Representation Theory (MAT440)

Lecture Notes

Preface

This series of lecture notes has been prepared for aiding students who took the BRAC University course **Representation Theory (MAT440)** in Summer 2024 semester. These notes were typeset under the supervision of mathematician **Dr. Syed Hasibul Hassan Chowdhury**. The main goal of this typeset is to have an organized digital version of the notes, which is easier to share and handle. If you see any mistakes or typos, please send me an email at atonuroychowdhury@gmail.com

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References:

- *Representation Theory: A First Course*, by **Joe Harris and William Fulton**
- *Representations of Finite and Compact Groups*, by **Barry Simon**
- *Introduction to Representation Theory*, by **Pavel Etingof et al.**

Contents

1 Representation of Finite Groups

§1.1 Definitions

Definition 1.1 (Representation). A **representation** of a finite group G on a finite dimensional complex vector space V is a homomorphism $\rho : G \to \mathrm{GL}(V)$ of G to the group of invertible linear transformations on V. We often say that such a homomorphism gives V the structure of a G-module. The dimension of V is sometimes called the **degree** of the representation ρ. We also sometimes call V itself a representation of G.

Definition 1.2. A **map** φ between two representations V and W of G is a linear map $\varphi : V \to W$ such that the following diagram commutes for every $g \in G$:

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "\rho(g)"'] & W \arrow[d, "\sigma(g)"] \\ V \arrow[r, "\varphi"'] & W \end{tikzcd}$$

In other words, $\varphi \circ \rho(g) = \sigma(g) \circ \varphi$. Here, $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$ are two group homomorphisms in question. We distinguish such a linear map $\varphi : V \to W$ between two representations of G from an ordinary linear map between vector spaces by calling it a **G-linear map**.

One can then define G-module structure on $\operatorname{Ker}\varphi$ and $\operatorname{im}\varphi$ by restricting the group homomorphisms $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$, namely,

$$\rho_1 : G \to \mathrm{GL}\left(\operatorname{Ker}\varphi\right) \text{ and } \sigma_1 : G \to \mathrm{GL}\left(\operatorname{im}\varphi\right).$$

Suppose $\mathbf{v} \in \operatorname{Ker}\varphi$. Then $\rho\left(g\right)\left(\mathbf{v}\right) \in \operatorname{Ker}\varphi$, because

$$\varphi\left(\rho\left(g\right)\left(\mathbf{v}\right)\right) = \sigma\left(g\right)\left(\varphi\left(\mathbf{v}\right)\right) = \sigma\left(g\right)\left(\mathbf{0}\right) = \mathbf{0}. \tag{1.1}$$

Also, let $\mathbf{w} \in \operatorname{im}\varphi$. Then $\mathbf{w} = \varphi(\mathbf{v})$ for some $\mathbf{v} \in V$. Then $\sigma\left(g\right)\left(\mathbf{w}\right) \in \operatorname{im}\varphi$, because

$$\sigma\left(g\right)\varphi(\mathbf{v}) = \varphi\left(\rho\left(g\right)\left(\mathbf{v}\right)\right) \in \operatorname{im}\varphi. \tag{1.2}$$

One can also give the quotient vector space $W/\operatorname{im}\varphi = \operatorname{Coker}\varphi$ a G-module structure by introducing the group homomorphism $\sigma_2 : G \to \mathrm{GL}(\operatorname{Coker}\varphi)$. Given $\mathbf{w} + \operatorname{im}\varphi \in \operatorname{Coker}\varphi$ and $g \in G$, one defines

$$\sigma_2\left(g\right)\left(\mathbf{w} + \operatorname{im}\varphi\right) = \sigma(g)\left(\mathbf{w}\right) + \operatorname{im}\varphi \in \operatorname{Coker}\varphi. \tag{1.3}$$

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The space of all G-linear maps from V to W is denoted $\operatorname{Hom}_G(V, W)$. It has a vector space structure. Suppose $\varphi, \psi \in \operatorname{Hom}_G(V, W)$ and $z \in \mathbb{C}$. Then we have the following commutative squares:

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "\rho(g)"'] & W \arrow[d, "\sigma(g)"] \\ V \arrow[r, "\varphi"'] & W \end{tikzcd} \qquad \begin{tikzcd} V \arrow[r, "\psi"] \arrow[d, "\rho(g)"'] & W \arrow[d, "\sigma(g)"] \\ V \arrow[r, "\psi"'] & W \end{tikzcd}$$

Then one can show that $z\varphi + \psi$ is also a G-linear map. Indeed,

$$\begin{aligned}\sigma(g) \circ (z\varphi + \psi)(\mathbf{v}) &= z\sigma(g)(\varphi(\mathbf{v})) + \sigma(g)(\psi(\mathbf{v})) \\ &= z\varphi(\rho(g)(\mathbf{v})) + \psi(\rho(g)(\mathbf{v})) \\ &= (z\varphi + \psi)(\rho(g)\mathbf{v}).\end{aligned}$$

This proves the commutativity of the following square:

$$\begin{tikzcd} V \arrow[r, "z\varphi+\psi"] \arrow[d, "\rho(g)"'] & W \arrow[d, "\sigma(g)"] \\ V \arrow[r, "z\varphi+\psi"'] & W \end{tikzcd}$$

Therefore, $z\varphi + \psi \in \mathrm{Hom}_G(V, W)$, i.e. $\mathrm{Hom}_G(V, W)$ is a complex vector space.

Definition 1.3 (Subrepresentation)**.** Suppose one is given a representation V of G with the help of the group homomorphism $\rho : G \to \mathrm{GL}(V)$ and $W \subset V$ be a vector subspace. One calls W **invariant** under the action of G if for all $g \in G$ and all $\mathbf{w} \in W$, one has $\rho(g)\mathbf{w} \in W$.

A **subrepresentation** of a representation V of G is a vector subspace W of V that is invariant under the action of G. A representation V of G is called **irreducible** if there is no proper nonzero invariant subspace W of V, i.e., there is no invariant subsapce $W \subset V$ such that $W \neq \{\mathbf{0}\}$ and $W \neq V$.

§1.2 Linear algebra revisited

Definition 1.4 (Tensor product)**.** The **tensor product** of two complex vector spaces V and W is another complex vector space $V \otimes W$ equipped with a bilinear map $\theta : V \times W \to V \otimes W$ that is *universal*: for any bilinear map $\beta : V \times W \to U$ to a complex vector space U, there exists a unique linear map $\alpha : V \otimes W \to U$ such that the following diagram commutes:

$$\begin{tikzcd} V \times W \arrow[r, "\theta"] \arrow[d, "\beta"'] & V \otimes W \arrow[dl, dashed, "\exists!\,\alpha"] \\ U & \end{tikzcd}$$

In other words, $\beta = \alpha \circ \theta$.

If we want the ground field $\mathbb{C}$ to be mentioned, we write the tensor product by $V \otimes_{\mathbb{C}} W$. If $\{\mathbf{e}_i\}$ and $\{\mathbf{f}_j\}$ are bases of V and W, respectively, $\{\mathbf{e}_i \otimes \mathbf{f}_j\}$ form a basis for $V \otimes W$. Similarly, one can form the tensor product $V_1 \otimes \cdots \otimes V_n$ of n vector spaces, with the universal (in the above sense) multilinear map

$$\begin{aligned}\theta : V_1 \times \cdots \times V_n &\to V_1 \otimes \cdots \otimes V_n \\ (\mathbf{v}_1, \ldots, \mathbf{v}_n) &\mapsto \mathbf{v}_1 \otimes \cdots \otimes \mathbf{v}_n.\end{aligned} \tag{1.4}$$

In particular, one can construct

$$V^{\otimes n} = \underbrace{V \otimes \cdots \otimes V}_{n\text{-copies}},$$

for a fixed complex vector space V. If $\{\mathbf{e}_i \mid i = 1, 2, \ldots, m\}$ is a basis for V, then the set

$$\{\mathbf{e}_{i_1} \otimes \mathbf{e}_{i_2} \otimes \cdots \otimes \mathbf{e}_{i_n} \mid i_1, \ldots, i_n \in \{1, 2, \ldots, m\}\} \tag{1.5}$$

is a basis for $V^{\otimes n}$. It follows that $\dim V^{\otimes n} = m^n$.

Let $\mathfrak{S}_n$ be the symmetric group on the set $\{1, 2, \ldots, n\}$. It is a finite group of order $n!$ that consists of all the permutations (i.e. bijections) on the set $\{1, 2, \ldots, n\}$. An alternating multilinear map $\beta : V \times \cdots \times V \to U$ satisfies

$$\beta\left(\mathbf{v}_{\sigma(1)}, \ldots, \mathbf{v}_{\sigma(n)}\right) = \operatorname{sgn}\sigma \beta\left(\mathbf{v}_1, \ldots, \mathbf{v}_n\right), \tag{1.6}$$

for every $\sigma \in \mathfrak{S}_n$.

Definition 1.5 (Exterior power)**.** The **exterior power** of a complex vector spaces V is another complex vector space $\Lambda^n V$ equipped with an alternating multilinear map

$$\begin{aligned} \kappa : V \times \cdots \times V &\to \Lambda^n V \\ (\mathbf{v}_1, \ldots, \mathbf{v}_n) &\mapsto \mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_n, \end{aligned}$$

that is *universal*: for any alternating multilinear map $\beta : V \times \cdots \times V \to U$ to a complex vector space U, there exists a unique linear map $\alpha : \Lambda^n V \to U$ such that the following diagram commutes:

$$\begin{tikzcd} V \times \cdots \times V \arrow[r, "\kappa"] \arrow[d, "\beta"'] & \Lambda^n V \arrow[dl, dashed, "\exists!\,\alpha"] \\ U & \end{tikzcd}$$

In other words, $\beta = \alpha \circ \kappa$.

If $\{\mathbf{e}_i \mid i = 1, 2, \ldots, m\}$ is a basis for V, then the set

$$\{\mathbf{e}_{i_1} \wedge \mathbf{e}_{i_2} \wedge \cdots \wedge \mathbf{e}_{i_n} \mid 1 \leq i_1 < i_2 < \cdots < i_n \leq m\} \tag{1.7}$$

is a basis for $\Lambda^n V$. It follows that $\dim \Lambda^n V = \binom{m}{n}$.

A symmetric multilinear map $\beta : V \times \cdots \times V \to U$ satisfies

$$\beta\left(\mathbf{v}_{\sigma(1)}, \ldots, \mathbf{v}_{\sigma(n)}\right) = \beta\left(\mathbf{v}_1, \ldots, \mathbf{v}_n\right), \tag{1.8}$$

for every $\sigma \in \mathfrak{S}_n$.

Definition 1.6 (Symmetric power)**.** The **symmetric power** of a complex vector spaces V is another complex vector space $\operatorname{Sym}^n V$ equipped with an symmetric multilinear map

$$\begin{aligned} \delta : V \times \cdots \times V &\to \operatorname{Sym}^n V \\ (\mathbf{v}_1, \ldots, \mathbf{v}_n) &\mapsto \mathbf{v}_1 \odot \cdots \odot \mathbf{v}_n, \end{aligned}$$

that is *universal*: for any symmetric multilinear map $\beta : V \times \cdots \times V \to U$ to a complex vector space U, there exists a unique linear map $\alpha : \operatorname{Sym}^n V \to U$ such that the following diagram commutes:

$$\begin{tikzcd} V \times \cdots \times V \arrow[r, "\delta"] \arrow[d, "\beta"'] & \operatorname{Sym}^n V \arrow[dl, dashed, "\exists!\,\alpha"] \\ U & \end{tikzcd}$$

In other words, $\beta = \alpha \circ \delta$.

If $\{\mathbf{e}_i \mid i = 1, 2, \dots, m\}$ is a basis for V, then the set

$$\{\mathbf{e}_{i_1} \odot \mathbf{e}_{i_2} \odot \cdots \odot \mathbf{e}_{i_n} \mid 1 \le i_1 \le i_2 \le \cdots \le i_n \le m\} \tag{1.9}$$

is a basis for $\operatorname{Sym}^n V$. It follows that $\dim \operatorname{Sym}^n V = \binom{m+n-1}{n}$.

§1.3 New representations from old ones

If V and W are representations of G, then so are the direct sum $V \oplus W$ and the tensor product $V \otimes W$. More explicitly, suppose $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$ are the relevant group homomorphisms. Then, one defines $\rho \oplus \sigma : G \to \mathrm{GL}(V \oplus W)$ by

$$(\rho \oplus \sigma)(g)(\mathbf{v} \oplus \mathbf{w}) = \rho(g)\mathbf{v} \oplus \sigma(g)\mathbf{w}, \tag{1.10}$$

for $g \in G$. Similarly, one can define the group homomorphism $\rho \otimes \sigma : G \to \mathrm{GL}(V \otimes W)$ by

$$(\rho \otimes \sigma)(g)(\mathbf{v} \otimes \mathbf{w}) = \rho(g)\mathbf{v} \otimes \sigma(g)\mathbf{w} \tag{1.11}$$

for $g \in G$.

For a representation V of G, the nth tensor power $V^{\otimes n}$ is again a representation of G:

$$(\rho \otimes \rho \otimes \cdots \otimes \rho)(g)(\mathbf{v}_1 \otimes \mathbf{v}_2 \otimes \cdots \otimes \mathbf{v}_n) = \rho(g)\mathbf{v}_1 \otimes \rho(g)\mathbf{v}_2 \otimes \cdots \otimes \rho(g)\mathbf{v}_n, \tag{1.12}$$

for $g \in G$. The exterior power $\Lambda^n(V)$ and the symmetric power $\operatorname{Sym}^n(V)$ are subrepresentations of $V^{\otimes n}$. Given the group homomorphism $\rho : G \to \mathrm{GL}(V)$, we defined the nth tensor power representation $\rho^{\otimes n} : G \to \mathrm{GL}\left(V^{\otimes n}\right)$ by (1.12). Now, the exterior power representation $\Lambda^n \rho : G \to \mathrm{GL}(\Lambda^n V)$, being a subrepresentation of $V^{\otimes n}$, can be defined as follows:

$$(\Lambda^n \rho)(g)(\mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \cdots \wedge \mathbf{v}_n) = \rho(g)\mathbf{v}_1 \wedge \rho(g)\mathbf{v}_2 \wedge \cdots \wedge \rho(g)\mathbf{v}_n. \tag{1.13}$$

One can now write down the group homomorphism $\operatorname{Sym}^n \rho : G \to \mathrm{GL}(\operatorname{Sym}^n V)$ associated with the subrepresentation $\operatorname{Sym}^n V$ of the representation $V^{\otimes n}$ of G:

$$(\operatorname{Sym}^n \rho)(g)(\mathbf{v}_1 \odot \mathbf{v}_2 \odot \cdots \odot \mathbf{v}_n) = \rho(g)\mathbf{v}_1 \odot \rho(g)\mathbf{v}_2 \odot \cdots \odot \rho(g)\mathbf{v}_n. \tag{1.14}$$

Now, let us define $\rho^* : G \to \mathrm{GL}(V^*)$, given $\rho : G \to \mathrm{GL}(V)$. Suppose $\{\mathbf{e}_i\}_{i=1}^m$ and $\{\widehat{\alpha}^i\}_{i=1}^m$ are bases of V and V^*, respectively. Here, $V^* = \operatorname{Hom}(V, \mathbb{C})$, the dual vector space of linear functionals on V. Any linear functional $\widehat{\omega} \in V^*$ can be written as

$$\widehat{\omega} = \sum_{i=1}^m \omega_i \widehat{\alpha}^i \tag{1.15}$$

Also, any vector $\mathbf{v} \in V$ can be written as

$$\mathbf{v} = \sum_{i=1}^m v^i \mathbf{e}_i. \tag{1.16}$$

In a given basis $\{\mathbf{e}_i\}_{i=1}^m$ of V and its dual basis $\{\widehat{\alpha}^i\}_{i=1}^m$ of V^*, $\omega \in V^*$ can be coordinated as a column vector $\begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_m \end{bmatrix}$, whereas a vector $\mathbf{v} \in V$ can be coordinated as $\begin{bmatrix} v^1 \\ v^2 \\ \vdots \\ v^m \end{bmatrix}$. We will simply denote the column vector $\begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_m \end{bmatrix}$ by $\widehat{\omega}$, and the column vector $\begin{bmatrix} v^1 \\ v^2 \\ \vdots \\ v^m \end{bmatrix}$ by $\mathbf{v}$. We then write the dual pairing

$$\langle \widehat{\omega}, \mathbf{v} \rangle = \widehat{\omega}(\mathbf{v}) = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_m \end{bmatrix}^T \begin{bmatrix} v^1 \\ v^2 \\ \vdots \\ v^m \end{bmatrix} = \widehat{\omega}^T \mathbf{v}. \tag{1.17}$$

Now, we want the dual representation V^* of V to satisfy

$$\langle \rho^* g(\widehat{\omega}), \rho(g)\mathbf{v} \rangle = \langle \widehat{\omega}, \mathbf{v} \rangle \tag{1.18}$$

for $g \in G$, $\mathbf{v} \in V$ and $\widehat{\omega} \in V^*$. Now, we claim that $\rho^* : V^* \to V^*$ defined by

$$\rho^*(g)(\widehat{\omega}) = \left[\rho\left(g^{-1}\right)\right]^T \widehat{\omega} \tag{1.19}$$

satisfies (1.18). Indeed,

$$\begin{aligned} \langle \rho^* g(\widehat{\omega}), \rho(g)\mathbf{v} \rangle &= \rho^* g(\widehat{\omega}) \left(\rho(g)\mathbf{v}\right) \\ &= \left[\rho\left(g^{-1}\right)\right]^T \widehat{\omega} \left[\rho(g)\mathbf{v}\right] \\ &= \widehat{\omega}\left(\rho\left(g^{-1}\right)\rho(g)\mathbf{v}\right) \\ &= \widehat{\omega}(\mathbf{v}) = \langle \widehat{\omega}, \mathbf{v} \rangle . \end{aligned}$$

Here we used the following definition of transpose: given a linear map $f : V \to W$, its transpose map $f^T : W^* \to V^*$ is defined as $f^T(\widehat{\omega})(\mathbf{v}) = \widehat{\omega}(f(\mathbf{v}))$. In light of this, we can also write (1.19) as

$$\rho^*(g)(\widehat{\omega})(\mathbf{v}) = \left[\rho\left(g^{-1}\right)\right]^T \widehat{\omega}(\mathbf{v}) = \widehat{\omega}\left(\rho\left(g^{-1}\right)\mathbf{v}\right). \tag{1.20}$$

Now, if V and W are representations of G, then so is $\mathrm{Hom}(V, W)$. In order to see this, we shall use the fact that

$$\mathrm{Hom}(V, W) \cong V^* \otimes W. \tag{1.21}$$

Note here that both V and W are finite dimensional complex vector spaces. Consider the group homomorphisms $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$. Now, the group homomorphism associated with dual representation on V^* of G is given by $\rho^* : G \to GL(V^*)$. Note that for $\widehat{\omega} \in V^*$, one has $\widehat{\omega}(\mathbf{e}_i) = \omega_i$, and for $\mathbf{v} \in V$, $\widehat{\alpha}^i(\mathbf{v}) = v^i$, where $\{\mathbf{e}_i\}_{i=1}^m$ is a basis for V and $\{\widehat{\alpha}^i\}_{i=1}^m$ is the dual basis for V^*. Note that $\widehat{\alpha}^i(\mathbf{e}_j) = \delta^i{}_j$.

Given $\varphi \in \mathrm{Hom}(V, W)$, define $\widetilde{g} : \mathrm{Hom}(V, W) \to V^* \otimes W$ by

$$\widetilde{g}(\varphi) = \sum_{i=1}^m \widehat{\alpha}^i \otimes \varphi(\mathbf{e}_i). \tag{1.22}$$

On the other hand, define $\widetilde{f} : V^* \otimes W \to \mathrm{Hom}(V, W)$ by

$$\widetilde{f}(\widehat{\kappa} \otimes \mathbf{w})(\mathbf{v}) = \widehat{\kappa}(\mathbf{v})\,\mathbf{w}, \tag{1.23}$$

where $\widehat{\kappa} \in V^*$, $\mathbf{v} \in V$, $\mathbf{w} \in W$. Then observe that $\widetilde{f}$ and $\widetilde{g}$ are inverses of each other. In fact,

$$\begin{aligned}
\widetilde{f}\left(\widetilde{g}\left(\varphi\right)\right)\left(\mathbf{v}\right) &= \widetilde{f}\left(\sum_{i=1}^{m} \widehat{\alpha}^{i} \otimes \varphi\left(\mathbf{e}_i\right)\right)\left(\mathbf{v}\right) \\
&= \sum_{i=1}^{m} \widetilde{f}\left(\widehat{\alpha}^{i} \otimes \varphi\left(\mathbf{e}_i\right)\right)\left(\mathbf{v}\right) \\
&= \sum_{i=1}^{m} \widehat{\alpha}^{i}\left(\mathbf{v}\right) \varphi\left(\mathbf{e}_i\right) \\
&= \sum_{i=1}^{m} v^{i} \varphi\left(\mathbf{e}_i\right) \\
&= \varphi\left(\sum_{i=1}^{m} v^{i} \mathbf{e}_i\right) \\
&= \varphi\left(\mathbf{v}\right).
\end{aligned}$$

Therefore,

$$\widetilde{f} \circ \widetilde{g} = \mathbb{1}_{\mathrm{Hom}(V,W)}\,. \tag{1.24}$$

Now, for a given $\widehat{\kappa} \otimes \mathbf{w} \in V^* \otimes W$,

$$\begin{aligned}
\widetilde{g}\left(\widetilde{f}\left(\widehat{\kappa} \otimes \mathbf{w}\right)\right) &= \sum_{i=1}^{m} \widehat{\alpha}^{i} \otimes \widetilde{f}\left(\widehat{\kappa} \otimes \mathbf{w}\right)\left(\mathbf{e}_i\right) \\
&= \sum_{i=1}^{m} \widehat{\alpha}^{i} \otimes \widehat{\kappa}\left(\mathbf{e}_i\right) \mathbf{w} \\
&= \sum_{i=1}^{m} \widehat{\kappa}\left(\mathbf{e}_i\right) \widehat{\alpha}^{i} \otimes \mathbf{w} \\
&= \sum_{i=1}^{m} \kappa_i \widehat{\alpha}^{i} \otimes \mathbf{w} \\
&= \widehat{\kappa} \otimes \mathbf{w}.
\end{aligned}$$

Therefore,

$$\widetilde{g} \circ \widetilde{f} = \mathbb{1}_{V^* \otimes W}\,. \tag{1.25}$$

(1.24) and (1.25) together imply that $\mathrm{Hom}\left(V, W\right) \cong V^* \otimes W$. We now define the representation of G on $\mathrm{Hom}(V, W)$ via the representation of G on $V^* \otimes W$. In fact, G acts on $V^* \otimes W$ via the map $\rho^* \otimes \sigma : G \to \mathrm{GL}\left(V^* \otimes W\right)$, so that $\left(\rho^* \otimes \sigma\right) g \left(\widehat{\kappa} \otimes \mathbf{w}\right) \in V^* \otimes W$. Then via $\widetilde{f} : V^* \otimes W \to \mathrm{Hom}(V, W)$, one has $\widetilde{f}\left(\left(\rho^* \otimes \sigma\right) g \left(\widehat{\kappa} \otimes \mathbf{w}\right)\right) \in \mathrm{Hom}(V, W)$. This is, by definition, the representation of G on $\mathrm{Hom}(V, W)$. In other words, $\gamma : G \to \mathrm{GL}\left(\mathrm{Hom}(V, W)\right)$ is defined by

$$\begin{aligned}
\gamma\left(g\right)\left(\widetilde{f}\left(\widehat{\kappa} \otimes \mathbf{w}\right)\right)\left(\mathbf{v}\right) &= \widetilde{f}\left(\left(\rho^* \otimes \sigma\right) g \left(\widehat{\kappa} \otimes \mathbf{w}\right)\right)\left(\mathbf{v}\right) \\
&= \widetilde{f}\left(\rho^*\left(g\right) \widehat{\kappa} \otimes \sigma(g)\mathbf{w}\right)\left(\mathbf{v}\right) \\
&= \left(\rho^*\left(g\right) \widehat{\kappa}\right)\left(\mathbf{v}\right) \sigma(g)\mathbf{w} \\
&= \widehat{\kappa}\left(\rho\left(g^{-1}\right)\mathbf{v}\right) \sigma(g)\mathbf{w} \\
&= \sigma\left(g\right)\left(\widehat{\kappa}\left(\rho\left(g^{-1}\right)\mathbf{v}\right)\mathbf{w}\right).
\end{aligned} \tag{1.26}$$

Now, let us write $\widetilde{f}\left(\widehat{\kappa} \otimes \mathbf{w}\right) = \varphi \in \mathrm{Hom}(V, W)$. So we have

$$\varphi\left(\mathbf{v}\right) = \widetilde{f}\left(\widehat{\kappa} \otimes \mathbf{w}\right)\left(\mathbf{v}\right) = \widehat{\kappa}\left(\mathbf{v}\right)\mathbf{w}. \tag{1.27}$$

As a result,

$$\varphi\left(\rho\left(g^{-1}\right)\mathbf{v}\right) = \widehat{\kappa}\left(\rho\left(g^{-1}\right)\mathbf{v}\right)\mathbf{w}. \tag{1.28}$$

(1.26) and (1.28) together imply that

$$(\gamma(g)\varphi)(\mathbf{v}) = \sigma(g)\left(\varphi\left(\rho\left(g^{-1}\right)\mathbf{v}\right)\right). \tag{1.29}$$

(1.29) can be expressed by means of the commutativity of the following diagram:

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "\rho(g)"'] & W \arrow[d, "\sigma(g)"] \\ V \arrow[r, "\gamma(g)\varphi"'] & W \end{tikzcd}$$

§1.4 Complete reducibility

Definition 1.7 (Hermitian inner product)**.** If V is a complex vector space, then a **Hermitian inner product** is a positive definite sesquilinear map $H : V \times V \to \mathbb{C}$ that satisfies the following:

(i) $H(a\mathbf{u} + b\mathbf{v}, \mathbf{w}) = \overline{a}H(\mathbf{u}, \mathbf{w}) + \overline{b}H(\mathbf{v}, \mathbf{w})$ and $H(\mathbf{w}, a\mathbf{u} + b\mathbf{v}) = aH(\mathbf{w}, \mathbf{u}) + bH(\mathbf{w}, \mathbf{v})$ for all $a, b \in \mathbb{C}$, $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$.

(ii) $H(\mathbf{u}, \mathbf{v}) = \overline{H(\mathbf{v}, \mathbf{u})}$, for all $\mathbf{u}, \mathbf{v} \in V$.

(iii) $H(\mathbf{u}, \mathbf{u}) > 0$, for every $\mathbf{u} \in V \setminus \{\mathbf{0}\}$ (positive definite).

If $W \subset V$ is a vector subspace of a complex vector space with a Hermitian inner product, we define the following subspace:

$$W^{\perp} = \{\mathbf{v} \in V \mid H(\mathbf{v}, \mathbf{w}) = 0, \text{ for all } \mathbf{w} \in W\}. \tag{1.30}$$

If V is a finite dimensional complex vector space, then we can write $V = W \oplus W^{\perp}$, i.e. $W^{\perp}$ is the the orthogonal complement of W. We also say that $W^{\perp}$ is the complementary subspace of V.

Definition 1.8. A Hermitian inner product H on a finite dimensional representation V of a finite group G ($\rho : G \to \mathrm{GL}(V)$) is said to be **preserved under group action** if

$$H(\rho(g)\mathbf{u}, \rho(g)\mathbf{w}) = H(\mathbf{u}, \mathbf{w}) \tag{1.31}$$

for all $g \in G$ and $\mathbf{u}, \mathbf{w} \in V$. H is then called a G**-invariant** Hermitian inner product.

If H is a G-invariant Hermitian inner product on a finite dimensional representation V of a finite group G, then we have

$$\begin{aligned} H(\rho(g)\mathbf{v}, \mathbf{w}) &= H\left(\rho(g)\mathbf{v}, \rho(g)\rho\left(g^{-1}\right)\mathbf{w}\right) \\ &= H\left(\mathbf{v}, \rho\left(g^{-1}\right)\mathbf{w}\right). \end{aligned} \tag{1.32}$$

Lemma 1.1

If $H : V \times V \to \mathbb{C}$ is a G-invariant Hermitian inner product on a finite dimensional representation V of a finite group G and $W \subset V$ is a subrepresentation, then $W^{\perp}$ is a G-invariant complement to W.

Proof. Since we are dealing with finite dimensional complex vector spaces, $W^{\perp}$ is a complement to W. It, therefore, suffices to show that $W^{\perp}$ is G-invariant.

Suppose $g \in G$, $\mathbf{u} \in W^\perp$, and $\mathbf{w} \in W$. Let us denote the group homomorphism associated with the finite dimensional complex representation by $\rho : G \to \mathrm{GL}(V)$. Since the Hermitian inner product $H : V \times V \to \mathbb{C}$ is G-invariant, one has

$$H(\rho(g)\mathbf{u}, \mathbf{w}) = H(\mathbf{u}, \rho(g^{-1})\mathbf{w}). \tag{1.33}$$

Since W is a subrepresentation of V, one must have $\rho(g^{-1})\mathbf{w} \in W$ for any $g \in G$ and $\mathbf{w} \in W$. Hence, $H(\mathbf{u}, \rho(g^{-1})\mathbf{w}) = 0$ in (1.33) leads to

$$H(\rho(g)\mathbf{u}, \mathbf{w}) = 0 \tag{1.34}$$

This is true for all $\mathbf{w} \in W$. Therefore, from the definition of $W^\perp$, one then must have $\rho(g)\mathbf{u} \in W^\perp$ for any $g \in G$, which then implies that the subspace $W^\perp$ is G-invariant. ■

Proposition 1.2

If V is a complex representation of a finite group G, then there is a G-invariant Hermitian inner product on V.

Proof. Pick a Hermitian inner product $H_0 : V \times V \to \mathbb{C}$ on the finite dimensional complex vector space V with respect to which a given basis of V is orthonormal, i.e., choose a basis $\{\mathbf{e}_i\}_{i=1}^m$ of V and define $H_0(\mathbf{e}_i, \mathbf{e}_j) = \delta_{ij}$ and extend H_0 to all of $V \times V$ sesquilinearly. Given $\mathbf{v} = \sum_{i=1}^m v^i \mathbf{e}_i$ and $\mathbf{w} = \sum_{j=1}^m w^j \mathbf{e}_j$, we then have

$$H_0(\mathbf{v}, \mathbf{w}) = H_0\left(\sum_{i=1}^m v^i \mathbf{e}_i, \sum_{j=1}^m w^j \mathbf{e}_j\right) = \sum_{i=1}^m \overline{v^i} w^i. \tag{1.35}$$

Then define a new Hermitian inner product $H_1 : V \times V \to \mathbb{C}$ by averaging over all of G via representation $\rho : G \to \mathrm{GL}(V)$:

$$H_1(\mathbf{v}, \mathbf{w}) = \frac{1}{|G|} \sum_{g \in G} H_0(\rho(g)\mathbf{v}, \rho(g)\mathbf{w}). \tag{1.36}$$

Using the Hermitian inner product properties of H_0, one can verify that H_1 is also a Hermitian inner product on V. Additionally,

$$\begin{aligned} H_1(\rho(h)\mathbf{v}, \rho(h)\mathbf{w}) &= \frac{1}{|G|} \sum_{g \in G} H_0(\rho(g)\rho(h)\mathbf{v}, \rho(g)\rho(h)\mathbf{w}) \\ &= \frac{1}{|G|} \sum_{g \in G} H_0(\rho(gh)\mathbf{v}, \rho(gh)\mathbf{w}) \\ &= \frac{1}{|G|} \sum_{g' \in G} H_0(\rho(g')\mathbf{v}, \rho(g')\mathbf{w}) \quad (\text{where } g' = gh) \\ &= H_1(\mathbf{v}, \mathbf{w}). \end{aligned} \tag{1.37}$$

Then (1.37) implies that the Hermitian inner product $H_1 : V \times V \to \mathbb{C}$ defined by (1.36) on V is G-invariant. ■

Corollary 1.3

If W is a subrepresentation of a finite dimensional complex representation V of a finite group G, then there exists a complementary invariant subspace $W^\perp$ of V so that $V = W \oplus W^\perp$.

Proof. Given that V is a complex representation of a finite group G, there is a G-invariant Hermitian inner product on V by Proposition 1.2. Now, if W is a subrepresentation of V, then by Lemma 1.1, the complementary subspace $W^\perp$ is G-invariant, i.e., $V = W \oplus W^\perp$. ■

Corollary 1.4 (Maschke's theorem)
Any complex representation of a finite group can be expressed as a direct sum of irreducible representations.

Remark 1.1. The property of a representation being expressed as a direct sum of irreducibles is called complete reducibility (semisimplicity). Maschke's theorem tells us that any complex representation of a finite group is semisimple. The additive group $\mathbb{R}$, being an infinite group, doesn't have this property; for example, the representation

$$a \mapsto \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$$

is not semisimple.

The extent to which the decomposition of an arbitrary complex representation into a direct sum of irreducibles is unique is one of the consequences of the following.

Lemma 1.5 (Schur's lemma)
Recall that $\mathrm{Hom}_G(V, W)$ is the vector space of G-linear maps between two finite dimensional complex representations V and W of the finite group G. Suppose V and W are irreducible complex representations of G. Then

(a) Every element of $\mathrm{Hom}_G(V, W)$ is either 0 or an isomorphism.

(b) $\dim_{\mathbb{C}} \mathrm{Hom}_G(V, W) = 0$ or 1.

Proof. (a) Let $\varphi : V \to W$ be a non-zero G-linear map. We have verified in (1.1) that $\mathrm{Ker}\, \varphi \subseteq V$ is a G-invariant subspace of V. Since V is irreducible, by hypothesis, one has

$$\mathrm{Ker}\, \varphi = \{\mathbf{0}\}\,, \tag{1.38}$$

because $\mathrm{Ker}\, \varphi \neq V$, as φ is chosen to be nonzero.

We also know from (1.2) that $\mathrm{im}\, \varphi \subseteq W$ is a G-invariant subspace of W, i.e., $\mathrm{Im}\varphi$ is a subrepresentation of W. Since W is also irreducible, by hypothesis, one must have

$$\mathrm{im}\, \varphi = W, \tag{1.39}$$

because $\mathrm{im}\, \varphi \neq \{\mathbf{0}\}$ as φ is chosen to be nonzero.

Now, $\mathrm{Ker}\, \varphi = \{\mathbf{0}\}$ and $\mathrm{im}\, \varphi = W$ together imply that $\varphi : V \to W$ is a bijective linear map from V to W, i.e., φ is an isomorphism between vector spaces.

(b) Suppose $\varphi_1, \varphi_2 \in \mathrm{Hom}_G(V, W)$ with both being nonzero. Then by (a), φ_1 and φ_2 are both isomorphisms. Since $\varphi_1^{-1} : W \to V$ and $\varphi_2 : V \to W$, one can compose them to obtain $\varphi = \varphi_1^{-1} \circ \varphi_2 \in \mathrm{Hom}_G(V, V)$.

Now, $\varphi : V \to V$ is a linear operator on the finite dimensional complex vector space V. Also, since $\mathbb{C}$ is algebraically closed, $\det(\varphi - \lambda\, \mathbb{1}_V) = 0$ has a solution (here $\varphi - \lambda\, \mathbb{1}_V$ is considered a square matrix) which implies that $\mathrm{Ker}(\varphi - \lambda\, \mathbb{1}_V) \neq \{\mathbf{0}\}$, i.e., $\varphi - \lambda\, \mathbb{1}_V$ is not an isomorphism belonging to the vector space $\mathrm{Hom}_G(V, V)$. Then, by (a), one concludes that $\varphi - \lambda\, \mathbb{1}_V$ must be the 0-map in $\mathrm{Hom}_G(V, V)$, i.e.,

$$\varphi = \varphi_1^{-1} \circ \varphi_2 = \lambda\, \mathbb{1}_V\,.$$

In other words, $\varphi_2 = \lambda \varphi_1$. Since this is true for any pair of G-linear maps $\varphi_1, \varphi_2 \in \mathrm{Hom}_G(V, W)$, we have $\dim_{\mathbb{C}} \mathrm{Hom}_G(V, W) = 1$.

∎

Lemma 1.6

Suppose V_1, V_2, W are finite dimensional complex representation of the finite group G. Then one has the following vector space isomorphisms:

$$\begin{aligned} \mathrm{Hom}_G(V_1 \oplus V_2, W) &\cong \mathrm{Hom}_G(V_1, W) \oplus \mathrm{Hom}_G(V_2, W), \\ \mathrm{Hom}_G(W, V_1 \oplus V_2) &\cong \mathrm{Hom}_G(W, V_1) \oplus \mathrm{Hom}_G(W, V_2). \end{aligned}$$

Proof. Following are the required linear maps that can easily be verified to be isomorphisms:

$$\begin{aligned} s : \mathrm{Hom}_G(V_1, W) \oplus \mathrm{Hom}_G(V_2, W) &\to \mathrm{Hom}_G(V_1 \oplus V_2, W), \\ s(\varphi_1, \varphi_2)(\mathbf{v}_1, \mathbf{v}_2) &= \varphi_1(\mathbf{v}_1) + \varphi_2(\mathbf{v}_2). \end{aligned} \tag{1.40}$$

$$\begin{aligned} u : \mathrm{Hom}_G(V_1 \oplus V_2, W) &\to \mathrm{Hom}_G(V_1, W) \oplus \mathrm{Hom}_G(V_2, W) \\ u(\varphi) &= (\varphi \circ i_1, \varphi \circ i_2), \end{aligned} \tag{1.41}$$

where $i_1 : V_1 \to V_1 \oplus V_2$ and $i_2 : V_2 \to V_1 \oplus V_2$ are the canoncial inclusions defined by

$$i_1(\mathbf{v}_1) = (\mathbf{v}_1, \mathbf{0}_{V_2}) \quad \text{and} \quad i_2(\mathbf{v}_2) = (\mathbf{0}_{V_1}, \mathbf{v}_2).$$

Now, one can check that $u \circ s = \mathbb{1}_{\mathrm{Hom}_G(V_1, W) \oplus \mathrm{Hom}_G(V_2, W)}$ and $s \circ u = \mathbb{1}_{\mathrm{Hom}_G(V_1 \oplus V_2, W)}$. Indeed,

$$\begin{aligned} (u \circ s)(\varphi_1, \varphi_2) &= u(s(\varphi_1, \varphi_2)) \\ &= (s(\varphi_1, \varphi_2) \circ i_1, s(\varphi_1, \varphi_2) \circ i_2). \end{aligned}$$

Now,

$$\begin{aligned} (s(\varphi_1, \varphi_2) \circ i_1)(\mathbf{v}_1) &= s(\varphi_1, \varphi_2)(i_1(\mathbf{v}_1)) \\ &= s(\varphi_1, \varphi_2)(\mathbf{v}_1, \mathbf{0}_{V_2}) \\ &= \varphi_1(\mathbf{v}_1) + \varphi_2(\mathbf{0}_{V_2}) \\ &= \varphi_1(\mathbf{v}_1). \end{aligned}$$

Therefore, $s(\varphi_1, \varphi_2) \circ i_1 = \varphi_1$. Similarly, $s(\varphi_1, \varphi_2) \circ i_2 = \varphi_2$. Hence,

$$(u \circ s)(\varphi_1, \varphi_2) = (s(\varphi_1, \varphi_2) \circ i_1, s(\varphi_1, \varphi_2) \circ i_2) = (\varphi_1, \varphi_2).$$

So we have

$$u \circ s = \mathbb{1}_{\mathrm{Hom}_G(V_1, W) \oplus \mathrm{Hom}_G(V_2, W)}. \tag{1.42}$$

On the other hand, given $\varphi \in \mathrm{Hom}_G(V_1 \oplus V_2, W)$,

$$\begin{aligned} [(s \circ u)(\varphi)](\mathbf{v}_1, \mathbf{v}_2) &= [s(\varphi \circ i_1, \varphi \circ i_2)](\mathbf{v}_1, \mathbf{v}_2) \\ &= (\varphi \circ i_1)(\mathbf{v}_1) + (\varphi \circ i_2)(\mathbf{v}_2) \\ &= \varphi(\mathbf{v}_1, \mathbf{0}_{V_2}) + \varphi(\mathbf{0}_{V_1}, \mathbf{v}_2) \\ &= \varphi(\mathbf{v}_1, \mathbf{v}_2). \end{aligned}$$

Therefore,

$$s \circ u = \mathbb{1}_{\mathrm{Hom}_G(V_1 \oplus V_2, W)}. \tag{1.43}$$

So $s : \mathrm{Hom}_G(V_1, W) \oplus \mathrm{Hom}_G(V_2, W) \to \mathrm{Hom}_G(V_1 \oplus V_2, W)$ is an isomorphism.

Now consider the following linear maps

$$\begin{aligned} t : \mathrm{Hom}_G(W, V_1) \oplus \mathrm{Hom}_G(W, V_2) &\to \mathrm{Hom}_G(W, V_1 \oplus V_2) \\ t(\varphi_1, \varphi_2)(\mathbf{w}) &= (\varphi_1(\mathbf{w}), \varphi_2(\mathbf{w})). \end{aligned} \tag{1.44}$$

$$\begin{aligned} v : \mathrm{Hom}_G(W, V_1 \oplus V_2) &\to \mathrm{Hom}_G(W, V_1) \oplus \mathrm{Hom}_G(W, V_2) \\ v(\varphi) &= (q_1 \circ \varphi, q_2 \circ \varphi), \end{aligned} \tag{1.45}$$

where $q_1 : V_1 \oplus V_2 \to V_1$ and $q_2 : V_1 \oplus V_2 \to V_2$ are the canonical projections, defined by

$$q_1(\mathbf{v}_1, \mathbf{v}_2) = \mathbf{v}_1 \quad \text{and} \quad q_2(\mathbf{v}_1, \mathbf{v}_2) = \mathbf{v}_2.$$

Now, one can check that $v \circ t = \mathbb{1}_{\mathrm{Hom}_G(W,V_1)\oplus\mathrm{Hom}_G(W,V_2)}$ and $t \circ v = \mathbb{1}_{\mathrm{Hom}_G(W,V_1\oplus V_2)}$. Indeed,

$$\begin{aligned} (v \circ t)(\varphi_1, \varphi_2) &= v(t(\varphi_1, \varphi_2)) \\ &= (q_1 \circ t(\varphi_1, \varphi_2), q_2 \circ t(\varphi_1, \varphi_2)). \end{aligned}$$

Now,

$$\begin{aligned} (q_1 \circ t(\varphi_1, \varphi_2))(\mathbf{w}) &= q_1 [t(\varphi_1, \varphi_2)\mathbf{w}] \\ &= q_1(\varphi_1(\mathbf{w}), \varphi_2(\mathbf{w})) \\ &= \varphi_1(\mathbf{w}). \end{aligned}$$

Therefore, $q_1 \circ t(\varphi_1, \varphi_2) = \varphi_1$. Similarly, $q_2 \circ t(\varphi_1, \varphi_2) = \varphi_2$. Hence,

$$(v \circ t)(\varphi_1, \varphi_2) = (q_1 \circ t(\varphi_1, \varphi_2), q_2 \circ t(\varphi_1, \varphi_2)) = (\varphi_1, \varphi_2).$$

So we have

$$v \circ t = \mathbb{1}_{\mathrm{Hom}_G(W,V_1)\oplus\mathrm{Hom}_G(W,V_2)}. \tag{1.46}$$

On the other hand, given $\varphi \in \mathrm{Hom}_G(W, V_1 \oplus V_2)$, let $\varphi(\mathbf{w}) = (\mathbf{v}_1, \mathbf{v}_2)$. Then

$$\begin{aligned} [(t \circ v)(\varphi)](\mathbf{w}) &= t(q_1 \circ \varphi, q_2 \circ \varphi)(\mathbf{w}) \\ &= ((q_1 \circ \varphi)(\mathbf{w}), (q_2 \circ \varphi)(\mathbf{w})) \\ &= (\mathbf{v}_1, \mathbf{v}_2) = \varphi(\mathbf{w}). \end{aligned}$$

Therefore,

$$t \circ v = \mathbb{1}_{\mathrm{Hom}_G(W,V_1\oplus V_2)}. \tag{1.47}$$

So $t : \mathrm{Hom}_G(W, V_1) \oplus \mathrm{Hom}_G(W, V_2) \to \mathrm{Hom}_G(W, V_1 \oplus V_2)$ is an isomorphism. ■

Now, let G be a finite group and V be a finite dimensional complex representation of G. Since V is a direct sum of irreducible representations by Maschke's theorem, up to isomorphism we can group together the isomorphic representations and say that

$$V \cong V_1^{r_1} \oplus \cdots \oplus V_m^{r_m} \tag{1.48}$$

Here $V_i^{r_i}$ is the shorthand for r_i fold direct sum of V_i with itself.

$$V_i^{r_i} = \underbrace{V_i \oplus V_i \oplus \cdots \oplus V_i}_{r_i\text{-fold direct sum}}. \tag{1.49}$$

Here, for distinct i and j, V_i and V_j are non-isomorphic, and the integers $r_i \geq 1$.

Remark 1.2. While grouping together in (1.48), we are grouping isomorphic representations together, NOT isomorphic vector spaces. V_1 and V_2 may be isomorphic as vector spaces, but we don't group them together unless they are isomorphic representations. In other words, if $\rho : G \to \mathrm{GL}(V)$ is the said representation of G into V, we group two irreducible subrepresentations W_1 and W_2 together while writing (1.48) if there exists a vector space isomorphism $\psi : W_1 \to W_2$ such that the following diagram commutes for every $g \in G$:

$$\begin{tikzcd} W_1 \arrow[r, "\psi"] \arrow[d, "\rho(g)|_{W_1}"'] & W_2 \arrow[d, "\rho(g)|_{W_2}"] \\ W_1 \arrow[r, "\psi"'] & W_2 \end{tikzcd}$$

When we say V_i and V_j are not isomorphic for $i \neq j$ in (1.48), we mean that they are not isomorphic as representations, i.e. there is no isomorphism in $\mathrm{Hom}_G(V_i, V_j)$. In principle, they can be isomorphic as vector spaces, but that's not our concern here.

Proposition 1.7

In (1.48), $r_i = \dim_{\mathbb{C}} \mathrm{Hom}_G(V_i, V) = \dim_{\mathbb{C}} \mathrm{Hom}_G(V, V_i)$.

Proof. By Lemma 1.6,

$$\mathrm{Hom}_G(V_i, V) \cong \mathrm{Hom}_G\left(V_i, \bigoplus_{j=1}^{m} V_j^{r_j}\right) \cong \bigoplus_{j=1}^{m} \mathrm{Hom}_G\left(V_i, V_j^{r_j}\right). \tag{1.50}$$

But $\mathrm{Hom}_G\left(V_i, V_j^{r_j}\right)$ is

$$\mathrm{Hom}_G\left(V_i, V_j^{r_j}\right) = \mathrm{Hom}_G\left(V_i, \underbrace{V_j \oplus \cdots \oplus V_j}_{r_j\text{-fold direct sum}}\right) \cong \underbrace{\mathrm{Hom}_G(V_i, V_j) \oplus \cdots \oplus \mathrm{Hom}_G(V_i, V_j)}_{r_j\text{-fold direct sum}}. \tag{1.51}$$

Since V_i's are pairwise non-isomorphic for $j \neq i$, we have $\mathrm{Hom}_G(V_i, V_j) = \{\mathbf{0}\}$, so that

$$\dim_{\mathbb{C}} \mathrm{Hom}_G(V_i, V_j) = 0 \quad \text{and} \quad \dim_{\mathbb{C}} \mathrm{Hom}_G(V_i, V_j) = 1. \tag{1.52}$$

So we have

$$\begin{aligned} \dim_{\mathbb{C}} \mathrm{Hom}_G(V_i, V) &= \dim_{\mathbb{C}} \left(\bigoplus_{j=1}^{m} \mathrm{Hom}_G\left(V_i, V_j^{r_j}\right)\right) \\ &= \sum_{j=1}^{m} \dim_{\mathbb{C}} \mathrm{Hom}_G\left(V_i, V_j^{r_j}\right) \\ &= \dim_{\mathbb{C}} \mathrm{Hom}_G\left(V_i, V_i^{r_i}\right) \\ &= \dim_{\mathbb{C}} \left(\underbrace{\mathrm{Hom}_G(V_i, V_i) \oplus \cdots \oplus \mathrm{Hom}_G(V_i, V_i)}_{r_i\text{-fold direct sum}}\right) \\ &= \underbrace{1 + 1 + \cdots + 1}_{r_i\text{-fold sum}} \\ &= r_i. \end{aligned} \tag{1.53}$$

Similarly, $\dim_{\mathbb{C}} \mathrm{Hom}_G(V, V_i) = r_i$. ∎

Proposition 1.8

The decomposition (1.48) is unique up to replacement of each V_i by an isomorphic representation.

Proof. Suppose

$$V \cong V_1^{r_1} \oplus \cdots \oplus V_m^{r_m} \cong W_1^{s_1} \oplus \cdots \oplus W_n^{s_n} \tag{1.54}$$

are two decompositions into non-isomorphic irreducible representations of G. By Proposition 1.7, for $i_0 \in \{1, 2, \ldots, m\}$,

$$\begin{aligned} r_{i_0} &= \dim_{\mathbb{C}} \operatorname{Hom}_G (V_{i_0}, V) \\ &= \dim_{\mathbb{C}} \operatorname{Hom}_G \left(V_{i_0}, \bigoplus_{j=1}^{n} W_j^{s_j} \right) \\ &= \dim_{\mathbb{C}} \left(\bigoplus_{j=1}^{n} \operatorname{Hom}_G \left(V_{i_0}, W_j^{s_j} \right) \right) \\ &= \sum_{j=1}^{n} s_j \dim \operatorname{Hom}_G (V_{i_0}, W_j) . \end{aligned} \tag{1.55}$$

Since $r_{i_0} > 0$, there must exist some $j_0 \in \{1, 2, \ldots, n\}$ such that $\operatorname{Hom}_G (V_{i_0}, W_{j_0}) \neq \{\mathbf{0}\}$, i.e. it is nontrivial. Then by Schur's lemma, $W_{j_0} \cong V_{i_0}$. The j_0 must also be unique because W_j's are pairwise non-isomorphic. In other words, the only nonvanishing contribution in the sum (1.55) is due to the unique value $j = j_0$, for which

$$\dim_{\mathbb{C}} \operatorname{Hom}_G (V_{i_0}, W_{j_0}) = 1 \quad \text{and} \quad \dim_{\mathbb{C}} \operatorname{Hom}_G (V_{i_0}, W_j) = 0 \text{ for } j \neq j_0. \tag{1.56}$$

Hence, by (1.55) and (1.56), $r_{i_0} = s_{j_0}$. Thus we have an injection $\sigma : \{1, 2, \ldots, m\} \to \{1, 2, \ldots, n\}$ such that $V_{i_0} \cong W_{j_0} = W_{\sigma(i_0)}$ and $r_{i_0} = s_{j_0} = s_{\sigma(i_0)}$ for each i_0.

In a similar manner, interchanging V_i and W_j throughout above, we have an injection $\tau : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, m\}$ such that $W_{j_0} \cong V_{\tau(j_0)}$ and $s_{j_0} = r_{\tau(i_0)}$ for each j_0. The first injection σ implies that $m \leq n$. The latter injection τ gives $n \leq m$. Therefore, $m = n$, and σ and τ are permutations, i.e. $\sigma \in \mathfrak{S}_n$. Hence, (1.48) is unique up to replacement of each V_{i_0} by an isomorphic representation W_{j_0}. ■

Corollary 1.9
The irreducible complex representations of a finite abelian group G are all 1-dimensional.

Proof. Let V be a complex irreducible representation of a finite group G and $\rho : G \to \mathrm{GL}(V)$ be the underlying group homomorphism. Then, for each $g \in G$, the map $\rho(g) : V \to V$ is G-linear:

$$\begin{tikzcd} V \arrow[r, "\rho(g)"] \arrow[d, "\rho(h)"'] & V \arrow[d, "\rho(h)"] \\ V \arrow[r, "\rho(g)"'] & V \end{tikzcd}$$

The diagram above is commutative for all $h \in G$ for a given $g \in G$. Indeed,

$$\rho(g)\rho(h) = \rho(gh) = \rho(hg) = \rho(h)\rho(g).$$

We, therefore, have $\rho(g) \in \operatorname{Hom}_G(V, V)$. By Schur's lemma, $\dim_{\mathbb{C}} \operatorname{Hom}_G(V, V) = 1$, so $\rho(g) = \lambda_g \mathbb{1}_V$ for some $\lambda_g \in \mathbb{C}$.

Now, choose a non-zero vector $\mathbf{v} \in V$ and consider the 1-dimensional subspace

$$\langle \mathbf{v} \rangle = \mathbb{C}\mathbf{v} \subset V,$$

by taking all complex multiples of the nonzero vector $\mathbf{v}$. Observe that $\langle \mathbf{v} \rangle$ is G-invariant. Indeed,

$$\rho(g)\mathbf{v} = \lambda_g \mathbb{1}_V \mathbf{v} = \lambda_g \mathbf{v} \in \langle \mathbf{v} \rangle ,$$

i.e. $\langle \mathbf{v} \rangle$ is a G-invariant subspace of V, i.e. a subrepresentation. But V is irreducible by hypothesis. Hence, $\langle \mathbf{v} \rangle = V$. In other words, V is 1-dimensional. ■

Definition 1.9 (Faithful representation)**.** A complex representation V of a finite group G is called **faithful** if the homomorphism $\rho : G \to \mathrm{GL}(V)$ is injective.

Corollary 1.10

If G has a faithful complex irreducible representation, then $Z(G)$ is cyclic.

Proof. Let $\rho : G \to \mathrm{GL}(V)$ be the injective group homomorphism associated with a faithful irreducible complex representation V of a finite group G. Now, let $z \in Z(G)$ so that $zg = gz$ for all $g \in G$. Now consider the map $\rho(z) : V \to V$. Since z commutes with all $g \in G$, the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "\rho(z)"] \arrow[d, "\rho(g)"'] & V \arrow[d, "\rho(g)"] \\ V \arrow[r, "\rho(z)"'] & V \end{tikzcd}$$

Hence, $\rho(z) \in \mathrm{Hom}_G(V, V)$. By Schur's lemma, $\dim_{\mathbb{C}} \mathrm{Hom}_G(V, V) = 1$, so $\rho(z) = \lambda_z \mathbb{1}_V$ for some $\lambda_z \in \mathbb{C}^\times := \mathbb{C} \setminus \{0\}$.

Now, the map $Z(G) \to \mathbb{C}^\times = \mathrm{GL}(\mathbb{C})$ given by $z \mapsto \lambda_z$ is a representation of the subgroup $Z(G)$ of G. Moreover, this representation is faithful, because

$$\begin{aligned} \lambda_z = \lambda_{z'} &\implies \lambda_z \mathbb{1}_V = \lambda_{z'} \mathbb{1}_V \\ &\implies \rho(z) = \rho(z') \\ &\implies z = z', \end{aligned}$$

since ρ is injective. Therefore, the map $Z(G) \to \mathbb{C}^\times = \mathrm{GL}(\mathbb{C})$ given by $z \mapsto \lambda_z$ is injective. So $Z(G)$ is isomorphic to a finite subgroup of $\mathbb{C}^\times$. Finite subgroups of the multiplicative group of a field is a cyclic group. Hence, $Z(G)$ is cyclic. ∎

One also knows from elementary group theory that every finite abelian group is isomorphic to a direct product of cyclic groups. In other words, if G is a finite abelian group, then we can write G as

$$G = C_{n_1} \times \cdots \times C_{n_r}, \tag{1.57}$$

where each C_{n_i} is a cyclic group of order n_i.

Proposition 1.11

A finite abelian group G has precisely $|G|$-many irreducible complex representations.

Proof. We write G as a direct product of cyclic groups as follows:

$$G = \langle x_1 \rangle \times \cdots \times \langle x_r \rangle, \tag{1.58}$$

where $|\langle x_j \rangle| = n_j$, and x_j generates the cyclic group $\langle x_j \rangle$. Suppose $\rho : G \to \mathbb{C}^\times$ is an irreducible representation of the finite abelian group G (which is 1-dimensional by Corollary **??**). Let

$$\rho(e_1, \ldots, e_{j-1}, x_j, e_{j+1}, \ldots, e_r) = \lambda_j \in \mathbb{C}^\times, \tag{1.59}$$

where e_k's are the identity elements of the cyclic group $C_{n_k} = \langle x_k \rangle$. Since $x_j^{n_j} = e_j$, and since $\rho : G \to \mathbb{C}^\times$ is a group homomorphism, one must have

$$1 = \rho(e_1, \ldots, e_r) = \rho\left(e_1, \ldots, e_{j-1}, x_j^{n_j}, e_{j+1}, \ldots, e_r\right) = \lambda_j^{n_j}. \tag{1.60}$$

Then $\lambda_j^{n_j} = 1$ gives us that λ_j is a n_j-th root of unity. Also, observe that

$$\rho\left(x_1^{j_1}, \dots, x_r^{j_r}\right) = \lambda_1^{j_1} \cdots \lambda_r^{j_r}, \tag{1.61}$$

for $1 \leq j_k \leq n_k$ for each k. Thus, the r-tuple $(\lambda_1, \dots, \lambda_r)$ completely determines the homomorphism $\rho : G \to \mathbb{C}^\times$. There are n_j many n_j-th root of unity, so there are n_j many choices for λ_j. Hence, there are total $n_1 \cdots n_r$ many choices for the r-tuple $(\lambda_1, \dots, \lambda_r)$. Therefore, there are $n_1 \cdots n_r$ many irreducible representations $\rho : G \to \mathbb{C}^\times$. But

$$|G| = |\langle x_1 \rangle \times \cdots \times \langle x_r \rangle| = \prod_{j=1}^{r} |\langle x_j \rangle| = \prod_{j=1}^{r} n_j. \tag{1.62}$$

Hence, there are $|G|$ many irreducible complex representation of the finite abelian group G. ■

Example 1.1 (Example of finite abelian group representations). (i) Consider the finite abelian group $G = C_2 \times C_2 = \langle x_1 \rangle \times \langle x_2 \rangle$, with $x_1^2 = e_1$ and $x_2^2 = e_2$.[1]

We are concerned with the 2nd roots of unity, namely 1 and -1. There are 4 possible choices for (λ_1, λ_2), they are $(1,1), (1,-1), (-1,1), (-1,-1)$. Corresponding to these 4 choices, there are 4 irreducible representations $\rho_1, \rho_2, \rho_3, \rho_4$. The way these 4 irreducible representations map is illustrates in the following table:

(λ_1, λ_2)	(e_1, e_2)	(x_1, e_2)	(e_1, x_2)	(x_1, x_2)
$\rho_1 \equiv (1,1)$	1	1	1	1
$\rho_2 \equiv (1,-1)$	1	1	-1	-1
$\rho_3 \equiv (-1,1)$	1	-1	1	-1
$\rho_4 \equiv (-1,-1)$	1	-1	-1	1

From this table, we can see that there is no irreducible faithful representation of G.

(ii) Now consider the cyclic group $G = C_4 = \langle x \rangle$. This group has 4 elements: e, x, x^2, x^3, and $x^4 = e$. There are 4 roots of unity, namely $1, -1, i, -i$. Corresponding to these 4 roots of unity, there are 4 irreducible representations $\rho_1, \rho_2, \rho_3, \rho_4$. The way these 4 irreducible representations map is illustrates in the following table:

λ	e	x	x^2	x^3
$\rho_1 \equiv 1$	1	1	1	1
$\rho_2 \equiv -1$	1	-1	1	-1
$\rho_3 \equiv i$	1	i	-1	$-i$
$\rho_4 \equiv -i$	1	$-i$	-1	i

From the table, we can see that ρ_3 and ρ_4 are faithful.

[1] This is the Klein four-group. Geometrically, it represents the group of all symmetries of a non-square rectangle.