) &= \frac{1}{120}\sum_{g\in\mathfrak{S}_5}\chi_{\mathrm{Sym}^2\rho_{\mathrm{std}}}(g) \\
&= \frac{1}{120}\Big[\chi_{\mathrm{Sym}^2\rho_{\mathrm{std}}}(1) + 30\,\chi_{\mathrm{Sym}^2\rho_{\mathrm{std}}}((1\ 2\ 3\ 4)) + 20\,\chi_{\mathrm{Sym}^2\rho_{\mathrm{std}}}((1\ 2\ 3)\,(4\ 5)) \\
&\quad + 20\,\chi_{\mathrm{Sym}^2\rho_{\mathrm{std}}}((1\ 2\ 3)) + 10\,\chi_{\mathrm{Sym}^2\rho_{\mathrm{std}}}((1\ 2)) \\
&\quad + 24\,\chi_{\mathrm{Sym}^2\rho_{\mathrm{std}}}((1\ 2\ 3\ 4\ 5)) + 15\,\chi_{\mathrm{Sym}^2\rho_{\mathrm{std}}}((1\ 2)\,(3\ 4))\Big] \\
&= \frac{1}{120}[10+0+20+20+40+0+30] = 1.
\end{aligned}$$

Therefore, the trivial representation σ is a direct summand of $\mathrm{Sym}^2\rho_{\mathrm{std}}$.

$$\begin{aligned}
\left(\chi_{\sigma'}, \chi_{\mathrm{Sym}^2\rho_{\mathrm{std}}}\right) &= \frac{1}{120}\sum_{g\in\mathfrak{S}_5}(\operatorname{sgn} g)\chi_{\mathrm{Sym}^2\rho_{\mathrm{std}}}(g) \\
&= \frac{1}{120}\Big[\chi_{\mathrm{Sym}^2\rho_{\mathrm{std}}}(1) - 30\,\chi_{\mathrm{Sym}^2\rho_{\mathrm{std}}}((1\ 2\ 3\ 4)) - 20\,\chi_{\mathrm{Sym}^2\rho_{\mathrm{std}}}((1\ 2\ 3)\,(4\ 5)) \\
&\quad + 20\,\chi_{\mathrm{Sym}^2\rho_{\mathrm{std}}}((1\ 2\ 3)) - 10\,\chi_{\mathrm{Sym}^2\rho_{\mathrm{std}}}((1\ 2)) \\
&\quad + 24\,\chi_{\mathrm{Sym}^2\rho_{\mathrm{std}}}((1\ 2\ 3\ 4\ 5)) + 15\,\chi_{\mathrm{Sym}^2\rho_{\mathrm{std}}}((1\ 2)\,(3\ 4))\Big] \\
&= \frac{1}{120}[10-0-20+20-40+0+30] = 0.
\end{aligned}$$

So, the alternating representation σ' is **NOT** a direct summand of $\operatorname{Sym}^2 \rho_{\text{std}}$.

$$\begin{aligned}
\left(\chi_{\rho_{\text{std}}}, \chi_{\operatorname{Sym}^2 \rho_{\text{std}}}\right) &= \frac{1}{120} \sum_{g \in \mathfrak{S}_5} \overline{\chi_{\rho_{\text{std}}}(g)}\, \chi_{\operatorname{Sym}^2 \rho_{\text{std}}}(g) \\
&= \frac{1}{120} \Big[\chi_{\operatorname{Sym}^2 \rho_{\text{std}}}(1) - 30\, \chi_{\operatorname{Sym}^2 \rho_{\text{std}}}((1\ 2\ 3\ 4)) - 20\, \chi_{\operatorname{Sym}^2 \rho_{\text{std}}}((1\ 2\ 3)\,(4\ 5)) \\
&\quad + 20\, \chi_{\operatorname{Sym}^2 \rho_{\text{std}}}((1\ 2\ 3)) - 10\, \chi_{\operatorname{Sym}^2 \rho_{\text{std}}}((1\ 2)) \\
&\quad + 24\, \chi_{\operatorname{Sym}^2 \rho_{\text{std}}}((1\ 2\ 3\ 4\ 5)) + 15\, \chi_{\operatorname{Sym}^2 \rho_{\text{std}}}((1\ 2)\,(3\ 4))\Big] \\
&= \frac{1}{120} [4 \cdot 10 + 30 \cdot 0 \cdot 0 + 20 \cdot (-1) \cdot 1 + 20 \cdot 1 \cdot 1 + 10 \cdot 2 \cdot 4 + 24 \cdot (-1) \cdot 0 + 15 \cdot 0 \cdot 2] \\
&= \frac{1}{120} [40 + 0 - 20 + 20 + 80 + 0 + 0] = 1.
\end{aligned}$$

Therefore, the standard representation ρ_{std} is a direct summand of $\operatorname{Sym}^2 \rho_{\text{std}}$. The other direct summand is a 5-dimensional irreducible representation, which we don't know of yet. Let's call that ψ. Then

$$\operatorname{Sym}^2 \rho_{\text{std}} \cong \sigma \oplus \rho_{\text{std}} \oplus \psi, \tag{3.31}$$

so that

$$\chi_\psi = \chi_{\operatorname{Sym}^2 \rho_{\text{std}}} - \chi_\sigma - \chi_{\rho_{\text{std}}}. \tag{3.32}$$

Using this formula, we fill out another row of the character table.

# $\mathfrak{S}_5$	1 1	30 $(1\ 2\ 3\ 4)$	20 $(1\ 2\ 3)\,(4\ 5)$	20 $(1\ 2\ 3)$	10 $(1\ 2)$	24 $(1\ 2\ 3\ 4\ 5)$	15 $(1\ 2)\,(3\ 4)$
σ	1	1	1	1	1	1	1
σ'	1	-1	-1	1	-1	1	1
ρ_{std}	4	0	-1	1	2	-1	0
$\rho_{\text{std}} \otimes \sigma'$	4	0	1	1	-2	-1	0
$\Lambda^2 \rho_{\text{std}}$	6	0	0	0	0	1	-2
ψ	5	-1	1	-1	1	0	1

By Lemma 3.4, $\psi \otimes \sigma'$ is also an irreducible representation. Its character is given by $\chi_\psi \chi_{\sigma'}$. Therefore, the complete character table is:

# $\mathfrak{S}_5$	1 1	30 $(1\ 2\ 3\ 4)$	20 $(1\ 2\ 3)\,(4\ 5)$	20 $(1\ 2\ 3)$	10 $(1\ 2)$	24 $(1\ 2\ 3\ 4\ 5)$	15 $(1\ 2)\,(3\ 4)$
σ	1	1	1	1	1	1	1
σ'	1	-1	-1	1	-1	1	1
ρ_{std}	4	0	-1	1	2	-1	0
$\rho_{\text{std}} \otimes \sigma'$	4	0	1	1	-2	-1	0
$\Lambda^2 \rho_{\text{std}}$	6	0	0	0	0	1	-2
ψ	5	-1	1	-1	1	0	1
$\phi = \psi \otimes \sigma'$	5	1	-1	-1	-1	0	1

§3.5 Properties of group from character table

In this section, we will examine what information about the group we can extract from the character table. Indeed, if two groups are isomorphic, then their irreducible representations are in a one-to-one correspondecne. Then the character values of those irreducible representations are also equal. Therefore, if two groups are isomorphic, then their character tables are identical. In this section, we will explore whether the converse is true or not.

Intuitively speaking, to know a group, it suffices to know all its irreducible representations. Because the regular representation $\mathbb{C}[G]$ is a faithful representation as we'll soon see. This regular representation contains all the irreducible representations as V subrepresentations, with multiplicity $\dim V$. Therefore, if we know all the irreducible representations, then we can construct the regular representations. If we know the regular representation, then we know the group.

The character table gives us the trace values of the irreducible representations. For a given group G, the characters determine a representation up to isomorphism. But given a character table, can we actually reconstruct the group up to isomorphism? This is the question we will try to answer in this section.

Proposition 3.7

If a value α appears in the character table of a group G then also $\overline{\alpha}$ appears in the character table

Proof. Let $\rho : G \to \mathrm{GL}(V)$ be an irreducible representation. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $\rho(g)$. All of these eigenvalues are $|G|$-th root of unities, since $g^{|G|} = e$. The eigenvalues of $\rho(g)^{-1}$ will be $\lambda_1^{-1}, \ldots, \lambda_n^{-1}$. Therefore,

$$\chi_\rho\left(g^{-1}\right) = \sum_{i=1}^{n} \lambda_i^{-1} = \sum_{i=1}^{n} \overline{\lambda_i} = \overline{\sum_{i=1}^{n} \lambda_i} = \overline{\chi_\rho(g)}. \tag{3.33}$$

So, if α appears in the row of ρ in the character table as $\chi_\rho(g)$, then $\overline{\alpha}$ also appears in the row of ρ in the character table as $\chi_\rho(g^{-1})$. ■

Corollary 3.8

If all the entries in the character table are real, then every element of the group is conjugate to its inverse.

Proof. If all the entries of the character table are real, then it means $\chi(g) = \chi(g^{-1})$ for every irreducible character χ and every $g \in G$. Since irreducible characters form a basis for the space of class functions, we have $c(g) = c(g^{-1})$ for every class function c. Now consider the class function c

$$c(h) = \begin{cases} 1 & \text{if } h \text{ is in the conjugacy class of } g, \\ 0 & \text{otherwise.} \end{cases} \tag{3.34}$$

Clearly, $c(g) = 1$. Therefore, $c(g^{-1}) = 1$, and hence g and g^{-1} are conjugate. ■

Proposition 3.9

For a finite group G,

$$Z(G) = \{g \in G \mid |\chi(g)| = \chi(e), \text{ for every irreducible character } \chi\}.$$

Proof. Take $g \in Z(G)$. Let $\rho : G \to \mathrm{GL}(V)$ be an irreducible representation, and χ_ρ be the corresponding irreducible character. Since $gx = xg$ for every $x \in G$, $\rho(g) : V \to V$ is a G-linear map. Indeed, the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "\rho(x)"] \arrow[d, "\rho(g)"'] & V \arrow[d, "\rho(g)"] \\ V \arrow[r, "\rho(x)"'] & V \end{tikzcd}$$

$$\rho(g) \circ \rho(x) = \rho(gx) = \rho(xg) = \rho(x) \circ \rho(g). \tag{3.35}$$

So $\rho(g) : V \to V$ is a G-linear map. Since V is an irreducible representation, by Schur's lemma, $\rho(g) = 0$ or $\rho(g)$ an isomorphism of representations. But since $\rho(g^{-1})$ is the inverse of $\rho(g)$, $\rho(g) = 0$ is not possible. Hence, $\rho(g)$ an isomorphism of representations.

Again, by Schur's lemma, $\dim \mathrm{Hom}_G(V,V) = 0$ or 1. But $\mathrm{Hom}_G(V,V)$ contains $\mathbb{1}_V$, so it is 1-dimensional. Therefore, all the elements of $\mathrm{Hom}_G(V,V)$ are scalar multiples of $\mathbb{1}_V$. In particular, $\rho(g) = c\,\mathbb{1}_V$ for some $c \in \mathbb{C}$.

$$\rho(g)^{|G|} = \rho\left(g^{|G|}\right) = \rho(1) = \mathbb{1}_V \implies c^{|G|}\,\mathbb{1}_V = \mathbb{1}_V \implies c^{|G|} = 1. \tag{3.36}$$

So we have, $|c| = 1$. Now,

$$|\chi_\rho(g)| = |\mathrm{Tr}[\rho(g)]| = |\mathrm{Tr}[c\,\mathbb{1}_V]| = |c \dim V| = |c| \dim V = \dim V = \chi_\rho(e). \tag{3.37}$$

So we have proved that if $g \in Z(G)$, then for any irreducible character χ_ρ, $|\chi_\rho(g)| = \chi_\rho(e)$.

Now, conversely, suppose $|\chi_\rho(g)| = \chi_\rho(e)$ for every irreducible character χ_ρ associated with the irreducible representation $\rho : G \to \mathrm{GL}(V)$. We need to show that $g \in Z(G)$.

Following the analysis of Lemma 3.5 in (3.11), $|\chi_\rho(g)| = \chi_\rho(e)$ implies that all the eigenvalues of $\rho(g)$ are equal, say λ_ρ. $\rho(g)$ is diagonalizable, and all its eigenvalues are $\lambda_\rho \in \mathbb{C}$. Therefore, $\rho(g) = \lambda_\rho\,\mathbb{1}_V$. This is true for any irreducible representation $\rho : G \to \mathrm{GL}(V)$. As a result, for any $x \in G$ and $\mathbf{v} \in V$,

$$\begin{aligned}
\rho_i(gx)\,\mathbf{v} &= \rho_i(g)\,\rho_i(x)\,\mathbf{v} \\
&= \lambda_{\rho_i}\rho_i(x)\,\mathbf{v} \\
&= \rho_i(x)\,(\lambda_{\rho_i}\mathbf{v}) \\
&= \rho_i(x)\,[\rho_i(g)\,\mathbf{v}] \\
&= \rho_i(xg)\,\mathbf{v}.
\end{aligned} \tag{3.38}$$

This holds for any $\mathbf{v} \in V_i$. Therefore, $\rho_i(gx) = \rho_i(xg)$. Let $\sigma : G \to \mathrm{GL}(\mathbb{C}[G])$ be the regular representation defined by

$$\sigma(h)\left(\sum_{x\in G} a_x\delta_x\right) = \sum_{x\in G} a_x\delta_{hx} = \sum_{x\in G} a_{h^{-1}x}\delta_x. \tag{3.39}$$

This is a faithful representation. Indeed, if $h \in \mathrm{Ker}\,\sigma$, then $\sigma(h)$ is the identity map on $\mathbb{C}[G]$.

$$\begin{aligned}
\sigma(h)\left(\sum_{x\in G} a_x\delta_x\right) = \sum_{x\in G} a_x\delta_x &\implies \sum_{x\in G} a_{h^{-1}x}\delta_x = \sum_{x\in G} a_x\delta_x \\
&\implies h^{-1}x = x \quad \forall x \in G \\
&\implies h = e.
\end{aligned} \tag{3.40}$$

Now, any irreducible representation $\rho : G \to \mathrm{GL}(V)$ appear in the regular representation $\dim V$ times. Suppose $\rho_i : G \to \mathrm{GL}(V_i)$, $i = 1, 2, \ldots, k$ are all the irreducible representations, with $\dim V_i = n_i$. Then

$$\mathbb{C}[G] \cong V_1^{n_1} \oplus V_2^{n_2} \oplus \cdots \oplus V_k^{n_k} = \bigoplus_{i=1}^{k} V_i^{n_i} =: W. \tag{3.41}$$

This is an isomorphism of representations. In other words,

$$\sigma \cong \bigoplus_{i=1}^{k} \rho_i^{\oplus n_i} =: \psi. \tag{3.42}$$

We have shown earlier that $\rho_i(g) = \lambda_{\rho_i}\,\mathbb{1}_{V_i}$ for any irreducible representation $\rho_i : G \to \mathrm{GL}(V_i)$. We have also showed that for any $x \in G$, $\rho_i(gx) = \rho_i(xg)$. Now we want to show that $\psi(gx) = \psi(xg)$ for any $x \in G$. A generic element from W looks like

$$\left(\mathbf{v}_1^{(1)}, \mathbf{v}_2^{(1)}, \ldots, \mathbf{v}_{n_1}^{(1)}, \mathbf{v}_1^{(2)}, \mathbf{v}_2^{(2)}, \ldots, \mathbf{v}_{n_2}^{(2)}, \ldots, \mathbf{v}_1^{(k)}, \mathbf{v}_2^{(k)}, \ldots, \mathbf{v}_{n_k}^{(k)}\right),$$

with $\mathbf{v}_j^{(i)} \in V_i$.

$$
\begin{aligned}
&\psi\left(gx\right)\left(\mathbf{v}_1^{(1)},\ldots,\mathbf{v}_{n_1}^{(1)},\mathbf{v}_1^{(2)},\ldots,\mathbf{v}_{n_2}^{(2)},\ldots,\mathbf{v}_1^{(k)},\ldots,\mathbf{v}_{n_k}^{(k)}\right)\\
&=\left[\bigoplus_{i=1}^{k}\rho_i^{\oplus n_i}\left(gx\right)\right]\left(\mathbf{v}_1^{(1)},\ldots,\mathbf{v}_{n_1}^{(1)},\mathbf{v}_1^{(2)},\ldots,\mathbf{v}_{n_2}^{(2)},\ldots,\mathbf{v}_1^{(k)},\ldots,\mathbf{v}_{n_k}^{(k)}\right)\\
&=\left(\rho_1\left(gx\right)\mathbf{v}_1^{(1)},\ldots,\rho_1\left(gx\right)\mathbf{v}_{n_1}^{(1)},\rho_2\left(gx\right)\mathbf{v}_1^{(2)},\ldots,\rho_2\left(gx\right)\mathbf{v}_{n_2}^{(2)},\ldots,\rho_k\left(gx\right)\mathbf{v}_1^{(k)},\ldots,\rho_k\left(gx\right)\mathbf{v}_{n_k}^{(k)}\right)\\
&=\left(\rho_1\left(xg\right)\mathbf{v}_1^{(1)},\ldots,\rho_1\left(xg\right)\mathbf{v}_{n_1}^{(1)},\rho_2\left(xg\right)\mathbf{v}_1^{(2)},\ldots,\rho_2\left(xg\right)\mathbf{v}_{n_2}^{(2)},\ldots,\rho_k\left(xg\right)\mathbf{v}_1^{(k)},\ldots,\rho_k\left(xg\right)\mathbf{v}_{n_k}^{(k)}\right)\\
&=\left[\bigoplus_{i=1}^{k}\rho_i^{\oplus n_i}\left(xg\right)\right]\left(\mathbf{v}_1^{(1)},\ldots,\mathbf{v}_{n_1}^{(1)},\mathbf{v}_1^{(2)},\ldots,\mathbf{v}_{n_2}^{(2)},\ldots,\mathbf{v}_1^{(k)},\ldots,\mathbf{v}_{n_k}^{(k)}\right)\\
&=\psi\left(xg\right)\left(\mathbf{v}_1^{(1)},\ldots,\mathbf{v}_{n_1}^{(1)},\mathbf{v}_1^{(2)},\ldots,\mathbf{v}_{n_2}^{(2)},\ldots,\mathbf{v}_1^{(k)},\ldots,\mathbf{v}_{n_k}^{(k)}\right).
\end{aligned}
$$

Therefore, $\psi\left(gx\right)=\psi\left(xg\right)$ for any $x \in G$. ψ is isomorphic to a faithful representation σ (the regular representation), so it is faithful as well. Therefore, $gx = xg$ for any $x \in G$. So $g \in Z(G)$. This proves the equality

$$Z(G) = \{g \in G \ : \ |\chi(g)| = \chi(e), \text{ for every irreducible character } \chi\}\,. \tag{3.43}$$

■

Definition 3.1 (Commutator subgroup). Given a group G, we define commutator in G as

$$[g,h] = g^{-1}h^{-1}gh, \tag{3.44}$$

for $g, h \in G$. The **commutator subgroup** of G, often denoted as G' or $[G,G]$ is the subgroup generated by all the commutator elements $[g,h]$.

Lemma 3.10

Given a group G, G/G' is abelian. Furthermore, if G/N is abelian, then N must contain G'.

Proof. The commutator subgroup G' is generated by all the commutators $[g,h] = g^{-1}h^{-1}gh$. Consider $g_1G', g_2G' \in G/G'$. Then

$$\left(g_1G'\right)\cdot\left(g_2G'\right) = \left(g_1g_2\right)G' = g_2g_1\left[g_1,g_2\right]G' = g_2g_1G' = \left(g_2G'\right)\cdot\left(g_1G'\right) \tag{3.45}$$

So G/G' is abelian.

Given a group homomorphism $f : G_1 \to G_2$,

$$f\left([g,h]\right) = f\left(g^{-1}h^{-1}gh\right) = f(g)^{-1}f(h)^{-1}f(g)f(h) = \left[f\left(g\right), f\left(h\right)\right]. \tag{3.46}$$

Now consider the quotient map $f : G \to G/N$. Given any $g, h \in G$, since G/N is abelian, $\left[f\left(g\right), f\left(h\right)\right] = e_{G/N}$, the identity of G/N. Therefore,

$$f\left([g,h]\right) = \left[f\left(g\right), f\left(h\right)\right] = e_{G/N}. \tag{3.47}$$

So $[g,h] \in \operatorname{Ker} f = N$. Therefore, all the commutators $[g,h] = g^{-1}h^{-1}gh$ are contained in N. So the subgroup generated by all the commutators is also contained in N. Hence, $G' \subseteq N$. ■

Proposition 3.11

Given a finite group G, $|G/G'|$ is the number of 1-dimensional representations of G.

Proof. Since G/G' is abelian, all its irreducible representations are 1-dimensional. Also, since it is abelian, all conjugacy classes are singletons. Therefore, there are $|G/G'|$-many 1-dimensional irreducible representations of G/G'.

$$G \xrightarrow{\pi} G/G' \xrightarrow{\rho} \mathbb{C}^{\times}$$

Given a 1-dimensional irreducible representation ρ of G/G', we have a 1-dimensional irreducible representation of G as $\rho \circ \pi$, where $\pi : G \to G/G'$ is the projection map. Therefore,

$$\text{\# of 1-dimensional irreducible representation of } G \geq \left|G/G'\right|. \tag{3.48}$$

Now, if $\sigma : G \to \mathbb{C}^{\times}$ is a 1-dimensional irreducible representations of G, then $G/\operatorname{Ker}\sigma \cong \sigma(G)$ is an abelian group. So $G' \subseteq \operatorname{Ker}\sigma$.

This gives us a representation of G/G': $\tau : G/G' \to \mathbb{C}^{\times}$, given by $\tau(gG') = \sigma(g)$. This is well-defined, for $gG' = hG'$, $h^{-1}g \in G' \subseteq \operatorname{Ker}\sigma$, so that $\sigma(h) = \sigma(g)$. Hence,

$$\tau(gG') = \sigma(g) = \sigma(h) = \tau(hG'). \tag{3.49}$$

Therefore, every 1-dimensional irreducible representations of G gives rise to a 1-dimensional representation of G/G'. So we have

$$\text{\# of 1-dimensional irreducible representation of } G \leq \left|G/G'\right|. \tag{3.50}$$

Combining these,

$$\frac{|G|}{|G'|} = \text{\# of 1-dimensional irreducible representation of } G. \tag{3.51}$$

■

Lemma 3.12

For a finite group G, $G' = \bigcap_{\rho \text{ 1-d irrep}} \operatorname{Ker}\rho$.

Proof. If $\rho : G \to \mathbb{C}^{\times}$ is a 1-dimensional representation, then $\operatorname{im}\rho = G/\operatorname{Ker}\rho$ is abelian, so $\operatorname{Ker}\rho$ contains G'. Since this is true for every ρ, therefore,

$$G' \subseteq \bigcap_{\rho \text{ 1-d irrep}} \operatorname{Ker}\rho. \tag{3.52}$$

On the other hand, suppose $g \in \operatorname{Ker}\rho$ for every 1-dimensional irreducible representations ρ. Then $\rho(g) = 1$ for every 1-dimensional irreducible representations ρ. Therefore,

$$\rho'(gG') = 1 \tag{3.53}$$

for every 1-dimensional irreducible representations ρ' of G/G'. G/G' has only 1-dimensional irreducible representations. Therefore, $\chi(gG') = 1$ for every irreducible character χ of G/G'. Hence, $c(gG') = 1$ for every class function c on G/G'. This proves that $gG' = eG'$, so that $g \in G'$. Therefore,

$$\bigcap_{\rho \text{ 1-d irrep}} \operatorname{Ker}\rho \subseteq G'. \tag{3.54}$$

Hence, $\bigcap_{\rho \text{ 1-d irrep}} \operatorname{Ker}\rho = G'$. ■

If two finite groups G and H have the same character table,

- Their order is also equal, because $|G| = \sum_{\rho} \chi_{\rho}(e)^2$.

- Their number of 1-dimensional irreducible representations is also equal. Hence, $|G/G'| = |H/H'|$. Then $|G| = |H|$ implies $|G'| = |H'|$. Therefore, if two groups have the identical character table, their commutator subgroups have the same order.

- There will be same number of elements such that $|\chi(g)| = \chi(e)$ for every irreducible character χ. So $|Z(G)| = |Z(H)|$, by Proposition 3.9.

So, the character table uniquely identifies these properties of the group. Can the character table uniquely identify the group (up to isomorphism)? Let's see a few examples.

Example 3.3. The quaternion group is $Q = \{\pm\mathbf{1}, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$ with $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -\mathbf{1}$ and $\mathbf{ij} = \mathbf{k}$, $\mathbf{jk} = \mathbf{i}$, $\mathbf{ki} = \mathbf{j}$, $\mathbf{ji} = -\mathbf{k}$, $\mathbf{kj} = -\mathbf{i}$, $\mathbf{ik} = -\mathbf{j}$. The conjugacy classes of Q are

$$\{\mathbf{1}\}, \quad \{-\mathbf{1}\}, \quad \{\mathbf{i}, -\mathbf{i}\}, \quad \{\mathbf{j}, -\mathbf{j}\}, \quad \{\mathbf{k}, -\mathbf{k}\}.$$

Suppose $\rho : Q \to \mathrm{GL}(\mathbb{C}) \equiv \mathbb{C}^\times$ is a 1-dimensional representation of Q.

$$\rho(\mathbf{k}) = \rho(\mathbf{ij}) = \rho(\mathbf{i})\rho(\mathbf{j}) = \rho(\mathbf{j})(\rho\mathbf{i}) = \rho(\mathbf{ji}) = \rho(-\mathbf{k}) = \rho(-\mathbf{1})\rho(\mathbf{k}). \tag{3.55}$$

Therefore, $\rho(-\mathbf{1}) = 1$. Since $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -\mathbf{1}$, this gives us that $\rho(\mathbf{i}), \rho(\mathbf{j}), \rho(\mathbf{k}) \in \{1, -1\}$. Furthermore, since $\rho(-\mathbf{1}) = 1$,

$$\rho(\mathbf{i}) = \rho(-\mathbf{i}), \quad \rho(\mathbf{j}) = \rho(-\mathbf{j}), \quad \rho(\mathbf{k}) = \rho(-\mathbf{k}). \tag{3.56}$$

If all of them are 1, then we have the trivial representation. If two of them are 1, then it forces the third one to be 1 as well because of $\mathbf{ij} = \mathbf{k}$, $\mathbf{jk} = \mathbf{i}$, $\mathbf{ki} = \mathbf{j}$.

Now, suppose only one of them is 1, and WLOG, $\rho(\mathbf{i}) = \rho(-\mathbf{i}) = 1$. Since $\mathbf{jk} = \mathbf{i}$, this means we must have either $\rho(\mathbf{j}) = \rho(\mathbf{k}) = 1$ or $\rho(\mathbf{j}) = \rho(\mathbf{k}) = -1$. This gives us a 1-dimensional representation other than the trivial one.

For the other cases $\rho(\mathbf{j}) = \rho(-\mathbf{j}) = 1$ and $\rho(\mathbf{k}) = \rho(-\mathbf{k}) = 1$, we get two other nontrivial 1-dimensional representations. These are the all four 1-dimensional representations of Q.

Q	$\mathbf{1}$	$-\mathbf{1}$	$\mathbf{i}$	$-\mathbf{i}$	$\mathbf{j}$	$-\mathbf{j}$	$\mathbf{k}$	$-\mathbf{k}$
ρ_1	1	1	1	1	1	1	1	1
ρ_2	1	1	1	1	-1	-1	-1	-1
ρ_3	1	1	-1	-1	1	1	-1	-1
ρ_4	1	1	-1	-1	-1	-1	1	1

There is a 2-dimensional irreducible representation. The embedding $Q \subset \mathrm{GL}_2(\mathbb{C})$ is given by

$$\begin{gathered}
\mathbf{1} \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad -\mathbf{1} \equiv \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \mathbf{i} \equiv \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad -\mathbf{i} \equiv \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}, \\
\mathbf{j} \equiv \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad -\mathbf{j} \equiv \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \mathbf{k} \equiv \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}, \quad -\mathbf{k} \equiv \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.
\end{gathered} \tag{3.57}$$

From the matrices above, we can see that

$$\chi(\mathbf{1}) = 2, \quad \chi(-\mathbf{1}) = -2, \quad \chi(\mathbf{i}) = \chi(-\mathbf{i}) = \chi(\mathbf{j}) = \chi(-\mathbf{j}) = \chi(\mathbf{k}) = \chi(-\mathbf{k}) = 0. \tag{3.58}$$

So $|\chi|^2$ is

$$|\chi|^2 = (\chi, \chi) = \frac{1}{|Q|} \sum_{g \in Q} \overline{\chi(g)}\, \chi(g) = \frac{1}{8}(2 \cdot 2 + (-2) \cdot (-2)) = 1. \tag{3.59}$$

So this is an irreducible representation as well. These are all the irreducible representations, since $1^2 + 1^2 + 1^2 + 1^2 + 2^2 = 8 = |Q_8|$. Then the character table of Q is

#	1	1	2	2	2
Q	$\mathbf{1}$	$\mathbf{-1}$	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
ρ_1	1	1	1	1	1
ρ_2	1	1	1	-1	-1
ρ_3	1	1	-1	1	-1
ρ_4	1	1	-1	-1	1
ρ_5	2	-2	0	0	0

Example 3.4. Consider the dihedral group, D_n, when n is even.

$$D_n = \left\{ s^i r^j \mid i = 0, 1; j = 0, \ldots, n-1; r^n = s^2 = e \text{ and } sr = r^{n-1} s \right\}.$$

D_4 has 5 conjugacy classes:

$$\{e\}, \quad \left\{r^2\right\}, \quad \left\{r, r^3\right\}, \quad \left\{s, sr^2\right\}, \quad \left\{sr, sr^3\right\}.$$

So, there are 5 irreducible representations of D_4. Suppose $\rho : D_4 \to \mathbb{C}^\times$ is a 1-dimensional representation. $\rho(r)^n = 1$, so $\rho(r)$ is one of the 4-th root of unities. $\rho(s)^2 = 1$, so $\rho(s) = \pm 1$. $sr = r^{-1}s$ gives us that

$$\rho(s)\rho(r) = \rho(r)^{-1}\rho(s) \implies \rho(r) = \rho(r)^{-1} \implies \rho(r) = \pm 1. \tag{3.60}$$

$\rho(s), \rho(r) \in \{1, -1\}$. Therefore, we have total 4 choices, and hence 4 1-dimensional representations.

D_4	e	r	r^2	r^3	s	sr	sr^2	sr^3
ρ_1	1	1	1	1	1	1	1	1
ρ_2	1	1	1	1	-1	-1	-1	-1
ρ_3	1	-1	1	-1	1	-1	1	-1
ρ_4	1	-1	1	-1	-1	1	-1	1

The remaining irreducible representation is a 2-dimensional representation, because there are exactly 5 of them, and 4 of them are 1-dimensional. If the remaining one has dimension d, then $8 = |D_4| = 1^2 + 1^2 + 1^2 + 1^2 + d^2$; so $d = 2$. Suppose $\sigma : D_n \to \mathrm{GL}\left(\mathbb{C}^2\right)$ is the 2-dimensional representation.

$C_4 = \{r^i \mid i = 0, \ldots, 3\}$ is an abelian subgroup of D_4. All the irreducible representations of C_4 are 1-dimensional, and there are exactly 4 of them. $1, i, -1, -i$ are the 4-th roots of unity. Then the irreducible representations of C_n are $\rho_j : C_n \to \mathbb{C}$, $\rho_j(r) = i^j$ for $j = 0, 1, 2, 3$.

σ defines a 2-dimensional representation of C_4, and it decomposes as direct sum of two 1-d irreducible representations. Therefore, in the matrix representation, $\sigma(r)$ should be a diagonal matrix:

$$\sigma(r) = \begin{bmatrix} i^k & 0 \\ 0 & i^l \end{bmatrix}, \tag{3.61}$$

for $k, l \in \{0, 1, 2, 3\}$. Since $s^2 = e$, $\sigma(s)$ is a 2×2 matrix of order 2. Therefore,

$$\sigma(s) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{3.62}$$

Since $rs = sr^{-1}$, i.e. $srs = r^{-1}$, one has

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} i^k & 0 \\ 0 & i^l \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} i^{-k} & 0 \\ 0 & i^{-l} \end{bmatrix} \implies \begin{bmatrix} i^l & 0 \\ 0 & i^k \end{bmatrix} = \begin{bmatrix} i^{-k} & 0 \\ 0 & i^{-l} \end{bmatrix} \tag{3.63}$$

Therefore, $k = -l$, i.e.

$$\sigma(r) = \begin{bmatrix} i^k & 0 \\ 0 & i^{-k} \end{bmatrix}. \tag{3.64}$$

If $k = 0$, or $k = 2$, $\sigma(r)$ is I or $-I$, respectively. This does not define an irreducible representation, since the 1-dimensional subspace $\{(z_1, z_2) \in \mathbb{C}^2 \mid z_1 = z_2\}$ is invariant under the action of both $\sigma(r)$ and $\sigma(s)$. For $k = 1$, we get

$$\sigma(r) = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}. \tag{3.65}$$

Then we calculate the following:

$$\sigma(e) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma(r) = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad \sigma\left(r^2\right) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \sigma\left(r^3\right) = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$$
$$\sigma(s) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma(sr) = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma\left(sr^2\right) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad \sigma\left(sr^3\right) = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}.$$

Then

$$(\chi_\sigma, \chi_\sigma) = \frac{1}{8} \sum_{g \in G} |\chi_\sigma(g)|^2 = \frac{1}{8} \left[2^2 + 0^2 + (-2)^2 + 0^2 + 0^2 + 0^2 + 0^2 + 0^2\right] = 1. \tag{3.66}$$

Therefore, σ is a 2-dimensional irreducible representation. Now, the character table of D_4 is

#	1	1	2	2	2
D_4	e	r^2	r	s	sr
ρ_1	1	1	1	1	1
ρ_2	1	1	1	-1	-1
ρ_3	1	1	-1	1	-1
ρ_4	1	1	-1	-1	1
σ	2	-2	0	0	0

The two character tables of the quaternion group Q and the dihedral group D_4 are identical. However, D_4 and Q are not isomorphic groups. There are 5 elements order 2 in D_4: r^2, s, sr, sr^2, sr^3. But the only element of order 2 in Q is $-\mathbf{1}$. Therefore, Q and D_8 are not isomorphic. So we have reached the conclusion that identical character table doesn't necessarily imply that the groups are isomorphic.

4 Induced Representation Theory

Suppose we have a finite group G and a subgroup $H \leq G$. Given a representation $\rho : H \to \mathrm{GL}(W)$ of the subgroup H, we want to construct a representation of G in a "canonical" way. This canonical representation of G will be called the induced representation

$$\mathrm{Ind}_H^G \rho : G \to \mathrm{GL}(?)\,.$$

§4.1 Induced representation

Definition 4.1. A representation $\rho_G : G \to \mathrm{GL}(V)$ is said to be **induced** by $\rho_H : H \to \mathrm{GL}(W)$ if

(a) $W \subseteq V$ is an H-invariant subspace of V, i.e. $\rho_G(h)(W) \subseteq W$ for every $h \in H$;

(b) for every left H-coset $\sigma = gH \in G/H$, there exists a vector subspace $W_\sigma \subseteq V$ such that

$$V = \bigoplus_{\sigma \in G/H} W_\sigma, \tag{4.1}$$

and $W = W_{eH}$;

(c) for $g \in G$, we have an action $\rho_G(g) : W_\sigma \to W_{g\sigma}$. Set theoretically, one writes

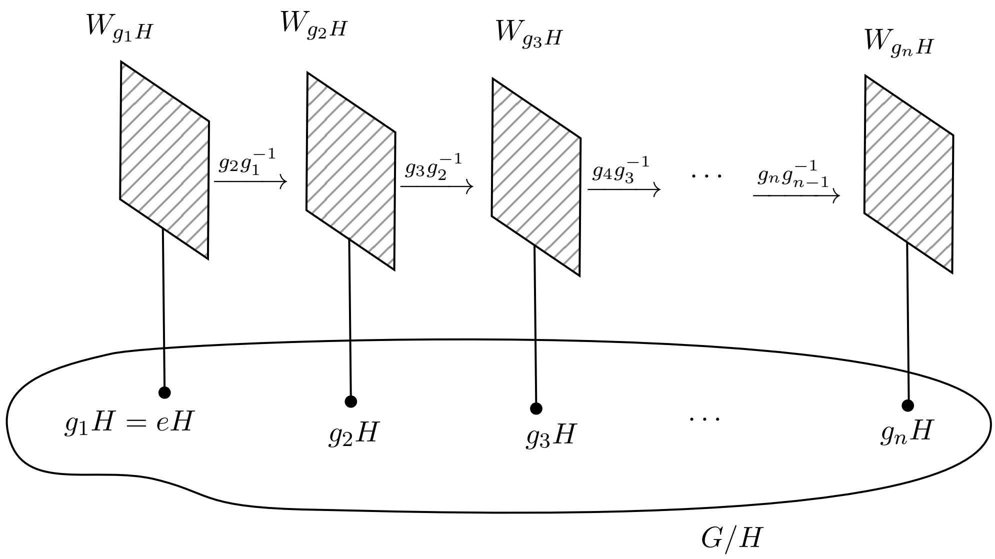
$$\rho_G(g)(W_{xH}) = W_{gxH}. \tag{4.2}$$

Remark 4.1. Note that G acts on G/H by left multiplication:

$$g \cdot (xH) = (gx)\,H. \tag{4.3}$$

The group action $\rho_G(g)$ on the subuspaces, on the other hand, is given by (4.2). Requirement (c) asks for these two actions to be equivariant, i.e. the following diagram commutes for every $g \in G$:

$$\begin{tikzcd}
G/H \arrow[r, "g"] \arrow[d] & G/H \arrow[d] \\
\{W_\sigma\}_{\sigma \in G/H} \arrow[r, "\rho_G(g)"'] & \{W_\sigma\}_{\sigma \in G/H}
\end{tikzcd}$$



Universal property of the induced representation:

Let W be an H-representation and V be a G-representation, with the corresponding group homomorphisms $\rho_H : H \to \mathrm{GL}(W)$ and $\rho_G : G \to \mathrm{GL}(V)$. One says that $W \xrightarrow{\alpha} V$ is an induction if it satisfies the following universal property:

(i) α is H-linear: for every $h \in H$, the following diagram commutes:

$$\begin{tikzcd} W \arrow[r, "\alpha"] \arrow[d, "\rho_H(h)"'] & V \arrow[d, "\rho_G\big|_H(h)"] \\ W \arrow[r, "\alpha"'] & V \end{tikzcd}$$

(ii) if Z is another representation of G, and $\beta : W \to Z$ is a H-linear map, then there exists a unique G-linear map $\overline{\beta} : V \to Z$ such that the following diagram commutes:

$$\begin{tikzcd} W \arrow[r, "\alpha"] \arrow[dr, "\forall\, \beta"'] & V \arrow[d, dashed, "\exists!\ \overline{\beta}"] \\ & Z \end{tikzcd}$$

i.e. $\beta = \overline{\beta} \circ \alpha$.

Lemma 4.1

If $V = \bigoplus_{\sigma \in G/H} W_\sigma$ as in Definition 4.1, then $W = W_{eH} \hookrightarrow V$ satisfies the universal property.

Proof. Let $\beta : W \to Z$ be any H-linear map with Z being a G-representation with the group homomorphism $\rho_Z : G \to \mathrm{GL}(Z)$.

Uniqueness of $\overline{\beta}$: Suppose the diagram

$$\begin{tikzcd} W \arrow[r, hook, "\alpha"] \arrow[dr, "\beta"'] & V \arrow[d, dashed, "\overline{\beta}"] \\ & Z \end{tikzcd} \tag{4.4}$$

commutes, with $\overline{\beta}$ being a G-linear map. We need to show that $\overline{\beta}$ is unique. Fix $\sigma = gH \in G/H$ and $\mathbf{v} \in W_\sigma$. We know how Gacts on the subspaces W_σ labelled by the cosets $\sigma \in G/H$ given by (4.2), from which one obtains

$$\rho_G\left(g^{-1}\right)(W_\sigma) = \rho_G\left(g^{-1}\right)(W_{gH}) = W_{g^{-1}gH} = W_{eH} = W. \tag{4.5}$$

Since $\mathbf{v} \in W_\sigma$, $\rho_G\left(g^{-1}\right)\mathbf{v} \in W$. By the commutativity of (4.4), one has $\beta = \overline{\beta} \circ \alpha$. Also, α is just the inclusion map $W \hookrightarrow V$. Hence,

$$\beta\left(\rho_G\left(g^{-1}\right)\mathbf{v}\right) = \overline{\beta} \circ \alpha\left(\rho_G\left(g^{-1}\right)\mathbf{v}\right) = \overline{\beta}\left(\rho_G\left(g^{-1}\right)\mathbf{v}\right). \tag{4.6}$$

$\overline{\beta}$ is G-linear, so the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "\overline{\beta}"] & Z \\ V \arrow[r, "\overline{\beta}"'] \arrow[u, "\rho_G(g^{-1})"] & Z \arrow[u, "\rho_Z(g^{-1})"'] \end{tikzcd}$$

Therefore,

$$\beta\left(\rho_G\left(g^{-1}\right)\mathbf{v}\right)=\overline{\beta}\left(\rho_G\left(g^{-1}\right)\mathbf{v}\right)=\rho_Z\left(g^{-1}\right)\overline{\beta}\left(\mathbf{v}\right). \tag{4.7}$$

Hence,

$$\overline{\beta}\left(\mathbf{v}\right)=\rho_Z\left(g\right)\beta\left(\rho_G\left(g^{-1}\right)\mathbf{v}\right), \tag{4.8}$$

for $\mathbf{v}\in W_\sigma=W_{gH}$. (4.8) depends on the choice of g, so we need to show the well-definedness of (4.8). Let $\sigma=gH=g'H$. Then $g^{-1}g'\in H$. Since $\beta:W\to Z$ is a H-linear map, the following diagram commutes:

$$\begin{tikzcd}
W \arrow[r, "\beta"] \arrow[d, "\rho_H(g^{-1}g')"'] & Z \arrow[d, "\rho_Z(g^{-1}g')"] \\
W \arrow[r, "\beta"'] & Z
\end{tikzcd}$$

In other words,

$$\beta\circ\rho_H\left(g^{-1}g'\right)=\rho_Z\left(g^{-1}g'\right)\circ\beta. \tag{4.9}$$

Composing by $\rho_Z\left(g\right)$ to the left, we have

$$\rho_Z\left(g\right)\circ\beta\circ\rho_H\left(g^{-1}g'\right)=\rho_Z\left(g'\right)\circ\beta. \tag{4.10}$$

As a result,

$$\begin{aligned}\rho_Z\left(g'\right)\beta\left(\rho_G\left(g'^{\,-1}\mathbf{v}\right)\right)&=\rho_Z\left(g\right)\beta\rho_H\left(g^{-1}g'\right)\rho_G\left(g'^{\,-1}\mathbf{v}\right)\\&=\rho_Z\left(g\right)\beta\rho_G\left(g^{-1}g'\right)\rho_G\left(g'^{\,-1}\mathbf{v}\right)\\&=\rho_Z\left(g\right)\beta\left(\rho_G\left(g^{-1}\right)\mathbf{v}\right).\end{aligned} \tag{4.11}$$

Here we used the fact that W is a subspace of V, and so $\rho_H=\rho_G|_H$ when acted on vectors of W. Hence, (4.8) is well-defined! (4.8) shows that $\overline{\beta}$ is uniquely determined by β on $V=\bigoplus_{\sigma\in G/H}W_\sigma$.

Existence of $\overline{\beta}$: For $\mathbf{v}\in W_\sigma=W_{gH}$, we define $\overline{\beta}$ by (4.8), and then extend it linearly to all of $V=\bigoplus_{\sigma\in G/H}W_\sigma$. We have shown that this definition is well-defined. We now need to show that $\beta=\overline{\beta}\circ\alpha$ and $\overline{\beta}$ is G-linear. Indeed, for $\mathbf{v}\in W=W_{eH}$,

$$\overline{\beta}\left(\mathbf{v}\right)=\rho_Z\left(e\right)\beta\left(\rho_G\left(e^{-1}\right)\mathbf{v}\right)=\beta\left(\mathbf{v}\right). \tag{4.12}$$

Since $\alpha:W\hookrightarrow V$ is the inclusion, $\alpha\left(\mathbf{v}\right)=\mathbf{v}$, so that from (4.12), we have

$$\beta\left(\mathbf{v}\right)=\overline{\beta}\left(\mathbf{v}\right)=\overline{\beta}\left(\alpha\left(\mathbf{v}\right)\right). \tag{4.13}$$

Therefore, $\beta=\overline{\beta}\circ\alpha$. In order to show that $\overline{\beta}$ is G-linear, we need to show the commutativity of the following diagram for all $x\in G$

$$\begin{tikzcd}
V \arrow[r, "\overline{\beta}"] \arrow[d, "\rho_G(x)"'] & Z \arrow[d, "\rho_Z(x)"] \\
V \arrow[r, "\overline{\beta}"'] & Z
\end{tikzcd}$$

Given $\mathbf{v}\in W_{gH}$, $\rho_G\left(x\right)\mathbf{v}\in\rho_G\left(x\right)\left(W_{gH}\right)=W_{xgH}$. Therefore,

$$\begin{aligned}\overline{\beta}\left(\rho_G\left(x\right)\mathbf{v}\right)&=\rho_Z\left(xg\right)\circ\beta\left[\rho_G\left(\left(xg\right)^{-1}\right)\rho_G\left(x\right)\mathbf{v}\right]\\&=\rho_Z\left(x\right)\rho_Z\left(g\right)\circ\beta\left[\rho_G\left(g^{-1}\right)\mathbf{v}\right]\\&=\rho_Z\left(x\right)\circ\overline{\beta}\left(\mathbf{v}\right).\end{aligned}$$

Therefore, $\overline{\beta}\circ\rho_G\left(x\right)=\rho_Z\left(x\right)\circ\overline{\beta}$ on each W_{gH}, and hence $\overline{\beta}$ is G-linear. ■

§4.2 Explicit construction of $\operatorname{Ind}_H^G W$

Given an H-representation W, we'll show that $\operatorname{Ind}_H^G W$ indeed exists! We construct $\operatorname{Ind}_H^G W$ in the following way: first we pick representatives

$$g_1 = e,\ g_2,\ g_3,\ \ldots,\ g_n \in G$$

of cosets in G/H, i.e. $G/H = \{g_1H, g_2H, \ldots, g_nH\}$. Define a vector space

$$V = W \oplus W \oplus \cdots \oplus W = W^n,$$

where the subspaces are labelled by the cosets, i.e.

$$V = W_{eH} \oplus W_{g_2H} \oplus \cdots \oplus W_{g_nH}. \tag{4.14}$$

Then we define a G-action on V in the following way: for $g \in G$ and $\mathbf{v} \in W_{g_iH}$, first find the unique j such that

$$g\,(g_iH) = (gg_i)\,H = g_jH.$$

In otheer words, there exists a unique j and a unique $h \in H$ such that

$$gg_i = g_jh. \tag{4.15}$$

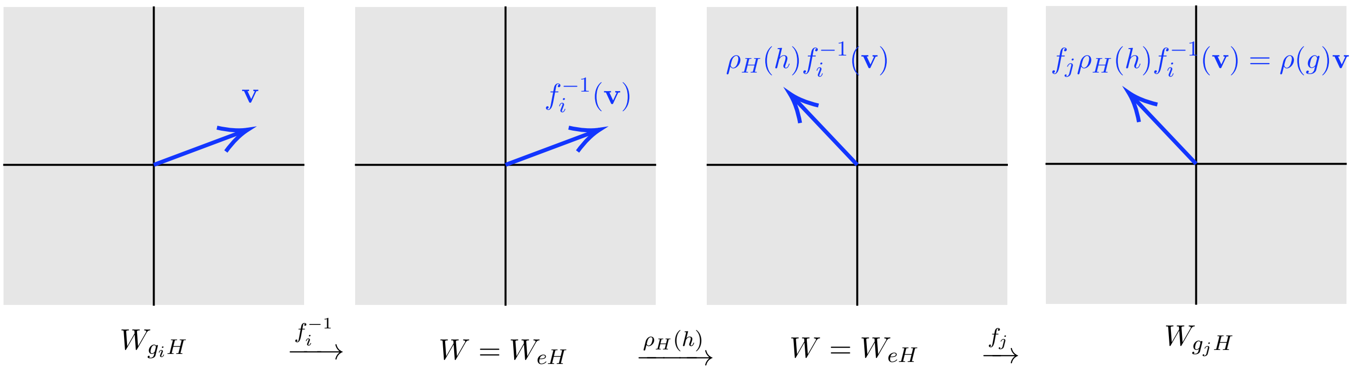
We need to consider vector space isomorphisms

$$f_i : W = W_{eH} \to W_{g_iH}.$$

W_{g_iH} is the same vector space as W, just labelled differently. So f_i is just the identity map. Then we define

$$\rho\,(g)\,\mathbf{v} = \left(f_j \circ \rho_H\,(h) \circ f_i^{-1}\right)\mathbf{v}, \tag{4.16}$$

for $\mathbf{v} \in W_{g_iH}$, and $gg_i = g_jh$.



Now we need to show that $\rho : G \to \mathrm{GL}(V)$ is, indeed, a representation.

Proposition 4.2

$\rho : G \to \mathrm{GL}(V)$ defined as in (4.16) is a representation, i.e. $\rho : G \to \mathrm{GL}(V)$ is a group homomorphism, where $V = \bigoplus_{i=1}^n W_{g_iH}$.

Proof. Let $g, g' \in G$, we need to show that $\rho\,(g'g) = \rho\,(g')\,\rho\,(g)$. Take $\mathbf{v} \in W_{g_iH}$. Choose the unique j and $h \in H$ such that $gg_i = g_jh$; then choose the unique k and $h' \in H$ such that $g'g_j = g_kh'$. Then $(g'g)\,g_i = g_k\,(h'h)$. As a result,

$$\begin{aligned}
\rho\,(g'g)\,\mathbf{v} &= \left[f_k\rho_H\,(h'h)\,f_i^{-1}\right]\mathbf{v} \\
&= \left[f_k\rho_H\,(h')\,\rho_H\,(h)\,f_i^{-1}\right]\mathbf{v} \\
&= \left[f_k\rho_H\,(h')\,f_j^{-1}\right]\left[f_j\rho_H\,(h)\,f_i^{-1}\right]\mathbf{v} \\
&= \rho\,(g')\,[\rho\,(g)\,\mathbf{v}]\,.
\end{aligned} \tag{4.17}$$

Therefore, $\rho\,(g'g) = \rho\,(g')\,\rho\,(g)$, and hence $\rho : G \to \mathrm{GL}(V)$ is a representation. ∎

Remark 4.2. From the construction, we can see that

$$\dim \operatorname{Ind}_H^G W = \dim \oplus_{\sigma \in G/H} W_\sigma = |G/H| \dim W = [G:H] \dim W, \tag{4.18}$$

where $[G:H]$ is the index of the subgroup H in G, i.e. the number of distinct left-cosets of H in G.

Example 4.1. Let $\mathbb{C}_{\text{triv}}$ be the 1-dimensional trivial representation on H. Then

$$\operatorname{Ind}_H^G (\mathbb{C}_{\text{triv}}) = \mathbb{C}[G/H]. \tag{4.19}$$

Here, the H-invariant subspace W is the 1-dimensional space $\mathbb{C}\mathbf{e}_H$. In this case, the subspaces W_σ, labelled by the cosets $\sigma \in G/H$, are all 1-dimensional, i.e. $W_\sigma = \mathbb{C}\mathbf{e}_\sigma$, for $\sigma \in G/H$. As a result,

$$\operatorname{Ind}_H^G (\mathbb{C}_{\text{triv}}) = \bigoplus_{\sigma \in G/H} W_\sigma = \bigoplus_{\sigma \in G/H} \mathbb{C}\mathbf{e}_\sigma = \mathbb{C}[G/H]. \tag{4.20}$$

In this case, the G-action on $V = \bigoplus_{\sigma \in G/H} \mathbb{C}\mathbf{e}_\sigma$ is given by

$$\rho(g)\,\mathbf{e}_\sigma = \mathbf{e}_{g\sigma}. \tag{4.21}$$

Here, $\mathbb{C}[G/H]$ is the permutation representation of G associated with its action on the set G/H by left multiplication. Hence, this permutation representation of G is induced from the trivial 1-dimensional representation of H.

Remark 4.3. Take $H = \{e\}$, the subgroup containing just the identity element. Then

$$\operatorname{Ind}_{\{e\}}^G (\mathbb{C}_{\text{triv}}) = \mathbb{C}[G]. \tag{4.22}$$

Hence, the regular representation of G is induced from the 1-dimensional trivial representation of $\{e\} \leq G$.

Example 4.2. Let $\mathbb{C}[H]$ be the representation space of the left-regular representation of H. Then

$$\operatorname{Ind}_H^G (\mathbb{C}[H]) = \mathbb{C}[G]. \tag{4.23}$$

Let $H = \{e, h_1, \ldots, h_k\}$. Then the representation space $\mathbb{C}[H]$ is

$$W = \mathbb{C}[H] = \mathbb{C}\mathbf{e}_e \oplus \mathbb{C}\mathbf{e}_{h_1} \oplus \cdots \oplus \mathbb{C}\mathbf{e}_{h_k}.$$

Clearly, W is H-invariant, since

$$\rho(h_i)\left(\mathbf{e}_{h_j}\right) = \mathbf{e}_{h_i h_j} \in W.$$

Now, let $\sigma = gH \in G/H$ be a coset. Then define

$$W_\sigma = \mathbb{C}\langle \mathbf{e}_{gh} \ : \ h \in H \rangle = \mathbb{C}\langle \mathbf{e}_x \ : \ x \in \sigma \rangle. \tag{4.24}$$

Then $\rho(g')\,\mathbf{e}_{gh} = \mathbf{e}_{g'gh} \in W_{g'gH}$, implying that

$$\rho(g')(W_{gH}) = W_{g'gH}.$$

Now,

$$\operatorname{Ind}_H^G W = \bigoplus_{\sigma \in G/H} W_\sigma = \bigoplus_{\sigma \in G/H} \mathbb{C}\langle \mathbf{e}_x \ : \ x \in \sigma \rangle = \bigoplus_{g \in G} \mathbb{C}\langle \mathbf{e}_g \rangle = \mathbb{C}[G]. \tag{4.25}$$

Remark 4.4. Since the regular representation of the group containing only the ideneity element is the 1-dimensional trvial representation, one has $\mathbb{C}[\{e\}] = \mathbb{C}_{\text{triv}}$. Plugging in $H = \{e\}$ in Example 4.2, we have

$$\operatorname{Ind}_{\{e\}}^G (\mathbb{C}_{\text{triv}}) = \mathbb{C}[G]. \tag{4.26}$$

(4.23) tells us that the regular representation of G is induced from the regular representation of H.

§4.3 Induction and restriction

Given a finite group G and its subgroup $H \leq G$ and a G-representation U with group homomorphism $\rho : G \to \mathrm{GL}(U)$, one writes $\operatorname{Res}_H^G \rho$ for the restriction of the group homomorphism ρ, i.e.

$$\operatorname{Res}_H^G \rho = \rho\big|_H : H \to \mathrm{GL}(U),$$

so that $\operatorname{Res}_H^G \rho$ is a H-representation.

Lemma 4.3

Restriction is transitive, i.e. if $H \leq K \leq G$ and $\rho : G \to \mathrm{GL}(V)$ is a G-representation, then $\operatorname{Res}_H^G \rho = \operatorname{Res}_H^K \left(\operatorname{Res}_K^G \rho\right)$.

Proof.

$$\operatorname{Res}_H^K \left(\operatorname{Res}_K^G \rho\right) = \left(\operatorname{Res}_K^G \rho\right)\big|_H = \left(\rho\big|_K\right)\big|_H = \rho\big|_H = \operatorname{Res}_H^G \rho. \tag{4.27}$$

■

Proposition 4.4

Induction is also transitive, i.e. if $H \leq K \leq G$ and $\rho : H \to \mathrm{GL}(W)$ is a H-representation, then $\operatorname{Ind}_H^G \rho \cong \operatorname{Ind}_K^G \left(\operatorname{Ind}_H^K \rho\right)$.

The proof is a bit lengthy and tedius. We can make our lives easier when we learn about induced class functions and induced characters. Then we'll give a proof of this.

Recall the universal mapping property of induced representation. For $H \leq G$, let W be a H-representation, and $\operatorname{Ind}_H^G W$ be the induced representation of G. Take the inclusion $\iota : W \to \operatorname{Ind}_H^G W = \bigoplus_{\sigma \in G/H} W_\sigma$. Then given any G-representation U and any H-linear map $\varphi : W \to U$, there exists a unique G-linear map $\widetilde{\varphi} : \operatorname{Ind}_H^G W \to U$ such that the following diagram commutes:

$$\begin{tikzcd} W \arrow[r, "\iota"] \arrow[dr, "\varphi"'] & \operatorname{Ind}_H^G W = \bigoplus_{\sigma \in G/H} W_\sigma \arrow[d, dashed, "\widetilde{\varphi}"] \\ & U \end{tikzcd} \tag{4.28}$$

i.e. $\widetilde{\varphi} \circ \iota = \varphi$.

Now, observe that φ is a H-linear map $W \to U$. Hence, $\varphi \in \operatorname{Hom}_H \left(W, \operatorname{Res}_H^G U\right)$, because U is a G-representation to begin with, and we can restrict to a H-representation. For every $\varphi \in \operatorname{Hom}_H \left(W, \operatorname{Res}_H^G U\right)$, there is a unique corresponding G-linear map $\widetilde{\varphi} : \operatorname{Ind}_H^G W \to U$, i.e. $\widetilde{\varphi} \in \operatorname{Hom}_G \left(\operatorname{Ind}_H^G W, U\right)$.

Similarly, obseve that $\iota : W \to \operatorname{Ind}_H^G W = \bigoplus_{\sigma \in G/H} W_\sigma$ is an H-linear map, because clearly the following diagram commutes:

$$\begin{tikzcd} W \arrow[r, "\iota"] \arrow[d, "\rho_H(h)"'] & \bigoplus_{\sigma \in G/H} W_\sigma \arrow[d, "\rho|_H(h) = \rho_H(h)"] \\ W \arrow[r] & \bigoplus_{\sigma \in G/H} W_\sigma \end{tikzcd}$$

Now, given any G-linear map $\widetilde{\varphi} \in \mathrm{Hom}_G\left(\mathrm{Ind}_H^G W, U\right)$, $\widetilde{\varphi}$ is automatically H-linear. Hence, the composition $\widetilde{\varphi} \circ \iota : W \to \mathrm{Res}_H^G U$ is a H-linear map. So given any $\widetilde{\varphi} \in \mathrm{Hom}_G\left(\mathrm{Ind}_H^G W, U\right)$, there exists a unique corresponding $\varphi = \widetilde{\varphi} \circ \iota \in \mathrm{Hom}_H\left(W, \mathrm{Res}_H^G U\right)$ subject to the commutativity of (4.28). It establishes the isomorphism of the two vector spaces

$$\mathrm{Hom}_G\left(\mathrm{Ind}_H^G W, U\right) \cong \mathrm{Hom}_H\left(W, \mathrm{Res}_H^G U\right). \tag{4.29}$$

Proposition 4.5

Suppose G is a finite group and $H \leq G$. Additionally, suppose U is a G-representation with group homomorphism $\rho_G : G \to \mathrm{GL}(U)$, and W is an H-representation with group homomorphism $\rho_H : H \to \mathrm{GL}(W)$. Then

$$\rho_G \otimes \mathrm{Ind}_H^G \rho_H = \mathrm{Ind}_H^G\left(\mathrm{Res}_H^G \rho_G \otimes \rho_H\right). \tag{4.30}$$

In terms of the representation space, this reads

$$U \otimes \mathrm{Ind}_H^G W = \mathrm{Ind}_H^G\left(\mathrm{Res}_H^G U \otimes W\right). \tag{4.31}$$

Proof. We'll construct a G-linear isomorphism

$$\varphi : U \otimes \left(\bigoplus_{\sigma \in G/H} W_\sigma\right) \to \bigoplus_{\sigma \in G/H} (U \otimes W)_\sigma .$$

Let $f_i : W = W_{eH} \to W_{g_i H}$ and $\overline{f}_i : U \otimes W = (U \otimes W)_{eH} \to (U \otimes W)_{g_i H}$ be the respective vector space isomorphisms (which are just identity maps, as we have discussed earlier). Here, f_i concerns $\mathrm{Ind}_H^G W = \bigoplus_{\sigma \in G/H} W_\sigma$, and $\overline{f}_i$ concerns $\mathrm{Ind}_H^G\left(\mathrm{Res}_H^G U \otimes W\right) = \bigoplus_{\sigma \in G/H} (U \otimes W)_\sigma$.

We define the i-th compoenent of φ to be

$$\varphi_i = \overline{f}_i \circ \left[\rho_G\left(g_i^{-1}\right) \otimes f_i^{-1}\right]. \tag{4.32}$$

It takes a vector $\mathbf{u} \otimes \mathbf{w} \in U \otimes W_{g_i H}$ to $\overline{f}_i\left(\rho_G\left(g_i^{-1}\right)\mathbf{u} \otimes f_i^{-1}\mathbf{w}\right) \in (U \otimes W)_{g_i H}$. Here, $\overline{f}_i$, $\rho_G\left(g_i^{-1}\right)$, f_i are all bijective linear maps. Hence, φ is a bijective linear map. Now we need to show the G-linearity of φ.

At this point, let's denote by $\rho(g)$ the group action of G on $U \otimes \left(\bigoplus_{\sigma \in G/H} W_\sigma\right)$, and by $\widetilde{\rho}(g)$ the group action of G on $\bigoplus_{\sigma \in G/H} (U \otimes W)_\sigma$. Now, take $\mathbf{u} \in U$ and $\mathbf{w} \in W_{g_i H}$, so that $\mathbf{u} \otimes \mathbf{w} \in U \otimes W_{g_i H}$. Given $g \in G$, take the unique j and $h \in H$ such that $g g_i = g_j h$. Then

$$\rho(g)(\mathbf{u} \otimes \mathbf{w}) = \rho_G(g)\mathbf{u} \otimes \left(\mathrm{Ind}_H^G \rho_H\right)(\mathbf{w}) = \rho_G(g)\mathbf{u} \otimes f_j\left(\rho_H(h) f_i^{-1}(\mathbf{w})\right). \tag{4.33}$$

This vector is in $U \otimes W_{g_j H}$, and the j-th component of φ will act on it. Hence,

$$\begin{aligned}
\varphi(\rho(g)(\mathbf{u} \otimes \mathbf{w})) &= \varphi_j(\rho(g)(\mathbf{u} \otimes \mathbf{w})) \\
&= \overline{f}_j\left[\rho_G\left(g_j^{-1}\right) \otimes f_j^{-1}\right]\left[\rho_G(g)\mathbf{u} \otimes f_j\left(\rho_H(h) f_i^{-1}(\mathbf{w})\right)\right] \\
&= \overline{f}_j\left[\rho_G\left(g_j^{-1} g\right)\mathbf{u} \otimes \rho_H(h) f_i^{-1}\mathbf{w}\right] \\
&= \overline{f}_j\left[\rho_G\left(h g_i^{-1}\right)\mathbf{u} \otimes \rho_H(h) f_i^{-1}\mathbf{w}\right];
\end{aligned} \tag{4.34}$$

because $gg_i = g_j h$ implies $g_j^{-1} g = h g_i^{-1}$. On the other hand,

$$\begin{aligned}
\widetilde{\rho}(g)\left[\varphi(\mathbf{u}\otimes\mathbf{w})\right] &= \widetilde{\varphi}(g)\left[\varphi_i(\mathbf{u}\otimes\mathbf{w})\right] \\
&= \widetilde{\rho}(g)\left[\overline{f}_i \circ \left[\rho_G\left(g_i^{-1}\right)\otimes f_i^{-1}\right]\right](\mathbf{u}\otimes\mathbf{w}) \\
&= \widetilde{\rho}(g)\left[\overline{f}_i\left(\rho_G\left(g_i^{-1}\right)\mathbf{u}\otimes f_i^{-1}\mathbf{w}\right)\right] \\
&= \left[\overline{f}_j \circ \left(\operatorname{Res}^G_H \rho_G \otimes \rho_H\right)(h)\circ \overline{f}_i^{-1}\right]\left[\overline{f}_i\left(\rho_G\left(g_i^{-1}\right)\mathbf{u}\otimes f_i^{-1}\mathbf{w}\right)\right] \\
&= \overline{f}_j \circ \left(\rho_G(h)\otimes\rho_H(h)\right)\left[\left(\rho_G\left(g_i^{-1}\right)\mathbf{u}\otimes f_i^{-1}\mathbf{w}\right)\right] \\
&= \overline{f}_j\left[\rho_G(h)\,\rho_G(g_i)\,\mathbf{u}\otimes\rho_H(h)\,f_i^{-1}\mathbf{w}\right].
\end{aligned} \tag{4.35}$$

Comparing (4.34) and (4.35), we have

$$\varphi\circ\rho(g) = \widetilde{\rho}(g)\circ\varphi. \tag{4.36}$$

In other words, the following diagram commutes:

$$\begin{tikzcd}
U\otimes\left(\bigoplus_{\sigma\in G/H} W_\sigma\right) \arrow[r, "\varphi"] \arrow[d, "\rho(g)"'] & \bigoplus_{\sigma\in G/H}(U\otimes W)_\sigma \arrow[d, "\widetilde{\rho}(g)"] \\
U\otimes\left(\bigoplus_{\sigma\in G/H} W_\sigma\right) \arrow[r, "\varphi"'] & \bigoplus_{\sigma\in G/H}(U\otimes W)_\sigma
\end{tikzcd}$$

This establishes the G-linearity of $\varphi : U\otimes\left(\bigoplus_{\sigma\in G/H} W_\sigma\right)\to\bigoplus_{\sigma\in G/H}(U\otimes W)_\sigma$. Hence, φ is a G-linear bijective linear map, i.e. an isomorphism of representations, as required. ■

§4.4 Induced class function

We have seen that given $H\leq G$ and a G-representation U with group homomorphism $\rho : G\to \mathrm{GL}(U)$, one writes $\operatorname{Res}^G_H \rho$ for the restruction of the group homomorphism ρ to the subgroup H, i.e. $\operatorname{Res}^G_H \rho = \rho|_H : H\to\mathrm{GL}(U)$.

Evidently, the representation space for both ρ and $\operatorname{Res}^G_H \rho$ is U. If χ is the character of G associated with the representation ρ, then by $\operatorname{Res}^G_H \chi$ we denote the character of the subgroup $H\leq G$ associated with the representation $\operatorname{Res}^G_H \rho$. One often writes $\chi|_H$ or simply χ_H for $\operatorname{Res}^G_H \chi$.

Lemma 4.6

Suppose $H\leq G$. If ψ is a nonzero character of H, then there exists an irreducible character χ of G such that

$$\left(\operatorname{Res}^G_H \chi,\psi\right)_H \neq 0. \tag{4.37}$$

(Here $(\alpha,\beta)_H$ denotes the inner product in the space $\mathbb{C}_{\text{class}}[H]$ of class functions on H.)

Proof. Let $\chi_1,\ldots,\chi_k$ be the irreducible characters of G. Let us denote the character associated with the regular representation of G by χ_{reg}. From (2.90), we know that

$$\chi_{\text{reg}}(h) = \begin{cases} |G| & \text{if } h = e, \\ 0 & \text{otherwise.}\end{cases} \tag{4.38}$$

Therefore,

$$\left(\chi_{\text{reg}}|_H,\psi\right)_H = \frac{1}{|H|}\sum_{g\in H}\overline{\chi_{\text{reg}}|_H(g)}\,\psi(g) = \frac{1}{|H|}|G|\,\psi(e). \tag{4.39}$$

By Corollary 2.9, every irreducible representation V is contained in the regular representation with multiplicity $\dim V$. Therefore, $\chi_{\text{reg}} = \sum_{i=1}^{k} n_i \chi_i$. So

$$\left(\chi_{\text{reg}}|_H, \psi\right)_H = \left(\sum_{i=1}^{k} n_i \chi_i|_H, \psi\right)_H = \sum_{i=1}^{k} \overline{n_i} \left(\chi_i|_H, \psi\right)_H = \sum_{i=1}^{k} n_i \left(\chi_i|_H, \psi\right)_H. \tag{4.40}$$

Since $\frac{1}{|H|} |G| \, \psi(e) \neq 0$, we have

$$\sum_{i=1}^{k} n_i \left(\chi_i|_H, \psi\right)_H \neq 0. \tag{4.41}$$

Therefore, at least one of the summands is nonzero, i.e. $\left(\chi_i|_H, \psi\right)_H \neq 0$ for some $i \in \{1, 2, \ldots, k\}$. ∎

Remark 4.5. $\left(\operatorname{Res}_H^G \chi, \psi\right) \neq 0$ is also expressed as $\psi \subset \operatorname{Res}_H^G \chi$, or ψ is a constituent of $\operatorname{Res}_H^G \chi$.

Lemma 4.7

Let χ be an irreducible character of G, and let $\operatorname{Res}_H^G \chi = \sum_i c_i \chi_i$, where χ_i's are irreducible characters of H, with $c_i \in \mathbb{Z}_{\geq 0}$. Then

$$\sum_i c_i^2 \leq [G : H], \tag{4.42}$$

with equality if and only if $\chi(g) = 0$ for every $g \in G \setminus H$.

(Here $[G : H]$ is the index of the subgroup H in the group G given by the formula $[G : H] = \frac{|G|}{|H|}$, which is the number of left H-cosets in G.)

Proof.

$$\sum_i c_i^2 = \left(\operatorname{Res}_H^G \chi, \operatorname{Res}_H^G \chi\right)_H = \frac{1}{|H|} \sum_{h \in H} |\chi(h)|^2. \tag{4.43}$$

But since χ is an irreducible character of G, one has

$$\begin{aligned} 1 = (\chi, \chi)_G &= \frac{1}{|G|} \sum_{g \in G} |\chi(g)|^2 \\ &= \frac{1}{|G|} \left[\sum_{h \in H} |\chi(h)|^2 + \sum_{g \in G \setminus H} |\chi(g)|^2 \right] \\ &= \frac{|H|}{|G|} \sum_i c_i^2 + \frac{1}{|G|} \sum_{g \in G \setminus H} |\chi(g)|^2 \\ &\geq \frac{|H|}{|G|} \sum_i c_i^2, \end{aligned} \tag{4.44}$$

with equality if and only if $\chi(g) = 0$ for every $g \in G \setminus H$. Therefore,

$$\sum_i c_i^2 \leq \frac{|G|}{|H|} = [G : H]. \tag{4.45}$$

∎

Definition 4.2 (Induction of class functions). Let $\psi \in \mathbb{C}_{\text{class}}(H)$. We define the **induced class function** $\operatorname{Ind}_H^G \psi$ on G by

$$\operatorname{Ind}_H^G \psi(g) = \frac{1}{|H|} \sum_{x \in G} \mathring{\psi}\left(x^{-1} g x\right), \tag{4.46}$$

where $\mathring{\psi}$ is a piecewise function defined on the whole of G as follows:

$$\mathring{\psi}(y) = \begin{cases} \psi(y) & \text{if } y \in H, \\ 0 & \text{if } y \notin H. \end{cases} \tag{4.47}$$

Lemma 4.8

If $\psi \in \mathbb{C}_{\text{class}}(H)$, then $\operatorname{Ind}_H^G \psi \in \mathbb{C}_{\text{class}}(G)$, and $\operatorname{Ind}_H^G (e) = [G : H]\, \psi(e)$.

Proof. Let g and g' be conjugate elements. Then there exists $y \in G$ such that $g' = y^{-1}gy$. Then

$$\begin{aligned} \operatorname{Ind}_H^G \psi(g') &= \frac{1}{|H|} \sum_{x \in G} \mathring{\psi}\left(x^{-1}g'x\right) \\ &= \frac{1}{|H|} \sum_{x \in G} \mathring{\psi}\left(x^{-1}y^{-1}gyx\right) \\ &= \frac{1}{|H|} \sum_{yx \in G} \mathring{\psi}\left((yx)^{-1}g(yx)\right) \\ &= \frac{1}{|H|} \sum_{z \in G} \mathring{\psi}\left(z^{-1}gz\right) = \operatorname{Ind}_H^G \psi(g). \end{aligned}$$

Therefore, $\operatorname{Ind}_H^G \psi$ is a class function on G.

$$\operatorname{Ind}_H^G \psi(e) = \frac{1}{|H|} \sum_{x \in G} \mathring{\psi}\left(x^{-1}ex\right) = \frac{1}{|H|}\, |G|\, \psi(e) = [G : H]\, \psi(e). \tag{4.48}$$

■

Let us now denote the index of the subgroup H in G by n, i.e. $n = [G : H]$. Then there are n-distinct left H-cosets in G. Let $g_1 = e, g_2, \ldots, g_n$ be a complete set of coset representatives, also known as **left transversal**. Then $g_1H(= eH = H), g_2H, g_3H, \ldots, g_nH$ are precisely the left cosets of H in G.

Lemma 4.9

Given a transversal as above,

$$\operatorname{Ind}_H^G \psi(g) = \sum_{i=1}^{n} \mathring{\psi}\left(g_i^{-1}gg_i\right). \tag{4.49}$$

Proof. Using the definition of induced class function,

$$\operatorname{Ind}_H^G \psi(g) = \frac{1}{|H|} \sum_{x \in G} \mathring{\psi}\left(x^{-1}gx\right) = \frac{1}{|H|} \sum_{i=1}^{n} \sum_{x \in g_iH} \mathring{\psi}\left(x^{-1}gx\right). \tag{4.50}$$

Now, for $x \in g_iH$, $x = g_ih$ for some $h \in H$. So

$$\mathring{\psi}\left(x^{-1}gx\right) = \mathring{\psi}\left(h^{-1}g_i^{-1}gg_ih\right). \tag{4.51}$$

If $g_i^{-1}gg_i \in H$, the RHS of (4.51) is equal to $\psi\left(h^{-1}g_i^{-1}gg_ih\right) = \psi\left(g_i^{-1}gg_i\right)$, since ψ is a class function on H. On the other hand, if $g_i^{-1}gg_i \notin H$, then $h^{-1}g_i^{-1}gg_ih \notin H$ as well. Then the RHS of (4.51) is 0, so is $\mathring{\psi}\left(g_i^{-1}gg_i\right)$. In either case, the RHS of (4.51) is $\mathring{\psi}\left(g_i^{-1}gg_i\right)$. Hence,

$$\operatorname{Ind}_H^G \psi(g) = \frac{1}{|H|} \sum_{i=1}^{n} \sum_{x \in g_iH} \mathring{\psi}\left(x^{-1}gx\right) = \frac{1}{|H|} \sum_{i=1}^{n} \sum_{x \in g_iH} \mathring{\psi}\left(g_i^{-1}gg_i\right) = \sum_{i=1}^{n} \mathring{\psi}\left(g_i^{-1}gg_i\right). \tag{4.52}$$

■

Theorem 4.10 (Frobenius reciprocity)
Let $\psi \in \mathbb{C}_{\text{class}}[H]$ and $\phi \in \mathbb{C}_{\text{class}}[G]$, where $H \leq G$. Then

$$\left(\operatorname{Res}_H^G \phi, \psi\right)_H = \left(\phi, \operatorname{Ind}_H^G \psi\right)_G. \tag{4.53}$$

Proof.

$$\left(\phi, \operatorname{Ind}_H^G \psi\right)_G = \frac{1}{|G|} \sum_{g\in G} \overline{\phi(g)} \operatorname{Ind}_H^G \psi(g) = \frac{1}{|G|\,|H|} \sum_{g\in G} \overline{\phi(g)} \sum_{x\in G} \mathring{\psi}\left(x^{-1}gx\right). \tag{4.54}$$

We use the change of variable $y = x^{-1}gx$. Then $g = xyx^{-1}$. For a fixed $x \in G$, as g varies in G, y also varies in G. Hence,

$$\left(\phi, \operatorname{Ind}_H^G \psi\right)_G = \frac{1}{|G|\,|H|} \sum_{x,y\in G} \overline{\phi\left(xyx^{-1}\right)}\, \mathring{\psi}(y) = \frac{1}{|G|\,|H|} \sum_{x,y\in G} \overline{\phi(y)}\, \mathring{\psi}(y), \tag{4.55}$$

since ϕ is a class function on G. The summands are x-independent, so the x-sum will yield $|G|$. Also, for $y \notin H$, $\mathring{\psi}(y) = 0$. So, the y-sum can be rewritten as a sum over $y \in H$. Hence,

$$\left(\phi, \operatorname{Ind}_H^G \psi\right)_G = \frac{|G|}{|G|\,|H|} \sum_{y\in H} \overline{\phi(y)}\, \psi(y) = \left(\operatorname{Res}_H^G \phi, \psi\right)_H. \tag{4.56}$$

■

Corollary 4.11
If ψ is a character of H, then $\operatorname{Ind}_H^G \psi$ is a character of G.

Proof. Let χ be an irreducible character of G. By Theorem 4.10,

$$\left(\chi, \operatorname{Ind}_H^G \psi\right)_G = \left(\operatorname{Res}_H^G \chi, \psi\right)_H \in \mathbb{Z}_{\geq 0}, \tag{4.57}$$

since ψ and $\operatorname{Res}_H^G \chi$ are both characters of H. Then (4.57) means that $\operatorname{Ind}_H^G \psi$ is a $\mathbb{Z}_{\geq 0}$-linear combination of irreducible characters of G. Therefore, $\operatorname{Ind}_H^G \psi$ is a character of G. ■

So we have proved that the induced class function of a character of H is a character of G, and that is precisely the character of the induced representation, as the following result states.

Theorem 4.12
Suppose $H \leq G$, and let χ be the character of the representation $\rho : H \to \mathrm{GL}(W)$. Then $\operatorname{Ind}_H^G \chi$ is the character of the induced representation $\operatorname{Ind}_H^G \rho$.

Proof. Let $\chi_{\operatorname{Ind}_H^G \rho}$ be the character of the induced representation $\operatorname{Ind}_H^G \rho$. We need to show that $\chi_{\operatorname{Ind}_H^G \rho} = \operatorname{Ind}_H^G \chi$. Let $g \in G$, then

$$\left(\operatorname{Ind}_H^G \rho\right)(g) : \bigoplus_{i=1}^n W_{g_iH} \to \bigoplus_{i=1}^n W_{g_iH},$$

where $g_1 = e, g_2, g_3, \ldots, g_n$ is a complete set of left transversals. Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a basis for W. Then

$$\begin{aligned}
\Big\{&\mathbf{v}_1^{(1)}, \mathbf{v}_2^{(1)}, \ldots, \mathbf{v}_k^{(1)},\\
&\mathbf{v}_1^{(2)}, \mathbf{v}_2^{(2)}, \ldots, \mathbf{v}_k^{(2)},\\
&\ldots,\\
&\mathbf{v}_1^{(n)}, \mathbf{v}_2^{(n)}, \ldots, \mathbf{v}_k^{(n)}\Big\}
\end{aligned}$$

is a basis for $\bigoplus_{i=1}^{n} W_{g_i H}$. Let A be the matrix representation of $\left(\operatorname{Ind}_H^G \rho\right)(g)$ in this basis. A_{ii} denotes the component of the i-th basis vector when $\left(\operatorname{Ind}_H^G \rho\right)(g)$ applied to the i-th basis vector.

By definition of $\left(\operatorname{Ind}_H^G \rho\right)(g)$, it takes $W_{g_i H}$ to $W_{g_j H}$, where j is the unique number such that $gg_i \in g_j H$. If $i \neq j$, $\left(\operatorname{Ind}_H^G \rho\right)(g)$ takes each of the vectors $\mathbf{v}_1^{(i)}, \mathbf{v}_2^{(i)}, \ldots, \mathbf{v}_k^{(i)}$ to some linear combinations of $\mathbf{v}_1^{(j)}, \mathbf{v}_2^{(j)}, \ldots, \mathbf{v}_k^{(j)}$. That contributes 0 to the diagonal entries of the matrix A.

On the contrary, when $i = j$, $\left(\operatorname{Ind}_H^G \rho\right)(g)$ takes $W_{g_i H}$ to itself. In that case $gg_i = g_i h$, for some unique $h \in H$. Then $h = g_i^{-1} g g_i$. Then the i-th component of $\left(\operatorname{Ind}_H^G \rho\right)(g)$ is

$$f_i \rho\left(h\right) f_i^{-1} = f_i \rho\left(g_i^{-1} g g_i\right) f_i^{-1} : W_{g_i H} \to W_{g_i H}.$$

Therefore,

$$\operatorname{Tr}\left[\left(\operatorname{Ind}_H^G \rho\right)(g)\right] = \sum_{i=j} \operatorname{Tr}\left[f_i \rho\left(h\right) f_i^{-1}\right] = \sum_{g_i^{-1} g g_i \in H} \operatorname{Tr}\left[\rho\left(g_i^{-1} g g_i\right)\right]. \tag{4.58}$$

The summand is nothing but $\chi\left(g_i^{-1} g g_i\right)$. This sum is the same as summing over all i, except when $g_i^{-1} g g_i \notin H$, we'll consider the summand to be 0. Therefore,

$$\begin{aligned}
\chi_{\operatorname{Ind}_H^G \rho}(g) &= \sum_{i=1}^{n} \begin{cases} \chi\left(g_i^{-1} g g_i\right) & \text{if } g_i^{-1} g g_i \in H, \\ 0 & \text{otherwise.} \end{cases} \\
&= \sum_{i=1}^{n} \mathring{\chi}\left(g_i^{-1} g g_i\right) \\
&= \operatorname{Ind}_H^G \chi\left(g\right),
\end{aligned} \tag{4.59}$$

by Lemma 4.9. Hence, $\chi_{\operatorname{Ind}_H^G \rho} = \operatorname{Ind}_H^G \chi$. ■

Let $\mathscr{C}_G(g)$ denote the conjugacy class of G containing g.

$$\mathscr{C}_G(g) = \left\{x^{-1} g x \mid x \in G\right\}.$$

For $H \leq G$, take the set $\mathscr{C}_G(g) \cap H$, which can be written as the disjoint union of some H-conjugacy classes. Note that $\mathscr{C}_G(g) \cap H$ cannot contain any H-conjugacy class partially. Indeed, let $x \in \mathscr{C}_G(g) \cap H$, and suppose x' is conjugate to x in H. Then $x' = h^{-1} x h$ for some $h \in H$, so $x' \in H$. Since $x \in \mathscr{C}_G(g)$, $x = y^{-1} g y$ for some $y \in G$. As a result, $x' = h^{-1} x h = h^{-1} y^{-1} g y h = (yh)^{-1} g\,(yh) \in \mathscr{C}_G(g)$. Hence, $x' \in \mathscr{C}_G(g) \cap H$. Therefore, $\mathscr{C}_G(g) \cap H$ cannot contain any H-conjugacy class partially.

A brief on double cosets: Let $H, K \leq G$. A double coset of H and K in G is a set of the form $HxK = \{hxk \mid h \in H \text{ and } k \in K\}$, for some $x \in G$. It's well known that two double cosets are either disjoint or equal, because double cosets are precisely the equivalence classes of the equivalence relation

$$x \sim y \iff \text{there exists } h \in H, k \in K \text{ such that } y = hxk.$$

The cardinality of a double coset HxK is

$$|HxK| = \frac{|H|\,|K|}{|H \cap xKx^{-1}|} = \frac{|H|\,|K|}{|x^{-1}Hx \cap K|}. \tag{4.60}$$

Indeed, there is a bijection between HxK and $HxKx^{-1}$. Therefore, these two sets have the same cardinality. The latter is a product of two subgroups. We know that if $H_1, H_2 \leq G$, then

$$|H_1 H_2| = \frac{|H_1|\,|H_2|}{|H_1 \cap H_2|}. \tag{4.61}$$

Therefore,

$$|HxK| = \left|HxKx^{-1}\right| = \frac{|H|\,|xKx^{-1}|}{|H \cap xKx^{-1}|} = \frac{|H|\,|K|}{|H \cap xKx^{-1}|}. \tag{4.62}$$

Similarly, the bijection between HxK and $x^{-1}HxK$ will give us

$$|HxK| = \frac{|H|\,|K|}{|x^{-1}Hx \cap K|}. \tag{4.63}$$

Proposition 4.13

Let ψ be a character of $H \leq G$, and let $g \in G$. Let

$$\mathscr{C}_G(g) \cap H = \bigcup_{i=1}^{m} \mathscr{C}_H(x_i)$$

where the $\mathscr{C}_H(x_i)$ are the pairwise disjoint H-conjugacy classes whose union is $\mathscr{C}_G(g) \cap H$. Then

$$\operatorname{Ind}_H^G \psi(g) = \begin{cases} 0 & \text{if } m = 0, \\ |C_G(g)| \sum\limits_{i=1}^{m} \frac{\psi(x_i)}{|C_H(x_i)|} & \text{otherwise;} \end{cases} \tag{4.64}$$

where $C_G(g) = \{x \in G \mid x^{-1}gx = g\}$ is the centralizer of g in G.

Proof. If $m = 0$, then $\mathscr{C}_G(g) \cap H = \varnothing$, i.e. $\{x \in G \mid x^{-1}gx \in H\}$ is an empty set. Then we have

$$\begin{aligned} \operatorname{Ind}_H^G \psi(g) &= \frac{1}{|H|} \sum_{x \in G} \mathring{\psi}\left(x^{-1}gx\right) = \frac{1}{|H|} \sum_{x \in G} \begin{cases} \psi(x^{-1}gx) & \text{if } x^{-1}gx \in H, \\ 0 & \text{otherwise} \end{cases} \\ &= 0. \end{aligned} \tag{4.65}$$

Now let $m > 0$. Let $X_i = \{x \in G \mid x^{-1}gx \in \mathscr{C}_H(x_i)\}$, for $1 \leq i \leq m$. Then X_i are pairwise disjoint, and their union is the set $\{x \in G \mid x^{-1}gx \in H\}$. By definition of the induced class function

$$\operatorname{Ind}_H^G \psi(g) = \frac{1}{|H|} \sum_{x \in G} \mathring{\psi}\left(x^{-1}gx\right) = \frac{1}{|H|} \sum_{i=1}^{m} \sum_{x \in X_i} \psi\left(x^{-1}gx\right). \tag{4.66}$$

For $x \in X_i$, $x^{-1}gx$ is H-conjugate to x_i, and $\psi \in \mathbb{C}_{\text{class}}[H]$, so $\psi(x^{-1}gx) = \psi(x_i)$. Therefore, there are $|X_i|$-many contributions each equal to $\psi(x_i)$. As a result,

$$\operatorname{Ind}_H^G \psi(g) = \sum_{i=1}^{m} \frac{|X_i|}{|H|} \psi(x_i). \tag{4.67}$$

Let's now calculate $\frac{|X_i|}{|H|}$. For each i, choose $g_i \in G$ such that $g_i^{-1}gg_i = x_i$. We claim that $C_G(g)\,g_iH = X_i$.

$$\begin{aligned} x \in X_i &\iff x^{-1}gx = h^{-1}x_ih = h^{-1}g_i^{-1}gg_ih \text{ for some } h \in H \\ &\iff g\left(xh^{-1}g_i^{-1}\right) = \left(xh^{-1}g_i^{-1}\right)g \text{ for some } h \in H \\ &\iff xh^{-1}g_i^{-1} \in C_G(g) \text{ for some } h \in H \\ &\iff x \in C_G(g)\,g_iH. \end{aligned}$$

Therefore, $C_G(g)\,g_iH = X_i$, and as a result, we have

$$|X_i| = |C_G(g)\,g_iH| = \frac{|C_G(g)|\,|H|}{\left|g_i^{-1}C_G(g)g_i \cap H\right|}. \tag{4.68}$$

Now,

$$\begin{aligned} x \in C_G(x_i) &\iff x x_i = x_i x \iff x g_i^{-1} g g_i = g_i^{-1} g g_i x \\ &\iff \left(g_i x g_i^{-1}\right) g = g \left(g_i x g_i^{-1}\right) \\ &\iff g_i x g_i^{-1} \in C_G(g)\, g_i \\ &\iff x \in g_i^{-1} C_G(g) g_i, \end{aligned}$$

proving that $g_i^{-1} C_G(g) g_i = C_G(x_i)$. So

$$|X_i| = \frac{|C_G(g)|\,|H|}{|C_G(x_i) \cap H|}. \tag{4.69}$$

Now,

$$h \in C_G(x_i) \cap H \iff h^{-1} x_i h = x_i \text{ and } h \in H \iff h \in C_H(x_i),$$

so that $C_G(x_i) \cap H = C_H(x_i)$. Therefore,

$$\frac{|X_i|}{|H|} = \frac{|C_G(g)|}{C_H(x_i)}. \tag{4.70}$$

Plugging it into (4.67), we have

$$\operatorname{Ind}_H^G \psi(g) = \sum_{i=1}^{m} \frac{|C_G(g)|}{C_H(x_i)} \psi(x_i) = |C_G(g)| \sum_{i=1}^{m} \frac{\psi(x_i)}{C_H(x_i)}. \tag{4.71}$$

■

We now give a proof of Theorem 4.4 as promised.

Proposition 4.14 (Previously Proposition 4.4)
Induction is transitive, i.e. if $H \le K \le G$ and $\rho : H \to \mathrm{GL}(W)$ is a H-representation, then $\operatorname{Ind}_H^G \rho \cong \operatorname{Ind}_K^G \left(\operatorname{Ind}_H^K \rho\right)$.

Proof. We show that the two associated characters are equal. Then by Corollary 2.6, we are done proving the isomorphism of representations. Let χ be the character associated with the H-representation $\rho : H \to \mathrm{GL}(W)$. By Theorem 4.12, induced character (induced as a class function) is the character of the induced representation. So we need to show that

$$\operatorname{Ind}_H^G \chi = \operatorname{Ind}_K^G \left(\operatorname{Ind}_H^K \chi\right). \tag{4.72}$$

Take $g \in G$. Then

$$\left(\operatorname{Ind}_H^G \chi\right)(g) = \frac{1}{|H|} \sum_{x \in G} \mathring{\chi}\left(x^{-1} g x\right). \tag{4.73}$$

On the other hand,

$$\begin{aligned} \left[\operatorname{Ind}_K^G \left(\operatorname{Ind}_H^K \chi\right)\right](g) &= \frac{1}{|K|} \sum_{\substack{y \in G \\ y^{-1} g y \in K}} \left(\operatorname{Ind}_H^K \chi\right)\left(y^{-1} g y\right) \\ &= \frac{1}{|K|} \frac{1}{|H|} \sum_{\substack{y \in G \\ y^{-1} g y \in K}} \sum_{\substack{z \in K \\ z^{-1}\left(y^{-1} g y\right) z \in H}} \chi\left(z^{-1}\left(y^{-1} g y\right) z\right). \end{aligned} \tag{4.74}$$

We can rewrite it as follows:

$$\left[\operatorname{Ind}_K^G \left(\operatorname{Ind}_H^K \chi\right)\right](g) = \frac{1}{|H|} \frac{1}{|K|} \sum_{y \in G, z \in K} \mathring{\chi}\left((yz)^{-1} g y z\right), \tag{4.75}$$

where $\mathring{\chi} = \chi$ on H, and $\mathring{\chi} = 0$ outside of H. Now, given any $x \in G$, there are exactly $|K|$-many solutions to the equation $x = yz$ where $y \in G$ and $z \in K$. Indeed, given any $z \in K$, there is a unique $y \in G$, namely $y = xz^{-1}$. Therefore,

$$\begin{aligned} \left[\operatorname{Ind}_K^G \left(\operatorname{Ind}_H^K \chi\right)\right](g) &= \frac{1}{|H|}\frac{1}{|K|}\sum_{y\in G, z\in K} \mathring{\chi}\left((yz)^{-1} gyz\right) \\ &= \frac{1}{|H|}\frac{1}{|K|}\sum_{x\in G} |K|\, \mathring{\chi}\left(x^{-1}gx\right) \\ &= \frac{1}{|H|}\sum_{x\in G} \mathring{\chi}\left(x^{-1}gx\right) \\ &= \left(\operatorname{Ind}_H^G \chi\right)(g)\,. \end{aligned} \tag{4.76}$$

Hence, $\operatorname{Ind}_H^G \chi = \operatorname{Ind}_K^G \left(\operatorname{Ind}_H^K \chi\right)$, and we are done! ■

§4.5 Other constructions of $\operatorname{Ind}_H^G W$

There are actually several other constructions of the induced representation. But they all are equivalent by the universal property of induced representation.

Theorem 4.15

Let $H \le G$, and let W be an H-representation. Suppose $W \xrightarrow{\alpha_1} V_1$ and $W \xrightarrow{\alpha_2} V_2$ are two inductions of W to the group G. Then V_1 and V_2 are isomorphic as G-representations.

Proof. By the universal property of induced representation, if Z is another representation of G, and $\beta : W_1 \to Z$ is a H-linear map, then there exists a unique G-linear map $\overline{\beta} : V_1 \to Z$ such that the following diagram commutes:

$$\begin{tikzcd} W \arrow[r, "\alpha_1"] \arrow[dr, "\forall\,\beta"'] & V_1 \arrow[d, dashed, "\exists!\ \overline{\beta}"] \\ & Z \end{tikzcd} \tag{4.77}$$

Similarly, there exists a unique G-linear map $\overline{\beta}' : V_2 \to Z$ such that the following diagram commutes:

$$\begin{tikzcd} W \arrow[r, "\alpha_2"] \arrow[dr, "\forall\,\beta"'] & V_2 \arrow[d, dashed, "\exists!\ \overline{\beta}'"] \\ & Z \end{tikzcd} \tag{4.78}$$

Taking $Z = V_2$ and $\beta = \alpha_2$ in (4.77), there exists a unique G-linear map $f : V_1 \to V_2$ subject to the commutativity of the following diagram:

$$\begin{tikzcd} W \arrow[r, "\alpha_1"] \arrow[dr, "\alpha_2"'] & V_1 \arrow[d, dashed, "\exists!\ f"] \\ & V_2 \end{tikzcd} \tag{4.79}$$

Similarly, taking $Z = V_1$ and $\beta = \alpha_1$ in (4.78), there exists a unique G-linear map $g : V_2 \to V_1$ subject to the commutativity of the following diagram:

$$
\begin{tikzcd}
W \arrow[r, "\alpha_2"] \arrow[dr, "\alpha_1"'] & V_2 \arrow[d, dashed, "\exists!\ g"] \\
 & V_1
\end{tikzcd}
\tag{4.80}
$$

Combining (4.79) and (4.80), we have the following commutative diagram:

$$
\begin{tikzcd}
 & V_2 \arrow[d, "g"] & & & \\
W \arrow[ur, "\alpha_2"] \arrow[r, "\alpha_1"] \arrow[dr, "\alpha_2"'] & V_1 \arrow[d, "f"] & \Longrightarrow & W \arrow[r, "\alpha_2"] \arrow[dr, "\alpha_2"'] & V_2 \arrow[d, "f\circ g"] \\
 & V_2 & & & V_2
\end{tikzcd}
\tag{4.81}
$$

But taking $Z = V_2$ and $\beta = \alpha_2$ in (4.78), there exists a unique G-linear map $\overline{\alpha_2} : V_2 \to V_2$, namely $\mathbb{1}_{V_2} : V_2 \to V_2$ subject to the commutativity of the following diagram:

$$
\begin{tikzcd}
W \arrow[r, "\alpha_2"] \arrow[dr, "\alpha_1"'] & V_2 \arrow[d, dashed, "\exists!\ \overline{\alpha_2}=\mathbb{1}_{V_2}"] \\
 & V_2
\end{tikzcd}
\tag{4.82}
$$

But by (4.81), taking $\overline{\alpha_2} = f \circ g$ also makes the diagram commute. Therefore, by the uniqueness, $f \circ g = \mathbb{1}_{V_2}$. Similarly, $g \circ f = \mathbb{1}_{V_1}$. f, g are G-linear maps. Therefore, $f : V_1 \to V_2$ is an isomorphism of G-representations. ■

§4.5.i Using right-cosets

We previously constructed $\operatorname{Ind}_H^G W$ as the direct sum of W_{g_iH}'s, where g_iH's make up a complete set of left-cosets of H. One can similarly construct $\operatorname{Ind}_H^G W$ using right-cosets as well.

Let $n = [G : H]$, so there are n right cosets of H in G. Take $g_1 = e, g_2, g_3, \ldots, g_n$ such that $Hg_1, Hg_2, \ldots, Hg_n$ are all the right-cosets of H in G, so that

$$G = Hg_1 \sqcup Hg_2 \sqcup \cdots \sqcup Hg_n. \tag{4.83}$$

(Here $\sqcup$ signifies disjoint union) As before, we take

$$V = W \oplus W \oplus \cdots \oplus W = W^n. \tag{4.84}$$

We label them with the cosets, so

$$V = W_{Hg_1} \oplus W_{Hg_2} \oplus \cdots \oplus W_{Hg_n}. \tag{4.85}$$

Suppose $\rho_H : H \to \mathrm{GL}(W)$ be the H-representation in question. We define the induced representation $\rho : G \to \mathrm{GL}(V)$ as follows: For $g \in G$, we are going to define $\rho(g)(\mathbf{v})$ for $\mathbf{v} \in W_{Hg_j}$. Suppose $gg_j^{-1} = g_i^{-1}h$ for some $h \in H$. Then

$$\rho(g)\,\mathbf{v} := \text{ the copy of } \rho_H(h)\,\mathbf{v} \text{ in } W_{Hg_i}. \tag{4.86}$$

More formally speaking, suppose $f_i : W = W_{He} \to W_{Hg_i}$ be the identification. Then for $\mathbf{v} \in W_{Hg_j}$ and $gg_j^{-1} = g_i^{-1}h$,

$$\rho(g)\,\mathbf{v} = \left[f_i \circ \rho_H(h) \circ f_j^{-1}\right]\mathbf{v}. \tag{4.87}$$

Then it's a routine check that this is, indeed, a representation. One can similarly check as in Lemma 4.1 that it satisfies the universal property of induced representation[1].

§4.5.ii Using function space

Let $\rho_H : H \to \mathrm{GL}(W)$ be a representation of H, and $H \le G$. Consider the function space

$$V = \{f : G \to W \mid f(hx) = \rho_H(h)\, f(x),\ \forall x \in G, h \in H\}. \tag{4.88}$$

We can define a representation $\rho : G \to \mathrm{GL}(V)$ of G on V as follows:

$$[\rho(g)\, f](x) = f(xg). \tag{4.89}$$

We need to check that $\rho(g)\, f$ is an element of V.

$$[\rho(g)\, f](hx) = f(hxg) = \rho_H(h)\, f(xg) = \rho_H(h)\,[\rho(g)\, f](x). \tag{4.90}$$

Therefore, $\rho(g)\, f \in V$. Given $g, g' \in G$,

$$[\rho(gg')\, f](x) = f(xgg').$$
$$[\rho(g)\,\rho(g')\, f](x) = [\rho(g')\, f](xg) = f(xgg').$$

Therefore, $\rho(gg')\, f = \rho(g)\,\rho(g')\, f$, and hence, $\rho : G \to \mathrm{GL}(V)$ is, indeed, a representation of G. We want to show that this is also an induced representation, induced by ρ_H.

The crucial observation is that any function $f \in V$ is completely determined by its values in the right-coset representatives. To be precise, take $g_1 = e, g_2, g_3, \ldots, g_n$ such that $Hg_1, Hg_2, \ldots, Hg_n$ are all the right-cosets of H in G, so that

$$G = Hg_1 \sqcup Hg_2 \sqcup \cdots \sqcup Hg_n.$$

Then any $g \in G$ can uniquely be written as $g = hg_i$ for some $h \in H$ and $i \in \{1, 2, \ldots, n\}$. Then

$$f(g) = f(hg_i) = \rho_H(h)\, f(g_i). \tag{4.91}$$

So, if we know the value of f on each g_i, then we know the value of f on the whole of G. Now we can define an isomorphism of vector spaces:

$$\psi : W^n = \bigoplus_{i=1}^{n} W_{Hg_i} \to V$$
$$(\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n) \mapsto f, \quad f(g_i) = \mathbf{w}_i\ \forall\, i.$$

This is an isomorphism of vector spaces, because as mentioned earlier, f is completely determined by its values on each g_i. We claim that this is also a G-linear map.

Take $\mathbf{w}_j \in W_{Hg_j}$. Then $\psi(\mathbf{w}) \in V$ is a function, such that $\psi(\mathbf{w}_j)(g_j) = \mathbf{w}_j$ and $\psi(\mathbf{w}_j)(g_k) = \mathbf{0}$ for $k \ne j$. We can write it as follows:

$$\psi(\mathbf{w}_j)(g_k) = \begin{cases} \mathbf{w}_j & \text{if } k = j, \\ \mathbf{0} & \text{if } k \ne j. \end{cases} \tag{4.92}$$

Let $\rho_1 : G \to \mathrm{GL}\left(\bigoplus_{i=1}^n W_{Hg_i}\right)$ and $\rho_2 : G \to \mathrm{GL}(V)$ be the relevant group homomorphisms of the representations. For $g \in G$,

$$[\rho_2(g)\,\psi(\mathbf{w}_j)](g_k) = \psi(\mathbf{w}_j)(g_k g). \tag{4.93}$$

[1]Check `https://atonurc.github.io/assets/catrep_talk_2.pdf` (slides 24 to 29) for a detailed proof that this construction satisfies the universal property of induced representation.

$g_k g$ belongs to a right H-coset, say Hg_l. Then $g_k g = h' g_l$. As a result,

$$[\rho_2(g)\psi(\mathbf{w}_j)](g_k) = \psi(\mathbf{w}_j)(h' g_l) = \rho_H(h')\psi(\mathbf{w}_j)(g_l). \tag{4.94}$$

For $l \neq j$, this is 0. Suppose, for $k = i$, we have $l = j$. Then $g_i g = h g_j$, i.e. $g g_j^{-1} = h g_i^{-1}$. Then we have

$$[\rho_2(g)\psi(\mathbf{w}_j)](g_i) = \rho_H(h)\psi(\mathbf{w}_j)(g_j) = \rho_H(h)\mathbf{w}_j. \tag{4.95}$$

To summarize,

$$[\rho_2(g)\psi(\mathbf{w}_j)](g_k) = \begin{cases} \rho_H(h)\mathbf{w}_j & \text{if } k = i, \text{ where } i \text{ satisfies } g g_j^{-1} = h g_i^{-1}, \\ 0 & \text{otherwise.} \end{cases} \tag{4.96}$$

Now, $\rho_1(g)\mathbf{w}_j$ gives us the copy of $\rho_H(h)\mathbf{w}_j$ in the i-th component W_{Hg_i}, where i satisfies $g g_j^{-1} = h g_i^{-1}$. Therefore, applying ψ on it gives

$$[\psi(\rho_1(g)\mathbf{w}_j)](g_k) = \begin{cases} \rho_H(h)\mathbf{w}_j & \text{if } k = i, \text{ where } i \text{ satisfies } g g_j^{-1} = h g_i^{-1}, \\ 0 & \text{otherwise.} \end{cases} \tag{4.97}$$

Therefore,

$$\psi \circ \rho_1(g) = \rho_2(g) \circ \psi, \tag{4.98}$$

i.e. the following diagram commutes for every $g \in G$:

$$\begin{tikzcd} \bigoplus_{i=1}^{n} W_{Hg_i} \arrow[r, "\psi"] \arrow[d, "\rho_1(g)"'] & V \arrow[d, "\rho_2(g)"] \\ \bigoplus_{i=1}^{n} W_{Hg_i} \arrow[r, "\psi"'] & V \end{tikzcd}$$

Therefore, ψ is an isomorphism of G-representations. We have already seen that $\bigoplus_{i=1}^{n} W_{Hg_i} \cong \operatorname{Ind}_H^G W$. As a result, $V \cong \operatorname{Ind}_H^G W$.

§4.5.iii By extension of scalars

We have seen that a G-representation V can be thought of as a G-module. The concept of module we know is that of a module over a ring. But G is not a ring. So "a module over G" doesn't make any sense. However, we can think of a G-representation V as a module over a ring as well. That ring is precisely the group algebra $\mathbb{C}[G]$.

Definition 4.3. The group algebra $\mathbb{C}[G]$ is

$$\mathbb{C}[G] = \operatorname{span}\{\delta_g \mid g \in G\}.$$

The multiplication rule is given by

$$\delta_g \cdot \delta_{g'} = \delta_{gg'}, \tag{4.99}$$

and extend it linearly. The multiplicative identity is δ_e, where $e \in G$ is the identity element.

If $\rho : G \to \mathrm{GL}(V)$ is a representation, then V is a $\mathbb{C}[G]$-module (module over the ring of $\mathbb{C}[G]$).

$$\left(\sum_{g \in g} a_g \delta_g\right) \cdot \mathbf{v} := \sum_{g \in g} a_g \rho(g)\mathbf{v}. \tag{4.100}$$

Then one can check that this satisfies all the axioms of a module over the ring $\mathbb{C}[G]$. Usually in a module, the scalars come from the ground ring. Here, the ring is $\mathbb{C}[G]$. In (4.100), $\sum_{g\in g} a_g\delta_g \in \mathbb{C}[G]$ is the scalar from the ring.

Similarly, a vector space V that has a $\mathbb{C}[G]$-module structure is a representation of G. The representation $\rho : G \to \mathrm{GL}(V)$ is defined as follows:

$$\rho(g)\,\mathbf{v} := \delta_g \cdot \mathbf{v}. \tag{4.101}$$

The multiplication $\delta_g \cdot \mathbf{v}$ on the RHS of (4.101) comes from the $\mathbb{C}[G]$-module structure of V. Here $\mathbf{v} \in V$, and $\delta_g \in \mathbb{C}[G]$. The multiplication $\delta_g \cdot \mathbf{v}$ is the scalar multiplication of the module element $\mathbf{v}$ by the scalar δ_g from the ground ring $\mathbb{C}[G]$.

So we have established that a representation of a group G is the same as a vector space V which has the structure of a module over the ring $\mathbb{C}[G]$. Now, let W be a representation of H, where $H \leq G$, with group homomorphism $\rho_H : H \to \mathrm{GL}(W)$. Then W is a $\mathbb{C}[H]$-module:

$$\delta_h \cdot \mathbf{w} := \rho_H(h)\,\mathbf{w}. \tag{4.102}$$

We want to make a $\mathbb{C}[G]$-module out of it. For that purpose, we need to extend the scalars of the module from $\mathbb{C}[H]$ to $\mathbb{C}[G]$. One way to do it is to take the tensor product

$$\mathbb{C}[G] \otimes_{\mathbb{C}[H]} W.$$

Here, both $\mathbb{C}[G]$ and W are representations of H, i.e. both are vector spaces that are $\mathbb{C}[H]$-modules. Then the tensor product $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} W$ is also a $\mathbb{C}[H]$-module. This has the structure of a $\mathbb{C}[G]$-module as well:

$$\delta_x \cdot \left(\delta_g \otimes_{\mathbb{C}[H]} \mathbf{w}\right) := \delta_{xg} \otimes_{\mathbb{C}[H]} \mathbf{w}. \tag{4.103}$$

Therefore, $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} W$ defines a representation of the group G. Indeed, this is the same as the induced representation $\mathrm{Ind}_H^G W$. To show this, first we fix representatives

$$g_1 = e,\ g_2,\ g_3,\ \ldots,\ g_n \in G$$

of left-cosets in G/H, i.e. $G/H = \{g_1H, g_2H, \ldots, g_nH\}$. Then we define the linear map

$$\psi : \bigoplus_{i=1}^{n} W_{g_iH} \to \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W$$

as follows: given $\mathbf{w} \in W_{g_iH}$ (remember that W and W_{g_iH} are the same vector space, just labelled differently), we define

$$\psi(\mathbf{w}) = \delta_{g_i} \otimes_{\mathbb{C}[H]} \mathbf{w}, \qquad \text{for } \mathbf{w} \in W_{g_iH}. \tag{4.104}$$

To be precise, let $f_i : W \to W_{g_iH}$ be the vector space isomorphisms. W_{g_iH} is the same vector space as W, just labelled differently. So f_i is just the identity map. Then for $\mathbf{w} \in W_{g_iH}$,

$$\psi(\mathbf{w}) = \delta_{g_i} \otimes_{\mathbb{C}[H]} f_i^{-1}\mathbf{w}. \tag{4.105}$$

This is a well-defined linear map. This is also an isomorphism of vector spaces, because the inverse map is given by

$$\phi : \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W \to \bigoplus_{i=1}^{n} W_{g_iH}.$$

It takes $\delta_g \otimes_{\mathbb{C}[H]} \mathbf{w}$, for some $g \in G$ and $\mathbf{w} \in W$, and maps it to $\bigoplus_{i=1}^{n} W_{g_iH}$. The way it is defined is as follows: first identify which coset g belongs to, i.e. write $g = g_ih$ for some $h \in H$. Then

$$\begin{aligned}\delta_g \otimes_{\mathbb{C}[H]} \mathbf{w} &= \delta_{g_ih} \otimes_{\mathbb{C}[H]} \mathbf{w} = \delta_{g_i}\delta_h \otimes_{\mathbb{C}[H]} \mathbf{w}\\ &= \delta_{g_i} \otimes_{\mathbb{C}[H]} \delta_h\mathbf{w} = \delta_{g_i} \otimes_{\mathbb{C}[H]} \delta_h\mathbf{w}\\ &= \delta_{g_i} \otimes_{\mathbb{C}[H]} \rho_H(h)\,\mathbf{w}.\end{aligned} \tag{4.106}$$

So ϕ maps it to

$$\phi\left(\delta_g \otimes_{\mathbb{C}[H]} \mathbf{w}\right) = \text{ the copy of } \rho_H(h)\,\mathbf{w} \text{ in } W_{g_iH}. \tag{4.107}$$

To be precise, for $g = g_i h$ and $\mathbf{w} \in W$,

$$\phi\left(\delta_g \otimes_{\mathbb{C}[H]} \mathbf{w}\right) = f_i\left[\rho_H(h)\,\mathbf{w}\right]. \tag{4.108}$$

Now, given $\mathbf{w} \in W_{g_iH}$,

$$\begin{aligned}\phi(\psi(w)) &= \phi\left(\delta_{g_i} \otimes_{\mathbb{C}[H]} f_i^{-1}\mathbf{w}\right) = f_i f_i^{-1}\mathbf{w}\\ &= \mathbf{w} \in W_{g_iH}.\end{aligned} \tag{4.109}$$

Therefore, $\phi \circ \psi = \mathbb{1}_{\bigoplus_{i=1}^n W_{g_iH}}$. On the other hand, given $g \in G$ and $\mathbf{w} \in W$,

$$\psi(\phi(\delta_g \otimes \mathbf{w})) = \psi\left[f_i \rho_H(h)\,\mathbf{w}\right] \tag{4.110}$$

where $g = g_i h$, for $h \in H$. Then

$$\begin{aligned}\psi(\phi(\delta_g \otimes \mathbf{w})) &= \psi\left[f_i \rho_H(h)\,\mathbf{w}\right]\\ &= \delta_{g_i} \otimes_{\mathbb{C}[H]} f_i^{-1}\left[f_i \rho_H(h)\,\mathbf{w}\right]\\ &= \delta_{g_i} \otimes_{\mathbb{C}[H]} \rho_H(h)\,\mathbf{w}\\ &= \delta_g \otimes \mathbf{w},\end{aligned} \tag{4.111}$$

by (4.106). Therefore, $\psi \circ \phi = \mathbb{1}_{\mathbb{C}\otimes_{\mathbb{C}[H]}W}$. So $\psi : \bigoplus_{i=1}^n W_{g_iH} \to \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W$ is an isomorphism of vector space. Now we show that it is a G-linear map.

Let $\rho_1 : G \to \mathrm{GL}\left(\bigoplus_{i=1}^n W_{g_iH}\right)$ and $\rho_2 : G \to \mathrm{GL}\left(\mathbb{C}[G] \otimes_{\mathbb{C}[H]} W\right)$ be the relevant group homomorphisms of the representations. To show that ψ is a homomorphism of representations, we need to show that the following diagram commutes for every $g \in G$.

$$\begin{tikzcd} \bigoplus_{i=1}^n W_{g_iH} \arrow[r, "\psi"] \arrow[d, "\rho_1(g)"'] & \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W \arrow[d, "\rho_2(g)"] \\ \bigoplus_{i=1}^n W_{g_iH} \arrow[r, "\psi"'] & \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W \end{tikzcd}$$

Take $\mathbf{w} \in W_{g_iH}$. Suppose $gg_i = g_j h$ for some $h \in H$.

$$\begin{aligned}\rho_2(g)\,\psi(\mathbf{w}) &= \rho_2(g)\left[\delta_{g_i} \otimes_{\mathbb{C}[H]} f_i^{-1}\mathbf{w}\right]\\ &= \delta_{gg_i} \otimes_{\mathbb{C}[H]} f_i^{-1}\mathbf{w}\\ &= \delta_{g_jh} \otimes_{\mathbb{C}[H]} f_i^{-1}\mathbf{w}\\ &= \delta_{g_j}\delta_h \otimes_{\mathbb{C}[H]} f_i^{-1}\mathbf{w}\\ &= \delta_{g_j} \otimes_{\mathbb{C}[H]} \delta_h\left(f_i^{-1}\mathbf{w}\right)\\ &= \delta_{g_j} \otimes_{\mathbb{C}[H]} \rho_H(h)\left(f_i^{-1}\mathbf{w}\right).\end{aligned} \tag{4.112}$$

On the other hand,

$$\begin{aligned}\psi(\rho_2(g)\,\mathbf{w}) &= \psi\left(f_j \rho_H(h)\, f_i^{-1}\mathbf{w}\right)\\ &= \delta_{g_j} \otimes_{\mathbb{C}[H]} f_j^{-1}\left(f_j \rho_H(h)\, f_i^{-1}\mathbf{w}\right)\\ &= \delta_{g_j} \otimes_{\mathbb{C}[H]} \rho_H(h)\left(f_i^{-1}\mathbf{w}\right).\end{aligned} \tag{4.113}$$

Therefore, ψ is a G-linear map, and hence it is an isomorphism of representations.

5 Young Tableaux

In this chapter, we'll see a very elegant description of irreducible representations of $\mathfrak{S}_n$ through Young tableaux.

§5.1 Young diagram

Definition 5.1. A **partition** of a positive integer n is a sequence of positive integer $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ satisfying $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l > 0$ with $n = \lambda_1 + \lambda_2 + \cdots + \lambda_l$. We write $\lambda \vdash n$ to denote that λ is a partition of n.

For instance the number 4 has five partitions: $(4), (3,1), (2,2), (2,1,1), (1,1,1,1)$. These partitions are pictorially represented by Young diagrams as follows.

Definition 5.2 (Young diagram)**.** A **Young diagram** is a finite collection of boxes arranged in left-justified rows with row size weakly decreasing (equal or less). The Young diagram associated with the partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ is the one that has l rows and the i-th row contains λ_i boxes.

For example, the Young diagrams corresponding to the partitions of 4 are:

$$\underset{}{\overset{(4)}{\begin{array}{|c|c|c|c|}\hline \; & \; & \; & \; \\ \hline \end{array}}} \qquad \overset{(3,1)}{\begin{array}{|c|c|c|}\hline \; & \; & \; \\ \hline \multicolumn{1}{|c|}{\;} \\ \cline{1-1} \end{array}} \qquad \overset{(2,2)}{\begin{array}{|c|c|}\hline \; & \; \\ \hline \; & \; \\ \hline \end{array}} \qquad \overset{(2,1,1)}{\begin{array}{|c|c|}\hline \; & \; \\ \hline \multicolumn{1}{|c|}{\;} \\ \cline{1-1} \multicolumn{1}{|c|}{\;} \\ \cline{1-1} \end{array}} \qquad \overset{(1,1,1,1)}{\begin{array}{|c|}\hline \; \\ \hline \; \\ \hline \; \\ \hline \; \\ \hline \end{array}}$$

Because of the 1-1 corresponsence between partitions and Young diagram, same symbol is used for both of them. Foor example, by $(3,1)$, one represents the second Young diagram above. A Young tableau is obtained by filling the boxes of a Young diagram with numbers.

Definition 5.3 (Young tableau)**.** Suppose $\lambda \vdash n$. A **Young tableau** t of shape λ is obtained by filling in the boxes of λ with each of $1, 2, \ldots, n$ exactly once. In this case, we say that t is a λ-tableau.

We can easily see that there are $n!$ such λ-tableaux. For example, corresponding to $\lambda = (2,1)$, there are $3! = 6$ such tableaux:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \multicolumn{1}{|c|}{3} \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline \multicolumn{1}{|c|}{3} \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline \multicolumn{1}{|c|}{2} \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|}\hline 3 & 1 \\ \hline \multicolumn{1}{|c|}{2} \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline \multicolumn{1}{|c|}{1} \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|}\hline 3 & 2 \\ \hline \multicolumn{1}{|c|}{1} \\ \cline{1-1} \end{array}$$

Definition 5.4 (Standard Young tableau)**.** A **standard Young tableau** is a Young tableau whose entries are increasing across each row and each column.

The only standard tableau for $\lambda = (2,1)$ are

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \multicolumn{1}{|c|}{3} \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline \multicolumn{1}{|c|}{2} \\ \cline{1-1} \end{array}$$

Here is another example of a standard tableau associated with $\lambda = (3,3,2,1) \vdash 9$:

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 & 6 \\ \hline 7 & 8 & \multicolumn{1}{c}{} \\ \cline{1-2} 9 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

We know from our previous discussions on character theory that the conjugacy classes of $\mathfrak{S}_n$ are characterized by the cycle types, and thus they correspond to partitions of n, which we have seen to be equivalent to Young diagrams of size n. We also have learned from the construction of character table that the number of irreducible representations of a finite group is equal to the number of its conjugacy classes. So it makes perfect sense to talk about constructing an irreducible representation of $\mathfrak{S}_n$ corresponding to each Young diagram of size n.

§5.2 Tabloid and permutation module M^λ

> **Definition 5.5** (Young tabloid)**.** Two λ-tableaux t_1 and t_2 are **row-equivalent**, denoted $t_1 \sim t_2$, if the corresponding rows of the two tableaux contain the same elements. A **tabloid** of shape λ, or λ-tabloid is such an equibalence class, denoted by $[t] = \{t_1 \mid t_1 \sim t\}$, where t is a λ-tabloid.

We draw the tabloid $[t]$ by removing the vertical bars separating the entries within each row. For instance, if

$$t = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array},$$

then $[t]$ is the tabloid drawn as

$$\begin{array}{cc}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array},$$

which represents the equivalence class containing the following tableaux:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \text{and} \quad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

Also, for example, notice that

$$\begin{array}{ccc}\hline 1 & 4 & 7 \\ \hline 3 & 6 & \\ \cline{1-2} 2 & 5 & \\ \cline{1-2}\end{array} = \begin{array}{ccc}\hline 4 & 7 & 1 \\ \hline 6 & 3 & \\ \cline{1-2} 2 & 5 & \\ \cline{1-2}\end{array} \neq \begin{array}{ccc}\hline 4 & 7 & 1 \\ \hline 6 & 2 & \\ \cline{1-2} 3 & 5 & \\ \cline{1-2}\end{array} \neq \begin{array}{ccc}\hline 4 & 7 & 1 \\ \hline 2 & 3 & \\ \cline{1-2} 6 & 5 & \\ \cline{1-2}\end{array}.$$

We now want to define a representation of $\mathfrak{S}_n$ on a vector space whose basis is the set of tabloids for a given shape. For this, we need to find a way for the elements of $\mathfrak{S}_n$ to act on the relevant tabloids. There is an obvious choice of letting the permutations permute the entries of the tabloid. For instance,

$$(1\ 2\ 3)\ \begin{array}{cc}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array} = \begin{array}{cc}\hline 2 & 3 \\ \hline 1 & \\ \cline{1-1}\end{array}. \tag{5.1}$$

This is a well-defined action. Indeed, if $t_1 \sim t_2$, then the rows of t_1 and t_2 contain the same elements. After performing the permutation $\pi \in \mathfrak{S}_n$, the rows will also have the same elements. In other words,

$$[t_1] = [t_2] \implies \pi\,[t_1] = \pi\,[t_2]\,,$$

i.e. the action of $\mathfrak{S}_n$ on the tabloids is well-defined.

> **Definition 5.6** (Permutation module). Suppose $\lambda \vdash n$. Let M^λ denote the vector space whose basis is the set of λ-tabloids. Then M^λ is a representation of $\mathfrak{S}_n$, known as the **permutation module** corresponding to λ.

Example 5.1. Consider $\lambda = (n)$. We see then $M^\lambda = M^{(n)}$ is the 1-dimesnional vector space spanned by the single tabloid

$$\begin{array}{c}\hline 1 \quad 2 \quad 3 \quad \cdots \quad n \\ \hline\end{array}$$

There is this single tabloid in the basis set of $M^{(n)}$ as there is only one row involved. Since this tabloid is fixed under the action of $\mathfrak{S}_n$ by permuting its entries, we see that $M^{(n)}$ is the 1-dimensional trivial representation.

Example 5.2. Consider $\lambda = (1, 1, \ldots, 1)$. Then a λ-tabloid is simply a permutation of $\{1, 2, 3, \ldots, n\}$. Here is an example of such a λ-tabloid:

$$\begin{array}{c}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \vdots \\ \hline n \\ \hline\end{array}$$

There are $n!$ such distinct λ-tabloids. Each λ-tabloid corrresponds to a unique $\sigma \in \mathfrak{S}_n$. $\pi \in \mathfrak{S}_n$ acting on the λ-tabloid corrresponding to $\sigma \in \mathfrak{S}_n$ gives us the λ-tabloid corresponding to $\pi\sigma \in \mathfrak{S}_n$. Therefore, it follows that $M^{(1,1,\ldots,1)}$ is isomorphic to the regular representation $\mathbb{C}[\mathfrak{S}_n]$.

Example 5.3. Consider $\lambda = (n-1, 1)$. Let $[t_i]$ be the λ-tabloid with i on the second row. Then M^λ has basis as $[t_1], [t_2], \ldots, [t_n]$. For example, for $n = 4$, $M^{(3,1)}$ has the following basis:

$$[t_1] = \begin{array}{l}\hline 2 \quad 3 \quad 4 \\ \hline 1 \\ \hline\end{array}, \quad [t_2] = \begin{array}{l}\hline 1 \quad 3 \quad 4 \\ \hline 2 \\ \hline\end{array}, \quad [t_3] = \begin{array}{l}\hline 1 \quad 2 \quad 4 \\ \hline 3 \\ \hline\end{array}, \quad [t_3] = \begin{array}{l}\hline 1 \quad 2 \quad 3 \\ \hline 4 \\ \hline\end{array}.$$

The action of $\pi \in \mathfrak{S}_n$ sends $[t_i]$ to $\left[t_{\pi(i)}\right]$. Hence, $M^{(n-1,1)}$ is isomorphic to the defining permutation representation of $\mathfrak{S}_n$ on $\mathbb{C}^n$.

Now we study the dimension and characters of M^λ.

> **Proposition 5.1**
> If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \vdash n$, then
> $$\dim M^\lambda = \frac{n!}{\lambda_1!\lambda_2!\cdots\lambda_l!}. \tag{5.2}$$

Proof. The first row of a λ-tabloid has λ_1-many entries, and there are $\binom{n}{\lambda_1}$-many ways to choose the λ_1-many entries of the first row. Then there are $\binom{n-\lambda_1}{\lambda_2}$-many ways to choose the λ_2-many entries of the second row. Continuing this way, there are $\binom{n-\lambda_1-\cdots-\lambda_{l-1}}{\lambda_l}$-many ways to choose the λ_l-many entries of the l-th row. Choosing the entries is enough, we don't need to arrange or permute the entries in a row. Therefore, the total number of λ-tabloids is

$$\binom{n}{\lambda_1}\binom{n-\lambda_1}{\lambda_2}\cdots\binom{n-\lambda_1-\cdots-\lambda_{l-1}}{\lambda_l} = \frac{n!}{\lambda_1!\lambda_2!\cdots\lambda_l!}. \tag{5.3}$$

The dimension of M^λ is the number of λ-tabloids. Therefore, $\dim M^\lambda = \frac{n!}{\lambda_1!\lambda_2!\cdots\lambda_l!}$. ■

Proposition 5.2

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \vdash n$, and $g \in \mathfrak{S}_n$. Let $(m_1, m_2, \ldots, m_r)$ be the cycle type of g (i.e. g is a product of an m_1-cycle, an m_2-cycle, and so forth in its the disjoint cycle product form). Then the character of the representation of $\mathfrak{S}_n$ on M^λ evaluated at g is equal to the coefficient of $x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_l^{\lambda_l}$ in the product

$$\prod_{i=1}^{r} \left(x_1^{m_i} + x_2^{m_i} + \cdots + x_l^{m_i}\right).$$

Proof. Since M^λ can be realized as a permutation representation on the set of λ-tabloids, by Lemma 2.2, the character value evaluated at $g \in \mathfrak{S}_n$ is the number of λ-tabloids fixed by g. Now, only those λ-tabloids will be g-fixed for which each cycle of g permutes the elements from a single row.

This is achieved by putting the m_i-cycles in a single row (for all the rows) so that under the action of that cycle, the tabloid remains invariant. There are r such steps where r comes from the fact that the cycle type of g is given by the r-tuple $(m_1, m_2, \ldots, m_r)$.

As we expand the polynomial given above (also known as generating function), the variable we select in each factor, say x_i, corresponds to the choice of which row in the tabloid we would like to put the cycle in. Specifically, choosing $x_j^{m_i}$ corresponds to placing a cycle of length m_i into the j-th row of the tabloid. Then for any term after the expansion is carried out, the exponent of x_j corresponds to the total number of elements placed in the j-th row which we expect to be λ_j.

So, the coefficient of the term $x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_l^{\lambda_l}$ in the given polynomial expression precisely refers to the number of tabloids of shape λ that emerges after this r-step process, i.e. the number of λ-tabloids fixed by a permutation of cycle type $(m_1, m_2, \ldots, m_r)$. ∎

Example 5.4. Let us calculate the full list of characters of the permutation modules for $\mathfrak{S}_4$. We know that the character at the identity element is equal to the dimension of the representation space.

$$\begin{aligned}
\dim M^{(4)} &= \frac{4!}{4!} = 1 \\
\dim M^{(3,1)} &= \frac{4!}{3!1!} = 4 \\
\dim M^{(2,2)} &= \frac{4!}{2!2!} = 6 \\
\dim M^{(2,1,1)} &= \frac{4!}{2!1!1!} = 12 \\
\dim M^{(1,1,1)} &= \frac{4!}{1!1!1!1!} = 24.
\end{aligned}$$

Suppose, we want to calculate the character of $\mathfrak{S}_4$ corresponding to $M^{(2,2)}$ at the permutation $(1\ 2) \in \mathfrak{S}_4$. Then $\lambda = (2, 2)$ and the cycle type is $(2, 1, 1)$. Hence, the generating polynomial is

$$\left(x_1^2 + x_2^2\right)(x_1 + x_2)(x_1 + x_2) = x_1^4 + 2x_1^3 x_2 + 2x_1^2 x_2^2 + 2x_1 x_2^3 + x_2^4. \tag{5.4}$$

The coefficient of $x_1^2 x_2^2$ in the above polynomial is 2. Hence, the character of $\mathfrak{S}_4$ associated with $M^{(2,2)}$ evaluated at $(1\ 2)$ is 2. From the same polynomial, we see the coefficient of $x_1^3 x_2$ is 2. So the character of $\mathfrak{S}_4$ associated with $M^{(3,1)}$ evaluated at $(1\ 2)$ is also 2. Similarly, we can compute other characters as well, which we can express in the following table:

Permutation Cycle type	1 $(1,1,1,1)$	$(1\ 2\ 3)$ $(3,1)$	$(1\ 2)$ $(2,1,1)$	$(1\ 2)(3\ 4)$ $(2,2)$	$(1\ 2\ 3\ 4)$ (4)
$M^{(4)}$	1	1	1	1	1
$M^{(3,1)}$	4	1	2	0	0
$M^{(2,2)}$	6	0	2	2	0
$M^{(2,1,1)}$	12	0	2	0	0
$M^{(1,1,1,1)}$	24	0	0	0	0

Note that this is **NOT** the character table of $\mathfrak{S}_4$, because the permutation modules M^λ are not, in general, irreducible representations.

Now we move forward to construct irreducible representations of $\mathfrak{S}_n$.

§5.3 Specht modules S^λ

In the previous section, we constructed representations M^λ of $\mathfrak{S}_n$, known as permutation modules. In this section, we construct an irreducible subrepresentation of M^λ that corresponds uniquely to λ.

Definition 5.7. For a tableau t of size n, the **row group** of t, denoted by R_t is the subgroup of $\mathfrak{S}_n$ consisting of permutations which only permutes the elements within each row of t. Similarly, the **column group** C_t is the subgroup of $\mathfrak{S}_n$ consisting of permutations which only permutes the elements within each column of t.

For example, if

$$t = \begin{array}{|c|c|c|}\hline 4 & 1 & 2 \\ \hline 3 & 5 \\ \cline{1-2}\end{array},$$

then

$$\begin{aligned} R_t &= \mathfrak{S}_{\{1,2,4\}} \times \mathfrak{S}_{\{3,5\}} \cong \mathfrak{S}_3 \times \mathfrak{S}_2, \\ C_t &= \mathfrak{S}_{\{3,4\}} \times \mathfrak{S}_{\{1,5\}} \times \mathfrak{S}_{\{2\}} \cong \mathfrak{S}_2 \times \mathfrak{S}_2. \end{aligned}$$

Let us select certain elements from the vector space M^λ to span a subspace.

Definition 5.8 (Polytabloid)**.** If t is a tableau, then the associated **polytabloid** is

$$\mathbf{e}_t = \sum_{\pi \in C_t} (\operatorname{sgn} \pi)\, \pi\, [t]\,. \tag{5.5}$$

For example, if

$$t = \begin{array}{|c|c|c|}\hline 4 & 1 & 2 \\ \hline 3 & 5 \\ \cline{1-2}\end{array},$$

then the associated polytabloid $\mathbf{e}_t$ is

$$\begin{aligned} \mathbf{e}_t &= e \begin{array}{ccc}\hline 4 & 1 & 2 \\ \hline 3 & 5 \\ \cline{1-2}\end{array} - (3\ 4) \begin{array}{ccc}\hline 4 & 1 & 2 \\ \hline 3 & 5 \\ \cline{1-2}\end{array} - (1\ 5) \begin{array}{ccc}\hline 4 & 1 & 2 \\ \hline 3 & 5 \\ \cline{1-2}\end{array} + (3\ 4)\,(1\ 5) \begin{array}{ccc}\hline 4 & 1 & 2 \\ \hline 3 & 5 \\ \cline{1-2}\end{array} \\ &= \begin{array}{ccc}\hline 4 & 1 & 2 \\ \hline 3 & 5 \\ \cline{1-2}\end{array} - \begin{array}{ccc}\hline 3 & 1 & 2 \\ \hline 4 & 5 \\ \cline{1-2}\end{array} - \begin{array}{ccc}\hline 4 & 5 & 2 \\ \hline 3 & 1 \\ \cline{1-2}\end{array} + \begin{array}{ccc}\hline 3 & 5 & 2 \\ \hline 4 & 1 \\ \cline{1-2}\end{array} \end{aligned}$$

Now, using the following lemma, we'll see that $\mathfrak{S}_n$ permutes the set of polytabloids.

Lemma 5.3

Let t be a tableau and $\pi \in \mathfrak{S}_n$. Then

$$\mathbf{e}_{\pi t} = \pi \mathbf{e}_t. \tag{5.6}$$

Proof. Let us denote the columns of the tableau t by $C_1, C_2, \ldots, C_k$ so that the column group can be written as

$$C_t = \mathfrak{S}_{C_1} \times \mathfrak{S}_{C_2} \times \cdots \times \mathfrak{S}_{C_k}.$$

Then

$$C_{\pi(t)} = \mathfrak{S}_{\pi(C_1)} \times \mathfrak{S}_{\pi(C_2)} \times \cdots \times \mathfrak{S}_{\pi(C_k)},$$

where $\pi\left(C_i\right)$ consists of the images of the column C_i under $\pi \in \mathfrak{S}_n$. Given $\sigma \in \mathfrak{S}_{C_i}$, if

$$\sigma = (a_1\ a_2\ \cdots\ a_k)\,(b_1\ b_2\ \cdots\ b_m)\cdots,$$

and if π sends x to x', then

$$\pi\sigma\pi^{-1} = (a_1'\ a_2'\ \cdots\ a_k')\,(b_1'\ b_2'\ \cdots\ b_m')\cdots. \tag{5.7}$$

Therefore, $\mathfrak{S}_{\pi(C_i)} = \pi\mathfrak{S}_{C_i}\pi^{-1}$. As a result,

$$\pi C_t \pi^{-1} = C_{\pi t}. \tag{5.8}$$

Then using the definition of polytabloid,

$$\begin{aligned}
\mathbf{e}_{\pi t} &= \sum_{\sigma \in C_{\pi t}} (\operatorname{sgn}\sigma)\,\sigma\,[\pi t] \\
&= \sum_{\sigma \in \pi C_t \pi^{-1}} (\operatorname{sgn}\sigma)\,\sigma\,[\pi t] \\
&= \sum_{\sigma' \in C_t} \left(\operatorname{sgn}\left(\pi\sigma'\pi^{-1}\right)\right)\pi\sigma'\pi^{-1}\ \pi\,[t] \\
&= \pi \sum_{\sigma' \in C_t} (\operatorname{sgn}\sigma')\,\sigma'\,[t] \\
&= \pi\mathbf{e}_t.
\end{aligned} \tag{5.9}$$

■

Now we are ready to extract an irreducible subrepresentation from M^λ.

> **Definition 5.9** (Specht module)**.** For any partition $\lambda \vdash n$, the corresponding **Specht module**, denoted by S^λ, is the submodule of M^λ spanned by the polytabloids $\mathbf{e}_t$, where t is taken over all tableaux of shape λ.

Let us pause a bit and loot at a few examples.

Example 5.5. Consider $\lambda = (n)$. Then there is only one polytabloid, namely

$$\begin{array}{c}\hline 1\ \ 2\ \ 3\ \cdots\ n \\ \hline\end{array}$$

as there is nothing to permute along the columns. Since the polytabloid is fixed under the action of any $\pi \in \mathfrak{S}_n$, we see that $S^{(n)}$ is the 1-dimensional trivial representation.

Example 5.6. Consider $\lambda = (1, 1, \ldots, 1)$. Let

$$t = \begin{array}{c}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \vdots \\ \hline n \\ \hline\end{array}\,.$$

There is only one column here, which we denote by C. $C_t = \mathfrak{S}_{\{1,2,\ldots,n\}} = \mathfrak{S}_n$. There are $n!$ many elements in C_t. Hence, there are $n!$ many λ-tabloids for this example. If t is such a λ tabloid, then one can write down the polytabloid $\mathbf{e}_t$ associated with the given λ-tabloid using (5.5).

If one writes down the polytabloid $\mathbf{e}_{t'}$ associated with another λ-tabloid t' among the $n!$ many choices, then one easily finds that

(i) if t' is obtained from t through an even permutation, $[t'] = \sigma\,[t]$, for $\operatorname{sgn}\sigma = 1$. Then

$$\mathbf{e}_{t'} = \sum_{\pi \in C_t} (\operatorname{sgn}\pi)\,\pi\,[t'] = \sum_{\pi \in C_t} (\operatorname{sgn}\pi)\,\pi\sigma\,[t] = \sum_{\pi\sigma \in C_t} (\operatorname{sgn}(\pi\sigma))\,\pi\sigma\,[t] = \mathbf{e}_t. \tag{5.10}$$

(ii) if t' is obtained from t through an odd permutation, $[t'] = \sigma\,[t]$, for $\operatorname{sgn}\sigma = -1$. Then

$$\mathbf{e}_{t'} = \sum_{\pi \in C_t} (\operatorname{sgn}\pi)\,\pi\,[t'] = \sum_{\pi \in C_t} (\operatorname{sgn}\pi)\,\pi\sigma\,[t] = -\sum_{\pi\sigma \in C_t} (\operatorname{sgn}(\pi\sigma))\,\pi\sigma\,[t] = -\mathbf{e}_t. \tag{5.11}$$

So there is only 1 linearly independent polytabloid, and hence S^λ is a 1-dimensional representation of $\mathfrak{S}_n$. From Lemma 5.3, it follows that

$$\pi\mathbf{e}_t = \mathbf{e}_{\pi t} = (\operatorname{sgn}\pi)\,\mathbf{e}_t. \tag{5.12}$$

Hence, $S^{(1,1,\ldots,1)}$ is the sign representation of $\mathfrak{S}_n$.

Example 5.7. Consider $\lambda = (n-1, 1)$. Let $[t_i]$ be the λ-tabloid whose entry in the second row is i. There are n such λ-tabloids. For a given tableau t_i of shape $\lambda = (n-1, 1)$,

$$t_i = \begin{array}{|c|c|c|c|}\hline j & a & b & \cdots \\ \hline i \\ \cline{1-1}\end{array}, \tag{5.13}$$

the polytabloid $\mathbf{e}_{t_i}$ reads

$$\mathbf{e}_{t_i} = \begin{array}{cccc}\hline j & a & b & \cdots \\ \hline i & & & \\ \cline{1-1}\end{array} - \begin{array}{cccc}\hline i & a & b & \cdots \\ \hline j & & & \\ \cline{1-1}\end{array} = [t_i] - [t_j]\,, \tag{5.14}$$

as there is only two element in the column group C_{t_i} which permutes i and j. Now, the number of linearly independent polytabloid $\mathbf{e}_{t_i}$ is actually $n-1$, namely

$$[t_1] - [t_2]\,, [t_2] - [t_3]\,, \ldots, [t_{n-1}] - [t_n]\,.$$

If we denote $[t_i]$ by $\mathbf{v}_i$, then $\mathbf{v}_1 - \mathbf{v}_2, \mathbf{v}_2 - \mathbf{v}_3, \ldots, \mathbf{v}_{n-1} - \mathbf{v}_n$ forms a basis for $S^{(n-1,1)}$. Then

$$S^{(n-1,1)} = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n \mid c_1 + c_2 + \cdots + c_n = 0\}\,. \tag{5.15}$$

Indeed, $\dim S^{(n-1,1)} = n-1$. This is an irreducible representation of $\mathfrak{S}_n$ known as the **standard representation**, which we studied earlier. The direct sum of the standard representation $S^{(n-1,1)}$ and the trivial representation $S^(n)$ is the defining representation of $\mathfrak{S}_n$, i.e. $S^{(n-1,1)} \oplus S^{(n)} \cong M^{(n-1,1)}$.

For $\mathfrak{S}_3$, we have seen that there ave exactly 3 irreducible representations of it, namely, trivial, sign, and standard representation. These correspond to the 3 Specht modules discussed in the last 3 examples, namely $S^{(3)}$, $S^{(1,1,1)}$, and $S^{(2,1)}$. So, we see that in this case the irreducible representations are precisely the specht modules. Amazingly, this is true, in general, as given by the following Theorem:

Theorem 5.4

The Specht modules S^λ for $\lambda \vdash n$ form a complete list of irreducible representations of $\mathfrak{S}_n$ over $\mathbb{C}$.

Before diving into the proof of Theorem 5.4, we'll need a few results.

Lemma 5.5

Let t and t' be two λ-tableaux. Then $\sum_{\pi \in C_t} (\operatorname{sgn}\pi)\,\pi\,[t'] = \pm\mathbf{e}_t$. This sign is positive if t' is obtained from t using an even permutation, and negative otherwise.

Proof. Suppose t' is obtained from t by applying $g \in \mathfrak{S}_n$, i.e. $[t'] = g\,[t]$. In fact, we can assume WLOG that $g \in C_t$. Indeed, if $[t'] = [t]$, then $g = e$. Otherwise, suppose $(i\ j)$ is a transposition in g. We can rearrange the rows of t' so that i and j are in the same column of t. Thus, we get $(i\ j) \in C_t$. Similarly, all the transpositions of g are in C_t. So we can assume $g \in C_t$. As a result, when π varies in C_t, πg also varies in C_t.

$$\begin{aligned}\sum_{\pi\in C_t} (\operatorname{sgn}\pi)\,\pi\,[t'] &= \sum_{\pi\in C_t} (\operatorname{sgn}\pi)\,\pi g\,[t] \\ &= (\operatorname{sgn} g) \sum_{\pi g\in C_t} (\operatorname{sgn}(\pi g))\,(\pi g)\,[t] \\ &= (\operatorname{sgn} g)\ \mathbf{e}_t. \end{aligned} \tag{5.16}$$

■

Definition 5.10 (Lexicographic ordering)**.** Let $\lambda = (\lambda_1, \ldots, \lambda_l)$ and $\mu = (\mu_1, \ldots, \mu_k)$ be two partitions of n with $\lambda \neq \mu$. Let i be the first index where λ and μ differ. If $\lambda_i < \mu_i$, we write $\lambda \prec \mu$. This is called the **lexicographic order** on the set of partitions of n.

The lexicographic ordering is a total order, as one can verify easily. In other words, given any $\lambda \neq \mu$, either $\lambda \prec \mu$ or $\mu \prec \lambda$.

Lemma 5.6 (Dominance lemma)

Let $\lambda = (\lambda_1, \ldots, \lambda_l)$ and $\mu = (\mu_1, \ldots, \mu_k)$ be two partitions of n with $\lambda \prec \mu$. Let t be a λ-tableau and t' be a μ-tableau. Then there exists a pair (x, y) such that x, y are in the same row of t' and in the same column of t.

Proof. Let i be the first index where λ and μ differ. By hypothesis, $\lambda_i < \mu_i$. Assume the contrary. Then for each j, the elements of the j-th row of t' are all in different columns of t. We sort the entries in the column of t so that the elements of the first j rows of t' all occur in the first j rows of t. We can do this because if there is a column of t that contains more that j entries from the first j rows of t', then we must have two entries in the column coming from the same row, which we assumed doesn't happen. Now,

$$\begin{aligned}\lambda_1 + \cdots + \lambda_i &= \text{ number of elements in the first } i \text{ rows of } t \\ &\geq \text{ number of elements of } t' \text{ in the first } i \text{ rows of } t \\ &= \text{ number of elements of in the first } i \text{ rows of } t' \\ &= \mu_1 + \cdots + \mu_i.\end{aligned}$$

This contradicts the fact that i is the first index where λ and μ differ, and $\lambda_i < \mu_i$. ■

Lemma 5.7

Let $\lambda = (\lambda_1, \ldots, \lambda_l)$ and $\mu = (\mu_1, \ldots, \mu_k)$ be two partitions of n with $\lambda \prec \mu$. Let t be a λ-tableau and t' be a μ-tableau. Then

$$\sum_{\pi\in C_t} (\operatorname{sgn}\pi)\,\pi\,[t'] = 0,$$

where the equality holds in M^μ.

Proof. By Dominance lemma, there exists a transposition $(x\ y) \in C_t$ and $(x\ y) \in R_{t'}$. Let's call the transposition $g := (x\ y)$. $\operatorname{sgn} g = -1$; and since $g \in C_t$, when π varies in C_t, πg also varies in C_t.

$$\begin{aligned}\sum_{\pi \in C_t} (\operatorname{sgn} \pi)\, \pi\, [t'] &= \sum_{\pi \in C_t} (\operatorname{sgn}(\pi g))\, (\pi g)\, [t'] \\ &= -\sum_{\pi \in C_t} (\operatorname{sgn} \pi)\, \pi\, [g \cdot t'] \\ &= -\sum_{\pi \in C_t} (\operatorname{sgn} \pi)\, \pi\, [t'] \,, \end{aligned} \tag{5.17}$$

since $g \in R_{t'}$. Therefore, $\sum\limits_{\pi \in C_t} (\operatorname{sgn} \pi)\, \pi\, [t'] = 0$. ■

Lemma 5.8

If $\lambda = (\lambda_1, \ldots, \lambda_l)$ and $\mu = (\mu_1, \ldots, \mu_k)$ are two partitions of n with $\lambda \neq \mu$, then S^λ and S^μ are not isomorphic representations.

Proof. WLOG, assume $\lambda \prec \mu$. Assume the contrary that $f : S^\lambda \to S^\mu$ is an isomorphism of representations. Then for any $\mathbf{v} \in S^\lambda$ and any $g \in \mathfrak{S}_n$,

$$f\,(g \cdot \mathbf{v}) = g \cdot f\,(\mathbf{v})\,. \tag{5.18}$$

Then it follows that for any λ-tableau t,

$$f\left(\sum_{\pi \in C_t} (\operatorname{sgn} \pi)\, \pi \cdot \mathbf{v}\right) = \sum_{\pi \in C_t} (\operatorname{sgn} \pi)\, \pi \cdot f\,(\mathbf{v})\,. \tag{5.19}$$

For any μ-tableau t', $\sum_{\pi \in C_t} (\operatorname{sgn} \pi)\, \pi\, [t'] = 0$ by Lemma 5.7. Therefore, the RHS of (5.19) is always 0. Now, take $\mathbf{v} = \mathbf{e}_t$.

$$\begin{aligned}\sum_{\pi \in C_t} (\operatorname{sgn} \pi)\, \pi \cdot \mathbf{e}_t &= \sum_{\pi \in C_t} (\operatorname{sgn} \pi)\, \pi \sum_{\sigma \in C_t} (\operatorname{sgn} \sigma)\, \sigma\, [t] \\ &= \sum_{\pi \in C_t} \sum_{\sigma \in C_t} (\operatorname{sgn} \pi)\, (\operatorname{sgn} \sigma)\, (\pi\sigma)\, [t] \\ &= \sum_{\pi \in C_t} \sum_{\pi\sigma \in C_t} (\operatorname{sgn}(\pi\sigma))\, (\pi\sigma)\, [t] \\ &= \sum_{\pi \in C_t} \mathbf{e}_t = |C_t|\, \mathbf{e}_t \neq 0. \end{aligned} \tag{5.20}$$

Since the argument of f on the LHS (5.19) is nonzero, the LHS must also be nonzero since f is an isomorphism. But we have seen that the RHS of (5.19) is always 0. Contradiction! Therefore, there is no such isomorphism. ■

Lemma 5.9

Given $\lambda \vdash n$, S^λ is an irreducible representation of $\mathfrak{S}_n$.

Proof. Let W be a nontrivial $\mathfrak{S}_n$-invariant subspace of S^λ. Then take a nonzero vector $\mathbf{w} \in W$. Then it is some complex linear combination of $[t_i]$'s where each t_i is a λ-tableau (S^λ is a subspace of M^λ, and M^λ is spanned by $[t_i]$'s for t_i being a λ-tableau). In other words,

$$\mathbf{w} = \sum_{i=1}^{N} c_i\, [t_i]\,. \tag{5.21}$$

Fix a λ-tableau t. Since W is a $\mathfrak{S}_n$-invariant subspace, $\sum_{\pi \in C_t} (\operatorname{sgn} \pi)\, \pi \cdot \mathbf{w}$ is also in W.

$$\sum_{\pi \in C_t} (\operatorname{sgn} \pi)\, \pi \cdot \sum_{i=1}^{N} c_i\, [t_i] = \sum_{i=1}^{N} c_i \sum_{\pi \in C_t} (\operatorname{sgn} \pi)\, \pi\, [t_i] = \sum_{i=1}^{N} \pm c_i \mathbf{e}_t, \tag{5.22}$$

by Lemma 5.5. This is a scalar multiple of $\mathbf{e}_t$, say $c\,\mathbf{e}_t$. Then we have two cases:

(i) $c = 0$ for every choice of nonzero $\mathbf{w} \in W$: By (5.20), $\sum_{\pi \in C_t} (\operatorname{sgn} \pi)\, \pi$ applied to $\mathbf{e}_t$ gives us $|C_t|\, \mathbf{e}_t$. Since $c = 0$ for every choice of nonzero $\mathbf{w} \in W$, this means $\mathbf{e}_t \notin W$. Not only that, the component of $\mathbf{e}_t$ in any vector of W must also be 0.

Hence, $\mathbf{e}_t \in W^\perp$. Since $W^\perp$ is also a $\mathfrak{S}_n$-invariant subspace, $\mathbf{e}_{\sigma t} = \sigma\, \mathbf{e}_t \in W^\perp$ for every $\sigma \in \mathfrak{S}_n$. As a result, every $\mathbf{e}_t$ is in $W^\perp$, as we can get to any λ-tableau by permuting the entries of t. Hence, $W^\perp$ is the whole S^λ. In other words, $W = 0$.

(ii) $c \neq 0$ for some nonzero $\mathbf{w} \in W$: In this case, since W is a subspace, we have $\mathbf{e}_t \in W$. Since W is a $\mathfrak{S}_n$-invariant subspace, $\mathbf{e}_{\sigma t} = \sigma\, \mathbf{e}_t \in W$ for every $\sigma \in \mathfrak{S}_n$. As a result, every $\mathbf{e}_t$ is in W, as we can get to any λ-tableau by permuting the entries of t. Hence, W is the whole S^λ.

Therefore, S^λ is irreducible. ■

Now we gather all the pieces of the puzzle to complete the proof of Theorem 5.4.

Proof of Theorem 5.4. The number of partitions of n is the same as the number of conjugacy classes of $\mathfrak{S}_n$, which is the number of irreducible representations of $\mathfrak{S}_n$. The collection $\left\{ S^\lambda \mid \lambda \vdash n \right\}$ is a collection of pairwise non-isomorphic irreducible representations. The number of irreducible representations in this collection is precisely the number of irreducible representations $\mathfrak{S}_n$ can have. Therefore, this is the collection of **all** irreducible representations $\mathfrak{S}_n$. ■

Polytabloids are, in general, linearly dependent, as we have seen in the examples before. We know that the Specht module S^λ is spanned by the polytabloids. We may ask how to select a basis for S^λ from the set of polytabloids. The answer is given in the following theorem.

Theorem 5.10

Let $\lambda \vdash n$. The set

$$\{\mathbf{e}_t \mid t \text{ is a standard } \lambda\text{-tableau } \}$$

forms a basis for S^λ as a vector space.

As before, we need a few machinaries to prove Theorem 5.10. For $\lambda \vdash n$, let f^λ denote the number of standard λ tabloids. We want to show that $\dim S^\lambda = f^\lambda$. Since S^λ's are all the irreducible representations, the sum of the square of their dimensions will be the cardinality of $\mathfrak{S}_n$. So we will have

$$\sum_\lambda \left(f^\lambda\right)^2 = n!$$

We will prove a combinatorial identity first, which proves that this is indeed the case.

Proposition 5.11 (Robinson–Schensted correspondence)

There is a one-to-one correspondence between elements of $\mathfrak{S}_n$ and pairs (P, Q) of standard tableaux, both of the same shape λ. In other words,

$$\sum_{\lambda \vdash n} \left(f^\lambda\right)^2 = n! \tag{5.23}$$

Proof. Let $\sigma \in \mathfrak{S}_n$ be a permutation:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ \sigma(1) & \sigma(2) & \sigma(3) & \cdots & \sigma(n) \end{pmatrix}.$$

From this permutation, we construct a sequence of standard tableaux $(P_0, Q_0), (P_1, Q_1), \ldots, (P_n, Q_n)$, with (P_n, Q_n) being the corresponding pair of standard tableaux, both of the same shape λ. (P_0, Q_0) is the empty tableaux, and at the i-th step, we add one element to each of (P_{i-1}, Q_{i-1}). Here is the algorithm:

1. Suppose we have constructed up to (P_{i-1}, Q_{i-1}). Now we will insert i into Q_i and $\sigma(i)$ into P_i.
2. Look at the first row of P_{i-1}. If all the numbers are smaller than $\sigma(i)$, then add $\sigma(i)$ to the end of the first row.
3. Otherwise, find the leftmost number x in the first row that is greater than $\sigma(i)$. Replace x by $\sigma(i)$.
4. Head over to the next row, repeat the previous step for x. That is, either add x to the end of the row, or find the leftmost number that is greater than x; replace that number with x; then do this again for that number. We have to keep on doing this until a new box is added at the end of some row. Thus we get P_i.
5. When a new box is created, create the same new box at the same spot in Q_{i-1}, and write i in that new box. Thus we get Q_i.

Clearly, each P_i is a standard tableau. Q is a record of where a box was added in each step of the construction of P. After the insertion of $\sigma(i)$, one new box is added to the shape of P_{i-1} to obtain the shape of P_i. We add a box to the shape of Q_{i-1} and write i in the box. Then each Q_i is also a standard tableau, because a new box is added at the end of a row, or it creates a new row. The new box always has a box above it (except in the first row), so the values in the columns and rows are always in an increasing order.

We illustrate the process using the example of

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 5 & 2 \end{pmatrix} \in \mathfrak{S}_5.$$

At the beginning P_0 and Q_0 are the empty tableaux.

1. $i = 1$, $\sigma(i) = 3$. So P_1 and Q_1 are:
$$P_1 = \begin{array}{|c|}\hline 3 \\ \hline\end{array}, \qquad Q_1 = \begin{array}{|c|}\hline 1 \\ \hline\end{array}.$$
2. $i = 2$, $\sigma(i) = 4$. We just add 4 to the end of P_1.
$$P_2 = \begin{array}{|c|c|}\hline 3 & 4 \\ \hline\end{array}, \qquad Q_2 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}.$$
3. $i = 3$, $\sigma(i) = 1$. The leftmost entry in the first row that is larger than $\sigma(i)$ is 3. So we replace 1 with 3, and move 3 to the next row.
$$P_3 = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}, \qquad Q_3 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}.$$
4. $i = 4$, $\sigma(i) = 5$. We just add 5 to the end of the first row of P_3.
$$P_4 = \begin{array}{|c|c|c|}\hline 1 & 4 & 5 \\ \hline 3 \\ \cline{1-1}\end{array}, \qquad Q_4 = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}.$$
5. $i = 5$, $\sigma(i) = 2$. The leftmost entry in the first row that is larger than $\sigma(i)$ is 4. So we replace 2 with 4, and move 4 to the next row. Then we can add 4 to the end of the next row.
$$P_5 = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2}\end{array}, \qquad Q_5 = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array}.$$

Thus, from a permutation in $\mathfrak{S}_n$, we get a pair of standard tableaux. The crucial fact about this algorithm is that it is perfectly reversible. At the i-th step, we add a box containing i to Q_{i-1} in the place where P_i got a new box. The reverse algorithm is as follows: Write the incomplete σ:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ * & * & * & \cdots & * \end{pmatrix}.$$

We will fill out this.

1. Delete from Q_i the box containing i. Locate that box in P_i and delete it. Suppose that box contains the number x.
2. Move to the above row. Find the rightmost number less than x. Replace x with that number y.
3. Do the same for y. That is, go to the previous row. Find the righmost number less than y. Replace that number with y. Then keep on repeating this process until finally some number k is bumped from the first row.
4. Add k as $\sigma(i)$ in the incomplete σ matrix:

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & i-1 & i & i+1 & \cdots & n \\ * & * & \cdots & * & k=\sigma(i) & \sigma(i+1) & \cdots & \sigma(n) \end{pmatrix}.$$

As before, we illustrate the process using the example of

$$P = \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2} \end{array}, \qquad Q = \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2} \end{array}.$$

1. Look up 5 from Q. The number in the corresponding place in P_5 is 4. The rightmost number in the previous row smaller than 4 is 2. Write 4 in place of 2. There is no previous row, and 2 is removed. Therefore, $\sigma(5) = 2$.

$$P_4 = \begin{array}{|c|c|c|} \hline 1 & 4 & 5 \\ \hline 3 \\ \cline{1-1} \end{array}, \qquad Q_4 = \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1} \end{array}, \qquad \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ * & * & * & * & 2 \end{pmatrix}.$$

2. Look up 4 from Q. The number in the corresponding place in P_4 is 5. There is no previous row, and 5 is removed. Therefore, $\sigma(4) = 5$.

$$P_3 = \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 3 \\ \cline{1-1} \end{array}, \qquad Q_3 = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}, \qquad \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ * & * & * & 5 & 2 \end{pmatrix}.$$

3. Look up 3 from Q. The number in the corresponding place in P_3 is 3. The rightmost number in the previous row smaller than 3 is 1. Write 3 in place of 1. There is no previous row, and 1 is removed. Therefore, $\sigma(3) = 1$.

$$P_2 = \begin{array}{|c|c|} \hline 3 & 4 \\ \hline \end{array}, \qquad Q_2 = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array}, \qquad \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ * & * & 1 & 5 & 2 \end{pmatrix}.$$

4. Look up 2 from Q. The number in the corresponding place in P_2 is 4. There is no previous row, and 4 is removed. Therefore, $\sigma(2) = 4$.

$$P_1 = \begin{array}{|c|} \hline 3 \\ \hline \end{array}, \qquad Q_1 = \begin{array}{|c|} \hline 1 \\ \hline \end{array}, \qquad \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ * & 4 & 1 & 5 & 2 \end{pmatrix}.$$

5. This step is now obvious.

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 5 & 2 \end{pmatrix}.$$

Therefore, we have established a one-to-one correspondence

$$\sigma \in \mathfrak{S}_n \longleftrightarrow \text{ pairs of standard tableaux } (P, Q)\,.$$

Therefore,

$$\sum_{\lambda \vdash n} \left(f^{\lambda}\right)^2 = n! \tag{5.24}$$

∎

> **Definition 5.11.** Given $\lambda \vdash n$, we define an ordering $<_{\text{row}}$ on the set of λ-tabloids as follows: let $[t]$ and $[t']$ be λ-tabloids with $[t] \neq [t']$. We say that $[t] <_{\text{row}} [t']$ if and only if there exists some i such that
>
> (a) for all $j > i$, j is in the same row of $[t]$ and $[t']$; and
>
> (b) i is in a higher row of $[t]$ than $[t']$.

In other words, for $[t] \neq [t']$, let i be the largest number which is not in the same row of $[t]$ and $[t']$. Then $[t] <_{\text{row}} [t']$ if and only if i is in a higher row of $[t]$ than $[t']$. (Here, higher row means that if we label the rows from top to bottom, the number of the row is smaller.) Given $[t] \neq [t']$, then either $[t] <_{\text{row}} [t']$ or $[t'] <_{\text{row}} [t]$ holds.

Also, this ordering is transitive: Suppose $[s] <_{\text{row}} [t]$ and $[t] <_{\text{row}} [u]$. Suppose i_1 is the largest number which is not in the same row of $[s]$ and $[t]$; and i_2 is the largest number which is not in the same row of $[t]$ and $[u]$. Let $i = \max\{i_1, i_2\}$. Then for $j > i$, j is in the same row of $[s]$, $[t]$, $[u]$. We have 3 cases:

1. If $i = i_1 = i_2$, then i is in a higher row of $[s]$ than $[t]$; and i is in higher row of $[t]$ than $[u]$. Therefore, i is in a higher row of $[s]$ than $[u]$, i.e. $[s] <_{\text{row}} [u]$.
2. Otherwise, if $i = i_1 > i_2$, $i = i_1$ is in a higher row of $[s]$ than $[t]$; and $i > i_2$ is in the same row of $[t]$ and $[u]$. Therefore, i is in a higher row of $[s]$ than $[u]$, i.e. $[s] <_{\text{row}} [u]$.
3. Similarly, if $i = i_2 > i_1$, $i > i_1$ is in the same row of $[s]$ and $[t]$; and $i = i_2$ is in a higher row of $[t]$ than $[u]$. Therefore, i is in a higher row of $[s]$ than $[u]$, i.e. $[s] <_{\text{row}} [u]$.

Therefore, $<_{\text{row}}$ is a total ordering on the set of all λ-tabloids. For instance, for $n = 5$, the number of $(3, 2)$-tabloids is $\frac{5!}{3!2!} = 10$. Here are all the $(3, 2)$-tabloids in the ascending order of $<_{\text{row}}$.

$$\begin{array}{ccc}\hline 3 & 4 & 5 \\ \hline 1 & 2 & \\ \cline{1-2}\end{array} <_{\text{row}} \begin{array}{ccc}\hline 2 & 4 & 5 \\ \hline 1 & 3 & \\ \cline{1-2}\end{array} <_{\text{row}} \begin{array}{ccc}\hline 1 & 4 & 5 \\ \hline 2 & 3 & \\ \cline{1-2}\end{array} <_{\text{row}} \begin{array}{ccc}\hline 2 & 3 & 5 \\ \hline 1 & 4 & \\ \cline{1-2}\end{array} <_{\text{row}} \begin{array}{ccc}\hline 1 & 3 & 5 \\ \hline 2 & 4 & \\ \cline{1-2}\end{array}$$

$$<_{\text{row}} \begin{array}{ccc}\hline 1 & 2 & 5 \\ \hline 3 & 4 & \\ \cline{1-2}\end{array} <_{\text{row}} \begin{array}{ccc}\hline 2 & 3 & 4 \\ \hline 1 & 5 & \\ \cline{1-2}\end{array} <_{\text{row}} \begin{array}{ccc}\hline 1 & 3 & 4 \\ \hline 2 & 5 & \\ \cline{1-2}\end{array} <_{\text{row}} \begin{array}{ccc}\hline 1 & 2 & 4 \\ \hline 3 & 5 & \\ \cline{1-2}\end{array} <_{\text{row}} \begin{array}{ccc}\hline 1 & 2 & 3 \\ \hline 4 & 5 & \\ \cline{1-2}\end{array}$$

> **Lemma 5.12**
>
> Let $\lambda \vdash n$, and let t be a standard λ-tableau. Then for $\pi \in C_t$ with $\pi \neq e$, $\pi[t] <_{\text{row}} [t]$.

Proof. Let i be the largest entry that is **NOT** fixed by π. Then all the entries $j > i$ are not permuted by π. So those entries are in the same row of $\pi[t]$ and $[t]$. Since all the entries $j > i$ are fixed, we must have $\pi(i) < i$. Since t is a standard tableau, $\pi(i)$ is in a higher row of $[t]$ than i. Therefore, after applying π, i will be in a higher row of $\pi[t]$ that it was in $[t]$.

To summarize, for $j > i$, j is in the same row of $\pi[t]$ and $[t]$; and i is in a higher row of $\pi[t]$ than $[t]$. Therefore, $\pi[t] <_{\text{row}} [t]$. ∎

Lemma 5.13

Let $\lambda \vdash n$. Then $\{\mathbf{e}_t \mid t \text{ is a standard } \lambda\text{-tableau}\}$ is a linearly independent subset of S^λ.

Proof. Let $t_1, t_2, \ldots, t_k$ be all the standard λ-tableaux. WLOG, we can assume

$$[t_1] <_{\text{row}} [t_2] <_{\text{row}} \cdots <_{\text{row}} [t_k].$$

Suppose $c_1, c_2, \ldots, c_k \in \mathbb{C}$ such that

$$c_1\, \mathbf{e}_{t_1} + c_2\, \mathbf{e}_{t_2} + \cdots + c_k\, \mathbf{e}_{t_k} = \mathbf{0}. \tag{5.25}$$

When we expand $\mathbf{e}_{t_k}$, we get

$$\mathbf{e}_{t_k} = [t_k] \pm \pi_1\, [t_k] \pm \pi_2\, [t_k] \pm \cdots$$

Hence,

$$\mathbf{0} = (c_k\, [t_k] \pm c_k\, \pi_1\, [t_k] \pm c_k\, \pi_2\, [t_k] \pm \cdots) + \sum_{i=1}^{k-1} c_i\, ([t_i] \pm \pi'\, [t_i] + \cdots)$$

In order to cancel out $[t_k]$, it must appear again in the sum. But the other terms are either $\pm c_k \pi\, [t_k]$, or $\pm c_i \pi'\, [t_i]$ for $i < k$. By Lemma 5.13,

$$\pi\, [t_k] <_{\text{row}} [t_k]. \tag{5.26}$$

Also, for $i < k$, Also, the possible value of $\pi'\, [t_i]$ is either $[t_i]$ itself (when π' is the identity), or $\pi'\, [t_i]$ for some nontrivial $\pi' \in C_{t_i}$. In either case, we have

$$[t_i] <_{\text{row}} [t_k] \qquad \text{or } \pi'\, [t_i] <_{\text{row}} [t_i] <_{\text{row}} [t_k]. \tag{5.27}$$

Therefore, $[t_k]$ doesn't appear in the sum. Hence, c_k must be 0. Inductively, all the c_i's must be 0. ■

Now we can finally give a proof of Theorem 5.10.

Proof of Theorem 5.10. By Lemma 5.13, $\{\mathbf{e}_t \mid t \text{ is a standard } \lambda\text{-tableau }\}$ is a linearly independent set. So $\dim S^\lambda \geq f^\lambda$. The order of a group is equal to the sum of square of the dimension of the irreducible representations. Therefore,

$$n! = |\mathfrak{S}_n| = \sum_{\lambda \vdash n} \left(\dim S^\lambda\right)^2 \geq \sum_{\lambda \vdash n} \left(f^\lambda\right)^2 = n!, \tag{5.28}$$

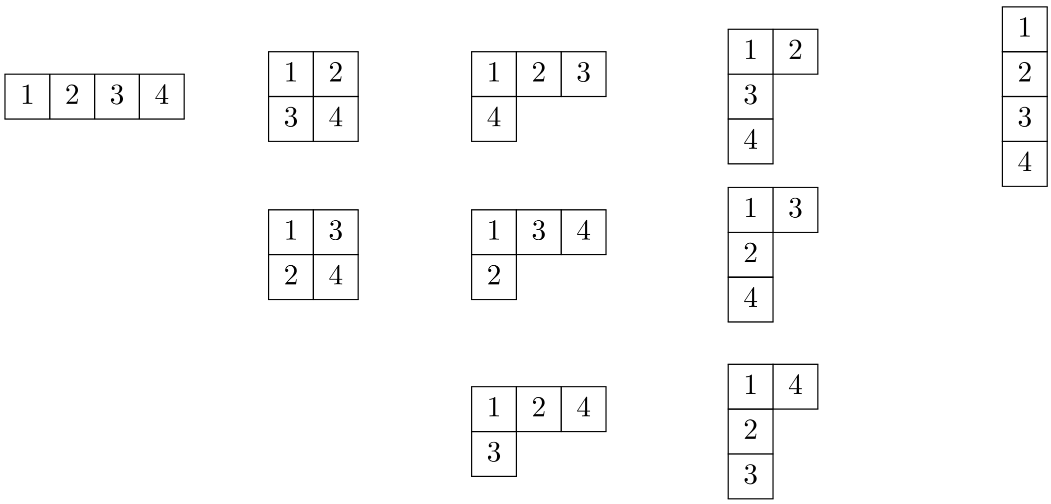
where the last equality follows from Robinson–Schensted correspondence. Therefore, $\dim S^\lambda = f^\lambda$ for every λ. Now, $\{\mathbf{e}_t \mid t \text{ is a standard } \lambda\text{-tableau }\}$ is a linearly independent set of cardinality $f^\lambda = \dim S^\lambda$. Therefore, it spans S^λ, i.e. it's a basis for S^λ. ■

Example 5.8. For $\mathfrak{S}_3$, we list standard λ-tableaux under each possible partitions:

(3)	$(2,1)$	$(1,1,1)$
$\begin{array}{\|c\|c\|c\|}\hline 1 & 2 & 3 \\ \hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$ $\begin{array}{\|c\|c\|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$	$\begin{array}{\|c\|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}$
$\dim S^{(3)} = 1$	$\dim S^{(2,1)} = 1$	$\dim S^{(1,1,1)} = 1$

For $\mathfrak{S}_4$, we list standard λ-tableaux under each possible partitions:

(3) (2, 2) (3, 1) (2, 1, 1) (1, 1, 1, 1)



$\dim S^{(4)} = 1$ $\dim S^{(2,2)} = 2$ $\dim S^{(3,1)} = 3$ $\dim S^{(2,1,1)} = 3$ $\dim S^{(1,1,1,1)} = 4$

Theorem 5.14 (Frobenius character formula)

Suppose $\lambda = (\lambda_1, \ldots, \lambda_l), \mu = (\mu_1, \ldots, \mu_m)$ are partitions of n. The character of S^λ evaluated at an element of $\mathfrak{S}_n$ with cycle type μ is equal to the coefficient of $x_1^{\lambda_1+l-1} x_2^{\lambda_2+l-2} \cdots x_l^{\lambda_l}$ in the following polynomial

$$\prod_{1\le i<j\le l} (x_i - x_j) \prod_{i=1}^{m} (x_1^{\mu_i} + x_2^{\mu_i} + \cdots + x_l^{\mu_i}) .$$