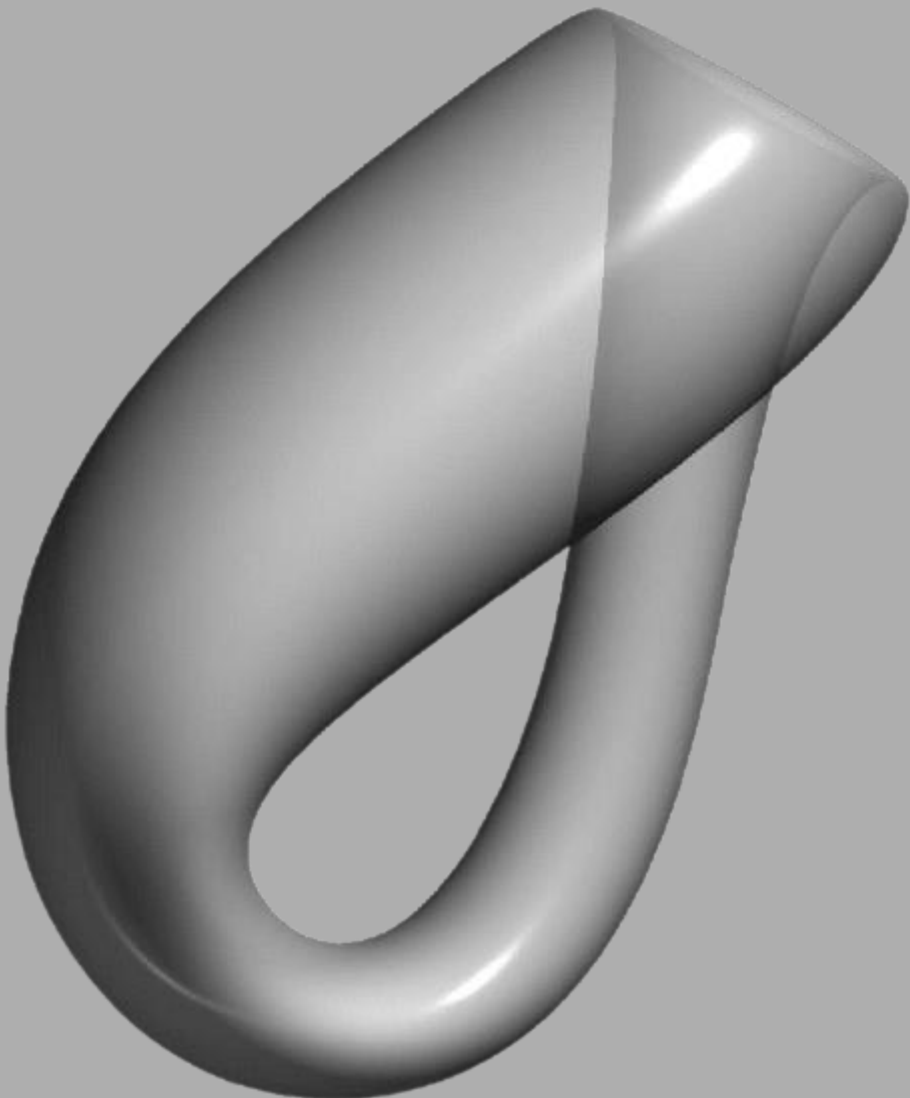
Lectures on
Differential Geometry



Department of Mathematics and Natural Sciences,
Brac University

Lectures on Differential Geometry

Preface

This series of lecture notes has been prepared for aiding students who took the BRAC University course **Differential Geometry I and II (MAT313 and MAT401)**. These notes were typeset by Atonu Roy Chowdhury under the supervision of Professor Syed Hasibul Hassan Chowdhury, PhD.

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I

Part 1

1 Manifolds

§1.1 Topological Manifolds

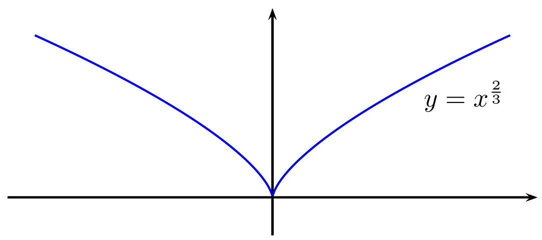
Definition 1.1 (Locally Euclidean Space)**.** A topological space M is **locally Euclidean** of dimension n if every point in M has a neighborhood U such that there is a homeomorphism φ from U onto an open subset of $\mathbb{R}^n$. We call the pair $(U, \varphi : U \to \mathbb{R}^n)$ a **chart**, U a **coordinate neighborhood** and φ a **coordinate system** on U. We also say that a chart (U, φ) is centered at $p \in U$ if $\varphi(p) = \vec{0}$.

Definition 1.2 (Topological Manifold)**.** A **topological manifold** of dimension n is a Hausdorff, second countable, locally Euclidean space of dimension n.

Example 1.1. The Euclidean space $\mathbb{R}^n$ is covered by a single chart $(\mathbb{R}^n, \mathbb{1}_{\mathbb{R}^n})$, where $\mathbb{1}_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n$ is the identity map. Every open subset U of $\mathbb{R}^n$ is also a topological manifold with the chart $(U, \mathbb{1}_U)$.

Recall that the hausdorff condition and second countability are "hereditary properties". That is, they are inherited by subspaces: a subspace of a Hausdorff space is also Hausdorff, and a subspace of a second countable space is also second countable. Hence, any subspace of $\mathbb{R}^n$ is Hausdorff and second countable.

Example 1.2 (The Cusp)**.** The graph of $y = x^{\frac{2}{3}}$ in $\mathbb{R}^2$ is a topological manifold.

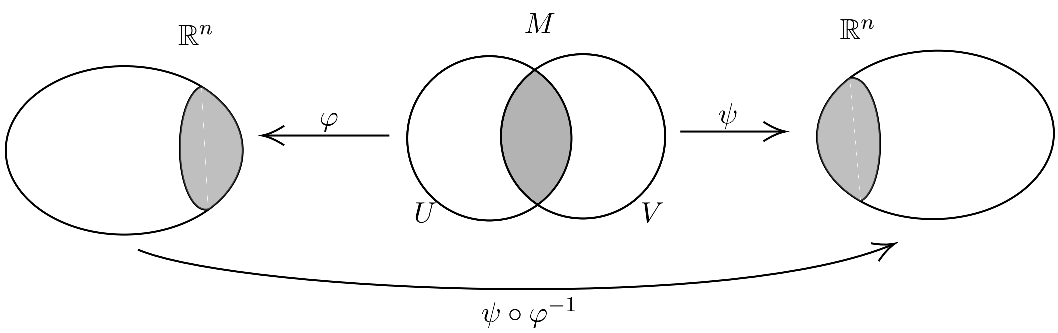


As a subspace of $\mathbb{R}^2$, it is Hausdorff and second countable. It is locally Euclidean, because it is homeomorphic to $\mathbb{R}$ via the map $\left(x, x^{2/3}\right) \mapsto x$. This map is continuous since it is just the projection onto first coordinate. The inverse map $x \mapsto \left(x, x^{2/3}\right)$ is continuous, as both $x \mapsto x$ and $x \mapsto x^{2/3}$ are continuous.

Definition 1.3 (Compatible Charts)**.** Two charts $(U, \varphi : U \to \mathbb{R}^n)$ and $(V, \psi : V \to \mathbb{R}^n)$ of a topological manifold are $\boldsymbol{C^\infty}$**-compatible** if the two maps

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V) \quad \text{and} \quad \varphi \circ \psi^{-1} : \psi(U \cap V) \to \varphi(U \cap V)$$

are both C^∞. These two maps are called **transition functions** between the charts. If $U \cap V$ is empty, then the two charts are automatically compatible.



> **Definition 1.4** (Atlas). A $\boldsymbol{C^\infty}$**-atlas** or simply an **atlas** on a locally Euclidean space M is a collection $\mathscr{U} = \{(U_\alpha, \varphi_\alpha)\}$ of pairwise C^∞-compatible charts that cover M. In other words,
>
> $$M = \bigcup_\alpha U_\alpha \,.$$

Example 1.3. The unit circle S^1 in the complex plane can be described as the set of points $\{e^{it} \in \mathbb{C} \mid 0 \le t < 2\pi\}$. Let U_1 and U_2 be the following two open subsets of S^1:

$$U_1 = \left\{e^{it} \in \mathbb{C} \mid -\pi < t < \pi\right\} \text{ and } U_2 = \left\{e^{it} \in \mathbb{C} \mid 0 < t < 2\pi\right\} .$$

Define $\varphi_i : U_i \to \mathbb{R}$ by

$$\begin{aligned} \varphi_1\left(e^{it}\right) &= t\,, \quad -\pi < t < \pi\,; \\ \varphi_2\left(e^{it}\right) &= t\,, \quad 0 < t < 2\pi\,. \end{aligned}$$

(U_1, φ_1) and (U_2, φ_2) are charts on S^1. Their intersection $U_1 \cap U_2$ consists of two disjoint subsets of S^1 denoted by A and B.

$$A = \left\{e^{it} \in \mathbb{C} \mid -\pi < t < 0\right\} \text{ and } B = \left\{e^{it} \in \mathbb{C} \mid 0 < t < \pi\right\} .$$

$U_1 \cap U_2 = A \sqcup B$. Now,

$$\begin{aligned} \varphi_1\left(U_1 \cap U_2\right) &= \varphi_1\left(A \sqcup B\right) = \varphi_1\left(A\right) \sqcup \varphi_1\left(B\right) = (-\pi, 0) \sqcup (0, \pi) \\ \varphi_2\left(U_1 \cap U_2\right) &= \varphi_2\left(A \sqcup B\right) = \varphi_2\left(A\right) \sqcup \varphi_2\left(B\right) = (\pi, 2\pi) \sqcup (0, \pi) \end{aligned}$$

Now, the transition function $\varphi_2 \circ \varphi_1^{-1} : \varphi_1\left(U_1 \cap U_2\right) \to \varphi_2\left(U_1 \cap U_2\right)$ is given by:

$$\left(\varphi_2 \circ \varphi_1^{-1}\right)(t) = \begin{cases} t + 2\pi & \text{for } t \in (-\pi, 0) \\ t & \text{for } t \in (0, \pi) \end{cases}$$

Similarly, the transition function $\varphi_1 \circ \varphi_2^{-1} : \varphi_2\left(U_1 \cap U_2\right) \to \varphi_1\left(U_1 \cap U_2\right)$ is given by:

$$\left(\varphi_1 \circ \varphi_2^{-1}\right)(t) = \begin{cases} t - 2\pi & \text{for } t \in (\pi, 2\pi) \\ t & \text{for } t \in (0, \pi) \end{cases}$$

These two transition functions are C^∞. Therefore, (U_1, φ_1) and (U_2, φ_2) are C^∞-compatible charts on S^1 and form an atlas.

Remark 1.1. Although the C^∞-compatibility of charts is clearly reflexive and symmetric, it is not transitive. The reason is as follows. Suppose (U_1, φ_1) is C^∞-compatible with (U_2, φ_2), and (U_2, φ_2) is C^∞-compatible with (U_3, φ_3). Note that the three coordinate functions are simultaneously defined only on the triple intersection $U_1 \cap U_2 \cap U_3$. Thus, the composite

$$\varphi_3 \circ \varphi_1^{-1} = \left(\varphi_3 \circ \varphi_2^{-1}\right) \circ \left(\varphi_2 \circ \varphi_1^{-1}\right)$$

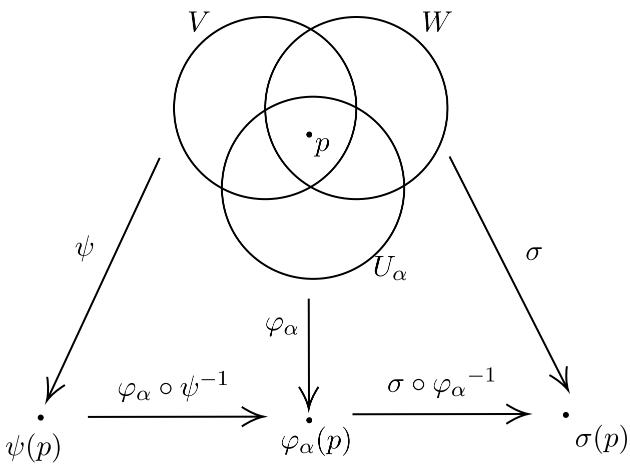
is C^∞, but only on $\varphi_1\left(U_1 \cap U_2 \cap U_3\right)$, not necessarily on $\varphi_1\left(U_1 \cap U_3\right)$. A priori we know nothing about $\varphi_3 \circ \varphi_1^{-1}$ on $\phi_1\left((U_1 \cap U_3) \setminus (U_1 \cap U_2 \cap U_3)\right)$ and so we cannot conclude that (U_1, ϕ_1) and (U_3, ϕ_3) are C^∞-compatible.

We say that a chart (V, ψ) is compatible with an atlas $\{(U_\alpha, \varphi_\alpha)\}$ if it is compatible with all the charts $(U_\alpha, \varphi_\alpha)$ of the atlas.

Lemma 1.1

Let $\{(U_\alpha, \varphi_\alpha)\}$ be an atlas on a locally Euclidean space M. If two charts (V, ψ) and (W, σ) are both compatible with the atlas $\{(U_\alpha, \varphi_\alpha)\}$, then they are compatible with each other.

Proof. Let $p \in V \cap W$. First, we need to show that $\sigma \circ \psi^{-1}$ is C^∞ at $\psi(p)$.



Since $\{(U_\alpha, \varphi_\alpha)\}$ is an atlas for M, $p \in U_\alpha$ for some α. Hence, $p \in V \cap W \cap U_\alpha$. By the remark above,

$$\sigma \circ \psi^{-1} = \left(\sigma \circ \varphi_\alpha^{-1}\right) \circ \left(\varphi_\alpha \circ \psi^{-1}\right)$$

is C^∞ on $\psi(V \cap W \cap U_\alpha)$, and hence at $\psi(p)$. Since p was an arbitrary point of $V \cap W$, this proves that $\sigma \circ \psi^{-1}$ is C^∞ on $\psi(V \cap W)$. Similarly, $\psi \circ \sigma^{-1}$ is C^∞ on $\sigma(V \cap W)$. ∎

Remark 1.2. In the equality $\sigma \circ \psi^{-1} = (\sigma \circ \varphi_\alpha^{-1}) \circ (\varphi_\alpha \circ \psi^{-1})$, the maps on the two sides of the equality sign have different domains. What the equality means is that the two maps are equal on their common domain.

§1.2 Smooth Manifold

Definition 1.5 (Maximal Atlas)**.** An atlas $\mathscr{M}$ on a locally Euclidean space is said to be **maximal** if it is not contained in a larger atlas. In other words, if $\mathscr{U}$ is any other atlas containing $\mathscr{M}$, then $\mathscr{U} = \mathscr{M}$.

Definition 1.6 (Smooth Manifold)**.** A **smooth** or C^∞ **manifold** is a topological manifold M together with a maximal atlas $\mathscr{M}$. To avoid confusion, we can denote it as a pair $(M, \mathscr{M})$ of a topological manifold M and a maximal atlas $\mathscr{M}$ on M. The maximal atlas is also called a *differentiable structure* on M.

Abuse of Notation. Often instead of writing $(M, \mathscr{M})$ is a smooth manifold, we shall say M is a smooth manifold. Whenever we say that we take a chart (U, φ) on a smooth manifold M, we mean that (U, φ) is contained in the differentiable structure (*i.e.*, maximal atlas) $\mathscr{M}$ on M.

In practice, to check that a topological manifold M is a smooth manifold, it is not necessary to exhibit a maximal atlas. The existence of *any* atlas on M will do, because of the following proposition.

Proposition 1.2

Any atlas $\mathscr{U} = \{(U_\alpha, \varphi_\alpha)\}$ on a locally Euclidean space is contained in a unique maximal atlas.

Proof. Adjoin to the atlas $\mathscr{U}$ all the charts (V_i, ψ_i) that are compatible with $\mathscr{U}$. By Lemma 1.1, the charts (V_i, ψ_i) are compatible with one another. So the enlarged collection of charts is an atlas. Can we enlarge this new atlas any further? Any chart compatible with the new atlas (that we wish to adjoin to the new atlas) must be compatible with the original atlas $\mathscr{U}$ and so by construction belongs to the new atlas. This proves that the new atlas is maximal.

Now we need to prove the uniqueness. Let $\mathscr{M}$ be the maximal atlas that we constructed in the preceeding paragraph. If $\mathscr{M}'$ is another maximal atlas containing $\mathscr{U}$, then all the charts in $\mathscr{M}'$ are compatible with $\mathscr{U}$ and so by construction must belong to $\mathscr{M}$. This proves that $\mathscr{M}' \subseteq \mathscr{M}$. $\mathscr{M}'$ is a maximal atlas contained in another atlas, so $\mathscr{M}$ and $\mathscr{M}'$ must be the same. Therefore, the maximal atlas containing $\mathscr{U}$ is unique. ■

In summary, to show that a topological space M is a smooth manifold, it suffices to check that

(i) M is Hausdorff and second countable,

(ii) M has a C^∞ atlas (not necessarily maximal).

Multiple Differentiable Structures on the Same Topological Manifold

From Proposition 1.2, if we have an atlas $\mathscr{U}$ on a topological manifold M, then $\mathscr{U}$ is contained in a unique maximal atlas $\mathscr{M}$. However, if we start with a different atlas $\mathscr{V}$ on M, $\mathscr{V}$ is contained in a unique maximal atlas $\mathscr{N}$. Then $\mathscr{M}$ and $\mathscr{N}$ are not, in general, the same. Therefore, $(M, \mathscr{M})$ and $(M, \mathscr{N})$ are two different smooth manifolds with the same underlying topological manifold.

For example, $\mathscr{U} = \{(\mathbb{R}, \mathbb{1}_{\mathbb{R}})\}$ is an atlas on $\mathbb{R}$ with a single chart. This atlas is contained in a maximal atlas, say $\mathscr{M}$. Then $(\mathbb{R}, \mathscr{M})$ is a smooth manifold with the usual differentiable structure. If we consider the map $\varphi : \mathbb{R} \to \mathbb{R}$ that sends x to x^3, then φ is a homeomorphism. Therefore, $\mathscr{V} = \{(\mathbb{R}, \varphi)\}$ is also an atlas on $\mathbb{R}$ with a single chart $(\mathbb{R}, \varphi)$. This $\mathscr{V}$ is contained in another maximal atlas $\mathscr{N}$. Then $(\mathbb{R}, \mathscr{N})$ is also a smooth manifold. This example is important as we will see an useful example using these two smooth manifolds in the following chapter.

Recall that we can put several topologies on a set, and then the set becomes different topological spaces under different topologies. In a similar spirit, we can have multiple maximal atlases (say $\mathscr{M}_1$ and $\mathscr{M}_2$) on a topological manifold M, and M can become different smooth manifolds $(M, \mathscr{M}_1)$ and $(M, \mathscr{M}_2)$ when equipped with different differentiable structures.

Some Notations

From now on, a "manifold" will mean a "smooth manifold". Also we shall use the terms "smooth" and "C^∞" interchangeably. Let $\vec{v} \in \mathbb{R}^n$ be a vector, or an n-tuple. The function $r^i : \mathbb{R}^n \to \mathbb{R}$ is defined as $r^i(\vec{v}) = v^i$. Let (U, φ) be a chart of the n-dimensional manifold M and let $p \in U$. Since $\varphi : U \to \mathbb{R}^n$, we write

$$\varphi(p) = \left(x^1(p), x^2(p), \ldots, x^n(p)\right),$$

with each component x^i of φ being a real valued function $x^i : U \to \mathbb{R}$ such that $x^i = r^i \circ \varphi$. The functions $x^1, x^2, \ldots, x^n$ are called *coordinates* or *local coordinates* on U. We sometimes write $\varphi = (x^1, x^2, \ldots, x^n)$ and the chart $(U, \varphi) = (U, x^1, x^2, \ldots, x^n)$.

Example 1.4 (Euclidean Space)**.** The Euclidean space $\mathbb{R}^n$ is a smooth manifold with a single chart $(\mathbb{R}^n, r^1, r^2, \ldots, r^n)$, where $r^1, r^2, \ldots, r^n$ are the standard coordinates on $\mathbb{R}^n$.

Example 1.5 (Open Subset of a Manifold)**.** Any open subset V of a manifold M is also a manifold. If $\{(U_\alpha, \varphi_\alpha)\}$ is an atlas for M, then

$$\mathscr{U}_V = \left\{\left(U_\alpha \cap V, \varphi_\alpha\big|_{U_\alpha \cap V}\right)\right\}$$

is an atlas for V. Notice that V, equipped with the subspace topology inherited from M, is indeed Hausdorff and second countable. It is a topological manifold because $U_\alpha \cap V$ is open in M, and φ_α is an open map; hence, as a restriction of a homeomorphism, $\varphi_\alpha\big|_{U_\alpha \cap V}$ is a homeomorphism mapping

$U_\alpha \cap V$ to an open subset of $\mathbb{R}^n$. Now we are left to show that any two charts in the collection $\mathscr{U}_V$ are compatible.

$$\varphi_\alpha|_{U_\alpha \cap V} \circ \varphi_\beta|_{U_\beta \cap V}^{-1} = \left(\varphi_\alpha \circ \varphi_\beta^{-1}\right)|_{\varphi_\beta(U_\alpha \cap U_\beta \cap V)} .$$

As a restriction of a C^∞ map, this is also a C^∞ map. Hence $\mathscr{U}_V$ is truly an atlas for V.

Example 1.6 (General Linear Groups)**.** For any two positive integers m and n, let $\mathbb{R}^{m \times n}$ be the vector space of all $m \times n$ matrices. Since $\mathbb{R}^{m \times n}$ is isomorphic to $\mathbb{R}^{mn}$, we give it the topology of $\mathbb{R}^{mn}$. The definition of general linear group $\mathrm{GL}(n, \mathbb{R})$ is as follows:

$$\mathrm{GL}(n, \mathbb{R}) := \left\{A \in \mathbb{R}^{n \times n} \mid \det A \neq 0\right\} .$$

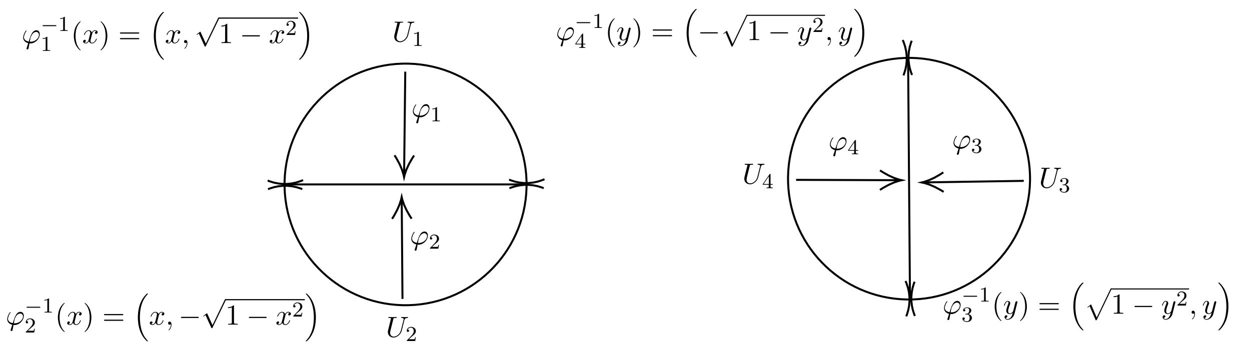
Consider the determinant function $\det : \mathbb{R}^{n \times n} \to \mathbb{R}$. It is a polynomial of the entries, hence continuous. In terms of this continuous function, the pre-image of $\mathbb{R} \setminus \{0\}$ is precisely $\mathrm{GL}(n, \mathbb{R})$.

$$\mathrm{GL}(n, \mathbb{R}) = \det^{-1}(\mathbb{R} \setminus \{0\}) .$$

Since det is a continuous function from $\mathbb{R}^{n \times n} \cong \mathbb{R}^{n^2}$ to $\mathbb{R}$, and $\mathbb{R} \setminus \{0\}$ is open in $\mathbb{R}$, $\det^{-1}(\mathbb{R} \setminus \{0\})$ will be open in $\mathbb{R}^{n^2}$. Therefore, by Example 1.5, $\mathrm{GL}(n, \mathbb{R})$ is a manifold.

Example 1.7 (Unit circle in the (x, y)-plane)**.** In Example 1.3, we found a C^∞ atlas with 2 charts on the unit circle S^1 in the complex plane $\mathbb{C}$. We'll now view S^1 as the unit circle in $\mathbb{R}^2$ with defining equation $x^2 + y^2 = 1$. We can cover S^1 with 4 open sets: the upper and lower semicirles U_1 and U_2, the right and left semicircles U_3 and U_4. The homeomorphisms are:

$$\varphi_i : U_i \to (-1, 1) \ , \ \varphi_i(x, y) = \begin{cases} x & \text{if } i = 1, 2 \\ y & \text{if } i = 3, 4 \end{cases}$$



Let us check that on $U_1 \cap U_3$,

$$\left(\varphi_3 \circ \varphi_1^{-1}\right)(\varphi_1(x, y)) = \left(\varphi_3 \circ \varphi_1^{-1}\right)(x) = \varphi_3\left(x, \sqrt{1 - x^2}\right) = \sqrt{1 - x^2} .$$

Since $(1, 0) \notin U_1 \cap U_3$, we can conclude that $\varphi_3 \circ \varphi_1^{-1}$ is C^∞. Also, on $U_2 \cap U_4$,

$$\left(\varphi_2 \circ \varphi_4^{-1}\right)(\varphi_4(x, y)) = \left(\varphi_2 \circ \varphi_4^{-1}\right)(y) = \varphi_2\left(-\sqrt{1 - y^2}, y\right) = -\sqrt{1 - y^2} .$$

Since $(0, -1) \notin U_2 \cap U_4$, we can conclude that $\varphi_2 \circ \varphi_4^{-1}$ is C^∞. In a similar manner, one can check that $\varphi_i \circ \varphi_j^{-1}$ is C^∞ for every i, j. Therefore, $\{(U_i, \varphi_i) \mid 1 \le i \le 4\}$ is indeed a C^∞ atlas on S^1.

If M and N are manifolds, it's natural to think that $M \times N$ should also be a manifold. Now we shall demonstrate it. $M \times N$ with its product topology is Hausdorff and second countable (Proposition A.22 and Corollary A.21). To show that $M \times N$ is a manifold, it remains to exhibit an atlas on it. Recall that the product of two set maps $f : X \to X'$ and $g : Y \to Y'$ is

$$f \times g : X \times Y \to X' \times Y' \ , \ (f \times g)(x, y) = (f(x), g(y)) .$$

Proposition 1.3 (Atlas for Product Manifold)
If $\{(U_\alpha, \varphi_\alpha)\}$ and $\{(V_i, \psi_i)\}$ are C^∞ atlases for the manifolds M and N of dimensions m and n, respectively, then the collection

$$\{(U_\alpha \times V_i, \varphi_\alpha \times \psi_i : U_\alpha \times V_i \to \mathbb{R}^m \times \mathbb{R}^n)\}$$

of charts is a C^∞ atlas on $M \times N$. Therefore, $M \times N$ is a C^∞ manifold of dimension $m+n$.

Proof. φ_α is a homeomorphism of U_α onto $\varphi_\alpha(U_\alpha) = \overline{U_\alpha} \subseteq \mathbb{R}^m$, and ψ_i is a homeomorphism of V_i onto $\psi_i(V_i) = \overline{V_i} \subseteq \mathbb{R}^n$. Now,

$$(\varphi_\alpha \times \psi_i)(a,b) = (\varphi_\alpha(a), \psi_i(b)) = ((\varphi_\alpha \circ \pi_1)(a,b), (\psi_i \circ \pi_2)(a,b)) ,$$

where π_1 and π_2 are projection on first and second coordinate, respectively. Both $\varphi_\alpha \circ \pi_1$ and $\psi_i \circ \pi_2$ are composition of continuous maps, hence continuous. Therefore, by Theorem A.23, $\varphi_\alpha \times \psi_i$ is continuous. One can show that

$$(\varphi_\alpha \times \psi_i)^{-1} = \varphi_\alpha^{-1} \times \psi_i^{-1} .$$

Using an analogous argument as above, $\varphi_\alpha^{-1} \times \psi_i^{-1}$ is continuous. Therefore, $\varphi_\alpha \times \psi_i : U_\alpha \times V_i \to \overline{U_\alpha} \times \overline{V_i} \subseteq \mathbb{R}^{m+n}$ is a homeomorphism. Furthermore,

$$\bigcup_{\alpha,i} (U_\alpha \times V_i) = \bigcup_\alpha \left(U_\alpha \times \left(\bigcup_i V_i \right) \right) = \bigcup_\alpha (U_\alpha \times N) = \left(\bigcup_\alpha U_\alpha \right) \times N = M \times N .$$

Now, we are only left to show that any two charts are compatible with each other. It suffices to show that $(\varphi_\alpha \times \psi_i) \circ (\varphi_\beta \times \psi_j)^{-1} : \overline{U_\beta} \times \overline{V_j} \to \overline{U_\alpha} \times \overline{V_i}$ is a C^∞ map.

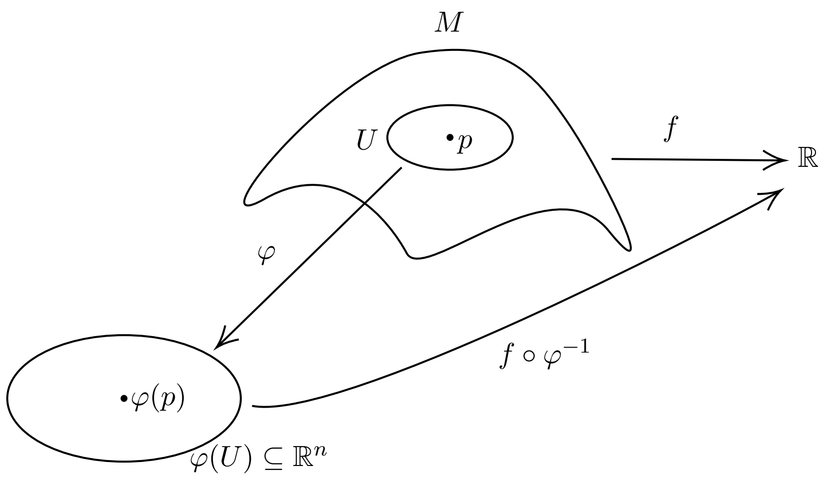
$$\begin{aligned}
(\varphi_\alpha \times \psi_i) \circ (\varphi_\beta \times \psi_j)^{-1} &= (\varphi_\alpha \times \psi_i) \circ \left(\varphi_\beta^{-1} \times \psi_j^{-1} \right) \\
\left((\varphi_\alpha \times \psi_i) \circ \left(\varphi_\beta^{-1} \times \psi_j^{-1} \right) \right)(x,y) &= (\varphi_\alpha \times \psi_i) \left(\varphi_\beta^{-1}(x), \psi_j^{-1}(y) \right) \\
&= \left(\left(\varphi_\alpha \circ \varphi_\beta^{-1} \right)(x), \left(\psi_i \circ \psi_j^{-1} \right)(y) \right) \\
\therefore (\varphi_\alpha \times \psi_i) \circ (\varphi_\beta \times \psi_j)^{-1} &= \left(\varphi_\alpha \circ \varphi_\beta^{-1} \right) \times \left(\psi_i \circ \psi_j^{-1} \right)
\end{aligned}$$

Both $\varphi_\alpha \circ \varphi_\beta^{-1}$ and $\psi_i \circ \psi_j^{-1}$ are C^∞ maps since $\{(U_\alpha, \varphi_\alpha)\}$ and $\{(V_i, \psi_i)\}$ are C^∞ atlases. Therefore, as a cartesian product of C^∞ maps, $(\varphi_\alpha \times \psi_i) \circ (\varphi_\beta \times \psi_j)^{-1}$ is also C^∞. This completes the proof. ■

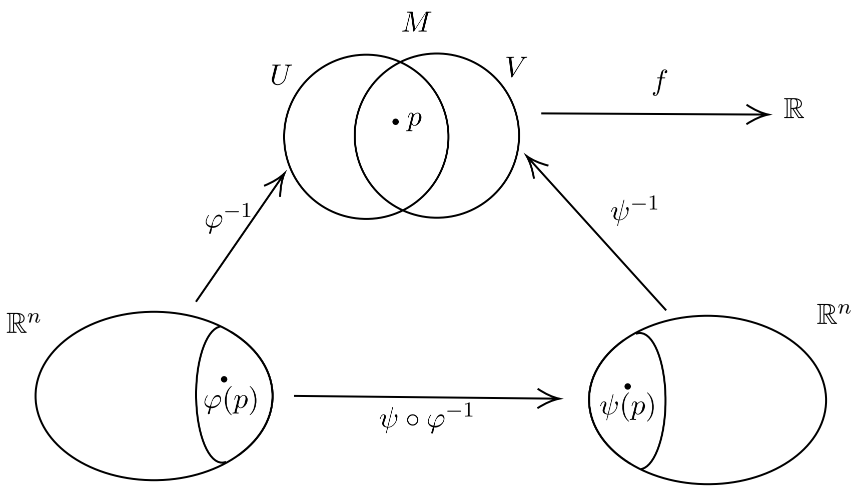
2 Smooth Maps on Manifold

§2.1 Smooth Functions on Manifold

> **Definition 2.1.** Let M be a smooth manifold of dimension n. A function $f : M \to \mathbb{R}$ is said to be C^∞ or smooth at a point $p \in M$ if there is a chart (U, φ) about p in M such that $f \circ \varphi^{-1} : \varphi(U) \subseteq \mathbb{R}^n \to \mathbb{R}$ is C^∞ at $\varphi(p)$. The function f is said to be C^∞ on M if it is C^∞ at every point of M.



Remark 2.1. The definition of the smoothness of a function f at a given point on the manifold is independent of the chart (U, φ). Let us check this.



Suppose that $f \circ \varphi^{-1}$ is C^∞ at $\varphi(p)$ for a given chart (U, φ) about $p \in M$. Let (V, ψ) be any other chart about p. Then on $\psi(U \cap V)$,

$$f \circ \psi^{-1} = \left(f \circ \varphi^{-1}\right) \circ \left(\varphi \circ \psi^{-1}\right).$$

$\varphi \circ \psi^{-1}$ is C^∞ by compatibility of charts. Therefore, as a composition of C^∞ maps, $f \circ \psi^{-1}$ is C^∞ at $\psi(p)$. This proves the independence of chart to determine the smoothness of a function at a given point.

Proposition 2.1

Let M be a manifold of dimension n, and $f : M \to \mathbb{R}$ a real-valued function on M. The following are equivalent:

(i) The function $f : M \to \mathbb{R}$ is C^∞.

(ii) The manifold M has an atlas such that for every chart (U, φ) in the atlas, $f \circ \varphi^{-1} : \varphi(U) \subseteq \mathbb{R}^n \to \mathbb{R}$ is C^∞.

(iii) For every chart (V, ψ) on M, the function $f \circ \psi^{-1} : \psi(V) \subseteq \mathbb{R}^n \to \mathbb{R}$ is C^∞.

Proof. **(ii)⇒(i):** Since (ii) holds, one can find for every $p \in M$, a coordinate neighborhood (U, φ) such that $f \circ \varphi^{-1}$ is C^∞ at $\varphi(p)$. Therefore, from the definition of C^∞ function on a manifold, $f : M \to \mathbb{R}$ is C^∞.

(i)⇒(iii): Let (V, ψ) be an arbitrary chart on M and $p \in V$. Since (i) holds, $f \circ \psi^{-1}$ is C^∞ at $\psi(p)$ (by the remark). Since p is an arbitrary point on V, $f \circ \psi^{-1}$ is C^∞ on $\psi(V)$.

(iii)⇒(ii): Obvious. ∎

Definition 2.2 (Pullback). Let $F : N \to M$ be a map and h a function on M. The **pullback** of h by F, denoted by F^*h, is the composite function $h \circ F$.

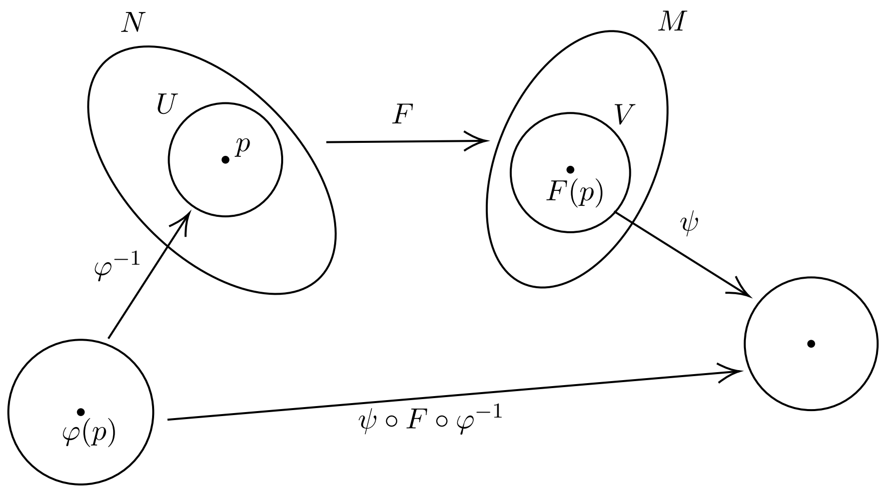
Using this terminology of pullback, a function f on M is C^∞ on a chart (U, φ) if and only if its pullback $(\varphi^{-1})^* f$ by φ^{-1} is C^∞ on the subset $\varphi(U)$ of Euclidean space.

§2.2 Smooth Maps Between Manifolds

Definition 2.3. Let N and M be manifolds of dimension n and m, respectively. A continuous map $F : N \to M$ is C^∞ at a point $p \in N$ if there are charts (V, ψ) about $F(p) \in M$ and (U, φ) about $p \in N$ such that the composition

$$\psi \circ F \circ \varphi^{-1} : \varphi\left(F^{-1}(V) \cap U\right) \subseteq \mathbb{R}^n \to \mathbb{R}^m$$

is C^∞ at $\varphi(p)$. The continuous map $F : N \to M$ is said to be C^∞ if it is C^∞ at every point of N.



Remark 2.2. Note that in the definition of smooth map between manifolds, one must have a continuous map to start with. We require $F : N \to M$ to be continuous so that $F^{-1}(V)$ is open and $\varphi\left(F^{-1}(V) \cap U\right)$ becomes an open subset of $\mathbb{R}^n$.

Proposition 2.2

Suppose $F : N \to M$ is C^∞ at $p \in N$. If (U, φ) is any chart about $p \in N$ and (V, ψ) is any chart about $F(p) \in M$, then $\psi \circ F \circ \varphi^{-1}$ is C^∞ at $\varphi(p)$.

Proof. Since F is C^∞ at $p \in N$, there are charts $(U_\alpha, \varphi_\alpha)$ about $p \in N$ and (V_β, ψ_β) about $F(p) \in M$ such that $\psi_\beta \circ F \circ \varphi_\alpha^{-1}$ is C^∞ at $\varphi_\alpha(p)$. By the C^∞ compatibility of charts in a differentiable structure, both $\varphi_\alpha \circ \varphi^{-1}$ and $\psi \circ \psi_\beta^{-1}$ are C^∞ on open subsets of Euclidean spaces. Hence, the composite

$$\psi \circ F \circ \varphi^{-1} = \left(\psi \circ \psi_\beta^{-1}\right) \circ \left(\psi_\beta \circ F \circ \varphi_\alpha^{-1}\right) \circ \left(\varphi_\alpha \circ \varphi^{-1}\right)$$

is C^∞ at $\varphi(p)$. ■

Proposition 2.3 (Smoothness of a map in terms of charts)

Let N and M be smooth manifolds, and $F : N \to M$ a continuous map. The following are equivalent:

(i) The map $F : N \to M$ is C^∞.

(ii) There are atlases $\mathscr{U}$ for N and $\mathscr{V}$ for M such that for every chart (U, φ) in $\mathscr{U}$ and (V, ψ) in $\mathscr{V}$, the map

$$\psi \circ F \circ \varphi^{-1} : \varphi\left(U \cap F^{-1}(V)\right) \to \mathbb{R}^m$$

is C^∞.

(iii) For every chart (U, φ) on N and (V, ψ) on M, the map

$$\psi \circ F \circ \varphi^{-1} : \varphi\left(U \cap F^{-1}(V)\right) \to \mathbb{R}^m$$

is C^∞.

Proof. **(ii)⇒(i):** Let $p \in N$. Suppose (U, φ) is a chart about p in $\mathscr{U}$ and (V, ψ) is a chart about $F(p)$ in $\mathscr{V}$. Now, (ii) implies that $\psi \circ F \circ \varphi^{-1}$ is C^∞ at $\varphi(p)$. By the definition of a C^∞ map, $F : N \to M$ is C^∞ at p. Since p was an arbitrary point of N, the map $F : N \to M$ is C^∞.

(i)⇒(iii): Suppose (U, φ) and (V, ψ) are charts on N and M, respectively, such that $U \cap F^{-1}(V) \neq \varnothing$. Let $p \in U \cap F^{-1}(V)$ so that $p \in U$ and $F(p) \in V$. Then (U, φ) is a chart about p and (V, ψ) is a chart about $F(p)$. By Proposition 2.2, $\psi \circ F \circ \varphi^{-1}$ is C^∞ at $\varphi(p)$. Since $\varphi(p)$ was an arbitrary point of $\varphi\left(U \cap F^{-1}(V)\right)$, the map $\psi \circ F \circ \varphi^{-1} : \varphi\left(U \cap F^{-1}(V)\right) \to \mathbb{R}^m$ is C^∞.

(iii)⇒(ii): Take $\mathscr{U}$ and $\mathscr{V}$ to be the maximal atlases of N and M, respectively. ■

Smoothness of a Map Depends on the Choice of Differentiable Structure

Consider the map $\varphi : \mathbb{R} \to \mathbb{R}$ that takes x to x^3. φ is continuous, so is its inverse $x \mapsto x^{1/3}$. Therefore, φ is a homeomorphism. So $(\mathbb{R}, \varphi)$ is a chart on $\mathbb{R}$. The collection $\mathscr{V} = \{(\mathbb{R}, \varphi)\}$ is an atlas on $\mathbb{R}$ with a single chart $(\mathbb{R}, \varphi)$. This $\mathscr{V}$ is contained in another maximal atlas $\mathscr{N}$. Then $M_1 = (\mathbb{R}, \mathscr{N})$ is also a smooth manifold.

Furthermore, $\mathscr{U} = \{(\mathbb{R}, \mathbb{1}_\mathbb{R})\}$ is an atlas on $\mathbb{R}$ with a single chart. This atlas is contained in a maximal atlas, say $\mathscr{M}$. Then $M_2 = (\mathbb{R}, \mathscr{M})$ is a smooth manifold with the usual differentiable structure. Although the underlying topological manifolds of M_1 and M_2 are the same, they are, nevertheless, different manifolds because the differentiable structures are not the same. That's why we denote them with different symbols.

Now we want to check whether $\varphi : \mathbb{R} \to \mathbb{R}$ is smooth. In order to do that, we need to choose which differentiable structure we put on the domain and range spaces. If we equip both the domain space and range space with the usual differentiable structure $\mathscr{M}$, then it's easy to check that $\varphi : M_2 \to M_2$

is smooth. However, in this case, φ^{-1} is not smooth. Because, if we take the charts $(\mathbb{R}, \mathbb{1}_{\mathbb{R}})$ from both manifolds, then

$$\mathbb{1}_{\mathbb{R}} \circ \varphi^{-1} \circ \mathbb{1}_{\mathbb{R}}^{-1} : \mathbb{R} \to \mathbb{R}$$

is just the map $\varphi^{-1} : \mathbb{R} \to \mathbb{R}$, $\varphi^{-1}(x) = x^{1/3}$, which, as a map between two Euclidean spaces, is not even C^1, let alone being C^∞. Therefore, $\varphi^{-1} : M_2 \to M_2$ is not smooth.

Now we consider $\varphi : M_1 \to M_2$. Recall that $M_1 = (\mathbb{R}, \mathscr{N})$ and $M_2 = (\mathbb{R}, \mathscr{M})$. Then φ is indeed smooth. Because if we take the atlas $\mathscr{V} = \{(\mathbb{R}, \varphi)\}$ from $\mathscr{N}$ and the atlas $\mathscr{U} = \{(\mathbb{R}, \mathbb{1}_{\mathbb{R}})\}$ from $\mathscr{M}$, then

$$\mathbb{1}_{\mathbb{R}} \circ \varphi \circ \varphi^{-1} = \mathbb{1}_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$$

is indeed a smooth map between Euclidean spaces. Therefore, Proposition 2.3 (ii)⇒(i) guarantees that φ is smooth. Furthermore, $\varphi^{-1} : M_2 \to M_1$ is also smooth. Because if we take the same atlases as above,

$$\varphi \circ \varphi^{-1} \circ \mathbb{1}_{\mathbb{R}}^{-1} = \mathbb{1}_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$$

is indeed a smooth map between Euclidean spaces. Therefore, Proposition 2.3 (ii)⇒(i) guarantees that φ^{-1} is smooth.

This might seem disturbing at first. Because the description of φ^{-1} does not change when we impose different differentiable structures on the domain and range spaces. $\varphi^{-1}(x) = x^{1/3}$ stays the same function. We have a hardwired notion that we cannot differentiate it at $x = 0$, that's why it is not C^1, let alone being C^∞. However, when we are talking about a map between two manifolds, the notion of smoothness depends solely on the differentiable structures on the domain and range spaces.

We have seen that depending on the choice of differentiable structures, the same map can be both smooth and non-smooth. Drawing the analogy with topology, in a calculus class, we say that the identity map $\mathbb{1}_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$ is continuous. But this is only true when the domain and range sets are equipped with the same topology. If we equip the range set with the discrete topology and the domain set with the usual topology, then $\mathbb{1}_{\mathbb{R}}$ no longer stays continuous. Thus, depending on the topologies on the domain and range sets, the same map can be both continuous and discontinuous. In a similar spirit, the same map can be smooth and non-smooth depending on the choice of differentiable structures.

Proposition 2.4 (Composition of C^∞ maps)

If $F : N \to M$ and $G : M \to P$ are C^∞ maps of manifolds, then the composite $G \circ F : N \to P$ is C^∞.

Proof. Let (U, φ), (V, ψ), and (W, σ) be charts on N, M, and P, respectively. Then

$$\sigma \circ (G \circ F) \circ \varphi^{-1} = \left(\sigma \circ G \circ \psi^{-1}\right) \circ \left(\psi \circ F \circ \varphi^{-1}\right).$$

Since F and G are C^∞, by Proposition 2.3 (i)⇒(iii), $\sigma \circ G \circ \psi^{-1}$ and $\psi \circ F \circ \varphi^{-1}$ are C^∞ maps on their respective domains. As a composite of C^∞ maps of open subsets of Euclidean spaces, $\sigma \circ (G \circ F) \circ \varphi^{-1}$ is C^∞. In particular,

$$\sigma \circ (G \circ F) \circ \varphi^{-1} : \varphi\left(U \cap F^{-1}(V)\right) \cap \psi\left(V \cap G^{-1}(W)\right) \to \mathbb{R}^p$$

is C^∞ provided N, M and P are of dimension n, m and p, respectively. By (iii)⇒(i) of Proposition 2.3, $G \circ F$ is C^∞. ■

Definition 2.4 (Diffeomorphism). A **diffeomorphism** of manifolds is a bijective C^∞ map $F : N \to M$ whose inverse F^{-1} is also C^∞.

Proposition 2.5

If (U, φ) is a chart on a manifold M of dimension n, then the coordinate map $\varphi : U \to \varphi(U) \subseteq \mathbb{R}^n$ is a diffeomorphism.

Proof. By definition, φ is a homeomorphism. So it suffices to check that both φ and φ^{-1} are smooth. In order to check the smoothness of $\varphi : U \to \varphi(U)$, we shall use the atlas $\{(U, \varphi)\}$ on the manifold U, and the atlas $\left\{\left(\varphi(U), \mathbb{1}_{\varphi(U)}\right)\right\}$ on the manifold $\varphi(U)$. Observe that,

$$\mathbb{1}_{\varphi(U)} \circ \varphi \circ \varphi^{-1} : \varphi(U) \to \varphi(U)$$

is just the identity map on $\varphi(U)$, hence C^∞. Therefore, by (ii)⇒(i) of Proposition 2.3, φ is C^∞.

We shall use the same atlas as above to show the smoothness of $\varphi^{-1} : \varphi(U) \to U$. Now,

$$\varphi \circ \varphi^{-1} \circ \mathbb{1}_{\varphi(U)}^{-1} = \mathbb{1}_{\varphi(U)} : \varphi(U) \to \varphi(U) \, .$$

Identity map is C^∞, hence by (ii)⇒(i) of Proposition 2.3, φ^{-1} is C^∞. ∎

Proposition 2.6

Let U be an open subset of a manifold M of dimension n. If $F : U \to F(U) \subseteq \mathbb{R}^n$ is a diffeomorphism onto an open subset of $\mathbb{R}^n$, then (U, F) is a chart in the maximal atlas of M.

Proof. For any chart $(U_\alpha, \varphi_\alpha)$ in the maximal atlas of M, both φ_α and φ_α^{-1} are C^∞ by Proposition 2.5. As compositions of C^∞ maps, $F \circ \varphi_\alpha^{-1}$ and $\varphi_\alpha \circ F^{-1}$ are C^∞ maps. Therefore, (U, F) is compatible with every chart of the maximal atlas. Hence, it is compatible with the maximal atlas. Therefore, by the maximality of the atlas, the chart (U, F) is in the maximal atlas. ∎

Proposition 2.7 (Smoothness of a vector-valued function)

Let N be a manifold and $F : N \to \mathbb{R}^m$ a continuous map. The following are equivalent:

(i) The map $F : N \to \mathbb{R}^m$ is C^∞.

(ii) The manifold N has an atlas such that for every chart (U, φ) in the atlas, the map $F \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^m$ is C^∞.

(iii) For every chart (U, φ) on N, the map $F \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^m$ is C^∞.

Proof. **(ii)⇒(i):** In Proposition 2.3(ii), take the atlas $\mathscr{V}$ of $\mathbb{R}^m$ to be $\{(\mathbb{R}^m, \mathbb{1}_{\mathbb{R}^m})\}$. Now, $\varphi(U) = \varphi(U \cap N) = \varphi(U \cap F^{-1}(\mathbb{R}^m))$. Therefore,

$$\mathbb{1}_{\mathbb{R}^m} \circ F \circ \varphi^{-1} : \varphi\left(U \cap F^{-1}(\mathbb{R}^m)\right) \to \mathbb{R}^m$$

is the same as $F \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^m$, which is C^∞. Hence by (ii)⇒(i) of Proposition 2.3, F is C^∞.

(i)⇒(iii): In Proposition 2.3(iii), let (V, ψ) be the chart $(\mathbb{R}^m, \mathbb{1}_{\mathbb{R}^m})$ on $\mathbb{R}^m$. Hence,

$$\mathbb{1}_{\mathbb{R}^m} \circ F \circ \varphi^{-1} : \varphi\left(U \cap F^{-1}(\mathbb{R}^m)\right) \to \mathbb{R}^m$$

is C^∞, which is the same as $F \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^m$.

(iii)⇒(ii): Choose the maximal atlas of N. ∎

Proposition 2.8 (Smoothness in terms of components)

Let N be a manifold. A vector-valued function $F : N \to \mathbb{R}^m$ is C^∞ if and only if its component functions $F^1, \ldots, F^m : N \to \mathbb{R}$ are all C^∞.

Proof. $F : N \to \mathbb{R}^m$ is C^∞ if and only if for every chart (U, φ) on N, $F \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^m$ is C^∞ (Proposition 2.7). Now, $F \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^m$ is C^∞ if and only if $r^i \circ (F \circ \varphi^{-1})$ is C^∞ for every $1 \le i \le m$.

$$r^i \circ \left(F \circ \varphi^{-1}\right) = F^i \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}\,.$$

Therefore, F being smooth is equivalent to each $F^i \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}$ being smooth for every chart (U, φ). By Proposition 2.1, this is equivalent to each $F^i : N \to \mathbb{R}$ being smooth. Therefore, $F : N \to \mathbb{R}^m$ is C^∞ if and only if each $F^i : N \to \mathbb{R}$ is C^∞. ■

Proposition 2.9 (Smoothness of a map in terms of vector-valued functions)

Let $F : N \to M$ be a continuous map between two manifolds of dimensions n and m respectively. The following are equivalent:

(i) The map $F : N \to M$ is C^∞.

(ii) The manifold M has an atlas such that for every chart $(V, \psi) = (V, y^1, \ldots, y^m)$ in the atlas, the vector-valued function $\psi \circ F : F^{-1}(V) \to \mathbb{R}^m$ is C^∞.

(iii) For every chart $(V, \psi) = (V, y^1, \ldots, y^m)$ on M, the vector-valued function $\psi \circ F : F^{-1}(V) \to \mathbb{R}^m$ is C^∞.

Proof. **(ii)⇒(i):** Let $\mathscr{V}$ be the atlas for M in (ii), and let $\mathscr{U} = \{(U, \varphi)\}$ be an arbitrary atlas for N. For each chart (V, ψ) in the atlas $\mathscr{V}$, the collection $\left\{\left(U \cap F^{-1}(V), \varphi|_{U \cap F^{-1}(V)}\right)\right\}$ is an atlas for $F^{-1}(V)$. Since $\psi \circ F : F^{-1}(V) \to \mathbb{R}^m$ is C^∞, by (i)⇒(ii) of Proposition 2.7,

$$\psi \circ F \circ \varphi^{-1} : \varphi\left(U \cap F^{-1}(V)\right) \to \mathbb{R}^m$$

is C^∞. It then follows from (ii)⇒(i) of Proposition 2.3 that $F : N \to M$ is C^∞.

(i)⇒(iii): ψ is C^∞ by Proposition 2.5. As a composition of smooth maps, $\psi \circ F$ is C^∞.

(iii)⇒(ii): Trivial. Just take the maximal atlas of M and N. ■

Proposition 2.9 and Proposition 2.8 altogether gives rise to the following proposition.

Proposition 2.10 (Smoothness of a map in terms of components)

Let $F : N \to M$ be a continuous map between two manifolds of dimensions n and m respectively. The following are equivalent:

(i) The map $F : N \to M$ is C^∞.

(ii) The manifold M has an atlas such that for every chart $(V, \psi) = (V, y^1, \ldots, y^m)$ in the atlas, the components $y^i \circ F : F^{-1}(V) \to \mathbb{R}$ of F relative to the chart are all C^∞.

(iii) For every chart $(V, \psi) = (V, y^1, \ldots, y^m)$ on M, the components $y^i \circ F : F^{-1}(V) \to \mathbb{R}$ of F relative to the chart are all C^∞.

Example 2.1. Let M and N be manifolds and $\pi : M \times N \to M$, $\pi\,(p, q) = p$ be the projection onto the first factor. We want to show that π is a C^∞ map.

Let (p, q) be an arbitrary point of $M \times N$. Suppose $(U, \varphi) = (U, x^1, \ldots, x^m)$ and $(V, \psi) = (V, y^1, \ldots, y^n)$ are coordinate neighborhoods of p and q in M and N, respectively. By Proposition 1.3,

$$(U \times V, \varphi \times \psi) = \left(U \times V, x^1, \ldots, x^m, y^1, \ldots, y^n\right)$$

is a coordinate neighborhood of (p, q). Therefore, given $(a^1, \ldots, a^m, b^1, \ldots, b^n) \in (\varphi \times \psi)\,(U \times V) \subseteq \mathbb{R}^{m+n}$,

$$\begin{aligned}\left(\varphi \circ \pi \circ (\varphi \times \psi)^{-1}\right)\left(a^1, \ldots, a^m, b^1, \ldots, b^n\right) &= (\varphi \circ \pi)\left(\varphi^{-1}\left(a^1, \ldots, a^m\right), \psi^{-1}\left(b^1, \ldots, b^n\right)\right) \\ &= \varphi\left(\varphi^{-1}\left(a^1, \ldots, a^m\right)\right) = \left(a^1, \ldots, a^m\right)\end{aligned}$$

Therefore, $\varphi \circ \pi \circ (\varphi \times \psi)^{-1} : (\varphi \times \psi)(U \times V) \subseteq \mathbb{R}^{m+n} \to \mathbb{R}^m$ is just the projection onto the first m coordinates, which is a C^∞ map. Hence, $\pi : M \times N \to M$ is C^∞ at (p, q). Since (p, q) was chosen arbitrarily from $M \times N$, $\pi : M \times N \to M$ is C^∞ on $M \times N$.

Lemma 2.11

Let M_1, M_2 and N be manifolds of dimensions m_1, m_2 and n, respectively. Prove that a map $(f_1, f_2) : N \to M_1 \times M_2$ is C^∞ if and only if $f_i : N \to M_i$, $i = 1, 2$ are both C^∞.

Proof. Let $(f_1, f_2) = f$, and $\pi_i : M_1 \to M_2 \to M_i$ be projection maps for $i = 1, 2$. Both π_i are smooth, as proved in Example 2.1. If f is smooth, then $f_i = \pi_i \circ f : N \to M_i$ is composition of smooth maps, hence smooth.

Conversely, suppose both $f_i : N \to M_i$ are smooth. Then both f_i are continuous, hence so is f (Theorem A.23). Let $p \in N$, and take coordinate neighborhoods (U, φ), (V_1, ψ_1), (V_2, ψ_2) of p, $f_1(p)$, $f_2(p)$, respectively. We can choose U sufficiently small so that $f(U) \subseteq V_1 \times V_2$. Since f_i is smooth,

$$\psi_i \circ f_i \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^{m_i}$$

is smooth at $\varphi(p)$. Now, $(V_1 \times V_2, \psi_1 \times \psi_2)$ is a coordinate neighborhood of $f(p) = (f_1(p), f_2(p))$. Given $a \in \varphi(U) \subseteq \mathbb{R}^n$,

$$\begin{aligned}
\left((\psi_1 \times \psi_2) \circ f \circ \varphi^{-1}\right)(a) &= (\psi_1 \times \psi_2)\left(\left(f_1 \circ \varphi^{-1}\right)(a), \left(f_2 \circ \varphi^{-1}\right)(a)\right) \\
&= \left(\left(\psi_1 \circ f_1 \circ \varphi^{-1}\right)(a), \left(\psi_2 \circ f_2 \circ \varphi^{-1}\right)(a)\right) \\
\therefore (\psi_1 \times \psi_2) \circ f \circ \varphi^{-1} &= \left(\psi_1 \circ f_1 \circ \varphi^{-1}, \psi_2 \circ f_2 \circ \varphi^{-1}\right)
\end{aligned}$$

Both $\psi_i \circ f_i \circ \varphi^{-1}$ are smooth at $\varphi(p)$. Therefore, $(\psi_1 \times \psi_2) \circ f \circ \varphi^{-1}$ is also smooth at $\varphi(p)$. In other words, f is smooth at p. Since p was chosen arbitrarily from N, f is smooth on N. ■

§2.3 Partial Derivatives

On a manifold M of dimension n, let (U, φ) be a chart and $f : M \to \mathbb{R}$ a C^∞ function. As a function into $\mathbb{R}^n$, φ has n components: $x^1, x^2, \ldots, x^n$. Let $r^1.r^2, \ldots, r^n$ be standard coordinates on $\mathbb{R}^n$. That is, if $\vec{v} \equiv (v^1, v^2, \ldots, v^n) \in \mathbb{R}^n$, then $r^i(\vec{v}) = v^i$ for $1 \le i \le n$.

Now, $x^i = r^i \circ \varphi$. For $p \in U$, one defines the partial derivative $\frac{\partial f}{\partial x^i}$ of f with respect to x^i at p to be

$$\left.\frac{\partial}{\partial x^i}\right|_p f := \frac{\partial f}{\partial x^i}(p) := \frac{\partial \left(f \circ \varphi^{-1}\right)}{\partial r^i}(\varphi(p)) = \left.\frac{\partial}{\partial r^i}\right|_{\varphi(p)} \left(f \circ \varphi^{-1}\right).$$

Since $p = \varphi^{-1}(\varphi(p))$, the equation can be rewritten as

$$\frac{\partial f}{\partial x^i}\left(\varphi^{-1}(\varphi(p))\right) = \frac{\partial \left(f \circ \varphi^{-1}\right)}{\partial r^i}(\varphi(p)) \implies \left(\frac{\partial f}{\partial x^i} \circ \varphi^{-1}\right)(\varphi(p)) = \frac{\partial \left(f \circ \varphi^{-1}\right)}{\partial r^i}(\varphi(p)).$$

Thus, as functions on $\varphi(U)$,

$$\frac{\partial f}{\partial x^i} \circ \varphi^{-1} = \frac{\partial \left(f \circ \varphi^{-1}\right)}{\partial r^i}.$$

The partial derivative $\frac{\partial f}{\partial x^i}$ is C^∞ on U because its pullback $\frac{\partial f}{\partial x^i} \circ \varphi^{-1}$ is C^∞ on $\varphi(U)$.

Proposition 2.12

Suppose $(U, x^1, \ldots, x^n)$ is a chart on a manifold. Then $\dfrac{\partial x^i}{\partial x^j} = \delta^i_j$.

Proof. At a point $p \in U$, using $x^i = r^i \circ \varphi$,

$$\frac{\partial x^i}{\partial x^j}(p) = \frac{\partial \left(x^i \circ \varphi^{-1}\right)}{\partial r^j}(\varphi(p)) = \frac{\partial \left(r^i \circ \varphi \circ \varphi^{-1}\right)}{\partial r^j}(\varphi(p)) = \frac{\partial r^i}{\partial r^j}(\varphi(p)) = \delta^i_j\,.$$

■

Definition 2.5. Let $F : N \to M$ be a smooth map, and let $(U, \varphi) = (U, x^1, \ldots, x^n)$ and $(V, \psi) = (V, y^1, \ldots, y^m)$ be charts on N and M respectively such that $F(U) \subset V$. Denote by

$$F^i := y^i \circ F = r^i \circ \psi \circ F : U \to \mathbb{R}$$

the i-th component of F in the chart (V, ψ). Then the $m \times n$ matrix $\left[\frac{\partial F^i}{\partial x^j}\right]$ is called the **Jacobian matrix** of F relative to the charts (U, φ) and (V, ψ). In case N and M have the same dimension, the determinant of the Jacobian matrix is called the **Jacobian determinant** of F relative to the two charts. The Jacobian determinant is also written as

$$\det\left[\frac{\partial F^i}{\partial x^j}\right] = \frac{\partial\left(F^1, \ldots, F^n\right)}{\partial\left(x^1, \ldots, x^n\right)}\,.$$

Example 2.2 (Jacobian matrix of a transition map)**.** Let $(U, \varphi) = (U, x^1, \ldots, x^n)$ and $(V, \psi) = (V, y^1, \ldots, y^n)$ be overlapping charts on a manifold M. The transition map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is a diffeomorphism of open subsets of $\mathbb{R}^n$. Then its Jacobian matrix $J\left(\psi \circ \varphi^{-1}\right)$ at $\varphi(p)$ is the matrix $\left[\frac{\partial y^i}{\partial x^j}\right]$ of partial derivatives at p.

Since $\psi \circ \varphi^{-1}$ is a map between two open subsets of Euclidean spaces, $J\left(\psi \circ \varphi^{-1}\right) = \left[\frac{\partial\left(\psi \circ \varphi^{-1}\right)^i}{\partial r^j}\right]$.

$$\begin{aligned}
\frac{\partial\left(\psi \circ \varphi^{-1}\right)^i}{\partial r^j}(\varphi(p)) &= \frac{\partial\left(r^i \circ \psi \circ \varphi^{-1}\right)}{\partial r^j}(\varphi(p)) \\
&= \frac{\partial\left(y^i \circ \varphi^{-1}\right)}{\partial r^j}(\varphi(p)) \\
&= \frac{\partial y^i}{\partial x^j}(p)
\end{aligned}$$

Definition 2.6. A C^∞ map $F : N \to M$ is **locally invertible** at $p \in N$ if p has a neighborhood U on which $F|_U : U \to F(U)$ is a diffeomorphism.

Theorem 2.13 (Inverse Function Theorem for Manifolds)

Let $F : N \to M$ be a C^∞ map between two manifolds of the same dimension, and $p \in N$. Suppose for some charts $(U, \varphi) = (U, x^1, \ldots, x^n)$ about $p \in N$ and $(V, \psi) = (V, y^1, \ldots, y^n)$ about $F(p) \in M$, $F(U) \subseteq V$. Set $F^i = y^i \circ F$. Then F is locally invertible at p if and only if its Jacobian determinant $\det\left[\frac{\partial F^i}{\partial x^j}(p)\right]$ is nonzero.

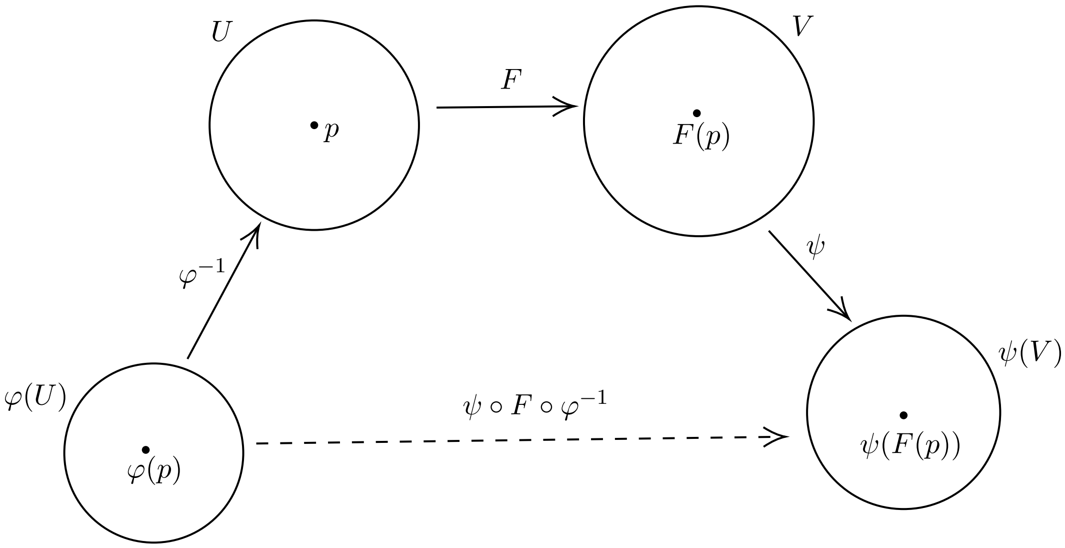
Proof. Since $F^i = y^i \circ F = r^i \circ \psi \circ F$, the Jacobian matrix of F relative to the charts (U, φ) and (V, ψ) is

$$\left[\frac{\partial F^i}{\partial x^j}(p)\right] = \left[\frac{\partial\left(r^i \circ \psi \circ F\right)}{\partial x^j}(p)\right] = \left[\frac{\partial\left(r^i \circ \psi \circ F \circ \varphi^{-1}\right)}{\partial r^j}(\varphi(p))\right] =$$

which is the Jacobian matrix of the map

$$\psi \circ F \circ \varphi^{-1} : \varphi(U) \subseteq \mathbb{R}^n \to \psi(V) \subseteq \mathbb{R}^n$$

between two open subsets of $\mathbb{R}^n$.



By Inverse Function Theorem for $\mathbb{R}^n$, $\psi \circ F \circ \varphi^{-1}$ is locally invertible at $\varphi(p)$ if and only if

$$\det\left[\frac{\partial F^i}{\partial x^j}(p)\right] = \det\left[\frac{\partial\left(\psi \circ F \circ \varphi^{-1}\right)^i}{\partial r^j}(\varphi(p))\right] \neq 0\,.$$

By Proposition 2.5, φ and ψ are diffeomorphisms. Therefore, local invertibility of $\psi \circ F \circ \varphi^{-1}$ at $\varphi(p)$ is equivalent to local invertibility of F at p. ■

Corollary 2.14

Let N be a manifold of dimension n. A set of n smooth functions $F^1, F^2, \ldots, F^n$ defined on a coordinate neighborhood $(U, x^1, x^2, \ldots, x^n)$ of a point $p \in N$ forms a coordinate system about p if and only if the Jacobian determinant $\det\left[\frac{\partial F^i}{\partial x^j}(p)\right]$ is nonzero.

Proof. **(⇒):** Let $F = (F^1, F^2, \ldots, F^n) : U \to \mathbb{R}^n$. If there exists a coordinate neighborhood $(W, F^1, F^2, \ldots, F^n)$ [1] about p in the maximal atlas of N, then F is a coordinate map, and hence $F : W \to F(W) \subseteq \mathbb{R}^n$ is a diffeomorphism by Proposition 2.5. In other words, F is locally invertible at p. Therefore, by Inverse Function Theorem for Manifolds, $\det\left[\frac{\partial F^i}{\partial x^j}(p)\right] \neq 0$.

(⇐): Since $\det\left[\frac{\partial F^i}{\partial x^j}(p)\right]$ is nonzero, $F : U \to \mathbb{R}^n$ is locally invertible at p (Inverse Function Theorem for Manifolds). In other words, there is a neighborhood W of $p \in N$ such that $F : W \to F(W) \subseteq \mathbb{R}^n$ is a diffeomorphism. Then by Proposition 2.6, there is a coordinate neighborhood $(W, F^1, F^2, \ldots, F^n)$ in the maximal atlas of N. ■

[1] Technically, it should be $F^i\big|_W$ instead of just F^i

3 Some Interesting Manifolds

§3.1 Real Projective Space

Define an equivalence relation $\sim$ on $\mathbb{R}^{n+1} \setminus \{0\}$ by

$$x \sim y \iff y = tx \text{ for some nonzero real number } t\,,$$

where $x, y \in \mathbb{R}^{n+1} \setminus \{0\}$. The **real projective space** is the quotient space of $\mathbb{R}^{n+1} \setminus \{0\}$ by this equivalence relation and is denoted by $\mathbb{R}P^n$. We denote the equivalence class of a point $(a^0, a^1, \ldots, a^n) \in \mathbb{R}^{n+1} \setminus \{0\}$ by $[a^0, a^1, \ldots, a^n]$ and let $\pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}P^n$ be the underlying projection map. We call $[a^0, a^1, \ldots, a^n]$ homogenous coordinates on $\mathbb{R}P^n$.

Geometrically, two nonzero points of $\mathbb{R}^{n+1}$ are equivalent if and only if they lie on the same line through the origin. So $\mathbb{R}P^n$ can be though of as the set of all lines through the origin in $\mathbb{R}^{n+1}$. A line through the origin in $\mathbb{R}^{n+1}$ is just a point in $\mathbb{R}P^n$.

Each line through the origin in $\mathbb{R}^{n+1}$ meets the unit sphere S^n in a pair of antipodal points. Conversely, a pair of antipodal points on S^n determines a unique line in $\mathbb{R}^{n+1}$.

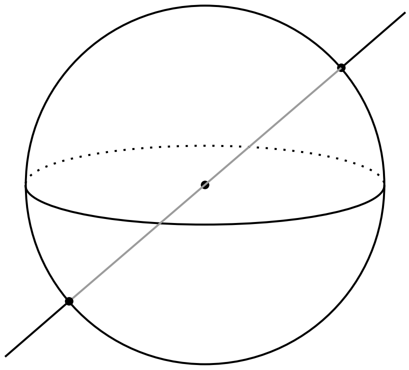


Figure 3.1: A line through 0 in $\mathbb{R}^3$ corresponds to a pair of antipodal points on S^2.

This suggests that we can define an equivalence relation $\sim$ on S^n by identifying the antipodal points:

$$x \sim y \iff x = \pm y\,, \quad x, y \in S^n\,.$$

We then have a bijection $\mathbb{R}P^n \leftrightarrow S^n/\!\sim$. We shall now see that this bijection is a homeomorphism.

Lemma 3.1
$\mathbb{R}P^n$ is homeomorphic to $S^n/\!\sim$.

Proof. Consider $f : \mathbb{R}^{n+1} \setminus \{0\} \to S^n$ defined by $f(x) = \frac{x}{\|x\|}$. Then f is continuous. Note that, for a nonzero real t,

$$f(tx) = \frac{tx}{\|tx\|} = \frac{t}{|t|}\frac{x}{\|x\|} = \begin{cases} f(x) & \text{if } t > 0 \\ -f(x) & \text{if } t < 0 \end{cases}$$

Now we define $\overline{f} : \mathbb{R}P^n \to S^n/\!\sim$ by $\overline{f}([x]) = [f(x)]$. This map is well-defined, since

$$\overline{f}([tx]) = [f(tx)] = [\pm f(x)] = [f(x)] = \overline{f}([x])\,.$$

Let $\pi_1 : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}P^n$ and $\pi_2 : S^n \to S^n/\!\sim$ be the respective projection maps. Now we have a commutative diagram.

$$\begin{tikzcd}
\mathbb{R}^{n+1} \setminus \{0\} \arrow[r, "f"] \arrow[d, "\pi_1"'] \arrow[dr, "\pi_2 \circ f"] & S^n \arrow[d, "\pi_2"] \\
\mathbb{R}P^n \arrow[r, "\overline{f}"'] & S^n/\sim
\end{tikzcd}$$

Now, $\pi_2 \circ f$ is the composition of two continuous maps, hence continuous. Therefore, by Proposition A.25, $\overline{f}$ is continuous.

Now, let $g : S^n \to \mathbb{R}^{n+1} \setminus \{0\}$ be the inclusion map given by $g(x) = x$. We know that g is continuous. It induces another map $\overline{g} : S^n/\sim \to \mathbb{R}P^n$ defined by $\overline{g}([x]) = [x]$. $\overline{g}$ is well-defined, because

$$\overline{g}([-x]) = [-x] = [x] = \overline{g}([x]) \ .$$

As before, we have another commutative diagram.

$$\begin{tikzcd}
S^n \arrow[r, "g"] \arrow[d, "\pi_2"'] \arrow[dr, "\pi_1 \circ g"] & \mathbb{R}^{n+1} \setminus \{0\} \arrow[d, "\pi_1"] \\
S^n/\sim \arrow[r, "\overline{g}"'] & \mathbb{R}P^n
\end{tikzcd}$$

$\pi_1 \circ g$ is the composition of two continuous maps, hence continuous. Therefore, by Proposition A.25, $\overline{g}$ is continuous. Now we are only left to show that $\overline{f}$ and $\overline{g}$ are inverses of one another. For $[x] \in \mathbb{R}P^n$,

$$\left(\overline{g} \circ \overline{f}\right)[x] = \overline{g}\left[\frac{x}{\|x\|}\right] = \left[\frac{x}{\|x\|}\right] = [x] \ ,$$

because $x \sim \frac{x}{\|x\|}$ in $\mathbb{R}^{n+1} \setminus \{0\}$, where the value of the nonzero real t is $\frac{1}{\|x\|}$. Furthermore, for $[x] \in S^n/\sim$, $x \in S^n$, so $\|x\| = 1$.

$$\left(\overline{f} \circ \overline{g}\right)[x] = \overline{f}[x] = \left[\frac{x}{\|x\|}\right] = [x] \ .$$

Hence, $\overline{g}$ is indeed the inverse of $\overline{f}$. Therefore, $\overline{f} : \mathbb{R}P^n \to S^n/\sim$ is a homeomorphism. ■

Proposition 3.2

The equivalence relation $\sim$ on $\mathbb{R}^{n+1} \setminus \{0\}$ is an open equivalence relation.

Proof. For an open set $U \subseteq \mathbb{R}^{n+1} \setminus \{0\}$, $\pi(U)$ is open in $\mathbb{R}P^n$ if and only if $\pi^{-1}(\pi(U))$ is open in $\mathbb{R}^{n+1} \setminus \{0\}$ (definition of Quotient Topology). Now, if we take an arbitrary point of U and then take its nonzero multiple, both the points will belong to the same equivalence class in $\mathbb{R}P^n$. In other words, for $x \in U$, tx and x will be mapped to the same point in $\pi(U)$. Hence,

$$\pi^{-1}(\pi(U)) = \bigcup_{t \in \mathbb{R}^\times} tU = \bigcup_{t \in \mathbb{R}^\times} \{tx \mid x \in U\} \ ,$$

where $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$. The map of multiplication by a nonzero real t is a homeomorphism from $\mathbb{R}^{n+1} \setminus \{0\}$ to itself. Hence, tU is open in $\mathbb{R}^{n+1} \setminus \{0\}$ for any nonzero t. Therefore, their union

$$\bigcup_{t \in \mathbb{R}^\times} tU = \pi^{-1}(\pi(U))$$

is also open in $\mathbb{R}^{n+1} \setminus \{0\}$. ■

Corollary 3.3
The real projective space $\mathbb{R}P^n$ is second countable.

Proof. Follows from the fact that $\mathbb{R}^{n+1} \setminus \{0\}$ is second countable and Corollary A.39. ∎

Proposition 3.4
$\mathbb{R}P^n$ is Hausdorff.

Proof. Let $S = \mathbb{R}^{n+1} \setminus \{0\}$. Now, consider the set

$$R = \left\{(x, y) \in S \times S \mid y = tx \text{ for some } t \in \mathbb{R}^\times\right\} = \left\{(x, y) \in S \times S \mid x \sim y\right\} .$$

R is the graph of $\sim$. We want to show that R is closed in $S \times S$. Consider the real valued function $f : S \times S \to \mathbb{R}$ defined by

$$f(x, y) = f\left(x^0, \ldots, x^n, y^0, \ldots, y^n\right) = \sum_{i \neq j} \left(x^i y^j - x^j y^i\right)^2 .$$

Note that f is continuous and vanishes if and only if $y = tx$ for some $t \in \mathbb{R}^\times$, since

$$\begin{aligned} f(x, y) = 0 &\iff \left(x^i y^j - x^j y^i\right)^2 \text{ for every } i \neq j \\ &\iff x^i y^j = x^j y^i \text{ for every } i \neq j \\ &\iff \frac{x^i}{y^i} = \frac{x^j}{y^j} \text{ for every } i \neq j \\ &\iff y = tx \text{ for some } t \in \mathbb{R}^\times \end{aligned}$$

Therefore, $R = f^{-1}(\{0\})$. $\{0\}$ is closed in $\mathbb{R}$ and f is continuous. Hence, R is closed in $S \times S$. Therefore, by Theorem A.36, $S/\sim = \mathbb{R}P^n$ is Hausdorff. ∎

The Standard Atlas on Real Projective Space

Let $[a^0, a^1, \ldots, a^n]$ be homogenous coordinates on projective space $\mathbb{R}P^n$. Consider the set

$$U_0 = \left\{\left[a^0, a^1, \ldots, a^n\right] \in \mathbb{R}P^n \mid a^0 \neq 0\right\} .$$

Let us denote by $\widetilde{U_0}$ the following set

$$\widetilde{U_0} = \left\{\left(a^0, a^1, \ldots, a^n\right) \in \mathbb{R}^{n+1} \setminus \{0\} \mid a^0 \neq 0\right\} .$$

The projection map $p_0 : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}$ onto first coordinate is continuous, and $\widetilde{U_0} = p_0^{-1}(\mathbb{R}^\times)$. $\mathbb{R}^\times$ is open in $\mathbb{R}$, so $\widetilde{U_0}$ is open in $\mathbb{R}^{n+1} \setminus \{0\}$. Note that $U_0 = \pi\left(\widetilde{U_0}\right)$. Since $\sim$ is an open equivalence relation and $\widetilde{U_0}$ is open, U_0 is also open in $\mathbb{R}P^n$. In a similar manner, we can also define the following open subsets of $\mathbb{R}P^n$ for each $i = 1, \ldots, n$.

$$U_i = \left\{\left[a^0, a^1, \ldots, a^n\right] \in \mathbb{R}P^n \mid a^i \neq 0\right\} .$$

It is trivial that

$$\bigcup_{i=0}^{n} U_0 = \mathbb{R}P^n .$$

Now, define $\widetilde{\varphi_0} : \widetilde{U_0} \to \mathbb{R}^n$ by

$$\varphi_0\left(a^0, a^1, \ldots, a^n\right) = \left(\frac{a^1}{a^0}, \frac{a^2}{a^0}, \ldots, \frac{a^n}{a^0}\right) .$$

$\widetilde{\varphi_0}$ is continuous since $a^0 \neq 0$. This induces a map $\varphi_0 : U_0 \to \mathbb{R}^n$ by

$$\varphi_0\left(\left[a^0, a^1, \ldots, a^n\right]\right) = \varphi_0\left(a^0, a^1, \ldots, a^n\right) = \left(\frac{a^1}{a^0}, \frac{a^2}{a^0}, \ldots, \frac{a^n}{a^0}\right).$$

This map is well-defined since

$$\varphi_0\left(\left[ta^0, ta^1, \ldots, ta^n\right]\right) = \left(\frac{ta^1}{ta^0}, \frac{ta^2}{ta^0}, \ldots, \frac{ta^n}{ta^0}\right) = \varphi_0\left(\left[a^0, a^1, \ldots, a^n\right]\right).$$

Due to Proposition A.25, continuity of $\widetilde{\varphi_0}$ implies continuity of φ_0. φ_0 has a continuous inverse $\varphi_0^{-1} : \mathbb{R}^n \to U_0$ given by

$$\varphi_0^{-1}\left(b^1, b^2, \ldots, b^n\right) = \pi\left(1, b^1, b^2, \ldots, b^n\right) = \left[1, b^1, b^2, \ldots, b^n\right].$$

φ_0^{-1} is continuous because it is the composition of π and the continuous map $(b^1, \ldots, b^n) \mapsto (1, b^1, b^2, \ldots, b^n)$. Now we shall check that φ_0^{-1} is indeed the inverse of φ_0.

$$\left(\varphi_0 \circ \varphi_0^{-1}\right)\left(b^1, b^2, \ldots, b^n\right) = \varphi_0\left(\left[1, b^1, b^2, \ldots, b^n\right]\right) = \left(b^1, b^2, \ldots, b^n\right).$$

$$\left(\varphi_0^{-1} \circ \varphi_0\right)\left[a^0, a^1, \ldots, a^n\right] = \varphi_0^{-1}\left(\frac{a^1}{a^0}, \frac{a^2}{a^0}, \ldots, \frac{a^n}{a^0}\right) = \left[1, \frac{a^1}{a^0}, \frac{a^2}{a^0}, \ldots, \frac{a^n}{a^0}\right] = \left[a^0, a^1, \ldots, a^n\right].$$

Hence, φ_0^{-1} is indeed the inverse of φ. Therefore, φ_0 is a homeomorphism. Similarly, there are homeomorphisms $\varphi_i : U_i \to \mathbb{R}^n$ for each $i = 1, \ldots, n$.

$$\varphi_i\left(\left[a^0, \ldots, a^n\right]\right) = \left(\frac{a^0}{a^i}, \ldots, \widehat{\frac{a^i}{a^i}}, \ldots, \frac{a^n}{a^i}\right),$$

where the caret sign $\widehat{}$ over $\frac{a^i}{a^i}$ means that this entry is to be omitted. This proves that $\mathbb{R}P^n$ is locally Euclidean with $(U_i.\varphi_i)$ as charts.

Now, on the intersection $U_0 \cap U_1$, there are two charts. For $[a^0, a^1, \ldots, a^n] \in U_0 \cap U_1$, we have $a_0 \neq 0$ and $a_1 \neq 0$.

$$\begin{tikzcd}
& \left[a^0, a^1, \ldots, a^n\right] \arrow[dl, mapsto, "\varphi_0"'] \arrow[dr, mapsto, "\varphi_1"] & \\
\left(\frac{a^1}{a^0}, \frac{a^2}{a^0}, \ldots, \frac{a^n}{a^0}\right) & & \left(\frac{a^0}{a^1}, \frac{a^2}{a^1}, \ldots, \frac{a^n}{a^1}\right)
\end{tikzcd}$$

On $U_0 \cap U_1$, one has $\varphi_0\left(U_0 \cap U_1\right) = \{(b^1, b^2, \ldots, b^n) \in \mathbb{R}^n \mid b^1 \neq 0\}$. Given $(b^1, b^2, \ldots, b^n) \in \varphi_0\left(U_0 \cap U_1\right)$, one obtains

$$\left(\varphi_1 \circ \varphi_0^{-1}\right)\left(b^1, b^2, \ldots, b^n\right) = \varphi_1\left(\left[1, b^1, b^2, \ldots, b^n\right]\right) = \left(\frac{1}{b^1}, \frac{b^2}{b^1}, \ldots, \frac{b^n}{b^1}\right).$$

This is a C^∞ map between open subsets of $\mathbb{R}^n$ since $b^1 \neq 0$ for $(b^1, b^2, \ldots, b^n) \in \varphi_0\left(U_0 \cap U_1\right)$. In a similar manner, one can show that $\varphi_i \circ \varphi_j^{-1}$ is C^∞ for every i, j. Therefore,

$$\{(U_i, \varphi_i) \mid 0 \leq i \leq n\}$$

is a C^∞ atlas on $\mathbb{R}P^n$, called the **standard atlas**. So we have shown that $\mathbb{R}P^n$ is second countable, Hausdorff locally Euclidean space equipped with a C^∞ atlas. Therefore, $\mathbb{R}P^n$ is a smooth manifold.

§3.2 The Grassmannian

The Grassmannianin $G(k,n)$ is the set of all k-planes through the origin in $\mathbb{R}^n$. Such a k-plane is a linear subspace of dimension k of $\mathbb{R}^n$ and has a basis consisting of k linearly independent vectors $a_1, a_2, \dots, a_k$ in $\mathbb{R}^n$. It is therefore completely specified by an $n \times k$ matrix

$$A = \begin{bmatrix} a_1 & a_2 & \cdots & a_k \end{bmatrix} \text{ such that } \operatorname{rank} A = k\,.$$

This matrix is called a matrix representative of the k-plane. Two bases $a_1, \dots, a_k$ and $b_1, \dots, b_k$ determine the same k-plane if there is a change-of-basis matrix $g = [g_{ij}] \in \mathrm{GL}(k, \mathbb{R})$ such that

$$b_j = \sum_i a_i g_{ij}\,.\text{ In matrix notation, } B = Ag\,.$$

Let $F(k,n)$ be the set of all $n \times k$ matrices of rank k, topologized as a subspace of $\mathbb{R}^{n\times k}$. We define an equivalence relation $\sim$ on $F(k,n)$ as follows:

$$A \sim B \iff \text{there is a matrix } g \in \mathrm{GL}(k,\mathbb{R}) \text{ such that } B = Ag\,.$$

There is a bijection between $G(k,n)$ and the quotient space $F(k,n)/\!\sim$. We give the Grassmannian $G(k,n)$ the quotient topology on $F(k,n)/\!\sim$. Let $\pi : F(k,n) \to F(k,n)/\!\sim$ be the quotient map.

Lemma 3.5

Let A be an $m \times n$ matrix (not necessarily square), and k a positive integer. Then $\operatorname{rank} A \geq k$ if and only if A has a nonsingular $k \times k$ submatrix. Equivalently, $\operatorname{rank} A \leq k-1$ if and only if all $k \times k$ minors of A vanish. (A $k \times k$ minor of a matrix A is the determinant of a $k \times k$ submatrix of A.)

Proof. **($\Rightarrow$):** Suppose $\operatorname{rank} A \geq k$. Then one can find k linearly independent columns, which we call $a_1, \dots, a_k$. Since the $m \times k$ matrix $B = \begin{bmatrix} a_1 & \cdots & a_k \end{bmatrix}$ has rank k, it has k linearly independent rows $b_1, \dots, b_k$. The submatrix C of B whose rows are $b_1, \dots, b_k$ is a $k \times k$ submatrix of A, and $\operatorname{rank} C = k$. In other words, C is a nonsingular $k \times k$ submatrix of A.

($\Leftarrow$): Suppose A has a nonsingular $k \times k$ submatrix B. Let $a_1, \dots, a_k$ be the columns of A such that the submatrix $\begin{bmatrix} a_1 & \cdots & a_k \end{bmatrix}$ contains B. Since $\begin{bmatrix} a_1 & \cdots & a_k \end{bmatrix}$ has k linearly independent rows, it also has k linearly independent columns. Thus, $\operatorname{rank} A \geq k$. ∎

Lemma 3.6

If $A \in F(k,n)$, then $Ag \in F(k,n)$ for every $g \in \mathrm{GL}(k,\mathbb{R})$.

Proof. Using the Linear Algebra fact that $\operatorname{rank}(XY) \leq \min\{\operatorname{rank} X, \operatorname{rank} Y\}$, we get that

$$\operatorname{rank}(Ag) \leq \min\{\operatorname{rank} A, \operatorname{rank} g\} = k\,.$$

Using the same result on $A = Ag\,g^{-1}$, we get

$$k = \operatorname{rank}(A) \leq \min\left\{\operatorname{rank}(Ag), \operatorname{rank} g^{-1}\right\} = \min\{\operatorname{rank}(Ag), k\} = \operatorname{rank}(Ag)\,.$$

Therefore, $\operatorname{rank}(Ag) = k$. ∎

So we get, the multiplication-by-g-map m_g that maps $A \in F(k,n)$ to Ag is truly a map from $F(k,n)$ to itself. m_g is continuous, because the components are nothing but polynomials in the entries. The inverse map of m_g is $m_{g^{-1}}$, since

$$\left(m_g \circ m_{g^{-1}}\right)(A) = m_g\left(Ag^{-1}\right) = Ag^{-1}g = A\,,$$

$$\left(m_{g^{-1}} \circ m_g\right)(A) = m_{g^{-1}}(Ag) = Agg^{-1} = A\,.$$

Therefore, $m_{g^{-1}} = m_g^{-1}$. It is also continuous by a similar reasoning. Therefore, m_g is a homeomorphism from $F(k,n)$ to itself.

Proposition 3.7

The equivalence relation $\sim$ on $F(k,n)$ is an open equivalence relation.

Proof. We shall mimic the proof of Proposition 3.2. For an open set $U \subseteq F(k,n)$, $\pi(U)$ is open in $G(k,n)$ if and only if $\pi^{-1}(\pi(U))$ is open in $F(k,n)$ (definition of Quotient Topology).

$$\pi^{-1}(\pi(U)) = \bigcup_{A\in U} [A] = \bigcup_{A\in U} \{Ag \mid g \in \mathrm{GL}(k,\mathbb{R})\} = \bigcup_{g\in \mathrm{GL}(k,\mathbb{R})} \{Ag \mid A \in U\} = \bigcup_{g\in \mathrm{GL}(k,\mathbb{R})} m_g(U) \,.$$

The map $m_g : F(k,n) \to F(k,n)$ is a homeomorphism, as shown above. Therefore, it is an open map. So $m_g(U)$ is open in $F(k,n)$ for every $g \in \mathrm{GL}(k,\mathbb{R})$. Hence, their union

$$\bigcup_{g\in \mathrm{GL}(k,\mathbb{R})} m_g(U) = \pi^{-1}(\pi(U))$$

is also open in $F(k,n)$. ■

Corollary 3.8

The Grassmannian $G(k,n)$ is second countable.

Proof. $F(k,n)$ is a subspace of the second countable space $\mathbb{R}^{n\times k}$, hence it is also second countable. $\sim$ is an open equivalence relation. Therefore, by Corollary A.39, $F(k,n)/\sim = G(k,n)$ is second countable. ■

Proposition 3.9

$G(k,n)$ is Hausdorff.

Proof. Let $S = F(k,n)$. Now, consider the set

$$R = \{(A,B) \in S\times S \mid B = Ag \text{ for some } g \in \mathrm{GL}(k,\mathbb{R})\} = \{(A,B)\in S\times S \mid A\sim B\}\,.$$

R is the graph of $\sim$. We want to show that R is closed. Take $A = \begin{bmatrix} a_1 & \cdots & a_k\end{bmatrix}$ and $B = \begin{bmatrix} b_1 & \cdots & b_k\end{bmatrix}$ from S. $A \sim B$ if and only if the columns of B can be expressed as a linear combination of the columns of A. In that case, we would have that the $n\times 2k$ matrix $M_{A,B} = \begin{bmatrix} A & B\end{bmatrix}$ has k linearly independent columns. In other words, rank $M_{A,B} = k \le k$. By Lemma 3.5, this is equivalent to **all** $(k+1)\times(k+1)$ minors of $M_{A,B}$ being 0.

Let $I = (i_0, i_1, \ldots, i_k)$ and $J = (j_0, j_1, \ldots, j_k)$ be $(k+1)$-tuples of integers such that

$$1 \le i_0 < i_1 < \cdots < i_k \le n \quad\text{and}\quad 1 \le j_0 < j_1 < \cdots < j_k \le 2k\,.$$

Define $f_{I,J} : S\times S \to \mathbb{R}$ such that it takes two matrices A and B and returns the $(k+1)\times(k+1)$ minor of $M_{A,B}$ corresponding to rows $i_0, i_1, \ldots, i_k$ and columns $j_0, j_1, \ldots, j_k$.

We have seen before that $A \sim B$ if and only if **all** the $(k+1)\times(k+1)$ minors of $M_{A,B}$ are 0. Therefore,

$$A\sim B \iff f_{I,J}(A,B) = 0 \text{ for every } I, J\,.$$

So, we can write R as

$$R = \bigcap_{I,J} f_{I,J}^{-1}(\{0\})\,.$$

$f_{I,J} : S\times S \to \mathbb{R}$ is continuous since determinant is nothing but a polynomial of the entries. $\{0\}$ is closed in $\mathbb{R}$. Therefore, $f_{I,J}^{-1}(\{0\})$ is closed in $S\times S$. There are only finitely many choices for I, J. In fact, there are total

$$\binom{n}{k+1}\binom{2k}{k+1}$$

ways one can choose I, J. Intersection of finitely many closed sets is closed. Therefore, R is closed in $S\times S$. Hence, by Theorem A.36, $S/\sim = G(k,n)$ is Hausdorff. ■

Next we want to find a C^{∞} atlas on the Grassmannian $G(k,n)$. For simplicity, we specialize to $G(2,4)$. For any 4×2 matrix A, let A_{ij} be the 2×2 submatrix consisting of its i-th row and j-th row. Define

$$V_{ij}=\{A\in F(2,4)\mid A_{ij}\text{ is nonsingular }\}=\{A\in F(2,4)\mid \det A_{ij}\neq 0\}\ .$$

The map $A\mapsto \det A_{ij}$ is a continuous real-valued function. V_{ij} is the pre-image of $\mathbb{R}\setminus\{0\}$ under this continuous function. So, we can conclude that V_{ij} is an open subset of $F(2,4)$.

Lemma 3.10

If $A\in V_{ij}$ then $Ag\in V_{ij}$ for every $g\in \mathrm{GL}(2,\mathbb{R})$.

Proof. Let A_i be the i-th row of A.

$$A=\begin{bmatrix}A_1\\A_2\\A_3\\A_4\end{bmatrix}\implies Ag=\begin{bmatrix}A_1g\\A_2g\\A_3g\\A_4g\end{bmatrix}\implies (Ag)_{ij}=\begin{bmatrix}A_ig\\A_jg\end{bmatrix}=A_{ij}g\,.$$

Therefore, $\det(Ag)_{ij}=\det A_{ij}\det g\neq 0$. So $Ag\in V_{ij}$. ∎

Define $U_{ij}=V_{ij}/\!\sim\,=\pi(V_{ij})$. Since $\sim$ is an open equivalence relation, U_{ij} is an open subset of $G(2,4)$. Also, the collection $\{V_{ij}\}$ covers $F(2,4)$, so $\{U_{ij}\}$ covers $G(2,4)$. Now, for $A\in V_{12}$, $A_{12}\in\mathrm{GL}(2,\mathbb{R})$.

$$A\sim AA_{12}^{-1}=\begin{bmatrix}\mathbb{I}_2\\A_{34}A_{12}^{-1}\end{bmatrix}$$

So we define a map $\widetilde{\varphi_{12}}:V_{12}\to\mathbb{R}^{2\times 2}$ by $\widetilde{\varphi_{12}}(A)=A_{34}A_{12}^{-1}$. This induces a map $\varphi_{12}:U_{12}\to\mathbb{R}^{2\times 2}$, $\varphi_{12}[A]=\widetilde{\varphi_{12}}(A)=A_{34}A_{12}^{-1}$. This map is well-defined, since

$$\varphi_{12}[Ag]=(Ag)_{34}(Ag)_{12}^{-1}=A_{34}g(A_{12}g)^{-1}=A_{34}gg^{-1}A_{12}^{-1}=A_{34}A_{12}^{-1}=\varphi_{12}[A]\ ,$$

for any $g\in\mathrm{GL}(2,\mathbb{R})$. $\widetilde{\varphi_{12}}$ is continuous since it is just rational function on the entries. In the light of Proposition A.25, continuity of $\widetilde{\varphi_{12}}$ implies the continuity of φ_{12}. φ_{12} has a continuous inverse $\varphi_{12}^{-1}:\mathbb{R}^{2\times 2}\to U_{12}$ given by

$$\varphi_{12}^{-1}(g)=\pi\left(\begin{bmatrix}\mathbb{I}_2\\g\end{bmatrix}\right)=\left[\begin{bmatrix}\mathbb{I}_2\\g\end{bmatrix}\right]$$

φ_{12}^{-1} is continuous because it is the composition of π and the continuous map that takes g to $\begin{bmatrix}\mathbb{I}_2\\g\end{bmatrix}$. Now we shall check that φ_{12}^{-1} is indeed the inverse of φ_{12}.

$$\left(\varphi_{12}\circ\varphi_{12}^{-1}\right)(g)=\varphi_{12}\left[\begin{bmatrix}\mathbb{I}_2\\g\end{bmatrix}\right]=g\mathbb{I}_2^{-1}=g\,.$$

$$\left(\varphi_{12}^{-1}\circ\varphi_{12}\right)[A]=\varphi_{12}^{-1}\left(A_{34}A_{12}^{-1}\right)=\left[\begin{bmatrix}\mathbb{I}_2\\A_{34}A_{12}^{-1}\end{bmatrix}\right]=[A]\ .$$

Hence, φ_{12}^{-1} is indeed the inverse of φ_{12}. Therefore, φ_{12} is a homeomorphism. Similarly, there are homeomorphisms $\varphi_{ij}:U_{ij}\to\mathbb{R}^{2\times 2}$ for every i,j. This proves $G(2,4)$ is locally Euclidean with (U_{ij},φ_{ij}) as charts.

Now, on the intersection $U_{12}\cap U_{23}$, there are two charts. For $[A]\in U_{12}\cap U_{23}$, both A_{12} and A_{23} are invertible. Take $\begin{bmatrix}a&b\\c&d\end{bmatrix}\in\varphi_{23}(U_{12}\cap U_{23})$. Then we have

$$\left(\varphi_{12}\circ\varphi_{23}^{-1}\right)\begin{bmatrix}a&b\\c&d\end{bmatrix}=\varphi_{12}\left[\begin{bmatrix}a&b\\1&0\\0&1\\c&d\end{bmatrix}\right]$$

Since $\left[\begin{bmatrix} a & b \\ 1 & 0 \\ 0 & 1 \\ c & d \end{bmatrix}\right] \in U_{12} \cap U_{23}$, $\begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix}$ is invertible. So b cannot be 0.

$$\left(\varphi_{12} \circ \varphi_{23}^{-1}\right) \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{b} & -\frac{a}{b} \\ \frac{d}{b} & c - \frac{da}{b} \end{bmatrix}$$

This is a C^∞ map between open subsets of $\mathbb{R}^{2\times 2} \cong \mathbb{R}^4$ since $b \neq 0$. In a similar manner, one can show that $\varphi_{ij} \circ \varphi_{pq}^{-1}$ is smooth for every i, j, p, q. Therefore,

$$\{(U_{ij}, \varphi_{ij}) \mid 1 \leq i < j \leq 4\}$$

is a C^∞ atlas on $G(2,4)$. So we have shown that $G(2,4)$ is second countable, Hausdorff locally Euclidean space equipped with a C^∞ atlas. Therefore, $G(2,4)$ is a smooth manifold.

Now it can be generalized in a similar manner. Let I be a strictly ascending multi-index $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. Let A_I be the $k \times k$ submatrix of A consisting of i_1-th, i_2-th, …, i_k-th rows of A. Define

$$V_I = \{A \in F(k,n) \mid \det A_I \neq 0\} \ .$$

V_I is an open subset of $F(k,n)$ because it is the pre-image of $\mathbb{R} \setminus \{0\}$ under the continuos real-valued function $A \mapsto \det A_I$. As before, it can be easily seen that $A \in V_I$ implies $Ag \in V_I$ for every $g \in \mathrm{GL}(k, \mathbb{R})$.

$$(Ag)_I = A_I g \implies \det (Ag)_I = \det A_I \det g \neq 0\,.$$

Let $U_I = V_I/{\sim} = \pi(V_I)$. Since $\sim$ is an open equivalence relation, $U_I = \pi(V_I)$ is open in $G(k,n)$. Also, the collection $\{V_I\}$ covers $F(k,n)$, so $\{U_I\}$ covers $G(k,n)$.

Next, we define $\widetilde{\varphi_I} : V_I \to \mathbb{R}^{(n-k)\times k}$ as follows

$$\widetilde{\varphi_I}(A) = \left(AA_I^{-1}\right)_{I'} ,$$

where $()_{I'}$ denotes the $(n-k) \times k$ submatrix obtained from the complement I' of the multi-index I. This induces a map $\varphi_I : U_I \to \mathbb{R}^{(n-k)\times k}$, $\varphi_I[A] = \widetilde{\varphi_I}(A) = \left(AA_I^{-1}\right)_{I'}$. One can easily check the well-definedness of φ_I.

$$\varphi_I[Ag] = \left(Ag\,(Ag)_I^{-1}\right)_{I'} = \left(Ag\,(A_I g)^{-1}\right)_{I'} = \left(Agg^{-1}A_I^{-1}\right)_{I'} = \left(AA_I^{-1}\right)_{I'} ,$$

for every $g \in \mathrm{GL}(k, \mathbb{R})$. $\widetilde{\varphi_I}$ is continuous. Therefore, by Proposition A.25, φ_I is continuous.

Let $\varphi_I^{-1} : \mathbb{R}^{(n-k)\times k} \to U_I$ be defined as follows: for $X \in \mathbb{R}^{(n-k)\times k}$, $\varphi_I^{-1}(X) = [A]$, where $A_I = \mathbb{I}_k$ and $A_{I'} = X$. Then φ_I^{-1} is easily seen to be continuous. Also, one can easily check that φ_I^{-1} is indeed the inverse of φ_I.

$$\left(\varphi_I \circ \varphi_I^{-1}\right)(X) = \varphi_I[A] = \left(AA_I^{-1}\right)_{I'} = \left(A\mathbb{I}_k^{-1}\right)_{I'} = A_{I'} = X\,.$$

$$\left(\varphi_I^{-1} \circ \varphi_I\right)[A] = \varphi_I^{-1}\left(AA_I^{-1}\right)_{I'} = [B] ,$$

where $B_I = \mathbb{I}_k$ and $B_{I'} = \left(AA_I^{-1}\right)_{I'}$. Therefore, $B = AA_I^{-1}$.

$$\left(\varphi_I^{-1} \circ \varphi_I\right)[A] = \left[AA_I^{-1}\right] = \left[AA_I^{-1}A_I\right] = [A]\ .$$

Therefore, φ_I^{-1} is indeed the inverse of φ_I. This completes the proof that φ_I is a homeomorphism. Hence, $G(k,n)$ is a locally Euclidean space with charts (U_I, φ_I).

Now, we want to show that $\varphi_I \circ \varphi_J^{-1}$ is a smooth map between open subsets of Euclidean space. For $[A] \in U_I \cap U_J$, both A_I and A_J are invertible. Now, take some $X \in \varphi_J (U_I \cap U_J)$. Then we have

$$\left(\varphi_I \circ \varphi_J^{-1}\right)(X) = \varphi_I [A] \ ,$$

with $A_J = \mathbb{I}_k$ and $A_{J'} = X$. Since $[A] \in U_I \cap U_J$, $\det A_I \neq 0$.

$$\left(\varphi_I \circ \varphi_J^{-1}\right)(X) = \left(AA_I^{-1}\right)_{I'} \ .$$

Now, the entries of $\left(AA_I^{-1}\right)_{I'}$ can be expressed as rational functions on the entries of X, with the denominator being $\det A_I \neq 0$. Therefore, we can conclude that $\varphi_I \circ \varphi_J^{-1}$ is a smooth map between open subsets of Euclidean space. Therefore,

$$\{(U_I, \varphi_I) \mid I \text{ is strictly asecnding multi-index } 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}$$

is a C^∞ atlas on $G(k,n)$. So we have shown that $G(k,n)$ is second countable, Hausdorff locally Euclidean space equipped with a C^∞ atlas. Therefore, $G(k,n)$ is a smooth manifold.

4 The Tangent Space

§4.1 The Tangent Space at a Point

We define a germ of a C^∞ function at $p \in M$ to be an equivalence class of C^∞ functions defined in some neighborhood of $p \in M$. Two functions of this sort are called equivalent if they agree on some, possibly smaller, neighborhood of p. In other words, two smooth functions $f : U \to \mathbb{R}$ and $g : V \to \mathbb{R}$ defined on some neighborhood of p are equivalent if there exists an open set $W \subseteq U \cap V$ containing p such that

$$f|_W = g|_W \, .$$

One can easily verify that this relation is indeed an equivalence relation. The set of germs of real-valued C^∞ functions at $p \in M$ is denoted by $C_p^\infty(M)$.

$C_p^\infty(M)$ is actually a set of equivalence classes. We denote the equivalence class of f as $[f]_p$. We can define addition, multiplication on $C_p^\infty(M)$ as follows:

$$[f]_p + [g]_p := [f+g]_p \ , [f]_p \cdot [g]_p := [f \cdot g]_p \ ,$$

where $f+g$ and $f \cdot g$ are defined pointwise, *i.e.* $(f+g)(x) = f(x) + g(x)$ and $(f \cdot g)(x) = f(x)\, g(x)$.

Now, we can check the well-definedness of addition and multiplication. Let $f \sim f'$ and $g \sim g'$. Then there exists neighborhoods U and V of p such that

$$f|_U = f'|_U \quad \text{and} \quad g|_V = g'|_V \, .$$

Now, on $U \cap V$ (which is also a neighborhood of p),

$$(f+g)(x) = f(x) + g(x) = f'(x) + g'(x) = (f'+g')(x) \implies f+g \sim f'+g' \, .$$

Therefore, addition in $C_p^\infty(M)$ is well-defined. One can similarly check that multiplication is also well-defined. With this addition and multiplication, $C_p^\infty(M)$ forms a ring. We can also define scalar multiplication by $\alpha \in \mathbb{R}$.

$$\alpha\, [f]_p := [\alpha f]_p \ ,$$

where $(\alpha f)(x) = \alpha f(x)$. The well-definedness of scalar multiplication can be checked in a similar manner. Let $f \sim f'$. So there exists a neighborhood U of p such that

$$f|_U = f'|_U \, .$$

For $x \in U$,

$$(\alpha f)(x) = \alpha f(x) = \alpha f'(x) = (\alpha f')(x) \implies \alpha f \sim \alpha f' \, .$$

Hence, scalar multiplication by a real number is well-defined. Therefore, with respect to the addition and scalar multiplication, $C_p^\infty(M)$ forms a vector space over $\mathbb{R}$. Since $C_p^\infty(M)$ is a ring and a vector space, and one can easily check that the following holds

$$\alpha\left([f]_p \cdot [g]_p\right) = \left(\alpha\, [f]_p\right) \cdot [g]_p = [f]_p \cdot \left(\alpha\, [g]_p\right) \ ,$$

$C_p^\infty(M)$ becomes an $\mathbb{R}$-algebra.

Abuse of Notation. Oftentimes we don't distinguish between an element of $C_p^\infty(M)$ (which is an equivalence class) and a representative of the class.

Definition 4.1 (Point-Derivation)**.** A point-derivation of $C_p^\infty(M)$ is a linear map $D_p : C_p^\infty(M) \to \mathbb{R}$ such that

$$D_p(fg) = (D_p f)\, g(p) + f(p)\,(D_p g)\ .$$

Definition 4.2 (Tangent Vector)**.** A tangent vector X_p at $p \in M$ is a point-derivation at p. The tangent vectors at $p \in M$ form a vector space denoted by T_pM.

Remark 4.1. If U is an open set containing $p \in M$, then the algebra $C_p^\infty(U)$ of germs of C^∞ functions in U at p is the same as $C_p^\infty(M)$. Hence, $T_pU = T_pM$.

We learned in chapter 4 that given the standard coordinates $r^1, r^2, \ldots, r^n$ on $\mathbb{R}^n$ and given a coordinate neighborhood $(U, \varphi) = (U, x^1, x^2, \ldots, x^n)$ about $p \in M$ with $x^i = r^i \circ \varphi$, one defines partial derivatives $\left.\frac{\partial}{\partial x^i}\right|_p$ at $p \in M$ as

$$\left.\frac{\partial}{\partial x^i}\right|_p f := \left.\frac{\partial}{\partial r^i}\right|_{\varphi(p)} \left(f \circ \varphi^{-1}\right) \in \mathbb{R}$$

with $f \in C_p^\infty(M)$. One can verify that $\left.\frac{\partial}{\partial x^i}\right|_p$ is a point-derivation.

$$\left.\frac{\partial}{\partial x^i}\right|_p (f \cdot g) = \left.\frac{\partial}{\partial r^i}\right|_{\varphi(p)} \left((f \cdot g) \circ \varphi^{-1}\right)\ .$$

Here, $f, g \in C_p^\infty(M)$. Now, for $a \in \varphi(U)$ with $\varphi^{-1}(a)$ belonging in the domain of both f and g,

$$\left((f \cdot g) \circ \varphi^{-1}\right)(a) = (f \cdot g)\left(\varphi^{-1}(a)\right) = f\left(\varphi^{-1}(a)\right) g\left(\varphi^{-1}(a)\right) = \left(\left(f \circ \varphi^{-1}\right) \cdot \left(g \circ \varphi^{-1}\right)\right)(a)\ .$$

Therefore, $(f \cdot g) \circ \varphi^{-1} = (f \circ \varphi^{-1}) \cdot (g \circ \varphi^{-1})$. As a result,

$$\begin{aligned}
\left.\frac{\partial}{\partial x^i}\right|_p (f \cdot g) &= \left.\frac{\partial}{\partial r^i}\right|_{\varphi(p)} \left(\left(f \circ \varphi^{-1}\right) \cdot \left(g \circ \varphi^{-1}\right)\right) \\
&= \left[\left.\frac{\partial}{\partial r^i}\right|_{\varphi(p)} \left(f \circ \varphi^{-1}\right)\right] \left(g \circ \varphi^{-1}\right)(\varphi(p)) + \left(f \circ \varphi^{-1}\right)(\varphi(p)) \left[\left.\frac{\partial}{\partial r^i}\right|_{\varphi(p)} \left(g \circ \varphi^{-1}\right)\right] \\
&= \left(\left.\frac{\partial}{\partial x^i}\right|_p f\right) g(p) + f(p) \left(\left.\frac{\partial}{\partial x^i}\right|_p g\right)
\end{aligned}$$

Furthermore, $\left.\frac{\partial}{\partial x^i}\right|_p$ is indeed linear. Therefore, it is a point-derivation.

Definition 4.3 (Differential of a Smooth Map)**.** Let $F : N \to M$ be a C^∞ map between manifolds. At $p \in N$, F induces a linear map between tangent spaces T_pN and $T_{F(p)}M$, called the **differential** of F at p, denoted by $F_{*,p} : T_pN \to T_{F(p)}M$ as follows. If $X_p \in T_pN$, then $F_{*,p}(X_p) \in T_{F(p)}M$ defined as

$$F_{*,p}(X_p)\, f := X_p(f \circ F)\ ,$$

where $f \in C_{F(p)}^\infty(M)$, so $f \circ F \in C_p^\infty(N)$.

Example 4.1 (Differential of a map between Euclidean Spaces)**.** Suppose $F : \mathbb{R}^n \to \mathbb{R}^m$ is smooth at $p \in \mathbb{R}^n$. Let $x^1, x^2, \ldots, x^n$ be the coordinates on $\mathbb{R}^n$ and $y^1, y^2, \ldots, y^m$ be the coordinates on $\mathbb{R}^m$. Then the tangent vectors

$$\left.\frac{\partial}{\partial x^1}\right|_p, \left.\frac{\partial}{\partial x^2}\right|_p, \ldots, \left.\frac{\partial}{\partial x^n}\right|_p$$

form a basis for the tangent space $T_p\mathbb{R}^n$, and the tangent vectors

$$\left.\frac{\partial}{\partial y^1}\right|_{F(p)}, \left.\frac{\partial}{\partial y^2}\right|_{F(p)}, \ldots, \left.\frac{\partial}{\partial y^m}\right|_{F(p)}$$

form a basis for the tangent space $T_{F(p)}\mathbb{R}^m$. The linear map $F_{*,p} : T_p\mathbb{R}^n \to T_{F(p)}\mathbb{R}^m$ is represented by a matrix $\left[a^i_j\,(p)\right]$ relative to these bases as follows:

$$F_{*,p}\left(\left.\frac{\partial}{\partial x^j}\right|_p\right) = \sum_{k=1}^{m} a^k_j\,(p)\,\left.\frac{\partial}{\partial y^k}\right|_{F(p)}\,,\, a^k_j\,(p) \in \mathbb{R}\,.$$

Let's evaluate both sides of the above equation on y^i.

$$\begin{aligned}\text{RHS} &= \sum_{k=1}^{m} a^k_j\,(p)\,\left.\frac{\partial}{\partial y^k}\right|_{F(p)} y^i = \sum_{k=1}^{m} a^k_j\,(p)\,\delta^i_k = a^i_j\,(p)\\ \text{LHS} &= F_{*,p}\left(\left.\frac{\partial}{\partial x^j}\right|_p\right) y^i = \left.\frac{\partial}{\partial x^j}\right|_p \left(y^i \circ F\right) = \left.\frac{\partial F^i}{\partial x^j}\right|_p\end{aligned}$$

where $F^i = y^i \circ F : \mathbb{R}^n \to \mathbb{R}$ is the i-th component of F. Therefore, the matrix representation of $F_{*,p}$ relative to the bases $\left\{\left.\frac{\partial}{\partial x^j}\right|_p\right\}$ and $\left\{\left.\frac{\partial}{\partial y^i}\right|_{F(p)}\right\}$ is $\left[\frac{\partial F^i}{\partial x^j}\,(p)\right]$. This is precisely the Jacobian matrix of the derivative DF_p of F at p as discussed in chapter 2.

Theorem 4.1 (The Chain Rule)
Let $F : N \to M$ and $G : M \to P$ be smooth maps of manifolds, and $p \in N$. Then,

$$(G \circ F)_{*,p} = G_{*,F(p)} \circ F_{*,p}\,.$$

In other words, the following diagram commutes:

$$\begin{tikzcd} T_pN \arrow[r, "F_{*,p}"] \arrow[rr, bend right, "(G\circ F)_{*,p}"'] & T_{F(p)}M \arrow[r, "G_{*,F(p)}"] & T_{G(F(p))}P \end{tikzcd}$$

Proof. Let $X_p \in T_pN$ and $f \in C^\infty_{G(F(p))}\,(P)$. Then

$$\left((G \circ F)_{*,p}\,X_p\right) f = X_p\,(f \circ G \circ F)\;.$$

Now, let $Y_{F(p)} = F_{*,p}X_p \in T_{F(p)}M$.

$$\begin{aligned}\left(\left(G_{*,F(p)} \circ F_{*,p}\right) X_p\right) f &= \left(G_{*,F(p)}\,(F_{*,p}X_p)\right) f = \left(G_{*,F(p)}\left(Y_{F(p)}\right)\right) f\\ &= Y_{F(p)}\,(f \circ G) = F_{*,p}X_p\,(f \circ G)\\ &= X_p\,(f \circ G \circ F)\end{aligned}$$

Therefore, $\left(G_{*,F(p)} \circ F_{*,p}\right) X_p = (G \circ F)_{*,p}\,X_p$, $\forall\, X_p \in T_pN$. Hence, $(G \circ F)_{*,p} = G_{*,F(p)} \circ F_{*,p}$. ■

Remark 4.2. Consider the identity map $\mathbb{1}_M : M \to M$ as a smooth map from M to itself. Then for a given $p \in M$, the differential

$$(\mathbb{1}_M)_{*,p} : T_pM \to T_pM$$

of $\mathbb{1}_M$ is the usual identity map on the tangent space T_pM, because for $X_p \in T_pM$ and $f \in C^\infty_p\,(M)$,

$$\left((\mathbb{1}_M)_{*,p}\,X_p\right) f = X_p\,(f \circ \mathbb{1}_M) = X_p f\,.$$

Therefore, $(\mathbb{1}_M)_{*,p}\,X_p = X_p$. Hence,

$$(\mathbb{1}_M)_{*,p} = \mathbb{1}_{T_pM} : T_pM \to T_pM\,.$$

Corollary 4.2

If $F : N \to M$ is a diffeomorphism of manifolds and $p \in N$, then $F_{*,p} : T_pN \to T_{F(p)}M$ is an isomorphism of vector spaces.

Proof. Since F is a diffeomorphism, it has a smooth inverse $G : M \to N$ such that $G \circ F = \mathbb{1}_N$ and $F \circ G = \mathbb{1}_M$. By Theorem 4.1 and the remark above,

$$G_{*,F(p)} \circ F_{*,p} = (G \circ F)_{*,p} = (\mathbb{1}_N)_{*,p} = \mathbb{1}_{T_pN} \ .$$

$$F_{*,p} \circ G_{*,F(p)} = F_{*,G(F(p))} \circ G_{*,F(p)} = (F \circ G)_{*,F(p)} = (\mathbb{1}_M)_{*,F(p)} = \mathbb{1}_{T_{F(p)}M} \ .$$

$G_{*,F(p)} \circ F_{*,p} = \mathbb{1}_{T_pN}$ and $F_{*,p} \circ G_{*,F(p)} = \mathbb{1}_{T_{F(p)}M}$ together imply that $F_{*,p} : T_pN \to T_{F(p)}M$ is an isomorphism of vector spaces. ∎

Corollary 4.3 (Invariance of Dimension)

If an open set $U \subseteq \mathbb{R}^n$ is diffeomorphic to an open set $V \subseteq \mathbb{R}^m$, then $n = m$.

Proof. Let $F : U \to V$ be a diffeomorphism and $p \in U$. By Corollary 4.2, $F_{*,p} : T_pU \to T_{F(p)}V$ is an isomorphism of vector spaces. There are vector space isoomorphisms $T_pU \simeq \mathbb{R}^n$ and $T_{F(p)}V \simeq \mathbb{R}^m$. Therefore, $\mathbb{R}^n$ is isomorphic to $\mathbb{R}^m$. This happens only when $n = m$. ∎

§4.2 Bases of the Tangent Space at a Point

We denote by $r^1, r^2, \ldots, r^n$ the standard coordinates on $\mathbb{R}^n$. Let (U, φ) be a chart about $p \in M$, where M is a smooth manifold of dimension n. We set $x^i = r^i \circ \varphi$. Since $\varphi : U \to \mathbb{R}^n$ is a diffeomorphism onto its image (Proposition 2.5), by Corollary 4.2,

$$\varphi_{*,p} : T_pM \to T_{\varphi(p)}\varphi(U) = T_{\varphi(p)}\mathbb{R}^n$$

is a vector space isomorphism. In particular, the tangent space T_pM is isomorphic to the vector space $T_{\varphi(p)}\mathbb{R}^n \simeq \mathbb{R}^n$, and hence T_pM has the same dimension n as the manifold.

Proposition 4.4

Let $(U, \varphi) = (U, x^1, x^2, \ldots, x^n)$ be a chart about a point $p \in M$. Then

$$\varphi_{*,p}\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = \left.\frac{\partial}{\partial r^i}\right|_{\varphi(p)} .$$

Proof. Let $f \in C^\infty_{\varphi(p)}(\mathbb{R}^n)$. Then

$$\varphi_{*,p}\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) f = \left.\frac{\partial}{\partial x^i}\right|_p (f \circ \varphi) = \left.\frac{\partial}{\partial r^i}\right|_{\varphi(p)} \left(f \circ \varphi \circ \varphi^{-1}\right) = \left.\frac{\partial}{\partial r^i}\right|_{\varphi(p)} f$$

Hence, $\varphi_{*,p}\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = \left.\frac{\partial}{\partial r^i}\right|_{\varphi(p)}$. ∎

Lemma 4.5

Let $T : V \to W$ be an isomorphism between the n-dimensional $\mathbb{F}$-vector spaces V and W. Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for V. Then $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_n)\}$ is a basis for W.

Proof. First, we want to show that $\mathcal{B} = \{T(\mathbf{v}_1), T(\mathbf{v}_2), \dots, T(\mathbf{v}_n)\}$ is a linearly independent set. Suppose for $c_1, c_2, \dots, c_n \in \mathbb{F}$,

$$c_1 T(\mathbf{v}_1) + c_2 T(\mathbf{v}_2) + \cdots + c_n T(\mathbf{v}_n) = \mathbf{0}_W \,.$$

Applying linearity on the LHS, we get

$$T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n) = \mathbf{0}_W = T(\mathbf{0}_V) \implies c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}_V \,.$$

Since $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for V, it's a linearly independent set. Therefore, $c_i = 0$ for each $i = 1, 2, \dots, n$. Therefore, $\mathcal{B}$ is a linearly independent set of vectors.

Now, since T is surjective, for $\mathbf{w} \in W$, $\mathbf{w} = T(\mathbf{v})$ for some $\mathbf{v} \in V$. We can write $\mathbf{v}$ as a linear combination of the basis vectors.

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n \,,$$

for $c_i \in \mathbb{F}$.

$$\mathbf{w} = T(\mathbf{v}) = T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n) = c_1 T(\mathbf{v}_1) + c_2 T(\mathbf{v}_2) + \cdots + c_n T(\mathbf{v}_n) \,.$$

Hence, $\mathcal{B}$ spans W. Therefore, $\mathcal{B}$ is a basis for W. ■

Proposition 4.6

If $(U, \varphi) = (U, x^1, x^2, \dots, x^n)$ is a chart containing $p \in M$, then the tangent space T_pM has basis

$$\left.\frac{\partial}{\partial x^1}\right|_p, \left.\frac{\partial}{\partial x^2}\right|_p, \dots, \left.\frac{\partial}{\partial x^n}\right|_p \,.$$

Proof. We have seen earlier that since $\varphi : U \to \mathbb{R}^n$ is a diffeomorphism onto its image, by Corollary 4.2,

$$\varphi_{*,p} : T_pM \to T_{\varphi(p)}\varphi(U) = T_{\varphi(p)}\mathbb{R}^n$$

is an isomorphism of vector spaces. By Proposition 4.4,

$$\varphi_{*,p}\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = \left.\frac{\partial}{\partial r^i}\right|_{\varphi(p)} \implies \varphi_{*,p}^{-1}\left(\left.\frac{\partial}{\partial r^i}\right|_{\varphi(p)}\right) = \left.\frac{\partial}{\partial x^i}\right|_p \,.$$

Since $\left\{\left.\frac{\partial}{\partial r_1}\right|_{\varphi(p)}, \left.\frac{\partial}{\partial r_2}\right|_{\varphi(p)}, \dots, \left.\frac{\partial}{\partial r_n}\right|_{\varphi(p)}\right\}$ is a basis for $T_{\varphi(p)}\mathbb{R}^n$, by Lemma 4.5,

$$\left\{\left.\frac{\partial}{\partial x^1}\right|_p, \left.\frac{\partial}{\partial x^2}\right|_p, \dots, \left.\frac{\partial}{\partial x^n}\right|_p\right\}$$

is a basis for T_pM. ■

Proposition 4.7 (Transition Matrix for Coordinate Vectors)

If $(U, x^1, x^2, \dots, x^n)$ and $(V, y^1, y^2, \dots, y^n)$ are two coordinate charts on a manifold M, then

$$\left.\frac{\partial}{\partial x^j}\right|_p = \sum_{i=1}^n \frac{\partial y^i}{\partial x^j}(p) \left.\frac{\partial}{\partial y^i}\right|_p \quad \text{for } p \in U \cap V \,.$$

Proof. At each point $p \in U \cap V$, the sets $\left\{\left.\frac{\partial}{\partial x^j}\right|_p\right\}$ and $\left\{\left.\frac{\partial}{\partial y^i}\right|_p\right\}$ are both bases for the tangent space T_pM. So there exists a change of basis matrix $\left[a_j^i(p)\right]$ of real numbers such that

$$\left.\frac{\partial}{\partial x^j}\right|_p = \sum_{k=1}^n a_j^k(p) \left.\frac{\partial}{\partial y^k}\right|_p \,.$$

Applying both sides of the equation to y^i, we get

$$\left.\frac{\partial}{\partial x^j}\right|_p y^i = \sum_{k=1}^{n} a_j^k(p) \left.\frac{\partial}{\partial y^k}\right|_p y^i = \sum_{k=1}^{n} a_j^k(p)\,\delta_k^i = a_j^i(p)\,.$$

Therefore, $a_j^i(p) = \left.\frac{\partial}{\partial x^j}\right|_p y^i = \frac{\partial y^i}{\partial x^j}(p)$. As a result,

$$\left.\frac{\partial}{\partial x^j}\right|_p = \sum_{k=1}^{n} \frac{\partial y^k}{\partial x^j}(p) \left.\frac{\partial}{\partial y^k}\right|_p\,.$$

■

Proposition 4.8 (Local Expression for the Differential)
Given a smooth map $F : N \to M$ of manifolds and a point $p \in N$, let $(U, x^1, x^2, \ldots, x^n)$ and $(V, y^1, y^2, \ldots, y^m)$ be coordinate charts about $p \in N$ and $F(p) \in M$, respectively. Relative to the bases $\left\{\left.\frac{\partial}{\partial x^j}\right|_p\right\}$ for T_pN and $\left\{\left.\frac{\partial}{\partial y^i}\right|_{F(p)}\right\}$ for $T_{F(p)}M$, the differential $F_{*,p} : T_pN \to T_{F(p)}M$ is represented by the matrix $\left[\frac{\partial F^i}{\partial x^j}(p)\right]$, where $F^i = y^i \circ F$ is the i-th component of F

Proof. By Proposition 4.6, $\left\{\left.\frac{\partial}{\partial x^j}\right|_p\right\}$ is a basis for T_pN and $\left\{\left.\frac{\partial}{\partial y^i}\right|_{F(p)}\right\}$ is a basis for $T_{F(p)}M$. $F_{*,p}\left(\left.\frac{\partial}{\partial x^j}\right|_p\right) \in T_{F(p)}M$, so it can be written as a linear combination of the basis vectors.

$$F_{*,p}\left(\left.\frac{\partial}{\partial x^j}\right|_p\right) = \sum_{k=1}^{m} a_j^k(p) \left.\frac{\partial}{\partial y^k}\right|_{F(p)}\,.$$

Applying both sides of the equation to y^i, we obtain

$$F_{*,p}\left(\left.\frac{\partial}{\partial x^j}\right|_p\right) y^i = \sum_{k=1}^{m} a_j^k(p) \left.\frac{\partial}{\partial y^k}\right|_{F(p)} y^i = \sum_{k=1}^{m} a_j^k(p)\,\delta_k^i = a_j^i(p)\,.$$

Using the definition of $F_{*,p}$,

$$F_{*,p}\left(\left.\frac{\partial}{\partial x^j}\right|_p\right) y^i = \left.\frac{\partial}{\partial x^j}\right|_p \left(y^i \circ F\right) = \frac{\partial F^i}{\partial x^j}(p) \implies a_j^i(p) = \frac{\partial F^i}{\partial x^j}(p)\,.$$

■

Remark 4.3. In terms of differentials, the inverse function theorem for manifolds can be described in the following coordinate free way: a C^∞ map $F : N \to M$ between two manifolds of same dimension is locally invertible at a point $p \in N$ if and only if the differential $F_{*,p} : T_pN \to T_{F(p)}M$ is an isomorphism of vector spaces.

Example 4.2 (The Chain Rule in Calculus Notation)**.** Suppose $w = G(x, y, z)$ is a C^∞ function $w : \mathbb{R}^3 \to \mathbb{R}$ and $F(t) = (\widetilde{x}(t), \widetilde{y}(t), \widetilde{z}(t))$ is a C^∞ function $F : \mathbb{R} \to \mathbb{R}^3$. Under composition

$$w = (G \circ F)(t) = G(x(t), y(t), z(t))$$

becomes a C^∞ function of $t \in \mathbb{R}$. (Here we abused notation $\widetilde{x}(t) = x, \widetilde{y}(t) = y, \widetilde{z}(t) = z$.) Now, let t_0 be fixed in $\mathbb{R}$. By Proposition 4.8, the matrix representation of $F_{*,t_0} : T_{t_0}\mathbb{R} \to T_{F(t_0)}\mathbb{R}^3$ is

$$\begin{bmatrix} \frac{\partial F^1}{\partial t}(t_0) \\ \frac{\partial F^2}{\partial t}(t_0) \\ \frac{\partial F^2}{\partial t}(t_0) \end{bmatrix} = \begin{bmatrix} \frac{\mathrm{d}\widetilde{x}}{\mathrm{d}t}(t_0) \\ \frac{\mathrm{d}\widetilde{y}}{\mathrm{d}t}(t_0) \\ \frac{\mathrm{d}\widetilde{z}}{\mathrm{d}t}(t_0) \end{bmatrix}.$$

Similarly, for $\mathbf{p}_0 = (x_0, y_0, z_0) \in \mathbb{R}^3$, the matrix representation of $G_{*,\mathbf{p}_0} : T_{\mathbf{p}_0}\mathbb{R}^3 \to T_{G(\mathbf{p}_0)}\mathbb{R}$ is

$$\left[\frac{\partial G}{\partial x}(\mathbf{p}_0) \quad \frac{\partial G}{\partial y}(\mathbf{p}_0) \quad \frac{\partial G}{\partial z}(\mathbf{p}_0)\right] = \left[\frac{\partial w}{\partial x}(\mathbf{p}_0) \quad \frac{\partial w}{\partial y}(\mathbf{p}_0) \quad \frac{\partial w}{\partial z}(\mathbf{p}_0)\right].$$

In a similar manner, the matrix representation of $(G \circ F)_{*,t_0} : T_{t_0}\mathbb{R} \to T_{G(F(t_0))}\mathbb{R}$ is

$$\left[\frac{\partial (G\circ F)}{\partial t}(t_0)\right] = \left[\frac{\mathrm{d}w}{\mathrm{d}t}(t_0)\right].$$

Now, the chain rule $(G \circ F)_{*,t_0} = G_{*,F(t_0)} \circ F_{*,t_0}$ is equivalent to

$$\begin{aligned}
\left[\frac{\mathrm{d}w}{\mathrm{d}t}(t_0)\right] &= \left[\frac{\partial w}{\partial x}(F(t_0)) \quad \frac{\partial w}{\partial y}(F(t_0)) \quad \frac{\partial w}{\partial z}(F(t_0))\right]\begin{bmatrix}\frac{\mathrm{d}\widetilde{x}}{\mathrm{d}t}(t_0)\\ \frac{\mathrm{d}\widetilde{y}}{\mathrm{d}t}(t_0)\\ \frac{\mathrm{d}\widetilde{z}}{\mathrm{d}t}(t_0)\end{bmatrix}\\
\iff \frac{\mathrm{d}w}{\mathrm{d}t}(t_0) &= \frac{\partial w}{\partial x}(F(t_0))\frac{\mathrm{d}x}{\mathrm{d}t}(t_0) + \frac{\partial w}{\partial y}(F(t_0))\frac{\mathrm{d}y}{\mathrm{d}t}(t_0) + \frac{\partial w}{\partial z}(F(t_0))\frac{\mathrm{d}z}{\mathrm{d}t}(t_0)
\end{aligned}$$

(Here also we abused notation $\widetilde{x}(t) = x, \widetilde{y}(t) = y, \widetilde{z}(t) = z$.) This is the usual form of chain rule taught in Calculus classes.

§4.3 Curves in a Manifold

Definition 4.4 (Smooth Curve)**.** A **smooth curve** in a manifold M is a smooth map $c : (a, b) \to M$ from some open interval (a, b) into M. Usually we assume $0 \in (a, b)$ and say that c is a curve starting at p if $c(0) = p$. The **velocity vector** $c'(t_0)$ of the curve c at time $t_0 \in (a, b)$ is defined to be

$$c'(t_0) := c_{*,t_0}\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0}\right) \in T_{c(t_0)}M.$$

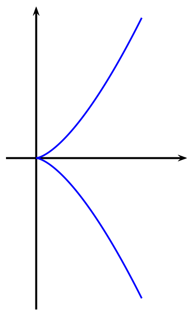
Notational Confusion

In the special case $c : (a, b) \to \mathbb{R}$, *i.e.*, when the target manifold is $\mathbb{R}$, by $c'(t)$ we mean a tangent vector at $c(t)$. Hence, $c'(t)$ is a real multiple of $\left.\frac{\mathrm{d}}{\mathrm{d}x}\right|_{c(t)}$. On the other hand, in calculus notation, $c'(t)$ is the derivative of the real valued function $c(t)$ and hence it is a scalar. In order to resolve the issue, we write $\dot{c}(t)$ for the calculus derivative (scalar).

Example 4.3. Define $c : \mathbb{R} \to \mathbb{R}^2$ by $c(t) = (t^2, t^3)$. This curve is known as a cuspidal cube.

$$c'(t_0) = c_{*,t_0}\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0}\right) \in T_{c(t_0)}\mathbb{R}^2 .$$

Then $c'(t_0)$ can be written as a linear combination of the basis vectors $\left.\frac{\partial}{\partial x}\right|_{c(t_0)}$ and $\left.\frac{\partial}{\partial y}\right|_{c(t_0)}$ of $T_{c(t_0)}\mathbb{R}^2$.



$$c_{*,t_0}\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0}\right) = a\left.\frac{\partial}{\partial x}\right|_{c(t_0)} + b\left.\frac{\partial}{\partial y}\right|_{c(t_0)} ,$$

where $a, b \in \mathbb{R}$. We now evaluate both sides on x to obtain:

$$c_{*,t_0}\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0}\right) x = a \implies a = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0}(x \circ c) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0} t^2 = 2t_0 .$$

Similarly, we evaluate both sides on y to obtain:

$$c_{*,t_0}\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0}\right) y = b \implies b = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0}(y \circ c) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0} t^3 = 3t_0^2 .$$

Therefore,

$$c_{*,t_0}\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0}\right) = 2t_0\left.\frac{\partial}{\partial x}\right|_{c(t_0)} + 3t_0^2\left.\frac{\partial}{\partial y}\right|_{c(t_0)} .$$

This means that in terms of the basis $\left\{\left.\frac{\partial}{\partial x}\right|_{c(t_0)}, \left.\frac{\partial}{\partial y}\right|_{c(t_0)}\right\}$ for $T_{c(t_0)}\mathbb{R}^2$,

$$c'(t_0) = \begin{bmatrix} 2t_0 \\ 3t_0^2 \end{bmatrix} .$$

The example above inspires the following proposition.

Proposition 4.9 (Velocity of a curve in local coordinates)
Let $c : (a, b) \to M$ be a smooth curve, and let $(U, x^1, x^2, \ldots, x^n)$ be a coordinate chart about $c(t)$. Let $c^i = x^i \circ c$ be the i-th component of the curve c in the chart. Then $c'(t)$ is given by

$$c'(t) = \sum_{i=1}^{n} \dot{c}^i(t)\left.\frac{\partial}{\partial x^i}\right|_{c(t)} .$$

Thus, relative to the basis $\left\{\left.\frac{\partial}{\partial x^i}\right|_{c(t)}\right\}$ for $T_{c(t)}M$, the velocity $c'(t)$ is represented by the column vector

$$\begin{bmatrix} \dot{c}^1(t) \\ \dot{c}^2(t) \\ \cdots \\ \dot{c}^n(t) \end{bmatrix}$$

Note: $\dot{c}^i(t) = \frac{\mathrm{d}c^i}{\mathrm{d}t}(t)$ is the derivative of the real valued function $c^i(t)$.

Proof. $c'(t) = c_{*,t}\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_t\right) \in T_{c(t)}M$. Since $\left\{\left.\frac{\partial}{\partial x^j}\right|_{c(t)}\right\}$ is a basis for $T_{c(t)}M$, we can write $c'(t)$ as a linear combination of basis vectors.

$$c'(t) = \sum_{i=1}^{n} a_j \left.\frac{\partial}{\partial x^j}\right|_{c(t)} .$$

Evaluating both sides on x^i, we get

$$c'(t)\, x^i = \sum_{i=1}^{n} a_j \left.\frac{\partial}{\partial x^j}\right|_{c(t)} x^i = \sum_{i=1}^{n} a_j \delta^i_j = a^i ,$$

because Proposition 2.12 tells us that $\frac{\partial x^i}{\partial x^j} = \delta^i_j$. Now, using the definition of $c'(t)$,

$$a^i = c'(t)\, x^i = c_{*,t}\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_t\right) x^i = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_t \left(x^i \circ c\right) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_t c^i = \frac{\mathrm{d}c^i}{\mathrm{d}t}(t) = \dot{c}^i(t) .$$

Therefore,

$$c'(t) = \sum_{i=1}^{n} \dot{c}^i(t) \left.\frac{\partial}{\partial x^i}\right|_{c(t)} .$$

■

Proposition 4.10 (Existence of a curve with a given initial vector)
For any point p in a manifold M and any tangent vector $X_p \in T_pM$, there are $\varepsilon > 0$ and a smooth curve $c : (-\varepsilon, \varepsilon) \to M$ such that $c(0) = p$ and $c'(0) = X_p$.

Proof. Let $(U, \varphi) = (U, x^1, x^2, \ldots, x^n)$ be a chart centered at p, i.e. $\varphi(p) = \mathbf{0} \in \mathbb{R}^n$. Since $X_p \in T_pM$, it can be written as a linear combination of the basis vectors.

$$X_p = \sum_{i=1}^{n} a^i \left.\frac{\partial}{\partial x^i}\right|_p .$$

Claim 1: There exists a curve $\alpha : (-\varepsilon, \varepsilon) \to \varphi(U)$ such that $\alpha(0) = \mathbf{0}$ and $\alpha'(0) = \sum\limits_{i=1}^{n} a^i \left.\frac{\partial}{\partial r^i}\right|_{\mathbf{0}}$.

Proof. We define $\alpha : \mathbb{R} \to \mathbb{R}^n$ by $\alpha(t) = (a^1 t, a^2 t, \ldots, a^n t)$. Then $\alpha(0) = \mathbf{0}$. We can choose ε sufficiently small such that $\alpha(t) \in \varphi(U)$ for $-\varepsilon < t < \varepsilon$. By Proposition 4.9,

$$\alpha'(0) = \sum_{i=1}^{n} \dot{\alpha}^i(0) \left.\frac{\partial}{\partial r^i}\right|_{\alpha(0)} = \sum_{i=1}^{n} a^i \left.\frac{\partial}{\partial r^i}\right|_{\mathbf{0}} ,$$

since $\dot{\alpha}^i(0) = \frac{\mathrm{d}\alpha^i}{\mathrm{d}t}(0) = \frac{\mathrm{d}(a^i t)}{\mathrm{d}t}(0) = a^i$. □

We define $c = \varphi^{-1} \circ \alpha : (-\varepsilon, \varepsilon) \to U \subseteq M$. Then $c(0) = \varphi^{-1}(\alpha(0)) = \varphi^{-1}(\mathbf{0}) = p$. Furthermore, using The Chain Rule,

$$c'(0) = c_{*,0}\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0\right) = \left(\left(\varphi^{-1}\right)_{*,\alpha(0)} \circ \alpha_{*,0}\right)\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0\right) = \left(\varphi^{-1}\right)_{*,\mathbf{0}} \alpha'(0) .$$

Using the expression for $\alpha'(0)$ and Proposition 4.4, we get

$$\begin{aligned}\left(\varphi^{-1}\right)_{*,\mathbf{0}}\alpha'(0) &= \left(\varphi^{-1}\right)_{*,\mathbf{0}}\left(\sum_{i=1}^{n} a^i \left.\frac{\partial}{\partial r^i}\right|_{\mathbf{0}}\right) = \sum_{i=1}^{n} a^i \left(\varphi^{-1}\right)_{*,\mathbf{0}}\left(\left.\frac{\partial}{\partial r^i}\right|_{\mathbf{0}}\right)\\ &= \sum_{i=1}^{n} a^i \left.\frac{\partial}{\partial x^i}\right|_{\varphi(0)} = \sum_{i=1}^{n} a^i \left.\frac{\partial}{\partial x^i}\right|_{p} = X_p\end{aligned}$$

Therefore, $c'(0) = X_p$. ■

Proposition 4.11

Suppose X_p is a tangent vector at a point p of a manifold M and $f \in C_p^\infty(M)$. If $c : (-\varepsilon, \varepsilon) \to M$ is a smooth curve starting at p with $c'(0) = X_p$, then

$$X_p f = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 (f \circ c) .$$

Proof. By definition of $c'(0)$ and $c_{*,0}$,

$$X_p f = c'(0) f = c_{*,0}\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0\right) f = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 (f \circ c) .$$

■

Computing the Differential Using Curves

Proposition 4.12

Let $F : N \to M$ be a smooth map of manifolds, $p \in N$, and $X_p \in T_p N$. If c is a smooth curve starting at $p \in N$ and with velocity X_p at p, then

$$F_{*,p}(X_p) = (F \circ c)'(0) .$$

In other words, $F_{*,p}(X_p)$ is the velocity vector of the curve $F \circ c$ at $(F \circ c)(0) = F(p)$.

Proof. By hypothesis, $c(0) = p$ and $c'(0) = X_p$. Now,

$$\begin{aligned}F_{*,p}(X_p) &= F_{*,p}(c'(0)) = F_{*,p}\left(c_{*,0}\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0\right)\right)\\ &= \left(F_{*,c(0)} \circ c_{*,0}\right)\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0\right) = (F \circ c)_{*,0}\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0\right)\\ &= (F \circ c)'(0)\end{aligned}$$

which is the velocity vector to the curve $F \circ c$ at $(F \circ c)(0) = F(p)$. ■

Example 4.4 (Differential of Left Multiplication)**.** $\mathrm{GL}(n, \mathbb{R})$ stands for the group of all $n \times n$ invertible matrices over $\mathbb{R}$. It is called the general linear group. Let $g \in \mathrm{GL}(n, \mathbb{R})$. Also, let $l_g : \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$ denote the left multiplication by the matrix g. In other words, for $B \in \mathrm{GL}(n, \mathbb{R})$, $l_g(B) = gB \in \mathrm{GL}(n, \mathbb{R})$.

Since we've seen earlier that $\mathrm{GL}(n, \mathbb{R})$ is an open subset of $\mathbb{R}^{n\times n} \simeq \mathbb{R}^{n^2}$, $T_g \mathrm{GL}(n, \mathbb{R})$ can be identified with $\mathbb{R}^{n\times n} \simeq \mathbb{R}^{n^2}$ for any $g \in \mathrm{GL}(n, \mathbb{R})$. Now, if I is the $n \times n$ identity matrix, then show that

$$(l_g)_{*,I} : T_I \mathrm{GL}(n, \mathbb{R}) \to T_g \mathrm{GL}(n, \mathbb{R})$$

is also left multiplication by g.

Solution. Since $\mathrm{GL}(n,\mathbb{R})$ is an open subset of the Euclidean space $\mathbb{R}^{n\times n} \equiv \mathbb{R}^{n^2}$, the only coordinate chart on $\mathrm{GL}(n,\mathbb{R})$ is $\left(\mathrm{GL}(n,\mathbb{R}), r^{11}, r^{12}, \dots, r^{nn}\right)$, where r^{ij}'s are the usual coordinates on $\mathbb{R}^{n\times n}$. There are vector space isomorphisms $\psi : T_I \,\mathrm{GL}(n,\mathbb{R}) \to \mathbb{R}^{n\times n}$ and $\varphi : T_g \,\mathrm{GL}(n,\mathbb{R}) \to \mathbb{R}^{n\times n}$.

$$\psi\left(\sum_{i,j=1}^{n} a^{ij} \left.\frac{\partial}{\partial r^{ij}}\right|_I\right) = \left[a^{ij}\right]_{i,j=1}^{n} \quad \text{and} \quad \varphi\left(\sum_{i,j=1}^{n} a^{ij} \left.\frac{\partial}{\partial r^{ij}}\right|_g\right) = \left[a^{ij}\right]_{i,j=1}^{n} .$$

Let $X \in T_I \,\mathrm{GL}(n,\mathbb{R})$. Then $(l_g)_{*,I} X \in T_g \,\mathrm{GL}(n,\mathbb{R})$. So we need to prove that

$$\varphi\left((l_g)_{*,I} X\right) = g\psi(X) .$$

Let us now compute $(l_g)_{*,I} X$. Choose a curve $c(t)$ in $\mathrm{GL}(n,\mathbb{R})$ with $c(0) = I$ and $c'(0) = X$. Let $\alpha = l_g \circ c$. By Proposition 4.12,

$$(l_g)_{*,I}(X) = (l_g \circ c)_{*,0}\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0\right) = (l_g \circ c)'(0) = \alpha'(0) .$$

Now, α is a map from an open interval to $\mathrm{GL}(n,\mathbb{R})$, and it is the composition of two smooth maps. Hence, α is a smooth curve in $\mathrm{GL}(n,\mathbb{R})$. $\alpha(0) = l_g(I) = g$. By Proposition 4.9,

$$\alpha'(0) = \sum_{i,j=1}^{n} \dot{\alpha}^{ij}(0) \left.\frac{\partial}{\partial r^{ij}}\right|_g \implies \varphi\left((l_g)_{*,I}(X)\right) = \varphi\left(\sum_{i,j=1}^{n} \dot{\alpha}^{ij}(0) \left.\frac{\partial}{\partial r^{ij}}\right|_g\right) = \left[\dot{\alpha}^{ij}(0)\right]_{i,j=1}^{n} .$$

Now, since $\alpha(t) = g\,c(t)$,

$$\begin{aligned}\alpha^{ij}(t) = \sum_{k=1}^{n} g^{ik} c^{kj}(t) \implies \dot{\alpha}^{ij}(0) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \left(\sum_{k=1}^{n} g^{ik} c^{kj}(t)\right) = \sum_{k=1}^{n} g^{ik} \left.\frac{\mathrm{d}c^{kj}(t)}{\mathrm{d}t}\right|_0 = \sum_{k=1}^{n} g^{ik} \dot{c}^{kj}(0) \\ &\implies \left[\dot{\alpha}^{ij}(0)\right]_{i,j=1}^{n} = g\left[\dot{c}^{ij}(0)\right]_{i,j=1}^{n}\end{aligned}$$

Now, since $X = c'(0)$, by Proposition 4.9,

$$X = c'(0) = \sum_{i,j=1}^{n} \dot{c}^{ij}(0) \left.\frac{\partial}{\partial r^{ij}}\right|_I \implies \psi(X) = \left[\dot{c}^{ij}(0)\right]_{i,j=1}^{n} .$$

$$\therefore \left[\dot{\alpha}^{ij}(0)\right]_{i,j=1}^{n} = g\left[\dot{c}^{ij}(0)\right]_{i,j=1}^{n} \implies \varphi\left((l_g)_{*,I}(X)\right) = g\psi(X) ,$$

as desired. ■

§4.4 Immersions and Submersions

Definition 4.5 (Immersion and Submersion)**.** A C^∞ map $F : N \to M$ is said to be an **immersion** at $p \in N$ if its differential $F_{*,p} : T_pN \to T_{F(p)}M$ is injective, and a **submersion** at p if $F_{*,p}$ is surjective. We call F an immersion if it is an immersion at every $p \in N$, and a submersion if it is a submersion ar every $p \in N$.

Remark 4.4. Suppose N and M are manifolds of dimension n and m, respectively. Then $\dim(T_pN) = n$ and $\dim\left(T_{F(p)}M\right) = m$. The injectivity of the differential $F_{*,p} : T_pN \to T_{F(p)}M$ immediately implies $m \geq n$. Similarly, surjectivity of $F_{*,p} : T_pN \to T_{F(p)}M$ implies that $n \geq m$. Thus, if $F : N \to M$ is an immersion at a point of N, then $m \geq n$ and if F is a submersion at a point of N, then $n \geq m$.

Example 4.5. The prototype of an immersion is the inclusion of $\mathbb{R}^n$ into a higher dimensional $\mathbb{R}^m$:

$$i\left(x^1, x^2, \ldots, x^n\right) = \left(x^1, x^2, \ldots, x^n, 0, \ldots, 0\right) .$$

The prototype of a submersion is the projection of $\mathbb{R}^n$ onto a lower dimensional $\mathbb{R}^m$:

$$\pi\left(x^1, x^2, \ldots, x^m, x^{m+1}, \ldots, x^n\right) = \left(x^1, x^2, \ldots, x^m\right) .$$

Example 4.6. Let U be an open subset of a manifold M and hence a manifold. The inclusion map $i : U \hookrightarrow M$ is injective and in genral, not surjective. The differential of i at $p \in U$, denoted by $i_{*,p} : T_pU \to T_pM$, is bijective. Indeed $\dim(T_pU) = \dim(T_pM)$ as $T_pU \simeq T_pM$ as vector spaces. Hence, the inclusion map $i : U \to M$ is both an immersion and a submersion. This is an example exhibiting the fact that a submersion need not be onto.

Rank, and Critical and Regular Points

Consider a smooth map $F : N \to M$ of manifolds. Its rank at a point $p \in N$, denoted by $\operatorname{rank} F(p)$, is defined as the rank of the differential $F_{*,p} : T_pN \to T_{F(p)}M$. Relative to the coordinate neighborhoods $(U, x^1, x^2, \ldots, x^n)$ at p and $(V, y^1, y^2, \ldots, y^m)$ at $F(p)$, the differential $F_{*,p}$ is represented by the Jacobian matrix $\left[\frac{\partial F^i}{\partial x^j}(p)\right]$ (Proposition 4.8), so

$$\operatorname{rank} F(p) = \operatorname{rank}\left[\frac{\partial F^i}{\partial x^j}(p)\right] .$$

Since the differential of a map is independent of coordinate charts, so is the rank of a Jacobian matrix.

> **Definition 4.6** (Critical and Regular Points)**.** A point $p \in N$ is a **critical point** of $F : N \to M$ if the differential $F_{*,p} : T_pN \to T_{F(p)}M$ is not surjective. It is a **regular point** of F if the differential $F_{*,p}$ is surjective. In other words, p is a regular point of F if and only if F is a submersion at p. A point in M is a **critical value** if it is the image of a critical point; otherwise it is a **regular value**.

Couple of important aspects of this definitions:

(i) We *do not* define regular value to be the image of a regular point. Any point of M that is not a critical value is defined as a regular value. Therefore, any point of M that is not in the image of F, or which doesn't have a preimage in N under F, is automarically a regular value.

(ii) A point $c \in M$ is a critical value if and only if some point in the preimage $F^{-1}(\{c\})$ is a critical point. A point c in the image of F is a regular value if and only if every point in the preimage $F^{-1}(\{c\})$ is a regular point.

Exercise 4.1. Show that there are 4 critical points for the height function of the 2-torus.

Solution. A 2-torus T^2 can be seen as a subset of $\mathbb{R}^3$ in the following way:

$$T^2 = \left\{\left(r\sin t, (R + r\cos t)\cos s, (R + r\cos t)\sin s\right) \in \mathbb{R}^3 \mid 0 \leq s, t < 2\pi\right\} ,$$

with $R > r$.

Let f be the height function $f : T^2 \to \mathbb{R}$, defined by $f(x, y, z) = z$. t, s are coordinate functions on T^2. Now, $p \in T^2$ is a critical point if and only if

$$\frac{\partial f}{\partial t}(p) = 0 = \frac{\partial f}{\partial s}(p) .$$

Let $p = \Big(r \sin t_0, (R + r\cos t_0)\cos s_0, (R + r\cos t_0)\sin s_0\Big)$

$$\frac{\partial f}{\partial t}(p) = \frac{\partial\left((R + r\cos t)\sin s\right)}{\partial t}(p) = -r\sin t_0 \sin s_0$$

$$\frac{\partial f}{\partial s}(p) = \frac{\partial\left((R + r\cos t)\sin s\right)}{\partial s}(p) = (R + r\cos t_0)\cos s_0$$

Since $R > r$, $R + r\cos t_0 = 0$ is not possible. Hence, $\cos s_0 = 0$. This gives us $\sin^2 s_0 = 1$. Therefore, we must have $\sin t_0 = 0$. Therefore,

$$\cos s_0 = 0 \implies s_0 \in \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\} \text{ and } \sin t_0 = 0 \implies t_0 \in \{0, \pi\} \ .$$

Therefore, plugging the values of s_0 and t_0, we get 4 critical points. The set of all the critical points of f is

$$\{(0,0,R+r), (0,0,-R-r), (0,0,R-r), (0,0,-R+r)\} \ .$$

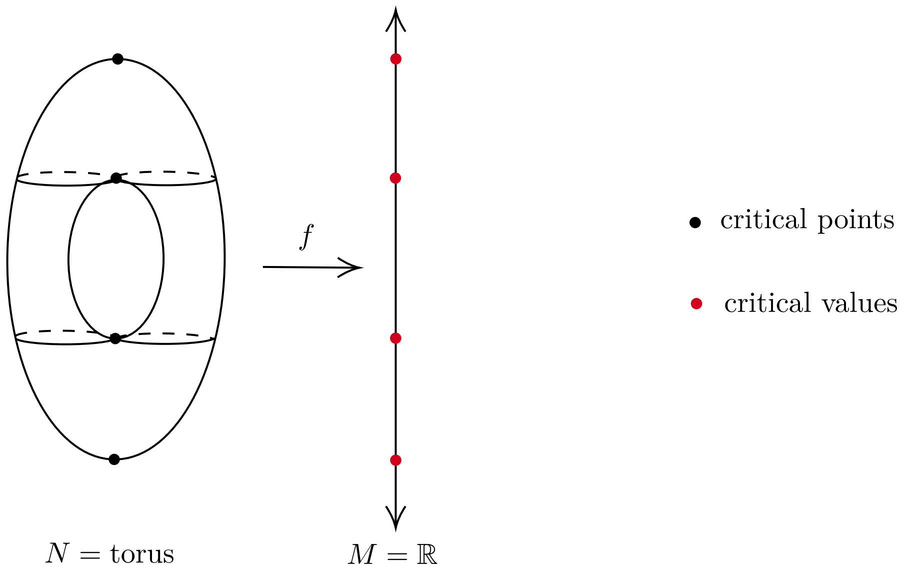


Figure 4.1: Critical points and critical values of the height function $f(x,y,z) = z$ of the 2-torus

Furthermore, the corresponding critical values are $R + r$, $-R - r$, $R - r$, $-R + r$. ■

Proposition 4.13

For a real valued function $f : M \to \mathbb{R}$, a point $p \in M$ is a critical point if and only if relative to some chart $(U, x^1, x^2, \ldots, x^n)$ contaning p, all the partial derivatives satisfy

$$\frac{\partial f}{\partial x^j}(p) = 0 \ , \ j = 1, 2, \ldots, n \, .$$

Proof. By Proposition 4.8, the differential $f_{*,p} : T_pM \to T_{f(p)}\mathbb{R} \simeq \mathbb{R}$ is represented by the matrix

$$\begin{bmatrix} \frac{\partial f}{\partial x^1}(p) & \frac{\partial f}{\partial x^2}(p) & \cdots & \frac{\partial f}{\partial x^n}(p) \end{bmatrix}$$

with respect to the basis $\left\{ \frac{\partial}{\partial x^1}\Big|_p, \ldots, \frac{\partial}{\partial x^n}\Big|_p \right\}$ of T_pM and $\left\{ \frac{\partial}{\partial y}\Big|_{f(p)} \right\}$ of $T_{f(p)}\mathbb{R}$.

Since the image of $f_{*,p}$ is a vector subspace of $T_{f(p)}\mathbb{R} \simeq \mathbb{R}$, it is either 0-dimensional or 1-dimensional. In other words, $f_{*,p} : T_pM \to T_{f(p)}\mathbb{R} \simeq \mathbb{R}$ is either the 0-map (everything is mapped to the 0-vector of the codomain) or a surjective map. Therefore, $f_{*,p}$ fails to be surjective if and only if the matrix representing it is the 0-matrix. In other words, all the partial derivatives $\frac{\partial f}{\partial x^j}(p)$ vanish. ■

5 Submanifolds

§5.1 Regular Submanifolds

Definition 5.1 (Regular Submanifold). A subset S of a manifold N of dimension n is a **regular submanifold** of dimension k if for every $p \in S$, there is a coordinate neighborhood $(U, x^1, x^2, \ldots, x^n)$ of p in the maximal atlas of N such that $U \cap S$ is defined by the vanishing of $n-k$ coordinate functions.

By renumbering the coordinates, we may assyme that these vanishing $n-k$ coordinate functions are $x^{k+1}, \ldots, x^n$. We call such a chart (U, φ) in N an **adapted chart** relative to S.

It is important to note that $U \cap S$ is supposed to be the maximal subset of U where $n-k$ coordinate functions vanish. By maximal, we mean that it is not contained in any other subset of U other than itself where $n-k$ coordinate functions vanish. This fact can be mathematically described as

$$U \cap S = \varphi^{-1}(*, \ldots, *, \underbrace{0, 0, \ldots, 0}_{(n-k) \text{ many}}).$$

On $U \cap S$, $\varphi = \left(x^1, x^2, \ldots, x^k, 0, 0, \ldots, 0\right)$. Let $\varphi_S : U \cap S \to \mathbb{R}^k$ be the restriction of the first k components of φ to $U \cap S$. In other words,

$$\varphi_S = \left(x^1, x^2, \ldots, x^k\right).$$

Note that, $(U \cap S, \varphi_S)$ is a chart for S in the subspace topology.

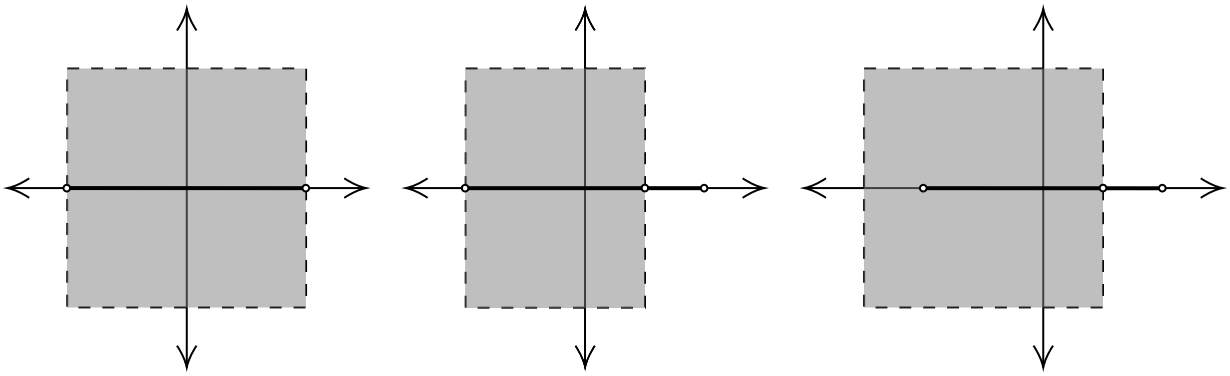
Definition 5.2 (Codimension). If S is a regular submanifold of dimension k in a manifold N of dimension n, then $n-k$ is said to be the **codimension** of S in N.

Remark 5.1. As a topological space, a regular submanifold of N is required to have the subspace topology. It's also noteworthy that the dimensioon k of the regular submanifold S may be equal to the dimension n of the manifold N. In this case, $U \cap S$ is defined by the vanishing of none of the coordinate functions, so $U \cap S = U$. Hence, $\varphi|_{U \cap S} = \varphi$. Therefore, an open subset of a manifold is a regular submanifold of the same dimension.

Example 5.1. The interval $S := (-1, 1)$ on the x-axis is a regular submanifold of the x-y plane ($\mathbb{R}^2$). As an adapted chart (for any point $p \in S$), we can choose the open square $U = (-1, 1) \times (-1, 1)$ with coordinates x, y of the plane $\mathbb{R}^2$. Then one immediately finds that $U \cap S$ is precisely the zero set of y on U.

Similarly, the open rectangle $V = (-1, 0.5) \times (-1, 1)$ with coordinates x, y of $\mathbb{R}^2$ is also an adapted chart relative to S. Because, the zero set of y on V is $(-1, 0.5)$, which is precisely $V \cap S$.

However, if we take $W = (-1.5, 0.5) \times (-1, 1)$, then (W, x, y) is not an adapted chart relative to S for any point $p \in (-1, 0.5) = W \cap S$. This is because the zero set of y on W is $(-1.5, 0.5) \neq W \cap S$.



$U = (-1, 1) \times (-1, 1)$ is an adapted chart relative to S

$V = (-1, 0.5) \times (-1, 1)$ is an adapted chart relative to S

$W = (-1.5, 0.5) \times (-1, 1)$ is not an adapted chart relative to S

Exercise 5.1. Let Γ be the graph of the function $f(x) = \sin\left(\frac{1}{x}\right)$ on $(0, 1)$, and I be the open interval on y axis:

$$I = \{(0, y) \mid -1 < y < 1\} .$$

Then show that $S = \Gamma \cup I$ is not a regular submanifold of $\mathbb{R}^2$.

Solution. Assume for the sake of contradiction that S is a regular submanifold of $\mathbb{R}^2$. Take $p = (0, y_p) \in I$. Then there exists a chart $(U, \varphi) = (U, x^1, x^2)$ about p in the maximal atlas of $\mathbb{R}^2$ such that $U \cap S$ is defined by the vanishing of k coordinate functions x^1, x^2 where k might be either 0 or 1 or 2. We can also assume that U is contained in the open ball $B(p, \varepsilon_0)$, where $0 < \varepsilon_0 < \min\{|1 - y_p|, |1 + y_p|\}$. Because, if U is not contained in this open ball, we can simply replace U by $U \cap B(p, \varepsilon_0)$.

Since U is contained in the open ball $B(p, \varepsilon_0)$, $U \cap \Gamma$ has multiple connected components. Therefore, $U \cap S$ is not connected. Furthermore, if U_0 is any open subset of U containing p, $U_0 \cap S$ is also disconnected.

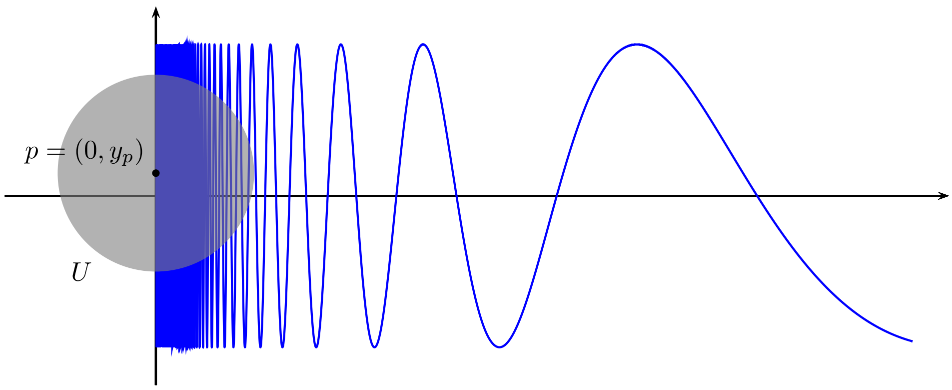


Figure 5.1: U intersects Γ in infinitely many components.

First of all, k cannot be 2, since there is only one point where both x^1 and x^2 vanish but $U \cap S$ contains more than one point.

If k is 0, then $U \cap S = U$. So U is contained in S. U is an open subset of $\mathbb{R}^2$ that contains p. So, there exists some $\varepsilon > 0$ such that $B(p, \varepsilon) \subseteq U \subseteq S$. However, this is also not possible, because for $0 < \delta < \varepsilon$,

$$\|(0, y_p) - (-\delta, y_p)\| = \delta < \varepsilon \implies (-\delta, y_p) \in B(p, \varepsilon) .$$

But S does not contain any point with negative x-coordinate. So, $(-\delta, y_p) \notin S$.

Now we are only left with the case $k = 1$. WLOG, we can assume that $U \cap S$ is defined by the vanishing of x^2. In other words, if X denotes the x-axis of $\mathbb{R}^2$, then $\varphi(U \cap S) = \varphi(U) \cap X$.

Claim 2: If V is an open subset of $\mathbb{R}$, then the connected components of V are also open in $\mathbb{R}$.

Proof. Let C be a connected component of V. Take $x \in C$. $x \in C \subseteq V$ and V is open in $\mathbb{R}$, so there exists $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subseteq V$. Now, $(x-\varepsilon, x+\varepsilon)$ is an interval, so it is connected. Therefore, $(x-\varepsilon, x+\varepsilon)$ intersects only one of the connected components of V. $x \in (x-\varepsilon, x+\varepsilon) \cap C$. Hence, $(x-\varepsilon, x+\varepsilon) \subseteq C$. For every $x \in C$, we can find such an ε. Therefore, C is open. $\square$

Now, X is homeomorphic to $\mathbb{R}$. $\varphi(U)$ is open in $\mathbb{R}^2$, so $V = \varphi(U) \cap X$ is open in X. So, the connected components of V are also open in X. Let C be the component of V that contains p. Then C is open in X. So $X \setminus C$ is closed in X. Also, X is closed in $\mathbb{R}^2$. Hence, $X \setminus C$ is closed in $\mathbb{R}^2$.

Now, $\varphi(U)$ is open in $\mathbb{R}^2$ and $X \setminus C$ is closed in $\mathbb{R}^2$. Therefore, $W = \varphi(U) \setminus (X \setminus C)$ is open in $\mathbb{R}^2$. Also, W is contained in $\varphi(U)$.

$$W = \varphi(U) \setminus (X \setminus C) = (\varphi(U) \setminus X) \cup C \implies W \cap X = C.$$

So, W intersects the x-axis in only one connected component C. Now choose $U' = \varphi^{-1}(W)$. Now, $U' \subseteq U$ since $W \subseteq \varphi(U)$. Also, U' is open in U, and hence in $\mathbb{R}^2$. U' contains p as $\varphi(p) \in C \subseteq W$.

$$\varphi(U' \cap S) = \varphi(U') \cap \varphi(U \cap S) = W \cap \varphi(U) \cap X = W \cap X = C.$$

Therefore, $\varphi(U' \cap S)$ is connected. Since φ is a homeomorphism, $U' \cap S$ is also connected. However, we proved earlier that if U_0 is an open subset of U containing p, $U_0 \cap S$ is disconnected. Thus we arrive at a contradiction. So S cannot be a regular submanifold of $\mathbb{R}^2$. $\blacksquare$

Proposition 5.1

Let S be a regular submanifold of N and let $\mathscr{U} = \{(U, \varphi)\}$ be a collection of compatible adapted charts of N that covers S. Then $\mathscr{U}_S = \{(U \cap S, \varphi_S)\}$ is an atlas for S. Therefore, a regular submanifold is itself a manifold. If N has dimension n and S is locally defined by the vanishing of $n-k$ coordinate functions, then $\dim S = k$.

Proof. Let $(U, \varphi) = (U, x^1, x^2, \dots, x^n)$ and $(V, \psi) = (V, y^1, y^2, \dots, y^n)$ be two adapted charts in the given collection $\mathscr{U}$. Assume that they intersect. From the definition of adapted chart relative to a submanifold S, it is possible to renumber the coordinates such that the last $n-k$ coordinate functions vanish on points of S intersected with the open set of the pertaining adapted chart. Therefore, for $p \in U \cap V \cap S$,

$$\varphi(p) = \left(x^1, x^2, \dots, x^k, 0, 0, \dots, 0\right) \quad \text{and} \quad \psi(p) = \left(y^1, y^2, \dots, y^k, 0, 0, \dots, 0\right).$$

So $\varphi_S(p) = \left(x^1, x^2, \dots, x^k\right)$ and $\psi_S(p) = \left(y^1, y^2, \dots, y^k\right)$. Therefore,

$$\left(\psi_S \circ \varphi_S^{-1}\right)\left(x^1, x^2, \dots, x^k\right) = \left(y^1, y^2, \dots, y^k\right),$$

where $\psi_S \circ \varphi_S^{-1} : \varphi_S(U \cap V \cap S) \subseteq \mathbb{R}^k \to \psi_S(U \cap V \cap S) \subseteq \mathbb{R}^k$. Note that since $\mathscr{U}$ is a C^∞ compatible collection of adapted charts of N, $\psi \circ \varphi^{-1} : \varphi(U \cap V) \subseteq \mathbb{R}^n \to \psi(U \cap V) \subseteq \mathbb{R}^n$ is C^∞. Let $i : \mathbb{R}^k \hookrightarrow \mathbb{R}^n$ be the inclusion map and $\pi : \mathbb{R}^n \to \mathbb{R}^k$ be the projection onto first k coordinates. Now,

$$\left(\psi \circ \varphi^{-1}\right)\left(x^1, x^2, \dots, x^k, 0, 0, \dots, 0\right) = \left(y^1, y^2, \dots, y^k, 0, 0, \dots, 0\right)$$

gives us the following identity:

$$\psi_S \circ \varphi_S^{-1} = \pi\big|_{\psi(U \cap V \cap S)} \circ \left(\psi \circ \varphi^{-1}\right)\Big|_{\varphi(U \cap V \cap S)} \circ i\big|_{\varphi_S(U \cap V \cap S)}.$$

Therefore, as a composition of smooth maps, $\psi_S \circ \varphi_S^{-1}$ is also smooth. Hence, any two charts in $\mathscr{U}_S$ are C^∞ compatible. Now, since the open sets in the collection $\mathscr{U}$ covers S, the open sets in $\mathscr{U}_S$ also covers S, proving that the collection $\mathscr{U}_S$ is a C^∞ atlas on S. $\blacksquare$

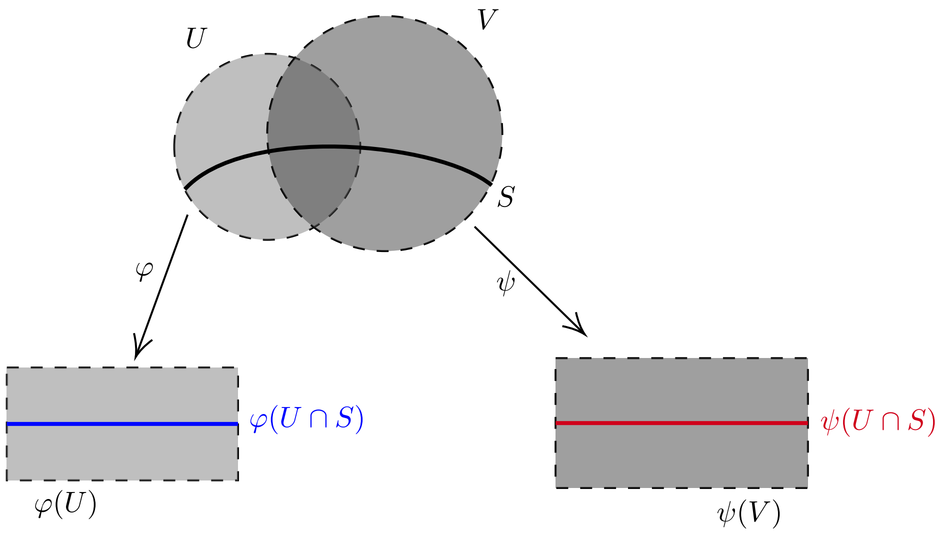


Figure 5.2: Overlapping adapted charts relative to a regular submanifold S.

§5.2 Level Sets of a Map

Definition 5.3. A **level set** of a map $F : N \to M$ is a subset

$$F^{-1}(\{c\}) = \{p \in N \mid F(p) = c\}$$

for some $c \in M$. The value $c \in M$ is called the **level** of the level set $F^{-1}(\{c\})$. If $F : N \to \mathbb{R}^m$, then $Z(F) := F^{-1}(\{\mathbf{0}\})$ is the **zero set** of F.

Recall that c is a regular value of F if either c is not in the image of F or at every point $p \in F^{-1}(\{c\})$, the differential $F_{*,p} : T_pN \to T_{F(p)}M$ is surjective.

Definition 5.4. The preimage $F^{-1}(\{c\})$ of a regular value c is called a **regular level set**. If $F : N \to \mathbb{R}^m$ and $\mathbf{0} \in \mathbb{R}^m$ is a regular value of F, then $F^{-1}(\{\mathbf{0}\})$ is called a **regular zero set**.

Remark 5.2. If a regular level set $F^{-1}(\{c\})$ is nonempty, then for $p \in F^{-1}(\{c\})$, the map $F : N \to M$ is a submersion at p. By Remark 4.4, $\dim N \geq \dim M$.

Example 5.2. The unit 2-sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ is the zero set of the function $f : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$f(x, y, z) = x^2 + y^2 + z^2 - 1\,.$$

In other words, $S^2 = f^{-1}(\{0\})$. Show that S^2 is a regular submanifold of $\mathbb{R}^3$.

Solution. Note that

$$\frac{\partial f}{\partial x} = 2x\,, \quad \frac{\partial f}{\partial y} = 2y\,, \quad \frac{\partial f}{\partial z} = 2z\,.$$

By Proposition 4.13, $\mathbf{p} \in \mathbb{R}^3$ is a critical point of f if and only if

$$\left.\frac{\partial f}{\partial x}\right|_{\mathbf{p}} = \left.\frac{\partial f}{\partial y}\right|_{\mathbf{p}} = \left.\frac{\partial f}{\partial z}\right|_{\mathbf{p}} = 0\,.$$

Hence, the only critical point of f is $\mathbf{0} \equiv (0, 0, 0)$. Since $\mathbf{0} \notin S^2 = f^{-1}(\{0\})$, 0 is a regular value of f.

Let us choose $p \in S^2$ such that $\frac{\partial f}{\partial x}(p) = 2x(p) \neq 0$. Then the Jacobian matrix of the map $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $\varphi(x, y, z) = (f(x, y, z), y, z)$ is as follows:

$$\begin{bmatrix} \frac{\partial \varphi^1}{\partial x} & \frac{\partial \varphi^1}{\partial y} & \frac{\partial \varphi^1}{\partial z} \\ \frac{\partial \varphi^2}{\partial x} & \frac{\partial \varphi^2}{\partial y} & \frac{\partial \varphi^2}{\partial z} \\ \frac{\partial \varphi^2}{\partial x} & \frac{\partial \varphi^2}{\partial y} & \frac{\partial \varphi^2}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The Jacobian determinant is $\frac{\partial f}{\partial x}$, which is nonzero at the point p. Then by Corollary 2.14, there is a neighborhood U_p of $p \in \mathbb{R}^3$ such that $(U_p, \varphi^1, \varphi^2, \varphi^3)$ is a chart in the maximal atlas of $\mathbb{R}^3$. Now, in the chart $(U_p, \varphi^1, \varphi^2, \varphi^3) = (U_p, f, y, z)$, the set $U_p \cap S^2$ is defined by the vanishing of the first coordinate $\varphi^1 = f$. Because S^2 is the zero set of f, so the maximal subset of U_p where f vanish must be $U_p \cap S^2$. Thus (U_p, f, y, z) is an adapted chart relative to S^2, and $(U_p \cap S^2, y, z)$ is a chart for S^2 (applies to those points $p \in S^2$ for which $\frac{\partial f}{\partial x}(p) \neq 0$).

Similarly, if one chooses $p \in S^2$ with $\frac{\partial f}{\partial y}(p) \neq 0$, then there is an adapted chart (V_p, x, f, z) containing p in which $V_p \cap S^2$ is defined by the vanishing of the second coordinate f. Then $(V_p \cap S^2, x, z)$ is a chart for S^2 (applies to those points $p \in S^2$ for which $\frac{\partial f}{\partial y}(p) \neq 0$).

In a similar manner, if now one chooses $p \in S^2$ with $\frac{\partial f}{\partial z}(p) \neq 0$, then there is an adapted chart (W_p, x, y, f) containing p in which $W_p \cap S^2$ is defined by the vanishing of the third coordinate f. Then $(W_p \cap S^2, x, y)$ is a chart for S^2 (applies to those points $p \in S^2$ for which $\frac{\partial f}{\partial z}(p) \neq 0$).

Now, for every $p \in S^2$, at least one of the partial derivatives $\frac{\partial f}{\partial x}(p)$, $\frac{\partial f}{\partial y}(p)$, $\frac{\partial f}{\partial z}(p)$ is nonzero. Hence, as p varies on all over S^2, one obtains a collection of adapted charts of $\mathbb{R}^3$ that covers S^2. Therefore, S^2 is a regular submanifold of $\mathbb{R}^3$. By Proposition 5.1, S^2 is a manifold of dimension $3 - 1 = 2$. ■

Lemma 5.2

Let $g : N \to \mathbb{R}$ be a C^∞ function. A regular level set $g^{-1}(\{c\})$ of level c of the function g is the regular zero level set $f^{-1}(\{0\})$ of the function $f = g - c$.

Proof. For $p \in N$,

$$g(p) = c \iff f(p) = g(p) - c = 0\,.$$

Hence, $g^{-1}(\{c\}) = f^{-1}(\{0\})$. Call this set S. Note that

$$f_{*,p} = g_{*,p}\,, \qquad \forall\, p \in N\,.$$

Hence, critical points of f and g are exactly the same. Since S is a regular level set of the function g, it does not contain any critical point of the function g. Hence, S does not contain any critical point of the function f either. In other words, $f^{-1}(\{0\}) = S$ is a regular zero set of the function f. ■

Theorem 5.3

Let $g : N \to \mathbb{R}$ be a C^∞ function on the manifold N. Then a non-empty regular level set $S = g^{-1}(\{c\})$ iis a regular submanifold of N of codimension 1.

Proof. Let $f = g - c$. By Lemma 5.2, $S = f^{-1}(\{0\})$ is a regular zero set of f. Let $p \in S$. Since p is a regular point of f, relative to any chart $(U, x^1, x^2, \dots, x^n)$ containing p,

$$\frac{\partial f}{\partial x^i}(p) \neq 0 \quad \text{for some } i.$$

Otherwise, p would be a critical point of f according to Proposition 4.13. By renumberiing the coordinate functions, we may assyme that $\frac{\partial f}{\partial x^1}(p) \neq 0$. Now, the Jacobian matrix of the C^∞ map $\varphi : U \to \mathbb{R}^n$ defined by

$$\varphi(p) = \Big(f(p), x^2(p), \dots, x^n(p)\Big)$$

is given by

$$\begin{bmatrix} \frac{\partial \varphi^1}{\partial x^1} & \frac{\partial \varphi^1}{\partial x^2} & \cdots & \frac{\partial \varphi^1}{\partial x^n} \\ \frac{\partial \varphi^2}{\partial x^1} & \frac{\partial \varphi^2}{\partial x^2} & \cdots & \frac{\partial \varphi^2}{\partial x^n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \varphi^n}{\partial x^1} & \frac{\partial \varphi^n}{\partial x^2} & \cdots & \frac{\partial \varphi^n}{\partial x^n} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x^1} & \frac{\partial f}{\partial x^2} & \cdots & \frac{\partial f}{\partial x^n} \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

So, the Jacobian determinant $\frac{\partial(f,x^2,\ldots,x^n)}{\partial(x^1,x^2,\ldots,x^n)}$ at p is $\frac{\partial f}{\partial x^1}(p) \neq 0$. Therefore, by Corollary 2.14, there is a neighborhood U_p of $p \in N$ on which $\varphi^1, \varphi^2, \ldots, \varphi^n$ form a coordinate system. In other words, $(U_p, f, x^2, \ldots, x^n)$ is a coordinate neighborhood containing p. In this chart, the set $U_p \cap S$ is defined by the vanishing of the first coordinate $\varphi^1 = f$. Because S is the zero set of f, so the maximal subset of U_p where f vanish must be $U_p \cap S$. Thus $(U_p, f, x^2, \ldots, x^n)$ is an adapted chart relative to S, and $(U_p \cap S, x^2, \ldots, x^n)$ is a chart for S containing p. Since p is arbitrary, S is a regular submanifold of dimension $n-1$ by Proposition 5.1. ■

The next step is to extend the result of Theorem 5.3 to a regular level set of a smooth map between manifolds.

Theorem 5.4 (Regular Level Set Theorem)

Let $F : N \to M$ be a C^∞ map of manifolds, with $\dim N = n$ and $\dim M = m$. Then a non-empty regular level set $S = F^{-1}(\{c\})$, with $c \in M$, is a regular submanifold of N of dimension $n - m$.

Proof. Choose a chart $(V, \psi) = (V, y^1, y^2, \ldots, y^m)$ of M centered at c, *i.e.* $\psi(c) = \mathbf{0} \in \mathbb{R}^m$. Then $F^{-1}(V)$ is open in N containing $F^{-1}(\{c\})$. Also, in $F^{-1}(V)$,

$$F^{-1}(\{c\}) = F^{-1}\left(\psi^{-1}(\{\mathbf{0}\})\right) = (\psi \circ F)^{-1}(\{\mathbf{0}\}) .$$

Hence, the level set $F^{-1}(\{c\})$ is the zero set of $\psi \circ F$.

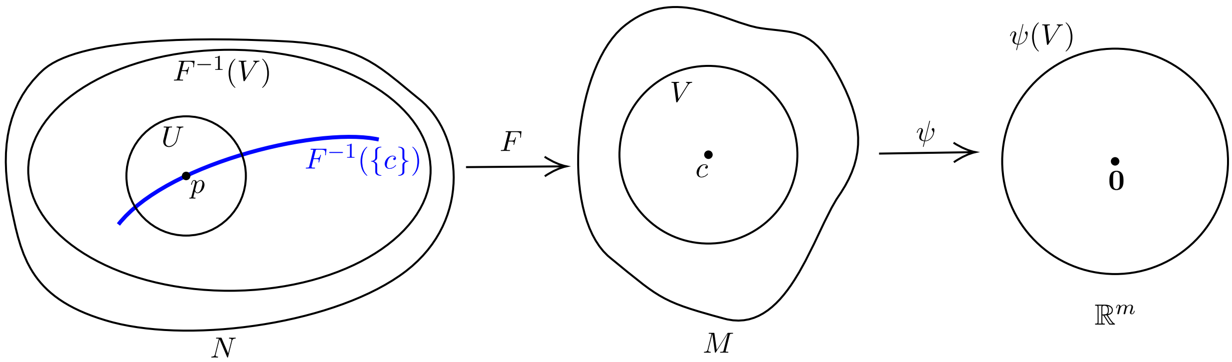


Figure 5.3: The level set $F^{-1}(\{c\})$ of F is the zero set of $\psi \circ F$.

Denote by $F^i = y^i \circ F = r^i \circ \psi \circ F = r^i \circ (\psi \circ F)$, with each $F^i : F^{-1}(V) \to \mathbb{R}$. Now for each $i \in \{1, 2, \ldots, m\}$,

$$F^i\left(F^{-1}(\{c\})\right) = r^i \circ \psi \circ F\left(F^{-1}(\{c\})\right) = \left(r^i \circ \psi\right)(c) = r^i(\mathbf{0}) = 0 .$$

Now we claim that, $F^{-1}(\{c\})$ is the maximal common zero set of the functions $F^1, F^2, \ldots, F^m$ on $F^{-1}(V)$. In other words, we want to show that

$$F^{-1}(\{c\}) = \bigcap_{i=1}^{m} \left(F^i\right)^{-1}(\{0\}) .$$

Assume for the sake of contradiction that there exists some $b \notin F^{-1}(\{c\})$ such that $F^i(b) = 0$ for every i. Then we have

$$F^i(b) = 0 \implies r^i(\psi(F(b))) = 0 \quad \forall i \implies \psi(F(b)) = \mathbf{0} = \psi(c) \implies F(b) = c .$$

This implies that $b \in F^{-1}(\{c\})$, contradiction! Therefore, $F^{-1}(\{c\})$ is the maximal common zero set of the functions $F^1, F^2, \ldots, F^m$ on $F^{-1}(V)$.

By hypothesis, the regular level set $F^{-1}(\{c\})$ is nonempty. So, by Remark 5.2, $n \geq m$. Now fix a point $p \in F^{-1}(\{c\})$ and let $(U, \varphi) = (U, x^1, x^2, \ldots, x^n)$ be a coordinate neighborhood of $p \in N$ contained in $F^{-1}(V)$. In other words, $p \in U \subseteq F^{-1}(V)$ (See Figure 5.3). Since $F^{-1}(\{c\})$ is a regular level set and $p \in F^{-1}(\{c\})$, p is a regular point of F. In other words, the differential $F_{*,p} : T_pN \to T_{F(p)}M$ is surjective. In other words, $\operatorname{rank} F_{*,p} = \dim T_{F(p)}M = m$. So, the matrix representation of $F_{*,p}$ also has rank m.

$$\operatorname{rank}\left[\frac{\partial F^i}{\partial x^j}(p)\right]_{1\le i\le m, 1\le j\le n} = m\,.$$

By renumbering F^i's and x^j's, we may assume that the first $m\times m$ block $\left[\frac{\partial F^i}{\partial x^j}(p)\right]_{1\le i,j\le m}$ is nonsingular. Now we replace the first m coordinates of the chart $(U,\varphi) = (U, x^1, x^2, \ldots, x^n)$ by $F^1, F^2, \ldots, F^m$.

Now we claim that there is a neighborhood U_p of p such that $(U_p, F^1, F^2, \ldots, F^m, x^{m+1}, \ldots, x^n)$ is a chart in the maximal atlas of N. It suffices to compute the pertinent Jacobian matrix at p. We write the $n\times n$ Jacobian matrix as block form:

$$J = \begin{bmatrix} \frac{\partial F^i}{\partial x^j} & \frac{\partial F^i}{\partial x^\beta} \\ \frac{\partial x^\alpha}{\partial x^j} & \frac{\partial x^\alpha}{\partial x^\beta} \end{bmatrix},$$

where $1 \le i,j \le m$ and $m+1 \le \alpha,\beta \le n$. Therefore,

$$J = \begin{bmatrix} \frac{\partial F^i}{\partial x^j} & * \\ 0_{(n-m)\times m} & I_{(n-m)\times(n-m)} \end{bmatrix}.$$

The determinant of J is $\det\left[\frac{\partial F^i}{\partial x^j}\right]_{1\le i,j\le m} \neq 0$ since $\left[\frac{\partial F^i}{\partial x^j}\right]_{1\le i,j\le m}$ is nonsingular. Therefore, by Corollary 2.14, there exists a neighborhood U_p of $p \in N$ such that there exists a coordinate neighborhood $(U_p, F^1, F^2, \ldots, F^m, x^{m+1}, \ldots, x^n)$ in the maximal atlas of N. In this chart, the set $U_p \cap S$ is defined by the vanishing of the first m coordinates $F^1, F^2, \ldots, F^m$. Because we proved earlier that S is the maximal common zero set of these m coordinate functions $F^1, F^2, \ldots, F^m$. So the maximal subset of U_p where all these m coordinates vanish must be $U_p \cap S$. Thus $(U_p, F^1, F^2, \ldots, F^m, x^{m+1}, \ldots, x^n)$ is an adapted chart relative to S, and $(U_p \cap S, x^{m+1}, \ldots, x^n)$ is a chart for S containing p. Since p is arbitrary, S is a regular submanifold of dimension $n-m$ by Proposition 5.1. ■

Following is a useful lemma that follows from the proof of the regular level set theorem.

Lemma 5.5

Let $F : N \to \mathbb{R}^m$ be a C^∞ map on a manifold N of dimension n and let S be the level set $F^{-1}(\{\mathbf{0}\})$. If relative to some coordinate chart $(U, x^1, x^2, \ldots, x^n)$ about $p \in S$, the Jacobian determinant

$$\frac{\partial\left(F^1, F^2, \ldots, F^m\right)}{\partial\left(x^{j_1}, x^{j_2}, \ldots, x^{j_m}\right)}(p)$$

is nonzero with $j_1, j_2, \ldots, j_m \in \{1, 2, \ldots, n\}$, then in some neighborhood of p, one may replace $x^{j_1}, x^{j_2}, \ldots, x^{j_m}$ by $F^1, F^2, \ldots, F^m$ to obtain an adapted chart of N relative to S.

Remark 5.3. The regular level set theorem gives a sufficient but not necessary condition for a subset of a manifold to be a regular submanifold — if the subset is a regular level set of some smooth map, then it is a regular submanifold. But there can be a regular submanifold of a manifold that fails to be a regular level set of some smooth map. Here is an example elucidating the fact: take $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x,y) = y^2$. This map is C^∞ and the zero set $Z(f)$ is the x-axis, a regular submanifold of $\mathbb{R}^2$. However, both $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ on the x-axis. In other words, every point of the x-axis is a critical point of f. Thus, although $Z(f)$ is a regular submanifold of $\mathbb{R}^2$, it is not a regular level set of f.

An Example of a Regular Submanifold

As a set, the special linear group $\mathrm{SL}(n,\mathbb{R})$ is the subset of $\mathrm{GL}(n,\mathbb{R})$ consisting of $n\times n$ real matrices of determinant 1. The product of two matrices with unit determminant is again a matrix with unit determinant. Furthermore, the inverse of a matrix with unit determminant is also a matrix with unit determminant. They follow from the following properties of the determinant function:

$$\det(AB) = (\det A)(\det B) \quad \text{and} \quad \det\left(A^{-1}\right) = \frac{1}{\det A}\,.$$

Hence, $\mathrm{SL}(n,\mathbb{R})$ is a subgroup of $\mathrm{GL}(n,\mathbb{R})$. Now let $f : \mathrm{GL}(n,\mathbb{R}) \to \mathbb{R}$ be given by $f(A) = \det A$. Notice that f is a C^∞ map and $\mathrm{SL}(n,\mathbb{R}) = f^{-1}(\{1\})$. We will now check that 1 is a regular value of the C^∞ map f, *i.e.* the matrices in $f^{-1}(\{1\})$ are all regular points of f.

Let a_{ij}, $1 \le i,j \le n$, be the standard coordinates on $\mathbb{R}^{n\times n}$, and let S_{ij} be the submatrix of $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ formed by deleting the i-th row and the j-th column of A. Then denote by $m_{ij}(A) := \det S_{ij}$, the (i,j) minor of A. Then, we have

$$f(A) = \det A = (-1)^{i+1} a_{i1} m_{i1}(A) + (-1)^{i+2} a_{i2} m_{i2}(A) + \cdots + (-1)^{i+n} a_{in} m_{in}(A)\,.$$

This is obtained by expanding along the i-th row. Therefore,

$$\frac{\partial f}{\partial a_{ij}}(A) = (-1)^{i+j} m_{ij}(A)\,.$$

By Proposition 4.13, a matrix B will be a critical point of f if and only if all the partial derivatives $\frac{\partial f}{\partial a_{ij}}(B)$ vanish. In other words,

$$\frac{\partial f}{\partial a_{ij}}(B) = (-1)^{i+j} m_{ij}(B) = 0\,, \qquad \text{for every } i,j\,.$$

Hence, a matrix $B = [b_{ij}] \in \mathrm{GL}(n,\mathbb{R})$ is a critical point of f if and only if all the $(n-1)\times(n-1)$ minors $m_{ij}(B)$ of B are 0. Then we have,

$$\det B = (-1)^{i+1} b_{i1} m_{i1}(B) + (-1)^{i+2} b_{i2} m_{i2}(B) + \cdots + (-1)^{i+n} b_{in} m_{in}(B) = 0\,.$$

Since every matrix in $\mathrm{SL}(n,\mathbb{R})$ has determinant 1, no matrix in $\mathrm{SL}(n,\mathbb{R})$ can be a critical point of f. In other words, all the matrices in $\mathrm{SL}(n,\mathbb{R})$ are regular points of f, and thus $\mathrm{SL}(n,\mathbb{R}) = f^{-1}(\{1\})$ is a regular level set. Then by Theorem 5.3, $\mathrm{SL}(n,\mathbb{R})$ is a regular submanifold of $\mathrm{GL}(n,\mathbb{R})$ of codimension 1. In other words,

$$\dim \mathrm{SL}(n,\mathbb{R}) = \dim \mathrm{GL}(n,\mathbb{R}) - 1 = n^2 - 1\,.$$

§5.3 Rank of a Smooth Map

Recall from the previous chapter that the rank of a smooth map $F : N \to M$ at $p \in N$ is defined as the rank of its differential at p, *i.e.* the rank of the linear map $F_{*,p} : T_pN \to T_{F(p)}M$. Here, $n = \dim N$ and $m = \dim M$. Now, we will study two situations related to rank of a smooth map:

(i) when F has maximal rank at $p \in N$,

(ii) F has constant rank in a neighborhood of $p \in N$.

If $F : N \to M$ has maximal rank at $p \in N$, then there are 3 mutually not exclusive possibilities:

(a) If $n = m$, then since $F_{*,p} : T_pN \to T_{F(p)}M$ is a linear map with maximal rank, one must have

$$\operatorname{rank} F_{*,p} = n = \dim T_pN\,.$$

By rank-nullity theorem,

$$\operatorname{rank} F_{*,p} + \operatorname{nullity} F_{*,p} = \dim T_pN = n \implies \operatorname{nullity} F_{*,p} = 0\,.$$

Hence, $\operatorname{Ker} F_{*,p} = \{\mathbf{0}\}$, yielding $F_{*,p}$ is non-singular. Since $\dim T_pN = \dim T_{F(p)}M$, and $F_{*,p}$ is non-singular, $F_{*,p}$ is bijective. Therefore, $F_{*,p}$ is a bijective linear transformation, *i.e.* an isomorphism between T_pN and $T_{F(p)}M$. Therefore, by Remark 4.3, $F : N \to M$ is locally invertible or a local diffeomorphism at $p \in N$.

(b) If $n \leq m$, then using the fact that $\operatorname{rank} F_{*,p} \leq \min\{m, n\}$, one obtains $\operatorname{rank} F_{*,p} \leq n$. Since $\operatorname{rank} F_{*,p}$ is maximal, $\operatorname{rank} F_{*,p} = n$. By rank-nullity theorem,

$$\operatorname{rank} F_{*,p} + \operatorname{nullity} F_{*,p} = \dim T_p N = n \implies \operatorname{nullity} F_{*,p} = 0 .$$

In other words, $\operatorname{Ker} F_{*,p} = \{\mathbf{0}\}$, so $F_{*,p} : T_p N \to T_{F(p)} M$ is injective. Therefore, $F : N \to M$ is an immersion at $p \in N$.

(c) If $n \geq m$, then again using $\operatorname{rank} F_{*,p} \leq \min\{m, n\}$, one obtains $\operatorname{rank} F_{*,p} \leq m$. Since $\operatorname{rank} F_{*,p}$ is maximal, $\operatorname{rank} F_{*,p} = m$. In other words, $\operatorname{im} F_{*,p}$ is a m-dimensional vector subspace of $T_{F(p)} M$, so $\operatorname{im} F_{*,p}$ must be $T_{F(p)} M$ itself. Hence, $F_{*,p} : T_p N \to T_{F(p)} M$ is surjective. Therefore, $F : N \to M$ is a submersion at $p \in N$.

Lemma 5.6

Suppose $T : V \to W$ is a linear transformation between finite dimensional vector spaces V and W. Let $I_W : W \to X$ and $I_V : V \to Y$ be vector space isomorphisms. Then $\operatorname{rank}(I_W \circ T) = \operatorname{rank} T$ and $\operatorname{rank}\left(T \circ I_V^{-1}\right) = \operatorname{rank} T$.

Proof. $I_W \circ T : V \to X$. $T(V)$ is a vector subspace of W. Then restricting I_W on $T(V)$ gives us an isomorphism $I_W|_{T(V)} : T(V) \to I_W(T(V))$. Therefore,

$$\operatorname{rank}(I_W \circ T) = \dim(I_W(T(V))) = \dim T(V) = \operatorname{rank} T .$$

Furthermore, $T \circ I_V^{-1} : X \to W$. Since $I_V^{-1} : Y \to V$ is a vector space isomorphism, $I_V^{-1}(Y) = V$. Therefore,

$$\operatorname{rank}\left(T \circ I_V^{-1}\right) = \dim\left(T\left(I_V^{-1}(Y)\right)\right) = \dim T(V) = \operatorname{rank} T .$$

■

Theorem 5.7 (Constant Rank Theorem for Manifolds)

Let N and M be manifolds of dimension n and m, respectively. Suppose $F : N \to M$ has constant rank k in a neighborhood of a point $p \in N$. Then there are charts (U, φ) centered at $p \in N$ ($\varphi(p) = \mathbf{0} \in \mathbb{R}^n$) and (V, ψ) centered at $F(p) \in M$ ($\psi(F(p)) = \mathbf{0} \in \mathbb{R}^m$) such that for $(r^1, r^2, \ldots, r^n) \in \varphi(U)$,

$$\left(\psi \circ F \circ \varphi^{-1}\right)\left(r^1, r^2, \ldots, r^n\right) = (r^1, r^2, \ldots, r^k, \underbrace{0, 0, \ldots, 0}_{(m-k) \text{ many } 0\text{s}}) .$$

Remark 5.4. Same as Remark B.1, we can add $F(U) \subseteq V$ in the statement of Constant Rank Theorem for Manifolds, because otherwise we can always find a smaller U such that $F(U)$ is contained in V. Here, too, the notation $\psi \circ F \circ \varphi^{-1}$ is a bit sloppy. What this composition actually means is the following:

$$\psi|_{F(U)} \circ F|_U \circ \varphi^{-1} : \varphi(U) \to \psi(F(U)) .$$

Proof of Constant Rank Theorem for Manifolds. Choose a chart $\left(\overline{U}, \overline{\varphi}\right)$ about $p \in N$ and $\left(\overline{V}, \overline{\psi}\right)$ about $F(p) \in M$ with $F\left(\overline{U}\right) \subseteq \overline{V}$. Then $\overline{\psi} \circ F \circ \overline{\varphi}^{-1} : \overline{\varphi}\left(\overline{U}\right) \subseteq \mathbb{R}^n \to \overline{\psi}\left(\overline{V}\right) \subseteq \mathbb{R}^m$ is a map between open subsets of Euclidean spaces. Because $\overline{\psi}$ and $\overline{\varphi}^{-1}$ are diffeomorphisms onto the respective images, the pertaining differentials are isomorphisms by Corollary 4.2. In other words, $\overline{\psi}_{*,F(p)}$ and $\left(\overline{\varphi}^{-1}\right)_{*,\overline{\varphi}(p)}$ are both vector space isomorphisms. Also, by The Chain Rule,

$$\left(\overline{\psi} \circ F \circ \overline{\varphi}^{-1}\right)_{*,\overline{\varphi}(p)} = \overline{\psi}_{*,F(p)} \circ F_{*,p} \circ \left(\overline{\varphi}^{-1}\right)_{*,\overline{\varphi}(p)} .$$

By Lemma 5.6, composition with isomorphism does not change the rank of a linear map. Therefore,

$$\operatorname{rank}\left(\overline{\psi}\circ F\circ\overline{\varphi}^{-1}\right)_{*,\overline{\varphi}(p)}=\operatorname{rank}F_{*,p}\,.$$

$\operatorname{rank}F_{*,p}$ is the rank of the smooth map $F:N\to M$ at $p\in N$, which is constant k in a neighborhood of p. Therefore, the map $\overline{\psi}\circ F\circ\overline{\varphi}^{-1}$ between open subsets of Euclidean spaces also has constant rank k in a neighborhood of $\overline{\varphi}(p)\in\mathbb{R}^n$. By Constant Rank Theorem for Euclidean Spaces, there are a diffeomorphism G of a neighborhood of $\overline{\varphi}(p)\in\mathbb{R}^n$ and a diffeomorphism H of a neighborhood of $\overline{\psi}(F(p))\in\mathbb{R}^m$ such that

$$\left(H\circ\overline{\psi}\circ F\circ\overline{\varphi}^{-1}\circ G^{-1}\right)\left(r^1,r^2,\ldots,r^n\right)=(r^1,r^2,\ldots,r^k,\ \underbrace{0,0,\ldots,0}_{(m-k)\text{ many }0\text{s}}\ )\,.$$

Setting $H\circ\overline{\psi}=\psi$ and $G\circ\overline{\varphi}=\varphi$, we obtain

$$\left(\psi\circ F\circ\varphi^{-1}\right)\left(r^1,r^2,\ldots,r^n\right)=(r^1,r^2,\ldots,r^k,\ \underbrace{0,0,\ldots,0}_{(m-k)\text{ many }0\text{s}}\ )\,.$$

Furthermore, Constant Rank Theorem for Euclidean Spaces guarantees that G sends $\overline{\varphi}(p)$ to $\mathbf{0}\in\mathbb{R}^n$ and H sends $\overline{\psi}(F(p))$ to $\mathbf{0}\in\mathbb{R}^m$. Therefore,

$$\varphi(p)=G(\overline{\varphi}(p))=\mathbf{0}\in\mathbb{R}^n\quad\text{and}\quad\psi(F(p))=H\left(\overline{\psi}(F(p))\right)=\mathbf{0}\in\mathbb{R}^m\,.$$

■

By a neighborhood of a subset A of a manifold M we mean an open set containing A.

Theorem 5.8 (Constant-Rank Level Set Theorem)
Let $f:N\to M$ be a C^∞ map of manifolds and $c\in M$. If f has constant rank k in a neighborhood of the level set $f^{-1}(\{c\})\subseteq N$, then $f^{-1}(\{c\})$ is a regular submanifold of N of codimension k.

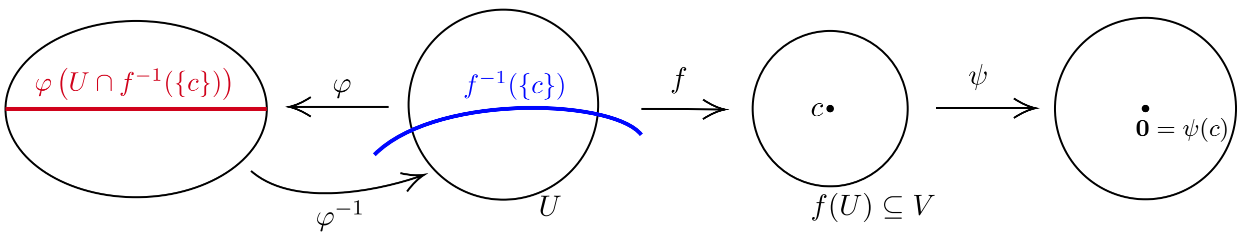
Proof. Let $p\in f^{-1}(\{c\})$. By hypothesis, there is a neighborhood p of N where f has constant rank. By Constant Rank Theorem for Manifolds, there are a coordinate chart $(U,\varphi)=(U,x^1,x^2,\ldots,x^n)$ centered at $p\in N$ $(\varphi(p)=\mathbf{0}\in\mathbb{R}^n)$ and a coordinate chart $(V,\psi)=(V,y^1,y^2,\ldots,y^m)$ centered at $f(p)=c\in N$ $(\psi(c)=\mathbf{0}\in\mathbb{R}^m)$ such that

$$\left(\psi|_{f(U)}\circ f|_U\circ\varphi^{-1}\right)\left(r^1,r^2,\ldots,r^n\right)=(r^1,r^2,\ldots,r^k,\ \underbrace{0,0,\ldots,0}_{(m-k)\text{ many }0\text{s}}\ )\,.$$

Let $\psi|_{f(U)}\circ f|_U\circ\varphi^{-1}=\widehat{f}$. Then we have,

$$\widehat{f}\left(r^1,r^2,\ldots,r^n\right)=\left(r^1,r^2,\ldots,r^k,0,0,\ldots,0\right)=\mathbf{0}\iff r^1=r^2=\cdots=r^k=0\,.$$

Hence, $\widehat{f}^{-1}(\{\mathbf{0}\})$ is defined by the vanishing of $r^1,r^2,\ldots,r^k$.



Now, observe that

$$\varphi\left(U \cap f^{-1}(\{c\})\right) = \varphi\left(f|_U^{-1}(\{c\})\right) = \varphi\left(f|_U^{-1}\left(\psi|_{f(U)}^{-1}(\{\mathbf{0}\})\right)\right)$$
$$= \left(\psi|_{f(U)} \circ f|_U \circ \varphi^{-1}\right)^{-1}(\{\mathbf{0}\}) = \widehat{f}^{-1}(\{\mathbf{0}\}) .$$

Hence, the image of the set $U \cap f^{-1}(\{c\})$ under φ is the level set $\widehat{f}^{-1}(\{\mathbf{0}\})$. One, therefore, obtains that a generic element of $\varphi\left(U \cap f^{-1}(\{c\})\right)$ has the first k coordinates vanishing.

Recall that $(U, \varphi) = (U, x^1, x^2, \ldots, x^n)$ is a chart in N centered at $p \in N$, where $x^i = r^i \circ \varphi$ with $i \in \{1, 2, \ldots, n\}$. Since $r^1, r^2, \ldots, r^k$ vanishes on $\varphi\left(U \cap f^{-1}(\{c\})\right)$, on $U \cap f^{-1}(\{c\})$, φ is given by the vanishing of the first k coordinates. In other words,

$$x^1 = r^1 \circ \varphi = 0\,, \; x^2 = r^2 \circ \varphi = 0\,, \; \ldots\,, \; x^k = r^k \circ \varphi = 0\,.$$

On the other hand, if $q \in U \setminus f^{-1}(\{c\})$, then since $\varphi\left(U \cap f^{-1}(\{c\})\right) = \widehat{f}^{-1}(\{\mathbf{0}\})$, $\varphi(q) \notin \widehat{f}^{-1}(\{\mathbf{0}\})$. Hence, not all the first k coordinate functions of $\varphi(q)$ are vanishing. Hence, the maximal subset of U where all the first k coordinates are vanishing is $U \cap f^{-1}(\{c\})$. Therefore, U is an adapted chart of N relative to $f^{-1}(\{c\})$ containing p. Since $p \in f^{-1}(\{c\})$ was an arbitrary point, $f^{-1}(\{c\})$ is a regular submanifold of N of codimension k. ■

Example 5.3. The orthogonal group $\mathrm{O}(n)$ is defined to be the subgroup of $\mathrm{GL}(n, \mathbb{R})$ consisting of matrices A with $A^T A = A A^T = I$, where I is the $n \times n$ identity matrix. Using the Constant-Rank Level Set Theorem, prove that $\mathrm{O}(n)$ is a regular submanifold of $\mathrm{GL}(n, \mathbb{R})$.

Solution. Define $f : \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$ by $f(A) = A^T A$. Then $\mathrm{O}(n) = f^{-1}(\{I\})$.

For $A, B \in \mathrm{GL}(n, \mathbb{R})$, there exists a unique matrix $C \in \mathrm{GL}(n, \mathbb{R})$ such that $B = AC$. Denote by $l_C, r_C : \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$, the left and right multiplication by C, respectively:

$$l_C(B) = CB \quad \text{and} \quad r_C(B) = BC\,.$$

Now, since $f(AC) = (AC)^T AC = C^T A^T AC = C^T f(A)\,C$, one has

$$(f \circ r_C)(A) = f(AC) = C^T f(A)\,C\,.$$

On the other hand,

$$(l_{C^T} \circ r_C \circ f)(A) = (l_{C^T} \circ r_C)(f(A))\, C^T f(A)\,C\,.$$

Therefore, $(f \circ r_C)(A) = (l_{C^T} \circ r_C \circ f)(A)$. Since this is true for every $A \in \mathrm{GL}(n, \mathbb{R})$,

$$f \circ r_C = l_{C^T} \circ r_C \circ f\,.$$

Now, by The Chain Rule,

$$(f \circ r_C)_{*,A} = (l_{C^T} \circ r_C \circ f)_{*,A} \implies f_{*,AC} \circ (r_C)_{*,A} = (l_{C^T})_{*,A^T AC} \circ (r_C)_{*,A^T A} \circ f_{*,A}\,.$$

Since left and right multiplications are local diffeomorphisms, the pertaining differentials are isomorphisms by Remark 4.3. Composition with isomorphism does not change the rank of a linear map (Lemma 5.6). Therefore,

$$\operatorname{rank} f_{*,A} = \operatorname{rank} f_{*,AC} = \operatorname{rank} f_{*,B}\,,$$

since $AC = B$. Since A and B are two arbitrary points of $\mathrm{GL}(n, \mathbb{R})$, we can conclude that f has constant rank on $\mathrm{GL}(n, \mathbb{R})$. Then by Constant-Rank Level Set Theorem, the level set $f^{-1}(\{I\}) = \mathrm{O}(n)$ is a regular submanifold of $\mathrm{GL}(n, \mathbb{R})$. ■

§5.4 The Immersion and Submersion Theorems

Consider a C^∞ map $f : N \to M$. Let $(U, \varphi) = (U, x^1, x^2, \dots, x^n)$ be a chart about $p \in N$ and $(V, \psi) = (V, y^1, y^2, \dots, y^m)$ be a chart about $f(p) \in M$. Write $f^i = y^i \circ f$. By Proposition 4.8, relative to the charts (U, φ) and (V, ψ), *i.e.*, relative to the basis $\left\{ \frac{\partial}{\partial x^i}\Big|_p \right\}_{i=1}^n$ of T_pN and the basis $\left\{ \frac{\partial}{\partial y^j}\Big|_{f(p)} \right\}_{j=1}^m$ of $T_{f(p)}M$, the linear map $f_{*,p} : T_pN \to T_{f(p)}M$ is represented by the matrix

$$\left[\frac{\partial f^i}{\partial x^j}(p) \right]_{1 \le i \le m; 1 \le j \le n} .$$

Hence,

$$\begin{aligned} f_{*,p} \text{ is injective } &\iff n \le m \text{ and } \operatorname{rank} \left[\frac{\partial f^i}{\partial x^j}(p) \right] = n\,, \\ f_{*,p} \text{ is surjective } &\iff m \le n \text{ and } \operatorname{rank} \left[\frac{\partial f^i}{\partial x^j}(p) \right] = m\,. \end{aligned}$$

The rank of a matrix is the number of linearly independent rows/columns of the matrix. Since the matrix $\left[\frac{\partial f^i}{\partial x^j}(p) \right]$ is of size $m \times n$, $\operatorname{rank} \left[\frac{\partial f^i}{\partial x^j}(p) \right] \le \min\{m, n\}$. Therefore, one finds that whenever $f : N \to M$ is an immersion or a submersion at $p \in N$, the matrix $\left[\frac{\partial f^i}{\partial x^j}(p) \right]$ is of maximal rank.

Having maximal rank at a point is an *open condition* in the following sense: the set

$$D_{\max}(f) := \{ p \in U \mid f_{*,p} \text{ has maximal rank at } p \}$$

is an open subset of U.

Claim 3: $D_{\max}(f)$ is an open subset of U.

Proof. Suppose k is the maximal rank of f. Then

$$\begin{aligned} \operatorname{rank} f_{*,p} = k &\iff \operatorname{rank} \left[\frac{\partial f^i}{\partial x^j}(p) \right] = k \\ &\iff \operatorname{rank} \left[\frac{\partial f^i}{\partial x^j}(p) \right] \ge k \end{aligned}$$

The last $\Rightarrow$ is obvious and $\Leftarrow$ holds since k is the maximal rank, so

$$\operatorname{rank} \left[\frac{\partial f^i}{\partial x^j}(p) \right] \ge k \implies k \ge \operatorname{rank} \left[\frac{\partial f^i}{\partial x^j}(p) \right] \ge k \implies \operatorname{rank} \left[\frac{\partial f^i}{\partial x^j}(p) \right] = k\,.$$

Hence, the complement $U \setminus D_{\max}(f)$ is defined by

$$U \setminus D_{\max}(f) = \left\{ p \in U \mid \operatorname{rank} \left[\frac{\partial f^i}{\partial x^j}(p) \right] < k \right\} .$$

By Lemma 3.5, $\operatorname{rank} \left[\frac{\partial f^i}{\partial x^j}(p) \right] < k$ is equivalent to the vanishing of **all** $k \times k$ minors of $\left[\frac{\partial f^i}{\partial x^j}(p) \right]$.

Now we shall use the fact that the common zero set of finitely many continuous functions is closed. In other words, if $f_1, f_2, \dots, f_n : X \to \mathbb{R}$ are continuous functions, then

$$\bigcap_{i=1}^n f_i^{-1}(\{0\})$$

is closed in X. This is because each $f_i^{-1}(\{0\})$ is closed in X, and intersection of finitely many closed sets is also closed.

Now, $U \setminus D_{\max}(f)$ is the collection of all $p \in U$ such that all the determinant functions on $k \times k$ submatrices of $\left[\frac{\partial f^i}{\partial x^j}(p)\right]$ vanish simultaneously. So we are looking at the common zero set of all the determinant functions on all $k \times k$ submatrices of $\left[\frac{\partial f^i}{\partial x^j}(p)\right]$. Since there are only finitely many $k \times k$ submatrices, we can conclude that $U \setminus D_{\max}(f)$ is closed in U. Therefore, $D_{\max}(f)$ is open in U. □

In particular, if f has maximal rank at p, it has maximal rank at all points in some neighborhood of p, which is denoted by $D_{\max}(f)$ here. We summarize all these results formally by means of the following proposition.

Proposition 5.9

Let N and M be manifolds of dimension n and m, respectively. If a C^∞ map $f : N \to M$ is an immersion at a point $p \in N$, then it has a constant rank n in a neighborhood of p (in this case $m \geq n$). If a C^∞ map $f : N \to M$ is a submersion at $p \in N$ (in which case $n \geq m$), then it has a constant rank m in a neighborhood of p.

Example 5.4. Although maximal rank at a point implies constant rank in a neighborhood, the converse is not true. The map $f : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $f(x, y) = (x, 0, 0)$ has Jacobian matrix

$$J = \begin{bmatrix} \frac{\partial f^1}{\partial x} & \frac{\partial f^1}{\partial y} \\ \frac{\partial f^2}{\partial x} & \frac{\partial f^2}{\partial y} \\ \frac{\partial f^3}{\partial x} & \frac{\partial f^3}{\partial y} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then J has rank 1 everywhere in $\mathbb{R}^2$. But this is not maximal since $\min\{2, 3\} = 2 \neq \operatorname{rank} J$.

Proposition 5.9 and Constant Rank Theorem for Manifolds imply the following theorem.

Theorem 5.10

Let N and M be manifolds of dimension n and m, respectively.

(i) **(Immersion theorem)** Suppose $f : N \to M$ is an immersion at $p \in N$ (then $n \leq m$). Then there are charts (U, φ) centered at $p \in N$ and (V, ψ) centered at $f(p) \in M$ such that in a neighborhood of $\varphi(p) = \mathbf{0} \in \mathbb{R}^n$,

$$\left(\psi \circ f \circ \varphi^{-1}\right)\left(r^1, r^2, \ldots, r^n\right) = (r^1, r^2, \ldots, r^n, \underbrace{0, 0, \ldots, 0}_{(m-n)\text{ 0s}}).$$

(ii) **(Submersion theorem)** Suppose $f : N \to M$ is a submersion at $p \in N$ (then $n \geq m$). Then there are charts (U, φ) centered at $p \in N$ and (V, ψ) centered at $f(p) \in M$ such that in a neighborhood of $\varphi(p) = \mathbf{0} \in \mathbb{R}^n$,

$$\left(\psi \circ f \circ \varphi^{-1}\right)\left(r^1, r^2, \ldots, r^m, r^{m+1}, \ldots, r^n\right) = \left(r^1, r^2, \ldots, r^m\right).$$

Corollary 5.11

A submersion $f : N \to M$ is an open map.

Proof. Let $W \subseteq N$ be open and $p \in W$ so that $f(p) \in f(W)$. Now, $f : N \to M$ is a submersion at p by hypothesis. Then by submersion theorem, there are charts (U, φ) centered at $p \in W \subseteq N$ and (V, ψ) centered at $f(p) \in f(W) \subseteq M$ such that in a neighborhood of $\varphi(p) = \mathbf{0} \in \mathbb{R}^n$, one has

$$\left(\psi \circ f \circ \varphi^{-1}\right)\left(r^1, r^2, \ldots, r^m, r^{m+1}, \ldots, r^n\right) = \left(r^1, r^2, \ldots, r^m\right).$$

We can take U small enough such that $U \subseteq W$.

Now, consider the subset $f(W) \subseteq M$. Any point in $f(W)$ is of the form $f(p)$ with $p \in W$. But from the previous argument, it follows that for $p \in W \subseteq N$ with W open, one can find an open set $U \subseteq W$ with respect to which f acts as a local projection. In other words,

$$\widehat{f} = \psi|_{f(U)} \circ f|_U \circ \varphi^{-1} : \varphi(U) \subseteq \mathbb{R}^n \to \psi(V) \subseteq \mathbb{R}^m$$

is projection onto the first m coordinates. By Proposition A.24, projection is an open map. Therefore,

$$\widehat{f}(\varphi(U)) = \psi|_{f(U)}(f(U)) = \psi(f(U))$$

is open in $\psi(V) \subseteq \mathbb{R}^m$. Since ψ is a diffeomorphism, $\psi^{-1} : \psi(V) \subseteq \mathbb{R}^m \to V$ is an open map. Therefore,

$$\psi^{-1}(\psi(f(U))) = f(U)$$

is open in V. Since V is open in M, $f(U)$ is open in M. Hence, for every $f(p) \in f(W) \subseteq M$, one can find $f(U)$ open in M such that

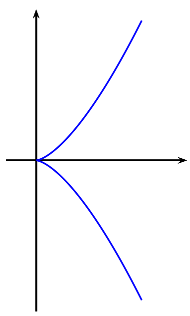
$$f(p) \in f(U) \subseteq f(W) \, .$$

This proves that $f(W)$ is open in M. ■

Example 5.5. Let $f : \mathbb{R} \to \mathbb{R}^2$ be defined by $f(t) = (t^2, t^3)$. Observe that

$$f(t_1) = f(t_2) \iff \left(t_1^2, t_1^3\right) = \left(t_2^2, t_2^3\right) \iff t_1 = t_2 \, .$$

The equality of the second component $t_1^3 = t_2^3$ forces $t_1 = t_2$, although $t_1^2 = t_2^2$ has 2 solutions $t_1 = \pm t_2$. So, the injectivity of the function t^3 forces the injectivity of $f(t) = (t^2, t^3)$. This f is represented by a cuspidal cubic.



We've seen in Example 4.3 that

$$f_{*,t_0}\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0}\right) = 2t_0 \left.\frac{\partial}{\partial x}\right|_{f(t_0)} + 3t_0^2 \left.\frac{\partial}{\partial y}\right|_{f(t_0)} \, .$$

Therefore, for $t_0 = 0$, we find that the differential $f_{*,0} : T_0\mathbb{R} \to T_{(0,0)}\mathbb{R}^2$ is the zero map, and hence it's not injective. Therefore, despite f being an injective map, it's not an immersion at 0.

6 Immersed vs Regular Submanifold

§6.1 Embedding

Example 6.1. Consider the smooth map $f : \mathbb{R} \to \mathbb{R}^2$ defined by $f(t) = \left(t^2 - 1, t^3 - t\right)$. This map is not injective as $f(1) = f(-1) = (0, 0)$. The matrix representation of f_{*,t_0} with respect to the standard coordinate of $\mathbb{R}$ and $\mathbb{R}^2$ is

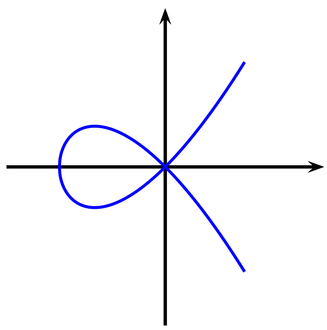
$$\begin{bmatrix} 2t_0 \\ 3t_0^2 - 1 \end{bmatrix} .$$

There is no t_0 such that $2t_0 = 3t_0^2 - 1 = 0$. So, rank $f_{*,t_0} = 1$ for every t_0. Therefore, f is an immersion, but it's not injective.

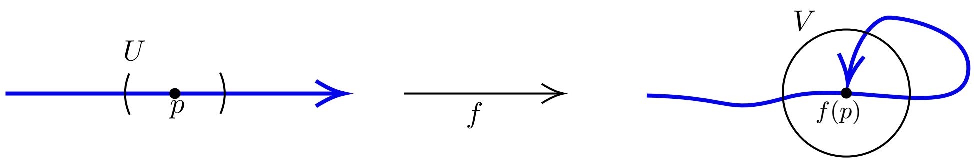
To find an equation for the image $f(\mathbb{R})$, let $x = t^2 - 1$ and $y = t^3 - t$. Then

$$y = t\left(t^2 - 1\right) = tx \implies y^2 = t^2x^2 = x^2(x + 1) .$$

Thus, the image of f is the nodal cubic $y^2 = x^2(x + 1)$.



Example 6.2. The map f shown in the following figure is an injective immersion but it's image, with respect to the subspace topology inherited from $\mathbb{R}^2$ is not homeomorphic to the domain $\mathbb{R}$, because there are points near $f(p)$ in the image that corresponds to points in $\mathbb{R}$ far away from p. Let us try to see it mathematically.



By Lemma A.19, f^{-1} is continuous if and only if for every $f(p) \in f(\mathbb{R})$ and for every neighborhood U of $f^{-1}(f(p)) = p$, there exists a neighborhood V of $f(p)$ such that $f^{-1}(V) \subseteq U$. If we choose U to be an interval around p, then there is no neighborhood $V \subseteq \mathbb{R}^2$ of $f(p)$ such that $f^{-1}(V) \subseteq U$. Because, no matter how small V is, it will contain points whose image under f^{-1} will be far away from p, let alone be contained in U. Therefore, f^{-1} is not continuous. In other words, $f : \mathbb{R} \to f(\mathbb{R})$ is not a homeomorphism.

> **Definition 6.1** (Embedding)**.** A C^∞ map $f : N \to M$ is called an **embedding** if
>
> (i) it is an injective immersion, and
>
> (ii) the image $f(N)$ with respect to the subspace topology is homeomorphic to N under f.

Remark 6.1. Here $f : N \to M$ is injective. In fact, the injectivity condition is redundant as the condition that $f : N \to f(N)$ is a homeomorphism already demands that f is injective.

One can equip the image $f(N)$ of N under f with not the subspace topology inherited from M, but the topology inherited from f. That is, a subset $f(U)$ of $f(N)$ is said to be open if and only if U is open in N. With this topology, $f : N \to f(N)$ is a homeomorphism.

Let $f(U) \subseteq f(N)$ be open. Then by the definition of the topology inherited from f, $U = f^{-1}(f(U))$ is open in N. Therefore, f is continuous. Now, consider $f^{-1} : f(N) \to N$. Let $U \subseteq N$ be open. Then $f(U) = \left(f^{-1}\right)^{-1}(U)$ is open. Therefore, f^{-1} is also continuous. Hence, with respect to the topology inherited by f, $f : N \to f(N)$ is a homeomorphism.

The image $f(N)$ of an injective immersion is called an **immersed submanifold**. The topology of an immersed submanifold is the one inherited from f. If the underlying set of an immersed submanifold is given the subspace topology, then the resulting space need not be a manifold at all. Contrary to an immersed submanifold, a regular submanifold of a manifold M us a subset S of M with the subspace topology such that every point of S has a neighborhood $U \cap S$ that is defined by the vanishing of the coordinate functions on U, where U is a chart in M.

Example 6.3. The figure eight is the image of the injective immersion

$$f : \left(-\frac{\pi}{2}, 3\frac{\pi}{2}\right) \to \mathbb{R}^2 \ , \ f(t) = (\cos t, \sin 2t) \ .$$

It is easily seen to be injective, since

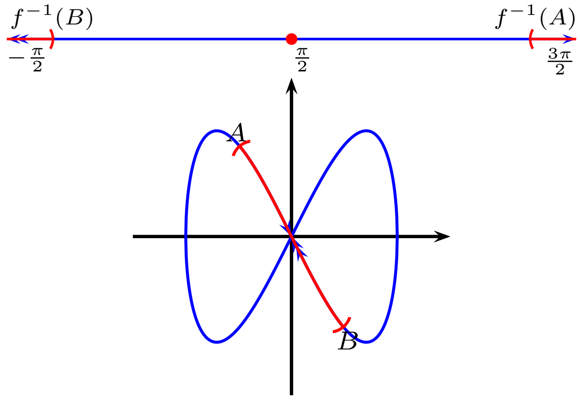
$$f(t_1) = f(t_2) \implies \cos t_1 = \cos t_2 \text{ and } \sin 2t_1 = \sin 2t_2 \implies \sin t_1 = \sin t_2 \, .$$

Therefore, either $\cos t_1 = \cos t_2 = 0$ or $t_1 - t_2 = 2n\pi$ for some integer n. There are only one $t \in \left(-\frac{\pi}{2}, 3\frac{\pi}{2}\right)$ with $\cos t = 0$. And, $t_1 - t_2 = 2n\pi$ is also not possible since the length of the interval is 2π, and the endpoints are not included. So f is injective.

To see that f is an immersion, the matrix representation of f_{*,t_0} is

$$\begin{bmatrix} -\sin t_0 \\ 2\cos 2t_0 \end{bmatrix} .$$

$\cos 2_{t_0} = 1 - 2\sin^2 t_0$, so $-\sin t_0$ and $2\cos 2t_0$ can't both be 0 simultaneously. Therefore, rank $f_{*,t_0} = 1$ for every t_0. So, f is an immersion.



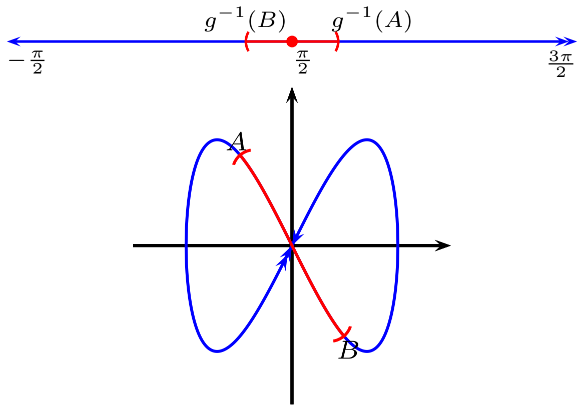
The set denoted by the segment AB in the x-y plane is **not** an open set relative to the topology inherited from f, as

$$f^{-1}(AB) = \left(-\frac{\pi}{2}, f^{-1}(B)\right) \cup \left(f^{-1}(A), 3\frac{\pi}{2}\right) \cup \left\{\frac{\pi}{2}\right\} .$$

$f^{-1}(AB)$ contains an isolated point and hence it's not open.

Figure eight is also the image of the injective immersion

$$g : \left(-\frac{\pi}{2}, 3\frac{\pi}{2}\right) \to \mathbb{R}^2 \ , \ g(t) = (\cos t, -\sin 2t) \ .$$



For this injective immersion, $g^{-1}(AB) = (g^{-1}(B), g^{-1}(A))$, which is open in $(-\frac{\pi}{2}, 3\frac{\pi}{2})$.

Hence, these two injective immersions f and g are distinct. The segment AB is not open in the topology inherited from f, while it is open in the topology inherited from g.

Why is the image of figure 8 not a manifold in subspace topology?

Let S be the image of figure 8. Assume the contrary that S is a manifold in subspace topology. Take $(0,0) \in S$. Since S is a manifold, there is an open set (in the subspace topology) U around $(0,0)$ such that U is homeomorphic to an open ball $B(\mathbf{a}, \varepsilon)$ of radius ε centered at $\mathbf{a} \in \mathbb{R}^n$. Denote this homeomorphism by φ, *i.e.* $\varphi(U) = B(\mathbf{a}, \varepsilon)$. Then $U \setminus \{(0,0)\}$ is homeomorphic to $B(\mathbf{a}, \varepsilon) \setminus \{\varphi(0,0)\}$. Therefore, they must have the same number of connected components. But $U \setminus \{(0,0)\}$ has 4 connected components, whereas $B(\mathbf{a}, \varepsilon) \setminus \{\varphi(0,0)\}$ has 1 or 2 connected components (depending on the dimension n[1]). Thus we arrive at a contradiction!

Theorem 6.1

If $f : N \to M$ is an embedding, then its image $f(N)$ is a regular submanifold of M.

Proof. Let $p \in N$. By the immersion theorem (Theorem 5.10), there are charts (U, φ) centered at $p \in N$ and (V, ψ) centered at $f(p) \in M$ such that in **any** neighborhood of $\varphi(p) = \mathbf{0} \in \mathbb{R}^n$ contained in $\varphi(U)$,

$$\left(\psi \circ f \circ \varphi^{-1}\right)\left(r^1, r^2, \ldots, r^n\right) = \left(r^1, r^2, \ldots, r^n, 0, \ldots, 0\right).$$

In fact, this is an abuse of notation as discussed in the previous chapter:

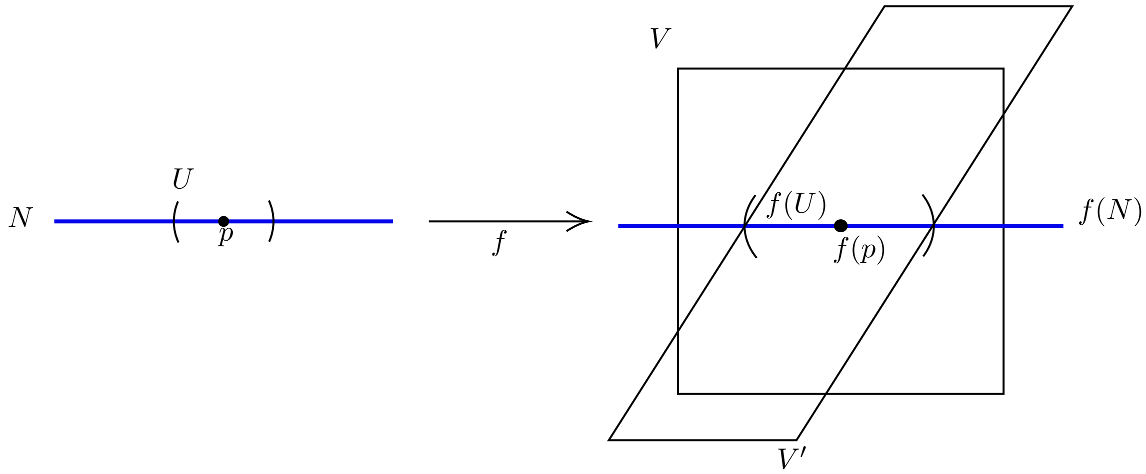
$$\psi|_{f(U)} \circ f|_U \circ \varphi^{-1}\left(r^1, r^2, \ldots, r^n\right) = \left(r^1, r^2, \ldots, r^n, 0, \ldots, 0\right).$$

So, evidently, we want $f(U) \subseteq V$. This means any element of $\psi(f(U))$ is of the form $(r^1, r^2, \ldots, r^n, 0, \ldots, 0)$. Thus, $f(U)$ is defined in V by the vanishing of the last $m-n$ coordinate functions:

$$r^{n+1} \circ \psi = y^{n+1} = 0, \ldots, r^m \circ \psi = y^m = 0.$$

Now, $V \cap f(N)$ can be larger than $f(U)$, and on $f(U)$ only the last $m-n$ coordinate functions vanish identically. It might be the case that on $V \cap f(N)$, the last $m-n$ corrdinate functions don't vanish, and hence V can't be an adapted chart about $f(p)$ relative to $f(N)$.

[1]If $n = 1$, the number of connected components is 2. For $n \geq 2$, it is 1.



We need to show that in some neighborhood of $f(p)$ in V, the set $f(N)$ is defined by the vanishing of the last $m-n$ coordinate functions.

Since $f : N \to M$ is an embedding, N is homeomorphic to $f(N)$ in the subspace topology. Hence, given an open subset U of N, $f(U)$ is open in $f(N)$. By the definition of subspace topology, there is an open set V' in M such that $f(U) = V' \cap f(N)$. Now,

$$V \cap V' \cap f(N) = V \cap f(U) = f(U) \ ,$$

and $f(U)$ is defined by the vanishing of the last $m-n$ coordinate functions on V. Therefore, $V \cap V'$ is a neighborhood of $f(p)$ in M such that $f(N)$ is defined by the vanishing of the last $m-n$ coordinate functions on $(V \cap V') \cap f(N)$. Thus,

$$\left(V \cap V', \psi|_{V \cap V'}\right) = \left(V \cap V', y^1, y^2, \ldots, y^m\right)$$

is an adapted chart containing $f(p)$ relative to $f(N)$. Since $f(p)$ is an arbitrary point of $f(N)$, this proves that $f(N)$ is a regular submanifold. ■

Theorem 6.2

If N is a regular submanifold of M, then the inclusion $i : N \hookrightarrow M$, $i(p) = p$ is an embedding.

Proof. Note that $i : N \to i(N) = N$ is the identity map, so it is a homeomorphism since both the domain and codomain space N are equipped with the same topology, *i.e.* subspace topology inherited from M. Furthermore, $N \subseteq M$ implies that $\dim N \leq \dim M$. So, in order to show that i is an embedding, it suffices to show that $i_{*,p}$ is of rank $n = \dim N$ for every $p \in N$.

Since N is a regular submanifold of M, choose an adapted chart $(V, y^1, y^2, \ldots, y^n, y^{n+1}, \ldots, y^m)$ for M about p relative to N such that $V \cap N$ is the zero set of the last $m-n$ coordinate functions $y^{n+1}, \ldots, y^m$. Hence, $(V \cap N, y^1, y^2, \ldots, y^n)$ is going to be a chart of the manifold N about p. Relative to these two charts, $i : N \hookrightarrow M$ is given by

$$\left(y^1, y^2, \ldots, y^n\right) \mapsto \left(y^1, y^2, \ldots, y^n, 0, \ldots, 0\right) .$$

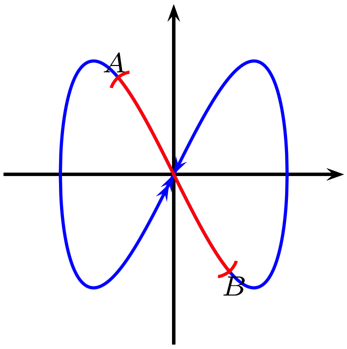
The corresponding differential $i_{*,p} : T_pN \to T_pM$ is represented by the following $m \times n$ matrix relative to the abovementioned charts:

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} I_{n\times n} \\ 0_{m\times n} \end{bmatrix} .$$

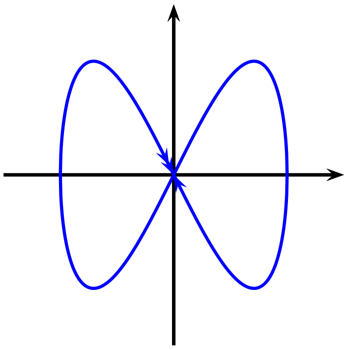
Hence, $\operatorname{rank} i_{*,p} = n$. This is true for every p. Hence, $i : N \to M$ is an immersion. We have already proved that $i : N \to i(N) = N$ is a homeomorphism. Therefore, i is an embedding. ■

§6.2 Smooth Maps into a Submanifold

We shall start with an observation. Consider the injective immersion $g : \left(-\frac{\pi}{2}, 3\frac{\pi}{2}\right) \to \mathbb{R}^2$ given by $g(t) = (\cos t, -\sin 2t)$. The image of g in $\mathbb{R}^2$ is the figure 8 given by the following image:



Let us denote by S the image of g, *i.e.* $S = \{g(t) \mid -\frac{\pi}{2} < t < 3\frac{\pi}{2}\}$. Now, consider the other injective immersion $f : \left(-\frac{\pi}{2}, 3\frac{\pi}{2}\right) \to \mathbb{R}^2$ given by $f(t) = (\cos t, \sin 2t)$, the image of which is given by the following image:



Now, one can show that $\operatorname{im} f \subseteq S$. In can be seen by proving that for every $t_1 \in \left(-\frac{\pi}{2}, 3\frac{\pi}{2}\right)$, there exists some $t_2 \in \left(-\frac{\pi}{2}, 3\frac{\pi}{2}\right)$ such that $f(t_1) = g(t_2)$. In fact, this can be achieved by choosing

$$t_2 = \begin{cases} -t_1 & \text{if } t_1 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \\ 2\pi - t_1 & \text{if } t_1 \in \left(\frac{\pi}{2}, 3\frac{\pi}{2}\right) \\ t_1 & \text{if } t_1 = \frac{\pi}{2} \end{cases} .$$

So there is an inclusion $\iota : \operatorname{im} f \to S \subseteq \mathbb{R}^2$. Denote by $\widetilde{f} = \iota \circ f$, so that $\widetilde{f} : \left(-\frac{\pi}{2}, 3\frac{\pi}{2}\right) \to S$. The map $\widetilde{f}$ is induced from the map $f : \left(-\frac{\pi}{2}, 3\frac{\pi}{2}\right) \to \mathbb{R}^2$ given by $f(t) = (\cos t, \sin 2t)$, which is a C^∞ map. However, the induced map $\widetilde{f}$ is not even continuous, let alone C^∞. Here we equipped S with the immersed submanifold topology inherited from the injective immersion g. In this topology, the segment AB is open in S. But

$$\widetilde{f}^{-1}(AB) = (\iota \circ f)^{-1}(AB) = f^{-1}\left(\iota^{-1}(AB)\right) = f^{-1}(AB)$$

is not open in $\left(-\frac{\pi}{2}, 3\frac{\pi}{2}\right)$ because it contains an isolated point as discussed earlier.

To summarize, although f is C^∞, the induced map $\widetilde{f}$ is not even continuous, let alone being C^∞. This is so because the set S containing the image of f is not a regular submanifold of $\mathbb{R}^2$. In fact, we have the following result.

Theorem 6.3

Suppose $f : N \to M$ is C^∞ and the image of f lies in a subset S of M. If S is a regular submanifold of M, then the induced map $\widetilde{f} : N \to S$ is C^∞.

Proof. Let $p \in N$. Also, let $n = \dim N$ and $m = \dim N$. By hypothesis, $\operatorname{im} f \subseteq S$. So $f(p) \in S \subseteq M$. Since S is a regular submanifold of M, there is an adopted chart $(V, \psi) = (V, y^1, y^2, \ldots, y^m)$ of M about $f(p)$ such that $S \cap V$ is the zero set of the last $m - s$ coordinate functions $y^{s+1}, \ldots, y^m$. If one denotes $\psi_S = (y^1, y^2, \ldots, y^s)$, then $(S \cap V, \psi_S)$ is a chart for the regular submanifold S. Now, take a chart (U, φ) about p. By choosing U sufficiently small, we can assume $f(U) \subseteq V$. This is possible since f is continuous. Since $\operatorname{im} f \subseteq S$, $f(U) \subseteq S \cap V$. Then, with respect to the charts $(S \cap V, \psi_S)$ about $f(p)$ and (U, φ) about p,

$$\psi_S \circ \widetilde{f} \circ \varphi^{-1} : \varphi(U) \subseteq \mathbb{R}^n \to \mathbb{R}^s$$

is given by

$$\left(\psi_S \circ \widetilde{f} \circ \varphi^{-1}\right)(a) = \left(y^1\left(\widetilde{f}\left(\varphi^{-1}(a)\right)\right), y^2\left(\widetilde{f}\left(\varphi^{-1}(a)\right)\right), \ldots, y^s\left(\widetilde{f}\left(\varphi^{-1}(a)\right)\right)\right)$$

for $a \in \varphi(U)$. Now, $\widetilde{f}(\varphi^{-1}(a)) = f(\varphi^{-1}(a))$. So

$$\left(\psi_S \circ \widetilde{f} \circ \varphi^{-1}\right)(a) = \left(y^1\left(f\left(\varphi^{-1}(a)\right)\right), y^2\left(f\left(\varphi^{-1}(a)\right)\right), \ldots, y^s\left(f\left(\varphi^{-1}(a)\right)\right)\right).$$

Since $f : N \to M$ is smooth, relative to the charts (U, φ) of N and (V, ψ) of M, $\psi \circ f \circ \varphi^{-1} : \varphi(U) \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is smooth. The components of $\psi \circ f \circ \varphi^{-1}$ are

$$r^i \circ \left(\psi \circ f \circ \varphi^{-1}\right) = y^i \circ f \circ \varphi^{-1},$$

and they are smooth. We've seen that the components of $\psi_S \circ \widetilde{f} \circ \varphi^{-1}$ are $y^i \circ f \circ \varphi^{-1}$ for $i = 1, 2, \ldots, s$. Therefore, $\psi_S \circ \widetilde{f} \circ \varphi^{-1}$ is smooth. This proves that $\widetilde{f}$ is smooth at p. This is true for every $p \in N$, so $\widetilde{f}$ is smooth. ■

Example 6.4 (Multiplication map of $\mathrm{SL}(n, \mathbb{R})$)**.** The multiplication map $\mu : \mathrm{GL}(n, \mathbb{R}) \times \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$ is given by

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

where a_{ik}, b_{kj} are the matrix entries of $A, B \in \mathrm{GL}(n, \mathbb{R})$, respectively. Since $\mathrm{GL}(n, \mathbb{R})$ is an open subset of $\mathbb{R}^{n^2}$, it is an n^2-dimensional manifold with a single chart $\left(\mathrm{GL}(n, \mathbb{R}), \mathbb{1}_{\mathrm{GL}(n,\mathbb{R})}\right)$. Similarly, $\mathrm{GL}(n, \mathbb{R}) \times \mathrm{GL}(n, \mathbb{R})$ is an open subset of $\mathbb{R}^{n^2+n^2} = \mathbb{R}^{2n^2}$, so it is a manifold with a single chart $\left(\mathrm{GL}(n, \mathbb{R}) \times \mathrm{GL}(n, \mathbb{R}), \mathbb{1}_{\mathrm{GL}(n,\mathbb{R}) \times \mathrm{GL}(n,\mathbb{R})}\right)$. Now, the map μ is smooth because each component of $\mu(AB)$ is a polynomial in the entries of the matrices A and B.

Now, since $\mathrm{SL}(n, \mathbb{R})$ is a regular submanifold of $\mathrm{GL}(n, \mathbb{R})$, $\mathrm{SL}(n, \mathbb{R}) \times \mathrm{SL}(n, \mathbb{R})$ is a regular submanifold of $\mathrm{GL}(n, \mathbb{R}) \times \mathrm{GL}(n, \mathbb{R})$. Hence, the inclusion map

$$i : \mathrm{SL}(n, \mathbb{R}) \times \mathrm{SL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R}) \times \mathrm{GL}(n, \mathbb{R})$$

is an embedding and hence C^∞ by Theorem 6.2. Hence,

$$\mu \circ i : \mathrm{SL}(n, \mathbb{R}) \times \mathrm{SL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$$

is also C^∞ as a composition of smooth maps. Since $\mathrm{SL}(n, \mathbb{R})$ is a subgroup of $\mathrm{GL}(n, \mathbb{R})$, the product of two matrices in $\mathrm{SL}(n, \mathbb{R})$ is also in $\mathrm{SL}(n, \mathbb{R})$. Hence, the image of $\mu \circ i$ lies in $\mathrm{SL}(n, \mathbb{R})$, which is a regular submanifold of $\mathrm{GL}(n, \mathbb{R})$. Therefore, the induced map

$$\widetilde{\mu} : \mathrm{SL}(n, \mathbb{R}) \times \mathrm{SL}(n, \mathbb{R}) \to \mathrm{SL}(n, \mathbb{R})$$

is C^∞ by Theorem 6.3.

7 The Tangent Bundle

§7.1 The Topology of the Tangent Bundle

Definition 7.1 (Tangent Bundle)**.** Let M be a smooth manifold. The **tangent bundle** TM of M is the *disjoint union* of all the tangent spaces

$$TM := \bigsqcup_{p\in M} T_pM\,.$$

In general, if $\{A_i\}_{i\in I}$ is a collection of subsets of a set S, then their *disjoint union* is defined as

$$\bigsqcup_{i\in I} A_i = \bigcup_{i\in I} \{p\} \times A_i\,.$$

So, a generic element of TM is of the form (p, X_p) where $X_p \in T_pM$. There is a natural map $\pi : TM \to M$ given by

$$\pi\left(p, X_p\right) = p\,.$$

TM as a set consists of ordered pairs (p, X_p) such that $p \in M$ and $X_p \in T_pM$.

Remark 7.1. T_pM consists of all the point-derivations at p. A point-derivation at p is certainly not a point-derivation at q, for $p \neq q$. Therefore, T_pM and T_qM are disjoint. Therefore, the union $\bigcup\limits_{p\in M} T_pM$ is (up to notation) the same as the disjoint union $\bigsqcup\limits_{p\in M} T_pM$, since for distinct points p and q in M, the tangent spaces T_pM and T_qM are already disjoint. That's why we sometimes write $TM = \bigcup\limits_{p\in M} T_pM$.

We now give the set TM a topology. Let $(U, \varphi) = (U, x^1, x^2, \ldots, x^n)$ be a chart on M. Since U is an open subset of M, by Remark 4.1, $T_pU = T_pM$. Let

$$TU = \bigsqcup_{p\in U} T_pU = \bigsqcup_{p\in U} T_pM\,.$$

At $p \in U$, let $X_p \in T_pM$ so that

$$X_p = \sum_{i=1}^{n} c^i\left(X_p\right) \left.\frac{\partial}{\partial x^i}\right|_p\,.$$

Now, we define the map $\widetilde{\varphi} : TU \to \varphi\left(U\right) \times \mathbb{R}^n$ by

$$\left(p, X_p\right) \mapsto \left(x^1\left(p\right), x^2\left(p\right), \ldots, x^n\left(p\right), c^1\left(X_p\right), c^2\left(X_p\right), \ldots, c^n\left(X_p\right)\right)\,.$$

It is easy to see that $\widetilde{\varphi}$ has an inverse given by

$$\left(\varphi\left(p\right), c^1, c^2, \ldots, c^n\right) \mapsto \left(p, \sum_{i=1}^{n} c^i \left.\frac{\partial}{\partial x^i}\right|_p\right)\,.$$

$\widetilde{\varphi}$ given as above is a bijection. We use $\widetilde{\varphi}$ to transfer the topology of $\varphi\left(U\right) \times \mathbb{R}^n$ to TU: a set $A \subseteq TU$ is open if and only if $\widetilde{\varphi}\left(A\right)$ is open in $\varphi\left(U\right) \times \mathbb{R}^n$, where $\varphi\left(U\right) \times \mathbb{R}^n$ is given its standard topology as an open subset of $\mathbb{R}^{2n}$. With this topology induced from $\widetilde{\varphi}$, TU and $\varphi\left(U\right) \times \mathbb{R}^n$ are homeomorphic to each other.

Lemma 7.1

Let $V \subseteq U$ be open in U. Then the topology on TV as a subspace of TU is the same as the one induced by the bijection $\widetilde{\varphi}|_{TV} : TV \to \varphi(V) \times \mathbb{R}^n \subseteq \varphi(U) \times \mathbb{R}^n$.

Proof. φ is a homeomorphism, and hence an open map. Therefore, $\varphi(V)$ is open in $\varphi(U)$. As a result, $\varphi(V) \times \mathbb{R}^n \subseteq \varphi(U) \times \mathbb{R}^n$ is open in the subspace topology inherited from $\mathbb{R}^{2n}$.

Now, consider the subspace topology on TV inherited from TU. In this topology, let $A \subseteq TV$ be open. Then there exists $B \subseteq TU$ open such that $A = B \cap TV$.

$$\widetilde{\varphi}(A) = \widetilde{\varphi}(B \cap TV) = \widetilde{\varphi}(B) \cap \widetilde{\varphi}(TV) = \widetilde{\varphi}(B) \cap (\varphi(V) \times \mathbb{R}^n) .$$

Since B is open in TU, $\widetilde{\varphi}(B)$ is open in $\varphi(U) \times \mathbb{R}^n$. Therefore, $\widetilde{\varphi}|_{TV}(A) = \widetilde{\varphi}(B) \cap (\varphi(V) \times \mathbb{R}^n)$ is open in $\varphi(V) \times \mathbb{R}^n$. Therefore, A is open in TV in the topology induced by the bijection $\widetilde{\varphi}|_{TV} : TV \to \varphi(V) \times \mathbb{R}^n$.

Now, let A be open in TV in the topology induced by the bijection $\widetilde{\varphi}|_{TV}$. Then $\widetilde{\varphi}|_{TV}(A) = \widetilde{\varphi}(A)$ is open in $\varphi(V) \times \mathbb{R}^n$. We have shown that $\varphi(V) \times \mathbb{R}^n$ is open in $\varphi(U) \times \mathbb{R}^n$. Therefore, $\widetilde{\varphi}(A)$ is open in $\varphi(U) \times \mathbb{R}^n$. This means that A is open in TU. Then $A = A \cap TV$ as $A \subseteq TV$. So A is open in TV in the subspace topology inherited from TU.

Therefore, one can conclude that the subspace topology on TV inherited from TU is the same as the one induced by the bijection $\widetilde{\varphi}|_{TV} : TV \to \varphi(V) \times \mathbb{R}^n \subseteq \varphi(U) \times \mathbb{R}^n$. ■

Now, let $\mathcal{B}$ be the collection of all open subsets of TU_α as U_α runs over all coordinate open sets in M. In other words, if $\mathscr{A} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ is the maximal atlas of M,

$$\begin{aligned} \mathcal{B} &= \bigcup_{\alpha \in I} \{A \mid A \subseteq TU_\alpha \text{ is open in } TU_\alpha\} \\ &= \{A \mid A \subseteq TU_\alpha \text{ is open in } TU_\alpha\,,\ \alpha \in I\} \end{aligned}$$

Now we shall show that $\mathcal{B}$ forms a basis for topology.

Lemma 7.2 (i) For any manifold M, the set M is the union of all $A \in \mathcal{B}$.

(ii) Let U and V be coordinate open sets in a manifold M. If A is open in TU and B is open in TV, then $A \cap B$ is open in $T(U \cap V)$.

Proof. (i) Let $\mathscr{A} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ be the maximal atlas for M. Then for every $\alpha \in I$, $T_pU_\alpha = T_pM$ since U_α is open in M for each α. Now,

$$\begin{aligned} TM &= \bigsqcup_{p \in M} T_pM = \bigcup_{p \in M} \{p\} \times T_pM = \bigcup_{p \in M} \{p\} \times T_pU_\alpha \\ &= \bigcup_{\alpha \in I} \left(\bigcup_{p \in M} \{p\} \times T_pU_\alpha \right) = \bigcup_{\alpha \in I} TU_\alpha \end{aligned}$$

Now, TU_α is open in itself. So $TU_\alpha \in \mathcal{B}$. Therefore,

$$TM = \bigcup_{\alpha \in I} TU_\alpha \subseteq \bigcup_{A \in \mathcal{B}} A .$$

Also, each $A \in \mathcal{B}$ is contained in some TU_α. Therefore, the union of all A's is contained in the union of all TU_α's. In other words,

$$\bigcup_{A \in \mathcal{B}} A \subseteq \bigcup_{\alpha \in I} TU_\alpha = TM .$$

Therefore, we can conclude that $TM = \bigcup_{\alpha \in I} TU_\alpha = \bigcup_{A \in \mathcal{B}} A$.

(ii) Note that since $U \cap V \subseteq U$, $T(U \cap V)$ is endowed with the subspace topology inherited from TU. Since $A \subseteq TU$ is open, $A \cap T(U \cap V)$ is open in $T(U \cap V)$ in the subspace topology. Similarly, for $B \subseteq TV$ open, $B \cap T(U \cap V)$ is open in $T(U \cap V)$ in the subspace topology.

Now, we want to show that $TU \cap TV = T(U \cap V)$.

$$\begin{aligned}
TU \cap TV &= \left(\bigsqcup_{p \in U} T_p U \right) \cap \left(\bigsqcup_{q \in V} T_q V \right) = \left(\bigcup_{p \in U} \{p\} \times T_p U \right) \cap \left(\bigcup_{q \in V} \{q\} \times T_q V \right) \\
&= \bigcup_{p \in U \cap V} \left((\{p\} \times T_p U) \cap (\{p\} \times T_p V) \right) \\
&= \bigcup_{p \in U \cap V} \{p\} \times (T_p U \cap T_p V) \\
&= \bigcup_{p \in U \cap V} \{p\} \times T_p(U \cap V) = T(U \cap V)
\end{aligned}$$

Since $A \subseteq TU$ and $B \subseteq TV$, $A \cap B \subseteq TU \cap TV = T(U \cap V)$. Hence,

$$A \cap B = A \cap B \cap T(U \cap V) = (A \cap T(U \cap V)) \cap (B \cap T(U \cap V)) .$$

We have previously shown that both $A \cap T(U \cap V)$ and $B \cap T(U \cap V)$ are open in $T(U \cap V)$. Therefore, their intersection $A \cap B$ is also open in $T(U \cap V)$.

■

Lemma 7.2 implies that $\mathcal{B}$ is a basis for some topology on TM. This is because of Proposition A.10. Now, we give TM the topology generated by the basis $\mathcal{B}$. We declare $A \subseteq TM$ to be open if and only if there exists $\{B_\lambda\} \subseteq \mathcal{B}$ such that

$$A = \bigcup_{\lambda} B_\lambda .$$

Lemma 7.3

A manifold M has a countable basis consisting of coordinate open sets.

Proof. Let $\mathscr{A} = \{(U_\alpha, \varphi_\alpha)\}_\alpha$ be the maximal atlas on M. M is second countable, so there exists a countable basis. Let $\mathfrak{B} = \{B_i\}_i$ be a countable basis for M.

For each U_α and $p \in U_\alpha$, choose $B_{p,\alpha} \in \mathfrak{B}$ such that

$$p \in B_{p,\alpha} \subseteq U_\alpha .$$

Such $B_{p,\alpha}$ exists because $\mathfrak{B}$ is a basis. Then $\{B_{p,\alpha}\}$ is a subcollection of $\mathfrak{B}$, and hence it is countable. Also for any open set $U \subseteq M$ and $p \in U$, let U_β be a coordinate open set about p. Then any open subset of U_β is also a coordinate open set. If we take $U_\alpha = U \cap U_\beta$, U_α is a coordinate open set. So

$$p \in B_{p,\alpha} \subseteq U_\alpha \subseteq U .$$

Therefore, $\{B_{p,\alpha}\}$ is a countable basis. Now, since any open subset of a coordinate open set is again a coordinate open set, and $B_{p,\alpha}$ is an open subset of U_α, we can conclude that $\{B_{p,\alpha}\}$ is a countable basis consisting of coordinate open sets. ■

Proposition 7.4

The tangent bundle TM is second countable.

Proof. Let $\{U_i\}_i$ be a countable basis for M consisting of coordinate open sets. Let φ_i be the coordinate map on U_i. We have shown that TU_i is homeomorphic to $\varphi_i(U_i) \times \mathbb{R}^n$, which is an open subset of $\mathbb{R}^n$.

Hence, $\varphi_i(U_i) \times \mathbb{R}^n$ is secoond countable. Now, homeomorphism preserves second countability, so TU_i is also second countable.

For each i, choose a countable basis $\{B_{i,j}\}_j$ for TU_i. Then $\{B_{i,j}\}_{i,j}$ is also countable. Now we need to show that $\{B_{i,j}\}_{i,j}$ is a basis for TM. Let $A \subseteq TM$ be open and take $(p, X_p) \in A$. We need to show the existence of $B_{i,j}$ such that $(p, X_p) \in B_{i,j} \subseteq A$.

Since $\{U_i\}$ is a basis for M, $p \in U_i$ for some i. Then

$$(p, X_p) \in \{p\} \times T_pM = \{p\} \times T_pU_i \subseteq \bigcup_{p \in U_i} \{p\} \times T_pU_i = TU_i\,.$$

Therefore, $(p, X_p) \in A \cap TU_i$. A is open in TM, and TU_i is open in TM. Therefore, $A \cap TU_i$ is also open in TM. Let $\widetilde{A} = A \cap TU_i$. We want to show that $\widetilde{A}$ is open in TU_i.

Since $\widetilde{A}$ is open in TM, it can be expressed as

$$\widetilde{A} = \bigcup_{\alpha \in J \subseteq I} \widetilde{A}_\alpha\,,$$

where $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ is the maximal atlas of M, and $\widetilde{A}_\alpha$ is open in TU_α.

$$\widetilde{A}_\alpha \subseteq \widetilde{A} \subseteq TU_i\,.$$

$\widetilde{A}_\alpha$ is open in TU_α, TU_i is open in TU_i. Therefore, $\widetilde{A}_\alpha \cap TU_i = \widetilde{A}_\alpha$ is open in $T(U_i \cap U_\alpha)$. Since $U_i \cap U_\alpha$ is open in U_i, $T(U_i \cap U_\alpha)$ is open in TU_i. Therefore, $\widetilde{A}_\alpha$ is open in TU_i. This is true for each $\alpha \in J$. Hence, $\widetilde{A}$ is open in TU_i.

Now, $\widetilde{A}$ is open in TU_i and $(p, X_p) \in \widetilde{A} = A \cap TU_i$. Since $\{B_{i,j}\}_j$ is a basis for TU_i, there exists some $B_{i,j}$ such that

$$(p, X_p) \in B_{i,j} \subseteq \widetilde{A} = A \cap TU_i \subseteq A \implies (p, X_p) \in B_{i,j} \subseteq A\,.$$

Therefore, the countable collection $\{B_{i,j}\}_{i,j}$ is a basis for TM. ■

Proposition 7.5

TM is Hausdorff.

Proof. Let (p, X_p) and (q, Y_q) be distinct points of TM.

Case 1: $p \neq q$.

Since M ia Hausdorff, there exists disjoint open subsets U_1 and V_1 of M that contain p and q, respectively. Furthermore, there exists coordinate open sets U_2 and V_2 around p and q, respectively. Then $U = U_1 \cap U_2$ and $V = V_1 \cap V_2$ are disjoint coordinate open sets that contain p and q, respectively.

$$(p, X_p) \in \{p\} \times T_pM = \{p\} \times T_pU \subseteq \bigcup_{p \in U} \{p\} \times T_pU = TU\,.$$

Similarly, $(q, Y_q) \in TV$. We have shown that $TU \cap TV = T(U \cap V)$. Since $U \cap V = \varnothing$, $TU \cap TV = \varnothing$. Therefore, TU and TV are the disjoint open subsets of TM that contain (p, X_p) and (q, Y_q), respectively.

Case 2: $p = q$.

Let (U, φ) be a coordinate chart containing p. Then (p, X_p) and (p, Y_p) are distinct points on TU, which is homeomorphic to $\varphi(U) \times \mathbb{R}^n$. $\varphi(U) \times \mathbb{R}^n$ is Hausdorff, hence so is TU. Therefore, (p, X_p) and (p, Y_p) can be separated by open subsets of TU, which are also open subset of TM.

Therefore, TM is Hausdoeff. ■

The Manifold Structure on TM

Proposition 7.6

Let $\{U_\alpha, \varphi_\alpha\}_{\alpha\in I}$ be an atlas for M. Then $\{TU_\alpha, \widetilde{\varphi}_\alpha\}_{\alpha\in I}$ is an atlas for TM.

Let's begin with an observation. Let $(U, x^1, x^2, \ldots, x^n)$ and $(V, y^1, y^2, \ldots, y^n)$ be two charts on M. Then for any $p \in U \cap V$, there are two bases for T_pM:

$$\left\{ \frac{\partial}{\partial x^j}\bigg|_p \right\}_{j=1}^n \quad \text{and} \quad \left\{ \frac{\partial}{\partial y^i}\bigg|_p \right\}_{i=1}^n .$$

So $X_p \in T_pM$ has two basis expansions

$$X_p = \sum_{j=1}^n a^j \frac{\partial}{\partial x^j}\bigg|_p = \sum_{i=1}^n b^i \frac{\partial}{\partial y^i}\bigg|_p .$$

By applying y^k on both sides, one obtains

$$b^k = \sum_{j=1}^n a^j \frac{\partial y^k}{\partial x^j}(p) .$$

Proof of Proposition 7.6. We have already shown that $\widetilde{\varphi}_\alpha : TU_\alpha \to \varphi_\alpha(U_\alpha) \times \mathbb{R}^n \subseteq \mathbb{R}^{2n}$ is a homeomorphism from an open subset of TM to an open subset of the Euclidean space $\mathbb{R}^{2n}$. Also, we have shown that

$$TM = \bigcup_{\alpha\in I} TU_\alpha .$$

So, it remains to check that on $TU_\alpha \cap TU_\beta$, $\widetilde{\varphi}_\alpha$ and $\widetilde{\varphi}_\beta$ are C^∞-compatible.

Let $U_{\alpha\beta} = U_\alpha \cap U_\beta$. Then $\widetilde{\varphi}_\beta \circ \widetilde{\varphi}_\alpha^{-1} : \varphi_\alpha(U_{\alpha\beta}) \times \mathbb{R}^n \to \varphi_\beta(U_{\alpha\beta}) \times \mathbb{R}^n$ is given by

$$\left(\varphi_\alpha(p), a^1, \ldots, a^n\right) \xmapsto{\widetilde{\varphi}_\alpha^{-1}} \left(\varphi_\alpha^{-1}(\varphi_\alpha(p)), \sum_{j=1}^n a^j \frac{\partial}{\partial x^j}\bigg|_p\right) \xmapsto{\widetilde{\varphi}_\beta} \left(\left(\varphi_\beta \circ \varphi_\alpha^{-1}\right)\varphi_\alpha(p), b^1, \ldots, b^n\right) ,$$

where $b^k = \sum\limits_{j=1}^n a^j \frac{\partial y^k}{\partial x^j}(p)$. Since $\{U_\alpha, \varphi_\alpha\}_\alpha$ is an atlas for M, $\varphi_\beta \circ \varphi_\alpha^{-1}$ is smooth. Now,

$$\begin{aligned} b^i &= \sum_{j=1}^n a^j \frac{\partial y^i}{\partial x^j}(p) = \sum_{j=1}^n a^j \frac{\partial\left(y^i \circ \varphi_\alpha^{-1}\right)}{\partial r^j}(\varphi_\alpha(p)) \\ &= \sum_{j=1}^n a^j \frac{\partial\left(r^i \circ \varphi_\beta \circ \varphi_\alpha^{-1}\right)}{\partial r^j}(\varphi_\alpha(p)) \\ &= \sum_{j=1}^n a^j \frac{\partial\left(\varphi_\beta \circ \varphi_\alpha^{-1}\right)^i}{\partial r^j}(\varphi_\alpha(p)) \end{aligned}$$

$\frac{\partial\left(\varphi_\beta \circ \varphi_\alpha^{-1}\right)^i}{\partial r^j}$ is C^∞ as $\varphi_\beta \circ \varphi_\alpha^{-1}$ is C^∞. Therefore, the map $\widetilde{\varphi}_\beta \circ \widetilde{\varphi}_\alpha^{-1}$ given by

$$\left(\varphi_\alpha(p), a^1, \ldots, a^n\right) \mapsto \left(\left(\varphi_\beta \circ \varphi_\alpha^{-1}\right)\varphi_\alpha(p), \sum_{j=1}^n a^j \frac{\partial y^1}{\partial x^j}(p), \sum_{j=1}^n a^j \frac{\partial y^2}{\partial x^j}(p), \ldots, \sum_{j=1}^n a^j \frac{\partial y^n}{\partial x^j}(p)\right)$$

is C^∞. ■

§7.2 Vector Bundle

On the tangent bundle TM is a smooth manifold M, there is a natural projection map $\pi : TM \to M$ with $\pi(p, X_p) = p$. This makes the tangent bundle into a C^∞ vector bundle that we will define now.

> **Definition 7.2.** Given any map $\pi : E \to M$ between two smooth manifolds, we call the preimage $\pi^{-1}(p) := \pi^{-1}(\{p\})$ of a point $p \in M$ the **fibre** at p. The fibre at p is often written E_p, i.e. $E_p = \pi^{-1}(p)$.

> **Definition 7.3.** For any two maps $\pi : E \to M$ and $\pi' : E' \to M$ with the same target space M, a map $\phi : E \to E'$ is said to be **fibre-preserving** if $\phi(E_p) \subseteq E'_p$ for every $p \in M$.

Exercise 7.1. Given two maps $\pi : E \to M$ and $\pi' : E' \to M$, a map $\phi : E \to E'$ is fibre-preserving if and only if the following diagram commutes:

$$\begin{tikzcd} E \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & E' \arrow[dl, "\pi'"] \\ & M & \end{tikzcd}$$

In other words, $\pi = \pi' \circ \phi$.

Solution. **($\Rightarrow$):** Let $\phi : E \to E'$ be fibre-preserving. Then $\phi(E_p) \subseteq E'_p$ for every $p \in M$. Given $x \in E$, $x \in E_p$ for some $p \in M$.

$$x \in E_p = \pi^{-1}(p) \implies \pi(x) = p\,.$$

Since $\phi(E_p) \subseteq E'_p$, $\phi(x) \in E'_p$. As a result, $\pi'(\phi(x)) = p$. So we obtain

$$\pi(x) = p = \pi'(\phi(x)) \implies \pi = \pi' \circ \phi\,.$$

($\Leftarrow$): Now, let $\pi = \pi' \circ \phi$. Take $x \in E_p$. Then $\pi(x) = p$. As a result,

$$\pi'(\phi(x)) = p \implies \phi(x) \in E'_p\,.$$

This is true for every $x \in E_p$. Therefore, $\phi(E_p) \subseteq E'_p$. ■

> **Definition 7.4.** A surjective smooth map $\pi : E \to M$ of manifolds is said to be **locally trivial of rank** r if
>
> (i) Each fibre $\pi^{-1}(p)$ has the structure if a vector space of dimension r.
>
> (ii) For each $p \in M$, there are open neighborhood U of p and a fibre-preversing diffeomorphism
>
> $$\phi : \pi^{-1}(U) \to U \times \mathbb{R}^r\,,$$
>
> such that for every $q \in U$, the restriction
>
> $$\phi\big|_{\pi^{-1}(q)} : \pi^{-1}(q) \to \{q\} \times \mathbb{R}^r$$
>
> is a vector space isomorphism. Such an open set U is called a **trivializing open set** for E, and ϕ is called a **trivialization** of E over U.

The collection $\{(U_\alpha, \phi_\alpha)\}_\alpha$ with

$$\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^r$$

being the diffeomorphism discussed above (called the trivialization of E over U_α) and $\{U_\alpha\}_\alpha$ an open cover of M, is called a **local trivilization** for E and $\{U_\alpha\}_\alpha$ is called a **trivializing open cover** of M for E.

> **Definition 7.5.** A C^∞ **vector bundle of rank** r is a triple (E, M, π) consisting of manifolds E and M and a surjective smooth map $\pi : E \to M$ that is locally trivial of rank r. The manifold E is called the total space of the vector bundle and M the base space.

Abuse of Notation. We sometimes say that E is a vector bundle over M. We also call the surjective smooth map $\pi : E \to M$ the vector bundle.

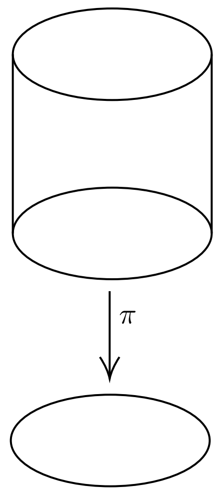
The tangent bundle of a manifold is the triple (TM, M, π) where TM is the total space of the tangent bundle.

Example 7.1 (Product Bundle)**.** Given a manifold M, let $\pi : M \times \mathbb{R}^r \to M$ be the projection onto the first factor. Then $M \times \mathbb{R}^r$ is a vector bundle of rank r, called the product bundle of rank r over M. The vector space structure on the fibre $\pi^{-1}(p) = \{(p, \mathbf{v}) \mid \mathbf{v} \in \mathbb{R}^r\}$ is the obvious one:

$$(p, \mathbf{u}) + (p, \mathbf{v}) = (p, \mathbf{u} + \mathbf{v}) \ , \quad \text{and } \alpha (p, \mathbf{v}) = (p, \alpha\mathbf{v}) \ \text{ for } \alpha \in \mathbb{R} \, .$$

A local trivialization on $M \times \mathbb{R}^r$ is given by the identity map $\mathbb{1}_{M \times \mathbb{R}^r}$.

The infinite cylinder $S^1 \times \mathbb{R}$ is the product bundle of rank 1 over the unit circle S^1.



Remark 7.2. In general, a generic element of the vector bundle E belongs in some E_p where p is a point in the base space M. Since E_p has a vector space structure, we can say that a generic element of E is $\mathbf{e}_p$, where $\mathbf{e}_p$ is a vector in E_p for some $p \in M$. However, in the case of tangent bundle, we saw that a generic element of TM is an ordered pair (p, X_p) where $X_p \in T_pM$. This is because in this case $\pi^{-1}(p) = E_p$ is not the same as T_pM. Rather, we have $E_p = \{p\} \times T_pM$.

Let $\pi : E \to M$ be a C^∞ vector bundle. Suppose $\left(W, \widetilde{\psi}\right) = (W, x^1, x^2, \ldots, x^n)$ is a chart on M and

$$\widetilde{\phi} : \pi^{-1}(V) \to V \times \mathbb{R}^r$$

is a trivialization of E over V, with $V \cap W \neq 0$. Sometimes we write $E\big|_V = \pi^{-1}(V)$. Then $U = V \cap W$ is a coordinate open set, with the chart $\left(U, \widetilde{\psi}\big|_U\right)$. Furthermore,

$$\widetilde{\phi}\big|_U : \pi^{-1}(U) \to U \times \mathbb{R}^r$$

is a trivialization of E over U. We write $\widetilde{\psi}\big|_U = \psi$, and $\widetilde{\phi}\big|_U = \phi$. Then (U, ψ) is a chart in the maximal atlas of M, and

$$\phi : \pi^{-1}(U) \to U \times \mathbb{R}^r$$

is a trivialization of E over U.

Since a generic element of $\pi^{-1}(U) \subseteq E$ is a vector $\mathbf{e}_p$ for some $p \in U \subseteq M$, the map ϕ is given by

$$\phi(\mathbf{e}_p) = \left(p, c^1(\mathbf{e}_p), c^2(\mathbf{e}_p), \ldots, c^r(\mathbf{e}_p)\right)$$

where $\mathbf{e}_p = \sum_{i=1}^{r} c^i(\mathbf{e}_p)\,\widehat{e}_i$ in terms of an ordered basis $\{\widehat{e}_i\}_{i=1}^r$ for E_p. Now, consider the map

$$(\psi \times \mathbb{1}_{\mathbb{R}^r}) \circ \phi : \pi^{-1}(U) \to \psi(U) \times \mathbb{R}^r \subseteq \mathbb{R}^{n+r}\,.$$

This map is given by

$$\mathbf{e}_p \mapsto \left(x^1(p), x^2(p), \ldots, x^n(p), c^1(\mathbf{e}_p), c^2(\mathbf{e}_p), \ldots, c^r(\mathbf{e}_p)\right)\,.$$

$\psi \times \mathbb{1}_{\mathbb{R}^r}$ is a diffeomorphism since both ψ and $\mathbb{1}_{\mathbb{R}^r}$ are diffoemorphisms. Also, ϕ is a diffeomorphism. Therefore, $(\psi \times \mathbb{1}_{\mathbb{R}^r}) \circ \phi$ is a diffeomorphism from $E|_U$ onto its image. Hence, $\left(E|_U, (\psi \times \mathbb{1}_{\mathbb{R}^r}) \circ \phi\right)$ is a chart on the total space E of the C^∞ vector bundle $\pi : E \to M$. We call $x^1, x^2, \ldots, x^n$ the **base coordinates** and $c^1, c^2, \ldots, c^r$ the **fibre coordinates** of the chart $\left(E|_U, (\psi \times \mathbb{1}_{\mathbb{R}^r}) \circ \phi\right)$ on E. Note that the fibre coordinates c^i depend on the trivialization ϕ of the bundle. c^i's don't depend on the coordinate map ψ on the base U.

Definition 7.6 (Bundle Map). Let $\pi_E : E \to M$ and $\pi_F : F \to N$ be two vector bundles. A **bundle map** from E to F is a pair of maps $\left(f, \widetilde{f}\right)$, $f : M \to N$ and $\widetilde{f} : E \to F$ such that

(i) The following diagram commutes

$$\begin{tikzcd} E \arrow[r, "\widetilde{f}"] \arrow[d, "\pi_E"'] & F \arrow[d, "\pi_F"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

In other words, $\pi_F \circ \widetilde{f} = f \circ \pi_E$.

(ii) $\widetilde{f} : E \to F$ is linear on each fibre, i.e. for every $p \in M$, $\widetilde{f}|_{E_p} : E_p \to F_{f(p)}$ is a linear map of vector spaces.

The collection of all vector bundles (as objects) together with bundle maps between them (as morphisms) forms a category[1].

Example 7.2. A smooth map $f : N \to M$ of manifolds induces a bundle map $\left(f, \widetilde{f}\right)$, where $\widetilde{f} : TN \to TM$ is given by

$$\widetilde{f}(p, X_p) = (f(p), f_{*,p}(X_p)) \in \{f(p)\} \times T_{f(p)}M \subseteq TM\,.$$

This gives rise to a *covariant functor* from the category of smooth manifolds and smooth maps to the category of vector bundles and bundle maps. To each manifold M, we associate its tangent bundle TM, and to each smooth map $f : N \to M$, we aasociate the bundle map $Tf = \left(f : N \to M, \widetilde{f} : TN \to TM\right)$.

If E and F are two C^∞ vector bundles over the same manifold M, then a bundle map $\left(\mathbb{1}_M, \widetilde{f}\right)$ from E to F over M is a bundle map in which the base map is the identity $\mathbb{1}_M$. Then we have the following commutative diagram:

$$\begin{tikzcd} E \arrow[r, "\widetilde{f}"] \arrow[d, "\pi_E"'] & F \arrow[d, "\pi_F"] \\ M \arrow[r, "\mathbb{1}_M"'] & M \end{tikzcd}$$

[1]See Appendix **??** if you're not familiar with the definition of categories and functors.

The commutativity of this diagram implies that $\widetilde{f}$ is a fibre-preserving map (Exercise 7.1).

For a fixed manifold M, we can also consider the category of all C^∞ vector bundles over M and C^∞ bundle maps (of the form $\left(\mathbb{1}_M, \widetilde{f}\right)$) over M. In this category, it makes sense to speak of an **isomorphism** of vector bundles over M. In this case, the linear map $\widetilde{f} : E_p \to F_{\mathbb{1}(p)} = F_p$ is an isomorphism of vector spaces. Any vector bundle over M isomorphic over M to the product bundle $M \times \mathbb{R}^r$ is called a trivial bundle.

§7.3 Smooth Sections

Definition 7.7 (Section)**.** A **section** of a vector bundle $\pi : E \to M$ is a map $s : M \to M$ such that $\pi \circ s = \mathbb{1}_M$, the identity map on M. We say that a section $s : M \to E$ is smooth if it is smooth as a map from M to E. A smooth section of E over U is a smooth map $s : U \to E\big|_U$ such that $\pi\big|_{E|_U} \circ s = \mathbb{1}_U$.

Definition 7.8 (Vector Field)**.** A **vector field** X on a manifold M is a map that assigns a tangent vector $X_p \in T_pM$ to each point $p \in M$. In terms of tangent bundle, a vector field on M is simply a section $X : M \to TM$ of the tangent bundle $\pi : TM \to M$. The vector field is smooth if X is a smooth map between manifolds.

Remark 7.3. When we say $X : M \to TM$ is a section of the tangent bundle $\pi : TM \to M$, we consider TM to be the union of T_pM across all $p \in M$, not disjoint union. Since there is a one-to-one correspondence between $\bigsqcup_{p\in M} T_pM$ and $\bigcup_{p\in M} T_pM$, we give $\bigcup_{p\in M} T_pM$ the topology inherited from $\bigsqcup_{p\in M} T_pM$ via the one-to-one correspondence. In other words, if $i : \bigcup_{p\in M} T_pM \to \bigsqcup_{p\in M} T_pM$ is the bijection given by $i(\mathbf{v}_p) = (p, \mathbf{v}_p)$, then a set $X \subseteq \bigcup_{p\in M} T_pM$ is open if and only if $i(X) \subseteq \bigsqcup_{p\in M} T_pM$ is open.

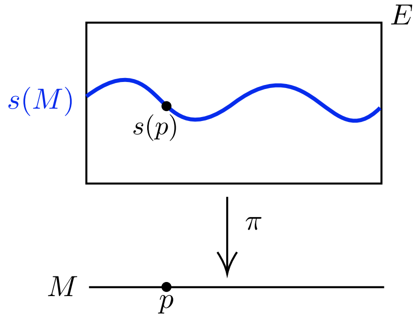
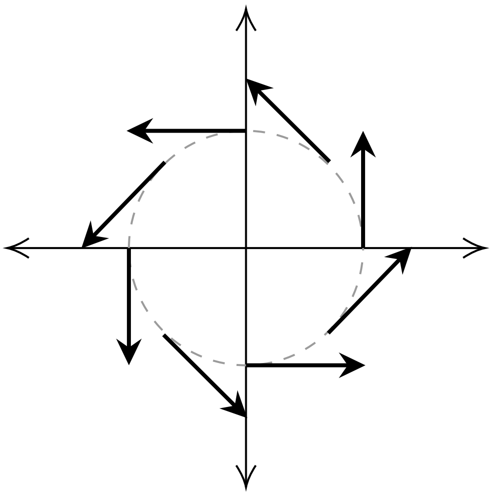


Figure 7.1: A section of a vector bundle $\pi : E \to M$ is a map $s : M \to E$.

Example 7.3. Consider the following vector field on $\mathbb{R}^2$

$$X_{(x,y)} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} = \begin{bmatrix} -y \\ x \end{bmatrix}.$$

It is a smooth vector field on $\mathbb{R}^2$. Visually, it's described as the following image:



Proposition 7.7

Let s and t be C^∞ sections of a C^∞ vector bundle $\pi : E \to M$, and let f be a C^∞ real-valued function on M. Then

(i) The sum $s + t : M \to E$ defined by

$$(s+t)(p) = s(p) + t(p) \in E_p \text{ for } p \in M,$$

is a C^∞ section of E.

(ii) The product $fs : M \to E$ defined by

$$(fs)(p) = f(p)\, s(p) \in E_p \text{ for } p \in M,$$

is a C^∞ section of E.

Proof. (i) It is clear that $s + t$ is a section of E. Indeed,

$$\pi \circ (s+t)(p) = \pi(s(p) + t(p)) = \pi(\mathbf{e}_p + \mathbf{v}_p) = p,$$

so that $\pi \circ (s+t) = \mathbb{1}_M$. Now it remains to show that $s + t$ is smooth. For this purpose, let $p \in M$ and let V be a trivializing open set for E containing $p \in M$, with the trivialization

$$\phi : \pi^{-1}(V) \to V \times \mathbb{R}^r.$$

Choose a chart $(U, \varphi) = (U, x^1, \ldots, x^n)$ about $p \in M$ such that $U \subseteq V$. Then $\left(E|_U, (\varphi \times \mathbb{1}_{\mathbb{R}^r}) \circ \phi|_U\right)$ is a chart on E. Let $q \in U$ and $s(q) = \mathbf{e}_q$ with $\mathbf{e}_q \in E_q$. If $\{\widehat{e}_i\}_{i=1}^r$ is a basis of E_q, and $\mathbf{e}_q = \sum\limits_{i=1}^r c^i(\mathbf{e}_q)\,\widehat{e}_i$. Then we have

$$(\phi \circ s)(q) = \phi(\mathbf{e}_q) = \left(q, c^1(\mathbf{e}_q), \ldots, c^r(\mathbf{e}_q)\right) = \left(q, \left(c^1 \circ s\right)(q), \ldots, (c^r \circ s)(q)\right).$$

Therefore, the map $(\varphi \times \mathbb{1}_{\mathbb{R}^r}) \circ \phi|_U \circ s \circ \varphi^{-1} : \varphi(U) \subseteq \mathbb{R}^n \to \varphi(U) \times \mathbb{R}^r \subseteq \mathbb{R}^{n+r}$ is given by

$$\varphi(q) = \left(x^1(q), x^2(q), \ldots, x^n(q)\right) \mapsto \left(x^1(q), x^2(q), \ldots, x^n(q), \left(c^1 \circ s\right)(q), \ldots, (c^r \circ s)(q)\right).$$

Since s is smooth, this map is smooth. In particular, all of it components are smooth. Therefore, $c^i \circ s$ is a smooth function on U. Similarly, let $t(q) = \mathbf{v}_q \in E_q$. Then $\mathbf{v}_q = \sum_{i=1}^{r} d^i(\mathbf{v}_q)\,\widehat{e}_i$. Then in a similar manner as above, the map $(\varphi \times \mathbb{1}_{\mathbb{R}^r}) \circ \phi|_U \circ t \circ \varphi^{-1} : \varphi(U) \subseteq \mathbb{R}^n \to \varphi(U) \times \mathbb{R}^r \subseteq \mathbb{R}^{n+r}$ is given by

$$\varphi(q) = \left(x^1(q), x^2(q), \ldots, x^n(q)\right) \mapsto \left(x^1(q), x^2(q), \ldots, x^n(q), \left(d^1 \circ t\right)(q), \ldots, (d^r \circ t)(q)\right).$$

Since t is smooth, this map is smooth. In particular, all of it components are smooth. Therefore, $d^i \circ t$ is a smooth function on U. Hence, $c^i \circ s + d^i \circ t$ is a smooth function on U.

Now, the map $(\varphi \times \mathbb{1}_{\mathbb{R}^r}) \circ \phi|_U \circ (s+t) \circ \varphi^{-1} : \varphi(U) \subseteq \mathbb{R}^n \to \varphi(U) \times \mathbb{R}^r \subseteq \mathbb{R}^{n+r}$ is given by

$$\begin{aligned}\varphi(q) = \left(x^1(q), x^2(q), \ldots, x^n(q)\right) \mapsto (&x^1(q), x^2(q), \ldots, x^n(q), \\ &\left(c^1 \circ s\right)(q) + \left(d^1 \circ t\right)(q), \ldots, (c^r \circ s)(q) + (d^r \circ t)(q)).\end{aligned}$$

Since all the components of this map are smooth, this map is smooth. Therefore, $s+t$ is smooth on U. In patricular, $s+t$ is smooth at p. Since p is chosen arbitrarily, $s+t$ is smooth on all of M.

(ii) We shall use the same setup as above. f is a smooth function on M, so it is smooth on U. $c^i \circ s$ is also smooth on U. Therefore, their product is smooth on U. Now, the map $(\varphi \times \mathbb{1}_{\mathbb{R}^r}) \circ \phi|_U \circ (fs) \circ \varphi^{-1} : \varphi(U) \subseteq \mathbb{R}^n \to \varphi(U) \times \mathbb{R}^r \subseteq \mathbb{R}^{n+r}$ is given by

$$\begin{aligned}\varphi(q) = \left(x^1(q), x^2(q), \ldots, x^n(q)\right) \mapsto (&x^1(q), x^2(q), \ldots, x^n(q), \\ &f(q)\left(c^1 \circ s\right)(q), \ldots, f(q)(c^r \circ s)(q)).\end{aligned}$$

Since all the components of this map are smooth, this map is smooth. Therefore, fs is smooth on U. In patricular, fs is smooth at p. Since p is chosen arbitrarily, fs is smooth on all of M.

■

Denote the set of all C^∞ sections of E by $\Gamma(E)$. Proposition 7.7 shows that $\Gamma(E)$ is not only a vector space over $\mathbb{R}$, but also a module over the ring $C^\infty(M)$ of C^∞ functions on M. For any open subset $U \subseteq M$, one can also consider the vector space $\Gamma(U, E)$ of C^∞ sections of E over U. Then $\Gamma(U, E)$ is both an $\mathbb{R}$-vector space and a $C^\infty(U)$-module. A section over the whole manifold is called a **global section**.

Exercise 7.2. Show that the image of a smooth section $s : M \to E$ is a regular submanifold of E.

Solution. This follows readily from Theorem 6.1. It suffices to show that s is an embedding. Firstly, $\pi|_{s(M)} : s(M) \to M$ is the inverse of s. Since π is continuous, so is its restriction $\pi|_{s(M)}$. Therefore, s is a homeomorphism onto its image. Now we need to show that s is an immersion. For $p \in M$,

$$\pi \circ s = \mathbb{1}_M \implies \pi_{*,s(p)} \circ s_{*,p} = (\mathbb{1}_M)_{*,p} = \mathbb{1}_{T_pM}.$$

Hence, $s_{*,p}$ is injective for every $p \in M$. Therefore, s is an embedding, and consequently, $s(M)$ is a regular submanifold of E. ■

§7.4 Smooth Frames

Definition 7.9 (Frame)**.** A **frame** for a vector bundle $\pi : E \to M$ over an open set U is a collection of sections $s_1, \ldots, s_r$ of E over U such that at each point $p \in U$, the elements $s_1(p), \ldots, s_r(p)$ form a basis for the r-dimensional vector space $E_p = \pi^{-1}(p)$. A frame $s_1, \ldots, s_r$ is said to be smooth if $s_1, \ldots, s_r$ are C^∞ as sections of E over U. A frame for the tangent bundle $TM \to M$ over an open set U is simply called a frame on U.

Example 7.4. The collection of vector fields

$$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$$

is a smooth frame on $\mathbb{R}^3$.

Example 7.5. Let M be a manifold and $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_r$ the standard basis on $\mathbb{R}^r$. In other words, $\mathbf{e}_i \in \mathbb{R}^r$ whose i-th component is 1 and rest of the components are all 0. Define $\overline{e}_i : M \to M \times \mathbb{R}^r$ by

$$\overline{e}_i(p) = (p, \mathbf{e}_i) .$$

Then $\overline{e}_1, \ldots, \overline{e}_r$ is a C^∞ frame for the product bundle $M \times \mathbb{R}^r \to M$.

Example 7.6 (The frame of a trivialization)**.** Let $\pi : E \to M$ be a smooth vector bundle of rank r. If $\phi : \pi^{-1}(U) \xrightarrow{\sim} U \times \mathbb{R}^r$ is a trivialization of E over an open set $U \subseteq M$, then ϕ^{-1} carries the frame $\overline{e}_1, \ldots, \overline{e}_r$ of the product bundle $U \times \mathbb{R}^r$ to a C^∞ frame $t_1, t_2, \ldots, t_r$ for E over U:

$$t_i(p) = \phi^{-1}(\overline{e}_i(p)) = \phi^{-1}(p, \mathbf{e}_i) ,$$

for $p \in U$. We call $t_1, \ldots, t_r$ the C^∞ frame over U of the trivialization ϕ.

Lemma 7.8

Let $\phi : E|_U \to U \times \mathbb{R}^r$ be a trivialization over an open set U of a C^∞ vector bundle $E \to M$, and $t_1, \ldots, t_r$ the C^∞ frame over U of the trivilization. Then a section $s = \sum_{i=1}^{r} b^i t_i$ of E over U is C^∞ if and only if its coefficients b^i relative to the frame $t_1, \ldots, t_r$ are C^∞. (Here $b^i : U \to \mathbb{R}$.)

Proof. **(⇐)** According to Proposition 7.7, each $b^i t_i$ is a C^∞ section, and hence their sum $s = \sum_{i=1}^{r} b^i t_i$ is also a C^∞ section.

(⇒) Suppose the section $s = \sum_{i=1}^{r} b^i t_i$ of E over U is C^∞. s is a map $s : U \to E|_U$, and so $\phi \circ s : U \to U \times \mathbb{R}^r$ is C^∞ as it's the composition of two C^∞ maps. Now, note that

$$(\phi \circ s)(p) = \phi\left(\sum_{i=1}^{r} b^i(p)\, t_i(p)\right) = \sum_{i=1}^{r} b^i(p)\, \phi(t_i(p)) ,$$

since ϕ is linear at each E_p. By Example 7.6, $\phi(t_i(p)) = (p, \mathbf{e}_i)$. Hence,

$$(\phi \circ s)(p) = \sum_{i=1}^{r} b^i(p)(p, \mathbf{e}_i) = \left(p, \sum_{i=1}^{r} b^i(p)\, \mathbf{e}_i\right) = \left(p, b^1(p), \ldots, b^r(p)\right) .$$

Let $P : U \times \mathbb{R}^r \to \mathbb{R}^r$ be the projection map. It is a smooth map. Therefore, $P \circ \phi \circ s$ is a smooth map on U.

$$(P \circ \phi \circ s)(p) = \left(b^1(p), \ldots, b^r(p)\right) .$$

So, b^i's are the components of $P \circ \phi \circ s$. Hence, by Proposition 2.8, b^i is smooth on U for every i. ∎

Proposition 7.9 (Characterization of C^∞ sections)

Let $\pi : E \to M$ be a C^∞ vector bundle and U an open subset of M. Suppose $s_1, \ldots, s_r$ is a C^∞ frame for E over U. Then a section $s = \sum_{j=1}^{r} c^j s_j$ of E over U is C^∞ if and only if the coefficients c^j are C^∞ functions on U.

Proof. If $s_1, \dots, s_r$ is the frame of a trivialization over U, then this proposition is exactly Lemma 7.8. We prove the general result by reducing it to this case.

($\Leftarrow$) According to Proposition 7.7, each $c^j s_j$ is a C^∞ section, and hence their sum $s = \sum_{i=1}^{r} c^j s_j$ is also a C^∞ section.

($\Rightarrow$) Suppose $s = \sum_{i=1}^{r} c^j s_j$ is a C^∞ section of E over U. Fix a point $p \in U$ and choose a trivializing open set $V \subseteq U$ for E containing p with trivialization $\phi : \pi^{-1}(V) \to V \times \mathbb{R}^r$. (There exists a trivializing open set V' containing p, and a trivialization $\Phi : \pi^{-1}(V') \to V' \times \mathbb{R}^r$. Then $V = U \cap V'$, and $\phi = \Phi\big|_{\pi^{-1}(V)}$.)

By Example 7.6, let $t_1, \dots, t_r$ denote the C^∞ frame of the trivialization ϕ. Now, we write the sections s and s_j by means of the frame $t_1, \dots, t_r$.

$$s\big|_V = \sum_{i=1}^{r} b^i t_i \quad \text{and} \quad s_j\big|_V = \sum_{i=1}^{r} a^i_j t_i \,.$$

Here, we need to restrict s and s_j's on V, because t_i's are sections of E over V. The coefficients b^i and a^i_j are C^∞ functions on V by Lemma 7.8. Now,

$$\sum_{i=1}^{r} b^i t_i = s\big|_V = \sum_{j=1}^{r} c^j\big|_V s_j\big|_V = \sum_{i,j=1}^{r} c^j\big|_V a^i_j t_i \,.$$

Comparing the coefficients of t_i yields

$$b^i = \sum_{j=1}^{r} c^j\big|_V a^i_j \,.$$

In matrix notation, if we denote $\left[a^i_j\right]_{i,j=1}^{r} = A$,

$$b = \begin{bmatrix} b^1 \\ \vdots \\ b^r \end{bmatrix} = A \begin{bmatrix} c^1\big|_V \\ \vdots \\ c^r\big|_V \end{bmatrix} = Ac \,.$$

At each point of V, being the transition matrix between two bases (t_i's and s_j's), A is invertible. By Cramer's rule for matrix inverse,

$$\left(A^{-1}\right)_{ij} = \frac{1}{\det A} (-1)^{i+j} \left(\ (j,i) \text{ minor of } A \ \right).$$

The (j,i) minor of A is a smooth function of a^i_j's, and a^i_j's are smooth function on V. Therefore, the entries of the inverse A^{-1} are C^∞ functions on V.

Now, $c = A^{-1}b$. We have already shown that b^i's are C^∞ functions on V. Hence, $c = A^{-1}b$ is a column vector of C^∞ functions on V. This proves that $c^1, \dots, c^r$ are smooth at $p \in V \subseteq U$. Since p is an arbitrary point of U, $c^1, \dots, c^r$ are smooth on all of U. ∎

8 Partition of Unity

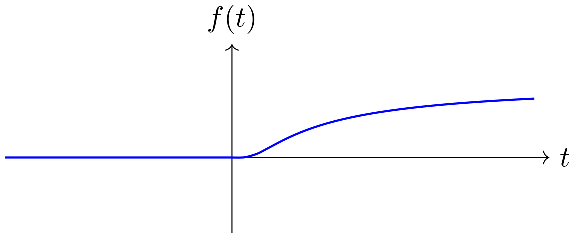
A partition of unity on a manifold is a collection of non-negative functions that sum to 1 (subjected to some other conditions that we will specify later). Usually, one demands, in addition, that the partition of unity be *subordinate* to an open cover $\{U_\alpha\}_{\alpha\in A}$ of the manifold M. What this means is that the partition of unity $\{\rho_\alpha\}_{\alpha\in A}$ is indexed by the same set as the open cover $\{U_\alpha\}_{\alpha\in A}$ and for each α in the index set A, the support of ρ_α (to be defined shortly) is contained in U_α. In particular, ρ_α vanishes outside U_α.

§8.1 Smooth Bump Functions

We introduce the function $f:\mathbb{R}\to\mathbb{R}$, defined by

$$f(t)=\begin{cases} e^{-\frac{1}{t}} & \text{for } t>0 \\ 0 & \text{for } t\le 0 \end{cases},$$

the graph of which looks like



Lemma 8.1

$f:\mathbb{R}\to\mathbb{R}$ is smooth.

Proof. It is clearly smooth in $\mathbb{R}\setminus\{0\}$ because of the exponential nature. So, one only needs to show that all the derivatives of f exist and are continuous at 0.

One first verifies that for any $k\ge 0$,

$$\lim_{t\to 0^+}\frac{e^{-\frac{1}{t}}}{t^k}=0\,.$$

In fact, $\lim\limits_{t\to 0^+}\frac{e^{-\frac{1}{t}}}{t^k}=\lim\limits_{t\to 0^+}\frac{t^{-k}}{e^{\frac{1}{t}}}$. So it suffices to show that $\lim\limits_{t\to 0^+}\frac{t^{-k}}{e^{\frac{1}{t}}}=0$. We shall prove it by inuction. The base case is $k=0$.

$$\lim_{t\to 0^+}\frac{t^0}{e^{\frac{1}{t}}}=\lim_{t\to 0^+}\frac{1}{e^{\frac{1}{t}}}=0\,.$$

By inductive hypothesis, this statement is true for some $k\ge 0$. Now we shall show it for $k+1$.

$$\lim_{t\to 0^+}\frac{t^{-(k+1)}}{e^{\frac{1}{t}}}\overset{\text{L'Hôpital}}{=}\lim_{t\to 0^+}\frac{-(k+1)\,t^{-k-2}}{-t^{-2}e^{\frac{1}{t}}}=(k+1)\lim_{t\to 0^+}\frac{t^{-k}}{e^{\frac{1}{t}}}=0\,.$$

Hence, for any $k\ge 0$, one has $\lim\limits_{t\to 0^+}\frac{e^{-\frac{1}{t}}}{t^k}=0$. Now, we show by induction that for $t>0$, the k-th derivative of f is of the form

$$f^{(k)}(t)=\frac{P_k(t)}{t^{2k}}e^{-\frac{1}{t}}\,,$$

for some polynomial P_k. It's clearly true for $k = 0$, i.e. $f(t) = \frac{P_0(t)}{1} e^{-\frac{1}{t}}$. Here, $P_0(t) = 1$. Suppose this statement is true for some $k \geq 0$. Now we shall show it for $k+1$.

$$\begin{aligned} f^{(k+1)}(t) &= \left(\frac{P_k(t)}{t^{2k}} e^{-\frac{1}{t}}\right)' = \frac{P_k'(t)}{t^{2k}} e^{-\frac{1}{t}} + \frac{P_k(t)}{t^{2k}} \frac{1}{t^2} e^{-\frac{1}{t}} - 2k \frac{P_k(t)}{t^{2k+1}} e^{-\frac{1}{t}} \\ &= \frac{t^2 P_k'(t) + P_k(t) - 2kt P_k(t)}{t^{2k+2}} e^{-\frac{1}{t}} = \frac{\widetilde{P_{k+1}}(t)}{t^{2k+2}} e^{-\frac{1}{t}} \end{aligned}$$

where $\widetilde{P_{k+1}} = t^2 P_k'(t) + P_k(t) - 2kt P_k(t)$ is a polynomial. So, we have proved by induction that $f^{(k)}(t) = \frac{P_k(t)}{t^{2k}} e^{-\frac{1}{t}}$. Now,

$$\lim_{t \to 0^+} f^{(k)}(t) = \lim_{t \to 0^+} \frac{P_k(t)}{t^{2k}} e^{-\frac{1}{t}} = P_k(0) \lim_{t \to 0^+} \frac{e^{-\frac{1}{t}}}{t^{2k}} = 0\,.$$

We now show that for each $k \geq 0$, $f^{(k)}(0) = 0$. Again, we use induction. For $k = 0$, $f(0) = 0$. Let us assume that $f^{(k)}(0) = 0$ for some $k \geq 0$. Now, we shall show that $f^{(k+1)}(0) = 0$.

$$\begin{aligned} f^{(k+1)}(0) &= \lim_{t \to 0^+} \frac{f^{(k)}(t) - f^{(k)}(0)}{t} = \lim_{t \to 0^+} \frac{\frac{P_k(t)}{t^{2k}} e^{-\frac{1}{t}} - 0}{t} \\ &= \lim_{t \to 0^+} \frac{P_k(t)}{t^{2k+1}} e^{-\frac{1}{t}} = P_k(0) \lim_{t \to 0^+} \frac{e^{-\frac{1}{t}}}{t^{2k+1}} \\ &= 0 \end{aligned}$$

Therefore, we have shown that

$$\lim_{t \to 0^+} f^{(k)}(t) = 0 = f^{(k)}(0)\ ,$$

proving that each $f^{(k)}$ is continuous at 0, and hence f is smooth at $t = 0$. ■

We now construct a smooth version of a step function denoted by $g(t)$ by dividing $f(t)$ by a positive function $l(t)$. The quotient will then be zero for $t \leq 0$ as follows from the definition of $f(t)$. We want the denominator function $l(t)$ to be equal to $f(t)$ for $t \geq 1$, which will then mean that $g(t) = 1$ for $t \geq 1$. This suggests that we choose $l(t) = f(t) + f(1-t)$. So we define

$$g(t) = \frac{f(t)}{f(t) + f(1-t)}\,.$$

Clearly, g is 0 when $t \leq 0$, and 1 when $t \geq 1$. Now, for $0 < t < 1$, $0 < 1-t < 1$. In this case, g is

$$g(t) = \frac{f(t)}{f(t) + f(1-t)} = \frac{e^{-1/t}}{e^{-1/t} + e^{-1/(1-t)}} = \frac{1}{1 + \frac{e^{1/t}}{e^{1/(1-t)}}}\,.$$

Thus, we obtain a piecewise formula for g.

$$g(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ \frac{1}{1+\frac{e^{1/t}}{e^{1/(1-t)}}} & \text{if } 0 < t < 1 \\ 1 & \text{if } t \geq 1 \end{cases}$$

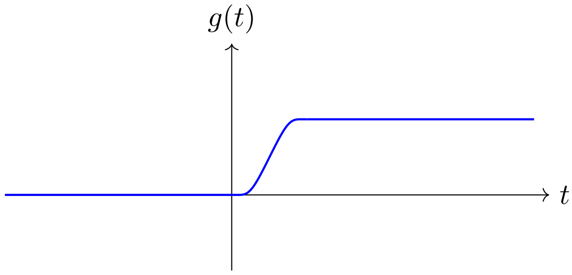
The function g is C^∞ on $\mathbb{R}$ since it is a quotient of C^∞ functions and the denominator is never 0. Next, we want to show that g is strictly increasing on $(0,1)$. For $t \in [0,1]$,

$$g(t) = \frac{1}{1 + \frac{e^{1/t}}{e^{1/(1-t)}}} = \frac{e^{1/(1-t)}}{e^{1/(1-t)} + e^{1/t}}\,.$$

Taking derivative, we get

$$\begin{aligned}\frac{\mathrm{d}g}{\mathrm{d}t}(t) &= \frac{\frac{1}{(1-t)^2}e^{1/(1-t)}\left(e^{1/(1-t)}+e^{1/t}\right)-e^{1/(1-t)}\left(\frac{1}{(1-t)^2}e^{1/(1-t)}-\frac{1}{t^2}e^{1/t}\right)}{\left(e^{1/(1-t)}+e^{1/t}\right)^2}\\ &= \frac{e^{\frac{1}{1-t}+\frac{1}{t}}}{\left(e^{1/(1-t)}+e^{1/t}\right)^2}\left(\frac{1}{(1-t)^2}+\frac{1}{t^2}\right)\end{aligned}$$

which is positive in $(0,1)$. Hence, g is strictly increasing on $(0,1)$. The graph of g is as follows:



We will now make a linear change of variables. Choose two positive real numbers $a < b$, and make a linear change of variables to map $\left[a^2, b^2\right]$ to $[0,1]$.

$$x \mapsto \widetilde{x} = \frac{x}{b^2-a^2} - \frac{a^2}{b^2-a^2}.$$

Then we have,

$$\begin{aligned} x \in \left(-\infty, a^2\right) &\implies \widetilde{x} \in (-\infty, 0)\\ x \in \left[a^2, b^2\right] &\implies \widetilde{x} \in [0,1]\\ x \in \left(b^2, \infty\right) &\implies \widetilde{x} \in (1, \infty)\end{aligned}$$

Now, set $h(x) = g(\widetilde{x}) = g\left(\frac{x-a^2}{b^2-a^2}\right)$. Since

$$g(\widetilde{x}) = \begin{cases} 0 & \text{for } \widetilde{x} \leq 0 \\ 1 & \text{for } \widetilde{x} \geq 1 \end{cases},$$

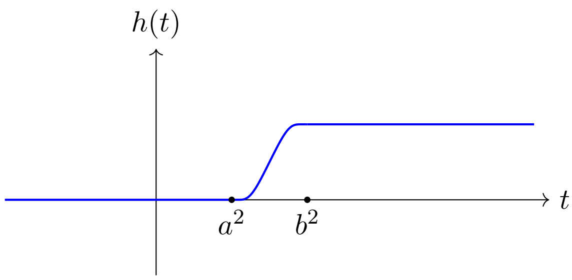
one has

$$h(x) = \begin{cases} 0 & \text{for } x \leq a^2 \\ 1 & \text{for } x \geq b^2 \end{cases}.$$

Furthermore,

$$\frac{\mathrm{d}h}{\mathrm{d}x}(x) = \frac{\mathrm{d}g}{\mathrm{d}\widetilde{x}}(\widetilde{x})\frac{\mathrm{d}\widetilde{x}}{\mathrm{d}x} = \frac{g'(\widetilde{x})}{b^2-a^2}.$$

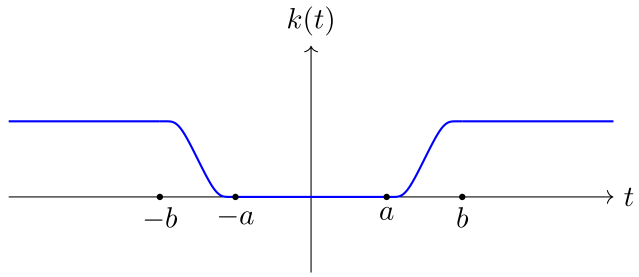
We have seen that $g'(\widetilde{x}) > 0$ when $\widetilde{x} \in (0,1)$. Hence, $h'(x) > 0$ when $x \in \left(a^2, b^2\right)$. So, the smooth function h is strrictly increasing on $\left(a^2, b^2\right)$. The graph of h is as follows:



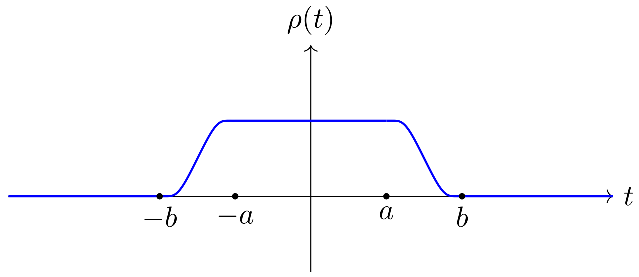
Now, let $k(x) = h(x^2)$, which makes k and even function of x and hence is symmetric about the x axis.

$$k(x) = h\left(x^2\right) = g\left(\frac{x^2 - a^2}{b^2 - a^2}\right).$$

One finds that whenever $x \in (-\infty, -b] \cup [b, \infty)$, $k(x) = 1$, and whenever $x \in [-a, a]$, $k(x) = 0$. One can verify as above that $k(x)$ is strictly decreasing in $(-b, -a)$, and strictly increasing in (a, b). Hence, the graph of k is as follows:



Finally, set $\rho(x) = 1 - k(x) = 1 - g\left(\frac{x^2-a^2}{b^2-a^2}\right)$, the graph of which is as follows:



This $\rho(x)$ is a C^∞ bump function at $0 \in \mathbb{R}$. We now give the definition of a bump function. Recall that $\mathbb{R}^\times$ denotes the set of nonzero real numbers.

Definition 8.1 (Support). The **support** of a real-valued function $f : M \to \mathbb{R}$ on a manifold M is defined to be the closure in M of the subset on which $f \neq 0$.

$$\operatorname{supp} f = \overline{\{q \in M \mid f(q) \neq 0\}} = \operatorname{cl}_M\left(f^{-1}\left(\mathbb{R}^\times\right)\right).$$

Definition 8.2 (Bump Function). Let $q \in M$ and U a neighborhood of q. By a **bump function** at q supported in U, we mean any continuous non-negative function $\rho : M \to \mathbb{R}$ that is 1 is a neighborhood of q with $\operatorname{supp}\rho \subseteq U$. We call it a **smooth bump function** if it is C^∞ as a map between manifolds.

We have previously constructed a C^∞ bump function at 0 in $\mathbb{R}$ that is identically 1 on $[-a, a]$ and has support in $[-b, b]$. By shifting the graph to the right, for any $q \in \mathbb{R}$, $\rho(x - q)$ is a C^∞ bump function at q.

One can easily extend this construction for a bump function on $\mathbb{R}^n$. To get a C^∞ bump function at $\mathbf{0} \in \mathbb{R}^n$ that is 1 on the closed ball $\overline{B}(\mathbf{0}, a)$ and has support in the closed ball $\overline{B}(\mathbf{0}, b)$, set

$$\sigma(\mathbf{x}) = \rho(\|\mathbf{x}\|) = 1 - g\left(\frac{\|\mathbf{x}\|^2 - a^2}{b^2 - a^2}\right).$$

σ is smooth, because

$$\mathbf{x} \mapsto \frac{\|\mathbf{x}\|^2 - a^2}{b^2 - a^2} \text{ and } t \mapsto 1 - g(t)$$

are both C^∞, and their composition is σ. To get a C^∞ bump function at $\mathbf{q} \in \mathbb{R}^n$, take $\sigma(\mathbf{x} - \mathbf{q})$.

Construction of a Smooth Bump Function on a Manifold

We have constructed a C^∞ bump function $\sigma(\mathbf{x}-\mathbf{q})$ at $\mathbf{q} \in \mathbb{R}^n$ from a C^∞ bump function $\sigma(\mathbf{x})$ at $\mathbf{0} \in \mathbb{R}^n$ whose supprt is contained in the closed ball $\overline{B}(\mathbf{0}, b)$. Now we want to extend this idea from $\mathbb{R}^n$ to a manifold M.

Exercise 8.1. Let M be an n-dimensional manifold and $q \in M$. Suppose U is any neighborhood of q. Construct a smooth bump function at q supprted in U.

Solution. There exists a coordinate chart (V, ψ) in the maximal atlas of M such that $q \in V \subseteq U$. Such a coordinate open set exists because if V' is a coordinate open set about q, we can just take $V = V' \cap U$.

Now, there exists a bump function $\rho : \mathbb{R}^n \to \mathbb{R}$ at $\psi(q)$ supported in $\psi(V) \subseteq \mathbb{R}^n$ that is identically 1 in $\overline{B}(\psi(q), a)$. Suppose the support of ρ is $\overline{B}(\psi(q), b)$ for some $b > 0$. Then

$$\overline{B}(\psi(q), a) \subseteq \overline{B}(\psi(q), b) \subseteq \psi(V) \subseteq \mathbb{R}^n .$$

Now, define a function $f : M \to \mathbb{R}$ by

$$f(p) = \begin{cases} \rho(\psi(p)) & \text{if } p \in V \\ 0 & \text{if } p \notin V \end{cases} .$$

For $p \in V$, f is the composition of two smooth maps ρ and ψ, and hence f is smooth at p. Now we need to check that f is smooth at $p \notin V$.

Note that, by the construction of $\rho : \mathbb{R}^n \to \mathbb{R}$, $\operatorname{supp} \rho = \overline{B}(\psi(q), b) \subseteq \psi(V) \subseteq \mathbb{R}^n$. Being a closed and bounded subset of $\mathbb{R}^n$, $\operatorname{supp} \rho$ is compact. Since ψ^{-1} is continuous, $\psi^{-1}(\operatorname{supp} \rho)$ is also a compact subspace of M. M is a manifold, hence it is Hausforff. Therefore, $\psi^{-1}(\operatorname{supp} \rho)$ is closed in M. As a result,

$$\begin{aligned} \operatorname{supp} f &= \operatorname{cl}_M\left(\psi^{-1}\left(\rho^{-1}\left(\mathbb{R}^\times\right)\right)\right) = \psi^{-1}\left(\operatorname{cl}_{\mathbb{R}^n}\left(\rho^{-1}\left(\mathbb{R}^\times\right)\right)\right) \\ &= \psi^{-1}(\operatorname{supp} \rho) \subseteq V \end{aligned}$$

Since $\operatorname{supp} f \subseteq V$, $p \notin V$ gives us

$$p \in M \setminus V \subseteq M \setminus \operatorname{supp} f \implies p \in M \setminus \operatorname{supp} f .$$

$\operatorname{supp} f$ is closed, hence $M \setminus \operatorname{supp} f$ is open. If (W', φ') is a chart about p, $W = W' \cap (M \setminus \operatorname{supp} f)$ is a coordinate open set about p contained in $M \setminus \operatorname{supp} f$. If we write $\varphi = \varphi'|_W$, then (W, φ) is a chart about p.

$$f \circ \varphi^{-1} : \varphi(W) \to \mathbb{R} .$$

For $\varphi(x) \in \varphi(W)$, $x \in W \subseteq M \setminus \operatorname{supp} f$. So $f(x) = 0$. Therefore, the map $f \circ \varphi^{-1}$ is identically 0, and hence smooth in $\varphi(W)$. As a result, f is smooth on $W \ni p$. In particular, f is smooth at p. Since $p \notin V$ is arbitrary, f is smooth at all $p \notin V$. Therefore, f is smooth on all of M. ■

In general, a C^∞ function on an open subset U of a manifold M cannot be extended to a C^∞ function on the whole of M. For instance, take $\sec(x)$ as a C^∞ function on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \subseteq \mathbb{R}$. We can't find a C^∞ function on $\mathbb{R}$ that agrees with $\sec(x)$ globally, i.e. on the whole of $\mathbb{R}$. However, if we require the C^∞ function on M (to be found) to agree with a given C^∞ function only on some neighborhood of a point in U, then such a C^∞ extension is possible.

Proposition 8.2 (C^∞ extension of a function)

Suppose f is a C^∞ function defined on a neighborhood U of a point $p \in M$. Then there is a C^∞ function $\widetilde{f}$ on M that agrees with f in some possibly smaller neighborhood of p (i.e., contained in U).

Proof. Choose a C^∞ bump function $\rho : M \to \mathbb{R}$ that is identically 1 in a neighborhood V of p with $V \subseteq U$ (see the previous construction) and $\operatorname{supp}\rho \subseteq U$. Now, define

$$\widetilde{f}(q) = \begin{cases} \rho(q) f(q) & \text{for } q \in U \\ 0 & \text{for } q \notin U \end{cases} .$$

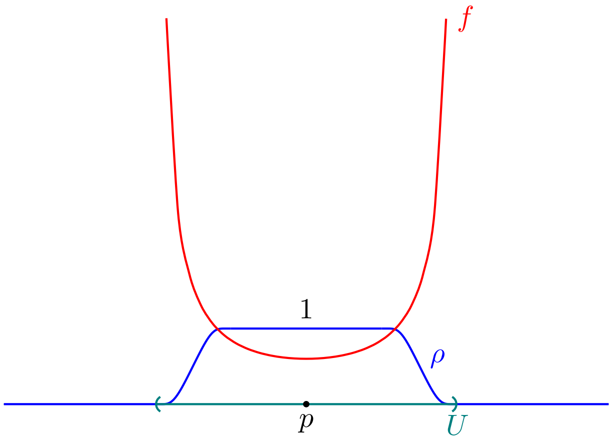


Figure 8.1: Extending the domain of a function by multiplying by a bump function.

By the definition, $\widetilde{f}$ agrees with f on $V \subset U$. As the product of two smooth functions on U, $\widetilde{f}$ is smooth on U. Now, using the definition of $\widetilde{f}$,

$$\widetilde{f}(p) \neq 0 \implies \rho(q) \neq 0 \text{ and } f(q) \neq 0 \implies \widetilde{f}^{-1}(\mathbb{R}^\times) \subseteq \rho^{-1}(\mathbb{R}^\times) .$$

$$\therefore \operatorname{cl}_M\left(\widetilde{f}^{-1}(\mathbb{R}^\times)\right) \subseteq \operatorname{cl}_M\left(\rho^{-1}(\mathbb{R}^\times)\right) \implies \operatorname{supp}\widetilde{f} \subseteq \operatorname{supp}\rho .$$

If $q \notin U$, then $q \notin \operatorname{supp}\widetilde{f}$ (since $\operatorname{supp}\widetilde{f} \subseteq \operatorname{supp}\rho \subseteq U$). Since $\operatorname{supp}\widetilde{f}$ is closed in M, one can find an coordinate neighborhood of q that is disjoint from $\operatorname{supp}\widetilde{f}$. On this open set, $\widetilde{f}$ is identically 0. Therefore, similarly as in the solution of previous exercise, $\widetilde{f}$ is smooth at q. Since $q \notin U$ is arbitrary, $\widetilde{f}$ is smooth at every $q \notin U$. ■

§8.2 Partitions of Unity

If $\{U_i\}_{i\in I}$, I being finite, is a finite open cover of M, a C^∞ partition of unity subordinate to $\{U_i\}_{i\in I}$ is a collection of non-negative C^∞ functions $\{\rho_i : M \to \mathbb{R}\}_{i\in I}$ such that $\operatorname{supp}\rho_i \subseteq U_i$, and

$$\sum_{i\in I} \rho_i = 1 .$$

When I is an infinite set, for the sum to make sense, we'll impose *local finiteness* condition.

Definition 8.3 (Local Finiteness)**.** A collection $\{A_\alpha\}_\alpha$ of subsets of a topological space S is said to be **locally finite** for every point $q \in S$ has a neighborhood that intersects only finitely many of the A_α's. In particular, every $q \in S$ is contained in only finitely many of the A_α's.

Example 8.1 (An open cover that is not locally finite)**.** Let $U_{r,n}$ be the open interval $\left(r - \frac{1}{n}, r + \frac{1}{n}\right)$ on the real line $\mathbb{R}$. The open cover $\{U_{r,n} \mid r \in \mathbb{Q}, \ n \in \mathbb{Z}^+\}$ of $\mathbb{R}$ is not locally finite.

Definition 8.4 (Partition of Unity)**.** A C^∞ **partition of unity** on a manifold M is a collection of non-negative C^∞ functions $\{\rho_\alpha : M \to \mathbb{R}\}_{\alpha\in A}$ such that

(i) The collection of supports $\{\operatorname{supp}\rho_\alpha\}_{\alpha\in A}$ is locally finite.

(ii) $\sum \rho_\alpha = 1$.

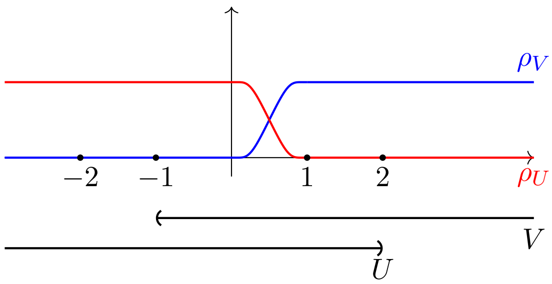
Given an open cover $\{U_\alpha\}_{\alpha \in A}$ of M, we say that a partition of unity $\{\rho_\alpha\}_{\alpha \in A}$ is **subordinate** to the open cover $\{U_\alpha\}_{\alpha \in A}$ if $\operatorname{supp} \rho_\alpha \subseteq U_\alpha$ for every $\alpha \in A$.

Remark 8.1. Since the collection of supports $\{\operatorname{supp} \rho_\alpha\}_{\alpha \in A}$ is locally finite, every point $q \in M$ lies in finitely many sets $\operatorname{supp} \rho_\alpha$. Hence, $\rho_\alpha (q) \neq 0$ for only finitely many α. Hence, the sum $\sum \rho_\alpha$ is a finite sum at every point.

Example 8.2. Let U and V be the open intervals $(-\infty, 2)$ and $(1, \infty)$ in $\mathbb{R}$, respectively. Define

$$\rho_V (t) = g (t) = \begin{cases} 0 & \text{if } t \leq 0 \\ \frac{e^{1/(1-t)}}{e^{1/(1-t)} + e^{1/t}} & \text{if } 0 < t < 1 \\ 1 & \text{if } t \geq 1 \end{cases} .$$

Define $\rho_U = 1 - \rho_V$.



Then $\{\rho_U, \rho_V\}$ is a partition of unity subordinate to the open cover $\{U, V\}$.

Existence of Partition of Unity

Lemma 8.3

If $\rho_1, \ldots, \rho_m$ are real-valued function on a manifold M, then

$$\operatorname{supp} \left(\sum_{i=1}^{m} \rho_i \right) \subseteq \bigcup_{i=1}^{m} \operatorname{supp} \rho_i .$$

Proof. Let $\rho = \sum_{i=1}^{m} \rho_m$, $A_i = \rho_i^{-1} (\mathbb{R}^\times)$ and $A = \rho^{-1} (\mathbb{R}^\times)$. If $p \in A$, $\rho (p) \neq 0$. Then $\rho_i (p) \neq 0$ for some i.

$$p \in A \implies p \in A_i \subseteq \bigcup_{i=1}^{m} A_i \implies A \subseteq \bigcup_{i=1}^{m} A_i .$$

Now, by Proposition A.6 and Proposition A.8,

$$\operatorname{cl}_M (A) \subseteq \operatorname{cl}_M \left(\bigcup_{i=1}^{m} A_i \right) = \bigcup_{i=1}^{m} \operatorname{cl}_M (A_i) .$$

$\operatorname{cl}_M (A) = \operatorname{supp} \rho = \operatorname{supp} (\sum_{i=1}^{m} \rho_m)$, and $\operatorname{cl}_M (A_i) = \operatorname{supp} \rho_i$. Therefore,

$$\operatorname{supp} \left(\sum_{i=1}^{m} \rho_i \right) \subseteq \bigcup_{i=1}^{m} \operatorname{supp} \rho_i .$$

∎

Proposition 8.4

Let M be a compact manifold and $\{U_\alpha\}_{\alpha\in A}$ and open cover of M. There exists a C^∞ partition of unity $\{\rho_\alpha\}_{\alpha\in A}$ subordinate to $\{U_\alpha\}_{\alpha\in A}$.

Proof. Let $q \in M$, find an open set U_α containing q fro the given cover, and let ψ_q be a C^∞ bump function at q supported in U_α. Since $\psi_q(q) > 0$, we can find a neighborhood W_q of q such that $\psi_q(p) > 0$ for every $p \in W_q$.

By the compactness of M, the open cover $\{W_q \mid q \in M\}$ has a finite subcover, say $\{W_{q_1}, \ldots, W_{q_m}\}$. Let $\psi_{q_1}, \ldots, \psi_{q_m}$ be the corresponding bump functions (eacg of these functions is supported in some U_α). Since $\{W_{q_1}, \ldots, W_{q_m}\}$ is a finite open cover of M, for any point $q \in M$, $q \in W_{q_i}$ for some $i \in \{1, 2, \ldots, m\}$. Hence,

$$\psi(q) := \sum_{i=1}^{m} \psi_{q_i}(q) > 0\,.$$

Define $\varphi_i = \frac{\psi_{q_i}}{\psi}$. In other words, for $q \in M$,

$$\varphi_i(q) = \frac{\psi_{q_i}(q)}{\psi(q)}\,.$$

This division is well-defined, since $\psi(q) > 0$.

$$\sum_{i=1}^{m} \varphi_i(q) = \sum_{i=1}^{m} \frac{\psi_{q_i}(q)}{\psi(q)} = \frac{\sum\limits_{i=1}^{m} \psi_{q_i}(q)}{\psi(q)} = \frac{\psi(q)}{\psi(q)} = 1\,.$$

Since $\psi > 0$, $\varphi_i(q) \neq 0$ if and only if $\psi_{q_i}(q)$. Therefore,

$$\operatorname{supp}\varphi_i \subseteq \operatorname{supp}\psi_{q_i} \subseteq U_\alpha \text{ for some } \alpha \in A\,.$$

Hence, $\{\varphi_i\}_{i=1}^{m}$ is a partition of unity such that for every $i \in \{1, 2, \ldots, m\}$, $\operatorname{supp}\varphi_i \subseteq U_\alpha$ for some $\alpha \in A$.

The next step is to make the index set of the partition of unity the same as that of the open cover. Now we shall define a function $\tau : \{1, 2, \ldots, m\} \to A$. For each $i \in \{1, 2, \ldots, m\}$, we define $\tau(i)$ to be an index $\alpha \in A$ such that

$$\operatorname{supp}\varphi_i \subseteq U_\alpha\,.$$

Note that, there might be multiple choices for $\tau(i)$. In other words, it might be the case that $\operatorname{supp}\varphi_i$ is contained in both U_α and U_β. In that case, we define $\tau(i)$ to be either of α or β.

Now, we group the collection of functions $\{\varphi_i\}$ into subcollections according to $\tau(i)$, i.e., all the φ_i's will be in the same subcollection if $\tau(i) = \alpha$ for some $\alpha \in A$. Let us define ρ_α as

$$\rho_\alpha = \sum_{\tau(i)=\alpha} \varphi_i\,.$$

If there is no $i \in \{1, 2, \ldots, m\}$ for which $\tau(i) = \alpha$, we simply define $\rho_\alpha = 0$. Then we have,

$$\sum_{\alpha\in A} \rho_\alpha = \sum_{\alpha\in A} \sum_{\tau(i)=\alpha} \varphi_i = \sum_{i=1}^{m} \varphi_i = 1\,.$$

If $\rho_\alpha = 0$, then $\operatorname{supp}\rho_\alpha = \varnothing \subseteq U_\alpha$. Otherwise, by Lemma 8.3,

$$\operatorname{supp}(\rho_\alpha) = \operatorname{supp}\left(\sum_{\tau(i)=\alpha} \varphi_i\right) \subseteq \bigcup_{\tau(i)=\alpha} \operatorname{supp}\varphi_i \subseteq \bigcup_{\tau(i)=\alpha} U_\alpha = U_\alpha\,.$$

τ is a map from a finite set $\{1, 2, \ldots, m\}$ to A. Therefore, $\operatorname{im}\tau$ is a finite subset of A. This means that there are only finitely many α for which ρ_α is not identically 0 on all of M. Hence, there are only finitely many α's for which $\operatorname{supp}\rho_\alpha \neq \varnothing$. Since there are only finitely many nonempty sets in the collection $\{\operatorname{supp}\rho_\alpha\}_{\alpha\in A}$, the collection is locally finite. Therefore, $\{\rho_\alpha\}_{\alpha\in A}$ is a partition of unity subordinate to the open cover $\{U_\alpha\}_{\alpha\in A}$. ∎

Now we shall state a generalization of Proposition 8.4 without proof.

Theorem 8.5 (Existence of a C^∞ partition of unity)
Let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of a manifold M.

(i) There is a C^∞ partition of unity $\{\varphi_k\}_{k=1}^\infty$ with every φ_k having compact support, such that for each k, $\operatorname{supp}\varphi_k \subseteq U_\alpha$ for some $\alpha \in A$.

(ii) If we relax the condition of having compact supprt, then there is a C^∞ partition of unity $\{\rho_\alpha\}_{\alpha\in A}$ subordinate to the open cover $\{U_\alpha\}_{\alpha\in A}$.

9 Vector Field

§9.1 Smoothness of a Vector Field

Recall from the definition of Vector Field that a vector field X on a manifold M is smooth if the map $X : M \to TM$ is smooth as a section of the tangent bundle $\pi : TM \to M$[1]. In a coordinate chart $(U, \varphi) = (U, x^1, \ldots, x^n)$ on M about p, the value of the vector field X at $p \in U$ is a linear combination

$$X_p = \sum_{i=1}^{n} a^i(p) \left.\frac{\partial}{\partial x^i}\right|_p .$$

As p varies in U, the coefficients a^i become functions on U. We've seen that the chart (U, φ) on the manifold M induces a chart $(TU, \widetilde{\varphi})$ on the tangent bundle TM. Here,

$$TU = \bigsqcup_{p \in U} T_pU\,, \text{ and } \widetilde{\varphi} : TU \to \varphi(U) \times \mathbb{R}^n$$

is a homeomorphism. The homeomorphism $\widetilde{\varphi}$ is given by

$$(p, \mathbf{v}_p) \mapsto \Big(x^1(p), \ldots, x^n(p), c^1(\mathbf{v}_p), \ldots, c^n(\mathbf{v}_p)\Big)\,,$$

where $\mathbf{v}_p \in T_pM$ with $\mathbf{v}_p = \sum\limits_{i=1}^{n} c^i(\mathbf{v}_p) \left.\frac{\partial}{\partial x^i}\right|_p$. Therefore,

$$X_p = \sum_{i=1}^{n} a^i(p) \left.\frac{\partial}{\partial x^i}\right|_p = \sum_{i=1}^{n} c^i(X_p) \left.\frac{\partial}{\partial x^i}\right|_p .$$

Writing X_p as $X(p)$ and equating the coefficients, we get

$$a^i(p) = \Big(c^i \circ X\Big)(p) \implies a^i = c^i \circ X$$

as functions on U. Since $X(U) \subseteq TU$, and $a^i = c^i \circ X$, one finds that c^i's are smooth functions on TU. Thus, if X is smooth and $(U, x^1, \ldots, x^n)$ is any chart on M, then the coefficients a^i of $X = \sum\limits_{i=1}^{n} a^i \frac{\partial}{\partial x^i}$ relative to the frame $\frac{\partial}{\partial x^i}$ are smooth functions on U (By Proposition 7.9). The converse is also true as provided by the following lemma.

Lemma 9.1 (Smoothness of a vector field on a chart)

Let $(U, \varphi) = (U, x^1, \ldots, x^n)$ be a chart on a manifold M. A vector field $X = \sum\limits_{i=1}^{n} a^i \frac{\partial}{\partial x^i}$ on U is smooth (i.e., the section $X : U \to \pi^{-1}(U) = TU$ of the tangent bundle $\pi : TM \to M$ over U is smooth) if and only if the coefficient functions a^i are all smooth on U.

Proof. This lemma is a special case of Proposition 7.9, where we take E to be the tangent bundle TM of M and $\{s_i\}_{i=1}^{r}$, the C^∞ frame for E over U to be the coordinate vector fields $\left\{\frac{\partial}{\partial x^i}\right\}_{i=1}^{n}$. ■

Proposition 9.2 (Smoothness of a vector field in terms of coefficients)

Let X be a vector field on a manifold M. The following are equivalent:

(i) The vector field X is smooth on M.

(ii) The manifold M has an atlas $\mathscr{U}$ such that on any chart $(U, \varphi) = (U, x^1, \ldots, x^n)$ of $\mathscr{U}$, the

[1] As discussed in Remark 7.3, we are considering $TM = \bigcup\limits_{p \in M} T_pM$.

coefficients a^i of $X = \sum_{i=1}^{n} a^i \frac{\partial}{\partial x^i}$ relative to the frame $\frac{\partial}{\partial x^i}$ are all smooth.

(iii) On any chart $(U, \varphi) = (U, x^1, \ldots, x^n)$ on the manifold M, the coefficients a^i of $X = \sum_{i=1}^{n} a^i \frac{\partial}{\partial x^i}$ relative to the frame $\frac{\partial}{\partial x^i}$ are all smooth.

Proof. **(ii)⇒(i):** Since (ii) holds, X is smooth on every chart (U, φ) of the atlas $\mathscr{U}$. Since $\mathscr{U}$ covers all of M, X is smooth on M.

(i)⇒(iii): If X is smooth on M, then it is smooth on U for any chart (U, φ). Then Lemma 9.1 implies (iii).

(iii)⇒(ii): Just take $\mathscr{U}$ to be the maximal atlas of M. ■

A vector field X on a manifold M induces a linear map on the algebra $C^\infty(M)$ of C^∞ functions on M: for $f \in C^\infty(M)$, define Xf to be the function

$$(Xf)(p) := X_p f\,, \qquad p \in M\,.$$

Proposition 9.3 (Smoothness of a vector field in terms of functions)

A vector field X on M is smooth if and only if for every smooth function f on M, the function Xf is smooth on M.

Proof. **(⇒):** Suppose X is smooth and $f \in C^\infty(M)$. By Proposition 9.2, on any chart $(U, x^1, \ldots, x^n)$ on M, the coefficients a^i of $X = \sum_{i=1}^{n} a^i \frac{\partial}{\partial x^i}$ relative to the frame $\frac{\partial}{\partial x^i}$ are all smooth.

$$Xf = \sum_{i=1}^{n} a^i \frac{\partial f}{\partial x^i}$$

$a^i \frac{\partial f}{\partial x^i}$ is smooth on U as the product of two smooth functions. Hence, their sum Xf is smooth on U. Since M can be covered by charts, Xf is C^∞ on M.

(⇐): Let $(U, \varphi) = (U, x^1, \ldots, x^n)$ be any chart on M. Suppose $X = \sum_{i=1}^{n} a^i \frac{\partial}{\partial x^i}$ is C^∞ on U and $p \in U$. Each x^k is a smooth function on U. Hence, by Proposition 8.2, each x^k can be extended to a C^∞ function $\tilde{x}^k$ on M that agrees with x^k in a neighborhood V of p that is contained in U. Xf is smooth for every $f \in C^\infty(M)$, taking $f = \tilde{x}^k$ we get that $X\tilde{x}^k$ is also smooth on M. Now, on $V \subseteq U$,

$$X\tilde{x}^k = \left(\sum_{i=1}^{n} a^i \frac{\partial}{\partial x^i}\right) \tilde{x}^k = \left(\sum_{i=1}^{n} a^i \frac{\partial}{\partial x^i}\right) x^k = a^k\,.$$

Therefore a^k is smooth at $p \in U$. Since p is an arbitrary point in U, a^k is smooth on U for any chart (U, φ). Therefore, by Proposition 9.2, X is smooth on M. ■

Proposition 9.4 (Smooth extension of a vector field)

Suppose X is a C^∞ vector field defined on a neighborhood U of a point p in a manifold M. Then there is a C^∞ vector field $\tilde{X}$ on X that agrees with X on some possibly smaller neighborhood of p, say $V \subseteq U$.

Proof. Choose a C^∞ bump function $\rho : M \to \mathbb{R}$ supported in U that is identically 1 in a neighborhood V of p. Define

$$\tilde{X}(q) = \begin{cases} \rho(q)\, X(q) & \text{for } q \in U \\ \mathbf{0}_{T_q M} & \text{for } q \notin U \end{cases},$$

where $\mathbf{0}_{T_qM}$ is the zero vector of T_qM. By the definition of $\widetilde{X}$, it agrees with X on V. By Proposition 7.7(ii), $\widetilde{X}$ is smooth on U. Now, let $q \notin U$. We want to show that $\widetilde{X}$ is smooth at q.

Since $\operatorname{supp}\rho \subseteq U$, $q \notin U$ implies $q \in M \setminus U \subseteq M \setminus \operatorname{supp}\rho$. Since $\operatorname{supp}\rho$ is closed, $M \setminus \operatorname{supp}\rho$ is open. Hence, we can find a coordinate chart (W, φ) about q such that $W \subseteq M \setminus \operatorname{supp}\rho$. Then, for $r \in W$, $\widetilde{X}(r) = \mathbf{0}_{T_rM}$. Also, $(TW, \widetilde{\varphi})$ is a chart on TM about $\mathbf{0}_{T_rM}$.

$$\left(\widetilde{\varphi} \circ \widetilde{X}\right)(r) = (\varphi(r), \underbrace{0, 0, \ldots, 0}_{n \text{ 0s}}) .$$

φ is smooth. Therefore, by Proposition 2.9, $\widetilde{X}$ is smooth on W. In particular, $\widetilde{X}$ is smooth at q. Since $q \notin U$ was arbitrary, $\widetilde{X}$ is smooth at every $q \notin U$. Therefore, $\widetilde{X}$ is smooth on all of M. ■

§9.2 Integral Curves

> **Definition 9.1** (Integral Curve)**.** Let X be a C^∞ vector field on M, and $p \in M$. An **integral curve** of X is a smooth curve $c : (a, b) \to M$ such that $c'(t) = X_{c(t)}$ for every $t \in (a, b)$.
>
> Usually, we assume that $0 \in (a, b)$. In this case, if $c(0) = p$, then we say that c is the integral curve starting at p, and call p the initial point of c.

To show the dependence of an integral curve on the initial point p, one also writes $c_t(p)$ instead of $c(t)$.

> **Definition 9.2.** An integral curve is **maximal** if its domain can't be extended to a larger domain.

Example 9.1. Recall the vector field $X_{(x,y)} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ on $\mathbb{R}^2$ (Example 7.3). We will find an integral curve $c(t)$ of X starting at the point $(1, 0) \in \mathbb{R}^2$. The condition for $c(t) = (x(t), y(t))$ to be an integral curve of X is

$$c'(t) = X_{c(t)} \implies \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} -y(t) \\ x(t) \end{bmatrix} .$$

We need to solve the system of first order ODEs

$$\dot{x}(t) = -y(t) \ , \ \dot{y}(t) = x(t)$$

with the initial conditoin $(x(0), y(0)) = (1, 0)$.

$$y(t) = -\dot{x}(t) \implies \dot{y}(t) = -\ddot{x}(t) \implies x(t) = -\ddot{x}(t) \ .$$

It's well-known that the general solution to this equation is

$$x(t) = A\cos t + B\sin t \, .$$

Hence, $y(t) = -\dot{x}(t) = A\sin t - B\cos t$. By plugging in the initial condition $(x(0), y(0)) = (1, 0)$, one obtains

$$x(0) = A = 1 \text{ and } y(0) = -B = 0 \, .$$

So, the integral curve starting at $(1, 0)$ is

$$c(t) \equiv (x(t), y(t)) = (\cos t, \sin t) \ ,$$

which parametrizes the unit circle.

More generally, if the initial point of the integral curve, corresponding to $t = 0$, is $\mathbf{p} = (x_0, y_0)$, then

$$x_0 = x(0) = A \text{ and } y_0 = y(0) = -B \, .$$

In that case, a general solution for $x(t)$ and $y(t)$ would be

$$x(t) = x_0\cos t - y_0\sin t \, , \ y(t) = x_0\sin t + y_0\cos t \, .$$

This can be written in matrix notation as

$$c(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \mathbf{p},$$

where $\mathbf{p} \in \mathbb{R}^2$ is the given initial point. This shows that the integral curve $c(t)$ of X starting at $\mathbf{p} \in \mathbb{R}^2$, i.e. $c(0) = \mathbf{p}$ can be obtained by rotating the point $\mathbf{p} \in \mathbb{R}^2$ counterclockwise about the origin through an angle t. Notice that

$$c_s(c_t(\mathbf{p})) = c_{s+t}(\mathbf{p}),$$

since a rotation through an angle t followed by a rotation through an angle s is the same as a rotation through an angle $s+t$. Also, notice that for each $t \in \mathbb{R}$, $c_t : \mathbb{R}^2 \to \mathbb{R}^2$ is a diffeomorphism with inverse c_{-t}. Indeed, for a fixed $t_0 \in \mathbb{R}$,

$$c_{t_0} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos t_0 & -\sin t_0 \\ \sin t_0 & \cos t_0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos t_0 - y \sin t_0 \\ x \sin t_0 + y \cos t_0 \end{bmatrix}.$$

$c_{t_0} : \mathbb{R}^2 \to \mathbb{R}^2$ is easily seen to be a smooth map, with the inverse $c_{-t_0} : \mathbb{R}^2 \to \mathbb{R}^2$, which is also smooth.

Diff (M) stands for the group of diffeomorphisms of a manifold M with itself, with the group operation being composition. A homomorphism $c : \mathbb{R} \to \text{Diff}(M)$ is called a **1-parameter group of diffeomorphisms** of M. In Example 9.1, the integral curves of the vector field $X_{(x,y)} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ on $\mathbb{R}^2$ gives rise to a 1-parameter group of diffeomorphisms of $\mathbb{R}^2$.

Exercise 9.1. Let $X = x^2 \frac{\mathrm{d}}{\mathrm{d}x}$ be a vector field on $\mathbb{R}$. Find the maximal integral curve of X starting at $x = 2$.

Solution. Denote the integral curve by $x(t)$. Then

$$x'(t) = X_{x(t)} \implies \dot{x}(t) \left.\frac{\mathrm{d}}{\mathrm{d}x}\right|_{x(t)} = x^2 \left.\frac{\mathrm{d}}{\mathrm{d}x}\right|_{x(t)}.$$

Therefore, it follows that

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) = x^2 \implies \frac{\mathrm{d}x}{x^2} = \mathrm{d}t \implies -\frac{1}{x} = t + C.$$

Now, using the condition that $x(0) = 2$, we get

$$-\frac{1}{2} = 0 + C \implies -\frac{1}{x} = t - \frac{1}{2} = \frac{2t-1}{2} \implies \boxed{x = \frac{2}{1-2t}}.$$

The maximal interval containing 0 on which $x(t)$ is defined is $\left(-\infty, \frac{1}{2}\right)$. This example exhibits the fact that it may not be possible to extend the domain of definition of an integral curce to the entire real line. ■

§9.3 Local Flows

In the previous two examples, we've seen that locally, finding an integral curve requires solving a system of first order ODEs with initial conditions. In general, if X is a smooth vector field on M, to find an integral curve $c(t)$ of X starting at $p \in M$, choose first a coordinate chart $(U, \varphi) = (U, x^1, \ldots, x^n)$ about p. In local coordinates,

$$X_{c(t)} = \sum_{i=1}^n a^i(c(t)) \left.\frac{\partial}{\partial x^i}\right|_{c(t)}.$$

By Proposition 4.9,

$$c'(t) = \sum_{i=1}^n \dot{c}^i(t) \left.\frac{\partial}{\partial x^i}\right|_{c(t)},$$

where $c^i = x^i \circ c$ is the i-th component of $c(t)$ in the chart (U, φ). The condition $X_{c(t)} = c'(t)$ is, thus, equivalent to

$$\dot{c}^i(t) = a^i(c(t))$$

for $i = 1, \ldots, n$. This is a system of ODSs. The initial condition $c(0) = p$ translates to

$$\left(x^i \circ c\right)(0) = x^i(p) \implies \left(c^1(0), c^2(0), \ldots, c^n(0)\right) = \left(p^1, p^2, \ldots, p^n\right),$$

where $p^i = x^i(p)$. By an existence and uniqueness theorem from the theory of ODE, such a system has a unique solution in the following sense.

Theorem 9.5

Let V be an open subset of $\mathbb{R}^n$, $p_0 \in V$, and $f : V \to \mathbb{R}^n$ a C^∞ map. Then the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(y), \quad y(0) = p_0,$$

has a unique C^∞ solution $y : (a(p_0), b(p_0)) \to V$, where $(a(p_0), b(p_0))$ is the maximal interval containing 0 on which y is defined.

Remark 9.1. The uniqueness of the solution means that if $z : (\delta, \varepsilon) \to V$ satisfies the same ODE

$$\frac{\mathrm{d}z}{\mathrm{d}t} = f(z), \quad z(0) = p_0,$$

then the domain (δ, ε) of the equation of z is a subset of $(a(p_0), b(p_0))$, and $z(t) = y(t)$ on the interval (δ, ε).

The map $y : (a(p_0), b(p_0)) \to V$ can actually be thought of as a map with two arguments: t and q, and the condition for y to be an integral curve starting at the point q is

$$\frac{\mathrm{d}y}{\mathrm{d}t}(t, q) = f(y(t, q)), \quad y(0, q) = q.$$

Theorem 9.6 (Smooth dependece of solution on the initial point)

Let V be an open subset of $\mathbb{R}^n$ and $f : V \to \mathbb{R}^n$ a C^∞ map on V. For each point $p_0 \in V$, there are a neighborhood W of p_0 in V and a number $\varepsilon > 0$, and a C^∞ map

$$y : (-\varepsilon, \varepsilon) \times W \to V$$

such that

$$\frac{\mathrm{d}y}{\mathrm{d}t}(t, q) = f(y(t, q)), \quad y(0, q) = q,$$

for all $(t, q) \in (-\varepsilon, \varepsilon) \times W$.

It follows from Theorem 9.6 that if X is any C^∞ vector field on a chart (U, φ) and $p \in U$, then there are a neighborhood W of p in U, an $\varepsilon > 0$, and a C^∞ map

$$F : (-\varepsilon, \varepsilon) \times W \to U$$

such that for every $q \in W$, the map $F(t, q)$ is an integral curve of X starting at q. In particular, $F(0, q) = q$. We usually write $F_t(q)$ for $F(t, q)$.

Suppose $s, t \in (-\varepsilon, \varepsilon)$ are such that $F_t(F_s(q))$ and $F_{t+s}(q)$ are defined. Then $F_t(F_s(q))$ as a map of argument t is an integral curve of X starting at $F_s(q)$ (due to $t = 0$). By the uniqueness of the integral curve for a given vector field starting at a point,

$$F_t(F_s(q)) = F_{t+s}(q).$$

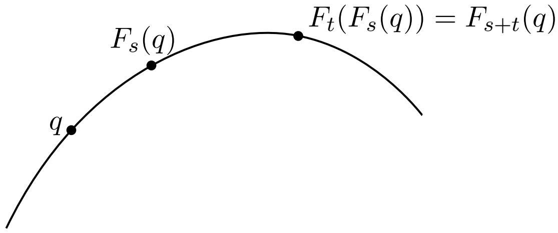


Figure 9.1: The flow line through q of a local flow.

The map F is called a **local flow** generated by the vector field X. For each $q \in U$, the map $F_t(q)$ of t is called the **flow line** of the local flow. Each flow line is an integral curve of X. If a local flow F is defined on $\mathbb{R} \times M$, then it's called a **global flow**.

Every smooth vector field has a local flow about any point, but not necessarily a global flow. A vector field having a global flow is called a **complete** vector field.

II

Part 2

10 Differential Forms on $\mathbb{R}^n$

Given an open set $U \subseteq \mathbb{R}^n$ and $p \in U$, T_pU is the set of tangent vectors at $p \in U$ is identified with the point derivations of C_p^∞ (germs of smooth functions at p), i.e. a tangent vector $X_p \in T_pU$ is a map $X_p : C_p^\infty \to \mathbb{R}$ such that X_p is $\mathbb{R}$-linear:

$$X_p(\alpha f + g) = \alpha(X_p f) + X_p g; \tag{10.1}$$

and satisfies the Leibniz condition:

$$X_p(fg) = (X_p f)\, g(p) + f(p)\,(X_p g)\,.$$

In contrast to the notion of point derivation, there is this notion of derivation of aa algebra. If X is a C^∞ vector field on an open subset $U \subseteq \mathbb{R}^n$, i.e. $X \in \mathfrak{X}(U)$, and f is a C^∞ function on U, i.e. $f \in C^\infty(U)$, then $Xf \in C^\infty(U)$ defined by

$$(Xf)(p) = X_p f.$$

Remember that f in (10.1) and (10) is a representative of an equivalence class, the equivalence class of germs of C^∞ functions at $p \in U$. These equivalence classes constitute $C_p^\infty(U)$. It is of course an $\mathbb{R}$-algebra. While in (10), $f \in C^\infty(U)$, the algebra of C^∞ functions on U with no reference of p whatsoever.

From the discussion above, a C^∞ vector field X gives rise to an $\mathbb{R}$-linear map $C^\infty(U) \to C^\infty(U)$ by $f \mapsto Xf$ that additionally has to satisfy the following Leibniz condition:

$$X(fg) = (Xf)\, g + f\,(Xg)\,. \tag{10.2}$$

Note that a derivation at p is not a derivation of the algebra C_p^∞. A derivation at p is a map from $C_p^\infty \to \mathbb{R}$ that satisfies (10), while a derivation of the algebra C_p^∞ is supposed to be a map from C_p^∞ to itself obeying Leibniz condition.

§10.1 1-form

From any C^∞ function $f : U \to \mathbb{R}$, one can construct a 1-form (dual notion of C^∞ vector field) $\mathrm{d}f$, the restriction of which to a given point $p \in U$ yields a covector $(\mathrm{d}f)_p \in T_p^*U$, the dual space of T_pU, in the following way:

$$(\mathrm{d}f)_p(X_p) = X_p f. \tag{10.3}$$

Proposition 10.1

If $x^1, x^2, \ldots, x^n$ are the standard coordinates on $\mathbb{R}^n$, then at each point $p \in \mathbb{R}^n$,

$$\left\{ \left(\mathrm{d}x^1\right)_p, \left(\mathrm{d}x^2\right)_p, \ldots, \left(\mathrm{d}x^n\right)_p \right\}$$

is the basis for the cotangent space $T_p^*\mathbb{R}^n$ dual to the basis $\left\{ \frac{\partial}{\partial x^1}\Big|_p, \ldots, \frac{\partial}{\partial x^n}\Big|_p \right\}$ for the tangent space $T_p\mathbb{R}^n$.

Proof. $\left(\mathrm{d}x^i\right)_p : T_p^*\mathbb{R}^n \to \mathbb{R}$ is a linear map for each i. Now,

$$\left(\mathrm{d}x^i\right)_p \left(\frac{\partial}{\partial x^j}\Big|_p \right) = \frac{\partial}{\partial x^j}\Big|_p \left(x^i\right) = \delta^i{}_j. \tag{10.4}$$

Therefore, $\left\{(\mathrm{d}x^1)_p, (\mathrm{d}x^2)_p, \ldots, (\mathrm{d}x^n)_p\right\}$ is the basis of $T_p^*\mathbb{R}^n$ dual to the basis $\left\{\frac{\partial}{\partial x^1}\Big|_p, \ldots, \frac{\partial}{\partial x^n}\Big|_p\right\}$ for $T_p\mathbb{R}^n$. ■

If ω is a 1-form on an open subset $U \subseteq \mathbb{R}^n$, then by Proposition 10.1, there is a linear combination

$$\omega_p = \sum_{i=1}^{n} a_i(p) \left(\mathrm{d}x^i\right)_p, \tag{10.5}$$

for some $a_i(p) \in \mathbb{R}$. As p varies over U, the coefficients a_i become functions on U, and we may write

$$\omega = \sum_{i=1}^{n} a_i \mathrm{d}x^i. \tag{10.6}$$

The 1-form ω is said to be C^∞ on U if the coefficient functions a_i are all C^∞ functions on U.

Proposition 10.2 (The differential in terms of coordinates)
If $f : U \to \mathbb{R}$ is a C^∞ function on an open set $U \subseteq \mathbb{R}^n$, then

$$\mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} \mathrm{d}x^i.$$

Proof. By Proposition 10.1, at each point $p \in U$,

$$(\mathrm{d}f)_p = \sum_{i=1}^{n} a_i(p) \left(\mathrm{d}x^i\right)_p, \tag{10.7}$$

for some constants $a_i(p)$ depending on p. Thus

$$\mathrm{d}f = \sum_{i=1}^{n} a_i \ \mathrm{d}x^i, \tag{10.8}$$

for some functions a_i on U. To evaluate a_j, apply both sides of (10.8) to the coordinate vector field $\frac{\partial}{\partial x^j}$:

$$\mathrm{d}f\left(\frac{\partial}{\partial x^j}\right) = \sum_{i=1}^{n} a_i \ \mathrm{d}x^i \left(\frac{\partial}{\partial x^j}\right) = \sum_{i=1}^{n} a_i {\delta^i}_j = a_j. \tag{10.9}$$

On the other hand, using $(\mathrm{d}f)_p (X_p) = X_p f = (Xf)(p)$, we get $(\mathrm{d}f)(X) = Xf$. So

$$\mathrm{d}f\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial f}{\partial x^j}. \tag{10.10}$$

Therefore, $a_j = \frac{\partial f}{\partial x^j}$. Hence,

$$\mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} \ \mathrm{d}x^i. \tag{10.11}$$

■

(10.11) tells us that $\mathrm{d}f$ will be a C^∞ 1-form if $\frac{\partial f}{\partial x^i}$ is C^∞ on U. Hence, it is sufficient to have f as a C^∞ function on U in order to have $\mathrm{d}f$ as a C^∞ 1-form.

§10.2 Differential k-forms

A differential form ω of degree k (or a k-form) on an open subset $U \subseteq \mathbb{R}^n$ is a map that assigns to each point $p \in U$, an alternating k-linear function on the tangent space $T_p\mathbb{R}^n$, i.e.

$$\omega_p \in A_k\left(T_p\mathbb{R}^n\right).$$

By Proposition C.15, a basis for $A_k\left(T_p\mathbb{R}^n\right)$ is

$$\left(\mathrm{d}x^I\right)_p = \left(\mathrm{d}x^{i_1}\right)_p \wedge \left(\mathrm{d}x^{i_2}\right)_p \wedge \cdots \wedge \left(\mathrm{d}x^{i_k}\right)_p,$$

where $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. Therefore, at each point $p \in U$, ω_p is a linear combination

$$\omega_p = \sum_I a_I(p)\left(\mathrm{d}x^I\right)_p, \tag{10.12}$$

and a k-form ω on U is a linear combination

$$\omega = \sum_I a_I \ \mathrm{d}x^I, \tag{10.13}$$

with function coefficients $a_I : U \to \mathbb{R}$. We say that a k-form ω is **smooth** on U if all the coeffieicnts a_I are C^∞ functions on U.

Denote by $\Omega^k(U)$ the vector space of C^∞ k-forms on U. A 0-form on U assigns to each point $p \in U$ an element of $A_0\left(T_p\mathbb{R}^n\right) = \mathbb{R}$. Thus a 0-form on U is simply a real-valued function on U, and $\Omega^0(U) = C^\infty(U)$.

Since one can multiply a C^∞ k-form by a C^∞ function on U from the left, the set $\Omega^k(U)$ of C^∞ k-forms on U is both a real vector space and a $C^\infty(U)$-module. With the wedge product as multiplication, the direct sum

$$\Omega^*(U) = \bigoplus_{k=0}^{n} \Omega^k(U)$$

becomes an algebra over $\mathbb{R}$ as well as a module over $C^\infty(U)$. As an algebra, it is anticommutative and associative.

Example 10.1. Let x, y, z be the coordinates on $\mathbb{R}^3$. The C^∞ 1-forms on $\mathbb{R}^3$ are

$$P(x,y,z)\ \mathrm{d}x + Q(x,y,z)\ \mathrm{d}y + R(x,y,z)\ \mathrm{d}z,$$

where P, Q, R range over all C^∞ functions on $\mathbb{R}^3$. The C^∞ 2-forms are

$$A(x,y,z)\ \mathrm{d}y \wedge \mathrm{d}z + B(x,y,z)\ \mathrm{d}x \wedge \mathrm{d}z + C(x,y,z)\ \mathrm{d}x \wedge \mathrm{d}y;$$

and the C^∞ 1-forms are

$$a(x,y,z)\ \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z.$$

Example 10.2 (A basis for 3-covectors)**.** Let x^1, x^2, x^3, x^4 be the standard coordinates on $\mathbb{R}^4$, and $p \in \mathbb{R}^4$. A basis for $A_3\left(T_p\mathbb{R}^4\right)$ is

$$\left\{\left(\mathrm{d}x^1\right)_p \wedge \left(\mathrm{d}x^2\right)_p \wedge \left(\mathrm{d}x^3\right)_p, \left(\mathrm{d}x^1\right)_p \wedge \left(\mathrm{d}x^2\right)_p \wedge \left(\mathrm{d}x^4\right)_p,\right.$$
$$\left.\left(\mathrm{d}x^1\right)_p \wedge \left(\mathrm{d}x^3\right)_p \wedge \left(\mathrm{d}x^4\right)_p, \left(\mathrm{d}x^2\right)_p \wedge \left(\mathrm{d}x^3\right)_p \wedge \left(\mathrm{d}x^4\right)_p\right\}.$$

So $\dim\left(A_3\left(T_p\mathbb{R}^n\right)\right) = 4$.

§10.3 Exterior Derivative

Before defining exterior derivative of a C^∞ k-form on an open subset $U \subseteq \mathbb{R}^n$, we first define it on 0-forms. The exterior derivative of a C^∞ function $f \in C^\infty(U)$ is its differential:

$$\mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} \mathrm{d}x^i \in \Omega^1(U).$$

Definition 10.1 (Exterior Derivative)**.** If $\omega = \sum_I a_I \ \mathrm{d}x^I \in \omega^K(U)$, then its exterior derivative is defined as follows:

$$\mathrm{d}\omega = \sum_I \mathrm{d}a_I \wedge \mathrm{d}x^I = \sum_I \left(\sum_{j=1}^{n} \frac{\partial a_I}{\partial x^j} \ \mathrm{d}x^j \right) \wedge \mathrm{d}x^I \in \Omega^{k+1}(U). \tag{10.14}$$

Example 10.3. Let ω be the 1 form $f\mathrm{d}x + g\mathrm{d}y$ on $\mathbb{R}^2$, where f and g are C^∞ functions on $\mathbb{R}^2$. Let us write $f_x = \frac{\partial f}{\partial x}$ and $f_y = \frac{\partial f}{\partial y}$. Then

$$\begin{aligned} \mathrm{d}\omega &= \mathrm{d}f \wedge \mathrm{d}x + \mathrm{d}g \wedge \mathrm{d}y \\ &= (f_x \mathrm{d}x + f_y \mathrm{d}y) \wedge \mathrm{d}x + (g_x \mathrm{d}x + g_y \mathrm{d}y) \wedge \mathrm{d}y \\ &= -f_y \ \mathrm{d}x \wedge \mathrm{d}y + g_x \ \mathrm{d}x \wedge \mathrm{d}y \\ &= (g_x - f_y)\, \mathrm{d}x \wedge \mathrm{d}y. \end{aligned}$$

Definition 10.2 (Graded Algebra)**.** An algebra A over a field $\mathbb{K}$ is said to be **graded** if it can be written as a direct sum

$$A = \bigoplus_{k=0}^{\infty} A^k$$

of vector spaces over $\mathbb{K}$ so that the multiplication map sends $A^k \times A^l$ to A^{k+l}.

The notation $A = \bigoplus_{k=0}^{\infty} A^k$ means that each element of A is uniquely a **finite sum**

$$a = a_{i_1} + a_{i_2} + \cdots + a_{i_m},$$

where $a_{i_j} \in A^{i_j}$.

Example 10.4. The polynomial algebra

$$\mathbb{R}[x, y] = \bigoplus_{k=0}^{\infty} A^k$$

with A^k being the vector space of homogenous polynomials of degree k in x and y. Observe that the 0 polynomial is trivially homogenous of any degree, and hence belongs to A^k for all $k \geq 0$. Multiplication of degree k homogenous polynomial with a degree l homogenous polynomial in x and y will result in a homogenous polynomial of degree $k + l$ in x and y.

Example 10.5. The algebra $\Omega^*(U)$ of C^∞ differential forms on U is also graded by the degree of differential forms. Each $\Omega^k(U)$ is a vector space. Multiplication of differential forms is defined by wedge product between them. The wedge product of a degree k differential form on U with a degree l differential form results in a degree $k + l$ differential form.

Definition 10.3 (Anti-derivation). Let $A = \bigoplus_{k=0}^{\infty} A^k$ be a graded algebra over a field $\mathbb{K}$. An **anti-derivation** of the graded algebra A is a $\mathbb{K}$-linear map $D : A \to A$ such that for $\omega \in A^k$ and $\tau \in A^l$, one has

$$D(\omega\tau) = (D\omega)\tau + (-1)^k \omega (D\tau). \tag{10.15}$$

If the antiderivation D sends $\omega \in A^k$ to $D\omega \in A^{k+m}$, we say that it is an antiderivation of degree m.

Proposition 10.3 (i) The exterior derivative $\mathrm{d} : \Omega^*(U) \to \Omega^*(U)$ is an antiderivation of degree 1:

$$\mathrm{d}(\omega \wedge \tau) = (\mathrm{d}\omega) \wedge \tau + (-1)^{\deg \omega} \omega \wedge \mathrm{d}\tau. \tag{10.16}$$

(ii) $\mathrm{d}^2 = 0$.

(iii) If $f \in \Omega^0(U) = C^\infty(U)$ and $X \in \mathfrak{X}(U)$ (the space of C^∞ vector fields), then $(\mathrm{d}f)(X) = Xf$.

Proof. (i) Since the exterior derivative operator $\mathrm{d} : \Omega^k(U) \to \Omega^{k+1}(U)$ is linear, it suffices to check the equality (10.16) for $\omega = f\, \mathrm{d}x^I$ and $\tau = g\, \mathrm{d}x^J$ with $I = (i_1, \ldots, i_k)$ and $J = (j_1, \ldots, j_l)$ being strictly ascending multi-indices.

$$\begin{aligned}
\mathrm{d}(\omega \wedge \tau) &= \mathrm{d}\left(fg \mathrm{d}x^I \wedge \mathrm{d}x^J\right) \\
&= \sum_{i=1}^n \frac{\partial (fg)}{\partial x^i} \mathrm{d}x^i \wedge \mathrm{d}x^I \wedge \mathrm{d}x^J \\
&= \sum_{i=1}^n \frac{\partial f}{\partial x^i} \cdot g \mathrm{d}x^i \wedge \mathrm{d}x^I \wedge \mathrm{d}x^J + \sum_{i=1}^n f \cdot \frac{\partial g}{\partial x^i} \mathrm{d}x^i \wedge \mathrm{d}x^I \wedge \mathrm{d}x^J \\
&= \sum_{i=1}^n \frac{\partial f}{\partial x^i} \mathrm{d}x^i \wedge \mathrm{d}x^I \wedge g \mathrm{d}x^J + \sum_{i=1}^n \frac{\partial g}{\partial x^i} \mathrm{d}x^i \wedge f\, \mathrm{d}x^I \wedge \mathrm{d}x^J.
\end{aligned} \tag{10.17}$$

Now, in the second sum in (10.17), one has to push $\frac{\partial g}{\partial x^i} \mathrm{d}x^i$ through the k-fold wedge product $\mathrm{d}x^I$ and hence in the process picks out a sign $(-1)^k$. Therefore,

$$\sum_{i=1}^n \frac{\partial g}{\partial x^i} \mathrm{d}x^i \wedge f\, \mathrm{d}x^I \wedge \mathrm{d}x^J = (-1)^k f\, \mathrm{d}x^I \wedge \frac{\partial g}{\partial x^i} \mathrm{d}x^i \wedge \mathrm{d}x^J. \tag{10.18}$$

Now, observe that

$$\mathrm{d}\omega = \mathrm{d}\left(f\, \mathrm{d}x^I\right) = \sum_{i=1}^n \frac{\partial f}{\partial x^i} \mathrm{d}x^i \wedge \mathrm{d}x^I, \text{ and} \tag{10.19}$$

$$\mathrm{d}\tau = \mathrm{d}\left(g\, \mathrm{d}x^J\right) \sum_{i=1}^n \frac{\partial g}{\partial x^i} \mathrm{d}x^i \wedge \mathrm{d}x^J. \tag{10.20}$$

Therefore,

$$\mathrm{d}(\omega \wedge \tau) = \mathrm{d}\omega \wedge \tau + (-1)^k \omega \wedge \mathrm{d}\tau. \tag{10.21}$$

(ii) Again, by $\mathbb{R}$-linearity of d, it suffices to show that $\mathrm{d}^2\omega = 0$ for $\omega = f\mathrm{d}x^I$.

$$\begin{aligned}
\mathrm{d}^2\left(f \mathrm{d}x^I\right) &= \mathrm{d}\left(\sum_{i=1}^n \frac{\partial f}{\partial x^i} \mathrm{d}x^i \wedge \mathrm{d}x^I\right) \\
&= \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 f}{\partial x^j \partial x^i} \mathrm{d}x^j \wedge \mathrm{d}x^i \wedge \mathrm{d}x^I.
\end{aligned} \tag{10.22}$$

If $i = j$, then $\mathrm{d}x^j \wedge \mathrm{d}x^i = 0$. If $i \neq j$, then $\frac{\partial^2 f}{\partial x^j \partial x^i}$ is symmetric in i and j, but $\mathrm{d}x^j \wedge \mathrm{d}x^i$ is alternating in i and j. Therefore, the terms with $i \neq j$ pair up and cancel out.

(iii) Let $X = \sum_{i=1}^{n} a^i \frac{\partial}{\partial x^i}$. Then

$$
\begin{aligned}
(\mathrm{d}f)(X) &= \left(\sum_{j=1}^{n} \frac{\partial f}{\partial x^j} \mathrm{d}x^j\right) \left(\sum_{i=1}^{n} a^i \frac{\partial}{\partial x^i}\right) \\
&= \sum_{j=1}^{n} \sum_{i=1}^{n} a^i \frac{\partial f}{\partial x^j} \mathrm{d}x^j \left(\frac{\partial}{\partial x^i}\right) \\
&= \sum_{j=1}^{n} \sum_{i=1}^{n} a^i \frac{\partial f}{\partial x^j} \delta^j{}_i \\
&= \sum_{i=1}^{n} a^i \frac{\partial f}{\partial x^i} = Xf.
\end{aligned} \tag{10.23}
$$

■

Proposition 10.4 (Characterization of exterior derivative)

The 3 properties of Proposition 10.3 uniquely characterize exterior derivative on an open set $U \subseteq \mathbb{R}^n$. In other words, if $D : \Omega^*(U) \to \Omega^*(U)$ is an antiderivation of degree 1 such that $D^2 = 0$ and for $f \in C^\infty(U)$ and $X \in \mathfrak{X}(U)$, $(Df)(X) = Xf$, then $D = \mathrm{d}$.

Proof. Since every k-form on U is a sum of terms such as $f \mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2} \wedge \cdots \wedge \mathrm{d}x^{i_k}$, by linearity of d, it suffices to show that $D = \mathrm{d}$ on a k-form of this type. Applying property (iii) for $f = x^i$, one has

$$
Dx^i(X) = X\left(x^i\right).
$$

Writing $X = \sum_{j=1}^{n} a^j \frac{\partial}{\partial x^j}$, we get

$$
Dx^i\left(\sum_{j=1}^{n} a^j \frac{\partial}{\partial x^j}\right) = \sum_{j=1}^{n} a^j \frac{\partial}{\partial x^j}\left(x^i\right) = a^i = \mathrm{d}x^i\left(\sum_{j=1}^{n} a^j \frac{\partial}{\partial x^j}\right).
$$

Therefore,

$$
Dx^i = \mathrm{d}x^i. \tag{10.24}
$$

Now,

$$
\begin{aligned}
D\left(f \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}\right) &= D\left(f Dx^{i_1} \wedge \cdots \wedge Dx^{i_k}\right) \\
&= Df \wedge \left(Dx^{i_1} \wedge \cdots \wedge Dx^{i_k}\right) + (-1)^0 f D\left(Dx^{i_1} \wedge \cdots \wedge Dx^{i_k}\right).
\end{aligned} \tag{10.25}
$$

Now, since $\mathrm{d}f(X) = Xf = Df(X)$ for any $X \in \mathfrak{X}(U)$, $\mathrm{d}f = Df$. Furthermore, $D\left(Dx^{i_1}\right) = 0$, and

$$
\begin{aligned}
D\left(Dx^{i_1} \wedge \cdots \wedge Dx^{i_k}\right) &= D^2 x^{i_1} \wedge Dx^{i_2} \wedge \cdots \wedge Dx^{i_k} - Dx^{i_1} \wedge D\left(Dx^{i_2} \wedge \cdots \wedge Dx^{i_k}\right) \\
&= -Dx^{i_1} \wedge D\left(Dx^{i_2} \wedge \cdots \wedge Dx^{i_k}\right).
\end{aligned} \tag{10.26}
$$

Therefore, by induction on k,

$$
D\left(Dx^{i_1} \wedge \cdots \wedge Dx^{i_k}\right) = 0. \tag{10.27}
$$

Hence, from (10.25),

$$
\begin{aligned}
D\left(f \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}\right) &= Df \wedge \left(Dx^{i_1} \wedge \cdots \wedge Dx^{i_k}\right) \\
&= \mathrm{d}f \wedge \left(\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}\right) \\
&= \mathrm{d}\left(f \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}\right).
\end{aligned} \tag{10.28}
$$

So $D = \mathrm{d}$ on $\Omega^*(U)$. ■

Closed Forms and Exact Forms

A k-form ω on U is **closed** if $\mathrm{d}\omega = 0$; it's **exact** if there is a $(k-1)$-form τ on U such that $\omega = \mathrm{d}\tau$. Since $\mathrm{d}^2 = 0$, every exact form is closed. But in general, a closed form may fail to be exact. We will see how non-exact closed forms capture the geometry of a manifold when we do de Rham cohomology on a manifold.

Example 10.6. Define a 1-form ω on $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ by

$$\omega = \frac{1}{x^2+y^2}\left(-y\mathrm{d}x + x\mathrm{d}y\right). \tag{10.29}$$

Then ω is closed.

A collection of vector spaces $\left\{V^k\right\}_{k=0}^{\infty}$ with linear maps $d_k : V^k \to V^{k+1}$ such that $d_{k+1} \circ d_k = 0$ is called a **differential complex** or a **cochain complex**. For aany open set $U \subseteq \mathbb{R}^n$, the exterior derivative d makes the vector space $\Omega^*(U)$ of C^∞ forms on U into a cochain complex, called the **de Rham complex** on U:

$$\Omega^0(U) \xrightarrow{\mathrm{d}} \Omega^1(U) \xrightarrow{\mathrm{d}} \Omega^2(U) \xrightarrow{\mathrm{d}} \cdots$$

The closed forms are precisely the elements of the kernel of d and the exact forms are the elements of the image of d. In the language of cohomology, d is also called the coboundary operator.

§10.4 Applications to Vector Calculus

The theory of differential forms unifies many theorems in vector calculus on $\mathbb{R}^3$. A vector valued function on $\mathbb{R}^3$ is the same as a vector field. Recall the 3 operators on scalar and vector-valued functions on $\mathbb{R}^3$.

$$\{\text{scalar function}\} \xrightarrow{\text{grad}} \{\text{vector function}\} \xrightarrow{\text{curl}} \{\text{vector function}\} \xrightarrow{\text{div}} \{\text{scalar function}\}.$$

Let f be a scalar function and $\begin{bmatrix} P \\ Q \\ R \end{bmatrix}$ be a vector field on $\mathbb{R}^3$, where each of P, Q, R is a scalar function on $\mathbb{R}^3$. Then

$$\begin{aligned} \operatorname{grad} f &= \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix}, \\ \operatorname{curl} \begin{bmatrix} P \\ Q \\ R \end{bmatrix} &= \begin{bmatrix} R_y - Q_z \\ P_z - R_x \\ Q_x - P_y \end{bmatrix}, \\ \operatorname{div} \begin{bmatrix} P \\ Q \\ R \end{bmatrix} &= P_x + Q_y + R_z. \end{aligned} \tag{10.30}$$

Then one has the following results.

Proposition 10.5

$$\operatorname{curl}\left(\operatorname{grad} f\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{10.31}$$

Proposition 10.6

$$\operatorname{div}\left(\operatorname{curl}\begin{bmatrix}P\\Q\\R\end{bmatrix}\right)=0. \tag{10.32}$$

Proposition 10.7

On $\mathbb{R}^3$, a vector field $\mathbf{F}(x,y,z)$ is the gradient of some scalar function if and only if $\operatorname{curl}\mathbf{F}=\mathbf{0}$.

A 1-form on $\mathbb{R}^3$ can be written as

$$P(x,y,z)\,\mathrm{d}x+Q(x,y,z)\,\mathrm{d}y+R(x,y,z)\,\mathrm{d}z.$$

This 1-form on $\mathbb{R}^3$ can be identified with the vector field $\begin{bmatrix}P\\Q\\R\end{bmatrix}$.

Similarly, the 2-forms on $\mathbb{R}^3$ given by

$$A(x,y,z)\,\mathrm{d}y\wedge\mathrm{d}z+B(x,y,z)\,\mathrm{d}z\wedge\mathrm{d}x+C(x,y,z)\,\mathrm{d}x\wedge\mathrm{d}y$$

can be identified with the vector field $\begin{bmatrix}A\\B\\C\end{bmatrix}$ on $\mathbb{R}^3$.

In terms of these identifications, the exterior derivative of a 0-form f (scalar function) is

$$\mathrm{d}f=\frac{\partial f}{\partial x}\mathrm{d}x+\frac{\partial f}{\partial y}\mathrm{d}y+\frac{\partial f}{\partial z}\mathrm{d}z,$$

which can be identified with the vector field

$$\begin{bmatrix}\frac{\partial f}{\partial x}\\\frac{\partial f}{\partial y}\\\frac{\partial f}{\partial z}\end{bmatrix}=\operatorname{grad}f.$$

The exterior derivative of a 1-form on $\mathbb{R}^3$ is

$$\begin{aligned}&\mathrm{d}(P\mathrm{d}x+Q\mathrm{d}y+R\mathrm{d}z)\\=&\frac{\partial P}{\partial y}\mathrm{d}y\wedge\mathrm{d}x+\frac{\partial P}{\partial z}\mathrm{d}z\wedge\mathrm{d}x+\frac{\partial Q}{\partial x}\mathrm{d}x\wedge\mathrm{d}y+\frac{\partial Q}{\partial z}\mathrm{d}z\wedge\mathrm{d}y+\frac{\partial R}{\partial x}\mathrm{d}x\wedge\mathrm{d}z+\frac{\partial R}{\partial y}\mathrm{d}y\wedge\mathrm{d}z\\=&\ (R_y-Q_z)\,\mathrm{d}y\wedge\mathrm{d}z+(P_z-R_x)\,\mathrm{d}z\wedge\mathrm{d}x+(Q_x-P_y)\,\mathrm{d}x\wedge\mathrm{d}y,\end{aligned}$$

which corresponds to

$$\begin{bmatrix}R_y-Q_z\\P_z-R_x\\Q_x-P_y\end{bmatrix}=\operatorname{curl}\begin{bmatrix}P\\Q\\R\end{bmatrix}.$$

The exterior derivative of a 2-form is

$$\begin{aligned}&\mathrm{d}(A\mathrm{d}y\wedge\mathrm{d}z+B\mathrm{d}z\wedge\mathrm{d}x+C\mathrm{d}x\wedge\mathrm{d}y)\\=&\ A_x\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z+B_y\mathrm{d}y\wedge\mathrm{d}z\wedge\mathrm{d}x+C_z\mathrm{d}z\wedge\mathrm{d}x\wedge\mathrm{d}y\\=&\ (A_x+B_y+C_z)\,\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z,\end{aligned}$$

which corresponds to

$$A_x+B_y+C_z=\operatorname{div}\begin{bmatrix}A\\B\\C\end{bmatrix}.$$

In summary, exterior derivative d on 0-forms is identified with **gradient**; exterior derivative d on 1-forms is identified with **curl**; exterior derivative d on 2-forms is identified with **divergence**. Using de Rham complex on U:

$$\Omega^0(U) \xrightarrow{\mathrm{d}} \Omega^1(U) \xrightarrow{\mathrm{d}} \Omega^2(U) \xrightarrow{\mathrm{d}} \Omega^3(U).$$

Using vector calculus language,

$$C^\infty(U) \xrightarrow{\text{grad}} \mathfrak{X}(U) \xrightarrow{\text{curl}} \mathfrak{X}(U) \xrightarrow{\text{div}} C^\infty(U).$$

Remark 10.1. Proposition 10.5 and Proposition 10.6 express the property $\mathrm{d}^2 = 0$ of exterior derivative. A vector field $\begin{bmatrix} P \\ Q \\ R \end{bmatrix}$ is the gradient of a C^∞ function f if and only if the corresponding 1-form $P\mathrm{d}x + Q\mathrm{d}y + R\mathrm{d}z$ is $\mathrm{d}f$. Proposition 10.7 expresses the fact that a 1-form on $\mathbb{R}^3$ is exact if and only if it is closed. It's worth remarking at this stage that Proposition 10.7 need not hold true on a region other than $\mathbb{R}^3$, as the following well-known example from calculus suggests.

Example 10.7. Suppose $U = \mathbb{R}^3 \setminus \{z\text{-axis}\}$, and $\mathbf{F}(x, y, z)$ is the vector field

$$\mathbf{F} = \begin{bmatrix} \frac{-y}{x^2+y^2} \\ \frac{x}{x^2+y^2} \\ 0 \end{bmatrix}$$

on U. Then $\operatorname{curl} \mathbf{F} = \mathbf{0}$. Indeed,

$$\begin{aligned} \operatorname{curl} \mathbf{F} &= \begin{bmatrix} \frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}\left(\frac{x}{x^2+y^2}\right) \\ \frac{\partial}{\partial z}\left(\frac{-y}{x^2+y^2}\right) - \frac{\partial}{\partial x}(0) \\ \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ \frac{(x^2+y^2)-x\cdot 2x}{(x^2+y^2)^2} - \frac{-(x^2+y^2)+y\cdot 2y}{(x^2+y^2)^2} \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0}. \end{aligned}$$

But $\mathbf{F}$ is not the gradient of a C^∞ function on U. Recall the theorem from vector calculus that the line integral of the gradient of a function along a curve gives the total change in the value of the function from the start to the end of the curve. In other words, if $\mathbf{r} : [a, b] \to \mathbb{R}^3$ is a curve and $f : \mathbb{R}^3 \to \mathbb{R}$ is a scalar function, then

$$\int_a^b (\boldsymbol{\nabla} f) \cdot \mathrm{d}\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)). \tag{10.33}$$

Then if $\mathbf{F}$ is the gradient of a smooth scalar function, then the line integral

$$\oint_C \frac{-y}{x^2+y^2}\,\mathrm{d}x + \frac{x}{x^2+y^2}\,\mathrm{d}y$$

over any closed curve would become 0. Let us take the closed curve to be the unit circle: $x = \cos t$,

$y = \sin t$, $z = 0$ for $t \in [0, 2\pi]$. Then

$$\begin{aligned}
&\oint_C \frac{-y}{x^2+y^2}\,\mathrm{d}x + \frac{x}{x^2+y^2}\,\mathrm{d}y \\
&= \int_0^{2\pi} -\sin t\,\mathrm{d}\left(\cos t\right) + \int_0^{2\pi} \cos t\,\mathrm{d}\left(\sin t\right) \\
&= \int_0^{2\pi} \sin^2 t\,\mathrm{d}t + \int_0^{2\pi} \cos^2 t\,\mathrm{d}t \\
&= 2\pi.
\end{aligned}$$

Hence, although $\operatorname{curl}\mathbf{F} = \mathbf{0}$, there is no C^∞ function f on U such that $\mathbf{F} = \operatorname{grad} f$. In the language of differential forms, the 1-form

$$\omega = \frac{-y}{x^2+y^2}\mathrm{d}x + \frac{x}{x^2+y^2}\mathrm{d}y$$

is closed but not exact.

It turns out that whether Proposition 10.7 is true for a region $U \subseteq \mathbb{R}^3$ depends on the topology of U. One measure of the failure of a closed k-form to be exact is the quotient vector space

$$H^k\left(U\right) = \frac{\{\text{closed } k\text{-forms on } U\}}{\{\text{exact } k\text{-forms on } U\}},$$

called the k-th de Rham cohomology of U. The generalization of Proposition 10.7 to any differential form on $\mathbb{R}^n$ is called the **Poincaré lemma**:

> For $k \geq 1$, every closed k-form on $\mathbb{R}^n$ is exact.

This statement is equivalent to the vanishing of the k-th de Rham cohomology $H^k\left(\mathbb{R}^n\right)$ for $k \geq 1$.

11 Differential Forms on Manifold

§11.1 Definition and Local Expression

Let M be a smooth manifold and $p \in M$. The **cotangent space** of M at p, denoted by T_p^*M is the dual space of the tangent space T_pM. An element in T_p^*M is called a covector at p. Thus, a covector $\omega_p \in T_p^*M$ is a linear function

$$\omega_p : T_pM \to \mathbb{R}.$$

A 1-form on M is a functiion that assigns to each $p \in M$, a covector at p.

Definition 11.1 (Differential of a function)**.** Let $f : M \to \mathbb{R}$ be a C^∞ function on a manifold M. Its **differential** is defined to be the 1-form $\mathrm{d}f$ on M such that for any $p \in M$ and $X_p \in T_pM$,

$$(\mathrm{d}f)_p(X_p) = X_pf. \tag{11.1}$$

Proposition 11.1

If $f : M \to \mathbb{R}$ is a C^∞ function, then for $p \in M$ and $X_p \in T_pM$,

$$f_{*,p}(X_p) = (\mathrm{d}f)_p(X_p) \left.\frac{\partial}{\partial x}\right|_{f(p)}.$$

Proof. Since $f_{*,p}(X_p) \in T_{f(p)}\mathbb{R}$, there is a real number c such that

$$f_{*,p}(X_p) = c \left.\frac{\partial}{\partial x}\right|_{f(p)}. \tag{11.2}$$

(Here the chart chosen on $\mathbb{R}$ is $(\mathbb{R}, \mathbb{1}_{\mathbb{R}})$ so that x is the coordinate of this chart, i.e. $x = \mathbb{1}_{\mathbb{R}}$.) To evaluate c, apply both sides of (11.2) to the function $x \in C^\infty(\mathbb{R})$. Then

$$f_{*,p}(X_p)(x) = c \left.\frac{\partial}{\partial x}\right|_{f(p)}(x) = c.$$

Therefore,

$$c = f_{*,p}(X_p)(x) = X_p(x \circ f) = X_pf = (\mathrm{d}f)_p(X_p), \tag{11.3}$$

since $x = \mathbb{1}_R$. Therefore, substituting the value of c into (11.2),

$$f_{*,p}(X_p) = (\mathrm{d}f)_p(X_p) \left.\frac{\partial}{\partial x}\right|_{f(p)}. \tag{11.4}$$

■

Let $(U, \varphi) \equiv (U, x^1, x^2, \ldots, x^n)$ be a coordinate chart on M. Here $x^i = r^i \circ \varphi$, where r^i is the i-th coordinate function of a vector in $\mathbb{R}^n$. Then the differentials $\mathrm{d}x^1, \mathrm{d}x^2, \ldots, \mathrm{d}x^n$ are 1-forms on U.

Proposition 11.2

At each point $p \in U$, the covectors $(\mathrm{d}x^1)_p, \ldots, (\mathrm{d}x^n)_p$ form a basis for the cotangent space T_p^*M, dual to the basis $\left\{ \left.\frac{\partial}{\partial x^1}\right|_p, \ldots, \left.\frac{\partial}{\partial x^n}\right|_p \right\}$ for the tangent space T_pM.

Proof. Observe that

$$\left(\mathrm{d}x^i\right)_p \left(\frac{\partial}{\partial x^j}\bigg|_p\right) = \frac{\partial}{\partial x^j}\bigg|_p \left(x^i\right) = \delta^i{}_j. \tag{11.5}$$

So $\left\{\left(\mathrm{d}x^1\right)_p, \ldots, \left(\mathrm{d}x^n\right)_p\right\}$ is the dual basis to $\left\{\frac{\partial}{\partial x^1}\Big|_p, \ldots, \frac{\partial}{\partial x^n}\Big|_p\right\}$. ■

Thus, every 1-form ω on U can be written as a linear combination

$$\omega = \sum_{i=1}^n a_i \, \mathrm{d}x^i,$$

where a_i are functions on U. In particular, if f is a C^∞ function on M, then the 1-form $\mathrm{d}f$, when restricted to U, must be a linear combination

$$\mathrm{d}f = \sum_{i=1}^n a_i \, \mathrm{d}x^i. \tag{11.6}$$

If we evaluate both sides of (11.6) on $\frac{\partial}{\partial x^j}$,

$$(\mathrm{d}f)\left(\frac{\partial}{\partial x^j}\right) = \sum_{i=1}^n a_i \, \mathrm{d}x^i \left(\frac{\partial}{\partial x^j}\right) = \sum_{i=1}^n a_i \delta^i{}_j = a_j.$$

Then

$$a_j = (\mathrm{d}f)\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial f}{\partial x^j}.$$

Therefore,

$$\mathrm{d}f = \sum_{i=1}^n \frac{\partial f}{\partial x^i} \, \mathrm{d}x^i. \tag{11.7}$$

§11.2 The Cotangent Bundle

The underlying set of the **cotangent bundle** is the disjoint union of the cotangent spaces at all points of M:

$$T^*M = \bigsqcup_{p\in M} T_p^*M = \bigcup_{p\in M} \{p\} \times T_p^*M. \tag{11.8}$$

Let us give T^*M a topology in the following way: let $\left(U, x^1, \ldots, x^n\right)$ be a chart on M and $p \in U$. Then each $\omega_p \in T_p^*M$ can be written uniquely as a linear combination

$$\omega_p = \sum_{i=1}^n c_i \left(\omega_p\right) \left(\mathrm{d}x^i\right)_p,$$

with $c_i\left(\omega_p\right) \in \mathbb{R}$. This gives rise to a bijection

$$\begin{aligned} \widetilde{\varphi} : T^*U &\to \varphi(U) \times \mathbb{R}^n \\ \left(p, \omega_p\right) &\mapsto \left(\varphi(p), c_1\left(\omega_p\right), c_2\left(\omega_p\right), \ldots, c_n\left(\omega_p\right)\right). \end{aligned}$$

We use this bijection $\widetilde{\varphi}$ to transfer the topology of $\varphi(U) \times \mathbb{R}^n$ to T^*U: a set $A \subseteq T^*U$ is said to be open if and only if $\widetilde{\varphi}(A)$ is open in $\varphi(U) \times \mathbb{R}^n$, where $\varphi(U) \times \mathbb{R}^n$ is given the subspace topology of $\mathbb{R}^{2n}$. Now, let $\mathscr{A} = \left\{\left(U_\alpha, \varphi_\alpha\right)\right\}_{\alpha\in I}$ be the maximal atlas of M. Now, let

$$\begin{aligned} \mathcal{B} &= \bigcup_{\alpha\in I} \left\{A \mid A \subseteq T^*U_\alpha \text{ is open in } A \subseteq T^*U_\alpha\right\} \\ &= \left\{A \mid A \subseteq T^*U_\alpha \text{ is open in } A \subseteq T^*U_\alpha, \ \alpha \in I\right\}. \end{aligned}$$

It can be shown using the same technique of tangent bundle that $\mathcal{B}$ forms a basis for topology. We give T^*M the topology generated by the basis $\mathcal{B}$. We declare $A \subseteq T^*M$ to be open if and only if there exists a subfamily $\{B_\lambda\}_\lambda \subseteq \mathcal{B}$ such that

$$A = \bigcup_\lambda B_\lambda.$$

Furthermore, T^*M has the structure of a C^∞ manifold. An atlas for T^*M is

$$\{(T^*U_\alpha, \widetilde{\varphi}_\alpha)\}_{\alpha \in I}.$$

If two coordinate open sets U_α and U_β intersect, suppose $U_{\alpha\beta} = U_\alpha \cap U_\beta$. Then for $p \in U_{\alpha\beta}$, each $\omega_p \in T_p^*M$ has two basis expansions:

$$\omega_p = \sum_{i=1}^n a_i \left(\mathrm{d}x^i\right)_p = \sum_{j=1}^n b_i \left(\mathrm{d}y^i\right)_p. \tag{11.9}$$

(Here $(U_\alpha, x^1, \ldots, x^n)$ and $(U_\beta, y^1, \ldots, y^n)$ are charts.) Now applying $\left.\frac{\partial}{\partial y^k}\right|_p$ to both sides of (11.9),

$$b_k = \sum_{i=1}^n a_i \left(\mathrm{d}x^i\right)_p \left(\left.\frac{\partial}{\partial y^k}\right|_p\right) = \sum_{i=1}^n a_i \left.\frac{\partial x^i}{\partial y^k}\right|_p.$$

Therefore, $\widetilde{\varphi}_\beta \circ \widetilde{\varphi}_\alpha^{-1} : \varphi_\alpha(U_{\alpha\beta}) \times \mathbb{R}^n \to \varphi_\beta(U_{\alpha\beta}) \times \mathbb{R}^n$ is given by

$$(\varphi_\alpha(p), a_1, \ldots, a_n) \mapsto \left(\left(\varphi_\beta \circ \varphi_\alpha^{-1}\right)(\varphi_\alpha(p)), \sum_{i=1}^n a_i \left.\frac{\partial x^i}{\partial y^1}\right|_p, \ldots, \sum_{i=1}^n a_i \left.\frac{\partial x^i}{\partial y^n}\right|_p\right).$$

$\varphi_\beta \circ \varphi_\alpha^{-1}$ is smooth, and each $\frac{\partial x^i}{\partial y^j}$ is smooth. Therefore, the transition map $\widetilde{\varphi}_\beta \circ \widetilde{\varphi}_\alpha^{-1}$ is smooth, making T^*M a smooth manifold.

T^*M is, in fact, a **vector bundle** of rank n over M. It has a natural projection $\pi : T^*M \to M$ given by $(p, \omega_p) \mapsto p$. In terms of cotangent bundle, a 1-form on M is simply a section of the cotangent bundle T^*M, i.e. it is a map $\omega : M \to T^*M$ such that $\pi \circ \omega = \mathbb{1}_M$. We say that a 1-form is **smooth** if it is C^∞ as a map $\omega : M \to T^*M$ between two manifolds.

§11.3 Characterization of Smooth 1-forms

By definition, a 1-form on an open set $U \subseteq M$ is C^∞ if it is C^∞ as a section of the cotangent bundle T^*M over U.

Lemma 11.3

Let $(U, \varphi) \equiv (U, x^1, \ldots, x^n)$ be a chart on a manifold M. A 1-form $\omega = \sum a_i \,\mathrm{d}x^i$ on U is smooth if and only if the coefficient functions a_i are all smooth on U.

Proof. This is a special case of *Proposition 9.4.2* of DG1 which states that:

> Let $\pi : E \to M$ be a C^∞ vector bundle and U an open subset of M. Suppose $s_1, \ldots, s_r$ is a C^∞ frame for E over U. Then a section $s = \sum_{j=1}^r c^j s_j$ of E over U is C^∞ if and only if the coefficients c^j are C^∞ functions on U.

Here we take E to be the cotangent bundle T^*M, and $\{s_i\}_{i=1}^r$ the C^∞ frame for E over U to be the coordinate 1-forms $\{(\mathrm{d}x^i)\}_{i=1}^n$. ■

Proposition 11.4

Let ω be a 1-form on a manifold M. Then the following are equivalent:

(i) ω is C^∞.

(ii) For every point $p \in M$, there is a chart $(U, x^1, \dots, x^n)$ about p such that if $\omega = \sum_{i=1}^n a_i \, dx^i$ on U, then the functions a_i are C^∞ on U.

(iii) For any chart $(U, x^1, \dots, x^n)$ on M, if $\omega = \sum_{i=1}^n a_i \, dx^i$ on U, then the functions a_i are C^∞ on U.

Proof. **(ii)⇒(i):** By Lemma 11.3, for every point $p \in M$, there is a chart $(U, x^1, \dots, x^n)$ about p such that ω is smooth on U. In particular, the section $\omega : M \to T^*M$ is smooth at p, for every $p \in M$. Therefore, $\omega : M \to T^*M$ is a smooth map between manifolds.

(i)⇒(iii): If $\omega : M \to T^*M$ is a smooth map between manifolds, ω is smooth at every chart of M. Therefore, by Lemma 11.3, if $\omega = \sum_{i=1}^n a_i \, dx^i$ on a chart $(U, x^1, \dots, x^n)$, each a_i is smooth on U.

(iii)⇒(ii): Obvious. ■

Proposition 11.5

A 1-form ω on a manifold M is C^∞ if and only if for every C^∞ vector field X, the function $\omega(X)$ is C^∞ on M.

Proof. **(⇒)** Suppose ω is a C^∞ 1-form and X is a C^∞ vector field. Let $(U, x^1, \dots, x^n)$ be a chart on M. Then

$$\omega = \sum_{i=1}^n a_i \, dx^i \quad \text{and} \quad X = \sum_{i=1}^n b^j \frac{\partial}{\partial x^j}, \tag{11.10}$$

for C^∞ functions a_i and b^j on U. Then on U, one has

$$\omega(X) = \left(\sum_{i=1}^n a_i \, dx^i\right)\left(\sum_{i=1}^n b^j \frac{\partial}{\partial x^j}\right) = \sum_{i=1}^n \sum_{j=1}^n a_i b^j \delta^i{}_j = \sum_{i=1}^n a_i b^i, \tag{11.11}$$

which is a C^∞ function on U. Since U was chosen to be an arbitrary coordinate open set, $\omega(X)$ is a smooth function on all of M.

(⇐) Suppose ω is a 1-form on M such that for every C^∞ vector field X on M, the function $\omega(X)$ is smooth on M. For a given $p \in M$, choose a coordinate neighborhood $(U, \varphi) \equiv (U, x^1, \dots, x^n)$ about p. Then one has

$$\omega = \sum_{i=1}^n a_i \, dx^i$$

on U. Now fix an integer $j \in \{1, 2, \dots, n\}$. We can extend the C^∞ vector field $\frac{\partial}{\partial x^j}$ on U to a C^∞ vector field X on the whole of M that agrees with $\frac{\partial}{\partial x^j}$ in a neighborhood V of p (not necessarily the whole of U, but possibly a smaller neighborhood) contained in U (*Proposition 11.1.4* of DG1). The extended vector field is defined in the following way: let $\sigma : M \to \mathbb{R}$ be a C^∞ bump function which is identically 1 on a neighborhood V of p and which has support contained in U. Now, define the vector field $q \mapsto X_q \in T_qM$, denoted by X, in terms of the bump function σ in the following way:

$$X_q = \begin{cases} \sigma(q) \left.\frac{\partial}{\partial x^j}\right|_q & \text{if } q \in U, \\ \mathbf{0} & \text{if } q \notin U. \end{cases} \tag{11.12}$$

The vector field X is smooth in the whole of M, as proved in *Proposition 11.1.4* of DG1. Now, by the hypothesis, $\omega(X)$ is C^∞ on M. In particular, $\omega(X)$ is smooth on V. Therefore,

$$\omega(X) = \left(\sum_{i=1}^n a_i \, dx^i\right)\left(\frac{\partial}{\partial x^j}\right) = \sum_{i=1}^n a_i \delta^i{}_j = a_j$$

is smooth on V. We, therefore, see that the coefficient functions a_i's appearing in $\omega = \sum_{i=1}^n a_i \,\mathrm{d}x^i$ are smooth on $V \subseteq U$. It means that for a given point p, we can find a chart $(V, \widetilde{x}^1, \ldots, \widetilde{x}^n)$, where

$$\widetilde{x}^i = r^i \circ \varphi|_V,$$

such that $\omega = \sum_{i=1}^n a_i|_V \,\mathrm{d}\widetilde{x}^i$ on V, with each $a_i|_V$ smooth on V. Therefore, by Proposition 11.4, ω is C^∞. ■

§11.4 Pullback of 1-forms

Recall that the differential of a smooth map $F : N \to M$ at $p \in N$ is a linear map $F_{*,p} : T_pN \to T_{F(p)}M$ defined by

$$[F_{*,p}(X_p)](f) = X_p(f \circ F), \tag{11.13}$$

where $f \in C^\infty_{F(p)}(M)$. Indeed, $f \circ F \in C^\infty_p(N)$. Analogously, the **codifferential** (the dual of a differential) at $F(p) \in M$ is a linear map

$$F^{*,p} : T^*_{F(p)}M \to T^*_pN.$$

One observes that the differential $F_{*,p}$ pushes forward a tangent vector at $p \in N$ while the codifferential $F^{*,p}$ pulls back a covector from $T^*_{F(p)}M$ at $F(p) \in M$ to T^*_pN.

Remark 11.1. Note that a vector field, in general, cannot be pushed forward under a smooth map $F : N \to M$. Suppose $F : N \to M$ is a smooth map of manifolds. Also suppose $F(p) = F(q) = z \in M$ so that F is not injective. Now, the differentials

$$F_{*,p} : T_pN \to T_zM \text{ and } F_{*,q} : T_qN \to T_zM$$

are linear maps. Now, let $X \in \mathfrak{X}(N)$ be a C^∞ vector field on N so that X_p under $F_{*,p}$ is pushed forward to $F_{*,p}(X_p) \in T_zM$ and X_q is pushed forward to $F_{*,q}(X_q) \in T_zM$ under $F_{*,q}$. There is no reason for $F_{*,p}(X_p)$ and $F_{*,q}(X_q)$ to be the same tangent vector in T_zM. In other words, in general,

$$F_{*,p}(X_p) \neq F_{*,q}(X_q),$$

so that $z \mapsto F_{*,p}(X_p) := Y_z \in T_zM$ and $z \mapsto F_{*,q}(X_q) := Y'_z \in T_zM$ are distinct vector fields on M, denoted by Y and Y', respectively. Therefore, if there were push forward of vector fields $F_* : \mathfrak{X}(N) \to \mathfrak{X}(M)$ associated with the non-injective smooth map $F : N \to M$, there is an ambiguity regarding which vector field X gets mapped to.

Furthermore, if F is not surjective, there is $z \in M$ such that $z \neq F(p)$ for any $p \in N$. In that case as well, defining the push forward vector field $F_*(X)$ at the point z is impossible. However, when $F : N \to M$ is a diffeomorphism, one can define the push forward of a vector field.

Contrary to the non-existence of push forward of a vector field associated with a generic smooth map $F : N \to M$, one can always talk about pullback of a 1-form ω on M:

$$(F^*\omega)_p(X_p) = \omega_{F(p)}(F_{*,p}(X_p)). \tag{11.14}$$

Here, $\omega \in \Omega^1(M)$, $X_p \in T_pN$, $p \in N$. Note that $(F^*\omega)_p$ is simply the image of the covector $\omega_{F(p)} \in T^*_{F(p)}M$ under the codifferential $F^{*,p} : T^*_{F(p)}M \to T^*_pN$. In other words,

$$(F^*\omega)_p = F^{*,p}\left(\omega_{F(p)}\right). \tag{11.15}$$

12 Differential k-forms

§12.1 Definition and Local Expression

We denoted by $A_k(V)$ the vector space of alternating k-tensors on V. We have also seen that if $\{\widehat{\alpha}^1, \ldots, \widehat{\alpha}^n\}$ is a basis for 1-tensors on V, then a basis element of $A_k(V)$ is

$$\widehat{\alpha}^{i_1} \wedge \widehat{\alpha}^{i_2} \wedge \cdots \wedge \widehat{\alpha}^{i_k},$$

where $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. We apply this construction to the tangent space T_pM of a manifold M at a point $p \in M$. The vector space $A_k(T_pM)$, usually denoted by $\Lambda^k\left(T_p^*M\right)$, is the space of all alternating k-tensors on the tangent space T_pM.

> **Definition 12.1** (Differential k-form). A **differential k-form** on a manifold M is a function ω that assigns to each point $p \in M$, a k-covector $\omega_p \in \Lambda^k\left(T_p^*M\right)$. An n-from on a manifold of dimension n is called a **top degree form**.

Example 12.1. On $\mathbb{R}^n$, at each point p, there is a standard basis for the tangent space $T_p\mathbb{R}^n$:

$$\left\{ \left.\frac{\partial}{\partial r^1}\right|_p, \left.\frac{\partial}{\partial r^2}\right|_p, \ldots, \left.\frac{\partial}{\partial r^n}\right|_p \right\}.$$

Let $\left\{(\mathrm{d}r^1)_p, \ldots, (\mathrm{d}r^n)_p\right\}$ be the dual basis of $T_p^*\mathbb{R}^n$.

$$\left(\mathrm{d}r^i\right)_p \left(\left.\frac{\partial}{\partial r^j}\right|_p \right) = \delta^i{}_j.$$

As p varies over $\mathbb{R}^n$, we get differential forms $\mathrm{d}r^1, \ldots, \mathrm{d}r^n$ on $\mathbb{R}^n$. By Proposition C.15, a basis element of alternating k-tensors $\Lambda^k\left(T_p^*\mathbb{R}^n\right)$ is

$$\left(\mathrm{d}r^{i_1}\right)_p \wedge \left(\mathrm{d}r^{i_2}\right)_p \wedge \cdots \wedge \left(\mathrm{d}r^{i_k}\right)_p,$$

where $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. If ω is a k-form on $\mathbb{R}^n$, then at each point $p \in \mathbb{R}^n$, ω_p is the following linear combination:

$$\omega_p = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} a_{i_1 \cdots i_k} \left(\mathrm{d}r^{i_1}\right)_p \wedge \left(\mathrm{d}r^{i_2}\right)_p \wedge \cdots \wedge \left(\mathrm{d}r^{i_k}\right)_p. \tag{12.1}$$

Omitting the point p, we write

$$\omega = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} a_{i_1 \cdots i_k} \, \mathrm{d}r^{i_1} \wedge \mathrm{d}r^{i_2} \wedge \cdots \wedge \mathrm{d}r^{i_k}. \tag{12.2}$$

In the expression above, $a_{i_1 \cdots i_k}$ are functions on $\mathbb{R}^n$. To simplify the notations, we use multi-indices to write (12.2) as

$$\omega = \sum_I a_I \, \mathrm{d}r^I, \tag{12.3}$$

where $\mathrm{d}r^I = \mathrm{d}r^{i_1} \wedge \mathrm{d}r^{i_2} \wedge \cdots \wedge \mathrm{d}r^{i_k}$, and $I = (i_1, i_2, \ldots, i_k)$ is a strictly ascending multi-index.

Suppose $(U, x^1, \ldots, x^n)$ is a coordinate chart on a manifold M. We have already defined the 1-forms $\mathrm{d}x^1, \ldots, \mathrm{d}x^n$ on U. Since at each point $p \in U$, $\left\{(\mathrm{d}x^1)_p, \ldots, (\mathrm{d}x^n)_p\right\}$ is a basis for $T_p^* M$, by Proposition C.15, a basis for $\Lambda^k\left(T_p^*\mathbb{R}^n\right)$ is

$$\left(\mathrm{d}x^{i_1}\right)_p \wedge \left(\mathrm{d}x^{i_2}\right)_p \wedge \cdots \wedge \left(\mathrm{d}x^{i_k}\right)_p,$$

where $1 \le i_1 < i_2 < \cdots < i_k \le n$. Thus, locally a k-form on U will be a linear combination

$$\omega = \sum_I a_I \, \mathrm{d}x^I, \tag{12.4}$$

where $\mathrm{d}x^I = \mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2} \wedge \cdots \wedge \mathrm{d}x^{i_k}$, $I = (i_1, i_2, \ldots, i_k)$ is a strictly ascending multi-index, and a_I are functions on U.

§12.2 The Bundle Point of View

Let V be a real vector space. Another common notation for the vector space $A_k(V)$ of alternating k-linear functionos on V is $\Lambda^k(V^*)$.

$$\begin{aligned}
\Lambda^0(V^*) &= A_0(V) = \mathbb{R}, \\
\Lambda^1(V^*) &= A_1(V) = V^*, \\
\Lambda^2(V^*) &= A_2(V),
\end{aligned}$$

and so on. Now, $\Lambda^k(T^*M)$ is defined to be the disjoint union of the vector spaces $\Lambda^k\left(T_p^*M\right)$ as p varies over M. So

$$\begin{aligned}
\Lambda^k(T^*M) &= \bigsqcup_{p\in M} \Lambda^k\left(T_p^*M\right) = \bigsqcup_{p\in M} A_k(T_pM) \\
&= \bigcup_{p\in M} \{p\} \times A_k(T_pM),
\end{aligned} \tag{12.5}$$

which is the set of all alternating k-tensors at all points of M. This set is called the k-th **exterior power** of the cotangent bundle T^*M.

If (U, φ) is a coordinate chart on M, then there is a bijection $\overline{\varphi} : \Lambda^k(T^*U) \to \varphi(U) \times \mathbb{R}^{\binom{n}{k}}$ defined as follows: a generic element of $\Lambda^k(T^*U)$ is (p, ω_p), where $\omega_p \in \Lambda^k\left(T_p^*U\right)$. Then ω_p is a unique linear combination

$$\omega_p = \sum_I a_I(p) \, \mathrm{d}x^I,$$

where I runs over the set of strictly ascending multi-indices of length k. There are $\binom{n}{k}$ many such multi-indices. If we fix a labeling of the multi-indices once and for all, then we have a $\binom{n}{k}$-tuple $(a_I)_I$. Then we define

$$\overline{\varphi}(p, \omega_p) = \left(\varphi(p), (a_I)_I\right) \in \varphi(U) \times \mathbb{R}^{\binom{n}{k}}.$$

Thus, $\Lambda^k(T^*U)$ is in a bijective correspondence with $\varphi(U) \times \mathbb{R}^{\binom{n}{k}}$. Using this bijective correspondence, one transfers the topology of $\varphi(U) \times \mathbb{R}^{\binom{n}{k}}$ to $\Lambda^k(T^*U)$. By varying the open set U in the charts contained in the maximal atlas of M, one can obtain a basis that generates the topology on the whole of $\Lambda^k(T^*M)$.

$\Lambda^k(T^*M)$ is defined to be the disjoint union of the vector spaces $\Lambda^k\left(T_p^*M\right)$ as p varies over M. So

$$\Lambda^k(T^*M) = \bigsqcup_{p\in M} \Lambda^k\left(T_p^*M\right). \tag{12.6}$$

If (U,φ) is a coordinate chart on M, then there is a bijection $\overline{\varphi} : \Lambda^k(T^*U) \to \varphi(U) \times \mathbb{R}^{\binom{n}{k}}$ defined as follows: a generic element of $\Lambda^k(T^*U)$ is (p,ω_p), where $\omega_p \in \Lambda^k\left(T_p^*U\right)$. Then ω_p is a unique linear combination

$$\omega_p = \sum_I a_I(p)\,\mathrm{d}x^I,$$

where I runs over the set of strictly ascending multi-indices of length k. There are $\binom{n}{k}$ many such multi-indices. If we fix a labeling of the multi-indices once and for all, then we have a $\binom{n}{k}$-tuple $(a_I)_I$. Then we define

$$\overline{\varphi}(p,\omega_p) = (\varphi(p),(a_I)_I) \in \varphi(U) \times \mathbb{R}^{\binom{n}{k}}.$$

Thus, $\Lambda^k(T^*U)$ is in a bijective correspondence with $\varphi(U) \times \mathbb{R}^{\binom{n}{k}}$. Using this bijective correspondence, one transfers the topology of $\varphi(U) \times \mathbb{R}^{\binom{n}{k}}$ to $\Lambda^k(T^*U)$. By varying the open set U in the charts contained in the maximal atlas of M, one can obtain a basis that generates the topology on the whole of $\Lambda^k(T^*M)$.

First, let us verify that $\Lambda^k(T^*M)$ is second countable. By *Lemma 9.1.3* of DG1, a manifold M has a countable basis consisting of coordinate open sets. Let $\{U_i\}_i$ be a countable basis for M consisting of coordinate open sets. Let φ_i be the coordinate map on U_i. We have shown that $\Lambda^k(T^*U_i)$ is homeomorphic to $\varphi_i(U_i) \times \mathbb{R}^{\binom{n}{k}}$, which is an open subset of $\mathbb{R}^{n+\binom{n}{k}}$. Hence, $\varphi_i(U_i) \times \mathbb{R}^{\binom{n}{k}}$ is secoond countable. Now, homeomorphism preserves second countability, so $\Lambda^k(T^*U_i)$ is also second countable.

For each i, choose a countable basis $\{B_{i,j}\}_j$ for $\Lambda^k(T^*U_i)$. Then $\{B_{i,j}\}_{i,j}$ is also countable. Now we need to show that $\{B_{i,j}\}_{i,j}$ is a basis for $\Lambda^k(T^*M)$. Let $A \subseteq \Lambda^k(T^*M)$ be open and take $(p,\omega_p) \in A$. We need to show the existence of $B_{i,j}$ such that $(p,\omega_p) \in B_{i,j} \subseteq A$.

Since $\{U_i\}$ is a basis for M, $p \in U_i$ for some i. Then

$$(p,\omega_p) \in \{p\} \times \Lambda^k\left(T_p^*U_i\right) \subseteq \bigcup_{p\in U_i} \{p\} \times \Lambda^k\left(T_p^*U_i\right) = \Lambda^k(T^*U_i)\,.$$

Therefore, $(p,\omega_p) \in A \cap \Lambda^k(T^*U_i)$.

We have used the topology on $\Lambda^k(T^*U_\alpha)$, for U_α being a coordinate open set of M, to define the topology on $\Lambda^k(T^*M)$. So $\Lambda^k(T^*U_\alpha)$ is a subspace of $\Lambda^k(T^*M)$. Since A is open in $\Lambda^k(T^*M)$, $\widetilde{A} := A \cap \Lambda^k(T^*U_i)$ is open in $\Lambda^k(T^*U_i)$. Now, $\widetilde{A}$ is open in $\Lambda^k(T^*U_i)$ and $(p,\omega_p) \in \widetilde{A} = A \cap \Lambda^k(T^*U_i)$. Since $\{B_{i,j}\}_j$ is a basis for $\Lambda^k(T^*U_i)$, there exists some $B_{i,j}$ such that

$$(p,\omega_p) \in B_{i,j} \subseteq \widetilde{A} = A \cap \Lambda^k(T^*U_i) \subseteq A \implies (p,\omega_p) \in B_{i,j} \subseteq A\,.$$

Therefore, the countable collection $\{B_{i,j}\}_{i,j}$ is a basis for TM.

Now we shall prove that $\Lambda^k(T^*M)$ is Hausdorff. Let (p,ω_p) and (q,τ_q) be distinct points of TM.

Case 1: $p \neq q$.

Since M is Hausdorff, there exist disjoint open subsets U_1 and V_1 of M that contain p and q, respectively. Furthermore, there exists coordinate open sets U_2 and V_2 around p and q, respectively. Then $U = U_1 \cap U_2$ and $V = V_1 \cap V_2$ are disjoint coordinate open sets that contain p and q, respectively.

$$(p,\omega_p) \in \{p\} \times \Lambda^k\left(T_p^*M\right) = \{p\} \times \Lambda^k\left(T_p^*U\right) \subseteq \Lambda^k(T^*U)\,.$$

Similarly, $(q,\tau_q) \in \Lambda^k(T^*V)$. Since $U \cap V = \varnothing$, $\Lambda^k(T^*U) \cap \Lambda^k(T^*V) = \varnothing$. Therefore, $\Lambda^k(T^*U)$ and $\Lambda^k(T^*V)$ are the disjoint open subsets of $\Lambda^k(T^*M)$ that contain (p,ω_p) and (q,τ_q), respectively.

Case 2: $p = q$.
Let (U, φ) be a coordinate chart containing p. Then (p, ω_p) and (p, τ_p) are distinct points on $\Lambda^k(T^*U)$, which is homeomorphic to $\varphi(U) \times \mathbb{R}^{\binom{n}{k}}$. $\varphi(U) \times \mathbb{R}^{\binom{n}{k}}$ is Hausdorff, hence so is $\Lambda^k(T^*U)$. Therefore, (p, ω_p) and (p, τ_p) can be separated by open subsets of $\Lambda^k(T^*U)$, which are also open subset of $\Lambda^k(T^*M)$.

Therefore, $\Lambda^k(T^*M)$ is Hausdorff.

So we have verified that $\Lambda^k(T^*M)$ is second countable, Hausdorff, and locally Euclidean. Now we just have to exhibit an atlas on $\Lambda^k(T^*M)$. Let $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in I}$ be an atlas for M. We are now going to show that $\left\{\left(\Lambda^k(T^*U_\alpha), \overline{\varphi}_\alpha\right)\right\}_{\alpha\in I}$ is an atlas for $\Lambda^k(T^*M)$. Clearly,

$$\begin{aligned}
\bigcup_{\alpha\in I} \Lambda^k(T^*U_\alpha) &= \bigcup_{\alpha\in I}\bigcup_{p\in U_\alpha} \{p\} \times \Lambda^k\left(T_p^*U_\alpha\right) \\
&= \bigcup_{\alpha\in I}\bigcup_{p\in U_\alpha} \{p\} \times \Lambda^k\left(T_p^*M\right) \\
&= \bigcup_{p\in\bigcup_{\alpha\in I}U_\alpha} \{p\} \times \Lambda^k\left(T_p^*M\right) \\
&= \bigcup_{M} \{p\} \times \Lambda^k\left(T_p^*M\right) = \Lambda^k(T^*M).
\end{aligned}$$

So $\left\{\left(\Lambda^k(T^*U_\alpha), \overline{\varphi}_\alpha\right)\right\}_{\alpha\in I}$ indeed covers the whole of $\Lambda^k(T^*M)$. Now we need to show that the charts are compatible. Let $(U, \varphi) = (U, x^1, \dots, x^n)$ and $(V, \psi) = (V, y^1, \dots, y^n)$ be two charts on M such that $U \cap V \neq \varnothing$. We need to show that $\left(\Lambda^k(T^*U), \overline{\varphi}\right)$ and $\left(\Lambda^k(T^*V), \overline{\psi}\right)$ are compatible charts on $\Lambda^k(T^*M)$, i.e. the map $\overline{\psi} \circ \overline{\varphi}^{-1}$ is C^∞.

$$\overline{\psi} \circ \overline{\varphi}^{-1} : \overline{\varphi}(U \cap V) = \varphi(U \cap V) \times \mathbb{R}^{\binom{n}{k}} \to \overline{\psi}(U \cap V) = \psi(U \cap V) \times \mathbb{R}^{\binom{n}{k}}.$$

Let's take a point $(\varphi(p), (a_I)_I) \in \varphi(U \cap V) \times \mathbb{R}^{\binom{n}{k}}$, where $p \in U \cap V$ and a_I are real numbers. Then $\overline{\varphi}^{-1}$ takes it to

$$\left(p, \sum_I a_I \left(\mathrm{d}x^I\right)_p\right) = \left(p, \sum_I a_I \left(\mathrm{d}x^{i_1}\right)_p \wedge \left(\mathrm{d}x^{i_2}\right)_p \wedge \cdots \wedge \left(\mathrm{d}x^{i_k}\right)_p\right).$$

Now, we can write the k-covector $\omega_p = \sum_I a_I \left(\mathrm{d}x^I\right)_p$ in the chart $(V, y^1, \dots, y^n)$ as follows:

$$\omega_p = \sum_I a_I \left(\mathrm{d}x^I\right)_p = \sum_J b_J \left(\mathrm{d}y^J\right)_p = \sum_J b_J \left(\mathrm{d}y^{j_1}\right)_p \wedge \left(\mathrm{d}y^{j_2}\right)_p \wedge \cdots \wedge \left(\mathrm{d}y^{j_k}\right)_p. \tag{12.7}$$

Now let us evaluate both sides of (12.7) to the tangent vectors $\left.\frac{\partial}{\partial y^{l_1}}\right|_p, \dots, \left.\frac{\partial}{\partial y^{l_k}}\right|_p$ to get:

$$\text{RHS} = \sum_J b_J \left(\mathrm{d}y^{j_1}\right)_p \wedge \cdots \wedge \left(\mathrm{d}y^{j_k}\right)_p \left(\left.\frac{\partial}{\partial y^{l_1}}\right|_p, \dots, \left.\frac{\partial}{\partial y^{l_k}}\right|_p\right) = \sum_J b_J \delta^J K = b_K, \tag{12.8}$$

where $K = (l_1, \dots, l_k)$ is a strictly ascending multi-index.

$$\text{LHS} = \sum_I a_I \left(\mathrm{d}x^{i_1}\right)_p \cdots \wedge \left(\mathrm{d}x^{i_k}\right)_p \left(\left.\frac{\partial}{\partial y^{l_1}}\right|_p, \dots, \left.\frac{\partial}{\partial y^{l_k}}\right|_p\right) = \sum_I a_I \det\left[\frac{\partial x^{i_{d_1}}}{\partial y^{l_{d_2}}}\right]_{d_1,d_2=1}^k. \tag{12.9}$$

Therefore,

$$b_K = \sum_I a_I \det\left[\frac{\partial x^{i_{d_1}}}{\partial y^{l_{d_2}}}\right]_{d_1,d_2=1}^k. \tag{12.10}$$

Now, in the action of $\overline{\psi} \circ \overline{\varphi}^{-1}$

$$\left(\varphi\left(p\right),\left(a_I\right)_I\right) \mapsto \left(\psi\left(p\right),\left(b_K\right)_K\right) = \left(\left(\psi \circ \varphi^{-1}\right)\left(\varphi\left(p\right)\right), \left(\sum_I a_I \det\left[\frac{\partial x^{i_{d_1}}}{\partial y^{l_{d_2}}}\right]^k_{d_1,d_2=1}\right)_K\right). \tag{12.11}$$

$\psi \circ \varphi^{-1}$ is smooth, the other components are also smooth, since they are just linear combination of smooth maps. Therefore, $\overline{\psi} \circ \overline{\varphi}^{-1}$ is C^∞, i.e. the charts $\left(\Lambda^k\left(T^*U\right), \overline{\varphi}\right)$ and $\left(\Lambda^k\left(T^*V\right), \overline{\psi}\right)$ are compatible. This proves that $\left\{\left(\Lambda^k\left(T^*U_\alpha\right), \overline{\varphi}_\alpha\right)\right\}_{\alpha\in I}$ is an atlas for $\Lambda^k\left(T^*M\right)$. So $\Lambda^k\left(T^*M\right)$ is a smooth manifold.

$\Lambda^k\left(T^*M\right)$ can, in fact, be shown to be a C^∞ vector bundle of rank $\binom{n}{k}$ over M, i.e. $\pi : \Lambda^k\left(T^*U\right) \to M$ is a C^∞ vector bundle of rank $\binom{n}{k}$ over M. Let $\pi : \Lambda^k\left(T^*M\right) \to M$ be the map that takes (p, ω_p) to $p \in M$. Then $\left(\Lambda^k\left(T^*M\right), M, \pi\right)$ is a vector bundle of rank $r = \binom{n}{k}$.

Here, $\pi^{-1}\left(p\right) = \{p\} \times \Lambda^k\left(T_p^*M\right)$, which is a vector space of dimension $\binom{n}{k}$. Each $p \in M$ is contained in a coordinate chart (U, φ), and we have a chart $\left(\Lambda^k\left(T^*U\right), \overline{\varphi}\right)$ on $\Lambda^k\left(T^*M\right)$. So we have a diffeomorphism

$$\overline{\varphi} : \pi^{-1}\left(U\right) = \Lambda^k\left(T^*U\right) \to \varphi\left(U\right) \times \mathbb{R}^{\binom{n}{k}}.$$

$\varphi\left(U\right)$ is diffeomorphic to U, via φ^{-1}. Therefore, we have the following diffeomorphism

$$\psi = \varphi^{-1} \times \mathbb{1}_{\mathbb{R}^{\binom{n}{k}}} \circ \overline{\varphi} : \pi^{-1}\left(U\right) \to U \times \mathbb{R}^{\binom{n}{k}}.$$

This diffeomorphism is fibre-preserving, since the following diagram commutes:

$$\begin{tikzcd}
\Lambda^k\left(T^*U\right) \arrow[rr, "\psi"] \arrow[dr, "\pi|_U"'] & & U \times \mathbb{R}^{\binom{n}{k}} \arrow[dl, "\pi'"] \\
& U &
\end{tikzcd}$$

Now, for every $q \in U$,

$$\psi|_{\pi^{-1}(q)} : \pi^{-1}(q) = \{q\} \times \Lambda^k\left(T_q^*U\right) \xrightarrow{\simeq} \{q\} \times \mathbb{R}^{\binom{n}{k}}$$

is a vector space isomorphism. Therefore, $\left(\Lambda^k\left(T^*M\right), M, \pi\right)$ is indeed a vector bundle of rank $r = \binom{n}{k}$.

A differential k-form is a section of this vector bundle. We define a k-form to be C^∞ if it is C^∞ as a section of the vector bundle $\Lambda^k\left(T^*M\right)$.

Notation. If $\pi : E \to M$ is a C^∞ vector bundle, then the vector space of C^∞ sections of E is denoted by $\Gamma(E)$, or $\Gamma(M, E)$. The vector space of all C^∞ k-forms, i.e. all C^∞ sections of the bundle $\Lambda^k\left(T^*M\right)$ is usually denoted by $\Omega^k\left(M\right)$. Thus,

$$\Omega^k\left(M\right) = \Gamma\left(\Lambda^k\left(T^*M\right)\right) = \Gamma\left(M, \Lambda^k\left(T^*M\right)\right).$$

Lemma 12.1

Let $\left(U, x^1, \ldots, x^n\right)$ be a chart on a manifold M. A k-form $\omega = \sum a_I dx^I$ on U is smooth if and only if the coefficient functions a_I are all smooth on U.

Proof. A k-form ω is just a section of this vector bundle. Now, given a chart $\left(U, x^1, \ldots, x^n\right)$ on M, the collection $\left\{\mathrm{d}x^I\right\}_I$ of sections (where I runs over the set of strictly ascending multi-indices of length k) is a smooth frame, since the collection $\left\{\left(\mathrm{d}x^I\right)_p\right\}_I$ forms a basis for $\Lambda^k\left(T_p^*M\right)$. Therefore, by *Proposition 9.4.2* of DG1, a section

$$\omega = \sum_I a_I \, \mathrm{d}x^I$$

of $\Lambda^k(T^*M)$ over U is C^∞ if and only if the coefficients a_I are C^∞ functions on U. ■

Proposition 12.2 (Characterization of a smooth k-form)
Let ω be a k-form on a manifold M. The following are equivalent:

(i) The k-form ω is C^∞ on M.

(ii) The manifold M has an atlas such that on every chart $(U,\phi)=(U,x^1,\ldots,x^n)$ in the atlas, the coefficients a_I of $\omega=\sum a_I\,\mathrm{d}x^I$ relative to the coordinate frame $\left\{\mathrm{d}x^I\right\}_I$ are all C^∞.

(iii) On every chart $(U,\phi)=(U,x^1,\ldots,x^n)$ on M, the coefficients a_I of $\omega=\sum a_I\,\mathrm{d}x^I$ relative to the coordinate frame $\left\{\mathrm{d}x^I\right\}_I$ are all C^∞.

(iv) For any k smooth vector fields $X_1,\ldots,X_k$ on M, the function $\omega\left(X_1,\ldots,X_k\right)$ is C^∞ on M.

Proof. **(ii)⇒(i):** By Lemma 12.1, for every point $p\in M$, there is a chart $(U,x^1,\ldots,x^n)$ about p such that ω is smooth on U. In particular, the section $\omega:M\to\Lambda^k\left(T^*M\right)$ is smooth at p, for every $p\in M$. Therefore, $\omega:M\to T^*M$ is a smooth map between manifolds.

(i)⇒(iii): If $\omega:M\to T^*M$ is a smooth map between manifolds, ω is smooth at every chart of M. Therefore, by Lemma 12.1, if $\omega=\sum_I a_I\,\mathrm{d}x^I$ on a chart $(U,x^1,\ldots,x^n)$, each a_i is smooth on U.

(iii)⇒(ii): Obvious.

(iii)⇒(iv): Given a chart $(U,\varphi)=(U,x^1,\ldots,x^n)$, $\omega=\sum_I a_I\,\mathrm{d}x^I$, and these coefficient functions a_I are all smooth. Suppose we are given any k smooth vector fields $X_1,X_2,\ldots,X_k$ on M. Then on U,

$$X_i=\sum_{j=1}^n b_i^j\frac{\partial}{\partial x^j}, \tag{12.12}$$

where each b_i^j are smooth functions on U. Therefore,

$$\begin{aligned}\omega\left(X_1,\ldots,X_k\right)&=\left(\sum_I a_I\,\mathrm{d}x^I\right)\left(X_1,\ldots,X_k\right)\\&=\left(\sum_I a_I\,\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_k}\right)\left(X_1,\ldots,X_k\right)\\&=\sum_I a_I\sum_{\sigma\in S_k}\left(\mathrm{d}x^{i_1}\otimes\cdots\otimes\mathrm{d}x^{i_k}\right)\left(X_{\sigma(1)},\ldots,X_\sigma(k)\right).\end{aligned}$$

Now, using (12.12),

$$\begin{aligned}\omega\left(X_1,\ldots,X_k\right)&=\sum_I a_I\sum_{\sigma\in S_k}\left(\mathrm{d}x^{i_1}\otimes\cdots\otimes\mathrm{d}x^{i_k}\right)\left(\sum_{j_1=1}^n b_{\sigma(1)}^{j_1}\frac{\partial}{\partial x^{j_1}},\ldots,\sum_{j_k=1}^n b_{\sigma(k)}^{j_k}\frac{\partial}{\partial x^{j_k}}\right)\\&=\sum_I a_I\sum_{\sigma\in S_k}\sum_{j_1,\ldots,j_k=1}^n b_{\sigma(1)}^{j_1}\cdots b_{\sigma(k)}^{j_k}\left(\mathrm{d}x^{i_1}\otimes\cdots\otimes\mathrm{d}x^{i_k}\right)\left(\frac{\partial}{\partial x^{j_1}},\ldots,\frac{\partial}{\partial x^{j_k}}\right)\\&=\sum_I a_I\sum_{\sigma\in S_k}\sum_{j_1,\ldots,j_k=1}^n b_{\sigma(1)}^{j_1}\cdots b_{\sigma(k)}^{j_k}\delta^{i_1}{}_{j_1}\cdots\delta^{i_k}{}_{j_k}\\&=\sum_I\sum_{\sigma\in S_k}a_I b_{\sigma(1)}^{i_1}\cdots b_{\sigma(k)}^{i_k},\end{aligned}$$

which is a sum of product of smooth functions, hence smooth. Therefore, $\omega\left(X_1,\ldots,X_k\right)$ is smooth on U. Since U is an arbitrary coordinate open set of M, $\omega\left(X_1,\ldots,X_k\right)$ is smooth on the whole M.

(iv)⇒(ii): Take $p\in M$, and let $(U,\varphi)=(U,x^1,\ldots,x^n)$ be a chart about p. For each $j=1,2,\ldots,n$, we can extend the vector field $\frac{\partial}{\partial x^j}$ to a C^∞ vector field X_j that agrees with $\frac{\partial}{\partial x^j}$ in a neighborhood V of p contained in U (V is not necessarily the whole of U, but possibly a smaller neighborhood).

On V, we can express ω as

$$\omega = \sum_I a_I \,\mathrm{d}\widetilde{x}^I, \tag{12.13}$$

where $\widetilde{x}^i = r^i \circ \varphi\big|_V = x^i\big|_V$. Fix a strictly ascending multi-index $J = (j_1, j_2, \ldots, j_k)$ of length k. Then $\omega\left(X_{j_1}, \ldots, X_{j_2}\right)$ is smooth on M, by hypothesis. Now, on V,

$$\begin{aligned}
\omega\left(X_{j_1}, \ldots, X_{j_2}\right) &= \left(\sum_I a_I \,\mathrm{d}\widetilde{x}^I\right)\left(\frac{\partial}{\partial x^{j_1}}, \ldots, \frac{\partial}{\partial x^{j_k}}\right) \\
&= \sum_I a_I \left(\mathrm{d}\widetilde{x}^{i_1} \wedge \cdots \wedge \mathrm{d}\widetilde{x}^{i_k}\right)\left(\frac{\partial}{\partial x^{j_1}}, \ldots, \frac{\partial}{\partial x^{j_k}}\right) \\
&= \sum_I a_I \left(\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}\right)\left(\frac{\partial}{\partial x^{j_1}}, \ldots, \frac{\partial}{\partial x^{j_k}}\right),
\end{aligned}$$

since $\widetilde{x}^i$ is nothing but x^i restricted to V. Now,

$$\omega\left(X_{j_1}, \ldots, X_{j_2}\right) = \sum_I a_I \left(\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}\right)\left(\frac{\partial}{\partial x^{j_1}}, \ldots, \frac{\partial}{\partial x^{j_k}}\right) = \sum_I a_I \delta^I{}_J = a_J. \tag{12.14}$$

Therefore, a_J is smooth on V. So on the chart $\left(V, \widetilde{x}^1, \ldots, \widetilde{x}^n\right)$, if we write $\omega = \sum_I a_I \,\mathrm{d}\widetilde{x}^I$, the coefficient functions a_I are all smooth. Around each point p, we can find such a chart $\left(V, \widetilde{x}^1, \ldots, \widetilde{x}^n\right)$. ■

Example 12.2. We defined the 0-tensors and the 0-covectors as constants, i.e. for a real vector space V, $A_0(V) = L_0(V) = \mathbb{R}$. Now, recall that

$$\Lambda^k\left(T^*M\right) = \bigcup_{p \in M} \{p\} \times \Lambda^k\left(T_p^*M\right).$$

Since $\Lambda^0\left(T_p^*M\right) = \mathbb{R}$ for every $p \in M$, one has

$$\Lambda^0\left(T^*M\right) = \bigcup_{p \in M} \{p\} \times \mathbb{R} = M \times \mathbb{R}. \tag{12.15}$$

Hence,

$$\Omega^0\left(M\right) = \Gamma\left(\Lambda^0\left(T^*M\right)\right) = \Gamma\left(M, M \times \mathbb{R}\right). \tag{12.16}$$

A C^∞ section of the 0-th exterior power of the tangent bundle T^*M is nothing but a C^∞ sectio of the globally trivial C^∞ vector bundle $M \times \mathbb{R}$ over M. Such a section maps $p \in M$ to a pair $(p, \sigma(p))$ with $\sigma(p) \in \mathbb{R}$. Therefore, such a section is nothing but a smooth assignment $p \mapsto \sigma(p)$, i.e. $\sigma \in C^\infty(M, \mathbb{R})$. So

$$\Omega^0\left(M\right) = \Gamma\left(M, M \times \mathbb{R}\right) = C^\infty\left(M, \mathbb{R}\right).$$

§12.3 Pullback of k-forms

Let $F : N \to M$ be a smooth map of manifolds. Recall that a 1-form $\omega \in \Omega^1(M)$ can be pulled back to $\Omega^1(N)$ via the pullback $F^* : \Omega^1(M) \to \Omega^1(N)$ defined by

$$\left(F^*\omega\right)_p\left(X_p\right) = \omega_{F(p)}\left(F_{*,p}\left(X_p\right)\right). \tag{12.17}$$

For 0-forms, i.e. functions, the pullback is defined by composition:

$$N \xrightarrow{F} M \xrightarrow{f} \mathbb{R}$$

Given $f \in C^\infty(M, \mathbb{R})$, its pullback is defined to be

$$F^*(f) = f \circ F \in C^\infty(N, \mathbb{R}), \tag{12.18}$$

so that indeed $F^* : \Omega^0(M) \to \Omega^0(N)$.

For a k-form ω on M, we define its pullback $F^*\omega$ as follows: if $p \in N$ and $X_p^1, X_p^2, \ldots, X_p^k \in T_pN$ are k tangent vectors, then

$$(F^*\omega)_p\left(X_p^1, X_p^2, \ldots, X_p^k\right) = \omega_{F(p)}\left(F_{*,p}\left(X_p^1\right), F_{*,p}\left(X_p^2\right), \ldots, F_{*,p}\left(X_p^k\right)\right). \tag{12.19}$$

Example 12.3. Let $U \subseteq M$ be open, and $\iota : U \to M$ be the inclusion map. For a smooth 0-form on M, i.e. a smooth function $f : M \to \mathbb{R}$, its pullback under ι^* is

$$\iota^* f = f \circ \iota = f\big|_U. \tag{12.20}$$

For a k-form ω on M, its pullback $\iota^*\omega$ is also given by restriction of domain. Indeed, for $p \in U$ and $X_p^1, X_p^2, \ldots, X_p^k \in T_pU = T_pM$, $\iota_{*,p}X_p^i = X_p^i$. So

$$\begin{aligned}(\iota^*\omega)_p\left(X_p^1, X_p^2, \ldots, X_p^k\right) &= \omega_{\iota(p)}\left(\iota_{*,p}\left(X_p^1\right), \ldots, \iota_{*,p}\left(X_p^k\right)\right)\\ &= \omega_p\left(X_p^1, X_p^2, \ldots, X_p^k\right).\end{aligned}$$

Therefore,

$$(\iota^*\omega)_p = \omega_p, \tag{12.21}$$

for $p \in U$. As a result, $\iota^*\omega = \omega\big|_U$.

Proposition 12.3 (Linearity of pullback)

Let $F : N \to M$ be a C^∞ map. If ω, τ are k-forms on M and α is a real number, then

(i) $F^*(\omega + \tau) = F^*\omega + F^*\tau$.

(ii) $F^*(\alpha\omega) = \alpha F^*\omega$.

Proof. Suppose $F : N \to M$ is C^∞. Then the pullback $F^* : \Omega^*(M) \to \Omega^*(N)$ is defined as follows: if $\omega \in \Omega^k(M)$, $F^*\omega \in \Omega^k(N)$ is defined as:

$$(F^*\omega)_p\left(X_p^1, \ldots, X_p^k\right) = \omega_{F(p)}\left(F_{*,p}X_p^1, \ldots, F_{*,p}X_p^k\right), \tag{12.22}$$

for $p \in N$, and $X_p^i \in T_pN$.

(a) For $\omega, \tau \in \Omega^k(M)$ and $X_p^1, \ldots, X_p^k \in T_pN$,

$$\begin{aligned}(F^*(\omega+\tau))_p\left(X_p^1, \ldots, X_p^k\right) &= (\omega+\tau)_{F(p)}\left(F_{*,p}X_p^1, \ldots, F_{*,p}X_p^k\right)\\ &= \left(\omega_{F(p)} + \tau_{F(p)}\right)\left(F_{*,p}X_p^1, \ldots, F_{*,p}X_p^k\right)\\ &= \omega_{F(p)}\left(F_{*,p}X_p^1, \ldots, F_{*,p}X_p^k\right) + \tau_{F(p)}\left(F_{*,p}X_p^1, \ldots, F_{*,p}X_p^k\right)\\ &= (F^*\omega)_p\left(X_p^2, \ldots, X_p^k\right) + (F^*\tau)_p\left(X_p^2, \ldots, X_p^k\right)\end{aligned}$$

Therefore,

$$F^*(\omega+\tau) = F^*\omega + F^*\tau. \tag{12.23}$$

(b) For $\alpha \in \mathbb{R}$ and $X_p^1, \ldots, X_p^k \in T_pN$,

$$\begin{aligned}
\left(F^*(\alpha\omega)\right)_p\left(X_p^1, \ldots, X_p^k\right) &= (\alpha\omega)_{F(p)}\left(F_{*,p}X_p^1, \ldots, F_{*,p}X_p^k\right) \\
&= \alpha\omega_{F(p)}\left(F_{*,p}X_p^1, \ldots, F_{*,p}X_p^k\right) \\
&= \alpha \cdot (F^*\omega)_p\left(X_p^1, \ldots, X_p^k\right).
\end{aligned}$$

Therefore,

$$F^*(\alpha\omega) = \alpha F^*\omega. \tag{12.24}$$

■

§12.4 The Wedge Product

If $\omega \in \Omega^k(M)$ and $\tau \in \Omega^l(M)$, then for any $p \in M$, $\omega_p \in \Lambda^k\left(T_p^*M\right)$ and $\tau_p \in \Lambda^l\left(T_p^*M\right)$ and $\omega_p \wedge \tau_p \in \Lambda^{k+l}\left(T_p^*M\right)$. Then we define the wedge product of ω and τ to be the $(k+l)$-form $\omega \wedge \tau$ such that

$$(\omega \wedge \tau)_p = \omega_p \wedge \tau_p. \tag{12.25}$$

> **Proposition 12.4**
> If ω and τ are C^∞ forms on M, then so is $\omega \wedge \tau$.

Proof. Let $(U, x^1, \ldots, x^n)$ be a chart on M. On U,

$$\omega = \sum_I a_I \,\mathrm{d}x^I, \quad \tau = \sum_J b_J \,\mathrm{d}x^J \tag{12.26}$$

for C^∞ functions a_I, b_J on U. Their Wedge product is

$$\begin{aligned}
\omega \wedge \tau &= \left(\sum_I a_I \,\mathrm{d}x^I\right) \wedge \left(\sum_J b_J \,\mathrm{d}x^J\right) \\
&= \sum_{I,J} a_I b_J \,\mathrm{d}x^I \wedge \mathrm{d}x^J \ldots
\end{aligned} \tag{12.27}$$

In (12.27), $\mathrm{d}x^I \wedge \mathrm{d}x^J = 0$ if I and J have at least an index in common. If I and J are disjoint, i.e., have none of their indices to be common, then

$$\mathrm{d}x^I \wedge \mathrm{d}x^J = \pm \mathrm{d}x^K, \tag{12.28}$$

where $K = I \cup J$ but reordered as an increasing sequence. Thus,

$$\omega \wedge \tau = \sum_K \left(\sum_{I \cup J = K} \pm a_I b_J\right) \mathrm{d}x^K. \tag{12.29}$$

Since the coefficients of $\mathrm{d}x^K$ in (12.29) are C^∞, by Proposition 12.2, $\omega \wedge \tau$ is C^∞ on M. ■

> **Proposition 12.5** (Pullback of wedge product)
> If $F : N \to M$ is a C^∞ map of manifolds and ω are τ are differential forms on M, then
>
> $$F^*(\omega \wedge \tau) = F^*(\omega) \wedge F^*(\tau). \tag{12.30}$$

Proof. If $\omega \in \Omega^k(M), \tau \in \Omega^l(M)$, and $X_p^1, \ldots, X_p^{k+l} \in T_pN$,

$$
\begin{aligned}
&(F^*(\omega \wedge T))_p \left(X_p^1, \ldots, X_p^{k+l}\right) \\
&= (\omega \wedge \tau)_{F(p)} \left(F_{*,p}X_p^1, \ldots, F_{*,p}X_p^{k+l}\right) \\
&= \left(\omega_{F(p)} \wedge \tau_{F(p)}\right) \left(F_{*,p}X_p^1, \ldots, F_{*,p}X_p^{k+l}\right) \\
&= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma)\, \omega_{F(p)} \left(F_{*,p}X_p^{\sigma(1)}, \ldots, F_{*,p}X_p^{\sigma(k)}\right) \tau_{F(p)} \left(F_{*,p}X_p^{\sigma(k+1)}, \ldots, F_{*,p}X_p^{\sigma(k+l)}\right) \\
&= \frac{1}{k!l!} \sum_{\sigma \in S_{k+1}} (\operatorname{sgn} \sigma) F^*\omega \left(X_p^{\sigma(1)}, \ldots, X_p^{\sigma(k)}\right) F^*\tau \left(X_p^{\sigma(k+1)}, \ldots, X_p^{\sigma(k+1)}\right) \\
&= \frac{1}{k!l!} \sum_{\sigma \in S_{k+1}} (\operatorname{sgn} \sigma) \left(F^*\omega \otimes F^*\tau\right) \left(X_p^{\sigma(1)}, \ldots, X_p^{\sigma(k+l)}\right) \\
&= \frac{1}{k!l!} \mathcal{A} \left(F^*\omega \otimes F^*\tau\right) \left(X_p^1, \ldots, X_p^{k+l}\right) \\
&= \left(F^*(\omega) \wedge F^*(\tau)\right) \left(X_p^1, \ldots, X_p^{k+l}\right).
\end{aligned}
$$

Therefore,

$$
F^*(\omega \wedge \tau) = F^*(\omega) \wedge F^*(\tau). \tag{12.31}
$$

■

We define the vector space $\Omega^*(M)$ of C^∞ differential forms on a manifold M of dimension n to be the direct sum

$$
\Omega^*(M) = \bigoplus_{k=0}^{n} \Omega^k(M). \tag{12.32}
$$

Each element of $\Omega^*(M)$ is uniquely a formal sum $\sum_{i=1}^r \omega_{k_i}$ with $\omega_{k_i} \in \Omega^{k_i}(M)$. With the wedge product, the vector space $\Omega^*(M)$ becomes a **graded algebra**, graded by the the degree of differential forms. Proposition 12.3 and Proposition 12.5 tells us that the pullback map $F^* : \Omega^*(M) \to \Omega^*(N)$ is a homomorphism of graded algebras[1].

[1]Note that we haven't yet proved that F^* prerserves smoothness of forms, so we don't yet know that F^* maps $\Omega^k(M)$ into $\Omega^k(N)$. But we shall soon prove this in Theorem 13.6, and once we do that we are all good with the notation.

13 Exterior Derivative

The basic objects in differential geometry are differential forms. Our goal will be to learn how we can differentiate and integrate differential forms on manifolds. Recall that an antiderivation on a graded algebra $A = \bigoplus_{k=0}^{\infty} A^k$ is an $\mathbb{R}$-linear map $D : A \to A$ such that

$$D(\omega \cdot \tau) = (D\omega) \cdot \tau + (-1)^k \omega \cdot (D\tau),$$

for $\omega \in A^k$ and $\tau \in A^l$, and $\cdot$ is the multiplication of the graded algebra. In the graded algebra A, an element of A^k is called a **homogenous element of degree** k. The antiderivation D is of degree m if

$$\deg(D\omega) = \deg\omega + m$$

for all homogenous elements $\omega \in A$.

Now, let M be a manifold and $\Omega^*(M)$ the graded algebra of C^∞ differential forms on M. Now, we'll see that on the graded algebra $\Omega^*(M)$, there is a uniquely and intrinsically defined anti-derivation called exterior derivative.

Definition 13.1 (Exterior derivative)**.** An **exterior derivative** on a manifold M is an $\mathbb{R}$-linear map

$$D : \Omega^*(M) \to \Omega^*(M)$$

such that

(i) D is an antiderivation of degree 1,

(ii) $D \circ D = 0$,

(iii) if f is a C^∞ function and X is a C^∞ vector field on M, then $(Df)(X) = Xf$.

Remark 13.1. Condition (iii) in the definition above says that on 0-forms, i.e. C^∞ functions on M, an exterior derivative agrees with the differential $\mathrm{d}f$ of a function f. We have learned earlier that in a coordinate chart $(U, x^1, \ldots, x^n)$, the 1-form $\mathrm{d}f$ can be expressed as

$$\mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}\,\mathrm{d}x^i.$$

Hence, in the chart $(U, x^1, \ldots, x^n)$,

$$Df = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}\,\mathrm{d}x^i.$$

We now prove the existence and uniqueness of the exterior differentiation on a manifold.

Lemma 13.1

Let $D : \Omega^*(M) \to \Omega^*(M)$ be an exterior derivative on M. If $f^1, \ldots, f^k$ are smooth functions on U, then

$$D\left(Df^1 \wedge Df^2 \wedge \cdots \wedge Df^k\right) = 0.$$

Proof. We prove it by induction on k. The base case $k = 1$ follows trivially from $D \circ D = 0$. Suppose

$D\left(Df^1 \wedge Df^2 \wedge \cdots \wedge Df^{k-1}\right) = 0$. Then

$$\begin{aligned} & D\left(Df^1 \wedge Df^2 \wedge \cdots \wedge Df^k\right) \\ &= D\left(\left(Df^1 \wedge Df^2 \wedge \cdots \wedge Df^{k-1}\right) \wedge Df^k\right) \\ &= D\left(Df^1 \wedge Df^2 \wedge \cdots \wedge Df^{k-1}\right) \wedge Df^k + (-1)^{k-1}\left(Df^1 \wedge Df^2 \wedge \cdots \wedge Df^{k-1}\right) \wedge D\left(Df^k\right) \\ &= 0. \end{aligned} \tag{13.1}$$

Therefore, $D\left(Df^1 \wedge Df^2 \wedge \cdots \wedge Df^k\right) = 0$ for any $k \geq 1$. ■

§13.1 Exterior Derivative on a Coordinate Chart

Suppose $(U, x^1, \ldots, x^n)$ is a coordinate chart on a manifold M. Then any k-form ω on U is uniquely a linear combination

$$\omega = \sum_I a_I \, \mathrm{d}x^I,$$

where $a_I \in C^\infty(U)$, and the sum runs over all strictly ascending multi-indices I of length k. The $\mathbb{R}$-linear map $\mathrm{d} : \Omega^*(U) \to \Omega^*(U)$ can be constructed to be an exterior derivative on U. In fact, d acts on a homogenous element $\omega \in \Omega^k(U)$ in the following way:

$$\begin{aligned} \mathrm{d}\omega = \mathrm{d}\left(\sum_I a_I \, \mathrm{d}x^I\right) &= \sum_I \mathrm{d}a_I \wedge \mathrm{d}x^I + (-1)^0 \sum_I a_I \, \mathrm{dd}x^I \\ &= \sum_I \mathrm{d}a_I \wedge \mathrm{d}x^I + \sum_I a_I \, \mathrm{d}\left(\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}\right) \\ &= \sum_I \mathrm{d}a_I \wedge \mathrm{d}x^I \\ &= \sum_I \sum_j \frac{\partial a_I}{\partial x^j} \mathrm{d}x^j \wedge \mathrm{d}x^I. \end{aligned} \tag{13.2}$$

(13.2) suggests that $\mathrm{d}\omega \in \Omega^{k+1}(U)$, and it can be written in the chart $(U, x^1, \ldots, x^n)$ using (13.2). This proves the existence of the exterior derivative $\mathrm{d} : \Omega^*(U) \to \Omega^*(U)$, on an open set U of M. The uniqueness of $\mathrm{d} : \Omega^*(U) \to \Omega^*(U)$ can be shown exactly the same way we proved it for the Euclidean case in Proposition 10.4.

Sometimes we write $\mathrm{d}_U\omega$ instead of $\mathrm{d}\omega$ to emphasize that it is the **unique** exterior derivative on the open set $U \subseteq M$. In other words, if (U, x^i) and (U, y^j) are two charts on M, and $\omega = \sum a_I \, \mathrm{d}x^I = \sum b_J \, \mathrm{d}y^J$, then

$$\mathrm{d}_U\omega = \sum_I \sum_i \frac{\partial a_I}{\partial x^i} \mathrm{d}x^i \wedge \mathrm{d}x^I = \sum_J \sum_i \frac{\partial b_J}{\partial y^i} \mathrm{d}y^i \wedge \mathrm{d}y^I. \tag{13.3}$$

This reveals that the expression $\mathrm{d}_U\omega$ is chart independent.

§13.2 Local Operators

An endomorphism of a vector space W (a linear transformation from W to itself) is often called an operator on W. For example, if $W = C^\infty(\mathbb{R})$, the vector space of C^∞ functions on $\mathbb{R}$, then $\frac{\mathrm{d}}{\mathrm{d}x}$ is an operator on W:

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) = f'(x).$$

The derivative has the desired property that the value of f' at a point p depends only on the values of f in a small neighborhood of p. More precisely, if $f = g$ on an open set $U \subseteq \mathbb{R}$, then $f' = g'$ on U. We say that the derivative is a local operator on $C^\infty(\mathbb{R})$.

Definition 13.2 (Local operator)**.** An operator $D : \Omega^*(M) \to \Omega^*(M)$ is said to be **local** if for all $k \geq 0$, whenever a k-form $\omega \in \Omega^k(M)$ restricts to 0 on an open set U (i.e. $\omega_p = 0$ at every $p \in U$), then $D\omega \equiv 0$ on U (i.e. $(D\omega)_p = 0$ at every $p \in U$).

An equivalent definition of local operator is that for all $k \geq 0$, whenever two k-forms $\omega, \tau \in \Omega^k(M)$ agree on an open set U, then $D\omega \equiv D\tau$ on U (i.e. $(D\omega)_p = (D\tau)_p$ at every $p \in U$).

Proposition 13.2

Any antiderivation D on $\Omega^*(M)$ is a local operator.

Proof. Suppose $\omega \in \Omega^*(M)$ and $\omega \equiv 0$ on an open subset U. Let $p \in U$. It suffices to show that $(D\omega)_p = 0$. Take a bump function f at p supported in U, i.e. $\operatorname{supp} f \subseteq U$. In particular, $f \equiv 1$ in a neighborhood V of p in U, so that $V \subset \operatorname{supp} f \subseteq U$. Then $f\omega \equiv 0$ on M. This can be seen by noting that if $q \in U$,

$$(f\omega)_q = f(q)\omega_q = 0,$$

since $\omega_q = 0$ by hypothesis. On the other hand, if $q \notin U$, then $q \notin \operatorname{supp} f$, so $f(q) = 0$, which yields

$$(f\omega)_q = f(q)\omega_q = 0.$$

Therefore, $f\omega \equiv 0$ on M. Applying D on $f\omega = f \wedge \omega$, we get

$$D(f\omega) = (Df) \wedge \omega + (-1)^0 f \wedge D\omega. \tag{13.4}$$

By the linearity of D, $D(f\omega) = 0$. Now, we evaluate the RHS of (13.4) at $p \in U$, and use the fact that $f(p) = 1$ and $\omega_p = 0$. As a result,

$$\begin{aligned} & (Df)_p \wedge \omega_p + f(p) \wedge (D\omega)_p = 0 \\ \implies & (D\omega)_p = 0. \end{aligned} \tag{13.5}$$

Since $p \in U$ is arbitrary, $D\omega \equiv 0$ on U. ■

Sometimes we are given a differential form τ that is defined only on an open subset U of a manifold M. We can use bump functions to extend τ to a global form $\tilde{\tau}$ on M that agrees with τ near some point.

Proposition 13.3

Suppose τ is a C^∞ differential k-form on an open subset U of M (such a differential form is called a local differential form). For any $p \in U$. There is a C^∞ global form $\tilde{\tau}$ on M (can be defined anywhere on M using its charts) that agrees with τ on a neighborhood of p contained in U.

Proof. Choose a smooth bump function f at p supported in U, i.e. $\operatorname{supp} f \subseteq U$. In particular, $f \equiv 1$ in a neighborhood V of p in U, so that $V \subset \operatorname{supp} f \subseteq U$. Then we define

$$\tilde{\tau}_q = \begin{cases} f(q)\,\tau_q & \text{if } q \in U, \\ \mathbf{0}_{\Lambda^k(T_q^*M)} & \text{if } q \notin U. \end{cases}$$

By the definition of $\tilde{\tau}$, it agrees with τ on V. By *Proposition 9.3.1(ii)* of DG1, $\tilde{\tau}$ is smooth on U. Now, let $q \notin U$. We want to show that $\tilde{\tau}$ is smooth at q.

Since $\operatorname{supp} f \subseteq U$, $q \notin U$ implies $q \in M \setminus U \subseteq M \setminus \operatorname{supp} f$. Since $\operatorname{supp} f$ is closed, $M \setminus \operatorname{supp} f$ is open. Hence, we can find a coordinate chart (W, φ) about q such that $W \subseteq M \setminus \operatorname{supp} f$. Then, for $r \in W$, $\tilde{\tau}_r = \mathbf{0}_{\Lambda^k(T_r^*M)}$. Also, $\left(\Lambda^k(T^*U), \overline{\varphi}\right)$ is a chart on $\Lambda^k(T^*M)$ about $\mathbf{0}_{\Lambda^k(T_r^*M)}$.

$$(\overline{\varphi} \circ \tilde{\tau})(r) = (\varphi(r), \underbrace{0, 0, \ldots, 0}_{\binom{n}{k} \text{ 0-s}}).$$

φ is smooth. Therefore, $\tilde{\tau}$ is smooth on W. In particular, $\tilde{\tau}$ is smooth at q. Since $q \notin U$ was arbitrary, $\tilde{\tau}$ is smooth at every $q \notin U$. Therefore, $\tilde{\tau}$ is smooth on all of M. ■

§13.3 Existence and Uniqueness of an Exterior Differentiation

To define an exterior derivative $\mathrm{d}:\Omega^*(M)\to\Omega^*(M)$, let $\omega\in\Omega^k(M)$ and $p\in M$. Choose a chart $(U,x^1,\ldots,x^n)$ about p. Suppose $\omega=\sum_I a_I\,\mathrm{d}x^I$ on U. Now, $\mathrm{d}\omega$ is supposed to be a $(k+1)$-form on M, i.e. $\mathrm{d}\omega\in\Omega^{k+1}(M)$. Define $\mathrm{d}\omega\in\Omega^{k+1}(M)$ such that at $p\in U$, $(\mathrm{d}\omega)_p$ is expressed as

$$(\mathrm{d}\omega)_p=\left(\sum_I \mathrm{d}a_I\wedge \mathrm{d}x^I\right)_p. \tag{13.6}$$

It needs to be proven that the definition (13.6) is independent of chart. If $(V,y^1,\ldots,y^n)$ is another chart about p, and $\omega=\sum_J b_J\,\mathrm{d}y^J$ on V, then on $U\cap V$,

$$\sum_I a_I\,\mathrm{d}_{U\cap V}x^I=\sum_J b_J\,\mathrm{d}_{U\cap V}y^J,$$

where $\mathrm{d}_{U\cap V}$ is the unique exterior derivative $\mathrm{d}_{U\cap V}:\Omega^*(U\cap V)\to\Omega^*(U\cap V)$. Then by the locality of exterior derivative,

$$\mathrm{d}_{U\cap V}\left(\sum_I a_I\,\mathrm{d}_{U\cap V}x^I\right)=\mathrm{d}_{U\cap V}\left(\sum_J b_J\,\mathrm{d}_{U\cap V}y^J\right). \tag{13.7}$$

Reading off the antiderivation $\mathrm{d}_{U\cap V}$ in the chart $(U\cap V,x^1,\ldots,x^n)$ using (13.6), the LHS of (13.7) can be recast into

$$\sum_I \mathrm{d}_{U\cap V}a_I\,\mathrm{d}_{U\cap V}x^I.$$

On the other hand, the antiderivation $\mathrm{d}_{U\cap V}$ in the chart $(U\cap V,y^1,\ldots,y^n)$ can be expressed using (13.6) to compute the RHS of (13.7):

$$\sum_J \mathrm{d}_{U\cap V}b_J\,\mathrm{d}_{U\cap V}y^J.$$

Therefore,

$$\sum_I \mathrm{d}_{U\cap V}a_I\,\mathrm{d}_{U\cap V}x^I=\sum_J \mathrm{d}_{U\cap V}b_J\,\mathrm{d}_{U\cap V}y^J, \tag{13.8}$$

on $U\cap V$. In particular, for $p\in U\cap V$,

$$\left(\sum_I \mathrm{d}_{U\cap V}a_I\,\mathrm{d}_{U\cap V}x^I\right)_p=\left(\sum_J \mathrm{d}_{U\cap V}b_J\,\mathrm{d}_{U\cap V}y^J\right)_p,$$

proving that the definition (13.6) is indeed chart independent. As p varies over all of M, (13.6) defines an operator

$$\mathrm{d}:\Omega^*(M)\to\Omega^*(M).$$

It's straightforward to verify that the 3 desired conditions of exterior derivative are fullfilled by the definition (13.6).

Now we prove the uniqueness of exterior derivative. Suppose $D:\Omega^*(M)\to\Omega^*(M)$ is an exterior derivative. We will now show that D coincides with the exterior derivative defined by (13.6).

Let $\omega\in\Omega^k(M)$, and $p\in M$. Choose a chart $(U,x^1,\ldots,x^n)$ about p, and suppose $\omega=\sum_I a_I\,\mathrm{d}x^I$ on U. Extend the functions $a_I,x^1,\ldots,x^n$ to C^∞ functions $\widetilde{a}_I,\widetilde{x}^1,\ldots,\widetilde{x}^n$ that agrees with $a_I,x^1,\ldots,x^n$ in a neighborhood V of p. Define

$$\widetilde{\omega}=\sum_I \widetilde{a}_I\,\mathrm{d}\widetilde{x}^I. \tag{13.9}$$

Then $\omega\equiv\widetilde{\omega}$ on V. Since D is a local operator, one must have $D\omega\equiv D\widetilde{\omega}$ on V. Thus,

$$(D\omega)_p=(D\widetilde{\omega})_p=\left[D\left(\sum_I \widetilde{a}_I\,\mathrm{d}\widetilde{x}^I\right)\right]_p. \tag{13.10}$$

Since D is an exterior derivative operator on $\Omega^* M$, and d is the exterior derivative operator defined by (13.6), for $f \in C^\infty(M)$,

$$(Df)(X) = Xf = (\mathrm{d}f)(X),$$

for any C^∞ vector field X. In particular,

$$D\widetilde{a}_I = \mathrm{d}\widetilde{a}_I, \text{ and } D\widetilde{x}^i = \mathrm{d}\widetilde{x}^i,$$

so that $D\widetilde{x}^I = d\widetilde{x}^I$, for a strictly ascending multi-index I of length k. Hence, (13.10) reduces to

$$\begin{aligned}(D\omega)_p &= \left[D\left(\sum_I \widetilde{a}_I\, \mathrm{d}\widetilde{x}^I\right)\right]_p \\ &= \left[D\left(\sum_I \widetilde{a}_I\, D\widetilde{x}^I\right)\right]_p \\ &= \left(\sum_I D\widetilde{a}_I \wedge D\widetilde{x}^I\right)_p \\ &= \left(\sum_I \mathrm{d}\widetilde{a}_I \wedge \mathrm{d}\widetilde{x}^I\right)_p .\end{aligned}$$

Now, since $\widetilde{a}_I = a_I$ and $\widetilde{x}^i = x^i$ in a neighborhood of p, we have $\mathrm{d}\widetilde{a}_I = \mathrm{d}a_I$ and $\mathrm{d}\widetilde{x}^I = \mathrm{d}x^I$ at p. Therefore,

$$(D\omega)_p = \left(\sum_I \mathrm{d}\widetilde{a}_I \wedge \mathrm{d}\widetilde{x}^I\right)_p = \left(\sum_I \mathrm{d}a_I \wedge \mathrm{d}x^I\right)_p = (\mathrm{d}\omega)_p . \tag{13.11}$$

So $D = d$, and hence the exterior derivative is unique.

The restriction of a k-form to a submanifold

Let S be a regular submanifold of a manifold M, and ω is a k-form on M, i.e. $\omega \in \Omega^k(M)$. Then the restriction of ω to S is the k-form $\omega|_S$ on S defined by

$$\left(\omega|_S\right)_p \left(X_p^1, \ldots, X_p^k\right) = \omega_p \left(X_p^1, \ldots, X_p^k\right), \tag{13.12}$$

for $X_p^1, \ldots, X_p^k \in T_pS \subseteq T_pM$. Thus, $\left(\omega|_S\right)_p$ is obtained from ω_p by restricting its domain to $T_pS \times T_pS \times \cdots \times T_pS$ (k-times).

Example 13.1. If S is a smooth curve in $\mathbb{R}^2$ defined by the non-constant function $f(x, y) = 0$ (f could be $x^2 + y^2 - 1$, defining the unit circle in $\mathbb{R}^2$), then

$$\mathrm{d}f = \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y$$

is a nonzero 1-form on $\mathbb{R}^2$. But since f is identically 0 on S, $(\mathrm{d}f)|_S = 0$. So a nonzero form on M can be restricted to a zero form on a submanifold S.

A form that is not identically zero is called a **nonzero form**. On the other hand, a form ω that is nowhere zero,i.e. $\omega_p \neq 0$ for all $p \in M$, is called a **nowhere vanishing form**.

Example 13.2 (A nowhere vanishing 1-form on S^1)**.** Let S^1 be the unit circle defined by $x^2 + y^2 = 1$ in $\mathbb{R}^2$. The 1-form $\mathrm{d}x$ restricts from $\mathbb{R}^2$ to a 1-form on S^1. When restricted to S^1, the domain of the covector $\left((\mathrm{d}x)|_{S^1}\right)_p$ is T_pS^1 instead of $T_p\mathbb{R}^2$:

$$\left((\mathrm{d}x)|_{S^1}\right)_p : T_pS^1 \to \mathbb{R}.$$

Now, from $x^2 + y^2 = 1$, one obtains

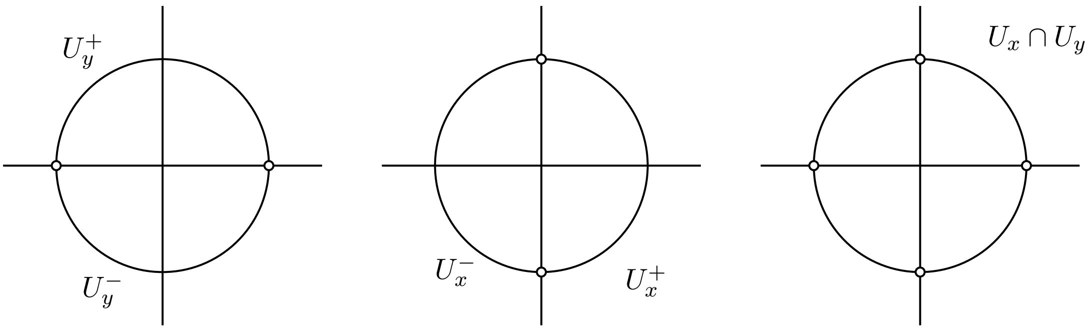
$$2x\,\mathrm{d}x + 2y\,\mathrm{d}y = 0. \tag{13.13}$$

At $p = (1, 0)$, (13.13) reduces to $(\mathrm{d}x)_p = 0$. It shows that although $\mathrm{d}x$ is a nowhere vanishing 1-form on $\mathbb{R}^2$, it vanishes at $(1, 0)$ when restricted to S^1.

To find a nowhere vanishing 1-form on S^1, we again take exterior derivative of both sides of the equation $x^2 + y^2 - 1 = 0$ to arrive at

$$2x\,\mathrm{d}x + 2y\,\mathrm{d}y = 0. \tag{13.14}$$

Let $U_x = \{(x, y) \in S^1 \mid x \neq 0\}$, and $U_y = \{(x, y) \in S^1 \mid y \neq 0\}$.



By (13.14), then one obtains on $U_x \cap U_y$,

$$\frac{\mathrm{d}y}{x} = -\frac{\mathrm{d}x}{y}. \tag{13.15}$$

Now we define a 1-form ω on S^1 by

$$\omega = \begin{cases} \frac{\mathrm{d}y}{x} & \text{on } U_x, \\ -\frac{\mathrm{d}x}{y} & \text{on } U_y. \end{cases} \tag{13.16}$$

Since $\frac{\mathrm{d}y}{x} = -\frac{\mathrm{d}x}{y}$ on $U_x \cap U_y$, ω is a well-defined 1-form on $S^1 = U_x \cup U_y$. To show that ω is C^∞ and nowhere vanishing, we need charts.

$$U_x^+ = \left\{(x, y) \in S^1 \mid x > 0\right\}, U_x^- = \left\{(x, y) \in S^1 \mid x < 0\right\},$$
$$U_y^+ = \left\{(x, y) \in S^1 \mid y > 0\right\}, U_y^- = \left\{(x, y) \in S^1 \mid y < 0\right\}.$$

On U_x^+, the local coordinates are the y-coordinates, so that $(\mathrm{d}y)_p$ is a basis for the cotangent space $T_p^*S^1$ at each $p \in U_x^+$. Now, since $\omega = \frac{\mathrm{d}y}{x}$ on U_x^+, ω is C^∞ and nowhere zero on U_x^+. Similarly, $\omega = \frac{\mathrm{d}y}{x}$ on U_x^- is also C^∞ and nowhere zero on U_x^-. One can show using similar argument that $\omega = -\frac{\mathrm{d}x}{y}$ is C^∞ and nowhere vanishing on U_y^+ and U_y^-. Hence, ω is C^∞ and nowhere zero on S^1.

It's easy to see that this nowhere vanishing smooth 1-form on S^1 is nothing but $x\,\mathrm{d}y - y\,\mathrm{d}x$. On U_x, $x \neq 0$; so using $x\,\mathrm{d}x + y\,\mathrm{d}y = 0$, we get

$$\begin{aligned} x\,\mathrm{d}y - y\,\mathrm{d}x &= x\,\mathrm{d}y - \frac{y}{x}\,x\,\mathrm{d}x = x\,\mathrm{d}y + \frac{y^2}{x}\,\mathrm{d}y \\ &= \left(x + \frac{y^2}{x}\right)\mathrm{d}y = \frac{x^2 + y^2}{x}\,\mathrm{d}y \\ &= \frac{\mathrm{d}y}{x}. \end{aligned} \tag{13.17}$$

On U_y, $y \neq 0$. Again using $x\,\mathrm{d}x + y\,\mathrm{d}y = 0$, we get

$$\begin{aligned} x\,\mathrm{d}y - y\,\mathrm{d}x &= \frac{x}{y}\,y\,\mathrm{d}y - y\,\mathrm{d}x = -\frac{x^2}{y}\,\mathrm{d}x - y\,\mathrm{d}x \\ &= -\left(\frac{x^2}{y} + y\right)\,\mathrm{d}x = -\frac{x^2+y^2}{y}\,\mathrm{d}x \\ &= -\frac{\mathrm{d}x}{y}. \end{aligned} \tag{13.18}$$

Therefore,

$$x\,\mathrm{d}y - y\,\mathrm{d}x = \omega = \begin{cases} \frac{\mathrm{d}y}{x} & \text{on } U_x, \\ -\frac{\mathrm{d}x}{y} & \text{on } U_y. \end{cases} \tag{13.19}$$

§13.4 Exterior Differentiation Under a Pullback

Theorem 13.4

Let $F : N \to M$ be a smooth map of manifolds. If $\omega \in \Omega^k(M)$, then

$$\mathrm{d}F^*\omega = F^*\mathrm{d}\omega.$$

Proof. Let us first check the case when $k = 0$, i.e. when ω is a 0-form (C^∞ function). We denote this smooth function with h. For $p \in N$ and $X_p \in T_pN$,

$$\left(\mathrm{d}F^*h\right)_p(X_p) = X_p\left(F^*h\right) = X_p\left(h \circ F\right), \tag{13.20}$$

since $\left(\mathrm{d}f\right)_p(X_p) = X_p f$ for $f \in C^\infty(M)$. On the other hand,

$$\left(F^*\mathrm{d}h\right)_p(X_p) = \left(\mathrm{d}h\right)_{F(p)}\left(F_{*,p}X_p\right) = \left(F_{*,p}X_p\right)(h) = X_p\left(h \circ F\right). \tag{13.21}$$

Combining (13.20) and (13.21), we get

$$\left(\mathrm{d}F^*h\right)_p = \left(F^*\mathrm{d}h\right)_p.$$

Since $p \in N$ is arbitrary,

$$\mathrm{d}F^*h = F^*\mathrm{d}h. \tag{13.22}$$

Now, consider the general case of a C^∞ k-form ω on M, i.e. $\omega \in \Omega^k(M)$. It suffices to verify that $\mathrm{d}F^*\omega = F^*\mathrm{d}\omega$ at an arbitrary point $p \in N$. This reduces the proof to a local computation. If $(V, y^1, \ldots, y^n)$ is a chart of M at $F(p)$, then on V,

$$\omega = \sum_I a_I\,\mathrm{d}y^I = \sum_I a_I\,\mathrm{d}y^{i_1} \wedge \mathrm{d}y^{i_2} \wedge \cdots \wedge \mathrm{d}y^{i_k},$$

for some C^∞ functions a_I on V. Now,

$$F^*\omega = \sum_I \left(F^*a_I\right)\left(F^*\mathrm{d}y^{i_1}\right) \wedge \left(F^*\mathrm{d}y^{i_2}\right) \wedge \cdots \wedge \left(F^*\mathrm{d}y^{i_k}\right).$$

Since $\mathrm{d}F^*h = F^*\mathrm{d}h$ for C^∞ function h, we have

$$\begin{aligned} F^*\omega &= \sum_I \left(a_I \circ F\right)\,\mathrm{d}\left(F^*y^{i_1}\right) \wedge \mathrm{d}\left(F^*y^{i_2}\right) \wedge \cdots \wedge \mathrm{d}\left(F^*y^{i_k}\right) \\ &= \sum_I \left(a_I \circ F\right)\,\mathrm{d}\left(y^{i_1} \circ F\right) \wedge \mathrm{d}\left(y^{i_2} \circ F\right) \wedge \cdots \wedge \mathrm{d}\left(y^{i_k} \circ F\right) \\ &= \sum_I \left(a_I \circ F\right)\,\mathrm{d}F^{i_1} \wedge \mathrm{d}F^{i_2} \wedge \cdots \wedge \mathrm{d}F^{i_k}. \end{aligned} \tag{13.23}$$

Therefore, from (13.23), one obtains

$$\mathrm{d}F^*\omega = \sum_I \mathrm{d}\left(a_I \circ F\right) \wedge \mathrm{d}F^{i_1} \wedge \mathrm{d}F^{i_2} \wedge \cdots \wedge \mathrm{d}F^{i_k}. \tag{13.24}$$

On the other hand,

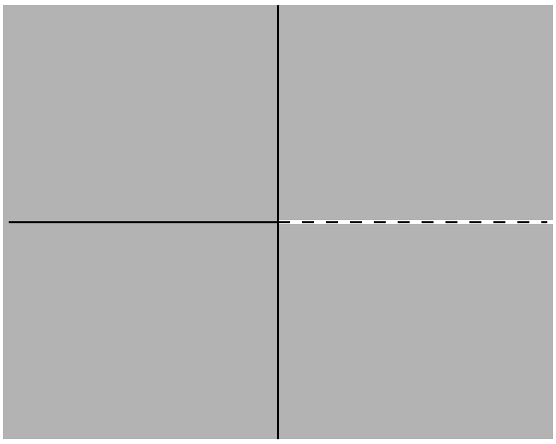
$$\begin{aligned} F^*\mathrm{d}\omega &= F^*\left(\sum_I \mathrm{d}a_I \wedge \mathrm{d}y^{i_1} \wedge \mathrm{d}y^{i_2} \wedge \cdots \wedge \mathrm{d}y^{i_k}\right) \\ &= \sum_I F^*\left(\mathrm{d}a_I\right) \wedge F^*\left(\mathrm{d}y^{i_1}\right) \wedge \cdots \wedge F^*\left(\mathrm{d}y^{i_k}\right) \\ &= \sum_I \mathrm{d}\left(F^*a_I\right) \wedge \mathrm{d}\left(F^*y^{i_1}\right) \wedge \cdots \wedge \mathrm{d}\left(F^*y^{i_k}\right) \\ &= \sum_I \mathrm{d}\left(a_I \circ F\right) \wedge \mathrm{d}\left(y^{i_1} \circ F\right) \wedge \cdots \wedge \mathrm{d}\left(y^{i_k} \circ F\right) \\ &= \sum_I \mathrm{d}\left(a_I \circ F\right) \wedge \mathrm{d}F^{i_1} \wedge \mathrm{d}F^{i_2} \wedge \cdots \wedge \mathrm{d}F^{i_k}. \end{aligned} \tag{13.25}$$

Comparing (13.24) and (13.25), one obtains

$$\mathrm{d}F^*\omega = F^*\mathrm{d}\omega, \tag{13.26}$$

on V. In particular, (13.26) holds at $p \in N$. Since $p \in N$ is arbitrary, (13.26) holds everywhere on N. ∎

Example 13.3. Let U be the open set $(0,\infty) \times (0, 2\pi)$ in the (r,θ) plane $\mathbb{R}^2$, i.e. U is $\mathbb{R}^2$ except the non-negative x-axis.



Define $F : U \to \mathbb{R}^2$ by

$$(x, y) = F\left(r, \theta\right) = \left(r\cos\theta, r\sin\theta\right).$$

Let us compute the pullback $F^*\left(\mathrm{d}x \wedge \mathrm{d}y\right)$.

$$F^*\mathrm{d}x = \mathrm{d}F^*x = \mathrm{d}\left(x \circ F\right) = \mathrm{d}\left(r\cos\theta\right) = \cos\theta\,\mathrm{d}r - r\sin\theta\,\mathrm{d}\theta; \tag{13.27}$$
$$F^*\mathrm{d}y = \mathrm{d}F^*y = \mathrm{d}\left(y \circ F\right) = \mathrm{d}\left(r\sin\theta\right) = \sin\theta\,\mathrm{d}r + r\cos\theta\,\mathrm{d}\theta. \tag{13.28}$$

Therefore,

$$\begin{aligned} F^*\left(\mathrm{d}x \wedge \mathrm{d}y\right) &= F^*\mathrm{d}x \wedge F^*\mathrm{d}y \\ &= \left[\cos\theta\,\mathrm{d}r - r\sin\theta\,\mathrm{d}\theta\right] \wedge \left[\sin\theta\,\mathrm{d}r + r\cos\theta\,\mathrm{d}\theta\right] \\ &= r\cos^2\theta\,\mathrm{d}r \wedge \mathrm{d}\theta - r\sin^2\theta\,\mathrm{d}\theta \wedge \mathrm{d}r \\ &= r\,\mathrm{d}r \wedge \mathrm{d}\theta. \end{aligned} \tag{13.29}$$

§13.5 Pullback Preserves Smoothness of Forms

In this section, we will prove that if ω is a smooth k-form on M, and $F : N \to M$ is smooth, then $F^*\omega$ is a smooth k-form on N. For that purpose, we need a lemma first.

Lemma 13.5

Let $(U, x^1, \ldots, x^n)$ be a chart on a manifold and $f^1, \ldots, f^k$ smooth functions on U. Then

$$\mathrm{d}f^1 \wedge \cdots \wedge \mathrm{d}f^k = \sum_I \frac{\partial\left(f^1, \ldots, f^k\right)}{\partial\left(x^{i_1}, \ldots, x^{i_k}\right)} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k},$$

where $I = (i_1, \ldots, i_k)$ is a strictly ascending multi-index of length k.

Proof. On U,

$$\mathrm{d}f^1 \wedge \cdots \wedge \mathrm{d}f^k = \sum_J c_J \, \mathrm{d}x^{j_1} \wedge \cdots \wedge \mathrm{d}x^{j_k}, \tag{13.30}$$

for some functions c_J. By the definition of the differential,

$$\mathrm{d}f^i\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial f^i}{\partial x^j}.$$

Applying both sides of (13.30) to the list of coordinate vector fields $\frac{\partial}{\partial x^{i_1}}, \ldots, \frac{\partial}{\partial x^{i_k}}$, we get

$$\begin{aligned}\text{LHS} &= \left(\mathrm{d}f^1 \wedge \cdots \wedge \mathrm{d}f^k\right)\left(\frac{\partial}{\partial x^{i_1}}, \ldots, \frac{\partial}{\partial x^{i_k}}\right) = \det\left[\frac{\partial f^i}{\partial x^{i_j}}\right] \\ &= \frac{\partial\left(f^1, \ldots, f^k\right)}{\partial\left(x^{i_1}, \ldots, x^{i_k}\right)},\end{aligned} \tag{13.31}$$

by Proposition C.13. On the other hand,

$$\text{RHS} = \sum_J c_J \left(\mathrm{d}x^{j_1} \wedge \cdots \wedge \mathrm{d}x^{j_k}\right)\left(\frac{\partial}{\partial x^{i_1}}, \ldots, \frac{\partial}{\partial x^{i_k}}\right) = \sum_J c_J \delta^J_I = c_I. \tag{13.32}$$

Hence, $c_I = \frac{\partial\left(f^1, \ldots, f^k\right)}{\partial\left(x^{i_1}, \ldots, x^{i_k}\right)}$. ■

Theorem 13.6

If $F : N \to M$ is a C^∞ map of manifolds and ω is a C^∞ k-form on M, then $F^*\omega$ is a C^∞ k-form on N.

Proof. It is enough to show that every point in N has a neighborhood on which $F^*\omega$ is C^∞. Fix $p \in N$ and choose a chart $(V, y^1, \ldots, y^m)$ on M about $F(p)$. Let $F^i = y^i \circ F$ be the i-th coordinate of the map F in this chart. By the continuity of F, there is a chart $(U, x^1, \ldots, x^n)$ on N about p such that $F(U) \subset V$. Since ω is C^∞, on V,

$$\omega = \sum_I a_I \, \mathrm{d}y^{i_1} \wedge \cdots \wedge \mathrm{d}y^{i_k}$$

for some C^∞ functions $a_I \in C^\infty(V)$. By properties of the pullback,

$$\begin{aligned}
F^*\omega &= F^*\left(\sum_I a_I \, \mathrm{d}y^{i_1} \wedge \cdots \wedge \mathrm{d}y^{i_k}\right) \\
&= \sum_I \left(F^* a_I\right) F^*\left(\mathrm{d}y^{i_1}\right) \wedge \cdots F^*\left(\mathrm{d}y^{i_k}\right) \\
&= \sum_I \left(a_I \circ F\right) \mathrm{d}F^* y^{i_1} \wedge \cdots \wedge \mathrm{d}F^* y^{i_k} \\
&= \sum \left(a_I \circ F\right) \mathrm{d}F^{i_1} \wedge \cdots \wedge \mathrm{d}F^{i_k} \\
&= \sum_{I,J} \left(a_I \circ F\right) \frac{\partial\left(F^{i_1}, \ldots, F^{i_k}\right)}{\partial\left(x^{j_1}, \ldots, x^{j_k}\right)} \mathrm{d}x^J.
\end{aligned} \tag{13.33}$$

Since the $a_I \circ F$ and $\frac{\partial\left(F^{i_1},\ldots,F^{i_k}\right)}{\partial\left(x^{j_1},\ldots,x^{j_k}\right)}$ are all C^∞, $F^*\omega$ is C^∞ on U. In particular, $F^*\omega$ is C^∞ at p. Since $p \in N$ is arbitrary, $F^*\omega$ is C^∞ on the whole of N. ■

Theorem 13.7

If $F : N \to M$ and $G : M \to K$ are smooth maps between manifolds, then

$$(G \circ F)^* = F^* \circ G^* : \Omega^*(K) \to \Omega^*(N). \tag{13.34}$$

Furthermore, if $\mathbb{1}_M$ is the identity map on M,

$$\left(\mathbb{1}_M\right)^* = \mathbb{1}_{\Omega^*(M)}. \tag{13.35}$$

Proof. Suppose $\mathbb{1}_M$ is the identity map on M. Take any $\omega \in \Omega^k(M)$. At any $p \in M$, for any $X_p^1, \ldots, X_p^k \in T_pM$,

$$\begin{aligned}
\left(\left(\mathbb{1}_M\right)^* \omega\right)_p \left(X_p^1, \ldots, X_p^k\right) &= \omega_{\mathbb{1}_M(p)} \left(\left(\mathbb{1}_M\right)_{*,p} X_p^1, \ldots, \left(\mathbb{1}_M\right)_{*,p} X_p^k\right) \\
&= \omega_p \left(\mathbb{1}_{T_pM} X_p^1, \ldots, \mathbb{1}_{T_pM} X_p^k\right) \\
&= \omega_p \left(X_p^1, \ldots, X_p^k\right),
\end{aligned} \tag{13.36}$$

since $\left(\mathbb{1}_M\right)_{*,p} = \mathbb{1}_{T_pM}$ by *Remark 6.1.2* of DG1. Therefore, $\left(\left(\mathbb{1}_M\right)^* \omega\right)_p = \omega_p$. Since $p \in M$ is arbitrary, $\left(\mathbb{1}_M\right)^* \omega = \omega$.

Now suppose $F : N \to M$ and $G : M \to K$ are smooth maps between manifolds. Take any $\omega \in \Omega^k(K)$. At any $p \in N$, for any $X_p^1, \ldots, X_p^k \in T_pN$,

$$\begin{aligned}
\left((G \circ F)^* \omega\right)_p \left(X_p^1, \ldots, X_p^k\right) &= \omega_{G(F(p))} \left((G \circ F)_{*,p} X_p^1, \ldots, (G \circ F)_{*,p} X_p^k\right) \\
&= \omega_{G(F(p))} \left(G_{*,F(p)} \left(F_{*,p} X_p^1\right), \ldots, G_{*,F(p)} \left(F_{*,p} X_p^k\right)\right),
\end{aligned} \tag{13.37}$$

since $(G \circ F)_{*,p} = G_{*,F(p)} \circ F_{*,p}$ by *Theorem 6.1.1* of DG1. Now on the other hand,

$$\begin{aligned}
\left(\left(F^* \circ G^*\right) \omega\right)_p \left(X_p^1, \ldots, X_p^k\right) &= \left(F^*\left(G^*\omega\right)\right)_p \left(X_p^1, \ldots, X_p^k\right) \\
&= \left(G^*\omega\right)_{F(p)} \left(F_{*,p} X_p^1, \ldots, F_{*,p} X_p^k\right) \\
&= \omega_{G(F(p))} \left(G_{*,F(p)} \left(F_{*,p} X_p^1\right), \ldots, G_{*,F(p)} \left(F_{*,p} X_p^k\right)\right).
\end{aligned} \tag{13.38}$$

Therefore,

$$\left((G \circ F)^* \omega\right)_p \left(X_p^1, \ldots, X_p^k\right) = \left(\left(F^* \circ G^*\right) \omega\right)_p \left(X_p^1, \ldots, X_p^k\right).$$

So we have $\left((G \circ F)^* \omega\right)_p = \left(\left(F^* \circ G^*\right) \omega\right)_p$. Since $p \in N$ is arbitrary,

$$(G \circ F)^* \omega = \left(F^* \circ G^*\right) \omega. \tag{13.39}$$

■

Remark 13.2. Theorem 13.6 tells us that F^* is indeed a map from $\Omega^k(M)$ to $\Omega^k(N)$. So we can think of it as a map between the graded algebras:

$$F^* : \Omega^*(M) \to F^*(N).$$

Previously we were writing this without really verifying that F^* preserves the smoothness of forms. Now, by Proposition 12.3 and Proposition 12.5, $F^* : \Omega^*(M) \to \Omega^*(N)$ is a homomorphism of graded algebras. This gives rise to a contravariant functor from the category **Man** of manifolds and smooth maps to the category **GrAlg** of graded algebras and graded algebra homomorphisms:

$$\mathcal{F} : \mathbf{Man} \to \mathbf{GrAlg}.$$

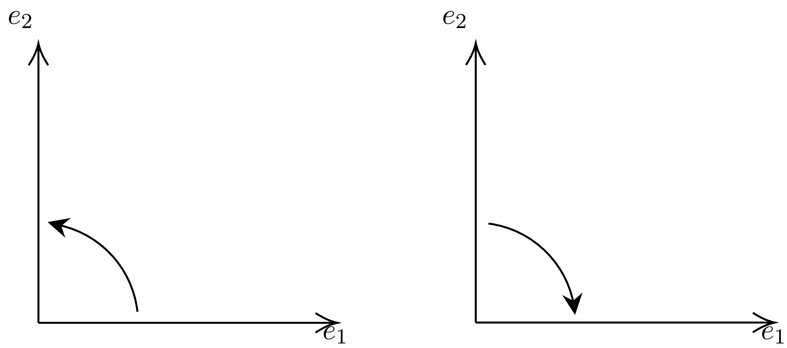
$\mathcal{F}$ takes an object of **Man**, a manifold M, to the graded algebra $\Omega^*(M)$; and it makes an arrow of **Man**, a smooth map $F : N \to M$, to the graded algebra homomorphism $F^* : \Omega^*(M) \to F^*(N)$. Since $\mathcal{F}$ reverses the direction of arrows, Theorem 13.7 ensures that it is a contravariant functor.

14 Orientation

§14.1 Orientations on a Vector Space

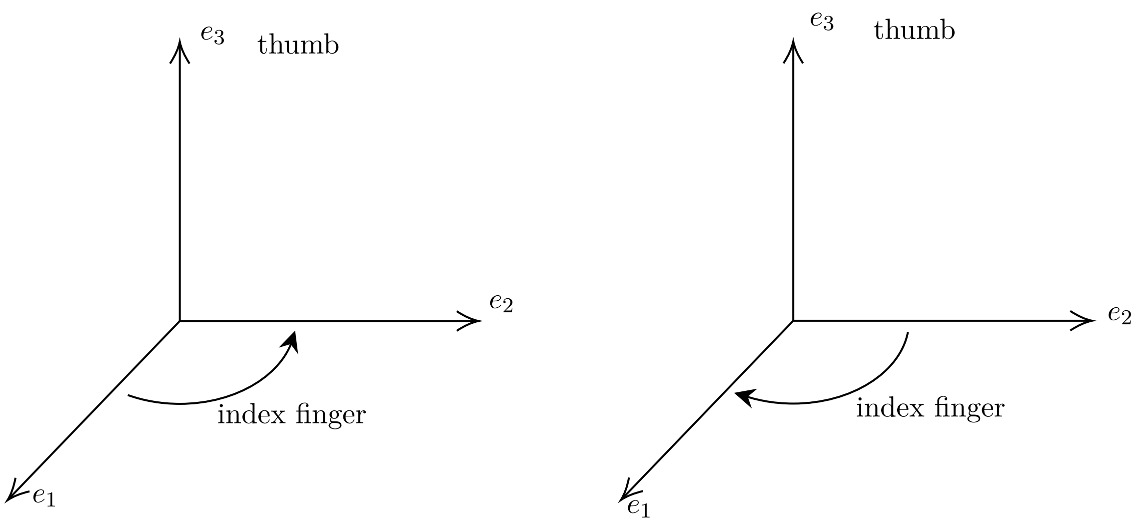
On $\mathbb{R}$, an orientation is one of the two possible directions:

On $\mathbb{R}^2$, an orientation is either counterclockwise or clockwise:



counterclockwise: (e_1, e_2) clockwise: (e_2, e_1)

On $\mathbb{R}^3$, an orientation is either right handed or left handed:



Right handed orientation: (e_1, e_2, e_3) Left handed orientation: (e_2, e_1, e_3)

Now, we want to define an orientation on $\mathbb{R}^4$, or more generally on $\mathbb{R}^n$. We do it through ordered basis for $\mathbb{R}^n$. Let $e_1, \ldots, e_n$ be the standard basis for $\mathbb{R}^n$. For $\mathbb{R}^1$, an orientation is given by e_1, or $-e_1$. For $\mathbb{R}^2$, counterclockwise orientation is (e_1, e_2), and clockwise orientation is (e_2, e_1). For $\mathbb{R}^3$, the right handed orientation is (e_1, e_2, e_3), and the left handed orientation is (e_2, e_1, e_3).

For any two ordered bases (u_1, u_2) and (v_1, v_2) of $\mathbb{R}^2$, there is a unique non-singular 2×2 matrix $A = [a_{ij}]$ such that

$$u_j = \sum_{i=1}^{2} v_i \, a_{ij}. \tag{14.1}$$

A is called the change of basis matrix from (v_1, v_2) to (u_1, u_2). In matrix notation, (14.1) can be written as

$$\begin{bmatrix} u_1 & u_2 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 \end{bmatrix} A. \tag{14.2}$$

We say that two ordered bases are **equivalent** if the change of basis matrix A has positive determinant. Then one can check that this is indeed an equivalence relation on the set of all ordered bases of $\mathbb{R}^2$. It, therefore, paritions the ordered bases into two equivalence classes. Each equivalence class is called an **orientation** on $\mathbb{R}^2$.

The equivalence class containing (e_1, e_2) is the counterclockwise orientation, and the equivalence class containing (e_2, e_1) is called clockwise orientation. Indeed,

$$\begin{bmatrix} e_1 & e_2 \end{bmatrix} = \begin{bmatrix} e_2 & e_1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

and

$$\det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1.$$

Similarly, for $\mathbb{R}^3$, the ordered bases (e_1, e_2, e_3) and (e_2, e_1, e_3) don't belong to the same equivalence class:

$$\begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} = \begin{bmatrix} e_2 & e_1 & e_3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and

$$\det \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = -1.$$

The general case for an n-dimensional vector space V is as follows:

Definition 14.1. Two ordered bases $u = \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix}$ and $v = \begin{bmatrix} v_1 & \cdots & u_n \end{bmatrix}$ of an n-dimensional vector space V are said to be **equivalent** if

$$u = vA,$$

for an $n \times n$ matrix A with $\det A > 0$. An **orientation** on V is an equivalence class of ordered bases.

The 0-dimensional vector space $\{\mathbf{0}\}$ is a special case as its basis is the empty set $\varnothing$. We define an orientation on $\{\mathbf{0}\}$ to be one of the two numbers ± 1.

§14.2 Orientations and n-covectors

Instead of using an ordered basis, we can also use an n-covector to specify an orientation on an n-dimensional vector space V. This is based onthe fact that the vector space $\Lambda^n(V^*)$ of n-covectors on V is 1-dimensional (so that it has 2 orientations).

Lemma 14.1

Let $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ be vectors in a vector space V. Suppose

$$u_j = \sum_{i=1}^{n} v_i a_{ij},$$

for a matrix $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ of real numbers. If ω is an n-covector on V, then

$$\omega(u_1, \ldots, u_n) = \det A \omega(v_1, \ldots, v_n).$$

Proof. By hypothesis, $u_j = \sum_{i=1}^n v_i a_{ij}$. Using linearity of ω, one arrives at

$$\begin{aligned}\omega(u_1, u_2, \ldots, u_n) &= \omega\left(\sum_{i_1=1}^n v_{i_1} a_{i_1 1}, \sum_{i_2=1}^n v_{i_2} a_{i_2 1}, \ldots, \sum_{i_n=1}^n v_{i_n} a_{i_n n}\right) \\ &= \sum_{i_1, i_2, \ldots, i_n = 1}^n a_{i_1 1} a_{i_2 2} \cdots a_{i_n n} \omega(v_{i_1}, v_{i_2}, \ldots, v_{i_n}). \end{aligned} \tag{14.3}$$

For $\omega(v_{i_1}, v_{i_2}, \ldots, v_{i_n})$ to be nonzero, $i_1, \ldots, i_n$ must all be different as ω is alternating. Therefore, $i_1, \ldots, i_n$ can be thought of as a permutation σ of $\{1, 2, \ldots, n\}$ that takes each j to i_j. From the alternating property of ω, one has

$$\omega\left(v_{\sigma(1)}, \ldots, v_{\sigma(n)}\right) = (\operatorname{sgn}\sigma)\, \omega(v_1, \ldots, v_n). \tag{14.4}$$

Therefore, using (14.3),

$$\begin{aligned}\omega(u_1, u_2, \ldots, u_n) &= \sum_{\sigma \in S_n} a_{\sigma(1)1} a_{\sigma(2)2} \cdots a_{\sigma(n)n} (\operatorname{sgn}\sigma)\, \omega(v_1, \ldots, v_n) \\ &= (\det A)\, \omega(v_1, \ldots, v_n). \end{aligned} \tag{14.5}$$

■

Corollary 14.2

If $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are ordered bases of a vector space V, then

$$\begin{aligned} & \omega(u_1, u_2, \ldots, u_n) \text{ and } \omega(v_1, \ldots, v_n) \text{ have the same sign} \\ \iff & \det A > 0 \\ \iff & u_1, \ldots, u_n \text{ and } v_1, \ldots, v_n \text{ are equivalent ordered bases.} \end{aligned}$$

We say that the n-covector represents the orientation $(v_1, \ldots, v_n)$ if $\omega(v_1, \ldots, v_n) > 0$. By Corollary 14.2, this notion is well-defined, i.e. independent of the choice of ordered basis $v_1, \ldots, v_n$ from the same equivalence class.

Remark 14.1. $\Lambda^n(V^*) \cong \mathbb{R}$, so that the set of nonzero n-covectors can be identified with $\mathbb{R} \setminus \{0\}$, which has 2 connected components. Two nonzero n-covectors ω and ω' on V are in the same component if and only if $\omega = a\omega'$ for some real number $a > 0$. Thus, each connected component of $\Lambda^n(V^*) \setminus \{0\}$ represents an orientation on V.

Example 14.1. Let e_1, e_2 be the standard basis of $\mathbb{R}^2$, and α^1, α^2 its dual basis. Then the 2-covector $\alpha^1 \wedge \alpha^2$ represents the counterclockwise orientation on $\mathbb{R}^2$, since

$$\left(\alpha^1 \wedge \alpha^2\right)(e_1, e_2) = 1 > 0.$$

Example 14.2. Let $\left.\frac{\partial}{\partial x}\right|_p, \left.\frac{\partial}{\partial y}\right|_p$ be the standard basis for the tangent space $T_p\mathbb{R}^2$, and $(\mathrm{d}x)_p, (\mathrm{d}y)_p$ be the dual basis, i.e. for the basis of $T_p^*\mathbb{R}^2$. Then $(\mathrm{d}x)_p \wedge (\mathrm{d}y)_p$ represents the counterclockwise orientation on $T_p\mathbb{R}^2$.

We define an equivalence relation on the nonzero n-covectors on the n-dimensional vector space V as follows:

$$\omega \sim \omega' \iff \omega = a\omega' \text{ for some } a > 0.$$

Then an orientation on V is also given by ana equivalence class of nonzero n-covectors on V.

§14.3 Orientations on a Manifold

Every vector space of dimension n has two orientations, corresponding to the two equivalence classes of ordered bases or the two equivalence classes of nonzero n-covectors. To orient a manifold M, we orient the tangent space at each point $p \in M$ in a coherent way so that the orientation doesn't change abruptly in a neighborhood of a point. The simplest way to guarantee this is to require that the n-form (or top degree form) on M specifying the orientation at each point be C^∞. We also want the n-form to be nowhere vanishing.

Definition 14.2. A manifold M of dimension n is **orientable** if it has a C^∞ nowhere vanishing n-form. If ω is a nowhere vanishing C^∞ n-form on M, then at each point $p \in M$, the n-covector ω_p picks out an equivalence class of ordered bases for the tangent space T_pM.

Example 14.3. The Euclidean space $\mathbb{R}^n$ is orientable as a manifold, because it has the nowhere vanishing n-form $\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \cdots \wedge \mathrm{d}x^n$.

If ω and ω' are both C^∞ nowhere vanishing n-forms on a manifold M of dimension n, then $\omega = f\omega'$ for a C^∞ nowhere vanishing function f on M. On a connoted manifold M, such a function f is either everywhere positive or everywhere negative. Thus, the C^∞ nowhere-vanishing n-forms on a connected manifold M can be partitioned into 2 equivalence class:

$$\omega \sim \omega' \iff \omega = f\omega' \text{ with } f > 0. \tag{14.6}$$

We call either equivalence class an orientation on the connected manifold M. Thus, by definition, a connected manifold has exactly 2 orientations. If the manifold M is not connected, then each connected component of M has one of the 2 possible orientations. We call a C^∞ nowhere-vanishing n-form on M that specifies an orientation of M an **orientation form**. An **oriented manifold** is a pair $(M, [\omega])$, where M is a manifold of dimension n and $[\omega]$ is an orientation on M, i.e. the equivalence class of nowhere vanishing C^∞ n-forms containing ω.

Remark 14.2 (Orientations on a 0-dimensional manifold). A zero dimensional manifold is a point, and by definition is always orientable. Its two orientations are represented by the numbers ± 1.

Definition 14.3. A diffeomorphism $F : (N, [\omega_N]) \to (M, [\omega_M])$ of oriented manifolds is said to be **orientation preserving** if $[F^*\omega_M] = [\omega_N]$. It's **orientation reversing** if $[F^*\omega_M] = [-\omega_N]$.

Proposition 14.3

Let U and V be open subsets of $\mathbb{R}^n$, A diffeomorphiam $F : U \to V$ is orientation-preserving if and only if the Jacobian determinant $\det\left[\frac{\partial F^i}{\partial x^j}\right]$ is everywhere positive on U.

Proof. Let $(x^1, \ldots, x^n)$ and $y', \ldots, y^n$ be standard coordinates on $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^n$, respectively.

$$\begin{aligned} F^*\left(\mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^n\right) &= F^*\left(\mathrm{d}y^1\right) \wedge \cdots \wedge F^*\left(\mathrm{d}y^n\right) \\ &= \mathrm{d}\left(F^*y^1\right) \wedge \cdots \wedge \wedge \mathrm{d}^*\left(F^*y^n\right) \\ &= \mathrm{d}\left(y^1 \circ F\right) \wedge \cdots \wedge \mathrm{d}\left(y^n \circ F\right) \\ &= \mathrm{d}F^1 \wedge \cdots \wedge \mathrm{d}F^n \\ &= \det\left[\frac{\partial F^i}{\partial x^j}\right] \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n, \end{aligned} \tag{14.7}$$

where the last equality follows from Lemma 13.5. Now, F is orientation preserving if and only if

$$F^*\left(\mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^n\right) \sim \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^k, \tag{14.8}$$

where $\sim$ is defined as (14.6). Using (14.7), we can conclude that (14.8) holds if and only if $\det\left[\frac{\partial F^i}{\partial x^j}\right]$ is everywhere positive on U. ■

§14.4 Orientation and Atlases

Definition 14.4 (Oriented atlas). An atlas on M is said to be **oriented** if for any two overlapping charts $(U, x^1, \ldots, x^n)$ and $(V, y^1, \ldots, y^n)$ of the atlas, the Jacobian determinant $\det\left[\frac{\partial y^i}{\partial x^j}\right]$ is everywhere positive on $U \cap V$.

Proposition 14.4

A manifold M of dimension n has a C^∞ nowhere vanishing n-form ω if and only if it has an oriented atlas.

Proof. ($\Leftarrow$) Supoose we are given an oriented atlas

$$\left\{\left(U_\alpha, x_\alpha^1, x_\alpha^2, \ldots, x_\alpha^n\right)\right\}_{\alpha \in A}.$$

Suppose $\{\rho_\alpha\}_{\alpha\in A}$ is a C^∞ partition of unity subordinate to the open cover $\{U_\alpha\}_{\alpha\in A}$. Define

$$\omega = \sum \rho_\alpha \, \mathrm{d}x_\alpha^1 \wedge \cdots \wedge \mathrm{d}x_\alpha^n. \tag{14.9}$$

Since $\{\operatorname{supp}\rho_\alpha\}$ is locally finite by deinition of partition of unity, for any $p \in M$, there is an open neighborhood U_p of p that intersects only finitely many of the sets $\operatorname{supp}\rho_\alpha$. Thus, (14.9) is a finite sum on U_p. This actually shows that ω is defined and C^∞ at every point of M.

Let $(U, x^1, \ldots, x^n)$ be one of the charts about p in the oriented atlas. On $U_\alpha \cap U$, by Lemma 13.5,

$$\mathrm{d}x_\alpha^1 \wedge \cdots \wedge \mathrm{d}x_\alpha^n = \det\left[\frac{\partial x_\alpha^i}{\partial x^j}\right] \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n. \tag{14.10}$$

By hypothesis, $\det\left[\frac{\partial x_\alpha^i}{\partial x^j}\right] > 0$ as the atlas is oriented. Then on $U_p \cap U$,

$$\omega = \sum \rho_\alpha \, \mathrm{d}x_\alpha^1 \wedge \cdots \wedge \mathrm{d}x_\alpha^n = \left(\sum \rho_\alpha \det\left[\frac{\partial x_\alpha^i}{\partial x^j}\right]\right) \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n. \tag{14.11}$$

The sum in (14.11) is a finite sum, since U_p intersects only finitely many of the sets $\operatorname{supp}\rho_\alpha$. Now, it's easy to see that the finite number in the parenthesis is actually positive at p. Indeed, $\det\left[\frac{\partial x_\alpha^i}{\partial x^j}\right] > 0$ at p, since the atlas is oriented. Furthermore, $\rho_\alpha(p) > 0$ for at least one $\alpha \in A$. Hence.

$$\omega_p = (\text{ positive number }) \times \left(\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n\right)_p \neq 0.$$

As p is an arbitrary point of M, the n-form ω is nowhere vanishing on M.

($\Rightarrow$) Suppose ω is a C^∞ nowhere vanishing n-form on M. Given an atlas on M, we will use ω to modify the atlas so that it becomes oriented. Without loss of generality, assume that all the open sets of the atlas are connected.

On a chart $(U, x^1, \ldots, x^n)$,

$$\omega = f \, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \tag{14.12}$$

for a C^∞ function f on U. Since ω is nowhere-vanishing and f is continuous, f is either everywhere positive or everywhere negative on U. If $f > 0$, we leave the chart as it is; if $f < 0$, we replace the chart by $(U, -x^1, x^2, \ldots, x^n)$. After all the charts have been checked and replaced if necessary, we have that on every chart $(V, y^1, \ldots, y^n)$

$$\omega = h \, \mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^n \tag{14.13}$$

with $h > 0$. This can be seen to be an oriented atlas, since if $(U, x^1, \ldots, x^n)$ and $(V, y^1, \ldots, y^n)$ are two charts, then on $U \cap V$

$$\omega = f\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n = h\, \mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^n, \tag{14.14}$$

with $f, h > 0$. From (14.14),

$$\mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^n = \frac{f}{h}\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n. \tag{14.15}$$

By Lemma 13.5,

$$\mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^n = \det\left[\frac{\partial y^i}{\partial x^j}\right] \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n. \tag{14.16}$$

Comparing (14.15) and (14.16),

$$\det\left[\frac{\partial y^i}{\partial x^j}\right] = \frac{f}{h} > 0 \tag{14.17}$$

on $U \cap V$. Hence, the modified atlas is oriented. ■

Example 14.4 (Non-orientability of the open Möbius band)**.** Let R be the rectangle

$$R = \left\{(x, y) \in \mathbb{R}^2 \mid 0 \le x \le 1, \text{ and } -1 < y < 1\right\}.$$

We define an equivalence relation $\sim$ on R as follows:

$$(0, y) \sim (1, -y), \tag{14.18}$$

for $y \in (-1, 1)$. Then $M = R/\sim$ is the open Möbius band. We want to show that M is not orientable.

Consider the following open sets on M:

$$\begin{aligned} U &= \{[x, y] \in M \mid 0 < x < 1\}, \\ V &= \left\{[x, y] \in M \mid x \neq \frac{1}{2}\right\}. \end{aligned} \tag{14.19}$$

(Here $[x, y]$ represents the equivalence class containing the point $(x, y) \in R$) Then we can define homeomorphisms $\varphi : U \to \varphi(U) \subset \mathbb{R}^2$ and $\psi : V \to \psi(V) \subset \mathbb{R}^2$:

$$\begin{aligned} \varphi([x, y]) &= (x, y), \\ \psi([x, y]) &= \begin{cases} (x, y) & \text{if } x < \frac{1}{2}, \\ (x - 1, -y) & \text{if } x > \frac{1}{2}. \end{cases} \end{aligned} \tag{14.20}$$

Then $\{(U, \varphi), (V, \psi)\}$ forms an atlas on M. Consider $(U, \varphi) \equiv (U, x^1, x^2)$ and $(V, \psi) = (V, y^1, y^2)$.

Assume for the sake of contradiction that M is orientable. Then there is a nowhere vanishing 2-form ω on M. Then on U,

$$\omega = f\, \mathrm{d}x^1 \wedge \mathrm{d}x^2, \tag{14.21}$$

for a C^∞ nowhere vanishing function f on U. Since U is connected, f is either positive, or negative. Similarly, on V,

$$\omega = g\, \mathrm{d}y^1 \wedge \mathrm{d}y^2, \tag{14.22}$$

for a C^∞ nowhere vanishing function g on V. Since V is connected, g is either positive, or negative. On $U \cap V$,

$$\omega = g\,\mathrm{d}y^1 \wedge \mathrm{d}y^2 = g \,\det\left[\frac{\partial y^i}{\partial x^i}\right] \,\mathrm{d}x^1 \wedge \mathrm{d}x^2, \tag{14.23}$$

using Lemma 13.5. Comparing (14.21) and (14.23), we get

$$f = g \,\det\left[\frac{\partial y^i}{\partial x^i}\right] \tag{14.24}$$

on $U \cap V$. Since f and g are either positive everywhere on $U \cap V$ or negative everywhere on $U \cap V$,

$$\det\left[\frac{\partial y^i}{\partial x^i}\right] = \frac{f}{g} \tag{14.25}$$

is also either positive everywhere on $U \cap V$ or negative everywhere on $U \cap V$. Let us now compute $\det\left[\frac{\partial y^i}{\partial x^i}\right]$.

$$\frac{\partial y^i}{\partial x^j} = \frac{\partial\left(r^i \circ \psi\right)}{\partial x^j} = \frac{\partial\left(\left(r^i \circ \psi\right) \circ \varphi^{-1}\right)}{\partial r^j} = \frac{\partial\left(\psi \circ \varphi^{-1}\right)^i}{\partial r^j}, \tag{14.26}$$

where r^j are coordinates of $\mathbb{R}^2$. Let A, B, C be the following open rectangles in $\mathbb{R}^2$:

$$A = \left(0, \frac{1}{2}\right) \times (-1,1)\,,\ B = \left(\frac{1}{2}, 1\right) \times (-1,1)\,,\ C = \left(-\frac{1}{2}, 0\right) \times (-1,1)\,.$$

Then $\varphi\,(U \cap V) = A \cup B$, $\psi\,(U \cap V) = A \cup C$. $\psi \circ \varphi^{-1} : A \cup B \to A \cup C$ is then

$$\left(\psi \circ \varphi^{-1}\right)\left(r^1, r^2\right) = \begin{cases} (r^1, r^2) & \text{if } (r^1, r^2) \in A, \\ (r^1 - 1, -r^2) & \text{if } (r^1, r^2) \in B. \end{cases} \tag{14.27}$$

So its Jacobian determinant $\det\left[\frac{\partial\left(\psi \circ \varphi^{-1}\right)^i}{\partial r^j}\right]$ is

$$\begin{aligned} \frac{\partial\left(\psi \circ \varphi^{-1}\right)^i}{\partial r^j} &= \begin{cases} \det\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \text{on } A, \\ \det\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} & \text{on } B. \end{cases} \\ &= \begin{cases} 1 & \text{on } A, \\ -1 & \text{on } B. \end{cases} \end{aligned} \tag{14.28}$$

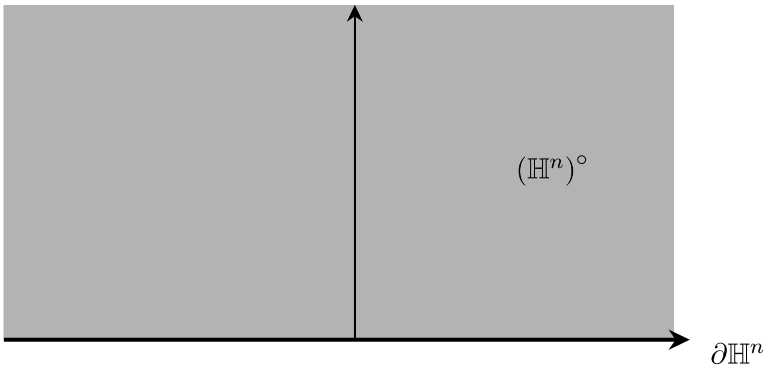
So $\frac{\partial y^i}{\partial x^j}$ is 1 on $\varphi^{-1}(A) \subseteq U \cap V$, and -1 on $\varphi^{-1}(B) \subseteq U \cap V$. But we have previously shown that $\det\left[\frac{\partial y^i}{\partial x^i}\right] = \frac{f}{g}$ is either positive everywhere on $U \cap V$ or negative everywhere on $U \cap V$. Thus we arrive at a contradiction! Hence, no nowhere vanishing 2-form on the open Möbius band M exists.

15 Manifolds with Boundary

The prototype of a manifold with boundary is the closed upper half plane

$$\mathbb{H}^n = \left\{ \left(x^1, \ldots, x^n \right) \in \mathbb{R}^n \mid x^n \geq 0 \right\},$$

with the subspace topology inherited from $\mathbb{R}^n$.



$(\mathbb{H}^n)^\circ = \{(x^1, \ldots, x^n) \in \mathbb{R}^n \mid x^n > 0\}$ is called the interior of $\mathbb{H}^n$; and $\partial\mathbb{H}^n = \{(x^1, \ldots, x^n) \in \mathbb{R}^n \mid x^n = 0\}$ is called the boundary of $\mathbb{H}^n$.

§15.1 Invariance of Domain

Definition 15.1. Let $S \subset \mathbb{R}^n$ be an arbitrary subset (not necessarily open). A function $f : S \to \mathbb{R}^m$ is **smooth** at a point $p \in S$ if there exists a neighborhood U of p in $\mathbb{R}^n$, and a C^∞ function $\widetilde{f} : U \to \mathbb{R}^m$ such that $\widetilde{f} = f$ on $U \cap S$. The function $f : S \to \mathbb{R}^m$ is said to be smooth on S if it is smooth at each point $p \in S$.

Lemma 15.1

A function $f : S \to \mathbb{R}^m$ with $S \subset \mathbb{R}^n$ is C^∞ if and only if there exists an open set $U \subseteq \mathbb{R}^n$ containing S and a C^∞ function $\widetilde{f} : U \to \mathbb{R}^m$ such that $\widetilde{f}\big|_S = f$.

Proof. ($\Leftarrow$) Suppose there is an open set $U \subseteq \mathbb{R}^n$ containing S, and a C^∞ function $\widetilde{f} : U \to \mathbb{R}^m$ such that $\widetilde{f}\big|_S = f$. Then for each $p \in S$, there is a open neighborhood of p, which is U itself, and a C^∞ function $\widetilde{f} : U \to \mathbb{R}^m$ such that $\widetilde{f}$ and f agre on $U \cap S$. In other words, $f : S \to \mathbb{R}^m$ is C^∞ at $p \in S$. Since $p \in S$ was chosen arbitrarily, $f : S \to \mathbb{R}^m$ is C^∞ everywhere on S.

($\Rightarrow$) Suppose $f : S \to \mathbb{R}^m$ is C^∞. Then for each $p \in S$, there is a neighborhood $U_p \subseteq \mathbb{R}^n$ and a C^∞ function $F_p : U_p \to \mathbb{R}^m$ such that $F_p = f$ at $U_p \cap S$. Take

$$U = \bigcup_{p \in S} U_p \subseteq \mathbb{R}^n.$$

Then U is an open subset of $\mathbb{R}^n$ that contains S. Since it is an open subset of an Euclidean space, it is a manifold; and $\{U_p\}_{p \in S}$ is an open cover of U. Therefore, there is a partition of unity $\{\rho_p\}_{p \in S}$ subordinate to the open cover $\{U_p\}_{p \in S}$. Now we define $\widetilde{f} : U \to \mathbb{R}^m$ as

$$\widetilde{f} = \sum_{p \in S} \rho_p F_p. \tag{15.1}$$

Given any $q \in S$, there is a neighborhood V_q of q in U that intersects only finitely many $\operatorname{supp} \rho_p$'s. Therefore, on V_q, the sum in (15.1) becomes a finite sum. Furthermore, as a finite sum and product of smooth functions, $\widetilde{f}$ is smooth on V_q. Therefore, $\widetilde{f} : U \to \mathbb{R}^m$ is smooth.

Now we need to verify that $\widetilde{f}$ agrees with f on S. Let's take any $q \in S$. For $q \in U_p$, since $F_p = f$ at $U_p \cap S$, we have

$$F_p(q) = f(q). \tag{15.2}$$

And for $q \notin U_p$, $q \notin \operatorname{supp} \rho_p$, since $\operatorname{supp} \rho_p \subseteq U_p$. Therefore,

$$1 = \sum_{p \in S} \rho_p(q) = \sum_{p \in S \text{ such that } q \in \operatorname{supp} \rho_p} \rho_p(q) = \sum_{p \in S \text{ such that } q \in U_p} \rho_p(q). \tag{15.3}$$

As a result,

$$\begin{aligned}
\widetilde{f}(q) &= \sum_{p \in S} \rho_p(q) F_p(q) = \sum_{p \in S \text{ such that } q \in U_p} \rho_p(q) F_p(q) \\
&= \sum_{p \in S \text{ such that } q \in U_p} \rho_p(q) f(q) \\
&= \left(\sum_{p \in S \text{ such that } q \in U_p} \rho_p(q) \right) f(q) \\
&= f(q).
\end{aligned} \tag{15.4}$$

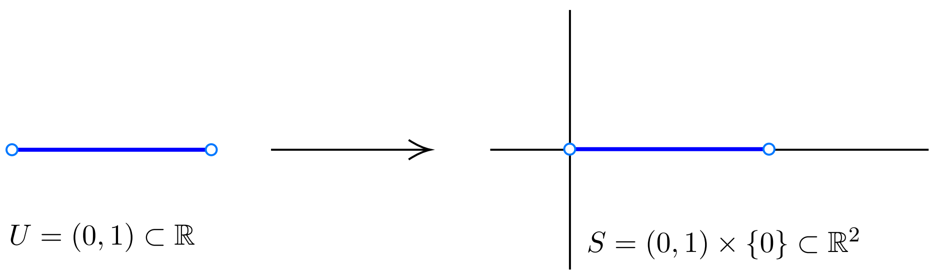
Therefore, $\widetilde{f}\big|_S = f$. ■

Remark 15.1. With the definition above, it now makes sense to speak about an arbitray set $S \subset \mathbb{R}^n$ being diffeomorphic to some set $T \subset \mathbb{R}^m$. This will be the case if and only if there are smooth maps (in the sense above) $f : S \to T \subset \mathbb{R}^m$ and $g : T \to S \subset \mathbb{R}^n$ that are inverses to each other.

Theorem 15.2

Let $U \subseteq \mathbb{R}^n$ be an open subset, $S \subset \mathbb{R}^n$ an arbitrary subset, and $f : U \to S$ a diffeomorphism. Then S is open in $\mathbb{R}^n$.

The diffeomorphism between U ans S forces S to be open in $\mathbb{R}^n$. Given that $f : U \to S$ is a diffeomorphism, we only know that an open subset of U is mapped to an open subset of S under f. Since U is open in itself, $f(U) = S$ is also open in S. We can't immediately conclude that $f(U) = S$ is open in $\mathbb{R}^n$. Besides, it's crucial that both U and S are subsets of the same Euclidean space $\mathbb{R}^n$. For example, there is a diffeomorphism between the open interval $(0,1) \subset \mathbb{R}$ and the open segment $S = (0,1) \times \{0\}$ in $\mathbb{R}^2$. But S is not open in $\mathbb{R}^2$.



Proof of Theorem 15.2. Let $f(p) \in S$ be an arbitrary point in S, with $p \in U$. Note that any point in S can be reached this way as f is onto. Since $f : U \to S$ is a diffeomorphism, $f^{-1} : S \to U$ is smooth with S being an arbitrary subset. By Lemma 15.1, there exist an open set $V \subseteq \mathbb{R}^n$ containing S and a C^∞ function $g : V \to \mathbb{R}^n$ such that $g\big|_S = f^{-1}$.

$$U \xrightarrow{f} V \xrightarrow{g} \mathbb{R}^n$$

g agrees f^{-1} on $f(U) = S$. Therefore, $g \circ f = \mathbb{1}_U : U \to U \subseteq \mathbb{R}^n$. Given $p \in U$, by the chain rule, one has

$$g_{*,f(p)} \circ f_{*,p} = \mathbb{1}_{T_pU} : T_pU \to T_pU, \tag{15.5}$$

the identity map on the tangent space T_pU. So $g_{*,f(p)}$ is the left inverse of $f_{*,p}$. The existence of left inverse implies injectivity, so $f_{*,p} : T_pU \to T_{f(p)}V$ is injective. Since U and V are open subsets of the same Euclidean space, they have the same dimension as manifolds. So T_pU and $T_{f(p)}V$ have the same dimension as vector spaces. Now, $f_{*,p} : T_pU \to T_{f(p)}V$ is an injective linear map between vector spaces of same dimension. So $f_{*,p} : T_pU \to T_{f(p)}V$ is invertible.

Now we recall *inverse function theorem*:

> A C^∞ map $F : N \to M$ between two manifolds of same dimension is locally invertible at $p \in N$ (i.e. p has a neighborhood U on which $F|_U : U \to F(U)$ is a diffeomorphism) if and only if the differential $F_{*,p} : T_pN \to T_{F(p)}M$ is an isomorphism of vector spaces.

We have that $f_{*,p} : T_pU \to T_{f(p)}V$ is an isomorphism of vector spaces. Hence, by *inverse function theorem*, $f : U \to V$ is locally invertible at p. This means that there are open neighborhoods U_p of $p \in U$ and $V_{f(p)}$ of $f(p) \in V$ such that

$$f|_{U_p} : U_p \to V_{f(p)}$$

is a diffeomorphism. Then it follows that

$$f(p) \in V_{f(p)} = f(U_p) \subseteq f(U) = S. \tag{15.6}$$

For every $f(p) \in S$, we can find an neighborhood $V_{f(p)} \ni f(p)$ open in $\mathbb{R}^n$ ($V_{f(p)}$ is open in V, V is open in $\mathbb{R}^n$; hence $V_{f(p)}$ is open in $\mathbb{R}^n$) that is contained in S. Therefore, S is open in $\mathbb{R}^n$. ■

Proposition 15.3

Let U and V be open subsets of the upper half space $\mathbb{H}^n$, and $f : U \to V$ be a diffeomorphism. Then f maps interior points to interior points, and boundary points to boundary points.

Proof. Let $p \in U$ be an interior point. Then there is an open ball B in $\mathbb{R}^n$ containing p, which is contained in U. Restriction of a diffeomorphism to an open subset is still a diffeomorphism. Hence,

$$f|_B : B \to f(B)$$

is a diffeomorphism, with B being open in $\mathbb{R}^n$. By Theorem 15.2, $f(B)$ is open in $\mathbb{R}^n$. Hence,

$$f(p) \in f(B) \subseteq f(U) = V \subseteq \mathbb{H}^n.$$

$f(B)$ is open in $\mathbb{R}^n$, and it is contained in $\mathbb{H}^n$. Therefore, $f(B) \subseteq (\mathbb{H}^n)^\circ$. In other words, $f(p) \in (\mathbb{H}^n)^\circ$, since $f(p) \in f(B)$. So f maps interior points to interior points.

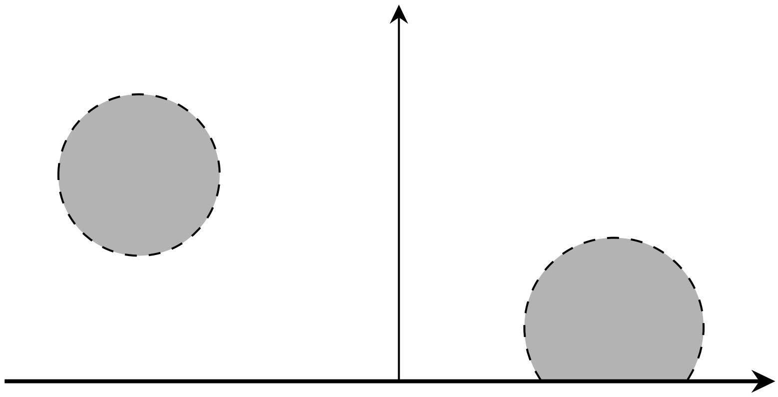
If p is a boundary point in $U \cap \partial\mathbb{H}^n$, then $f^{-1}(f(p)) = p$ is a boundary point. Since $f^{-1} : V \to U$ is a diffeomorphism, by the previous argument, f^{-1} takes interior points to interior points. If $f(p)$ were an interior point, then f^{-1} would have mapped it to an interior point. But f^{-1} maps $f(p)$ to a boundary point. So $f(p)$ cannot be an interior point. Therefore, $f(p)$ is a boundary point. ■

Remark 15.2. Replacing Euclidean spaces by manifolds throughout, one can prove in exactly the same way the **smooth invariance of domain for manifolds**:

> Suppose N and M are n-dimensional manifolds. Let $U \subseteq N$ be open, and $S \subset M$ be any arbitrary subset. If there is a diffeomorphism $F : U \to S$, then S is open in M.

§15.2 Manifolds with Boundary

In the upper half space $\mathbb{H}^n$, there are 2 types of open sets as seen in the following diagram:



In the left one, the set is disjoint from the boundary of $\mathbb{H}^n$, while in the right one the open set has nontrivial intersection with $\partial\mathbb{H}^n$. We say that a topological space M is locally $\mathbb{H}^n$ if every point $p \in M$ has a neighborhood U homeomorphic to an open subset of $\mathbb{H}^n$.

Definition 15.2. A **topological n-manifold with boundary** is a second countable, Hausdorff topological space that is locally $\mathbb{H}^n$.

Let M be a topological n-manifold with boundary. For $n \geq 2$, a chart on M is defined to be a pair (U, φ) consisting of an open set $U \subseteq M$ and a homeomorphism

$$\varphi : U \to \varphi(U) \subseteq \mathbb{H}^n$$

of U with an open subset $\varphi(U)$ of $\mathbb{H}^n$. A slight modification is necessary for the definition of a chart in the case $n = 1$.

Note that $\mathbb{H}^1 = [0, \infty)$ is the right half-line. We also need the left half line $\mathbb{L}^1 = (-\infty, 0]$ to model a 1-manifold with boundary locally. A chart (U, φ) in dimension 1 consists of an open set $U \subseteq M$ and a homeomorphism φ of U with an open subset of $\mathbb{H}^1$ or $\mathbb{L}^1$. With this slight modification of definition of chart in dimension 1, it can be seen that if $(U, x^1, x^2, \ldots, x^n)$ is a chart of an n-dimensional manifold with boundary, then so is $(U, -x^1, x^2, \ldots, x^n)$ for $n \geq 1$. A manifold with boundary has dimension at least 1, since a manifold of dimension 0, being a discrete set of points, necessarily has empty boundary.

Definition 15.3. A collection $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in A}$ of charts is a C^∞ **atlas** for the topological manifold M with boundary if

$$\bigcup_{\alpha\in A} U_\alpha = M,$$

and for any two charts $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β), the transition map

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

is a diffeomorphism. A C^∞ **manifold with boundary** is a topological manifold with boundary together with a maximal C^∞ atlas.

A point $p \in M$ is called an **interior point** in some chart (U, φ) if the point $\varphi(p)$ is an interior point of $\mathbb{H}^n$, i.e. $\varphi(p) \in (\mathbb{H}^n)^\circ$. Similarly, $p \in M$ is a **boundary point** if $\varphi(p)$ is a boundary point of $\mathbb{H}^n$, i.e. $\varphi(p) \in \partial\mathbb{H}^n$. These concepts are independent of charts. Suppose (V, ψ) is another chart about p. Then the diffeomorphism $\psi \circ \varphi^{-1}$ maps $\varphi(p)$ to $\psi(p)$. By Proposition 15.3, $\varphi(p)$ and $\psi(p)$ are both interior points, or both boundary points.

The set of all boundary points of M is denoted by ∂M. On the contrary, the set of all interior points of M is denoted by M°.

In contrast to the geometric notion of the interior and boundary of a manifold, there is the topological notion of the interior and boundary of a subset A of a topological space S. A point $p \in S$ is said to be an interior point of A if there exists an open subset $U \subseteq S$ such that

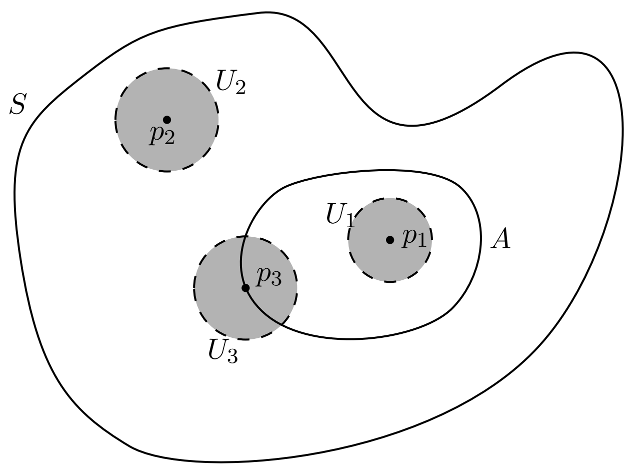
$$p \in U \subseteq A.$$

The point $p \in S$ is an exterior point of A if there exists an open set U of S such that

$$p \in U \subseteq A.$$

Finally $p \in S$ is a boundary point of A if every neighborhood of p contains both a point of A and a point not in A. One denotes by $\mathrm{int}_S(A), \mathrm{ext}_S(A), \mathrm{bd}_S(A)$ the sets of interior, exterior and boundary points of A in S, respectively. Clearly, the topological space S is the disjoint union

$$S = \mathrm{int}_S(A) \sqcup \mathrm{ext}_S(A) \sqcup \mathrm{bd}_S(A). \tag{15.7}$$



In the above diagram, $p_1 \in \mathrm{int}_S(A)$, $p_2 \in \mathrm{ext}_S(A)$, $p_3 \in \mathrm{bd}_S(A)$.

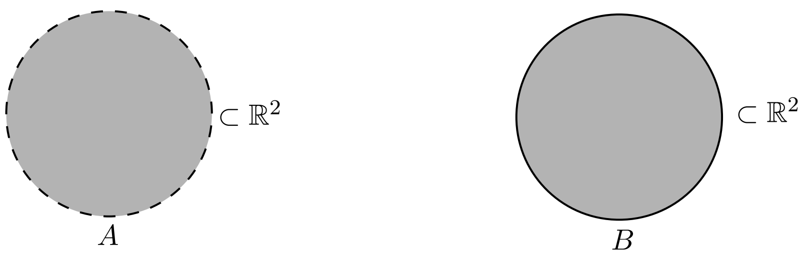
In case the subset $A \subseteq S$ is a manifold with boundary, we call $\mathrm{int}_S(A)$ the topological interior and $\mathrm{bd}_S(A)$ the topological boundary of A, to distinguish them from the manifold interior A° and the manifold boundary ∂A.

Note that the topological interior and the topological boundary of a set depend on an ambient space, while the manifold interior and the manifold boundary are both intrinsic.

Example 15.1 (Topological boundary vs. manifold boundary)**.** Let A be the open unit disk in $\mathbb{R}^2$:

$$A = \left\{ \mathbf{x} \in \mathbb{R}^2 \mid \|\mathbf{x}\| < 1 \right\}.$$

Then its topological boundary $\mathrm{bd}_{\mathbb{R}^2} A$ in $\mathbb{R}^2$ is the unit circle, while A being a 2-dimensional manifold (without boundary) has its manifold boundary to be the empty set $\varnothing$.



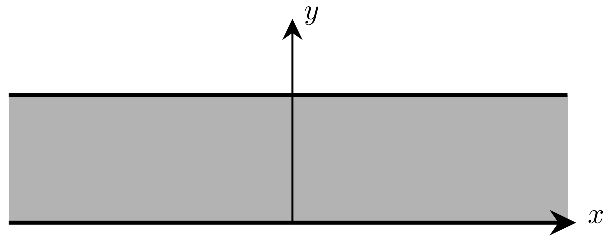
Now, consider B to be the closed unit disk in $\mathbb{R}^2$:

$$B = \left\{ \mathbf{x} \in \mathbb{R}^2 \mid \|\mathbf{x}\| \leq 1 \right\}.$$

It is a 2-dimensional manifold with boundary, with its manifold boundary ∂B being the unit circle. The topological boundary $\mathrm{bd}_{\mathbb{R}^2}(B)$ of B in $\mathbb{R}^2$ is also the unit circle and hence ∂B and $\mathrm{bd}\, B$ coincide with each other.

Example 15.2 (Topological interior vs. manifold interior)**.** Let S be the upper half-plane $\mathbb{H}^2$, and A be the subset

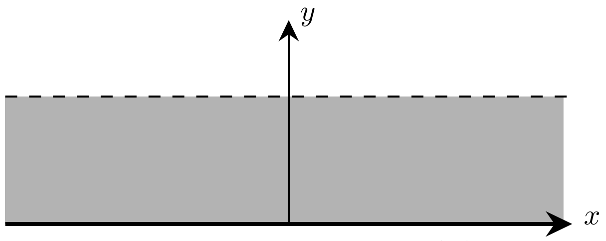
$$A = \left\{(x, y) \in \mathbb{H}^2 \mid y \leq 1\right\}.$$



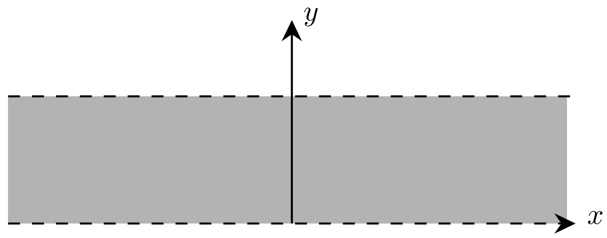
The topological interior $\operatorname{int}_{\mathbb{H}^2}(A)$ of A in $\mathbb{H}^2$ is the set

$$\operatorname{int}_{\mathbb{H}^2}(A) = \left\{(x, y) \in \mathbb{H}^2 \mid 0 \leq y < 1\right\},$$

containing the x-axis.



Topological interior $\operatorname{int}_{\mathbb{H}^2}(A)$



Manifold interior A°

On the other haand, the manifold interior A° of the 2-dimensional manifold with boundary A is the set

$$A^\circ = \left\{(x, y) \in \mathbb{H}^2 \mid 0 < y < 1\right\},$$

not containing the x-axis.

Let us now consider the same set A, but now as a subset of $\mathbb{R}^2$ instead of $\mathbb{H}^2$:

$$A = \left\{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq 1\right\}.$$

Then the topological interior $\operatorname{int}_{\mathbb{R}^2}(A)$ of A in $\mathbb{R}^2$ is the set

$$\operatorname{int}_{\mathbb{R}^2}(A) = \left\{(x, y) \in \mathbb{H}^2 \mid 0 < y < 1\right\},$$

which coincides with A°.

The boundary of a manifold with boundary

Let M be a manifold of dimension n with boundary ∂M. If (U, φ) is a chart of M, we denote by

$$\varphi' = \varphi|_{U \cap \partial M},$$

the restriction of the coordinate map φ to the boundary ∂M. Since φ maps boundary points to boundary points,

$$\varphi' = \varphi|_{U \cap \partial M} : U \cap \partial M \to \partial \mathbb{H}^n = \mathbb{R}^{n-1}.$$

Additionally, if (U, φ) and (V, ψ) are two charts on M, then

$$\psi' \circ (\varphi')^{-1} : \varphi(U \cap V \cap \partial M) \to \psi(U \cap V \cap \partial M)$$

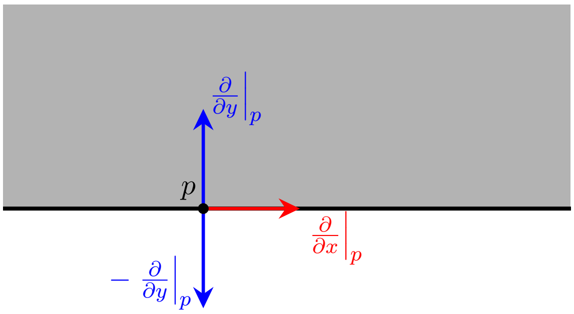
is C^∞. Thus, an atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ induces an atlas $\left\{\left(U_\alpha \cap \partial M, \varphi_\alpha|_{U_\alpha \cap \partial M}\right)\right\}_{\alpha \in A}$ for ∂M, making ∂M into a C^∞ manifold of dimension $n - 1$ (without boundary).

§15.3 Tangent Vectors, Differential Forms, and Orientations

Let M be a manifold with boundary and $p \in \partial M$. Let us first understand what $C_p^\infty(M)$ is. Two C^∞ functions $f : U \to \mathbb{R}$ and $g : V \to \mathbb{R}$ defined on neighborhoods U and V of $p \in M$ and are said to be equivalent if f and g agree on some neighborhood W of p contained in $U \cap V$. It can easily be verified that the relation defined above is an equivalence relation. Under this equivalence relation, a germ of C^∞ functions at p is an equivalence class of such functions. With the usual pointwise addition, multiplication and scalar multiplication, the set $C_p^\infty(M)$ of germs of C^∞ functions at p is an $\mathbb{R}$-algebra.

The **tangent space** T_pM at p is then defined to be the vector space of all point derivations on the algebra $C_p^\infty(M)$ [Recall that a point derivation of $C_p^\infty(M)$ is a linear map $D_p : C_p^\infty(M) \to \mathbb{R}$ obeying Leibniz condition].

Let us now take the example where $p \in \partial\mathbb{H}^2$.



$\frac{\partial}{\partial x}\Big|_p$ and $\frac{\partial}{\partial y}\Big|_p$ are both point derivations on $C_p^\infty\left(\mathbb{H}^2\right)$. The tangent space $T_p\mathbb{H}^2$ is represented by a 2-dimensional vector space with origin at p and spanned by the tangent vectors $\frac{\partial}{\partial x}\Big|_p$ and $\frac{\partial}{\partial y}\Big|_p$. Since $T_p\mathbb{H}^2$ is a vector space and $\frac{\partial}{\partial y}\Big|_p \in T_p\mathbb{H}^2$, we have $-\frac{\partial}{\partial y}\Big|_p \in \mathbb{H}^2$.

The **cotangent space** T_p^*M to the point $p \in \partial M$ of the manifold M with boundary ∂M is defined to be the dual of the tangent space T_pM:

$$T_p^*M = \operatorname{Hom}\left(T_pM, \mathbb{R}\right). \tag{15.8}$$

By taking the disjoint union of the cotangent spaces T_p^*M for all points $p \in M$, i.e. over all interior and boundary points of M, one arrives at the cotangent bundle T^*M of the manifold with boundary. Define the vector bundle.

$$\Lambda^k\left(T^*M\right) = \bigsqcup_{P\in M} \Lambda^k\left(T_p^*M\right). \tag{15.9}$$

Then a **differential k-form** on M is a section of the vector bundle $\Lambda^k\left(T^*M\right)$. A differential k-form is C^∞ if it is C^∞ as a section of the vector bundle $\Lambda^k\left(T^*M\right)$. For example, $\mathrm{d}x \wedge \mathrm{d}y$ is a C^∞ 2-form on $\mathbb{H}^2$.

An **orientation** on an n-dimensional manifold M with boundary is again a C^∞ nowhere vanishing n-form on M. We've seen in Proposition 14.4 that the orientability of a manifold without boundary (or equivalently the existence of a C^∞ nowhere-vanishing top degree form by the definition of orientability of a manifold) is equivalent to the existence of an oriented atlas. The same goes for manifold with boundary.

In the proof of Proposition 14.4 for establishing the necessary and sufficient condition for the orientability of a manifold without boundary, it was necessary to replace the chart $(U, x^1, \ldots, x^n)$ by $(U, -x^1, \ldots, x^n)$. This would not be possible to carry out in the case $n = 1$ for manifold with boundary if we had not allowed the left-half line $\mathbb{L}^1$ as a local model in the definition of a chart on a 1-dimensional manifold with boundary. It would be better understood if we look at the following example.

Example 15.3. The closed interval $[0, 1]$ is a C^∞ manifold with boundary. It has an atlas with 2 charts (U_1, ϕ_1) and (U_2, ϕ_2), where $U_1 = [0, 1), \phi_1(x) = x$, and $U_2 = (0, 1]$, $\phi_2(x) = 1 - x$.

With $\mathrm{d}x$ as the orientation form, $[0,1]$ should be an oriented manifold with boundary. However, $\{(U_1,\phi_1),(U_2,\phi_2)\}$ is not an oriented atlas as the transition map

$$\left(\phi_2 \cdot \phi_1^{-1}\right)(x) = 1 - x$$

has negative Jacobian determinant on $\phi_1(0,1) = (0,1)$. If one flips the sign of ϕ_2, then $\{(U_1,\phi_1),(U_2,-\phi_2)\}$ becomes an oriented atlas as

$$\left(\phi_2 \cdot \phi_1^{-1}\right)(x) = x - 1$$

has positive Jacobian determinant on $\phi_1(0,1) = (0,1)$. It's important to note that $-\phi_2(x) = x - 1$ maps $U_2 = (0,1]$ to $(-1,0]$ being an open set of the left half-line $\mathbb{L}^1 = (-\infty, 0]$. If we had allowed only $\mathbb{H}^1$ as a local model for a 1-dimensional manifold with boundary, the closed interval $[0,1]$ wouldn't have an oriented atlas.

§15.4 Outward-Pointing Vector Fields

Definition 15.4. Let M be a manifold with boundary ∂M, and $p \in \partial M$. We say that a tangent vector $X_p \in T_pM$ is **inward-pointing** if $X_p \notin T_p(\partial M)$, and there are a positive real number ε and a curve $c : [0,\varepsilon) \to M$ such that $c(0) = p$, $c(0,\varepsilon) \in M^\circ$, and $c'(0) = X_p$. A vector $X_p \in T_pM$ is **outward-pointing** if $-X_p$ is inward-pointing.

For example, on the upper half-plane $\mathbb{H}^2$, the tangent vector $\left.\frac{\partial}{\partial y}\right|_p$ is inward-pointing, and $-\left.\frac{\partial}{\partial y}\right|_p$ is outward-pointing at $p \in \partial\mathbb{H}^2$.

A vector field along ∂M is a map that assigns to each point $p \in \partial M$, a tangent vector $X_p \in T_pM$ (as opposed to $T_p(\partial M)$). We say that a vector field X along the boundary ∂M is **outward-pointing** if for all $p \in \partial M$, $X_p \in T_pM$ is outward-pointing.

In a coordinate neighborhood $(U, x^1, \ldots, x^n)$ of p in M, a vector field X along ∂M can be written as a linear combination

$$X_q = \sum_i a^i(q) \left.\frac{\partial}{\partial x^i}\right|_q, \tag{15.10}$$

for $q \in \partial M$. The vector field X along ∂M is said to be smooth at $p \in M$ if there exists a coordinate neighborhood of p for which the functions a^i on ∂M are C^∞ at p; it is said to be smooth if it is smooth at every point p.

Lemma 15.4

Let M be a manifold with boundary and let $p \in \partial M$. Suppose $X_p \in T_pM$ is expressed as a linear combination of basis vectors on a chart $(U, x^1, \ldots, x^n)$ as follows:

$$X_p = \sum_i a^i(p) \left.\frac{\partial}{\partial x^i}\right|_p.$$

Then X_p is outward pointing if and only if $a^n(p) < 0$.

Proof. ($\Rightarrow$) Suppose X_p is outward pointing, i.e. $Y_p = -X_p$ is inward pointing. Then $Y_p \notin T_p(\partial M)$ and there is a curve $c : [0,\varepsilon) \to M$ such that

$$c(0) = p,\ c'(0) = Y_p, \text{ and } c((0,\varepsilon)) \subseteq M^\circ.$$

Since $(U,\varphi) \equiv (U, x^1, \ldots, x^n)$ is a chart in the manifold with boundary M, U is diffeomorphic with an open subset of $\mathbb{H}^n$ via φ. Therefore, $x^n \geq 0$ on U. Since $p \in \partial M$, $x^n(p) = 0$. So if we take the curve $\varphi \circ c = (c^1, \ldots, c^n)$, where $c^i = x^i \circ c$, we have $c^n(0) = x^n(p) = 0$, and $c^n(t) \geq 0$ for $0 < t < \varepsilon$. Therefore,

$$\dot{c}^n(0) = \lim_{t \to 0^+} \frac{c^n(t) - c^n(0)}{t} = \lim_{t \to 0^+} \frac{c^n(t)}{t} \geq 0. \tag{15.11}$$

If $\dot{c}^n(0) = 0$, then $c^n(t) = 0$ for all $0 \le t \le \epsilon'$ for some $\epsilon' \le \varepsilon$. This would mean that

$$x^n(c(t)) = 0, \tag{15.12}$$

i.e. $c(t) \in \partial M$, which is not possible. Therefore, $\dot{c}^n(0) > 0$. Since $c'(0) = Y_p$, using *Proposition 6.3.1* of DG1,

$$Y_p = c'(0) = \sum_{i=1}^{n} \dot{c}^i(0) \left.\frac{\partial}{\partial x^i}\right|_p. \tag{15.13}$$

Since $Y_p = -X_p$, $-a^n(p) = \dot{c}^n(0) > 0$, i.e. $a^n(p) < 0$.

(⇐) Let $Y_p = -X_p$. By hypothesis, $-a^n(p) > 0$. We define $\alpha : \mathbb{R} \to \mathbb{R}^n$ as

$$\alpha(t) = \left(-a^1(p) \cdot t + p^1, -a^2(p) \cdot t + p^2, \ldots, -a^n(p) \cdot t + p^n\right), \tag{15.14}$$

where $\varphi(p) = (p^1, \ldots, p^n)$.

$p \in \partial M$, so $p^n = 0$. $\alpha(0) = \varphi(p) \in \varphi(U)$. Since $-a^n(p) > 0$, there exists $\varepsilon > 0$ such that $\alpha(t) \in \varphi(U) \subseteq \mathbb{H}^n$ for each $0 \le t < \varepsilon$. So we define the curve $c : [0, \varepsilon) \to U \subseteq M$ as

$$c(t) = \varphi^{-1}(\alpha(t)) = \varphi^{-1}\left(-a^1(p) \cdot t + p^1, -a^2(p) \cdot t + p^2, \ldots, -a^n(p) \cdot t\right). \tag{15.15}$$

Then clearly $c(0) = p$. For $0 < t < \varepsilon$, $-a^n(p) \cdot t > 0$, so $\alpha(t) \in (\mathbb{H}^n)^\circ$. As a result, $c(t) \in U^\circ \subseteq M^\circ$. Furthermore, $c'(0)$ is given by

$$\begin{aligned} c'(0) &= \sum_{i=1}^{n} \dot{c}^i(0) \left.\frac{\partial}{\partial x^i}\right|_p \\ &= \sum_{i=1}^{n} \frac{\mathrm{d}\left(-a^i(p) \cdot t + p^i\right)}{\mathrm{d}t}(0) \left.\frac{\partial}{\partial x^i}\right|_p \\ &= -\sum_i a^i(p) \left.\frac{\partial}{\partial x^i}\right|_p = -X_p. \end{aligned} \tag{15.16}$$

Therefore, $-X_p$ is inward pointing, i.e. X_p is outward pointing. ■

Proposition 15.5

On a manifold M with boundary ∂M, there is a smooth outward-pointing vector field along ∂M.

Proof. Let $\{(U_\alpha, x_\alpha^1, x_\alpha^2, \ldots, x_\alpha^n)\}_{\alpha \in A}$ be an atlas for the manifold M. Let $\{\rho_\alpha\}_{\alpha \in A}$ be a partition of unity subordinate to the open cover $\{U_\alpha\}_{\alpha \in A}$. On each U_α, we take the vector field $-\frac{\partial}{\partial x_\alpha^n}$, and then we attach them using the partition of unity:

$$X = -\sum_\alpha \rho_\alpha \frac{\partial}{\partial x_\alpha^n}. \tag{15.17}$$

Now we show that X is an outward pointing vector field. First, we are going to show that X is smooth. Given any $q \in M$, there is a coordinate open set U that intersects only finitely many $\operatorname{supp} \rho_\alpha$'s due to the local finiteness of $\{\operatorname{supp} \rho_\alpha\}_\alpha$. Now, on the chart $(U, x^1, \ldots, x^n)$,

$$X = -\sum_\alpha \rho_\alpha \frac{\partial}{\partial x_\alpha^n} = -\sum_\alpha \rho_\alpha \sum_{i=1}^{n} \frac{\partial x^i}{\partial x_\alpha^n} \frac{\partial}{\partial x^i} = -\sum_{i=1}^{n} \sum_\alpha \rho_\alpha \frac{\partial x^i}{\partial x_\alpha^n} \frac{\partial}{\partial x^i}. \tag{15.18}$$

Here we swapped the order of summation, because they are finite sums. $\frac{\partial x^i}{\partial x_\alpha^n}$ is smooth, since the charts are C^∞-compatible. ρ_α are also smooth. Therefore, X is smooth on U. In particular, X is smooth at q. Since $q \in M$ was arbitrary, X is smooth on all of M.

Now we are going to show that X_p is outward-pointing for each $p \in \partial M$. There is a coordinate neighborhood V of p that intersects only finitely many $\operatorname{supp} \rho_\alpha$'s due to the local finiteness of $\{\operatorname{supp} \rho_\alpha\}_\alpha$. Suppose $(V, y^1, \ldots, y^n)$ be a coordinate chart. Since $-\left.\frac{\partial}{\partial x_\alpha^n}\right|_p$ is an outward pointing vector in $T_p(V \cap U_\alpha)$, if we write

$$-\left.\frac{\partial}{\partial x_\alpha^n}\right|_p = \sum_{i=1}^n a_\alpha^i(p) \left.\frac{\partial}{\partial y^i}\right|_p, \tag{15.19}$$

we would have $a_\alpha^n(p) < 0$. Now, X_p is

$$X_p = -\sum_\alpha \rho_\alpha(p) \left.\frac{\partial}{\partial x_\alpha^n}\right|_p = \sum_\alpha \rho_\alpha(p) \sum_{i=1}^n a_\alpha^i(p) \left.\frac{\partial}{\partial y^i}\right|_p = \sum_{i=1}^n \sum_\alpha \rho_\alpha(p) a_\alpha^i(p) \left.\frac{\partial}{\partial y^i}\right|_p. \tag{15.20}$$

Here we swapped the order of summation, because they are finite sums. $\rho_\alpha(p) \geq 0$, and it is positive for at least one α since $\sum \rho_\alpha = 1$. Therefore, the coefficient of $\left.\frac{\partial}{\partial y^n}\right|_p$ in (15.20) is

$$\sum_\alpha \rho_\alpha(p) a_\alpha^n(p), \tag{15.21}$$

which is negative, since $a_\alpha^n(p) < 0$ for each α. Hence, X_p is outward pointing by Lemma 15.4. ■

§15.5 Interior Multiplication

If β is a k-covector on a vector space V, and $\mathbf{v} \in V$, for $k \geq 2$, the **interior multiplication** or **contraction** of β with $\mathbf{v}$ is the $(k-1)$-covector $\iota_{\mathbf{v}}\beta$ defined by

$$(\iota_{\mathbf{v}}\beta)(\mathbf{v}_2, \ldots, \mathbf{v}_k) = \beta(\mathbf{v}, \mathbf{v}_2, \ldots, \mathbf{v}_k), \tag{15.22}$$

with $\mathbf{v}_2, \ldots, \mathbf{v}_k \in V$. When β is a 1-covector, then $\iota_{\mathbf{v}}\beta$ is supposed to be a constant real number (i.e. a 0-covector). This is defined by

$$\iota_{\mathbf{v}}\beta = \beta(\mathbf{v}) \in \mathbb{R}. \tag{15.23}$$

Finally, when β is a 0-covector on V (i.e. a constant real number), we define

$$\iota_{\mathbf{v}}\beta = 0. \tag{15.24}$$

Interior multiplication on a manifold is defined pointaise. If X is a smooth vector field on M and $\omega \in \Omega^k(M)$, then $\iota_X\omega \in \Omega^{k-1}(M)$, defined by

$$(\iota_X\omega)_p = \iota_{X_p}\omega_p. \tag{15.25}$$

The right side of (15.25) makes sense according to (15.22), (15.23), and (15.24). Indeed, for $(k-1)$ many tangent vectors $X_p^2, \ldots, X_p^k$ with $k \geq 2$, (15.25) can be recast into the following using (15.22) as

$$(\iota_X\omega)_p\left(X_p^2, \ldots, X_p^k\right) = (\iota_{X_p}\omega_p)\left(X_p^2, \ldots, X_p^k\right) = \omega_p\left(X_p, X_p^2, \ldots, X_p^k\right). \tag{15.26}$$

If $X, X^2, \ldots, X^k$ are k-many smooth vector fields on M, then the RHS of (15.26) is $\left[\omega\left(X, X^2, \ldots, X^k\right)\right](p)$ while the LHS of (15.26) is $\left[(\iota_X\omega)\left(X^2, \ldots, X^k\right)\right](p)$. Therefore, for $k \geq 2$, one has

$$(\iota_X\omega)\left(X^2, \ldots, X^k\right) = \omega\left(X, X^2, \ldots, X^k\right), \tag{15.27}$$

for $(k-1)$ many C^∞ vector fields $X^2, \ldots, X^k$ on M. Now, since ω is a smooth k-form, for any smooth vector fields $X, X^2, \ldots, X^k$ on M, $\omega\left(X, X^2, \ldots, X^k\right)$ is a smooth function on M. As a result, for any C^∞ vector fields $X^2, \ldots, X^k$, $(\iota_X\omega)\left(X^2, \ldots, X^k\right)$ is a smooth function on M. Therefore, $\iota_X\omega$ is a smooth $(k-1)$-form on M.

Proposition 15.6

For 1-covectors $\alpha^1, \dots, \alpha^k$ on a vector space V and $\mathbf{v} \in V$,

$$\iota_{\mathbf{v}}\left(\alpha^1 \wedge \cdots \wedge \alpha^k\right) = \sum_{i=1}^{k} (-1)^{i-1} \alpha^i(\mathbf{v})\, \alpha^1 \wedge \cdots \wedge \widehat{\alpha^i} \wedge \cdots \wedge \alpha^k,$$

where the caret $\widehat{\ }$ on the i-th covector α^i means that α^i is omitted from the wedge product.

Proof. Consider $k \geq 2$. For any $\mathbf{v}_2, \dots, \mathbf{v}_k \in V$,

$$\begin{aligned}
\left[\iota_{\mathbf{v}}\left(\alpha^1 \wedge \cdots \wedge \alpha^k\right)\right](\mathbf{v}_2, \dots, \mathbf{v}_k) &= \left(\alpha^1 \wedge \cdots \wedge \alpha^k\right)(\mathbf{v}, \mathbf{v}_2, \dots, \mathbf{v}_k) \\
&= \det \begin{bmatrix} \alpha^1(\mathbf{v}) & \alpha^1(\mathbf{v}_2) & \cdots & \alpha^1(\mathbf{v}_k) \\ \alpha^2(\mathbf{v}) & \alpha^2(\mathbf{v}_2) & \cdots & \alpha^2(\mathbf{v}_k) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha^k(\mathbf{v}) & \alpha^k(\mathbf{v}_2) & \cdots & \alpha^k(\mathbf{v}_k) \end{bmatrix} \\
&= \sum_{i=1}^{k} (-1)^{i+1} \alpha^i(\mathbf{v}) \det \left[\alpha^l(\mathbf{v}_j)\right]_{\substack{1 \leq l \leq k, l \neq i \\ 2 \leq j \leq k}} \\
&= \sum_{i=1}^{k} (-1)^{i-1} \alpha^i(\mathbf{v}) \left(\alpha^1 \wedge \cdots \wedge \widehat{\alpha^i} \wedge \cdots \wedge \alpha^k\right)(\mathbf{v}_2, \dots, \mathbf{v}_k). \qquad (15.28)
\end{aligned}$$

Therefore,

$$\iota_{\mathbf{v}}\left(\alpha^1 \wedge \cdots \wedge \alpha^k\right) = \sum_{i=1}^{k} (-1)^{i-1} \alpha^i(\mathbf{v})\, \alpha^1 \wedge \cdots \wedge \widehat{\alpha^i} \wedge \cdots \wedge \alpha^k. \qquad (15.29)$$

If $k = 1$,

$$\iota_{\mathbf{v}}\left(\alpha^1\right) = \alpha^1(\mathbf{v}) = (-1)^{1-1} \alpha^1(\mathbf{v}). \qquad (15.30)$$

So, the equality holds for $k = 1$ as well. ■

Lemma 15.7

The interior multiplication $\iota_X \omega$ of a smooth k-form ω on M with a smooth vector field X on M has the following properties:

(i) $\iota_{fX}\omega = f\, \iota_X \omega$;

(ii) $\iota_X(f\omega) = f\, \iota_X \omega$;

for $f \in C^\infty(M)$.

Proof. (i) Suppose $k \geq 2$. For any $p \in M$, and any $X_p^2, \dots, X_p^k$,

$$\begin{aligned}
(\iota_{fX}\omega)_p\left(X_p^2, \dots, X_p^k\right) &= \left(\iota_{f(p)X_p}\omega_p\right)\left(X_p^2, \dots, X_p^k\right) \\
&= \omega_p\left(f(p)X_p, X_p^2, \dots, X_p^k\right) \\
&= f(p)\omega_p\left(X_p, X_p^2, \dots, X_p^k\right) \\
&= f(p)\, (\iota_X\omega)_p\left(X_p, X_p^2, \dots, X_p^k\right). \qquad (15.31)
\end{aligned}$$

Since $p \in M$ is arbitrary, we have

$$\iota_{fX}\omega = f\, \iota_X \omega. \qquad (15.32)$$

Now consider the case $k = 1$.

$$(\iota_{fX}\omega)_p = \omega_p(f(p)\, X_p) = f(p)\, [\omega(X)]_p = (f\, \iota_X\omega)_p. \qquad (15.33)$$

Therefore, for $k = 1$ as well, since $p \in M$ is arbitrary,

$$\iota_{fX}\omega = f\, \iota_X \omega. \tag{15.34}$$

(ii) Again, let us first consider the case $k \geq 2$. For any $p \in M$, and any $X_p^2, \ldots, X_p^k$,

$$\begin{aligned}(\iota_X(f\omega))_p \left(X_p^2, \ldots, X_p^k\right) &= (\iota_{X_p}(f(p)\omega_p)) \left(X_p^2, \ldots, X_p^k\right) \\ &= f(p)\omega_p \left(X_p, X_p^2, \ldots, X_p^k\right) \\ &= f(p)\ (\iota_X\omega)_p \left(X_p, X_p^2, \ldots, X_p^k\right).\end{aligned} \tag{15.35}$$

Since $p \in M$ is arbitrary, we have

$$\iota_X (f\omega) = f\, \iota_X \omega. \tag{15.36}$$

Now consider the case $k = 1$.

$$(\iota_X(f\omega))_p = \iota_{X_p} (f\,(p)\,\omega_p) = f\,(p)\,\omega_p\,(X_p) = f\,(p)\,[\omega\,(X)]_p = (f\,\iota_X\omega)_p\,. \tag{15.37}$$

Therefore, for $k = 1$ as well, since $p \in M$ is arbitrary,

$$\iota_X (f\omega) = f\, \iota_X \omega. \tag{15.38}$$

■

§15.6 Boundary Orientation

Now, we show that the boundary of an orientable manifold M with boundary is an orientable manifold without boundary.

Proposition 15.8

Let M be an orientable n-manifold with boundary ∂M. If ω is an orientation form on M and X is a smooth outward-pointing vector field along ∂M, then $\iota_X\omega$ is a smooth nowhere vanishing $(n-1)$ form on ∂M. Hence, ∂M is orientable.

Proof. Since ω is smooth on M, ω is also smooth on ∂M. By hypothesis, X is smooth on ∂M. Hence, the contraction $\iota_X\omega$ is also smooth on ∂M. We will now prove by contradiction that $\iota_X\omega$ is nowhere-vanishing on ∂M. Suppose. $\iota_X\omega$ vanishes at some $p \in \partial M$. It means that

$$(\iota_X\omega)_p \left(X_p^1, \ldots, X_p^{n-1}\right) = 0, \tag{15.39}$$

for any $X_p^1, \ldots, X_p^{n-1} \in T_p(\partial M)$. Let $Y_p^1, \ldots, Y_p^{n-1}$ be a basis for $T_p(\partial M)$. Since X is a smooth outward-pointing vector field along ∂M, $X_p \notin T_p\,(\partial M)$. Now,

$$\dim T_pM = \dim T_p\,(\partial M) + 1,$$

since ∂M is a manifold of dimension $n-1$ (without boundary). Since $X_p \notin T_p(\partial M)$ and $Y_p^1, \ldots, Y_p^{n-1}$ is a basis for $T_p(\partial M)$, one finds that $X_p, Y_p^1, \ldots, Y_p^{n-1}$ is a basis for T_pM. Hence,

$$\omega_p \left(X_p, Y_p^1, \ldots, Y_p^{n-1}\right) = (\iota_X\omega)_p \left(X_p^1, \ldots, X_p^{n-1}\right) = 0. \tag{15.40}$$

Now, by Lemma 14.1, since $X_p, Y_p' \cdots Y_p^{n-1}$ forms a basis for T_pM,

$$\omega_p \left(Z_p^1, \ldots, Z_p^n\right) = 0, \tag{15.41}$$

for all $Z_p^1, \ldots, Z_p^n \in T_pM$. In other words, $\omega_p \equiv 0$ on T_pM, a contradiction. ■

Example 15.4 (The boundary orientation on $\partial\mathbb{H}^n$)**.** An orientation form for the standard orientation on the upper half-space $\mathbb{H}^n$ is $\omega = \mathrm{d}x^1 \wedge \cdots \mathrm{d}x^n$. A smooth outward pointing vector field on $\partial\mathbb{H}^n$ is $-\frac{\partial}{\partial x^n}$. By Proposition 15.8, an orientation form on $\partial\mathbb{H}^n$ is given by the contraction

$$\iota_{-\frac{\partial}{\partial x^n}}(\omega) = \iota_{-\frac{\partial}{\partial x^n}}\left(\mathrm{d}x^1 \wedge \cdots \mathrm{d}x^n\right). \tag{15.42}$$

Using Proposition 15.6 and Lemma 15.7(i), we get

$$\begin{aligned}\iota_{-\frac{\partial}{\partial x^n}}(\omega) &= -\sum_{i=1}^{n}(-1)^{i-1}\left[\mathrm{d}x^i\left(\frac{\partial}{\partial x^n}\right)\right]\mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^n \\ &= \sum_{i=1}^{n}(-1)^i\,\delta^i{}_n\,\mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^{n-1} \wedge \mathrm{d}x^n \\ &= (-1)^n\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^{n-1}.\end{aligned} \tag{15.43}$$

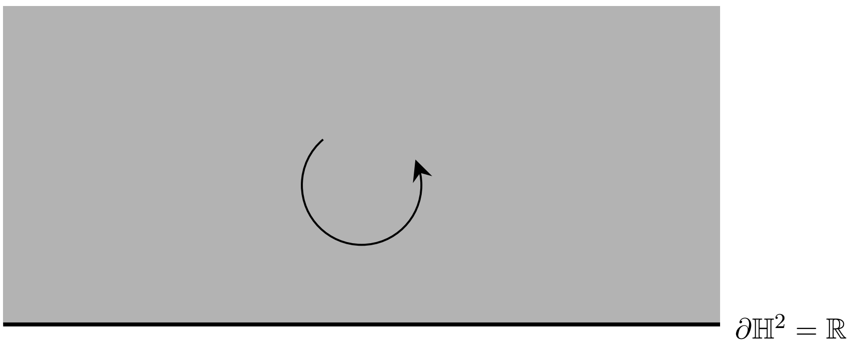
The $n = 1$ case needs a separate treatment. Recall that for a 1-covector β on a vector space V, and any vector $\mathbf{v} \in V$, the interior multiplication of β with $\mathbf{v}$ is defined as

$$\iota_{\mathbf{v}}\beta = \beta(\mathbf{v}) \in \mathbb{R}.$$

Now, for $\mathbb{H}^1 = [0,\infty)$, $\partial\mathbb{H}^1 = \{0\}$. Fix the orientation 1-form $\mathrm{d}x$ corresponding to the orientation directed from left to right. An outward pointing vector field on $\partial\mathbb{H}^1 = \{0\}$ is given by $-\frac{\partial}{\partial x}$. By Proposition 15.8, an orientation form on $\partial\mathbb{H}^1$ is given by the contraction

$$\iota_{-\frac{\partial}{\partial x}}(\mathrm{d}x) = -1. \tag{15.44}$$

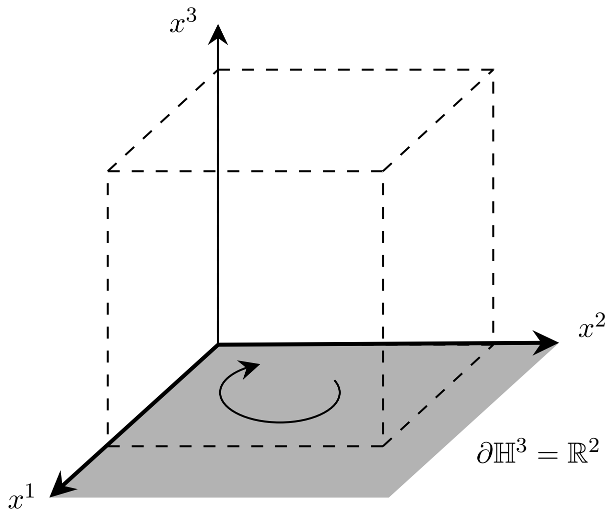
Let us now consider the case $n = 2$.



By (15.43), the boundary orientation on $\partial\mathbb{H}^2 = \mathbb{R}$ is given by the contraction

$$\iota_{-\frac{\partial}{\partial x^2}}\left(\mathrm{d}x^1 \wedge \mathrm{d}x^2\right) = \mathrm{d}x^1.. \tag{15.45}$$

Now consider $n = 3$.

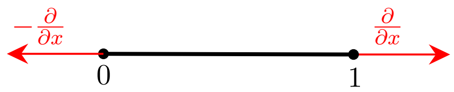


Here the manifold is $\mathbb{H}^3$ with boundary $\partial\mathbb{H}^3 = \mathbb{R}^2$, the x^1-x^2 plane (correspondint to $x^3 = 0$). According to (15.43), the boundary orientation on $\partial\mathbb{H}^3 = \mathbb{R}^2$ is given by the 2-form

$$(-1)^3 \, \mathrm{d}x^1 \wedge \mathrm{d}x^2 = -\mathrm{d}x^1 \wedge \mathrm{d}x^2. \tag{15.46}$$

This yields the anti-clockwise orientation on the x^1-x^2 plane.

Example 15.5. Consider the closed interval $[0, 1]$ in $\mathbb{R}$. One has $\partial [0, 1] = \{0, 1\}$. Also consider the orientation 1-form $\mathrm{d}x$ on $[0, 1]$ corresponding to the standard orientation directed from left to right.



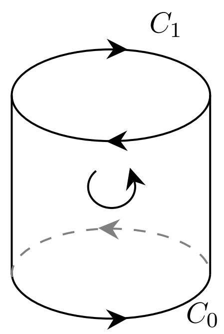
At the right boundary point 1, an outward pointing vector field reads $\frac{\partial}{\partial x}$. Hence, by Proposition 15.8, the boundary orientation at 1 is given by

$$\iota_{\frac{\partial}{\partial x}} (\mathrm{d}x) = 1. \tag{15.47}$$

On the other hand, at the left boundary point 0, an outward pointing vector field reads $-\frac{\partial}{\partial x}$. Hence, by Proposition 15.8, the boundary orientation at 0 is given by the contraction

$$\iota_{-\frac{\partial}{\partial x}} (\mathrm{d}x) = -1. \tag{15.48}$$

Example 15.6. Let M be the cylinder $S^1 \times [0, 1]$ with the counterclockise orientation when viewed from the exterior. Let us determine the boundary orientation on $C_0 = S^1 \times \{0\}$ and $C_1 = S^1 \times \{1\}$.



The counterclock arise orientation on M is given by the orientation form $\omega = \mathrm{d}\theta \wedge \mathrm{d}t$. An outward-pointing vector field on C_0 is given by $-\frac{\partial}{\partial t}$, so that the relevant contraction of ω with $-\frac{\partial}{\partial t}$ reads

$$\iota_{-\frac{\partial}{\partial t}} (\mathrm{d}\theta \wedge \mathrm{d}t) = -\frac{\partial}{\partial t} (\mathrm{d}t) (-1)^{2-1} \, \mathrm{d}\theta = \mathrm{d}\theta. \tag{15.49}$$

Hence, the boundary orientation on C_0 is given by the 1-form $\mathrm{d}\theta$.

Now, to determine the boundary orientation on $C_1 = S^1 \times \{1\}$, let us compute the contraction of ω on with an outward-pointing vector field $\frac{\partial}{\partial t}$ on C_1 :

$$\iota_{\frac{\partial}{\partial t}} (\mathrm{d}\theta \wedge \mathrm{d}t) = \frac{\partial}{\partial t} (\mathrm{d}t) (-1)^{2-1} \, \mathrm{d}\theta = -\mathrm{d}\theta. \tag{15.50}$$

So the boundary orientation on C_1 is given by the 1-form $-\mathrm{d}\theta$. Therefore, on C_0, the orientation is given by the counterclockwise orientation and on C_1, the orientation is given by the clockwise orientation viewed from the top.

Remark 15.3. Recall the nowhere-vanishing 1 form on S^1 from Example 13.2:

$$\omega = \begin{cases} \frac{\mathrm{d}y}{x} & \text{on } U_x = \{(x,y) \in S^1 \mid x \neq 0\}\,, \\ -\frac{\mathrm{d}x}{y} & \text{on } U_y = \{(x,y) \in S^1 \mid y \neq 0\}\,. \end{cases} \tag{15.51}$$

If we go back to polar coordinates, i.e. $x = \cos\theta$ and $y = \sin\theta$, then

$$\frac{\mathrm{d}y}{x} = \frac{\mathrm{d}\left(\sin\theta\right)}{\cos\theta} = \frac{\cos\theta\,\mathrm{d}\theta}{\cos\theta} = \mathrm{d}\theta, \tag{15.52}$$

$$-\frac{\mathrm{d}x}{y} = -\frac{\mathrm{d}\left(\cos\theta\right)}{\sin\theta} = -\frac{-\sin\theta\,\mathrm{d}\theta}{\sin\theta} = \mathrm{d}\theta. \tag{15.53}$$

Therefore, this nowhere vanishing 1-form ω is, in fact, $\mathrm{d}\theta$.

16 Integration on Manifolds

§16.1 Riemann Integral Review

Let us first recall the subject of Riemann integration over a closed rectangle in Euclidean space $\mathbb{R}^n$. A closed rectangle in $\mathbb{R}^n$ is a cartesian product

$$R = \left[a^1, b^1\right] \times \ldots \times [a^n, b^n]$$

of closed intervals in $\mathbb{R}$ with $a^i, b^i \in \mathbb{R}$. Let f be a bounded function $f : R \to \mathbb{R}$ defined on a closed rectangle R. The volume vol(R) of the closed rectangle is defined to be

$$\text{vol}\,(R) = \prod_{i=1}^{n} (b_i - a_i)\,. \tag{16.1}$$

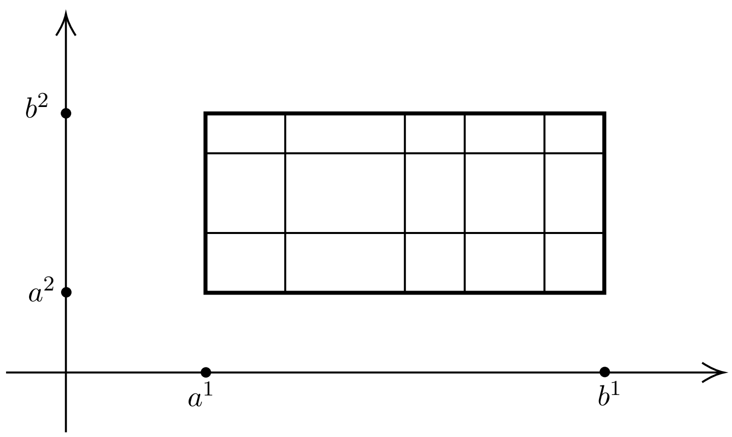
A partition of the closed interval $[a, b]$ is a set of real numbers $\{p_0, \ldots, p_n\}$ arranged in ascending order, i.e.

$$a = p_0 < p_1 < \ldots < p_n = b \tag{16.2}$$

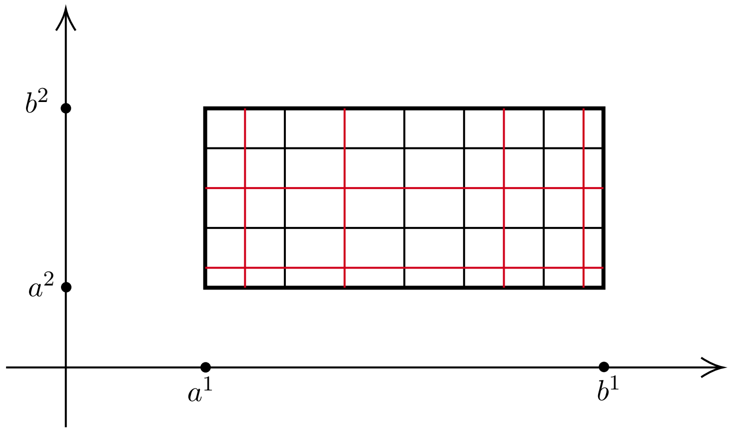
A partition of the rectangle R is a collection $P = \{P_1, \ldots, P_n\}$ such that each P_i is a partition of $[a^i, b^i]$. In other words. each P_i is an increasing sequence of real numbers

$$a^i_i = p^i_0 < p^i_1 < \ldots\ldots < p^i_{k^i} = b^i \tag{16.3}$$

for $i = 1, 2, \ldots\ldots n$ This way, the partition P of the closed rectangle R divides it into closed subrectangles, which we denote by R_j. On possible partition of a closed rectangle $[a^1, b^1] \times [a^2, b^2]$ is pictured below in $\mathbb{R}^2$:



A partition $P' = \{P'_1, \ldots, P'_n\}$ of the same rectangle R is called a **refinement** of the partition $P = \{P_1, \ldots, P_n\}$ of R if $P_i \subset P'_i$ for each $i = 1, 2, \ldots, n$. For example, the following partition of $[a^1, b^1] \times [a^2, b^2]$ is a refinement of the partition shown above. (The original partition P is drawn in black, while the new lines arising in the refined partition are drawn in red.)



We immediately see that each subrectangle R_j of P is subdivided into subrectangles R'_{jk} of the refinement P'. It's now time to define the lower and upper sum of the bounded function $f : R \to \mathbb{R}$ wirh respect to the partition P:

$$L(f, P) := \sum_{R_j} \left(\inf_{R_j} f \right) \operatorname{vol}(R_j), \tag{16.4}$$

$$U(f, P) := \sum_{R_j} \left(\sup_{R_j} f \right) \operatorname{vol}(R_j). \tag{16.5}$$

It's clear that for any partition P,

$$L(f, P) \leq U(f, P). \tag{16.6}$$

Now, suppose P' is a refinement of the partition P. Then each subrectangle R_j of P is subdivided into subrectangles R'_{jk} of the refinement P'. Furthermore, $\operatorname{vol}(R_j) = \sum_k \operatorname{vol}\left(R'_{jk}\right)$. Now, since $R'_{jk} \subset R_j$, one has

$$\inf_{R_j} f \leq \inf_{R'_{jk}} f \text{ and } \sup_{R_j} f \geq \sup_{R'_{jk}} f. \tag{16.7}$$

Now, from (16.4),

$$\begin{aligned} L(f, P) &= \sum_{R_j} \left(\inf_{R_j} f \right) \operatorname{vol}(R_j) \\ &\leq \sum_{R_j} \left(\inf_{R'_{jk}} f \right) \sum_k \operatorname{vol}\left(R'_{jk}\right) \\ &\leq \sum_{R'_{jk}} \left(\inf_{R'_{jk}} f \right) \operatorname{vol}\left(R'_{jk}\right) = L(f, P'). \end{aligned}$$

In other words,

$$L(f, P) \leq L(f, P'). \tag{16.8}$$

Similarly,

$$U(f, P) \geq U(f, P'). \tag{16.9}$$

Any two partitions P and P' of the rectangle R have a common refinement $Q = \{Q_1, \ldots, Q_n\}$ with $Q_i = P_i \cup P'_i$. Then by (16.8) and (16.9),

$$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P').$$

Therefore, for any two partitions P and P',

$$L(f, P) \leq U(f, P'). \tag{16.10}$$

From (16.10), one sees that $U(f, P')$ is an upper bound of $L(f, P)$ for any partition P of the rectangle R. As a result,

$$\sup_P L(f, P) \leq U(f, P'). \tag{16.11}$$

Again, from (16.11), one sees that $\sup_P L(f, P)$ is a lower bound of $U(f, P')$ for any partition P' of the rectangle R. Hence,

$$\sup_P L(f, P) \leq \inf_{P'} U(f, P'). \tag{16.12}$$

The supremum of the lower-sum $L(f, P)$ as P varies over all paritions of the rectangle is called the **lower integral**, and is denoted by

$$\underline{\int}_R f := \sup_P L(f, P) \tag{16.13}$$

On the contrary, the infimum of the upper-sum $U(f, P)$ as P varies over all paritions of the rectangle is called the **upper integral**, and is denoted by

$$\overline{\int}_R f := \inf_P U(f, P). \tag{16.14}$$

Using these notations (16.12) reads

$$\underline{\int}_R f \leq \overline{\int}_R f. \tag{16.15}$$

Definition 16.1 (Riemann integrability)**.** Let R be a closed rectangle in $\mathbb{R}^n$. A bounded function $f : R \to \mathbb{R}$ is said to be **Riemann integrable** if $\underline{\int}_R f = \overline{\int}_R f$. In this case, the **Riemann integral** of f is this common value, usually denoted by

$$\int_R f(\mathbf{x})\, \mathrm{d}x^1 \mathrm{d}x^2 \cdots \mathrm{d}x^n,$$

where $x^1, \ldots, x^n$ are the standard coordinates on $\mathbb{R}^n$.

Definition 16.2. If $f : A \subset \mathbb{R}^n \to R$, then the **extension of f by zero** is the function $\overline{f} : \mathbb{R}^n \to \mathbb{R}$ such that

$$\overline{f}(x) = \begin{cases} f(x) & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

If f is a bounded function on a bounded set A, then one encloses A in a closed rectangle $R \subset \mathbb{R}^n$. The the Riemann integral of $f : A \to \mathbb{R}$ over A is defined to be

$$\int_A f(\mathbf{x})\, \mathrm{d}x^1 \mathrm{d}x^2 \cdots \mathrm{d}x^n := \int_R \overline{f}(\mathbf{x})\, \mathrm{d}x^1 \mathrm{d}x^2 \cdots \mathrm{d}x^n, \tag{16.16}$$

provided the RHS exists, i.e. the extension $\overline{f}$ of f by zero is Riemann integrable over the closed rectangle R enclosing A. The **volume** $\operatorname{vol} A$ of a subset $A \subset \mathbb{R}^n$ is defined to be the integral $\int_A 1\, \mathrm{d}x^1 \mathrm{d}x^2 \cdots \mathrm{d}x^n$ if the integral exists.

Integrability conditions

Definition 16.3. A set $A \subset \mathbb{R}^n$ is said to have **measure zero** if for every $\varepsilon > 0$, there is a countable collection of closed rectangles $\{R_i\}_{i=1}^{\infty}$ such that $A \subset \bigcup_{i=1}^{\infty} R_i$ and

$$\sum_{i=1}^{\infty} \operatorname{vol}(R_i) < \varepsilon.$$

Theorem 16.1 (Lebesgue's theorem)

A bounded function $f : A \to \mathbb{R}$ on a bounded subset $A \subset \mathbb{R}^n$ is Riemann integrable if and only if the set Disc $\left(\overline{f}\right)$ of discontinuities of the extended function $\overline{f}$ has measure zero.

Proposition 16.2

If a continuous function $f : U \to \mathbb{R}$ defined on an open subset U of $\mathbb{R}^n$ has compact support, then f is Riemann integrable on U.

Proof. $f : U \to \mathbb{R}$ is continuous. On $U \setminus \operatorname{supp} f$, f is zero. Since f is continuous on $\operatorname{supp} f \subset U$, and $\operatorname{supp} f$ is compact (by hypothesis), f is bounded on $\operatorname{supp} f$. Also, f is zero on $U \setminus \operatorname{supp} f$. Hence, $f : U \to \mathbb{R}$ is a bounded continuous function on the open subset $U \subset \mathbb{R}^n$. We claim that the extension $\overline{f} : \mathbb{R}^n \to \mathbb{R}$ is continuous.

By the definition of extension of a function by zero, $\overline{f}$ agrees witt f on U, and hence $\overline{f}$ is continuous on U. It remains to show that f is continuous on the complement $\mathbb{R}^n \setminus U$ of U. Since $\operatorname{supp} f \subset U$, if $p \notin U$, then $P \notin \operatorname{supp} f$.

$\operatorname{supp} f$ being a compact subset of $\mathbb{R}^n$ is closed and bounded. Hence. $\mathbb{R}^n \setminus \operatorname{supp} f$ is open and $p \in \mathbb{R}^n \setminus \operatorname{supp} f$. Therefore, there exist an open ball B such that

$$p \in B \subset \mathbb{R}^n \setminus \operatorname{supp} f,$$

i.e. an open ball containing p and disjoint from $\operatorname{supp} f$. On this open ball B containing p, $f \equiv 0$. Therefore, f is

$$\overline{f}(x) = \begin{cases} f(x) & \text{if } x \in U, \\ 0 & \text{if } x \notin U. \end{cases} \tag{16.17}$$

On $B \setminus U$, $\overline{f}$ is clearly 0 . Since $B \cap U \cap \operatorname{supp} f = \varnothing$, and on $B \cap U$, f and $\overline{f}$ agree with other, one must have, $\overline{f} = 0$ on $B \cap U$. Hence, on B, $\overline{f} \equiv 0$. This implies that $\overline{f}$ is continuous at $p \in U$. We, therefore, have, $\overline{f}$ to be continuous on the whole of $\mathbb{R}^n$.

Note that $\overline{f}$, defined by (16.17) is also the zero extension of $f|_{\operatorname{supp} f} : \operatorname{supp} f \to \mathbb{R}$ with $\operatorname{supp} f$ being a bounded subset of $\mathbb{R}^n$. Now, we are all good to apply Lebesgue's theorem by which $f|_{\operatorname{supp} f} : \operatorname{supp} f \to \mathbb{R}$ is Riemann integravle integrable. Since f is zero on $U \setminus \operatorname{supp} f$, $f : U \to \mathbb{R}$ is also Riemann integrable. ■

Definition 16.4 (Domain of integration). A subset $A \subset \mathbb{R}^n$ is called a **domain of integration** if it is bounded and its topological boundary $\operatorname{bd} A$ is a set of measure zero.

Familiar plane figures, such as triangles, rectangles, disks are all domains of integration in $\mathbb{R}^2$.

Proposition 16.3

Every bounded continuous function f defined on a domain of integration A in $\mathbb{R}^n$ is Riemann integrable over A.

Proof. Let $\overline{f} : \mathbb{R}^n \to \mathbb{R}$ be the extension of f by zero, ie.,

$$\overline{f}(x) = \begin{cases} f(x) & \text{if } x \in A, \\ 0 & \text{if } x \notin U. \end{cases} \tag{16.18}$$

Since f is continuous on A by hypothesis, $\overline{A}$ is necessarily continuous in the open set $\operatorname{int}(A)$. One also observes that if p is an exterior point of A, i.e., if $p \in \mathbb{R}^n \setminus \overline{A}$, being an open set, there is an open set $U \subset \mathbb{R}^n$ such that

$$p \in U \subseteq \mathbb{R}^n \setminus \overline{A}.$$

Since $U \cap A = \varnothing$, $\overline{f} \equiv 0$ on U. Hence, $\overline{f}$ is continuous at p. One, thus, verifies that $\overline{f}$ is continuous at all interior and exterior points of A. Therefore, the set $\operatorname{Disc}\left(\overline{f}\right)$ of discontinuities of $\overline{f}$ is a subset of $\operatorname{bd}(A)$, a set of measure zero. By Lebergues' theorem, f is Riemann integrable on A. ■

§16.2 Integral of an n-form on $\mathbb{R}^n$

Definition 16.5. Let $\omega = f(\mathbf{x})\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$ be a C^∞ n-form on an open subset $U \subset \mathbb{R}^n$, with standard coordinates $x^1, \ldots, x^n$. Its integral over a subset $A \subset U$ is defined to be the Riemann integral of $f(\mathbf{x})$

$$\int_A \omega = \int_A f(\mathbf{x})\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n := \int_A f(\mathbf{x})\, \mathrm{d}x^1 \cdots \mathrm{d}x^n, \tag{16.19}$$

if the Riemann integral exists.

If f is a bounded continuous function on a domain of integration A in $\mathbb{R}^n$, then the integral $\int_A f\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$ exists by Proposition 16.3.

Let us now see how the integral of an n-form $\omega = f\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$ on an open subset $U \subseteq \mathbb{R}^n$ transform under change of variables. A change of variables on $U \subseteq \mathbb{R}^n$ is given by a diffeomorphism

$$T : V \subseteq \mathbb{R}^n \to U \subseteq \mathbb{R}^n.$$

Let $x^1, \ldots, x^n$ be the standard coordinates on U and $y^1, \ldots, y^n$ the standard coordinates on V. One, therefore, has $\left\{ \frac{\partial}{\partial x^1}\Big|_{T(p)}, \ldots, \frac{\partial}{\partial x^n}\Big|_{T(p)} \right\}$ to be a basis of $T_{T(p)}\mathbb{R}^n$ while $\left\{ \frac{\partial}{\partial y^1}\Big|_{p}, \ldots, \frac{\partial}{\partial y^n}\Big|_{p} \right\}$ is a basis of $T_p\mathbb{R}^n$. for $p \in V$. Now, the differential $DT(p) : T_p\mathbb{R}^n \to T_{T(p)}\mathbb{R}^n$ at $p \in V$ is represented by the following $n \times n$ matrix:

$$DT(p) = \begin{bmatrix} \frac{\partial T^1}{\partial y^1}(p) & \frac{\partial T^1}{\partial y^2}(p) & \cdots & \frac{\partial T^1}{\partial y^n}(p) \\ \frac{\partial T^2}{\partial y^1}(p) & \frac{\partial T^2}{\partial y^2}(p) & \cdots & \frac{\partial T^2}{\partial y^n}(p) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial T^n}{\partial y^1}(p) & \frac{\partial T^n}{\partial y^2}(p) & \cdots & \frac{\partial T^n}{\partial y^n}(p) \end{bmatrix}$$

The deterimant of the matrix DT is precisely the Jacobian determinant denoted by $\det(J(T))$, i.e. $\det(J(T)) = \det(DT)$, that arises in the change of variable formula for integration in multivariable calculus:

$$\int_U f\, \mathrm{d}x^1 \cdots \mathrm{d}x^n = \int_V (f \circ T)\, |\det(DT)|\, \mathrm{d}y^1 \cdots \mathrm{d}y^n, \tag{16.20}$$

with $f : U \subset \mathbb{R}^n \to \mathbb{R}$ being a bounded continuous function and $f \circ T : V \subset \mathbb{R}^n \to \mathbb{R}$. Here, we assume that U and V are both connected. By Lemma 13.5, one has

$$\mathrm{d}T^1 \wedge \cdots \mathrm{d}T^n = \det\left[\frac{\partial T^i}{\partial y^j}\right] \mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^n, \tag{16.21}$$

where $T^i = x^i \circ T = T^*\left(x^i\right)$ is the i-th component of T. Hence, for $T : V \subset \mathbb{R}^n \to U \subset \mathbb{R}^n$ and ω

being an n-form on U,

$$\begin{aligned}\int_V T^*\omega &= \int_V T^*\left(f\,\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^n\right)\\ &= \int_V (T^*f)\,T^*\mathrm{d}x'\wedge\cdots\wedge T^*\mathrm{d}x^n\\ &= \int_V (f\circ T)\ \mathrm{d}\left(T^*x^1\right)\wedge\cdots\wedge\mathrm{d}\left(T^*x^n\right)\\ &= \int_V (f\circ T)\,\mathrm{d}T^1\wedge\cdots\wedge\mathrm{d}T^n\\ &= \int_V (f_0T)\det(J(T))\,\mathrm{d}y^1\wedge\cdots\wedge\mathrm{d}y^n.\end{aligned}\tag{16.22}$$

Using (16.20),

$$\int_U\omega=\int_U f\,\mathrm{d}x^1\cdots\mathrm{d}x^n=\int_V (f\circ T)\left|\det J(T)\right|\,\mathrm{d}y^1\cdots\mathrm{d}y^n.\tag{16.23}$$

Since (16.22) and (16.23) differ by the sign of $\det(J(T))$, one has

$$\int_V T^*\omega=\pm\int_U\omega,\tag{16.24}$$

depending on whether the Jacobian determinant $\det\left(J(T)\right)$ is positive or negative. By Proposition 14.3, a diffeomorphism $T:V\subseteq\mathbb{R}^n\to U\in\mathbb{R}^n$ is orientation-preserving if and only if its Jacobian determinant

$$\det J\left(T\right)=\det\left[\frac{\partial T^i}{\partial y^j}\right]$$

is everywhere positive on V. Equation (16.24) tells us that the integral of differential form ω is not necessarily invariant under an arbitrary diffeomorphism $T:V\to U$. The integral of a differential form ω is only invariant ($\int_V T^*\omega=\int_U\omega$) if and only if the diffeomorphism $T:V\to U$ is orientation preserving.

§16.3 Integral of a differential form over a manifold

Our approach to integration over a general manifold has the following distinguishing features:

(a) The manifold must be oriented.

(b) On a manifold of dimension n, one can only integrate n-forms, not functions (which are 0-forms).

(c) The n-forms must have compact support.

Let M be an oriented manifold of dimension n, with an oriented atlas $\{(U_\alpha,\varphi_\alpha)\}_{\alpha\in A}$ giving the orientation of M. If $\omega\in\Omega^k(M)$, then

$$\operatorname{supp}\omega=\overline{\{p\in M\mid\omega_p\neq 0\}}=\mathrm{cl}_M\left(\{p\in M\mid\omega_p\neq 0\}\right).\tag{16.25}$$

Lemma 16.4

If (U,φ) is a chart in a manifold M (of dimension n) and ω is an n-form on U,

$$\operatorname{supp}\left[\left(\varphi^{-1}\right)^*\omega\right]=\varphi\left(\operatorname{supp}\omega\right).$$

Proof. Here $\varphi : U \to \varphi(U)$ is a diffeomorphism. In particular, it is a homeomorphism. Therefore. $\varphi\left(\overline{A}\right) = \overline{\varphi(A)}$. Now,

$$\begin{aligned}\operatorname{supp}\left[\left(\varphi^{-1}\right)^*\omega\right] &= \overline{\left\{q = \varphi(p) \in \varphi(U) \mid [(\varphi^{-1})^*\omega]_q \neq 0\right\}} \\ &= \overline{\varphi\left(\left\{p \in U \mid [(\varphi^{-1})^*\omega]_{\varphi(p)} \neq 0\right\}\right)} \\ &= \varphi\left(\overline{\left\{p \in U \mid [(\varphi^{-1})^*\omega]_{\varphi(p)} \neq 0\right\}}\right). \end{aligned} \tag{16.26}$$

Now, $\left[(\varphi^{-1})^*\omega\right]_{\varphi(p)} \in \Lambda^k\left(T^*_{\varphi(p)}\varphi(U)\right)$. It is 0 if and only if it yields 0 when applied to any basis vectors. Therefore,

$$\begin{aligned}\left[\left(\varphi^{-1}\right)^*\omega\right]_{\varphi(p)} = 0 &\iff \left[\left(\varphi^{-1}\right)^*\omega\right]_{\varphi(p)}\left(\left.\frac{\partial}{\partial r^{i_1}}\right|_{\varphi(p)}, \dots, \left.\frac{\partial}{\partial r^{i_k}}\right|_{\varphi(p)}\right) = 0 \text{ for all } I = (i_1, \dots, i_k) \\ &\iff \omega_{\varphi^{-1}(\varphi(p))}\left(\left(\varphi^{-1}\right)_{*,\varphi(p)}\left.\frac{\partial}{\partial r^{i_1}}\right|_{\varphi(p)}, \dots, \left(\varphi^{-1}\right)_{*,\varphi(p)}\left.\frac{\partial}{\partial r^{i_k}}\right|_{\varphi(p)}\right) = 0 \\ &\qquad\qquad \text{for all } I = (i_1, \dots, i_k) \\ &\iff \omega_p\left(\left.\frac{\partial}{\partial x^{i_1}}\right|_p, \dots, \left.\frac{\partial}{\partial x^{i_k}}\right|_p\right) = 0 \text{ for all } I = (i_1, \dots, i_k) \\ &\iff \omega_p = 0. \end{aligned} \tag{16.27}$$

Now using (16.26), we get

$$\begin{aligned}\operatorname{supp}\left[\left(\varphi^{-1}\right)^*\omega\right] &= \varphi\left(\overline{\left\{p \in U \mid [(\varphi^{-1})^*\omega]_{\varphi(p)} \neq 0\right\}}\right) \\ &= \varphi\left(\overline{\{p \in U \mid \omega_p \neq 0\}}\right) \\ &= \varphi(\operatorname{supp}\omega). \end{aligned} \tag{16.28}$$

■

Lemma 16.5
If $\omega, \tau \in \Omega^*(M)$, then

(a) $\operatorname{supp}(\omega + \tau) \subseteq \operatorname{supp}\omega \cup \operatorname{supp}\tau$.

(b) $\operatorname{supp}(\omega \wedge \tau) \subseteq \operatorname{supp}\omega \cap \operatorname{supp}\tau$.

Proof. (a) If $(\omega+\tau)_p \neq 0$, then $\omega_p \neq 0$ or $\tau_p \neq 0$. Therefore,

$$\left\{p \in M \mid (\omega+\tau)_p \neq 0\right\} \subseteq \{p \in M \mid \omega_p \neq 0\} \cup \{p \in M \mid \tau_p \neq 0\}. \tag{16.29}$$

Taking closure on both sides, and using the fact that $\overline{A \cup B} = \overline{A} \cup \overline{B}$, we get

$$\overline{\left\{p \in M \mid (\omega+\tau)_p \neq 0\right\}} \subseteq \overline{\{p \in M \mid \omega_p \neq 0\}} \cup \overline{\{p \in M \mid \tau_p \neq 0\}}. \tag{16.30}$$

In other words,

$$\operatorname{supp}(\omega+\tau) \subseteq \operatorname{supp}\omega \cup \operatorname{supp}\tau. \tag{16.31}$$

(b) If $(\omega \wedge \tau)_p \neq 0$, then $\omega_p \neq 0$ and $\tau_p \neq 0$. Therefore,

$$\left\{p \in M \mid (\omega \wedge \tau)_p \neq 0\right\} \subseteq \{p \in M \mid \omega_p \neq 0\} \cap \{p \in M \mid \tau_p \neq 0\}. \tag{16.32}$$

Taking closure on both sides, and using the fact that $\overline{A\cap B}\subseteq \overline{A}\cap\overline{B}$, we get

$$\overline{\left\{p\in M \mid (\omega\wedge\tau)_p\neq 0\right\}}\subseteq \overline{\{p\in M\mid \omega_p\neq 0\}}\cap\overline{\{p\in M\mid \tau_p\neq 0\}}. \tag{16.33}$$

In other words,

$$\operatorname{supp}(\omega\wedge\tau)\subseteq \operatorname{supp}\omega\cap\operatorname{supp}\tau. \tag{16.34}$$

■

Let $\Omega_{\mathrm{c}}^k(M)$ denote the vector space of C^∞ k-forms with compact support on M. Suppose (U,φ) is a chart in the atlas $\{(U_\alpha,\varphi_\alpha)\}_{\alpha\in A}$.

If $\omega\in\Omega_{\mathrm{c}}^n(U)$ is an n-form with compact support on U, since φ being a diffeomorphism is continuous, $\varphi(\operatorname{supp}\omega)$ is compact in $\varphi(U)$. Then by Lemma 16.4, supp $\left[(\varphi^{-1})^*\omega\right]$ is compact in $\varphi(U)\subseteq\mathbb{R}^n$. We define the integral of ω on U by

$$\int_U\omega=\int_{\varphi(u)\subset\mathbb{R}^n}\left(\varphi^{-1}\right)^*\omega. \tag{16.35}$$

If (U,ψ) is another chart in the oriented atlas with the same open set U, then $\varphi\circ\psi^{-1}:\psi(U)\to\varphi(U)$ is an orientation preserving diffeomorphism (i.e. with positive Jacobian determinant), so it preserves integral of n-form on open subset of $\mathbb{R}^n$. Therefore,

$$\begin{aligned}\int_{\varphi(U)}\left(\varphi^{-1}\right)^*\omega&=\int_{\psi(U)}\left(\varphi\circ\psi^{-1}\right)^*\left[\left(\varphi^{-1}\right)^*\omega\right]\\&=\int_{\psi(U)}\left[\left(\psi^{-1}\right)^*\circ\varphi^*\circ\left(\varphi^{-1}\right)^*\right]\omega\\&=\int_{\psi(U)}\left[\left(\psi^{-1}\right)^*\circ\left(\varphi^{-1}\circ\varphi\right)^*\right]\omega\\&=\int_{\psi(U)}\left(\psi^{-1}\right)^*\omega,\end{aligned} \tag{16.36}$$

proving the chart independence of the definition (16.35). By the linearity of integral on $\mathbb{R}^n$ and linearity of pullback, if $\omega,\tau\in\Omega_{\mathrm{c}}^n(U)$, then

$$\int_U(\omega+\tau)=\int_U\omega+\int_U\tau. \tag{16.37}$$

Now, let $\omega\in\Omega_{\mathrm{c}}^n(M)$. Choose a partition of unity $\{\rho_\alpha\}_\alpha$ subordinate to the open cover $\{U_\alpha\}_\alpha$. From the definition of partition of unity $\{\operatorname{supp}\rho_\alpha\}_\alpha$ is locally finite. Let $p\in\operatorname{supp}\omega$. There is a neighborhood W_p of p in M that intersects only finitely many of the sets $\operatorname{supp}\rho_\alpha$ (from the local finiteness of the set $\{\operatorname{supp}\rho_\alpha\}_\alpha$). The collection $\{W_p\mid p\in\operatorname{supp}\omega\}$ obviously covers supp ω. Since $\operatorname{supp}\omega$ is compact, there is a finite subcover of $\{W_p\mid p\in\operatorname{supp}\omega\}$ of $\operatorname{supp}\omega$. Let us denote this subcover by $\{W_{p_1},\ldots,W_{p_m}\}$. In other words,

$$\operatorname{supp}\omega\subseteq\bigcup_{i=1}^m W_{p_i} \tag{16.38}$$

Since each W_{p_i} intersects finitely many $\operatorname{supp}\rho_\alpha$ in $\{\operatorname{supp}\rho_\alpha\}_\alpha$, $\operatorname{supp}\omega$ must intersect only finitely many $\operatorname{supp}\rho_\alpha$. By Lemma 16.5(b),

$$\operatorname{supp}(\rho_\alpha\omega)\subseteq\operatorname{supp}\rho_\alpha\cap\operatorname{supp}\omega. \tag{16.39}$$

Thus for all but finitely many α, $\operatorname{supp}(\rho_\alpha\omega)$ is empty, i.e., $\rho_\alpha\omega\equiv 0$. Therefore, $\sum_\alpha\rho_\alpha\omega$ is a **finite** sum, and

$$\sum_\alpha\rho_\alpha\omega=\omega, \tag{16.40}$$

since $\sum_\alpha\rho_\alpha=1$. By (16.39), $\operatorname{supp}(\rho_\alpha\omega)\subseteq\operatorname{supp}\omega$, i.e. $\operatorname{supp}(\rho_\alpha\omega)$ is a closed subset of a compact set $\operatorname{supp}\omega$. Hence, $\operatorname{supp}(\rho_\alpha\omega)$ is compact. As a result, $\rho_\alpha\omega$ is an n-form compactly supported in the chart U_α, because

$$\operatorname{supp}(\rho_\alpha\omega)\subseteq\operatorname{supp}\rho_\alpha\cap\operatorname{supp}\omega\subseteq\operatorname{supp}\rho_\alpha\subseteq U_\alpha. \tag{16.41}$$

Hence, the integral $\int_{U_\alpha} \rho_\alpha \omega$ is defined, using (16.35). Therefore, we can define the inyehral of ω over M to be the finite sum

$$\int_M \omega := \sum_\alpha \int_{U_\alpha} \rho_\alpha \omega. \tag{16.42}$$

(This is a finite sum, because $\rho_\alpha \omega$ is a nonzero form on U_α for only finitely many α) Now, for the integral (16.42) to be well-defined, we must show that $\int_M \omega$ is independent of the choices of oriented atlas and partition of unity. Let $\{(V_\beta, \psi_\beta)\}_{\beta \in B}$ be another oriented atlas specifying the same orientation as that of $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$. Suppose $\{\chi_\beta\}_{\beta \in B}$ be a partition of unity subordinate to the open cover $\{V_\beta\}_{\beta \in B}$. Then

$$\left\{\left(U_\alpha \cap V_\beta, \varphi_\alpha|_{U_\alpha \cap V_\beta}\right)\right\}_{\alpha,\beta} \text{ and } \left\{\left(U_\alpha \cap V_\beta, \psi_\beta|_{U_\alpha \cap V_\beta}\right)\right\}_{\alpha,\beta}$$

are two new atlases of M specifying the same orientation on M. Then one has

$$\sum_\alpha \int_{U_\alpha} \rho_\alpha \omega = \sum_\alpha \int_{U_\alpha} \rho_\alpha \sum_\beta \chi_\beta \omega, \tag{16.43}$$

since $\sum_\beta \chi_\beta = 1$. Now, the sum $\sum_\beta \chi_\beta \omega$ is, in fact, a finite sum, because $\chi_\beta \omega$ is a nonzero form on V_β for only finitely many β. Therefore, we can take the sum in front of the integral.

$$\sum_\alpha \int_{U_\alpha} \rho_\alpha \omega = \sum_\alpha \sum_\beta \int_{U_\alpha} \rho_\alpha \chi_\beta \omega. \tag{16.44}$$

Now,

$$\operatorname{supp}(\rho_\alpha \chi_\beta) \subseteq \operatorname{supp} \rho_\alpha \cap \operatorname{supp} \chi_\beta \subseteq U_\alpha \cap V_\beta. \tag{16.45}$$

Using (16.45), we can rewrite (16.44) as

$$\sum_\alpha \int_{U_\alpha} \rho_\alpha \omega = \sum_\alpha \sum_\beta \int_{U_\alpha \cap V_\beta} \rho_\alpha \chi_\beta \omega. \tag{16.46}$$

Similarly,

$$\sum_\beta \int_{V_\beta} \chi_\beta \omega = \sum_\beta \sum_\alpha \int_{V_\beta \cap U_\alpha} \chi_\beta \rho_\alpha \omega. \tag{16.47}$$

In (16.46) and (16.47), we can actually interchange the α and β sums, since they are finite sums. Therefore, we can conclude that

$$\sum_\alpha \int_{U_\alpha} \rho_\alpha \omega = \sum_\beta \int_{V_\beta} \chi_\beta \omega. \tag{16.48}$$

Therefore, the definition of the integral of a compactly supported smooth n-form on M given by (16.42) is independent of the choices of oriented atlas and the partition of unity subordinate to that atlas.

Proposition 16.6

Let ω be an n-form with compact support on an oriented manifold M of dimension n. If $-M$ is the same manifold with the opposite orientation, then

$$\int_{-M} \omega = -\int_M \omega. \tag{16.49}$$

Proof. By the definition an integral (16.35) and (16.10), it is enough to show that for every chart $(U, \varphi) = (U, x^1, x^2, \ldots, x^n)$ in the oriented atlas $\{(U_\alpha, \varphi_\alpha)\}_\alpha$, [we're dropping subscript α for notational clarity] and differential form $\tau \in \Omega_c^n(U)$, if $(U, \widetilde{\varphi}) = (U, -x^1, x^2, \ldots, x^n)$ is the chart with the opposite orientation, then

$$\int_{\widetilde{\varphi}(U)} \left(\widetilde{\varphi}^{-1}\right)^* \tau = -\int \varphi(U) \left(\varphi^{-1}\right)^* \tau. \tag{16.50}$$

Let $r^1, \ldots, r^n$ be the standard coordinates on $\mathbb{R}^n$. Then one has

$$x^i = r^i \circ \varphi \text{ and } r^i = x^i \circ \varphi^{-1}. \tag{16.51}$$

When one is dealing with the chart $(U, \widetilde{\varphi})$, (16.51) still remains true for $i = 2, 3, \ldots, n$. Only the formula for $i = 1$, changes by a sign.

$$\begin{aligned} -x^1 &= r^1 \circ \widetilde{\varphi} \text{ and } r^1 = -x^1 \circ \widetilde{\varphi}^{-1}, \\ x^i &= r^i \circ \varphi \text{ and } r^i = x^i \circ \varphi^{-1}, \end{aligned} \tag{16.52}$$

for $i = 2, 3, \ldots, n$. Now suppose,

$$\tau = f \, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$$

on U. Then

$$\begin{aligned} \left(\widetilde{\varphi}^{-1}\right)^* \tau &= \left(\widetilde{\varphi}^{-1}\right)^* \left(f \, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n\right) \\ &= \left(f \circ \widetilde{\varphi}^{-1}\right) \mathrm{d}\left(x^1 \circ \widetilde{\varphi}^{-1}\right) \wedge \cdots \wedge \mathrm{d}\left(x^n \circ \widetilde{\varphi}^{-1}\right) \\ &= -\left(f \circ \widetilde{\varphi}^{-1}\right) \mathrm{d}r^1 \wedge \cdots \wedge \mathrm{d}r^n. \end{aligned} \tag{16.53}$$

Similarly,

$$\left(\varphi^{-1}\right)^* \tau = \left(f \circ \varphi^{-1}\right) \mathrm{d}r^1 \wedge \cdots \wedge \mathrm{d}r^n. \tag{16.54}$$

Now take the map $\varphi \circ \widetilde{\varphi}^{-1} : \widetilde{\varphi}(U) \to \varphi(U)$. Take $(a^1, \ldots, a^n) \in \widetilde{\varphi}(U)$. Let us compute $(r^i \circ \varphi \circ \widetilde{\varphi}^{-1})(a^1, \ldots, a^n)$. For $i = 1$,

$$\begin{aligned} \left(r^1 \circ \varphi \circ \widetilde{\varphi}^{-1}\right)\left(a^1, \ldots, a^n\right) &= \left(x^1 \circ \widetilde{\varphi}^{-1}\right)\left(a^1, \ldots, a^n\right) \\ &= -r^1\left(a^1, \ldots, a^n\right) = -a^1. \end{aligned} \tag{16.55}$$

For $i = 2, 3, \ldots, n$,

$$\begin{aligned} \left(r^i \circ \varphi \circ \widetilde{\varphi}^{-1}\right)\left(a^1, \ldots, a^n\right) &= \left(x^i \circ \widetilde{\varphi}^{-1}\right)\left(a^1, \ldots, a^n\right) \\ &= r^i\left(a^1, \ldots, a^n\right) = a^i. \end{aligned} \tag{16.56}$$

Therefore,

$$\left(\varphi \circ \widetilde{\varphi}^{-1}\right)\left(a^1, a^2, \ldots, a^n\right) = \left(-a^1, a^2, \ldots, a^n\right). \tag{16.57}$$

So, the Jacobian matrix of $\varphi \circ \widetilde{\varphi}^{-1}$ will be a diagonal matrix, with entries $-1, 1, \ldots, 1$. Hence,

$$\left|\det J\left(\varphi \circ \widetilde{\varphi}^{-1}\right)\right| = |-1| = 1. \tag{16.58}$$

Therefore,

$$\begin{aligned} \int_{\widetilde{\varphi}(U)} \left(\widetilde{\varphi}^{-1}\right)^* \tau &= -\int_{\widetilde{\varphi}(U)} \left(f \circ \widetilde{\varphi}^{-1}\right) \mathrm{d}r^1 \cdots \mathrm{d}r^n \\ &= -\int_{\widetilde{\varphi}(U)} \left(f \circ \varphi^{-1}\right) \circ \left(\varphi \circ \widetilde{\varphi}^{-1}\right) \left|\det J\left(\varphi \circ \widetilde{\varphi}^{-1}\right)\right| \mathrm{d}r^1 \cdots \mathrm{d}r^n \\ &= -\int_{\varphi(U)} \left(f \circ \varphi^{-1}\right) \mathrm{d}r^1 \cdots \mathrm{d}r^n \\ &= -\int \varphi(U) \left(\varphi^{-1}\right)^* \tau. \end{aligned}$$

Therefore, (16.50) holds. By the linearity of integration, this proves (16.49). ■

In practical computation, the definition of integral of an n-form over an oriented n-manifold using partition of unity is not very useful. To calculate explicitly integrals over an oriented n-manifold M, it's best to consider integrals over a parametrized set.

Definition 16.6 (Paramtrized set)**.** A **parametrized set** in an oriented n-manifold M is a subset $A \subseteq M$ together with a C^∞ map $F : D \to M$ from a compact domain of integration $D \subset \mathbb{R}^n$ to M such that $A = F(D)$ and F restricts to an orientation preserving diffeomorphism from $\operatorname{int}(D)$ to $F(\operatorname{int} D)$. Note that by smooth invariance of domain for manifolds (Remark 15.2), $F(\operatorname{int} D)$ is an open subset of M. The C^∞ map $F : D \to A$ is called a **parametrization** of A.

If $A \subseteq M$ is a parametrized set with parametrization $F : D \to A$ and ω is a C^∞ n-form on M, not necessarily with compact support, then we define $\int_A \omega$ to be $\int_D F^*\omega$.

$$\int_A \omega := \int_D F^*\omega. \tag{16.59}$$

One can show that this definition is parametrization independent. Indeed, if there is a C^∞ map $\widetilde{F} : \widetilde{D} \to M$ from a compact domain of integration $\widetilde{D} \subset \mathbb{R}^n$ to M such that $A = \widetilde{F}(\widetilde{D})$ and $\widetilde{F}$ restricts to an orientation preserving diffeomorphism from $\operatorname{int}\left(\widetilde{D}\right)$ to $\widetilde{F}\left(\operatorname{int}\widetilde{D}\right)$, them

$$\int_{\widetilde{D}} \widetilde{F}^*\omega = \int_D F^*\omega.$$

It can be seen by showing that there is a smooth map $G : \widetilde{D} \to D$ which restricts to an orientation preserving diffeomorphism from $\operatorname{int}\left(\widetilde{D}\right)$ to $\operatorname{int}(D)$, such that

$$F \circ G = \widetilde{F}. \tag{16.60}$$

Then by Theorem 13.7,

$$\widetilde{F}^* = G^* \circ F^*. \tag{16.61}$$

Then by the definition of integration over parametrized set,

$$\int_D F^*\omega = \int_{\widetilde{D}} G^*\left(F^*\omega\right) = \int_{\widetilde{D}} \widetilde{F}^*\omega. \tag{16.62}$$

It's important to note that $\int_D F^*\omega$ and $\int_{\widetilde{D}} \widetilde{F}^*\omega$ are equal and do not differ by a sign is because $G : \widetilde{D} \to D$ restricts to an orientation preserving diffecomorphism from $\operatorname{int}(\widetilde{D})$ to $G(\operatorname{int}(\widetilde{D})) = \operatorname{int} D$.

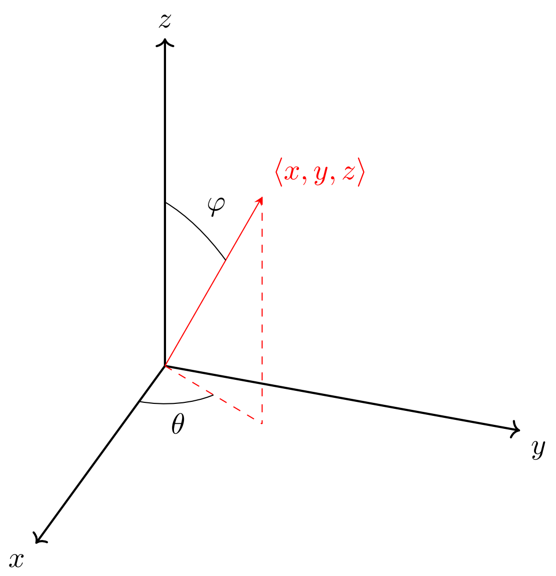
In case the parametrized set $A = F(D) \subseteq M$ is a manifold, then $\int_A \omega = \int_{D\subset\mathbb{R}^n} F^*\omega$ and the definition (16.35) $\int_U \omega = \int_{\phi(U)\subset\mathbb{R}^n} \left(\phi^{-1}\right)^* \omega$ coincide by looking at the smooth maps $F : D \subset \mathbb{R}^n \to F(D) = A$ and $\phi^{-1} : \phi(U) \subset \mathbb{R}^n \to U$. In the former case $D \subset \mathbb{R}^n$ is taken to be compact so that we don't want ω to be compactly supported in this case while in the latter case we have $\phi(U) \subset \mathbb{R}^n$ to be open or in other words U to be open so that in this case we required ω to be compactly supported inside the open subset U of M.

The theory of integration using parametrizing sets is computationally handy. We refer the interested reader to the treatment in *Analysis on Manifolds* by James Munkres (there again Munkres used parametrized open sets in contrast to the compact set $A = F(D)$ we used and hence Munkres needed the n-form ω to be compactly supported inside the parametrized open set). We try to content ourselves with an example.

Example 16.1 (Integral over a sphere)**.** Given unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ along the x-axis, y-axis and z-axis, respectively, the vector $\langle x, y, z\rangle$ from the origin $(0,0,0)$ to the point (x,y,z) is nothing but $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Then we take

$$r = \sqrt{x^2+y^2+z^2} = \|\langle x,y,z\rangle\| = \|x\mathbf{i} + y\mathbf{j} + z\mathbf{k}\|\,. \tag{16.63}$$

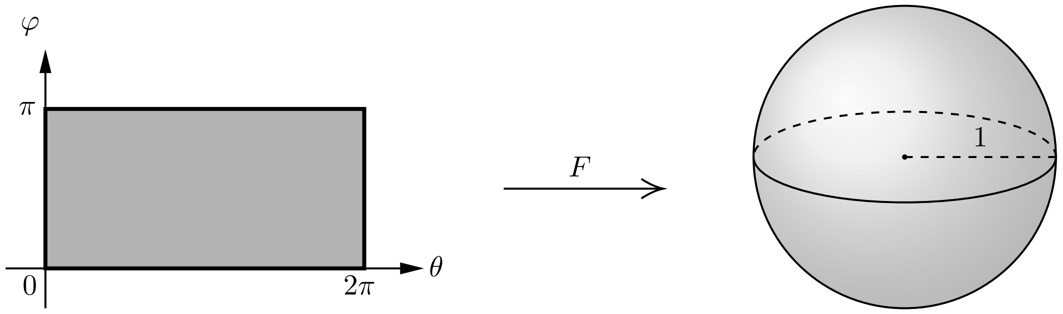
The set of all points in $\mathbb{R}^3$ obeying $\|\langle x,y,z\rangle\| = r$ for a given positive real r is the sphere of radius r centered at the origin $(0,0,0)$. Let us denote by φ the angle $\langle x,y,z\rangle$ makes with the positive z-axis, and denote by θ the angle that the vector $\langle x,y\rangle$ makes with the positive x-axis.



Let S^2 be the unit sphere in $\mathbb{R}^3$ $(r = 1)$. Let ω be the 2-form on S^2 given by

$$\omega = \begin{cases} \frac{\mathrm{d}y\wedge\mathrm{d}z}{x} & \text{if } x \neq 0, \\ \frac{\mathrm{d}z\wedge\mathrm{d}x}{y} & \text{if } y \neq 0, \\ \frac{\mathrm{d}x\wedge\mathrm{d}y}{z} & \text{if } z \neq 0. \end{cases} \tag{16.64}$$

We want to calculate $\int_{S^2} \omega$.



In this problem, the compact domain $D \subset \mathbb{R}^2$ of integration is given by

$$D = \left\{(\theta, \varphi) \in \mathbb{R}^2 \;\middle|\; 0 \leq \theta \leq 2\pi, 0 \leq \varphi \leq \pi\right\}. \tag{16.65}$$

Here, $S^2 \subset \mathbb{R}^3$ is the parametrized set which happens to be a 2-dimensional manifold, and the parametrization is the smooth map $F : D \to S^2$ given by

$$F(\theta, \varphi) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi) = (x, y, z) \in S^2 \subset \mathbb{R}^3. \tag{16.66}$$

Here x, y, z are 3 coordinate functions (0-forms) on S^2, so that the pullbacks F^*x, F^*y and F^*z under the smooth map F given by are expected to be functions in D. Indeed,

$$\begin{aligned} F^*x &= x \circ F = \sin\varphi\cos\theta, \\ F^*y &= y \circ F = \sin\varphi\sin\theta, \\ F^*z &= z \circ F = \cos\varphi. \end{aligned} \tag{16.67}$$

Thus, we have

$$F^*(\mathrm{d}x) = \mathrm{d}(F^*x) = \mathrm{d}(\sin\varphi\cos\theta) = \cos\varphi\cos\theta\,\mathrm{d}\varphi - \sin\theta\sin\varphi\,\mathrm{d}\theta. \tag{16.68}$$

$$F^*(\mathrm{d}y) = \mathrm{d}(F^*y) = \mathrm{d}(\sin\varphi\sin\theta) = \cos\varphi\sin\theta\,\mathrm{d}\varphi + \sin\varphi\cos\theta\,\mathrm{d}\theta. \tag{16.69}$$

$$F^*(\mathrm{d}z) = \mathrm{d}(F^*z) = \mathrm{d}(\cos\theta) = -\sin\theta\,\mathrm{d}\theta. \tag{16.70}$$

Now, for $x \neq 0$,

$$F^*\omega = \frac{F^*\mathrm{d}y \wedge F^*\mathrm{d}z}{F^*x} = \frac{(\cos\varphi\sin\theta\,\mathrm{d}\varphi + \sin\varphi\cos\theta\,\mathrm{d}\theta) \wedge (-\sin\theta\,\mathrm{d}\theta)}{\sin\varphi\cos\theta}$$
$$= \frac{\sin^2\theta\cos\theta\,\mathrm{d}\varphi \wedge \mathrm{d}\theta}{\sin\varphi\cos\theta} = \sin\theta\,\mathrm{d}\varphi \wedge \mathrm{d}\theta.$$

For $y \neq 0$,

$$F^*\omega = \frac{F^*\mathrm{d}z \wedge F^*\mathrm{d}x}{F^*y} = \frac{(-\sin\theta\,\mathrm{d}\theta) \wedge (\cos\varphi\cos\theta\,\mathrm{d}\varphi - \sin\theta\sin\varphi\,\mathrm{d}\theta)}{\sin\varphi\sin\theta}$$
$$= \frac{\sin^2\varphi\sin\theta\,\mathrm{d}\varphi \wedge \mathrm{d}\theta}{\sin\varphi\sin\theta} = \sin\theta\,\mathrm{d}\varphi \wedge \mathrm{d}\theta.$$

For $z \neq 0$,

$$F^*\omega = \frac{F^*\mathrm{d}x \wedge F^*\mathrm{d}y}{F^*z} = \frac{(\cos\varphi\cos\theta\,\mathrm{d}\varphi - \sin\theta\sin\varphi\,\mathrm{d}\theta) \wedge (\cos\varphi\sin\theta\,\mathrm{d}\varphi + \sin\varphi\cos\theta\,\mathrm{d}\theta)}{\cos\varphi}$$
$$= \frac{(\sin\varphi\cos\varphi\cos^2\theta + \sin\varphi\cos\varphi\sin^2\theta)\,\mathrm{d}\varphi \wedge \mathrm{d}\theta}{\cos\varphi}$$
$$= \frac{\sin\varphi\cos\varphi\,\mathrm{d}\varphi \wedge \mathrm{d}\theta}{\cos\varphi} = \sin\theta\,\mathrm{d}\varphi \wedge \mathrm{d}\theta.$$

Therefore,

$$F^*\omega = \sin\theta\,\mathrm{d}\varphi \wedge \mathrm{d}\theta, \tag{16.71}$$

everywhere. Now using the definition of integral over a parametrized set,

$$\int_{S^2} \omega = \int_D F^*\omega = \int_0^{2\pi}\int_0^{\pi} \sin\varphi\,\mathrm{d}\varphi\ \mathrm{d}\theta$$
$$= 2\pi\,[-\cos\varphi]\Big|_{\varphi=0}^{\varphi=\pi} = 4\pi, \tag{16.72}$$

which is the surface area of the unit sphere S^1.

§16.4 Stokes' Theorem

Theorem 16.7 (Stokes' Theorem)
Let M be an oriented smooth n-manifold with boundary, and let ω be a compactly supported smooth $(n-1)$-form on M. Then

$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega. \tag{16.73}$$

∂M is a smooth $(n-1)$-manifold without any boundary as we've sen in the previous lectures. There is an indued orientation on ∂M. The $(n-1)$ form ω appearing on the right side of (16.73) is to be interpreted as $\iota_{\partial M}^*\omega$, where

$$\iota_{\partial M} : \partial M \to M$$

is the canonical inclusion of the boundary ∂M into the n-manifold M. If $\partial M = \varnothing$, then the right side is to be interpreted as zero. When M is 1-dimensional, the right hand integral is just a finite sum in the following sense:

Integration over a zero-dimensional manifold

A compact oriented manifold of dimension 0 is a finite collection of points, each point oriented by +1 or −1 We write this fact as

$$M = \sum_{i=1}^{r} p_i - \sum_{j=1}^{s} q_j, \tag{16.74}$$

which means that each element of the collection $\{p_1, p_2, \ldots, p_r\}$ has orientation +1 while each element of the collection $\{q_1, q_2, \ldots, q_s\}$ has orientation −1. The object that is to be integrated over this oriented 0-dimensions manifold M is a 0-form $f : M \to \mathbb{R}$. The integral of this 0-form is defined to be the sum

$$\int_M f := \sum_{i=1}^{r} f(p_i) - \sum_{j=1}^{s} f(q_j) \tag{16.75}$$

We need the following result to prove Stokes' Theorem.

Lemma 16.8

Suppose M is a smooth manifold with or without boundary and $S \subseteq M$ is an immersed submanifold with or without boundary. Let $\iota : S \to M$ be the relevant inclusion. Then

$$\mathrm{d}\left(f\big|_S\right) = \iota^*(\mathrm{d}f).$$

Furthermore, the pullback of $\mathrm{d}f$ to S is zero if and only if f is constant on each component of S.

Proof. $f\big|_S = f \circ \iota : S \to \mathbb{R}$. Therefore,

$$\mathrm{d}\left(f\big|_S\right) = \mathrm{d}(f \circ \iota) = \mathrm{d}(\iota^* f) = \iota^*(\mathrm{d}f). \tag{16.76}$$

The pullback of $\mathrm{d}f$ to S is $\iota^*(\mathrm{d}f)$, which is equal to $\mathrm{d}\left(f\big|_S\right)$ as we have just proved. This will be zero if and only if $\mathrm{d}\left(f\big|_S\right) = 0$.

Let $g = f\big|_S$. We need to show that $\mathrm{d}g = 0$ if and only if g is constant on each component of S. Suppose g is constant on each component of S. Let $(U, x^1, \ldots, x^m)$ be a (connected) chart on S. Then on U,

$$\mathrm{d}g = \sum_{i=1}^{m} \frac{\partial g}{\partial x^i}\,\mathrm{d}x^i = 0. \tag{16.77}$$

So $\mathrm{d}g = 0$ on all of S, since U was an arbitrary connected coordinate open set.

Conversely, suppose $\mathrm{d}g = 0$. Let $(U, \varphi) \equiv (U, x^1, \ldots, x^m)$ be a (connected) chart on S. Then on U,

$$0 = \mathrm{d}g = \sum_{i=1}^{m} \frac{\partial g}{\partial x^i}\,\mathrm{d}x^i. \tag{16.78}$$

This gives us that

$$\frac{\partial g}{\partial x^i} = 0, \tag{16.79}$$

for each $i = 1, \ldots, m$. Then we get

$$\frac{\partial}{\partial r^i}\left(g \circ \varphi^{-1}\right) = 0 \tag{16.80}$$

on $\varphi(U)$. Since $\varphi(U)$ is a connected open subset of $\mathbb{R}^m$, we can conclude that $g \circ \varphi^{-1}$ is constant on $\varphi(U)$. As a result, g is constant on U.

Let V be another connected coordinate open subset of S, belonging to the same connected component of S as U, such that $U \cap V \neq \varnothing$ (if there doesn't exist such V, then U is a connected component of S). Using the same argument as above, we conclude that g is constant on V as well. Since the constants must agree on $U \cap V$, we must have g to be constant on $U \cup V$. Continuing like this, we conclude that g is constant on the connected component that contains U. Therefore, g is constant on each connected compoent of S. ∎

Proof of Stokes' Theorem. We begin with the special case when M is the upper half-space $\mathbb{H}^n$ itself, for $n > 1$. Since ω is a compactly supported $(n-1)$-form in $\mathbb{H}^n$, there is a real number $K > 0$ such that $\operatorname{supp}\omega$ is contained in the rectangle

$$R = [-K, K] \times \cdots \times [-K, K] \times [0, K]\,.$$

We can write ω using the standard coordinates of $\mathbb{H}^n$ as

$$\omega = \sum_{i=1}^{n} \omega_i \, \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^n, \tag{16.81}$$

where, as before, the caret stands for omission. Hence we have

$$\begin{aligned} \mathrm{d}\omega &= \sum_{i=1}^{n} \mathrm{d}\omega_i \wedge \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^n \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial \omega_i}{\partial x^j} \, \mathrm{d}x^j \wedge \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^n \\ &= \sum_{i=1}^{n} \frac{\partial \omega_i}{\partial x^i} \, \mathrm{d}x^i \wedge \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^n. \end{aligned} \tag{16.82}$$

Here in the j-sum, the $i \neq j$ terms will vanish since $\mathrm{d}x^k \wedge \mathrm{d}x^j = 0$ for $k \neq j$. In order to reinstate the first $\mathrm{d}x^i$ in its original position (where the caret occurs), it has to be pushed through the wedge product $(i-1)$-times and hence (16.82) reduces to

$$\mathrm{d}\omega = \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial \omega_i}{\partial x^i} \, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^i \wedge \cdots \wedge \mathrm{d}x^n. \tag{16.83}$$

Therefore,

$$\begin{aligned} \int_{\mathbb{H}^n} \mathrm{d}\omega &= \sum_{i=1}^{n} (-1)^{i-1} \int_R \frac{\partial \omega_i}{\partial x^i} \, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \\ &= \sum_{i=1}^{n} (-1)^{i-1} \int_0^K \int_{-K}^K \cdots \int_{-K}^K \frac{\partial \omega_i}{\partial x^i} \, \mathrm{d}x^1 \cdots \mathrm{d}x^n. \end{aligned} \tag{16.84}$$

The contribution of the terms with $i \neq n$ in this sum are:

$$\begin{aligned} &\int_0^K \int_{-K}^K \cdots \int_{-K}^K \frac{\partial \omega_i}{\partial x^i} \, \mathrm{d}x^1 \cdots \mathrm{d}x^n \\ &= \int_0^K \int_{-K}^K \cdots \int_{-K}^K \left[\omega_i\left(x\right)\right]_{x^i=-K}^{x^i=K} \, \mathrm{d}x^1 \cdots \widehat{\mathrm{d}x^i} \cdots \mathrm{d}x^n \\ &= 0, \end{aligned} \tag{16.85}$$

since we can choose K large enough so that $\omega = 0$ when $x^i = \pm K$. As a result,

$$\begin{aligned} \int_{\mathbb{H}^n} \mathrm{d}\omega &= (-1)^{n-1} \int_0^K \int_{-K}^K \cdots \int_{-K}^K \frac{\partial \omega_n}{\partial x^n} \, \mathrm{d}x^1 \cdots \mathrm{d}x^n \\ &= (-1)^{n-1} \int_{-K}^K \cdots \int_{-K}^K \left[\omega_n\left(x\right)\right]_{x^n=0}^{x^n=K} \, \mathrm{d}x^1 \cdots \mathrm{d}x^{n-1} \\ &= (-1)^n \int_{-K}^K \cdots \int_{-K}^K \left[\omega_n\left(x^1, \ldots, x^{n-1}, 0\right)\right]_{x^n=0}^{x^n=K} \, \mathrm{d}x^1 \cdots \mathrm{d}x^{n-1}, \end{aligned} \tag{16.86}$$

again by choosing K large enough so that $\omega = 0$ when $x^n = K$.

Let us now compute $\int_{\partial\mathbb{H}^n} \omega$.

$$\int_{\partial\mathbb{H}^n} \omega = \sum_{i=1}^{n} \int_{R \cap \partial\mathbb{H}^n} \omega_i\left(x^1, \ldots, x^{n-1}, 0\right) \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^n. \tag{16.87}$$

On $\partial\mathbb{H}^n$, $x^n = 0$, i.e. x^n is constant on $\partial\mathbb{H}^n$. Therefore, by Lemma 16.8,

$$\iota^* \left(\mathrm{d}x^n\right) = \mathrm{d}\left(x^n|_{\partial\mathbb{H}^n}\right) = 0, \tag{16.88}$$

where $\iota : \partial\mathbb{H}^n \to \mathbb{H}^n$ is the inclusion map. Hence, in (16.87), for the sum over all i, only the term corresponding to $i = n$ survives with $\mathrm{d}x^n$ being omitted in that term. Therefore,

$$\int_{\partial\mathbb{H}^n} \omega = \int_{R\cap\partial\mathbb{H}^n} \omega_n\left(x^1, \ldots, x^{n-1}, 0\right) \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^{n-1}. \tag{16.89}$$

By Example 15.4, the induced orientation on $\partial\mathbb{H}^n$ is given by the $(n-1)$-form $(-1)^n\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^{n-1}$. Using this orientation, (16.89) reads

$$\int_{\partial\mathbb{H}^n} \omega = (-1)^n \int_{-K}^{K} \cdots \int_{-K}^{K} \omega_n\left(x^1, \ldots, x^{n-1}, 0\right)\, \mathrm{d}x^1 \cdots \mathrm{d}x^{n-1}. \tag{16.90}$$

Comparing (16.86) and (16.90), we get

$$\int_{\mathbb{H}^n} \mathrm{d}\omega = \int_{\partial\mathbb{H}^n} \omega. \tag{16.91}$$

Now we consider the special case when $M = \mathbb{H}^1$ or $\mathbb{L}^1$. Let f be a smooth compactly supported 0-form on $\mathbb{H}^1$. Since f is compactly supported, there exists some $K > 0$ such that $\operatorname{supp} f \subseteq [0, K]$. Now,

$$\mathrm{d}f = \frac{\partial f}{\partial x}\,\mathrm{d}x. \tag{16.92}$$

So we have

$$\int_{\mathbb{H}^1} \mathrm{d}f = \int_{\mathbb{H}^1} \frac{\partial f}{\partial x} \wedge \mathrm{d}x = \int_0^K \frac{\partial f}{\partial x}\,\mathrm{d}x = f(K) - f(0) = -f(0), \tag{16.93}$$

since we can choose K large enough so that $f(K) = 0$. Now, the canonical orientation form $\mathrm{d}x$ on $\mathbb{H}^1$ induces an orientation on $\partial\mathbb{H}^1 = \{0\}$, which is -1 (by Example 15.4). Therefore, by the definition of integral of 0-forms on 0-dimensional manifold,

$$\int_{\partial\mathbb{H}^1} f = -f(0). \tag{16.94}$$

Comparing (16.93) and (16.94), we get

$$\int_{\mathbb{H}^1} \mathrm{d}f = \int_{\partial\mathbb{H}^1} f. \tag{16.95}$$

Now consider $M = \mathbb{L}^1$. The canonical orientation form on $\mathbb{L}^1$ is $\mathrm{d}x$. A smooth outward pointing vector field on $\mathbb{L}^1$ is $\frac{\partial}{\partial x}$. Therefore, the canoncial boundary orientation on $\partial\mathbb{L}^1 = \{0\}$ is given by the contraction

$$\iota_{\frac{\partial}{\partial x}}\left(\mathrm{d}x\right), \tag{16.96}$$

by Proposition 15.8. Since $\iota_{\mathbf{v}}\beta = \beta(\mathbf{v}) \in \mathbb{R}$, we have

$$\iota_{\frac{\partial}{\partial x}}\left(\mathrm{d}x\right) = \mathrm{d}x\left(\frac{\partial}{\partial x}\right) = 1. \tag{16.97}$$

Since f is compactly supported, there exists some $K > 0$ such that $\operatorname{supp} f \subseteq [-K, 0]$. Now,

$$\mathrm{d}f = \frac{\partial f}{\partial x}\,\mathrm{d}x. \tag{16.98}$$

So we have

$$\int_{\mathbb{L}^1} \mathrm{d}f = \int_{\mathbb{L}^1} \frac{\partial f}{\partial x} \wedge \mathrm{d}x = \int_{-K}^{0} \frac{\partial f}{\partial x}\,\mathrm{d}x = f(0) - f(-K) = f(0), \tag{16.99}$$

since we can choose K large enough so that $f(-K) = 0$. Since the orientation on $\partial\mathbb{L}^1 = \{0\}$ is $+1$, by the definition of integral of 0-forms on 0-dimensional manifold,

$$\int_{\partial\mathbb{L}^1} f = f(0). \tag{16.100}$$

Comparing (16.99) and (16.100), we get

$$\int_{\mathbb{L}^1} \mathrm{d}f = \int_{\partial\mathbb{L}^1} f. \tag{16.101}$$

Next we consider the special case $M = \mathbb{R}^n$. Since ω is a compactly supported $(n-1)$-form on $\mathbb{R}^n$, there exists some $K > 0$ such that $\operatorname{supp}\omega \subseteq R = [-K, K]^n$. Then we write

$$\omega = \sum_{i=1}^{n} \omega_i \,\mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^n. \tag{16.102}$$

Then

$$\begin{aligned} \mathrm{d}\omega &= \sum_{i=1}^{n} \frac{\partial \omega_i}{\partial x^i} \,\mathrm{d}x^i \wedge \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^n \\ &= \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial \omega_i}{\partial x^i} \,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^i \wedge \cdots \wedge \mathrm{d}x^n. \end{aligned} \tag{16.103}$$

Therefore,

$$\begin{aligned} \int_{\mathbb{R}^n} \mathrm{d}\omega &= \sum_{i=1}^{n} (-1)^{i-1} \int_R \frac{\partial \omega_i}{\partial x^i} \,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \\ &= \sum_{i=1}^{n} (-1)^{i-1} \int_{-K}^{K} \cdots \int_{-K}^{K} \frac{\partial \omega_i}{\partial x^i} \,\mathrm{d}x^1 \cdots \mathrm{d}x^n. \end{aligned} \tag{16.104}$$

Let us now compute the integrals:

$$\begin{aligned} &\int_{-K}^{K} \cdots \int_{-K}^{K} \frac{\partial \omega_i}{\partial x^i} \,\mathrm{d}x^1 \cdots \mathrm{d}x^n \\ &= \int_{-K}^{K} \cdots \int_{-K}^{K} \left[\omega_i(x)\right]_{x^i=-K}^{x^i=K} \,\mathrm{d}x^1 \cdots \widehat{\mathrm{d}x^i} \cdots \mathrm{d}x^n \\ &= 0, \end{aligned} \tag{16.105}$$

since we can choose K large enough so that $\omega = 0$ when $x^i = \pm K$. Therefore,

$$\int_{\mathbb{R}^n} \mathrm{d}\omega = 0. \tag{16.106}$$

Since $\mathbb{R}^n$ has empty boundary, i.e. $\partial\mathbb{R}^n = \varnothing$,

$$\int_{\partial\mathbb{R}^n} \omega = \int_{\varnothing} \omega = 0. \tag{16.107}$$

Hence,

$$\int_{\mathbb{R}^n} \mathrm{d}\omega = \int_{\partial\mathbb{R}^n} \omega. \tag{16.108}$$

So we have proved Stokes' Theorem for the special cases $M = \mathbb{H}^n, \mathbb{L}^1, \mathbb{R}^n$. Now let M be an arbitrary smooth manifold with boundary ∂M. Choose an atlas $\{(U_\alpha, \varphi_\alpha)\}_\alpha$ for M in which each U_α is diffeomorphic to either $\mathbb{R}^n$ or $\mathbb{H}^n$ (or $\mathbb{L}^1$ in dimension 1) via an orientation preserving diffeomorphism. This is possible since any open disk is diffeomorphic to $\mathbb{R}^n$ and any half-disk containing its boundary diameter is diffeomorphic to $\mathbb{H}^n$ (or $\mathbb{L}^1$ in dimension 1). For instance, the open ball

$$B = \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\| < 1\} \tag{16.109}$$

is diffeomorphic to $\mathbb{R}^n$ via the map $F : B \to \mathbb{R}^n$ defined as

$$F(\mathbf{x}) = \frac{1}{\sqrt{1 - \|\mathbf{x}\|^2}} \mathbf{x}. \tag{16.110}$$

If this map is not orientation preserving, we just take the map $\widetilde{F} = (-F^1, F^2, \ldots, F^n)$. Then the first row of $\left[\frac{\partial \widetilde{F}^i}{\partial x^j}\right]_{i,j=1}^n$ is the negative of the first row of $\left[\frac{\partial F^i}{\partial x^j}\right]_{i,j=1}^n$, and all the other rows stay the same. Therefore,

$$\det\left[\frac{\partial \widetilde{F}^i}{\partial x^j}\right]_{i,j=1}^n = -\det\left[\frac{\partial F^i}{\partial x^j}\right]_{i,j=1}^n. \tag{16.111}$$

So $\widetilde{F}$ is orientation preserving. In the same way, the half-disk containing its boundary diameter $B \cap \mathbb{H}^n$ is diffeomorphic to $\mathbb{H}^n$ (or $\mathbb{L}^1$) via an orientation preserving diffeomorphism.

Let $\{\rho_\alpha\}_\alpha$ be a C^∞ partition of unity subordinate to $\{U_\alpha\}$. From (16.41), $\operatorname{supp} \rho_\alpha \omega \subseteq U_\alpha$. Furthermore, by (16.39), $\operatorname{supp}(\rho_\alpha \omega) \subseteq \operatorname{supp} \omega$, i.e. $\operatorname{supp}(\rho_\alpha \omega)$ is a closed subset of a compact set $\operatorname{supp} \omega$. Hence, $\operatorname{supp}(\rho_\alpha \omega)$ is compact. In other words, $\rho_\alpha \omega$ has compact support in U_α. Furthermore, $\omega = \sum_\alpha \rho_\alpha \omega$ is, in fact, a finite sum, as proven earlier.

Since Stokes' Theorem holds for $\mathbb{R}^n$ and $\mathbb{H}^n$ (and $\mathbb{L}^1$), it holds for all the charts U_α in our atlas, with each U_α being diffeomorphic to $\mathbb{R}^n$ or $\mathbb{H}^n$ (or $\mathbb{L}^1$) and preserving orientation. Furthermore, $\partial M \cap U_\alpha = \partial U_\alpha$. Therefore,

$$\begin{aligned}
\int_{\partial M} \omega &= \int_{\partial M} \sum_\alpha \rho_\alpha \omega \\
&= \sum_\alpha \int_{\partial M} \rho_\alpha \omega && \left[\textstyle\sum_\alpha \rho_\alpha \omega \text{ is a finite sum}\right] \\
&= \sum_\alpha \int_{\partial U_\alpha} \rho_\alpha \omega && [\operatorname{supp}(\rho_\alpha \omega) \subseteq U_\alpha] \\
&= \sum_\alpha \int_{U_\alpha} \mathrm{d}(\rho_\alpha \omega) && [\text{Stokes' Theorem for } U_\alpha] \\
&= \sum_\alpha \int_M \mathrm{d}(\rho_\alpha \omega) && [\operatorname{supp} \mathrm{d}(\rho_\alpha \omega) \subseteq \operatorname{supp}(\rho_\alpha \omega) \subseteq U_\alpha] \\
&= \int_M \mathrm{d}\left(\sum_\alpha \rho_\alpha \omega\right) && \left[\textstyle\sum_\alpha \rho_\alpha \omega \text{ is a finite sum}\right] \\
&= \int_M \mathrm{d}\omega.
\end{aligned}$$

Therefore, for any manifold M,

$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega. \tag{16.112}$$

■

17 de Rham Cohomology

§17.1 Definitions

A differential form ω on a manifold M is said to be **closed** if $\mathrm{d}\omega = 0$, and **exact** if $\omega = \mathrm{d}\tau$ for some form τ of one degree less. Let us denote by $Z^k(M)$ the vector space of all closed k-forms on M, and by $B^k(M)$ the vector space of all exact k-forms on M. Since the exterior derivative operator d satisfies $\mathrm{d}^2 = 0$, $B^K(M)$ is a subspace of $Z^k(M)$. The quotient vector space

$$H^k(M) = \frac{Z^k(M)}{B^k(M)}$$

measures the extent to which closed k-forms fail to be exact, and is called the **de Rham cohomology** of M in degree k.

The quotient vector space construction introduces an equivalence relation on $Z^k(M)$: two closed k-forms on M are said to be equivalent if they differ by an exact k-form. In other words, $\omega' \sim \omega$ in $Z^k(M)$ if and only if $\omega' - \omega \in B^k(M)$, i.e.

$$\omega' - \omega = \mathrm{d}\tau, \tag{17.1}$$

for some $\tau \in \Omega^{k-1}(M)$. The equivalence class of a closed form ω is called its **cohomology class**, and is denoted by $[\omega]$. When (17.1) holds, we say that ω and ω' are **cohomologous**.

Proposition 17.1

If the manifold M has r connected components, then its de Rham cohomology in degree 0 is $H^0(M) = \mathbb{R}^r$. An element of $H^0(M)$ is specified by an ordered r tuple of real numbers, each real number representing a constant function on a connected component of M.

Proof. Since there are no nonzero exact 0-forms (smooth functions),

$$H^0(M) = Z^0(M) = \{\text{closed 0-forms }\}. \tag{17.2}$$

suppose f is a closed 0-form on M, i.e. $f \in C^\infty(M)$ such that $\mathrm{d}f = 0$. On any chart $(U, x^1, \ldots, x^4)$,

$$\mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}\,\mathrm{d}x^i. \tag{17.3}$$

Thus $\mathrm{d}f = 0$ if and only if all the partial derivatives $\frac{\partial f}{\partial x^i}$ vanish identically on U. This implies that f is locally constant on U. It means that closed 0-forms on M are the locally constant functions on M. Such a function has to be constant on each connected component of M. If M has r connected components, then such a locally constant function is represented by an ordered r tuple of real numbers. In other words, $Z^0(M) = \mathbb{R}^r$. ■

Proposition 17.2

On a manifold M of dimension n, the de Rham cohomology $H^k(M)$ vanishes for $k > n$.

Proof. Given $p \in M$, the tangent space T_pM is a vector space of dimension n. If ω is a k-form on M, then $\omega_p \in A_k(T_pM)$, a k-covector or an alternating k-linear function on the vector space T_pM. By Corollary C.17, if $k > n$, then $A_k(T_pM) = 0$. Hence, for $k > n$, the only k-form is just the zero form. ■

Example 17.1 (de Rham cohomology of the real line $\mathbb{R}$)**.** Since the real line $\mathbb{R}$ is connected, by Proposition 17.1, $H^0(\mathbb{R}) = \mathbb{R}$. By Proposition 17.2, $H^k(\mathbb{R}) = 0$ for $k \geq 2$. It remains to find $H^1(\mathbb{R})$.

There are no nonzero 2-forms on $\mathbb{R}$. Any 1-form on $\mathbb{R}$ can be expressed as $\omega = f\,\mathrm{d}x$, where $f \in C^\infty(\mathbb{R})$. One then has

$$\mathrm{d}\omega = \frac{\mathrm{d}f}{\mathrm{d}x}\,\mathrm{d}x \wedge \mathrm{d}x = 0. \tag{17.4}$$

In other words, every 1-form on $\mathbb{R}$ is closed. Now, a 1-form $\omega = f\,\mathrm{d}x$ on $\mathbb{R}$ is exact if and only if there is a C^∞ function g on $\mathbb{R}$ such that

$$\omega = f\,\mathrm{d}x = \mathrm{d}g = g'\,\mathrm{d}x, \tag{17.5}$$

where g' is the calculus derivative of g with respect to x. Such a function g is simply an anti derivative of f. By fundamental theorem of calculus, one indeed finds from the following

$$g(x) = \int_0^x f(t)\,\mathrm{d}t, \tag{17.6}$$

that $g'(x) = f(x)$. Hence, every 1-form $\omega = f\,\mathrm{d}x$ on $\mathbb{R}$ is exact. Hence, $Z^1(\mathbb{R}) = B'(\mathbb{R})$, resulting in

$$H^1(\mathbb{R}) = \frac{Z'(\mathbb{R})}{B^1(\mathbb{R})} = 0. \tag{17.7}$$

One, therefore, has

$$H^k(\mathbb{R}) = \begin{cases} \mathbb{R} & \text{for } k = 0, \\ 0 & \text{for } k \geq 1. \end{cases} \tag{17.8}$$

Example 17.2 (de Rham cohomology of a circle)**.** Let S^1 be the circle in the x-y place. Since S^1 is connected, by Proposition 17.1, $H^0(S^1) = \mathbb{R}$. Since S^1 is a one-dimensional manifold, by Proposition 17.2, $H^k(S^1) = 0$ for $k \geq 2$. Now we need to compute $H^1(S^1)$.

Let us first consider the map $h : \mathbb{R} \to S^1$ defined by

$$h(t) = (\cos t, \sin t). \tag{17.9}$$

Let $i : [0, 2\pi] \to \mathbb{R}$ be the inclusion map. Restriction of the domain of $h : \mathbb{R} \to S^1$ is obtained by the following composition $F = h \circ i : [0, 2\pi] \to S^1$, which is a parametrization of the circle. In Example 13.2, we found a nowhere vanishing 1-form $\omega = -y\,\mathrm{d}x + x\,\mathrm{d}y$ on S^1. Now,

$$\begin{aligned} h^*(-y\,\mathrm{d}x + x\,\mathrm{d}y) &= -(h^*y)\,\mathrm{d}(h^*x) + (h^*x)\,\mathrm{d}(h^*y) \\ &= -\sin t\,\mathrm{d}(\cos t) + \cos t\,\mathrm{d}(\sin t) \\ &= \sin^2 t\,\mathrm{d}t + \cos^2 t\,\mathrm{d}t = \mathrm{d}t. \end{aligned} \tag{17.10}$$

So $h^*\omega = \mathrm{d}t$. Now,

$$F^*\omega = (h \circ i)^*\omega = i^*h^*\omega = i^*(\mathrm{d}t) = \mathrm{d}t. \tag{17.11}$$

$F : [0, 2\pi] \to S^1$ is a parametrization of $S^1 = F([0, 2\pi])$, with $[0, 2\pi]$ being the compact domain of integration in $\mathbb{R}$. Therefore,

$$\int_{S^1} \omega = \int_{[0,2\pi]} F^*\omega = \int_0^{2\pi} \mathrm{d}t = 2\pi. \tag{17.12}$$

Note that S^1 is a 1-dimensional smooth manifold. If there were any non-closed 1-form on S^1, its exterior derivative would be nonzero, resulting in a nontrivial 2-form on S^1. But there is no nontrivial 2-form on S^1 as it is a 1-dimension manifold. Hence, $\Omega^1(S^1) = Z^1(S^1)$. Now, consider the following linear map

$$\begin{aligned} \varphi : \Omega^1\left(S^1\right) = Z^1\left(S^1\right) &\to \mathbb{R}, \\ \alpha &\mapsto \int_{S^1} \alpha. \end{aligned}$$

By (17.12), $\varphi(\omega) = 2\pi \neq 0$. Choose any nonzero $r \in \mathbb{R}$ and take the one form $\frac{r}{2\pi}\omega \in \Omega^1\left(S^1\right)$. With these choices made, one immediately finds that

$$\varphi\left(\frac{r}{2\pi}\omega\right) = r, \tag{17.13}$$

by linearity of φ. In other words, for every $r \in \mathbb{R}$, there exists an element in $\Omega^1\left(S^1\right)$, namely $\frac{r}{2\pi}\omega$, such that $\varphi\left(\frac{r}{2\pi}\omega\right) = r$. Hence, $\varphi : Z^1\left(S^1\right) \to \mathbb{R}$ is surjective.

Now, suppose $\beta \in B^1\left(S^1\right)$. Hence, there exists $f \in C^\infty\left(S^1\right)$ such that $\beta = \mathrm{d}f$. Then

$$\int_{S^1} \beta = \int_{S^1} \mathrm{d}f = \int_{\partial S^1} f = \int_{\varnothing} f = 0. \tag{17.14}$$

So $\varphi\left(\beta\right) = 0$, and as a result, $\beta \in \operatorname{Ker}\varphi$. Hence, $B^1\left(S^1\right) \subseteq \operatorname{Ker}\varphi$. Let us now prove that $\operatorname{Ker}\varphi \subseteq B^1\left(S^1\right)$.

Since $\omega = -y\,\mathrm{d}x + x\,\mathrm{d}y$ is a nowhere vanishing smooth 1-form on S^1, any 1-form $\alpha \in \Omega^1\left(S^1\right)$ can be written as $\alpha = f\omega$, with $f \in C^\infty\left(S^1\right)$. Now suppose $\alpha = f\omega \in \operatorname{Ker}\varphi$. Also, let $\overline{f} = h^* f = f \circ h \in C^\infty\left(\mathbb{R}\right)$. From (17.9) one easily finds that $h\left(t + 2\pi\right) = h\left(t\right)$, i.e. h is periodic with period 2π. Hence,

$$\overline{f}\left(t + 2\pi\right) = f\left(h\left(t + 2\pi\right)\right) = f\left(h\left(t\right)\right) = \overline{f}\left(t\right). \tag{17.15}$$

Therefore, $\overline{f}$ is also periodic of period 2π. Then

$$\begin{aligned} 0 = \varphi\left(\alpha\right) &= \int_{S^1} \alpha \\ &= \int_{[0,2\pi]} F^*\alpha = \int_{[0,2\pi]} F^*\left(f\omega\right) \\ &= \int_{[0,2\pi} F^*\left(f\right) F^*\left(\omega\right) \\ &= \int_0^{2\pi} \left(f \circ h \circ i\right)\,\mathrm{d}t \\ &= \int_0^{2\pi} \left(\overline{f} \circ i\right)\,\mathrm{d}t \\ &= \int_0^{2\pi} \overline{f}\left(t\right)\,\mathrm{d}t. \end{aligned} \tag{17.16}$$

Lemma 17.3

Suppose $\overline{f} \in C^\infty\left(\mathbb{R}\right)$ is a periodic function of period 2π and $\int_0^{2\pi} \overline{f}\left(u\right)\,\mathrm{d}u = 0$. Then $\overline{f}\,\mathrm{d}t = \mathrm{d}g$ for a C^∞ periodic function $\overline{g}$ of period 2π on $\mathbb{R}$.

Proof. Define $\overline{g} \in \Omega^0(\mathbb{R})$ by

$$\overline{g}(t) = \int_0^t \overline{f}(u)\,\mathrm{d}u. \tag{17.17}$$

By the fundamental theorem of calculus, $\overline{g}' = \overline{f}$. Since by hypothesis $\int_0^{2\pi} \overline{f}\left(u\right)\,\mathrm{d}u = 0$, and $\overline{f}$ is 2π periodic, one has

$$\overline{g}(t + 2\pi) = \int_0^{t+2\pi} \overline{f}(u)\,\mathrm{d}u = \int_0^{2\pi} \overline{f}(u)\,\mathrm{d}u + \int_{2\pi}^{t+2\pi} \overline{f}(u)\,\mathrm{d}u \tag{17.18}$$

$$= \int_{2\pi}^{t+2\pi} \overline{f}(u)\,\mathrm{d}u = \int_0^t \overline{f}(u)\,\mathrm{d}u = \overline{g}(t), \tag{17.19}$$

proving that $\overline{g}$ is indeed periodic of period 2π on $\mathbb{R}$. Moreover,

$$\mathrm{d}\overline{g} = \overline{g}'\,\mathrm{d}t = \overline{f}\,\mathrm{d}t. \tag{17.20}$$

■

Proposition 17.4

For $k=0,1$, under the pullback map $h^*:\Omega^*\left(S^1\right)\to\Omega^*(\mathbb{R})$, smooth k-forms on S^1 are identified with smooth periodic k-forms of period 2π on $\mathbb{R}$.

Proof. If $f\in\Omega^0\left(S^1\right)$, then since $h:\mathbb{R}\to S^1$ is periodic of period 2π, the pullback $h^*f=f\circ h\in\Omega^0(\mathbb{R})$ is periodic of period 2π.

Conversely, suppose $\bar{f}\in\Omega^0(\mathbb{R})$ is periodic of period 2π. Let $p=h\left(t_0\right)\in S^1$. Then let (V,ψ) be a chart around p, with V being a small open circular arc, and ψ takes $(\cos x,\sin x)$ to x. Then with respect to the canonical basis on $T_{t_0}\mathbb{R}$ and T_pS^1, the matrix representation of $h_{*,t_0}:T_{t_0}\to T_pS^1$ is given by (using *Proposition 6.2.5* of DG1)

$$\left[\frac{\partial\left(x\circ h\right)}{\partial x}\left(t_0\right)\right]=[1]\,. \tag{17.21}$$

Therefore, $h_{*,t_0}:T_{t_0}\to T_pS^1$ is an isomorphism of vector spaces. As a result, by the *Inverse Function Theorem*, h is a local diffeomorphism. In other words, for every $p\in S^1$, there is a neighborhood U of p where $s:U\to(a,b)\subseteq\mathbb{R}$ is the smooth inverse of $h\big|_{(a,b)}$.

Then we define $f=\bar{f}\circ s$ on U. To show that f is well-defined, let s_1 and s_2 be two inverses of h over U. By the periodic properties of sine and cosine, $s_1=s_2+2\pi n$ for some $n\in\mathbb{Z}$. Because $\bar{f}$ is periodic of period 2π, we have $\bar{f}\circ s_1=\bar{f}\circ s_2$. This proves that f is well-defined on U. Moreover,

$$\bar{f}=f\circ s^{-1}=f\circ h=h^*f\qquad\text{on } h^{-1}(U). \tag{17.22}$$

As p varies over S^1, we obtain a well-defined C^∞ function f on S^1 such that $\bar{f}=h^*f$. Thus, the image of $h^*:\Omega^0\left(S^1\right)\to\Omega^0(\mathbb{R})$ consists precisely of the C^∞ periodic functions of period 2π on $\mathbb{R}$.

As for 1-forms, note that $\Omega^1\left(S^1\right)=\Omega^0\left(S^1\right)\omega$, where $\omega=-y\,\mathrm{d}x+x\,\mathrm{d}y$, and $\Omega^1(\mathbb{R})=\Omega^0(\mathbb{R})\mathrm{d}t$. The pullback $h^*:\Omega^1\left(S^1\right)\to\Omega^1(\mathbb{R})$ is given by $h^*(f\omega)=(h^*f)\,\mathrm{d}t$, so the image of $h^*:\Omega^1\left(S^1\right)\to\Omega^1(\mathbb{R})$ consists of C^∞ periodic 1-forms of period 2π. ■

Now let $\bar{g}$ the periodic function of period 2π on $\mathbb{R}$ as in Lemma 17.3. One then has $\bar{g}\in\operatorname{im}h^*$ for $k=0$. Hence, there is a C^∞ function g on S^1 such that $h^*g=\bar{g}$. Then

$$\mathrm{d}\bar{g}=\mathrm{d}\left(h^*g\right)=h^*\left(\mathrm{d}g\right). \tag{17.23}$$

On the other hand,

$$\bar{f}\,\mathrm{d}t=h^*\left(f\right)\,h^*\left(\omega\right)=h^*\left(f\omega\right)=h^*\alpha. \tag{17.24}$$

Then from (17.20), (17.23), (17.24), one has

$$h^*\left(\mathrm{d}g\right)=h^*\alpha. \tag{17.25}$$

Now we claim that $h^*:\Omega^1\left(S^1\right)\to\Omega^1\left(\mathbb{R}\right)$ is injective. Let $\tau=j\omega\in\operatorname{Ker}h^*$, where $j\in C^\infty\left(S^1\right)$. Then $h^*\tau=h^*j\,\mathrm{d}t$. Since $\tau\in\operatorname{Ker}h^*$, we must have $h^*j=0$. In other words, for any $t\in\mathbb{R}$,

$$0=\left(h^*j\right)\left(t\right)=j\left(h\left(t\right)\right). \tag{17.26}$$

Since $h:\mathbb{R}\to S^1$ is surjective, $j=0$ on S^1. Hence, $\tau=j\omega=0$, proving the injectivity of $h^*:\Omega^1\left(S^1\right)\to\Omega^1\left(\mathbb{R}\right)$. Now, using (17.25) and the injectivity of h^*, we get

$$\alpha=\mathrm{d}g. \tag{17.27}$$

As a result, we get $\alpha\in B^1\left(S^1\right)$. Therefore, $\operatorname{Ker}\varphi\subseteq B^1\left(S^1\right)$. Earlier we proved $B^1\left(S^1\right)\subseteq\operatorname{Ker}\varphi$. So we have

$$B^1\left(S^1\right)=\operatorname{Ker}\varphi. \tag{17.28}$$

Now, $\varphi:Z^1\left(S^1\right)\to\mathbb{R}$ is a surjective linear map with kernel $B^1\left(S^1\right)$. Therefore, by the first isomorphism theorem,

$$\frac{Z^1\left(S^1\right)}{B^1\left(S^1\right)}=\frac{Z^1\left(S^1\right)}{\operatorname{Ker}\varphi}\cong\mathbb{R}. \tag{17.29}$$

So we have

$$H^k(S^1)\cong\begin{cases}\mathbb{R} & \text{for } k=0,1\\ 0 & \text{for } k\geq 2.\end{cases} \tag{17.30}$$

§17.2 Diffeomorphism Invariance

Lemma 17.5

The pullback map $F^* : \Omega^*(M) \to \Omega^*(N)$ of the smooth map $F : N \to M$ sends closed forms to closed forms, and sends exact forms to exact forms.

Proof. Suppose $\omega \in \Omega^*(M)$ is closed. By the commutativity of F^* with d,

$$\mathrm{d}F^*\omega = F^*(\mathrm{d}\omega) = 0. \tag{17.31}$$

Hence, $F^*\omega$ is also closed. Next suppose $\omega = \mathrm{d}\tau$ is exact. Then

$$F^*\omega = F^*(\mathrm{d}\tau) = \mathrm{d}F^*\tau \tag{17.32}$$

Hence, $F^*\omega$ is exact. ■

If we restrict ourselves to k-forms only, then F^* sends $Z^k(M)$ to $Z^k(N)$, and $B^k(M)$ to $B^k(N)$, and thus induces a linear map of quotient spaces, denoted by $\left(F^\#\right)^k$:

$$\left(F^\#\right)^k : \frac{Z^k(M)}{B^k(M)} \to \frac{Z^k(N)}{B^k(N)}, \quad \left(F^\#\right)^k([\omega]) = [F^*\omega].$$

This map $\left(F^\#\right)^k : H^k(M) \to H^k(N)$ is a map in cohomology, called the pullback map in cohomology.

From Remark 13.2, we know that the pullback map $F^* : \Omega^*(M) \to \Omega^*(N)$ corresponding to the smooth map $F : N \to M$ is associated with a contravariant functor from the category **Man** of manifolds and smooth maps to the category **GrAlg** of graded algebras and graded algebra homomorphisms. The functorial properties (Theorem 13.7) of F^* on differential forms induce the same functorial properties for the induced map $\left(F^\#\right)^k$ in cohomology.

Theorem 17.6

The following holds:

(a) If $\mathbb{1}_M : M \to M$ is the identity map, then $\left((\mathbb{1}_M)^\#\right)^k : H^k(M) \to H^k(M)$ is also the identity map on the vector space $H^k(M)$.

(b) If $F : N \to M$ and $G : M \to P$ are smooth snaps of manifolds, then

$$\left((G \circ F)^\#\right)^k = \left(F^\#\right)^k \circ \left(G^\#\right)^k$$

Proof. We are going to use Theorem 13.7. Given a smooth map $F : N \to M$, the linear map $\left(F^\#\right)^k : H^k(M) \to H^k(N)$ is defined as

$$\left(F^\#\right)^k [\omega] = [F^*\omega]. \tag{17.33}$$

(a) Let $[\omega] \in H^k(M)$. Then

$$\left((\mathbb{1}_M)^\#\right)^k [\omega] = [(\mathbb{1}_M)^* \omega] = [\omega]. \tag{17.34}$$

Therefore,

$$\left((\mathbb{1}_M)^\#\right)^k = \mathbb{1}_{H^k(M)}. \tag{17.35}$$

(b) Suppose $[\omega] \in H^k(P)$. Then

$$\left((G \circ F)^{\#}\right)^k [\omega] = [(G \circ F)^* \omega] = [(F^* \circ G^*) \omega] = [F^* G^* \omega] . \tag{17.36}$$

On the other hand,

$$\left(F^{\#}\right)^k \circ \left(G^{\#}\right)^k [\omega] = \left(F^{\#}\right)^k [G^* \omega] = [F^* G^* \omega] . \tag{17.37}$$

Therefore,

$$\left((G \circ F)^{\#}\right)^k = \left(F^{\#}\right)^k \circ \left(G^{\#}\right)^k . \tag{17.38}$$

■

From Theorem 17.6, it immediately follows that $\left(H^k(-), \left(F^{\#}\right)^k\right)$ is a contravariant functor from the category **Man** of smooth manifolds and smooth maps to the category $\mathbf{Vect}_{\mathbb{R}}$ of real vector spaces and linear maps.

Now, if F is an isomorphism in the former category, i.e. a diffeomorphism in the category **Man**, then $\left(F^{\#}\right)^k$ will be an isomorphism in the latter category. In other words, $\left(F^{\#}\right)^k : H^k(M) \to H^k(N)$ will be an isomorphism of vector spaces.

§17.3 The Ring Structure on de Rham Cohomology

The wedge product $\wedge$ of differential forms endows the vector space $\Omega^*(M)$ with a product structure in cohomology: if $[\omega] \in H^k(M)$ and $[\tau] \in H^l(M)$, then we definee

$$[\omega] \wedge [\tau] = [\omega \wedge \tau] . \tag{17.39}$$

For the product in (17.39) to be well-defined, we need to check the following:

(i) The wedge product $\omega \wedge \tau$ is a closed form.

(ii) The class $[\omega \wedge \tau]$ is independent of the choice of the representative for $[\omega]$ or $[\tau]$. In other words, we need to show that $(\omega + \mathrm{d}\alpha) \wedge (\tau + \mathrm{d}\beta)$ is cohomologous to $\omega \wedge \tau$. This would prove that

$$[\omega + \mathrm{d}\alpha] \wedge [\tau + \mathrm{d}\beta] = [(\omega + \mathrm{d}\alpha) \wedge (\tau + \mathrm{d}\beta)] = [\omega \wedge \tau] = [\omega] \wedge [\tau] .$$

We have

$$(\omega + \mathrm{d}\alpha) \wedge (\tau + \mathrm{d}\beta) = \omega \wedge \tau + \omega \wedge \mathrm{d}\beta + \mathrm{d}\alpha \wedge \tau + \mathrm{d}\alpha \wedge d\beta. \tag{17.40}$$

Using the antiderivation property of d, we have

$$\mathrm{d}(\omega \wedge \beta) = \mathrm{d}\omega \wedge \beta + (-1)^k \omega \wedge \mathrm{d}\beta = (-1)^k \omega \wedge \mathrm{d}\beta, \tag{17.41}$$

since ω is closed. Therefore, $\omega \wedge \mathrm{d}\beta$ is an exact form. In a similar manner,

$$\mathrm{d}(\alpha \wedge (\tau + \mathrm{d}\beta)) = \mathrm{d}\alpha \wedge (\tau + d\beta) + (-1)^{k-1} \alpha \wedge \mathrm{d}(\tau + \mathrm{d}\beta) = \mathrm{d}\alpha \wedge (\tau + \mathrm{d}\beta). \tag{17.42}$$

Hence, $\alpha \wedge (\tau + \mathrm{d}\beta)$ is also exact. Therefore, $(\omega + \mathrm{d}\alpha) \wedge (\tau + \mathrm{d}\beta)$ is cohomologous to $\omega \wedge \tau$. Hence, $\wedge$ is well-defined.

Now, if M is a manifold of dimension n, we set

$$H^*(M) = \bigoplus_{k=0}^{n} H^k(M) . \tag{17.43}$$

(17.43) means that an element of $H^*(M)$ is uniquely a finite formal sum of cohomology classes in $H^k(M)$ as k varies:

$$\alpha = \alpha_0 + \cdots + \alpha_n, \tag{17.44}$$

with $\alpha_k \in H^k(M)$. Now, one can easily verify that with the formal addition and wedge product, $H^*(M)$ satisfies all the properties of a ring. We call this ring the **cohomology ring** of M.

A ring $(A, +, \times)$ is **graded** if it can be written as a direct sum $A = \bigoplus_{k=0}^{\infty} A^k$ so that the ring multiplication $\times$ sends $A^k \times A^l$ to A^{k+l}. A graded ring $A = \bigoplus_{k=0}^{\infty} A^k$ is said to be **anticommutative** if for all $a \in A^k$ and $b \in A^l$,

$$a \times b = (-1)^{kl}\, b \times a. \tag{17.45}$$

Since the wedge product of differential forms is defined pointwise, i.e. for $\omega, \tau \in \Omega^*(M)$,

$$(\omega \wedge \tau)_p = \omega_p \wedge \tau_p,$$

we have $\omega \wedge \tau = (-1)^{kl}\, \tau \wedge \omega$ whenever $\omega \in \Omega^k(M)$ and $\tau \in \Omega^l(M)$. This way, $H^*(M)$ also becomes an anticommutative graded ring. Indeed, for $[\omega] \in H^k(M)$ and $[\tau] \in H^l(M)$,

$$\begin{aligned} [\omega] \wedge [\tau] &= [\omega \wedge \tau] = \left[(-1)^{kl}\, \tau \wedge \omega\right] \\ &= (-1)^{kl}\, [\tau \wedge \omega] = (-1)^{kl}\, [\tau] \wedge [\omega]. \end{aligned} \tag{17.46}$$

This way $H^*(M)$ becomes an anticommutative graded ring. Since $H^*(M)$ is also a real vector space, it is, in fact, an anticommutative graded algebra over $\mathbb{R}$.

Suppose $F : N \to M$ is a smooth map of manifolds. Since $F^*(\omega \wedge \tau) = F^*\omega \wedge F^*\tau$ for $\omega, \tau \in \Omega^*(M)$, the linear map

$$F^{\#} : H^*(M) \to H^*(N),$$

is a ring homomorphism. Then $\left(H^*(-), F^{\#}\right)$ becomes a contravariant functor from the category of smooth manifolds and smooth maps to the category of anticommutative graded rings and ring homomorphisms. If $F : N \to M$ is an isomorphism in the former category, i.e. if $F : N \to M$ is a diffeomorphism, then it is an isomorphism in the latter category as well, i.e. $F^{\#} : H^*(M) \to H^*(N)$ is a ring isomorphism.

18 The Long Exact Sequence of Cohomology

Definition 18.1 (Cochain complex)**.** A **cochain complex** $\mathcal{C}$ is a collection of vector spaces $\left\{C^k\right\}_{k\in\mathbb{Z}}$ together with a sequence of linear maps $d_k : C^k \to C^k$

$$\cdots \longrightarrow C^{-1} \xrightarrow{d_{-1}} C^0 \xrightarrow{d_0} C^1 \xrightarrow{d_1} C^2 \longrightarrow \cdots$$

such that for all $k \in \mathbb{Z}$,

$$d_k \circ d_{k-1} = 0. \tag{18.1}$$

We call the linear maps in the collection $\{d_k\}_{k\in\mathbb{Z}}$, the **differentials** of the cochain complex $\mathcal{C}$.

The vector space $\Omega^*(M)$ of differential forms on a manifold M together with the exterior derivative d is a cochain complex, called the de Rham complex of M :

$$0 \longrightarrow \Omega^0(M) \xrightarrow{\mathrm{d}} \Omega^1(M) \xrightarrow{\mathrm{d}} \Omega^2(M) \xrightarrow{\mathrm{d}} \Omega^3(M) \longrightarrow \cdots,$$

and $\mathrm{d} \circ \mathrm{d} = 0$. Many of the results on the de Rham cohomology of a manifold depend not on the topological properties of the manifold but on the algebraic properties of the de Rham complex. To better study de Ream cohomology, it is useful to isolate these algebraic properties that we do in this chapter.

§18.1 Exact Sequences

Definition 18.2 (Exact sequence)**.** A sequence of homomorphism of vector spaces $A \xrightarrow{f} B \xrightarrow{g} C$ is said to be **exact** at B if $\operatorname{im} f = \operatorname{Ker} g$. A sequence of homomorphisms

$$A^0 \xrightarrow{f_0} A^1 \xrightarrow{f_1} A^2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A^n$$

that is exact at every term except the first and the last term is called an **exact sequence**. A fiver term exact sequence of the form

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is said to be **short exact**.

Remark 18.1. When $A = 0$ in the three-term exact sequence $A \xrightarrow{f} B \xrightarrow{g} C$, i.e. $0 \xrightarrow{f} B \xrightarrow{g} C$ is exact if and only if $\operatorname{Ker} g = \operatorname{im} f = 0$, so that g is injective.

Similarly, when $C = 0$ in the three-term exact sequence $A \xrightarrow{f} B \xrightarrow{g} C$, i.e. $A \xrightarrow{f} B \xrightarrow{g} 0$ is exact if and only if $\operatorname{im} f = \operatorname{Ker} g = B$, so that f is surjective.

Proposition 18.1

Suppose $A \xrightarrow{f} B \xrightarrow{g} C$ is an exact sequence. Then

(i) the map f is surjective if and only if g is the zero map;

(ii) the map g is injective if and only if f is the zero map.

Proof. (i) Since the sequence is exact, we have $\operatorname{im} f = \operatorname{Ker} g$. $f : A \to B$ is surjective if and only if $\operatorname{im} f = B$. Therefore, the surjectivity of f is equivalent to $\operatorname{Ker} g = B$. $\operatorname{Ker} g = B$ means it takes all of B to $0 \in C$, i.e. g is the zero map.

(ii) $g : B \to C$ is injective if and only if $\operatorname{Ker} g = 0$. Therefore, the injectivity of g is equivalent to $\operatorname{im} f = 0$. $\operatorname{im} f = 0$ means it takes all of A to $0 \in B$, i.e. f is the zero map.

■

Proposition 18.2

The following hold:

(i) The 4-term sequence

$$0 \longrightarrow A \xrightarrow{\ f\ } B \longrightarrow 0$$

is exact if and only if $f : A \to B$ is an isomorphism.

(ii) If the following

$$A \xrightarrow{\ f\ } B \longrightarrow C \longrightarrow 0$$

is an exact sequence of vector spaces, then there is a linear isomorphism

$$C \cong \operatorname{Coker} f := \frac{B}{\operatorname{im} f}.$$

Proof. (i) Suppose that the 4-term sequence

$$0 \longrightarrow A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } 0$$

is exact. Then by Proposition 18.1, by the exactness of $0 \to A \xrightarrow{f} B$, we get f is injective. Again, using Proposition 18.1 and the exactness of $A \xrightarrow{f} B \to 0$, f is surjective. Therefore, f is bijective. The inverse of a bijective linear map is also linear. Hence, f is an isomorphism.

Conversely, suppose $f : A \to B$ is an isomorphism of vector spaces. Hence, f is injective and surjective. Since f is injective, we have

$$\operatorname{Ker} f = 0 = \operatorname{im}(0 \to A)\,. \tag{18.2}$$

Again, f is surjective, so

$$\operatorname{im} f = B = \operatorname{Ker}(B \to 0)\,. \tag{18.3}$$

Therefore, the 4-term sequence

$$0 \longrightarrow A \xrightarrow{\ f\ } B \longrightarrow 0$$

is exact.

(ii) Suppose the 4-term sequence

$$0 \longrightarrow A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } 0$$

is an exact sequence of vector spaces. By Proposition 18.1, g is surjective. By exactness of this sequence at B, $\operatorname{im} f = \operatorname{Ker} g$. Now, applying the first isomorphism theorem for the surjective linear map $g : B \to C$, we get an an isomorphism

$$C = \operatorname{im} g \cong \frac{B}{\operatorname{Ker} g} = \frac{B}{\operatorname{im} f} = \operatorname{Coker} f. \tag{18.4}$$

■

§18.2 Cohomology of cochain complexes

Recall that a cochain complex $\mathcal{C}$ is a collection of vector spaces $\left\{C^k\right\}_{k\in\mathbb{Z}}$ together with a sequence of linear maps $d_k : C^k \to C^k$

$$\cdots \longrightarrow C^{k-1} \xrightarrow{d_{k-1}} C^k \xrightarrow{d_k} C^{k+1} \longrightarrow \cdots$$

such that for all $k \in \mathbb{Z}$, $d_k \circ d_{k-1} = 0$. This implies that

$$\operatorname{im} d_{k-1} \subseteq \operatorname{Ker} d_k. \tag{18.5}$$

One can, therefore, form the quotient vector space

$$H^k(\mathcal{C}) := \frac{\operatorname{Ker} d_k}{\operatorname{im} d_{k-1}}, \tag{18.6}$$

which is called the k-th **cohomology vector space** of the cochain complex $\mathcal{C}$. It is a measure of the extent to which the cochain complex $\mathcal{C}$ fails to he exact at C^k. Elements of the vector space C^k are called cochains of degree k or k-cochains. A k-cochain in $\operatorname{Ker} d_k$ is called a k-cocycle; a k-cochain in $\operatorname{im} d_{k-1}$ is called a k-coboundary. The equivalence, class $[c] \in\in H^k(\mathcal{C})$ of a k-cocycle $c \in \operatorname{Ker} d_k$ is called its cohomology class. We denote these 2 subspaces of C^k by $Z^k(\mathcal{C})$ (subspace of k-cocycles) and by $B^k(\mathcal{C})$ (subspace of k-coboundaries).

Example 18.1. In the de Rham complex of a manifold M, a cocylce is a closed form and a coboundary is an exact form.

Definition 18.3 (Cochain map)**.** If $\mathcal{A}$ and $\mathcal{B}$ are 2 cochain complexes with differentials $\left\{d_k^{\mathcal{A}}\right\}_{k\in\mathbb{Z}}$ and $\left\{d_k^{\mathcal{B}}\right\}_{k\in\mathbb{Z}}$, respectively. A **cochain map** $\varphi : \mathcal{A} \to \mathcal{B}$ is a collection of linear maps $\varphi_k : A^k \to B^k$ such that

$$d_k^{\mathcal{B}} \circ \varphi_k = \varphi_{k+1} \circ d_k^{\mathcal{A}}. \tag{18.7}$$

In other words, the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r] & A^{k-1} \arrow[r, "d_{k-1}^{\mathcal{A}}"] \arrow[d, "\varphi_{k-1}"'] & A^k \arrow[r, "d_k^{\mathcal{A}}"] \arrow[d, "\varphi_k"] & A^{k+1} \arrow[r] \arrow[d, "\varphi_{k+1}"] & \cdots \\
\cdots \arrow[r] & B^{k-1} \arrow[r, "d_{k-1}^{\mathcal{B}}"'] & B^k \arrow[r, "d_k^{\mathcal{B}}"'] & B^{k+1} \arrow[r] & \cdots
\end{tikzcd}$$

Observe that a cochain map $\varphi : \mathcal{A} \to \mathcal{B}$ takes cocycles to cocycles and coboundaries to coboundaries. Consider the following commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & A^{k-1} \arrow[r, "d_{k-1}^{\mathcal{A}}"] \arrow[d, "\varphi_{k-1}"'] & A^k \arrow[r, "d_k^{\mathcal{A}}"] \arrow[d, "\varphi_k"] & A^{k+1} \arrow[r] \arrow[d, "\varphi_{k+1}"] & \cdots \\
\cdots \arrow[r] & B^{k-1} \arrow[r, "d_{k-1}^{\mathcal{B}}"'] & B^k \arrow[r, "d_k^{\mathcal{B}}"'] & B^{k+1} \arrow[r] & \cdots
\end{tikzcd} \tag{18.8}$$

(i) For $a \in Z^k(\mathcal{A})$, $d_k^{\mathcal{A}} a = 0$. Then by the commutativity of the right hand square in (18.8),

$$d_k^{\mathcal{B}}(\varphi_k(a)) = \varphi_{k+1}\left(d_k^{\mathcal{A}} a\right) = 0. \tag{18.9}$$

Therefore, $\varphi_k(a) \in \operatorname{Ker} d_k^{\mathcal{B}} = Z^k(\mathcal{B})$. In other words, the cochain map $\varphi : \mathcal{A} \to \mathcal{B}$ takes cocycles to cocycles.

(ii) Suppose $a \in B^k(\mathcal{A})$. Then $a = d^{\mathcal{A}}_{k-1} a'$ for some $a' \in A^{k-1}$. Then by the commutativity of the left hand square in (18.8),

$$\varphi_k(a) = \varphi_k\left(d^{\mathcal{A}}_{k-1} a'\right) = d^{\mathcal{B}}_{k-1}\left(\varphi_{k-1} a'\right). \tag{18.10}$$

Therefore, $\varphi_k(a) \in \operatorname{im} d^{\mathcal{B}}_{k-1} = B^k(\mathcal{B})$. In other words, the cochain map $\varphi : \mathcal{A} \to \mathcal{B}$ takes coboundaries to coboundaries.

Hence, we see that the coochain map $\varphi : \mathcal{A} \to \mathcal{B}$ naturally induces a linear map in cohomology:

$$\begin{aligned} (\varphi^*)^k : H^k(\mathcal{A}) &\to H^k(\mathcal{B}), \\ [a] &\mapsto [\varphi_k(a)]. \end{aligned} \tag{18.11}$$

Example 18.2. For a smooth map $F : N \to M$ between manifolds, the pullback map $F^* : \Omega^*(M) \to \Omega^*(N)$ on differential forms is a cochain map, because F^* commutes with d. By the discussion above, there is an induced map $\left(F^{\#}\right)^k : H^k(M) \to H^k(N)$ in cohomology.

§18.3 Zig-Zag Lemma

A sequence of cochain complexes

$$0 \longrightarrow \mathcal{A} \xrightarrow{\ i\ } \mathcal{B} \xrightarrow{\ j\ } \mathcal{C} \longrightarrow 0$$

is **short exact** if i and j are cochain maps, and for each k,

$$0 \longrightarrow A^k \xrightarrow{\ i_k\ } B^k \xrightarrow{\ j_k\ } C^k \longrightarrow 0$$

is a short exact sequence of vector spaces. In other words, the following is a commutative diagram with exact rows, for each k:

$$\begin{tikzcd} 0 \arrow[r] & A^{k+1} \arrow[r, "i_{k+1}"] & B^{k+1} \arrow[r, "j_{k+1}"] & C^{k+1} \arrow[r] & 0 \\ 0 \arrow[r] & A^k \arrow[u, "d^{\mathcal{A}}_k"] \arrow[r, "i_k"'] & B^k \arrow[u, "d^{\mathcal{B}}_k"] \arrow[r, "j_k"'] & C^k \arrow[u, "d^{\mathcal{C}}_k"] \arrow[r] & 0 \end{tikzcd} \tag{18.12}$$

Given a short exact sequence as above, we can constricy a linear map

$$(d^*)_k : H^k(\mathcal{C}) \to H^{k+1}(\mathcal{A}),$$

called the **connecting homomorphism** as follows: consider the short exact sequences in dimensions k and $k+1$ associated with the short exact sequence $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ of cochain complexes:

$$\begin{tikzcd} 0 \arrow[r] & A^{k+1} \arrow[r, "i_{k+1}"] & B^{k+1} \arrow[r, "j_{k+1}"] & C^{k+1} \arrow[r] & 0 \\ 0 \arrow[r] & A^k \arrow[u, "d^{\mathcal{A}}_k"] \arrow[r, "i_k"'] & B^k \arrow[u, "d^{\mathcal{B}}_k"'] \arrow[r, "j_k"'] & C^k \arrow[u, "d^{\mathcal{C}}_k"'] \arrow[r] & 0 \end{tikzcd} \tag{18.13}$$

We start with $[c] \in H^k(\mathcal{C})$, for some $c \in \operatorname{Ker} d^{\mathcal{C}}_k \subseteq C^k$. By the exactness of the bottom row, we have that j_k is surjective. So there is some $b \in B^k$ such that $j_k(b) = c$. By the commutativity of the right square,

$$j_{k+1}\left(d^{\mathcal{B}}_k b\right) = d^{\mathcal{C}}_k(j_k b) = d^{\mathcal{C}}_k(c) = 0. \tag{18.14}$$

Therefore, $d_k^{\mathcal{B}} b \in \operatorname{Ker} j_{k+1} = \operatorname{im} i_{k+1}$. So $d_k^{\mathcal{B}} b = i_{k+1}(a)$ for some $a \in A^{k+1}$. Now consider the diagram (18.13) for $k+1$ in place of k.

$$\begin{tikzcd}
0 \arrow[r] & A^{k+2} \arrow[r, "i_{k+2}"] & B^{k+2} \arrow[r, "j_{k+2}"] & C^{k+2} \arrow[r] & 0 \\
0 \arrow[r] & A^{k+1} \arrow[u, "d^{\mathcal{A}}_{k+1}"] \arrow[r, "i_{k+1}"'] & B^{k+1} \arrow[u, "d^{\mathcal{B}}_{k+1}"'] \arrow[r, "j_{k+1}"'] & C^{k+1} \arrow[u, "d^{\mathcal{C}}_{k+1}"'] \arrow[r] & 0
\end{tikzcd} \tag{18.15}$$

Now,

$$i_{k+2}\left(d_{k+1}^{\mathcal{A}} a\right) = d_{k+1}^{\mathcal{B}}\left(i_{k+1} a\right) = d_{k+1}^{\mathcal{B}}\left(d_k^{\mathcal{B}} b\right) = 0. \tag{18.16}$$

So $d_{k+1}^{\mathcal{A}} a \in \operatorname{Ker} i_{k+2}$. But i_{k+2} is injective by the exactness of the top row of (18.15). Therefore,

$$d_{k+1}^{\mathcal{A}} a = 0. \tag{18.17}$$

So we define

$$(d^*)_k [c] = [a]. \tag{18.18}$$

The recipe for defining the connecting homomorphism $(d^*)_k$ is best summarized by the following Zig-Zag diagram:

$$\begin{tikzcd}
a \arrow[r, tail, "i_{k+1}"] & d_k^{\mathcal{B}} b & \\
& b \arrow[u, "d_k^{\mathcal{B}}"] \arrow[r, two heads, "j_k"] & c
\end{tikzcd}$$

Note that there were choices involved in this definition. We chose the cocycle c to represent the cohomology class $[c]$. One could've chosen a cohomologous cocycle c' representing the same cohomology class $[c]$. Furthermore, we chose an element $b \in B^k$ such that $j_k(b) = c$ holds. Since j_k is surjective, and not necessarily injective, the choice for b is not unique. So we need to show that this definition of $(d^*)_k : H^k(\mathcal{C}) \to H^{k+1}(\mathcal{A})$ is well-defined.

Let $[c] = [c']$. Then

$$c - c' = d_{k-1} c'', \tag{18.19}$$

for some $c'' \in C^{k-1}$. As before, we choose some $b' \in B^k$ such that $j_k(b') = c'$, and then finally $d_k^{\mathcal{B}} b' = i_{k+1}(a')$. We need to show that $[a] = [a']$.

$$\begin{tikzcd}
0 \arrow[r] & A^{k+1} \arrow[r, "i_{k+1}"] & B^{k+1} \arrow[r, "j_{k+1}"] & C^{k+1} \arrow[r] & 0 \\
0 \arrow[r] & A^{k} \arrow[u, "d^{\mathcal{A}}_{k}"] \arrow[r, "i_{k}"'] & B^{k} \arrow[u, "d^{\mathcal{B}}_{k}"'] \arrow[r, "j_{k}"'] & C^{k} \arrow[u, "d^{\mathcal{C}}_{k}"'] \arrow[r] & 0 \\
0 \arrow[r] & A^{k-1} \arrow[u, "d^{\mathcal{A}}_{k-1}"] \arrow[r, "i_{k-1}"'] & B^{k-1} \arrow[u, "d^{\mathcal{B}}_{k-1}"'] \arrow[r, "j_{k-1}"'] & C^{k-1} \arrow[u, "d^{\mathcal{C}}_{k-1}"'] \arrow[r] & 0
\end{tikzcd}$$

Since j_{k-1} is surjective, there exists $b'' \in B^{k-1}$ such that $j_{k-1}(b'') = c''$. Then

$$\begin{aligned}
j_k\left(b - b' - d_{k-1}^{\mathcal{B}} b''\right) &= j_k(b) - j_k(b') - j_k\left(d_{k-1}^{\mathcal{B}} b''\right) \\
&= j_k(b) - j_k(b') - d_{k-1}^{\mathcal{C}}\left(j_{k-1} b''\right) \\
&= c - c' - d_{k-1} c'' = 0.
\end{aligned} \tag{18.20}$$

As a result, $b - b' - d_{k-1}^{\mathcal{B}} b'' \in \operatorname{Ker} j_k = \operatorname{im} i_k$. So

$$b - b' - d_{k-1}^{\mathcal{B}} b'' = i_k \left(a''\right) \tag{18.21}$$

for some $a'' \in A^k$. Now,

$$\begin{aligned} i_{k+1}\left(d_k^{\mathcal{A}} a''\right) &= d_k^{\mathcal{B}}\left(i_k a''\right) = d_k^{\mathcal{B}}\left(b - b' - d_{k-1}^{\mathcal{B}} b''\right) \\ &= d_k^{\mathcal{B}}\left(b\right) - d_k^{\mathcal{B}}\left(b'\right) = i_{k+1}\left(a\right) - i_{k+1}\left(a'\right) \\ &= i_{k+1}\left(a - a'\right). \end{aligned} \tag{18.22}$$

Since i_{k+1} is injective, we have

$$a - a' = d_k^{\mathcal{A}} a''. \tag{18.23}$$

In other words, $[a] = [a']$, i.e. $(d^*)_k$ is well-defined.

It's easy to show that $(d^*)_k$ is linear. Given $[c], [c'] \in H^k(\mathcal{C})$ and $\alpha \in \mathbb{R}$, $[c] + \alpha [c'] = [c + \alpha c']$. Suppose $c = j_k(b)$, $c' = j_k(b')$; and $d_k^{\mathcal{B}} b = i_{k+1}(a)$, $d_k^{\mathcal{B}} b' = i_{k+1}(a')$. Then

$$(d^*)_k [c] = [a], \text{ and } (d^*)_k [c'] = [a']. \tag{18.24}$$

Now, by the linearity of j_k,

$$j_k\left(b + \alpha b'\right) = j_k\left(b\right) + \alpha j_k\left(b'\right) = c + \alpha c'. \tag{18.25}$$

Furthermore, using the linearity of $d_k^{\mathcal{B}}$ and i_{k+1},

$$i_{k+1}\left(a + \alpha a'\right) = i_{k+1}\left(a\right) + \alpha i_{k+1}\left(a'\right) = d_k^{\mathcal{B}} b + \alpha d_k^{\mathcal{B}} b' = d_k^{\mathcal{B}}\left(b + \alpha b'\right). \tag{18.26}$$

Therefore,

$$(d^*)_k [c + \alpha c'] = [a + \alpha a'] = [a] + \alpha [a']. \tag{18.27}$$

In other words,

$$(d^*)_k \left([c] + \alpha [c']\right) = (d^*)_k [c] + \alpha (d^*)_k [c]. \tag{18.28}$$

Hence, $(d^*)_k$ is a linear map.

Theorem 18.3 (The Zig-Zag Lemma)

Given a short exact sequence of cochain complexes

$$0 \longrightarrow \mathcal{A} \xrightarrow{\ i\ } \mathcal{B} \xrightarrow{\ j\ } \mathcal{C} \longrightarrow 0,$$

one has a long exact sequence in cohomology:

$$\begin{tikzcd} H^{k+1}(\mathcal{A}) \arrow[r, "(i^*)^{k+1}"] & \cdots & \\ H^k(\mathcal{A}) \arrow[r, "(i^*)^k"] & H^k(\mathcal{B}) \arrow[r, "(j^*)^k"'] & H^k(\mathcal{C}) \arrow[ull, "(d^*)_k"'] \\ & \cdots \arrow[r, "(j^*)^{k-1}"'] & H^{k-1}(\mathcal{C}), \arrow[ull, "(d^*)_{k-1}"'] \end{tikzcd} \tag{18.29}$$

where $(i^*)^k$ ad $(j^*)^k$ are the maps in cohomology induced from the cochain maps i and j; and $(d^*)_k$ is the connecting homomorphism defined earlier.

Proof. To prove this theorem, one needs to check the exactness of the above sequence (18.29) at $H^k(\mathcal{A})$, $H^k(\mathcal{B})$ and $H^k(\mathcal{C})$ for each k. We shall make use of the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & A^{k+1} \arrow[r, "i_{k+1}"] & B^{k+1} \arrow[r, "j_{k+1}"] & C^{k+1} \arrow[r] & 0 \\
0 \arrow[r] & A^k \arrow[u, "d_k^{\mathcal{A}}"] \arrow[r, "i_k"'] & B^k \arrow[u, "d_k^{\mathcal{B}}"'] \arrow[r, "j_k"'] & C^k \arrow[u, "d_k^{\mathcal{C}}"'] \arrow[r] & 0 \\
0 \arrow[r] & A^{k-1} \arrow[u, "d_{k-1}^{\mathcal{A}}"] \arrow[r, "i_{k-1}"'] & B^{k-1} \arrow[u, "d_{k-1}^{\mathcal{B}}"'] \arrow[r, "j_{k-1}"'] & C^{k-1} \arrow[u, "d_{k-1}^{\mathcal{C}}"'] \arrow[r] & 0
\end{tikzcd} \tag{18.30}$$

Exactness at $H^k(\mathcal{A})$: Let $[a] \in \text{Ker}\,(i^*)_k$.

$$(i^*)_k\,[a] = [i_k\,(a)] = [0] \in H^k\,(\mathcal{B})\,. \tag{18.31}$$

In other words,

$$i_k\,(a) = d_{k-1}^{\mathcal{B}} b, \tag{18.32}$$

for some $b \in B^{k-1}$. Let $c = j_{k-1}\,(b)$. Then

$$d_{k-1}^{\mathcal{C}}\,(c) = d_{k-1}^{\mathcal{C}}\,(j_{k-1}\,(b)) = j_k\left(d_{k-1}^{\mathcal{B}} b\right) = j_k\,(i_k\,(a)) = 0, \tag{18.33}$$

since $j_k \circ i_k = 0$. Therefore, $c \in \text{Ker}\, d_{k-1}^{\mathcal{C}}$, i.e. $[c] \in H^{k-1}\,(\mathcal{C})$. $b \in B^{k-1}$ is such that $j_{k-1}\,(b) = c$. Furthermore, $d_{k-1}^{\mathcal{B}} b = i_k\,(a)$.

$$\begin{tikzcd}
a \arrow[r, tail, "i_k"] & d_{k-1}^{\mathcal{B}} b & \\
& b \arrow[u, "d_{k-1}^{\mathcal{B}}"] \arrow[r, two heads, "j_{k-1}"] & c
\end{tikzcd}$$

Therefore, $[a] = (d^*)_{k-1}\,[c]$. In other words,

$$\text{Ker}\,(i^*)_k \subseteq \text{im}\,(d^*)_{k-1}\,. \tag{18.34}$$

Now suppose $[a] \in \text{im}\,(d^*)_{k-1}$, i.e. $[a] = (d^*)_{k-1}\,[c]$ for some $[c] \in H^{k-1}\,(\mathcal{C})$. Then $c = j_{k-1}\,(b)$ for some $b \in B^{k-1}$, and $i_k\,(a) = d_{k-1}^{\mathcal{B}} b$. Now,

$$(i^*)^k\,[a] = [i_k\,(a)] = \left[d_{k-1}^{\mathcal{B}} b\right] = [0] \in H^k\,(\mathcal{B}) = \frac{\text{Ker}\, d_k^{\mathcal{B}}}{\text{im}\, d_{k-1}^{\mathcal{B}}}. \tag{18.35}$$

Therefore, $[a] \in \text{Ker}\,(i^*)^k$. So

$$\text{im}\,(d^*)_{k-1} \subseteq \text{Ker}\,(i^*)_k\,. \tag{18.36}$$

As a result of (18.34) and (18.36), we have

$$\text{Ker}\,(i^*)_k = \text{im}\,(d^*)_{k-1}\,. \tag{18.37}$$

So (18.29) is exact at $H^k\,(\mathcal{A})$.

Exactness at $H^k\,(\mathcal{B})$: Given $[a] \in H^k\,(\mathcal{A})$,

$$(j^*)^k \circ (i^*)^k\,[a] = (j^*)^k\,[i_k\,(a)] = [j_k\,(i_k\,(a))] = 0, \tag{18.38}$$

since $j_k \circ i_k = 0$. Therefore,

$$\text{im}\,(i^*)^k \subseteq \text{Ker}\,(j^*)^k\,. \tag{18.39}$$

Now, suppose $[b] \in \operatorname{Ker}(j^*)^k$. Then

$$(j^*)^k [b] = [j_k b] = [0] \in H^k(\mathcal{C}). \tag{18.40}$$

So we have

$$j_k(b) = d_{k-1}^{\mathcal{C}} c \tag{18.41}$$

for some $c \in C^{k-1}$. Since j_{k-1} is surjective, we have

$$c = j_{k-1}(b') \tag{18.42}$$

for some $b' \in B^{k-1}$. Then

$$\begin{aligned} j_k\left(b - d_{k-1}^{\mathcal{B}} b'\right) &= j_k(b) - j_k\left(d_{k-1}^{\mathcal{B}} b'\right) \\ &= d_{k-1}^{\mathcal{C}} c - d_{k-1}^{\mathcal{C}}(j_{k-1} b') \\ &= d_{k-1}^{\mathcal{C}} c - d_{k-1}^{\mathcal{C}} c = 0. \end{aligned} \tag{18.43}$$

So $b - d_{k-1}^{\mathcal{B}} b' \in \operatorname{Ker} j_k = \operatorname{im} i_k$, i.e. $b - d_{k-1}^{\mathcal{B}} b' = i_k(a)$ for some $a \in A^k$. Now,

$$i_{k+1}\left(d_k^{\mathcal{A}} a\right) = d_k^{\mathcal{B}}(i_k a) = d_k^{\mathcal{B}}\left(b - d_{k-1}^{\mathcal{B}} b'\right) = d_k^{\mathcal{B}} b = 0, \tag{18.44}$$

since $[b] \in H^k(\mathcal{B})$. Now, i_{k+1} is injective. As a result, $d_k^{\mathcal{A}} a = 0$. Now,

$$(i^*)^k [a] = [i_k(a)] = \left[b - d_{k-1}^{\mathcal{B}} b'\right] = [b], \tag{18.45}$$

i.e. $[b] \in \operatorname{im}(i^*)^k$. So

$$\operatorname{Ker}(j^*)^k \subseteq \operatorname{im}(i^*)^k. \tag{18.46}$$

As a result of (18.39) and (18.46), we have

$$\operatorname{Ker}(j^*)^k = \operatorname{im}(i^*)^k. \tag{18.47}$$

Hence, (18.29) is exact at $H^k(\mathcal{B})$.

Exactness at $H^k(\mathcal{C})$: First we prove that $\operatorname{im}(j^*)^k \subseteq \operatorname{Ker}(d^*)_k$. For $[b] \in H^k(\mathcal{B})$, we have

$$(d^*)_k\left((j^*)^k [b]\right) = (d^*)_k [j_k(b)]. \tag{18.48}$$

In the recipe for defining $(d^*)_k : H^k(\mathcal{C}) \to H^{k+1}(\mathcal{A})$, we can choose the element $b \in B^k$ that maps to $j_k(b)$. Then $d_k^{\mathcal{B}} b \in B^{k+1}$. Since $[b]$ is a k-th cohomology class, $b \in \operatorname{Ker} d_k^{\mathcal{B}}$. Therefore, $d_k^{\mathcal{B}} b = 0$. Following the Zig-Zag diagram,

$$\begin{tikzcd} 0 \arrow[r, tail, "i_{k+1}"] & d_k^{\mathcal{B}} b = 0 & \\ & b \arrow[u, "d_k^{\mathcal{B}}"] \arrow[r, two heads, "j_k"] & j_k(b), \end{tikzcd}$$

we see that since $i_{k+1}(0) = 0 = d_k^{\mathcal{B}} b$, by the definition of $(d^*)^k$, we must have

$$(d^*)^k [j_k(b)] = 0. \tag{18.49}$$

Therefore, $(j^*)^k [b] \in \operatorname{Ker}(d^*)_k$, proving the inclusion

$$\operatorname{im}(j^*)^k \subseteq \operatorname{Ker}(d^*)_k. \tag{18.50}$$

Now, let $[c] \in \operatorname{Ker}(d^*)_k \subseteq H^k(\mathcal{C})$. Then

$$(d^*)_k [c] = [a] = 0. \tag{18.51}$$

So a is a $(k+1)$-coboundary, i.e.

$$a = d_k^{\mathcal{A}} a', \tag{18.52}$$

for some $a' \in A^k$. The calculation for $(d^*)_k [c]$ can be representative by the following Zig-Zag diagram:

$$\begin{tikzcd}
d_k^{\mathcal{B}} a' = a \arrow[r, tail, "i_{k+1}"] & d_k^{\mathcal{B}} b & \\
a' \arrow[u, "d_k^{\mathcal{A}}"] & b \arrow[u, "d_k^{\mathcal{B}}"] \arrow[r, two heads, "j_k"] & c = j_k(b)
\end{tikzcd}$$

Here b is an element in B^k such that $j_k(b) = c$, and $i_{k+1}(a) = d_k^{\mathcal{B}} b$. $a' \in A^k$, $i_k : A^k \to B^k$. So both $i_k(a')$ and b are in B^k. Now,

$$\begin{aligned}
d_k^{\mathcal{B}}(b - i_k(a')) &= d_k^{\mathcal{B}} b - d_k^{\mathcal{B}}(i_k(a')) \\
&= d_k^{\mathcal{B}} b - i_{k+1}\left(d_k^{\mathcal{A}} a'\right) \\
&= d_k^{\mathcal{B}} b - i_{k+1}(a) = 0.
\end{aligned} \tag{18.53}$$

Therefore, $b - i_k(a') \in \operatorname{Ker} d_k^{\mathcal{B}}$, i.e. $b - i_k(a')$ is cocycle in B^k. Now,

$$j_k(b - i_k(a')) = j_k(b) - j_k(i_k(a')) = j_k(b) = c, \tag{18.54}$$

since $j_k \circ i_k = 0$ by the exactness of $0 \to A^k \xrightarrow{i_k} B^k \xrightarrow{j_k} C \to 0$. Therefore, $b - i_k(a')$ is a cocycle in B^k that gets mapped to c under j_k. Therefore,

$$[c] = [j_k(b - i_k(a'))] = (j^*)^k [b - i_k(a')]. \tag{18.55}$$

So $[c] \in \operatorname{im}(j^*)^k$. As a result,

$$\operatorname{Ker}(d^*)_k \subseteq \operatorname{im}(j^*)^k. \tag{18.56}$$

Combining (18.50) and (18.56), we have

$$\operatorname{im}(j^*)^k = \operatorname{Ker}(d^*)_k, \tag{18.57}$$

proving the exactness of (18.29) at $H^k(\mathcal{C})$. ■

Corollary 18.4 (The Snake Lemma)
A commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & A^1 \arrow[r] & B^1 \arrow[r] & C^1 \arrow[r] & 0 \\
0 \arrow[r] & A^0 \arrow[r] \arrow[u, "\alpha"] & B^0 \arrow[r] \arrow[u, "\beta"] & C^0 \arrow[r] \arrow[u, "\gamma"] & 0
\end{tikzcd}$$

induces a long exact sequence

$$\begin{tikzcd}
& \operatorname{cok}\alpha \arrow[r] & \operatorname{cok}\beta \arrow[r] & \operatorname{cok}\gamma \arrow[r] & 0 \\
0 \arrow[r] & \operatorname{Ker}\alpha \arrow[r] & \operatorname{Ker}\beta \arrow[r] & \operatorname{Ker}\gamma \arrow[ull] &
\end{tikzcd}$$

Proof. Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be the following cochain complexes:

$$\begin{array}{ll}
\mathcal{A} & 0 \longrightarrow A^0 \xrightarrow{\alpha} A^1 \longrightarrow 0 \\
\mathcal{B} & 0 \longrightarrow B^0 \xrightarrow{\beta} B^1 \longrightarrow 0 \\
\mathcal{C} & 0 \longrightarrow C^0 \xrightarrow{\gamma} C^1 \longrightarrow 0
\end{array}$$

So, the given commutative diagram with exact rows is a short exact sequence of cochain complexes:

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C} \longrightarrow 0$$

Therefore, by The Zig-Zag Lemma, there is a long exact sequence at the cohomology level:

$$\begin{tikzcd}
H^{k+1}(\mathcal{A}) \arrow[r] & \cdots & \\
H^k(\mathcal{A}) \arrow[r] & H^k(\mathcal{B}) \arrow[r] & H^k(\mathcal{C}) \arrow[ull] \\
& \cdots \arrow[r] & H^{k-1}(\mathcal{C}) \arrow[ull]
\end{tikzcd} \tag{18.58}$$

Notice that, for $k \neq 0, 1$

$$H^k(\mathcal{A}) = H^k(\mathcal{B}) = H^k(\mathcal{C}) = 0, \tag{18.59}$$

since the cochain groups are trivial for $k \neq 0, 1$. For $k = 0$,

$$H^0(\mathcal{A}) = \frac{\operatorname{Ker}\alpha}{\operatorname{im}(0 \to A^0)} = \operatorname{Ker}\alpha, \tag{18.60}$$

$$H^0(\mathcal{B}) = \frac{\operatorname{Ker}\beta}{\operatorname{im}(0 \to B^0)} = \operatorname{Ker}\beta, \tag{18.61}$$

$$H^0(\mathcal{C}) = \frac{\operatorname{Ker}\gamma}{\operatorname{im}(0 \to C^0)} = \operatorname{Ker}\gamma. \tag{18.62}$$

For $k = 1$,

$$H^1(\mathcal{A}) = \frac{\operatorname{Ker}(A^1 \to 0)}{\operatorname{im}\alpha} = \frac{A^1}{\operatorname{im}\alpha} = \operatorname{cok}\alpha, \tag{18.63}$$

$$H^1(\mathcal{B}) = \frac{\operatorname{Ker}(B^1 \to 0)}{\operatorname{im}\beta} = \frac{B^1}{\operatorname{im}\beta} = \operatorname{cok}\beta, \tag{18.64}$$

$$H^1(\mathcal{C}) = \frac{\operatorname{Ker}(C^1 \to 0)}{\operatorname{im}\gamma} = \frac{A^1}{\operatorname{im}\gamma} = \operatorname{cok}\gamma. \tag{18.65}$$

So (18.58) becomes

$$\begin{tikzcd}
& & \operatorname{cok}\alpha \arrow[r] & \operatorname{cok}\beta \arrow[r] & \operatorname{cok}\gamma \arrow[r] & 0 \arrow[r] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & \operatorname{Ker}\alpha \arrow[r] & \operatorname{Ker}\beta \arrow[r] & \operatorname{Ker}\gamma \arrow[ull] & &
\end{tikzcd}$$

where the dots represent trivial vector spaces (containing only the zero vector). ■

§18.4 The Mayer–Vietoris Sequence

Let $\{U, V\}$ be an open cover of a manifold M, and let $i_U : U \hookrightarrow M$, $i_U(p) = p$, be the inclusion map. Then the pullback

$$(i_U^*)_k : \Omega^k(M) \to \Omega^k(U)$$

is the restriction map that restricts the domain of a k-form on M to U (Example 12.3). In other words,

$$(i_U^*)_k\,\omega = \omega\big|_U, \tag{18.66}$$

for $\omega \in \Omega^k(M)$. Similarly,

$$(i_V^*)_k \, \omega = \omega|_V, \tag{18.67}$$

In fact, there are 4 relevant inclusion maps forming a commutative diagram:

$$\begin{tikzcd}
 & U \arrow[dr, hook, "i_U"] & \\
U \cap V \arrow[ur, hook, "j_U"] \arrow[rr, hook, "i_U \circ j_U = i_V \circ j_V"] \arrow[dr, hook, "j_V"'] & & U \cup V = M \\
 & V \arrow[ur, hook, "i_V"'] &
\end{tikzcd}$$

These inclusions induce the following commutative diagram of vector spaces:

$$\begin{tikzcd}
 & \Omega^k(U) \arrow[dl, "(j_U^*)_k"'] & \\
\Omega^k(U \cap V) & & \Omega^k(M) \arrow[ul, "(i_U^*)_k"'] \arrow[dl, "(i_V^*)_k"] \\
 & \Omega^k(V) \arrow[ul, "(j_V^*)_k"] &
\end{tikzcd}$$

Similarly as (18.67), $(j_U^*)_k$ and $(j_V^*)_k$ also restricts the domain of the smooth k-form. In other words,

$$(j_U^*)_k \, \omega = \omega|_{U\cap V} \text{ and } (j_V^*)_k \, \tau = \tau|_{U\cap V}. \tag{18.68}$$

We then define the following linear maps between vector spaces:

$$i_k : \Omega^k(M) \to \Omega^k(U) \oplus \Omega^k(V) \text{ and } j_k : \Omega^k(U) \oplus \Omega^k(V) \to \Omega^k(U \cap V),$$

defined by

$$i_k(\sigma) = ((i_U^*)_k \, \sigma, (i_V^*)_k \, \sigma) = \left(\sigma|_U, \sigma|_V\right); \tag{18.69}$$

$$j_k(\omega, \tau) = (j_V^*)_k \, \tau - (j_U^*)_k \, \omega = \tau|_{U\cap V} - \omega|_{U\cap V}. \tag{18.70}$$

If $U \cap V$ is empty, then we define $\Omega^k(U \cap V) = 0$, and in this case $j_k : \Omega^k(U) \oplus \Omega^k(V) \to \Omega^k(U \cap V)$ is simply the zero map. We call i_k the **restriction map** and j_k the **difference map**. The exterior derivative $\widetilde{d}$ on $\Omega^*(U) \oplus \Omega^*(V)$ is given by

$$\widetilde{\mathrm{d}}(\omega, \tau) = (\mathrm{d}_U \omega, \mathrm{d}_V \tau), \tag{18.71}$$

for $\omega \in \Omega^k(U)$ and $\tau \in \Omega^k(V)$, where d_U and d_V are exterior derivative operators on the open subsets U and V, respectively. We can interpret $\Omega^k(U) \oplus \Omega^k(V)$ as $\Omega^k(U \sqcup V)$, where $U \sqcup V$ is the disjoint union of U and V.

Proposition 18.5

Both the restriction map i_k and the difference map j_k commute with exterior derivatives, i.e. $\{i_k\}$ and $\{j_k\}$ are cochain maps.

Proof. Consider the pullback maps $(i_U^*)_k : \Omega^k(M) \to \Omega^k(U)$ and $(i_V^*)_k : \Omega^k(M) \to \Omega^k(V)$. Since exterior derivative commutes with pullback, the following diagrams commute:

$$\begin{tikzcd}
\Omega^k(M) \arrow[r, "(i_U^*)_k"] \arrow[d, "\mathrm{d}"'] & \Omega^k(U) \arrow[d, "\mathrm{d}_U"] \\
\Omega^{k+1}(M) \arrow[r, "(i_U^*)_{k+1}"'] & \Omega^{k+1}(U)
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega^k(M) \arrow[r, "(i_V^*)_k"] \arrow[d, "\mathrm{d}"'] & \Omega^k(V) \arrow[d, "\mathrm{d}_V"] \\
\Omega^{k+1}(M) \arrow[r, "(i_V^*)_{k+1}"'] & \Omega^{k+1}(V)
\end{tikzcd}$$

In other words,

$$\mathrm{d}_U \circ (i_U^*)_k = (i_U^*)_{k+1} \circ \mathrm{d}, \text{ and } \mathrm{d}_V \circ (i_V^*)_k = (i_V^*)_{k+1} \circ \mathrm{d}. \tag{18.72}$$

Now, for $\sigma \in \Omega^k(M)$,

$$\begin{aligned}
\left(\widetilde{\mathrm{d}} \circ i_k\right) \sigma &= \widetilde{\mathrm{d}}\left((i_U^*)_k \sigma, (i_V^*)_k \sigma\right) \\
&= \left(\mathrm{d}_U (i_U^*)_k \sigma, \mathrm{d}_V (i_V^*)_k \sigma\right) \\
&= \left((i_U^*)_{k+1} \mathrm{d}\sigma, (i_V^*)_{k+1} \mathrm{d}\sigma\right) \\
&= (i_{k+1} \circ \mathrm{d})(\sigma).
\end{aligned}$$

Therefore,

$$\widetilde{\mathrm{d}} \circ i_k = i_{k+1} \circ \mathrm{d}. \tag{18.73}$$

Again, from the commutativity of pullback with exterior derivative operator, we have the following commutative diagrams:

$$\begin{tikzcd}
\Omega^k(U) \arrow[r, "(j_U^*)_k"] \arrow[d, "\mathrm{d}_U"'] & \Omega^k(U \cap V) \arrow[d, "\mathrm{d}_{U\cap V}"] \\
\Omega^{k+1}(U) \arrow[r, "(j_U^*)_{k+1}"'] & \Omega^{k+1}(U \cap V)
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega^k(V) \arrow[r, "(j_V^*)_k"] \arrow[d, "\mathrm{d}_V"'] & \Omega^k(U \cap V) \arrow[d, "\mathrm{d}_{U\cap V}"] \\
\Omega^{k+1}(V) \arrow[r, "(j_V^*)_{k+1}"'] & \Omega^{k+1}(U \cap V)
\end{tikzcd}$$

In other words,

$$\mathrm{d}_{U\cap V} \circ (j_U^*)_k = (j_U^*)_{k+1} \circ \mathrm{d}_U, \text{ and } \mathrm{d}_{U\cap V} \circ (j_V^*)_k = (j_V^*)_{k+1} \circ \mathrm{d}_V. \tag{18.74}$$

Now, for $(\omega, \tau) \in \Omega^k(U) \oplus \Omega^k(V)$,

$$\begin{aligned}
(\mathrm{d}_{U\cap V} \circ j_k)(\omega, \tau) &= \mathrm{d}_{U\cap V}\left((j_V^*)_k \tau - (j_U^*)_k \omega\right) \\
&= \mathrm{d}_{U\cap V} (j_V^*)_k \tau - \mathrm{d}_{U\cap V} (j_U^*)_k \omega \\
&= (j_V^*)_{k+1} \mathrm{d}_V \tau - (j_U^*)_{k+1} \mathrm{d}_U \omega \\
&= j_k (\mathrm{d}_U \omega, \mathrm{d}_V \tau) \\
&= \left(j_k \circ \widetilde{\mathrm{d}}\right)(\omega, \tau).
\end{aligned}$$

Therefore,

$$\mathrm{d}_{U\cap V} \circ j_k = j_k \circ \widetilde{\mathrm{d}}. \tag{18.75}$$

∎

Proposition 18.6

For each $k \geq 0$, the sequence

$$0 \longrightarrow \Omega^k(M) \xrightarrow{i_k} \Omega^k(U) \oplus \Omega^k(V) \xrightarrow{j_k} \Omega^k(U \cap V) \longrightarrow 0 \tag{18.76}$$

is exact.

Proof. The above sequence can easily be seen to be exact at $\Omega^k(M)$ by noticing that $\operatorname{Ker} i_k = 0$. Indeed, if

$$i_k(\sigma) = \left(\sigma|_U, \sigma|_V\right) = (0, 0), \tag{18.77}$$

then $\sigma = 0$ on $U \cup V = M$. Therefore, $\operatorname{Ker} i_k = 0$.

Now we are going to prove that $\operatorname{im} i_k = \operatorname{Ker} j_k$. Let's take $(\omega, \tau) \in \operatorname{Ker} j_k \subseteq \Omega^k(U) \oplus \Omega^k(V)$. Then

$$0 = j_k(\omega, \tau) = \tau|_{U\cap V} - \omega|_{U\cap V}. \tag{18.78}$$

So ω and τ agree on $U \cap V$. So we can define $\sigma \in \Omega^k(M)$ as

$$\sigma_p = \begin{cases} \omega_p & \text{if } p \in U, \\ \tau_p & \text{if } p \in V. \end{cases} \tag{18.79}$$

σ is well-defined since ω and τ agree on $U \cap V$. Then we have

$$i_k(\sigma) = \left(\sigma|_U, \sigma|_V\right) = (\omega, \tau). \tag{18.80}$$

So $(\omega, \tau) \in \operatorname{im} i_k$, proving

$$\operatorname{Ker} j_k \subseteq \operatorname{im} i_k. \tag{18.81}$$

On the other hand, for any $\sigma \in \Omega^k(M)$,

$$j_k(i_k\sigma) = j_k\left(\sigma|_U, \sigma|_V\right) = \left(\sigma|_V\right)|_{U\cap V} - \left(\sigma|_U\right)|_{U\cap V} = \sigma|_{U\cap V} - \sigma|_{U\cap V} = 0. \tag{18.82}$$

Therefore,

$$\operatorname{im} i_k \subseteq \operatorname{Ker} j_k. \tag{18.83}$$

From (18.81) and (18.83), we have

$$\operatorname{im} i_k = \operatorname{Ker} j_k. \tag{18.84}$$

Now we are only left to prove that $j_k : \Omega^k(U) \oplus \Omega^k(V) \to \Omega^k(U \cap V)$ is surjective. Let $\{\rho_U, \rho_V\}$ be a partition of unity subordinate to the open cover $\{U, V\}$ of M. Suppose $\omega \in \Omega^k(U \cap V)$. Then we define $\omega_U \in \Omega^k(U)$ and $\omega_V \in \Omega^k(V)$ as follows:

$$\omega_U = \begin{cases} \rho_V\omega & \text{on } U \cap V, \\ 0 & \text{on } U \setminus (U \cap V). \end{cases} \qquad \omega_V = \begin{cases} \rho_U\omega & \text{on } U \cap V, \\ 0 & \text{on } V \setminus (U \cap V). \end{cases} \tag{18.85}$$

ω_U is called the **extension by zero** of $\rho_V\omega$ from $U \cap V$ to U; and similarly, ω_V is called the extension by zero of $\rho_U\omega$ from $U \cap V$ to V. We now need to show that ω_U and ω_V are smooth.

Clearly, ω_U is smooth on $U \cap V$. Suppose $q \in U \setminus (U \cap V) = U \setminus V$. Since $\operatorname{supp} \rho_V \subseteq V$, $q \in U \setminus \operatorname{supp} \rho_V$. Since $\operatorname{supp} \rho_V$ is closed, $U \setminus \operatorname{supp} \rho_V$ is open. So we can find a coordinate neighborhood (W, φ) about q such that $W \subseteq U \setminus \operatorname{supp} \rho_V$. Now, since W is disjoint from ρ_V, $\omega_U = 0$ on W. Therefore, ω_U is smooth on W. In particular, ω_U is smooth at $q \in U \setminus (U \cap V)$. Since $q \in U \setminus (U \cap V)$ is arbitrary, ω_U is smooth on all of $U \setminus (U \cap V)$. Therefore, ω_U is smooth. Similarly, ω_V is also smooth.

Now, since ω_U and ω_V are smooth, $\omega_U \in \Omega^k(U)$ and $\omega_V \in \Omega^k(V)$. Now,

$$j_k(-\omega_U, \omega_V) = \omega_V|_{U\cap V} + \omega_U|_{U\cap V} = \rho_U\omega + \rho_V\omega = \omega. \tag{18.86}$$

Therefore, j_k is surjective. Hence, (18.76) is a short exact sequence. ■

Lemma 18.7

The k-th cohomology vector space of $U \sqcup V$ is isomorphic to $H^k(U) \oplus H^k(V)$.

Proof. The cochain complex $\Omega^*(U) \oplus \Omega^*(V)$ is

$$\cdots \longrightarrow \Omega^{k-1}(U) \oplus \Omega^{k-1}(V) \xrightarrow{\widetilde{\mathrm{d}}_{k-1}} \Omega^{k}(U) \oplus \Omega^{k}(V) \xrightarrow{\widetilde{\mathrm{d}}_{k}} \Omega^{k+1}(U) \oplus \Omega^{k+1}(V) \longrightarrow \cdots$$

k-th cohomology vector space of the cochain complex $\Omega^{*}(U) \oplus \Omega^{*}(V)$ is

$$H^{k}(U \sqcup V) = \frac{\operatorname{Ker} \widetilde{\mathrm{d}}_{k}}{\operatorname{im} \widetilde{\mathrm{d}}_{k-1}}. \tag{18.87}$$

For $\omega \in \Omega^{k}(U)$ and $\tau \in \Omega^{k}(V)$, $\widetilde{\mathrm{d}}_{k}(\omega, \tau) = ((\mathrm{d}_U)_k \omega, (\mathrm{d}_V)_k \tau)$. So

$$\begin{aligned}(\omega, \tau) \in \operatorname{Ker} \widetilde{\mathrm{d}}_{k} &\iff (\mathrm{d}_U)_k \omega = (\mathrm{d}_V)_k \tau = 0 \\ &\iff \omega \in \operatorname{Ker}(\mathrm{d}_U)_k \text{ and } \tau \in \operatorname{Ker}(\mathrm{d}_V)_k \\ &\iff (\omega, \tau) \in \operatorname{Ker}(\mathrm{d}_U)_k \oplus \operatorname{Ker}(\mathrm{d}_V)_k .\end{aligned}$$

Therefore,

$$\operatorname{Ker} \widetilde{\mathrm{d}}_{k} = \operatorname{Ker}(\mathrm{d}_U)_k \oplus \operatorname{Ker}(\mathrm{d}_V)_k \tag{18.88}$$

Again,

$$\begin{aligned}(\omega, \tau) \in \operatorname{im} \widetilde{\mathrm{d}}_{k-1} &\iff (\omega, \tau) = \widetilde{\mathrm{d}}_{k-1}(\alpha, \beta) \text{ for some } (\alpha, \beta) \in \Omega^{k-1}(U) \oplus \Omega^{k-1}(V) \\ &\iff \omega = (\mathrm{d}_U)_{k-1} \alpha \text{ and } \tau = (\mathrm{d}_V)_{k-1} \beta, \text{ for some } \alpha \in \Omega^{k-1}(U), \ \beta \in \Omega^{k-1}(V) \\ &\iff \omega \in \operatorname{im}(\mathrm{d}_U)_{k-1} \text{ and } \tau \in \operatorname{im}(\mathrm{d}_V)_{k-1} \\ &\iff (\omega, \tau) \in \operatorname{im}(\mathrm{d}_U)_{k-1} \oplus \operatorname{im}(\mathrm{d}_V)_{k-1} .\end{aligned}$$

Therefore,

$$\operatorname{im} \widetilde{\mathrm{d}}_{k-1} = \operatorname{im}(\mathrm{d}_U)_{k-1} \oplus \operatorname{im}(\mathrm{d}_V)_{k-1} . \tag{18.89}$$

As a result,

$$H^{k}(U \sqcup V) = \frac{\operatorname{Ker} \widetilde{\mathrm{d}}_{k}}{\operatorname{im} \widetilde{\mathrm{d}}_{k-1}} = \frac{\operatorname{Ker}(\mathrm{d}_U)_k \oplus \operatorname{Ker}(\mathrm{d}_V)_k}{\operatorname{im}(\mathrm{d}_U)_{k-1} \oplus \operatorname{im}(\mathrm{d}_V)_{k-1}}. \tag{18.90}$$

Let us now consider the surjective linear map $\psi : \operatorname{Ker}(\mathrm{d}_U)_k \oplus \operatorname{Ker}(\mathrm{d}_V)_k \to H^k(U) \oplus H^k(V)$ defined by

$$\psi(\omega, \tau) = ([\omega], [\tau]) . \tag{18.91}$$

Now,

$$\begin{aligned}(\omega, \tau) \in \operatorname{Ker} \psi &\iff [\omega] = [0] \in H^k(U) \text{ and } [\tau] = [0] \in H^k(V) \\ &\iff \omega = \mathrm{d}_U \alpha \text{ and } \tau = \mathrm{d}_V \beta, \text{ for some } \alpha \in \Omega^{k-1}(U), \ \beta \in \beta \in \Omega^{k-1}(V) \\ &\iff (\omega, \tau) \in \operatorname{im}(\mathrm{d}_U)_{k-1} \oplus \operatorname{im}(\mathrm{d}_V)_{k-1} .\end{aligned}$$

Therefore, $\operatorname{Ker} \psi = \operatorname{im}(\mathrm{d}_U)_{k-1} \oplus \operatorname{im}(\mathrm{d}_V)_{k-1}$. Hence, by the first isomorphism theorem, we have the following isomorphism of vector spaces:

$$H^k(U) \oplus H^k(V) = \operatorname{im} \psi \cong \frac{\operatorname{Ker}(\mathrm{d}_U)_k \oplus \operatorname{Ker}(\mathrm{d}_V)_k}{\operatorname{Ker} \psi} = \frac{\operatorname{Ker}(\mathrm{d}_U)_k \oplus \operatorname{Ker}(\mathrm{d}_V)_k}{\operatorname{im}(\mathrm{d}_U)_{k-1} \oplus \operatorname{im}(\mathrm{d}_V)_{k-1}}. \tag{18.92}$$

So $H^k(U \sqcup V)$ is isomorphic to $H^k(U) \oplus H^k(V)$. ■

Theorem 18.8 (The Mayer–Vietoris Sequence)
Let U and V be open subsets of M such that $U \cup V = M$. Then there is a long exact sequence in cohomology:

$$\cdots \longrightarrow H^k(M) \longrightarrow H^k(U) \oplus H^k(V) \longrightarrow H^k(U \cap V) \longrightarrow H^{k+1}(M) \longrightarrow \cdots$$

called the **Mayer–Vietoris sequence**.

Proof. By Proposition 18.6, we have a short exact sequence of cochain complexes:

$$0 \longrightarrow \Omega^*(M) \xrightarrow{i} \Omega^*(U) \oplus \Omega^*(V) \xrightarrow{j} \Omega^*(U \cap V) \longrightarrow 0. \tag{18.93}$$

Then by The Zig-Zag Lemma, (18.93) induces a long exact sequence in cohomology:

$$\begin{tikzcd} H^{k+1}(M) \arrow[r, "(i^{\#})^{k+1}"] & \cdots & \\ H^k(M) \arrow[r, "(i^{\#})^k"] & H^k(U) \oplus H^k(V) \arrow[r, "(j^{\#})^k"'] & H^k(U \cap V) \arrow[ull, "(\mathrm{d}^{\#})_k"'] \\ & \cdots \arrow[r, "(j^{\#})^{k-1}"'] & H^{k-1}(U \cap V), \arrow[ull, "(\mathrm{d}^{\#})_{k-1}"'] \end{tikzcd} \tag{18.94}$$

■

In this sequence (18.94), $\left(i^{\#}\right)^k$ and $\left(j^{\#}\right)^k$ are induced from i_k and j_k:

$$\left(i^{\#}\right)^k [\sigma] = [i_k(\sigma)] = \left(\left[\sigma|_U\right], \left[\sigma|_V\right]\right), \tag{18.95}$$

$$\left(j^{\#}\right)^k ([\omega], [\tau]) = [j_k(\omega, \tau)] = \left[\tau|_{U\cap V} - \omega|_{U\cap V}\right]. \tag{18.96}$$

The connecting homomorphism $\left(\mathrm{d}^{\#}\right)^k : H^k(U \cap V) \to H^{k+1}(M)$ is cooked up in 3 steps using the same recipe as we did in § 18.3.

$$\begin{tikzcd} \Omega^{k+1}(M) \arrow[r, tail, "i_{k+1}"] & \Omega^{k+1}(U) \oplus \Omega^{k+1}(V) & \\ & \Omega^k(U) \oplus \Omega^k(V) \arrow[u, "\widetilde{\mathrm{d}}"] \arrow[r, two heads, "j_k"] & \Omega^k(U \cap V) \end{tikzcd}$$

$$\begin{tikzcd} \alpha \arrow[r, tail, "i_{k+1}"] & (-\mathrm{d}_U \xi_U, \mathrm{d}_V \xi_V) \arrow[r, "j_{k+1}"] & 0 \\ & (-\xi_U, \xi_V) \arrow[u, "\widetilde{\mathrm{d}}"] \arrow[r, two heads, "j_k"] & \xi \arrow[u, "\mathrm{d}_{U\cap V}"'] \end{tikzcd}$$

(i) We start with a closed k-form $\xi \in \Omega^k(U \cap V)$ and using a partition of unity $\{\rho_U, \rho_V\}$ subordinate to the open cover $\{U, V\}$, one can extend $\rho_U \xi$ by zero from $U \cap V$ to a k-form ξ_V on V, and extend $\rho_V \xi$ by zero from $U \cap V$ to a k-form ξ_U on U. Then

$$j_k(-\xi_U, \xi_V) = \xi_V|_{U\cap V} + \xi_U|_{U\cap V} = \rho_U \xi + \rho_V \xi = \xi. \tag{18.97}$$

(ii) Since $\{j_k\}$ is a cochain map, the following square commutes:

$$\begin{tikzcd} \Omega^{k+1}(U) \oplus \Omega^{k+1}(V) \arrow[r, "j_{k+1}"] & \Omega^{k+1}(U \cap V) \\ \Omega^k(U) \oplus \Omega^k(V) \arrow[u, "\widetilde{\mathrm{d}}"] \arrow[r, "j_k"'] & \Omega^k(U \cap V) \arrow[u, "\mathrm{d}_{U\cap V}"'] \end{tikzcd}$$

Hence,

$$\begin{aligned} j_{k+1}\left(-\mathrm{d}_U\xi_U, \mathrm{d}_V\xi_V\right) &= \left(j_{k+1}\circ\widetilde{\mathrm{d}}\right)\left(-\xi_U,\xi_V\right) = \left(\mathrm{d}_{U\cap V}\circ j_k\right)\left(-\xi_U,\xi_V\right) \\ &= \mathrm{d}_{U\cap V}\xi = 0. \end{aligned}$$

(iii) So $\left(-\mathrm{d}_U\xi_U, \mathrm{d}_V\xi_V\right) \in \operatorname{Ker} j_{k+1} = \operatorname{im} i_{k+1}$. Therefore, $-\mathrm{d}_U\xi_U$ on U and $\mathrm{d}_V\xi_V$ on V patch together to yield a global $(k+1)$-form $\alpha \in \Omega^{k+1}(M)$ such that

$$i_{k+1}(\alpha) = \left(-\mathrm{d}_U\xi_U, \mathrm{d}_V\xi_V\right). \tag{18.98}$$

Since $\{i_k\}$ is a chain map, the following square commutes:

$$\begin{tikzcd} \Omega^{k+2}(M) \arrow[r, tail, "i_{k+2}"] & \Omega^{k+2}(U)\oplus\Omega^{k+2}(V) \\ \Omega^{k+1}(M) \arrow[r, "i_{k+1}"'] \arrow[u, "\mathrm{d}"] & \Omega^{k+1}(U)\oplus\Omega^{k+1}(V) \arrow[u, "\widetilde{\mathrm{d}}"'] \end{tikzcd}$$

So we have

$$i_{k+2}(\mathrm{d}\alpha) = \widetilde{\mathrm{d}}\left(i_{k+1}\alpha\right) = \widetilde{\mathrm{d}}\left(-\mathrm{d}_U\xi_U, \mathrm{d}_V\xi_V\right) = (0,0).$$

Since i_{k+2} is injective, $\mathrm{d}\alpha = 0$, i.e. $\alpha \in \Omega^{k+1}(M)$ is also a closed form. So we define

$$\left(\mathrm{d}^\#\right)^k[\xi] = [\alpha] \in H^{k+1}(M). \tag{18.99}$$

Since $\Omega^k(M) = 0$ for $k<0$, The Mayer–Vietoris Sequence starts with

$$0 \longrightarrow H^0(M) \longrightarrow H^0(U)\oplus H^0(V) \longrightarrow H^0(U\cap V) \longrightarrow H^1(M) \longrightarrow \cdots$$

Proposition 18.9

In The Mayer–Vietoris Sequence, if U, V, and $U\cap V$ are connected and nonempty, then

(i) M is connected, and

$$0 \longrightarrow H^0(M) \xrightarrow{\left(i^\#\right)^0} H^0(U)\oplus H^0(V) \xrightarrow{\left(j^\#\right)^0} H^0(U\cap V) \longrightarrow 0$$

is exact;

(ii) we may start The Mayer–Vietoris Sequence with

$$0 \longrightarrow H^1(M) \xrightarrow{\left(i^\#\right)^1} H^1(U)\oplus H^1(V) \xrightarrow{\left(j^\#\right)^1} H^1(U\cap V) \xrightarrow{\left(\mathrm{d}^\#\right)^1} H^2(M) \longrightarrow \cdots$$

Proof. (i) The connectedness of M follows from the connectedness of U and V and that U and V are not disjoint using point set topological argument. But let us try to deduce it using The Mayer–Vietoris Sequence.

On a nonempty connected open set, the de Rham cohomology in dimension 0 is simply the vector space of constant functions (Proposition 17.1). The constant functions are characterized by real numbers. Additionally, if $u \in \mathbb{R}$ represents a constant function on U, then on $U\cap V$, it is the same constant function u, i.e. $u|_{U\cap V} = u$[1]. Therefore, the map $\left(j^\#\right)^0 : H^0(U)\oplus H^0(V) \to H^0(U\cap V)$ is given by

$$\left(j^\#\right)^0(u,v) = v|_{U\cap V} - u|_{U\cap V} = v - u. \tag{18.100}$$

Clearly, $\left(j^\#\right)^0$ is surjective.

[1]Here we are abusing the notation by denoting the constant function and its value, which is a real number, by the same symbol u.

$$0 \longrightarrow H^0(M) \xrightarrow{(i^\#)^0} H^0(U) \oplus H^0(V) \xrightarrow{(j^\#)^0} H^0(U \cap V) \xrightarrow{(\mathrm{d}^\#)^0} H^1(M) \longrightarrow \cdots$$

Surjectivity of $\left(j^\#\right)^0$ implies that im $\left(j^\#\right)^0 = H^0(U \cap V)$. Exactness of the Mayer–Vietoris sequence above implies

$$\operatorname{Ker}\left(\mathrm{d}^\#\right)^0 = \operatorname{im}\left(j^\#\right)^0 = H^0(U \cap V). \tag{18.101}$$

So $\left(\mathrm{d}^\#\right)^0 : H^0(U \cap V) \to H^1(M)$ is the zero map. Thusm the Mayer–Vietoris sequence starts with

$$0 \longrightarrow H^0(M) \xrightarrow{(i^\#)^0} \mathbb{R} \oplus \mathbb{R} \xrightarrow{(j^\#)^0} \mathbb{R} \xrightarrow{(\mathrm{d}^\#)^0} 0 \tag{18.102}$$

The above sequence is short exact, sice the Mayer–Vietoris sequence is exact. Exactness at $H^0(M)$ implies Ker $\left(i^\#\right)^0 = 0$, i.e. $\left(i^\#\right)^0$ is injective. Therefore, by the first isomorphism theorem,

$$\operatorname{im}\left(i^\#\right)^0 \cong \frac{H^0(M)}{\operatorname{Ker}\left(i^\#\right)^0} = H^0\left(M\right). \tag{18.103}$$

Exactness of (18.102) at $\mathbb{R} \oplus \mathbb{R}$ implies

$$\operatorname{im}\left(i^\#\right)^0 = \operatorname{Ker}\left(j^\#\right)^0. \tag{18.104}$$

$\left(j^\#\right)^0 (u, v)\,(u, v) = v - u$, so

$$\operatorname{Ker}\left(j^\#\right)^0 = \{(u, v) \in \mathbb{R} \oplus \mathbb{R} \mid v - u = 0\} = \{(u, u) \in \mathbb{R} \oplus \mathbb{R}\} \cong \mathbb{R}. \tag{18.105}$$

Therefore, combining (18.103), (18.104) and (18.105), we get

$$H^0\left(M\right) \cong \operatorname{im}\operatorname{im}\left(i^\#\right)^0 = \operatorname{Ker}\left(j^\#\right)^0 \cong \mathbb{R}. \tag{18.106}$$

So $H^0(M) \cong \mathbb{R}$, i.e. M is connected.

(ii) We have deduced earlier that $\left(\mathrm{d}^\#\right)^0 : H^0(U \cap V) \to H^1(M)$ is the zero map. Thus, in the Mayer–Vietoris sequence, the sequence of the following two maps

$$H^0(U \cap V) \xrightarrow{(\mathrm{d}^\#)^0} H^1(M) \xrightarrow{(i^\#)^1} H^1(U) \oplus H^1(V)$$

may be replaced by

$$0 \longrightarrow H^1(M) \xrightarrow{(i^\#)^1} H^1(U) \oplus H^1(V)$$

without affecting exactness. In other words, no information of the Mayer–Vietoris sequence is lost if we have

$$0 \longrightarrow H^0(M) \xrightarrow{(i^\#)^0} H^0(U) \oplus H^0(V) \xrightarrow{(j^\#)^0} H^0(U \cap V) \longrightarrow 0$$

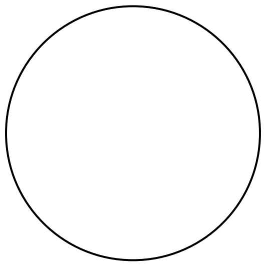
to be short exact, and we have a long exact sequence as follows:

$$0 \longrightarrow H^1(M) \xrightarrow{(i^\#)^1} H^1(U) \oplus H^1(V) \xrightarrow{(j^\#)^1} H^1(U \cap V) \xrightarrow{(\mathrm{d}^\#)^1} H^2(M) \longrightarrow \cdots$$

■

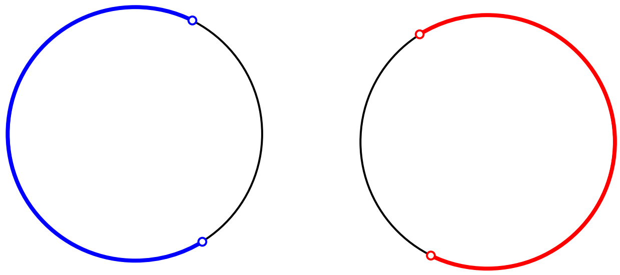
Example 18.3 (Cohomology of circle using Mayer–Vietoris sequence)**.** Let S^1 be the circle in $\mathbb{R}^2$,

$$S^1 = \left\{\mathbf{x} \in \mathbb{R}^2 \mid \|\mathbf{x}\| = 1\right\}.$$



S^1

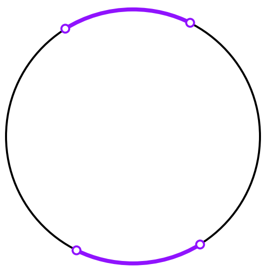
Since S^1 is connected, by Proposition 17.1, $H^0(S^1) = \mathbb{R}$. Since S^1 is a one-dimensional manifold, by Proposition 17.2, $H^k(S^1) = 0$ for $k \geq 2$. Now we want to compute $H^1(S^1)$ using The Mayer–Vietoris Sequence. We cover S^1 by open sets U and V as follows:



U V

U and V are open arcs on the circle S^1. Their intersection is the following (disjoint) union of two open arcs A and B:

A



B

Open arcs are diffeomorphic to open intervals on $\mathbb{R}$, which are diffeomorphic to the whole $\mathbb{R}$. Therefore, by diffeomorphism invariance,

$$H^1(U) \cong H^1(V) \cong H^1(A) \cong H^1(B) \cong H^1(\mathbb{R}) = 0. \tag{18.107}$$

As a result,

$$H^1(U \cap V) = H^1(A \sqcup B) \cong H^1(A) \oplus H^1(B) = 0. \tag{18.108}$$

Furthermore,

$$H^1(U) \oplus H^1(V) = 0. \tag{18.109}$$

In dimension 0, since open arcs are diffeomorphic to $\mathbb{R}$,

$$H^0(U) \cong H^0(V) \cong H^0(A) \cong H^0(B) \cong H^0(\mathbb{R}) = \mathbb{R}. \tag{18.110}$$

As a result,

$$H^0(U \cap V) = H^0(A \sqcup B) \cong H^0(A) \oplus H^0(B) = \mathbb{R} \oplus \mathbb{R}. \tag{18.111}$$

Furthermore,

$$H^0(U) \oplus H^1(V) = \mathbb{R} \oplus \mathbb{R}. \tag{18.112}$$

So, the Mayer–Vietoris sequence

$$\begin{tikzcd}
 & H^1\left(S^1\right) \arrow[r, "\left(i^{\#}\right)^1"] & H^1\left(U\right) \oplus H^1\left(V\right) \arrow[r] & \cdots \\
0 \arrow[r] & H^0\left(S^1\right) \arrow[r, "\left(i^{\#}\right)^0"'] & H^0\left(U\right) \oplus H^0\left(V\right) \arrow[r, "\left(j^{\#}\right)^0"'] & H^0\left(U \cap V\right) \arrow[ull, "\left(\mathrm{d}^{\#}\right)^0"']
\end{tikzcd}$$

becomes

$$0 \longrightarrow \mathbb{R} \xrightarrow{\left(i^{\#}\right)^0} \mathbb{R} \oplus \mathbb{R} \xrightarrow{\left(j^{\#}\right)^0} \mathbb{R} \oplus \mathbb{R} \xrightarrow{\left(\mathrm{d}^{\#}\right)^0} H^1\left(S^1\right) \longrightarrow 0. \tag{18.113}$$

By the rank-nullity theorem, if $f : V_1 \to V_2$ is a linear map between finite dimensional vector spaces, then

$$\dim V_1 = \operatorname{rank} f + \operatorname{nullity} f = \dim \operatorname{im} f + \dim \operatorname{Ker} f. \tag{18.114}$$

Since (18.113) is exact, $\left(i^{\#}\right)^0$ is injective. So $\dim \operatorname{Ker} \left(i^{\#}\right)^0 = 0$. Hence, $\dim \operatorname{im} \left(i^{\#}\right)^0 = \dim \mathbb{R} = 1$. As a result,

$$\dim \operatorname{Ker} \left(j^{\#}\right)^0 = \dim \operatorname{im} \left(i^{\#}\right)^0 = 1. \tag{18.115}$$

So we have

$$\dim \operatorname{im} \left(j^{\#}\right)^0 = \dim \left(\mathbb{R} \oplus \mathbb{R}\right) - \dim \operatorname{Ker} \left(j^{\#}\right)^0 = 1. \tag{18.116}$$

Since $\operatorname{Ker} \left(\mathrm{d}^{\#}\right)^0 = \operatorname{im} \left(j^{\#}\right)^0$, we have $\dim \operatorname{Ker} \left(\mathrm{d}^{\#}\right)^0 = 1$. By the exactness of (18.113), $\left(\mathrm{d}^{\#}\right)^0$ is surjective. Hence,

$$\dim H^1\left(S^1\right) = \dim \operatorname{im} \left(\mathrm{d}^{\#}\right)^0 = \dim \left(\mathbb{R} \oplus \mathbb{R}\right) - \dim \operatorname{Ker} \left(\mathrm{d}^{\#}\right)^0 = 1. \tag{18.117}$$

So $H^1\left(S^1\right) \cong \mathbb{R}$. Therefore,

$$H^k(S^1) \cong \begin{cases} \mathbb{R} & \text{for } k = 0, 1 \\ 0 & \text{for } k \geq 2. \end{cases} \tag{18.118}$$

III

Appendix

A Topology Review

§A.1 Euclidean Space $\mathbb{R}^n$

Before embarking on the concept of general topological space, let us look at the Euclidean space $\mathbb{R}^n$. $\mathbb{R}^n$ is equipped with the notion of distance between 2 points p and q.

Definition A.1 (Distance)**.** Let the coordinates of p and q be $(p^1, p^2, ...p^n)$ and $(q^1, q^2, ...q^n)$, respectively. The distance between p and q is given by

$$d(p,q) = \left[\sum_{i=1}^{n} (p^i - q^i)^2\right]^{\frac{1}{2}}$$

Definition A.2 (Open ball)**.** An open ball $B(p,r)$ in $\mathbb{R}^n$ with center $p \in \mathbb{R}^n$ and radius $r > 0$ is defined as the set

$$B(p,r) = \{x \in \mathbb{R}^n \ : \ d(x,p) < r\}$$

A set equipped with the notion of distance between its elements is called a metric space[1]. Thus the Euclidean space $\mathbb{R}^n$ is a metric space. And we can talk about open balls in $\mathbb{R}^n$ using this metric. We can define open sets in $\mathbb{R}^n$ using open balls $B(p,r)$ defined above.

Definition A.3 (Open Set in $\mathbb{R}^n$)**.** A set U in $\mathbb{R}^n$ is said to be open if for every p in U, there is an open ball $B(p,r)$ such that $B(p,r) \subseteq U$.

Proposition A.1

The union of an arbitrary collection of $\{U_\alpha\}$ of open sets is open. The intersection of finite collection of open sets is open.

Proof. Trivial. ∎

Example A.1. The intervals $\left(-\frac{1}{n}, \frac{1}{n}\right)$, $n = 1, 2, 3, ...$ are all open in $\mathbb{R}$ but their intersection

$$\bigcap_{n \in \mathbb{N}} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$$

is not open.

The metric d in $\mathbb{R}^n$ allows us to define open sets in $\mathbb{R}^n$. In other words, given a subset of $\mathbb{R}^n$, we can tell if it is open or not. This situation is a special case called **metric topology in** $\mathbb{R}^n$.

§A.2 Topology

Definition A.4 (Topology)**.** A topology on a set S is a collection $\mathcal{T}$ of subsets of S containing both the empty set $\varnothing$ and the S such that $\mathcal{T}$ is closed under arbitrary union and finite intersection. In other words,

- If $U_\alpha \in \mathcal{T}$ for all α in an index set A, then $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}$

[1] There are some properties that a metric (distance) function should have. We won't go into much details

- If $U_i \in \mathcal{T}$ for $i \in \{1, 2, ..., n\}$, then $\bigcap_{i=1}^{n} U_i \in \mathcal{T}$

The elements of $\mathcal{T}$ are called open sets.

Definition A.5 (Topological Space). The pair $(S, \mathcal{T})$ consisting of a set S together with a topology $\mathcal{T}$ on S is called a **topological space**.

Abuse of Notation. We shall often say "S is a topological space" in short. But there is always a topology $\mathcal{T}$ on S, which we recall when necessary.

Definition A.6 (Neighborhood). A **neighbourhood** of a point $p \in S$ is called an open set U containing p.

Definition A.7 (Closed Set). The complement of an open set is called a **closed set**.

Proposition A.2

The union of a finite collection of closed sets is closed. The intersection of an arbitrary collection of closed sets is closed.

Proof. Let $\{F_i\}_{i=1}^{n}$ be a finite collection of closed sets. Then, $\{S \setminus F_i\}_{i=1}^{n}$ is a finite collection of open sets. The intersection of a finite collection of open sets is open, therefore $\bigcap_{i=1}^{n} (S \setminus F_i)$ is open. By De Morgan's law,

$$\bigcap_{i=1}^{n} (S \setminus F_i) = S \setminus \left(\bigcup_{i=1}^{n} F_i \right) \quad \text{is open} \implies \bigcup_{i=1}^{n} F_i \quad \text{is closed}$$

Therefore, the union of a finite collection of closed sets is closed.
Now, let $\{F_\alpha\}_{\alpha \in A}$ be an arbitrary collection of closed sets with A being an index set. Then $\{S \setminus F_\alpha\}_{\alpha \in A}$ is an arbitrary collection of open sets. We know that the union of an arbitrary collection of open sets is open, therefore $\bigcup_{\alpha \in A} (S \setminus F_\alpha)$ is open. By De Morgan's law,

$$\bigcup_{\alpha \in A} (S \setminus F_\alpha) = S \setminus \left(\bigcap_{\alpha \in A} F_\alpha \right) \quad \text{is open} \implies \bigcap_{\alpha \in A} F_\alpha \quad \text{is closed}$$

Therefore, the intersection of an arbitrary collection of closed sets is closed. ∎

Definition A.8 (Subspace Topology). Let $(S, \mathcal{T})$ be a topological space and A a subset of S. Define $\mathcal{T}_A$ to be the collection of subsets

$$\mathcal{T}_A = \{U \cap A \mid U \in \mathcal{T}\}$$

$\mathcal{T}_A$ is called the **subspace topology** of A in S.

It is not hard to see that $\mathcal{T}_A$ satisfies the conditions of a Topology. Firstly, $\mathcal{T}_A$ contains both $\varnothing$ and A. For these, taking $U = \varnothing$ and $U = S$, respecitvely, suffices. By the distributive property of union and intersection

$$\bigcup_{\alpha} (U_\alpha \cap A) = \left(\bigcup_{\alpha} U_\alpha \right) \cap A \text{ and } \bigcap_{i=1}^{n} (U_i \cap A) = \left(\bigcap_{i=1}^{n} U_i \right) \cap A$$

which shows that $\mathcal{T}_A$ is closed under arbitrary union and finite intersection. So $\mathcal{T}_A$ is a Topology indeed.

Example A.2. Consider the subset $A = [0, 1]$ of $\mathbb{R}$. In the subspace topology, the half-open interval $\left[0, \frac{1}{2}\right)$ is an open subset of A, because $\left[0, \frac{1}{2}\right) = \left(-\frac{1}{2}, \frac{1}{2}\right) \cap [0, 1]$

Lemma A.3

Let Y be a subspace of X (that is Y has the subspace topology inherited from X). If U is open in Y and Y is open in X, then U is open in X.

Proof. Since U is open in Y, $U = Y \cap V$ for some V open in X. Both Y and V are open in X, hence $Y \cap V = U$ is also open in X. ∎

The same conclusion holds if you replace "open" by "closed".

Lemma A.4

Let Y be a subspace of X. If F is closed in Y and Y is closed in X, then F is closed in X.

Proof. Since F is closed in Y, $F = Y \cap K$ for some K closed in X. Both Y and K are closed in X, hence $Y \cap K = F$ is also closed in X. ∎

Definition A.9 (Closure)**.** Let S be a topological space and A a subset of S. The **closure** of A in S, denoted by $\overline{A}$ or $\mathrm{cl}_S(A)$, is defined to be the intersection of all the closed sets containing A.

As an intersection of closed sets, $\overline{A}$ is a closed set. It is the smallest closed set containing A in the sense that any closed set containing A contains $\overline{A}$.

Proposition A.5

A is closed if and only if $\overline{A} = A$.

Proof. If $A = \overline{A}$, then A is closed because $\overline{A}$ is closed. Now, suppose A is closed. Then A is a closed set containing A, so $\overline{A} \subseteq A$. Clearly, $A \subseteq \overline{A}$. Therefore, $A = \overline{A}$. ∎

Proposition A.6

If $A \subseteq B$ in a topological space S, then $\overline{A} \subseteq \overline{B}$.

Proof. Since $\overline{B}$ contains B, it also contains A. As a closed subset of S containing A, $\overline{B}$ also contains $\overline{A}$. ∎

Lemma A.7

Let A be a subset of a topological space S. Then $x \in \overline{A}$ if and only if every open set U containing x intersects A.

Proof. We shall prove the contrapositive statements in both directions. So we need to show that

$$x \notin \overline{A} \iff \exists\, U \ni x \text{ such that } U \text{ is open, and } U \cap A = \varnothing\,.$$

Let $x \notin \overline{A}$. We take $U = X \setminus \overline{A}$. This set is open, contains x, and does not intersect A.
Now conversely, suppose U is a open set containing x, and it does not intersect A. Then $X \setminus U$ is closed and it contains A. $\overline{A}$ is the intersection of all closed sets containing A, therefore $\overline{A} \subseteq X \setminus U$. That's why $x \notin \overline{A}$. ∎

Proposition A.8

$\overline{A} \cup \overline{B} = \overline{A \cup B}$.

Proof. $A \subseteq A \cup B$, so by Proposition A.6, $\overline{A} \subseteq \overline{A \cup B}$. Similarly, $\overline{B} \subseteq \overline{A \cup B}$. Therefore, $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$. $A \subseteq \overline{A}$, and $B \subseteq \overline{B}$. So, $A \cup B \subseteq \overline{A} \cup \overline{B}$. Therefore, $\overline{A \cup B} \subseteq \overline{\overline{A} \cup \overline{B}}$. But $\overline{A} \cup \overline{B}$ is closed, so $\overline{\overline{A} \cup \overline{B}} = \overline{A} \cup \overline{B}$. Hence, $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$. Therefore, we have proved that $\overline{A} \cup \overline{B} = \overline{A \cup B}$. ∎

§A.3 Bases and Countability

Definition A.10 (Basis and Basic Open Sets). A subcollection $\mathcal{B}$ of a topology $\mathcal{T}$ is a **basis** for $\mathcal{T}$ if given an open set U and a point p in U, there is an open set $\mathrm{B} \in \mathcal{B}$ such that $p \in \mathrm{B} \subseteq U$. An element of $\mathcal{B}$ is called a **basic open set**.

Example A.3. The collection of all open balls $B(p, r)$ in $\mathbb{R}^n$ with $p \in \mathbb{R}^n$ and $r > 0$ is a basis for the standard topology (metric topology) on $\mathbb{R}^n$.

Proposition A.9

A collection $\mathcal{B}$ of open sets of S is a basis if and only if every open set in S is a union of sets in $\mathcal{B}$.

Proof. $(\Rightarrow)$ We are given a collection of $\mathcal{B}$ of open sets of S that is a basis. U is any open set in S. Also, let $p \in U$. Therefore, there is a basic open set $B_p \in \mathcal{B}$ such that $p \in B_p \subseteq U$. Hence, one can show that $U = \bigcup\limits_{p \in U} B_p$.

$(\Leftarrow)$ Suppose, every open set in S is a union of open sets in $\mathcal{B}$. Now, given an open set U and a point $p \in U$, since $U = \bigcup\limits_{B_\alpha \in \mathcal{B}} B_\alpha$, there is a $B_\alpha \in \mathcal{B}$, such that $p \in B_\alpha \subseteq U$. Hence $\mathcal{B}$ is a basis. ∎

Proposition A.10

A collection $\mathcal{B}$ of subsets of a set S is a basis for some topology $\mathcal{T}$ on S if and only if

(i) S is the union of all the sets in $\mathcal{B}$, and

(ii) given any two sets B_1 and $B_2 \in \mathcal{B}$ and a point $p \in B_1 \cap B_2$, there is a set $B \in \mathcal{B}$ such that $p \in B \subset B_1 \cap B_2$.

Proof. **(⇒)** (i) follows from Proposition A.9.

(ii) If $\mathcal{B}$ is a basis, then B_1 and B_2 are open sets and hence so is $B_1 \cap B_2$. By the definition of a basis, there is a $B \in \mathcal{B}$ such that $p \in B \subseteq B_1 \cap B_2$.

(⇐) Define $\mathcal{T}$ to be the collection consisting of all sets that are unions of sets in $\mathcal{B}$. Then the empty set $\varnothing$ and the set S are in $\mathcal{T}$ and $\mathcal{T}$ is clearly closed under arbitrary union. To show that $\mathcal{T}$ is closed under finite intersection, let $U = \bigcup_\mu B_\mu$ and $V = \bigcup_\nu B_\nu$ be in $\mathcal{T}$, where $B_\mu, B_\nu \in \mathcal{B}$. Then

$$U \cap V = \left(\bigcup_\mu B_\mu \right) \cap \left(\bigcup_\nu B_\nu \right) = \bigcup_{\mu,\nu} (B_\mu \cap B_\nu) \ .$$

Thus, any p in $U \cap V$ is in $B_\mu \cap B_\nu$ for some μ, ν. By (ii) there is a set B_p in $\mathcal{B}$ such that $p \in B_p \subseteq B_\mu \cap B_V$. Therefore,

$$U \cap V = \bigcup_{p \in U \cap V} B_p \in \mathcal{T} \, .$$

Therefore, $\mathcal{B}$ generates a topology on S. ∎

We say that a point in $\mathbb{R}^n$ is rational if all of its coordinates are rational numbers. Let $\mathbb{Q}$ be the set of rational numbers and $\mathbb{Q}^+$ the set of positive rational numbers.

Lemma A.11

Every open set in $\mathbb{R}^n$ contains a rational point.

Proof. An open set U in $\mathbb{R}^n$ contains an open ball $B(p,r)$ which, in turn, contains an open cube $\prod_{i=1}^{n} I_i$ where I_i is the open interval $\left(p^i - \frac{r}{\sqrt{n}}, p^i + \frac{r}{\sqrt{n}}\right)$. Here is a visual example for $n=2$.

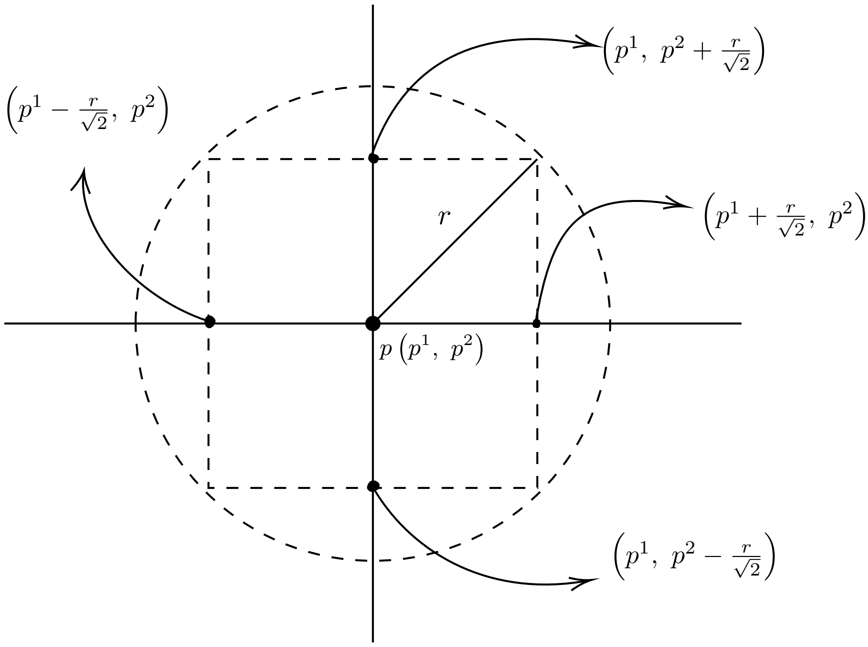


Figure A.1: $B(p,r)$ contains $\left(p^1 - \frac{r}{\sqrt{n}}, p^1 + \frac{r}{\sqrt{n}}\right) \times \left(p^2 - \frac{r}{\sqrt{n}}, p^2 + \frac{r}{\sqrt{n}}\right)$

Now back to general n. For each i, let q^i be a rational number in I_i. Then $(q^1, q^2, ..., q^n)$ is a rational point in $\prod_{i=1}^{n} I_i \subseteq B(p,r)$. Therefore, every open set contains a rational point. ■

Proposition A.12

The collection $\mathcal{B}_{\mathbb{Q}}$ of all open balls in $\mathbb{R}^n$ with rational centers and rational radii is a basis for $\mathbb{R}^n$.

Proof. Given an open set U in $\mathbb{R}^n$ and $p \in U$, there is an open ball $B(p,r')$ with positive real radius r' such that $p \in B(p,r') \subseteq U$. Take a rational number $r \in (0,r')$. Then we have

$$p \in B(p,r) \subseteq B(p,r') \subseteq U$$

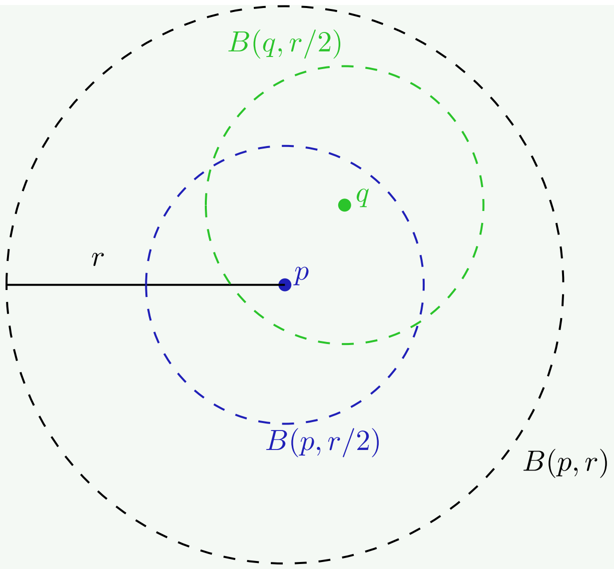
By Lemma A.11 , there is a rational point in the smaller ball $B\left(p, \frac{r}{2}\right)$.

Claim 4: $p \in B\left(q, \frac{r}{2}\right) \subseteq B(p,r)$

Proof. Since $d(p,q) < \frac{r}{2}$, we have $p \in B\left(q, \frac{r}{2}\right)$. Next, if $x \in B\left(q, \frac{r}{2}\right)$, then by triangle inequality

$$d(x,p) \le d(x,q) + d(q,p) < \frac{r}{2} + \frac{r}{2} = r$$

Therefore, $x \in B(p,r)$.



So, $p \in B\left(q, \frac{r}{2}\right)$ and $B\left(q, \frac{r}{2}\right) \subseteq B(p, r)$. □

As a result, $p \in B\left(q, \frac{r}{2}\right) \subseteq B(p, r) \subseteq B\left(p, r'\right) \subseteq U$. Hence we proved,

$$p \in B\left(q, \frac{r}{2}\right) \subseteq U$$

In other words, the collection $\mathcal{B}_{\mathbb{Q}}$ of open balls with rational centers and rational radii is a basis for $\mathbb{R}^n$. ■

Both the sets $\mathbb{Q}$ and $\mathbb{Q}^+$ are countable. Since the centers of the open balls in $\mathcal{B}_{\mathbb{Q}}$ are indexed by $\mathbb{Q}^n$, a countable set, and the radii are indexed by $\mathbb{Q}^+$, also a countable set, the collection $\mathcal{B}_{\mathbb{Q}}$ is countable.

Definition A.11 (Second Countable). A topological space is said to be second countable if it has a countable basis.

Proposition A.12 shows that $\mathbb{R}^n$ with its standard topology is second countable.

Proposition A.13

Let $\mathcal{B} = \{B_\alpha\}$ be a basis for S, and A a subspace of S. Then $\{B_\alpha \cap A\}$ is a basis for A.

Proof. Let U' be any open set in A and $p \in U'$. By the definition of subspace topology, $U' = U \cap A$ for some open set U in S. Since $p \in U \cap A \subset U$, there is a basic open set B_α such that $p \in B_\alpha \subset U$. Then

$$p \in B_\alpha \cap A \subset U \cap A = U',$$

which proves that the collection $\{B_\alpha \cap A \mid B_\alpha \in \mathcal{B}\}$ is a basis for A. ■

Corollary A.14

Subspace of a second countable space is also second countable.

Definition A.12 (Neighborhood Basis). Let S be a topological space and p be a point in S. A **basis of neighbourhoods** or a **neighbourhood basis** at p is a collection $\mathcal{B} = \{B_\alpha\}$ of neighbourhoods of p such that for any neighbourhood U of p there is a $B_\alpha \in \mathcal{B}$ such that $p \in B_\alpha \subseteq U$.

Definition A.13 (First Countable). A topological space S is first countable if it has a countable basis of neighbourhoods at every point $p \in S$.

Example A.4. For $p \in \mathbb{R}^n$, let $B\left(p, \frac{1}{n}\right)$ be the open ball of center p and radius $\frac{1}{n}$ in $\mathbb{R}^n$. Then $\left\{B\left(p, \frac{1}{n}\right)\right\}_{n=1}^{\infty}$ is a neighbourhood basis at p. Thus $\mathbb{R}^n$ is first countable.

An important note: An uncountable discrete topological space is first countable but not second countable. A second countable topological space is always first countable.

§A.4 Hausdorff Space

Definition A.14 (Hausdorff Space). A topological space S is Hausdorff if given any 2 distinct points x, y in S there exist disjoint open sets U, V such that $x \in U$ and $y \in V$.

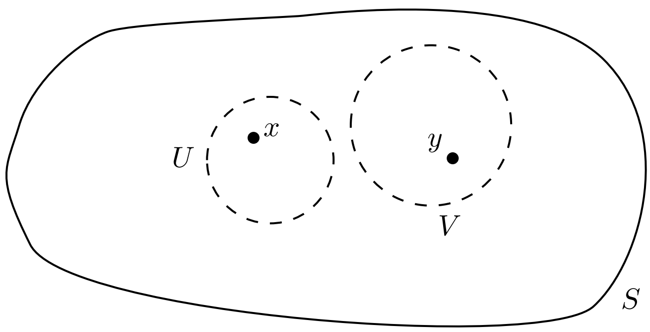


Figure A.2: Here S is a Hausdorff space, U and V are disjoint open sets containing x and y respectively.

Proposition A.15

Every singleton set (a one-point set) in a Hausdorff space S is closed.

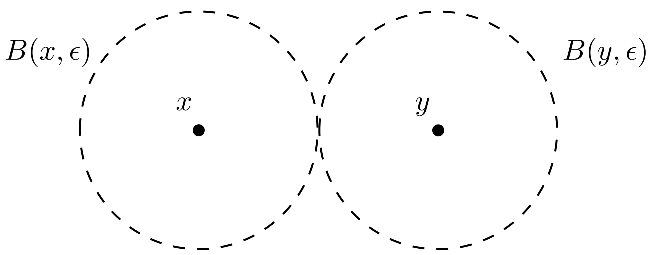
Proof. Let $x \in S$. We want to prove that $\{x\}$ is closed, i.e. $S \setminus \{x\}$ is open.

Let $y \in S \setminus \{x\}$. Since S is Hausdorff, we can find disjoint open sets U_y and V_y such that $x \in U_y$ and $y \in V_y$. No such V_y contains x. Therefore

$$S \setminus \{x\} = \bigcup_{y \in S \setminus \{x\}} V_y$$

So $S \setminus \{x\}$ is union of open sets, hence open. So $\{x\}$ is closed. ■

Example A.5. The Euclidean space $\mathbb{R}^n$ (equipped with standard/ metric topology) is Hausdorff, for given distinct points x, y in $\mathbb{R}^n$, if $\epsilon = \frac{1}{2}d(x, y)$, then the open balls $B(x, \epsilon)$ and $B(y, \epsilon)$ will be disjoint.



In a similar manner, one can show that every metric space is Hausdorff.

Lemma A.16

Let A be a subspace of X. If X is a Hausdorff space, then so is A.

Proof. Take $x, y \in A \subseteq X$ with $x \neq y$. As X is Hausdorff, we can find disjoint open sets U and V in X, such that $U \ni x$ and $V \ni y$. $x \in A$ and $x \in U$, so $x \in A \cap U$. Similarly, $y \in A \cap V$.

Now, both $A \cap U$ and $A \cap V$ are open in A, with respect to the subspace topology. Furthermore, $(A \cap U) \cap (A \cap V) = A \cap (U \cap V) = \varnothing$. Therefore, for $x, y \in A$ we've found disjoint open sets $A \cap U$ and $A \cap V$, containing x and y respectively. So A is Hausdorff. ■

§A.5 Continuity and Homeomorphism

Definition A.15 (Continuous Maps)**.** Let $f : X \to Y$ be a map of topological spaces. f is said to be continuous if for each open subset V of Y, the set $f^{-1}(V)$ is an open subset of X.

Proposition A.17

$f : X \to Y$ is continuous if and only if for every closed subset B of Y, the set $f^{-1}(B)$ will be closed in X.

Proof. ($\Rightarrow$) Suppose f is continuous. B is closed, so $Y \setminus B$ is open in Y. Therefore, by the continuity of f, $f^{-1}(Y \setminus B) = X \setminus f^{-1}(B)$ is open in X, so $f^{-1}(B)$ is closed.

($\Leftarrow$) Suppose $f^{-1}(B)$ is closed in X for any closed $B \subseteq Y$. Take any open set U in Y. Choose $B = Y \setminus U$. Then by the assumption $f^{-1}(Y \setminus U) = X \setminus f^{-1}(U)$ is closed in X. This gives us $f^{-1}(U)$ is open. So f is continuous. ■

Definition A.16 (Homeomorphism)**.** Let X and Y be topological spaces; let $f : X \to Y$ be a bijection. If both f and the inverse function $f^{-1} : Y \to X$ are continuous, then f is called a **homeomorphism**.

Example A.6. The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 3x + 1$ is a homeomorphism. We define $g : \mathbb{R} \to \mathbb{R}$ by $g(y) = \frac{1}{3}(y - 1)$. Then we have

$$f(g(y)) = y \quad \text{and} \quad g(f(x)) = x \qquad \forall\, x, y \in \mathbb{R}$$

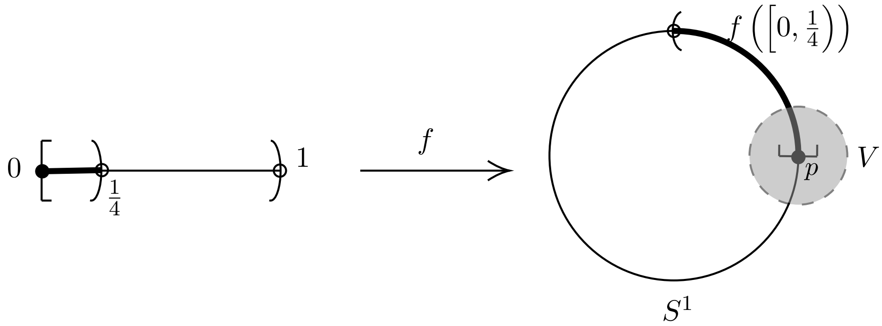
This proves $g = f^{-1}$. It is easy to see that both f and g are continuous functions. Therefore f is a homeomorphism.

However, a bijective function can be continuous without being a homeomorphism.

Example A.7. Let S^1 denote the unit circle in $\mathbb{R}^2$; that is $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$, considered as a **subspace**[2] of the space $\mathbb{R}^2$. Let $f : [0, 1) \to S^1$ be the

$$f(t) = (\cos 2\pi t, \sin 2\pi t)$$

It is left as an exercise for the reader to show that f is a continuous bijective function. But the function f^{-1} is not continuous.



[2]Subset of $\mathbb{R}^2$ equipped with subspace topology.

$U = \left[0, \frac{1}{4}\right)$ is an open set in $[0, 1)$ according to the subspace topology. We want to show that $f(U)$ is not open in S^1. That would prove the discontinuity of f^{-1}.
Let p be the point $f(0)$. And $p \in f(U)$. We need to find an open set of S^1 in subspace topology containing $p = f(0)$ and contained in $f(U)$ to show that $f(U)$ is open in S^1, i.e we have to find an open set in V of $\mathbb{R}^2$ such that $f(0) = p \in V \cap S^1 \subseteq f(U)$. But it is impossible as is evident from the figure above. No matter what V we choose, some part of $V \cap S^1$ would lie outside $f(U)$.

Lemma A.18 (Pasting Lemma)

Let $X = A \cup B$, where A and B are closed in X. Let $f : A \to Y$ and $g : B \to Y$ be continuous. If $f(x) = g(x)$ for every $x \in A \cap B$, then f and g combine to give a continuous function $h : X \to Y$ defined by

$$h(x) = \begin{cases} f(x) & \text{if } \; x \in A \\ g(x) & \text{if } \; x \in B \end{cases}$$

Proof. Let C be a closed subset of Y. Now,

$$h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$$

Since f is continuous, $f^{-1}(C)$ is closed in A, hence closed in X. Similarly, $g^{-1}(C)$ is closed in X. So $h^{-1}(C)$ is the union of two closed sets in X, hence it is closed in X. Therefore, h is continuous. ■

Lemma A.19

Let X, Y be topological spaces and $f : X \to Y$. Then f is continuous if and only if for every $x \in X$ and each neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subseteq V$.

Proof. $(\Rightarrow)$ Suppose f is continuous. Let $x \in X$ and $V \ni f(x)$ is open in Y. We take $U = f^{-1}(V)$. Since f is open and U is preimage of open set, so U is open. Also,

$$f(x) \in V \implies x \in f^{-1}(V) = U \text{ and } f(U) = f\left(f^{-1}(V)\right) \subseteq V$$

$(\Leftarrow)$ Let $V \subseteq Y$ be open. We need to show that $f^{-1}(V)$ is open. Take $x \in f^{-1}(V)$. Then $f(x) \in V$, so V is a neighborhood of $f(x)$. By assumption, there exists open $U \ni x$ such that

$$f(U) \subseteq V \implies U \subseteq f^{-1}(V)$$

So for every $x \in f^{-1}(V)$, there exists a neighborhood of x that is contained in $f^{-1}(V)$. So $f^{-1}(V)$ is open, and hence f is continuous. ■

§A.6 Product Topology

The Cartesian product of two sets A and B is the set $A \times B$ of all ordered pairs (a, b) with $a \in A$ and $b \in B$. Given two topological spaces X and Y, consider the collection $\mathcal{B}$ of subsets of $X \times Y$ of the form $U \times V$, with U open in X and V open in Y. We will call elements of $\mathcal{B}$ basic open sets in $X \times Y$. If $U_1 \times V_1$ and $U_2 \times V_2$ are in $\mathcal{B}$, then

$$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \times V_2) \; ,$$

which is also in $\mathcal{B}$. From this, it follows easily that $\mathcal{B}$ satisfies the conditions of Proposition A.10 for a basis and generates a topology on $X \times Y$, called the **product topology**. Unless noted otherwise, this will always be the topology we assign to the product of two topological spaces.

Proposition A.20

Let $\{U_i\}$ and $\{V_j\}$ be bases for the topological spaces X and Y, respectively. Then $\{U_i \times V_j\}$ is a basis for $X \times Y$.

Proof. Given an open set W in $X \times Y$ and point $(x, y) \in W$, we can find a basic open set $U \times V$ in $X \times Y$ such that $(x, y) \in U \times V \subset W$. Since U is open in X and $\{U_i\}$ is a basis for X, $x \in U_i \subset U$ for some U_i. Similarly, $y \in V_j \subset V$ for some V_j. Therefore,

$$(x, y) \in U_i \times V_j \subset U \times V \subset W.$$

By the definition of a basis, $\{U_i \times V_j\}$ is a basis for $X \times Y$. ■

Corollary A.21

The product of two second-countable spaces is second countable.

Proposition A.22

The product of two Hausdorff spaces X and Y is Hausdorff.

Proof. Given two distinct points $(x_1, y_1), (x_2, y_2)$ in $X \times Y$, without loss of generality we may assume that $x_1 \neq x_2$. Since X is Hausdorff, there exist disjoint open sets U_1, U_2 in X such that $x_1 \in U_1$ and $x_2 \in U_2$. Then $U_1 \times Y$ and $U_2 \times Y$ are disjoint neighborhoods of (x_1, y_1) and (x_2, y_2), so $X \times Y$ is Hausdorff. ■

The product topology can be generalized to the product of an arbitrary collection $\{X_\alpha\}_{\alpha \in \mathrm{A}}$ of topological spaces. Whatever the definition of the product topology, the projection maps

$$\pi_{\alpha_i} : \prod_\alpha X_\alpha \to X_{\alpha_i}\ , \pi_{\alpha_i}\left(\prod x_\alpha\right) = x_{\alpha_i}$$

should all be continuous. Thus, for each open set U_{α_i} in X_{α_i}, the inverse image $\pi_{\alpha_i}^{-1}(U_{\alpha_i})$ should be open in $\prod_\alpha X_\alpha$. By the properties of open sets, a finite intersection $\bigcap_{i=1}^r \pi_{\alpha_i}^{-1}(U_{\alpha_i})$ should also be open. Such a finite intersection is a set of the form $\prod_{\alpha \in A} U_\alpha$, where U_α is open in X_α and $U_\alpha = X_\alpha$ for all but finitely many $\alpha \in A$. We define the product topology on the Cartesian product $\prod_{\alpha \in \mathrm{A}} X_\alpha$ to be the topology with basis consisting of sets of this form.

Theorem A.23

Let $f : A \to \prod_{\alpha \in J} X_\alpha$ be given by the equation

$$f(a) = (f_\alpha(a))_{\alpha \in J}\,,$$

where $f_\alpha : A \to X_\alpha$ for each α. Let $\prod X_\alpha$ have the product topology. Then the function f is continuous if and only if each function f_α is continuous.

Proof. ($\Rightarrow$) Suppose f is continuous. Then $f_\alpha = \pi_\alpha \circ f$ is the composition of two continuous maps, hence continuous.

($\Leftarrow$) Now suppose f_α is continuous for every α. Let $U \subseteq \prod_{\alpha \in J} X_\alpha$ be a basic open set. Then U is of the form $\prod U_\alpha$ where U_α is open in X_α for every α, and $U_\alpha \neq X_\alpha$ for only fintely many α's. Then we have

$$\begin{aligned}
f^{-1}(U) = \bigcap_{\alpha} f_\alpha^{-1}(U_\alpha) &= \left(\bigcap_{U_\alpha \neq X_\alpha} f_\alpha^{-1}(U_\alpha)\right) \cap \left(\bigcap_{U_\alpha = X_\alpha} f_\alpha^{-1}(U_\alpha)\right) \\
&= \left(\bigcap_{U_\alpha \neq X_\alpha} f_\alpha^{-1}(U_\alpha)\right) \cap \left(\bigcap_{U_\alpha = X_\alpha} A\right) \\
&= \bigcap_{U_\alpha \neq X_\alpha} f_\alpha^{-1}(U_\alpha)
\end{aligned}$$

Since each f_α is continuous, $f_\alpha^{-1}(U_\alpha)$ is open in A. Therefore, as a finite intersection of open sets, $f^{-1}(U)$ is open, proving the continuity of f. ■

Proposition A.24

If X and Y are topological spaces, then the projection map $\pi : X \times Y \to X$, $\pi(x, y) = x$ is an open map (it maps open sets to open sets).

Proof. Let $\{U_\alpha\}$ and $\{V_\beta\}$ be bases for the topological spaces X and Y, respectively. Then, by Proposition A.20, $\mathcal{B} = \{U_\alpha \times V_\beta\}$ is a basis for $X \times Y$. Therefore, if W is an open subset of $X \times Y$, then W can be expressed as the union of some basic open sets.

$$W = \bigcup_{i,j} (U_i \times V_j) \ .$$

Then we have,

$$\pi(W) = \pi\left(\bigcup_{i,j} (U_i \times V_j)\right) = \bigcup_{i,j} \pi(U_i \times V_j) = \bigcup_i U_i \, .$$

Since U_i are basic open sets of X, $\bigcup_i U_i$ is an open subset of X. In other words, for W open in $X \times Y$, $\pi(W)$ is open in X. Therefore, π is an open map. ■

§A.7 Quotient Topology

Quotient topology is defined using an equivalence relation. An equivalence relation is a binary relation on a set that has some properties.

Definition A.17 (Equivalence Relation and Equivalence Class)**.** An equivalence relation $\sim$ on a set S is a binary relation which is reflexive, symmetric and transitive. That is

(i) $a \sim a$ for every $a \in S$

(ii) $a \sim b \implies b \sim a$

(iii) $a \sim b \ , \ b \sim c \implies a \sim c$

The equivalence class $[x]$, if $x \in S$, is the set of all elements in S equivalent to x.

An equivalence relation on S partitions S into disjoint equivalence classes. We denote the set of all equivalence classes with $S/\!\sim$ and call this the quotient of S by the equivalence relation $\sim$. There is a natural projection map $\pi : S \to S/\!\sim$ which projects $x \in S$ to its own equivalence class $[x] \in S/\!\sim$.

Abuse of Notation. Ideally $[x]$ denotes a point in $S/\!\sim$. But we will use the same notation $[x]$ to identify a set in S whose elements are all equivalent to each other under the given equivalence relation.

Definition A.18 (Quotient Topology)**.** Let S be a topological space. We define a topology called **quotient topology** on $S/\sim$ by declaring a set U in $S/\sim$ to be open if and only if $\pi^{-1}(U)$ is open in S.

It's not hard to see that quotient topology is a well defined topology. Note that $\pi^{-1}(\varnothing) = \varnothing$ and $\pi^{-1}(S/\sim) = S$ and hence $\varnothing$ and $S/\sim$ are both open sets in quotient topology. Now let $\{U_\alpha\}_{\alpha \in A}$ be an arbitrary collection of open sets in $S/\sim$. Then $\{\pi^{-1}(U_\alpha)\}_{\alpha \in A}$ is an an arbitrary collection of open sets in S. So,

$$\bigcup_{\alpha \in A} \pi^{-1}(U_\alpha) = \pi^{-1}\left(\bigcup_{\alpha \in A} U_\alpha\right) \text{ is open in } S \implies \bigcup_{\alpha \in A} U_\alpha \text{ open in } S/\sim$$

So arbitrary union of open sets is open in $S/\sim$. Now for a finite collection of open sets $\{U_i\}_{i=1}^n$ in $S/\sim$, $\{\pi^{-1}(U_i)\}_{i=1}^n$ is a finite collection of open sets in S. So,

$$\bigcap_{i=1}^{n} \pi^{-1}(U_i) = \pi^{-1}\left(\bigcap_{i=1}^{n} U_\alpha\right) \text{ is open in } S \implies \bigcap_{i=1}^{n} U_\alpha \text{ open in } S/\sim$$

So finite intersection of open sets is open in $S/\sim$. Therefore, we've verified that the open sets defined on $S/\sim$ indeed form a topology.

Continuity on Quotient Topology

Let $\sim$ be a equivalence relation on the topological space S and give $S/\sim$ the quotient topology. Suppose that the function $f : S \to Y$ is continuous from S to another topological space Y. Further assume that f is constant on each equivalence class. Then f induces a map

$$\bar{f} : S/\sim \to Y \ ; \ \bar{f}([p]) = f(p) \quad \forall\, p \in S$$

Note that this latter function $\bar{f}$ wouldn't be well-defined had we not assumed f to be constant on each equivalence class in $S/\sim$.

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "\pi"'] & Y \\ S/\sim \arrow[ur, "\bar{f}"'] & \end{tikzcd} \qquad \begin{array}{c} f = \overline{f} \circ \pi \\ f(p) = \overline{f}(\pi(p)) = f([p]) \end{array}$$

Proposition A.25

The induced map $\bar{f} : S/\sim \to Y$ is continuous if and only if the map $f : S \to Y$ is continuous.

Proof. ($\Rightarrow$). Suppose $f : S \to Y$ is continuous. Let V be open in Y. Then $f^{-1}(V) = \pi^{-1}\left(\bar{f}^{-1}(V)\right)$ is open in S. Therefore, by the definition of quotient topology, then $\bar{f}^{-1}(V)$ is open in $S/\sim$. Hence, we've shown that for a given open set V in Y, $\bar{f}^{-1}(V)$ is open in $S/\sim$. So, $\bar{f} : S/\sim \to Y$ is continuous.

($\Leftarrow$). If $\bar{f} : S/\sim \to Y$ is continuous, then $f = \bar{f} \circ \pi$ is the composition of two continuous maps, hence continuous. ■

Identification of a subset to a point

If A is a subspace of a topological space S, we can define a relation $\sim$ on S by declaring

$$x \sim x \ , \ \forall\, x \in S \qquad \text{and} \qquad x \sim y \ , \ \forall\, x, y \in A$$

It is immediate that $\sim$ is an equivalence relation. We say that the quotient space $S/\sim$ is obtained from S by identifying A to a point.

§A.8 Compactness

Definition A.19 (Open Cover). Let S be a topological space. A collection $\{U_\alpha\}_{\alpha\in I}$ of open subsets of S is said to be an open cover of S if

$$S \subseteq \bigcup_{\alpha\in I} U_\alpha$$

Since the open sets are in the topology of S and consequently $U_\alpha \subseteq S$ for every $\alpha \in I$, one has $\bigcup_{\alpha\in I} U_\alpha \subseteq S$. Therefore, the open cover condition in this case reduces to $S = \bigcup_{\alpha\in I} U_\alpha$.

With the subspace topology, a subset A of a topological space S is a topological space by its own right. The subspace A can be covered by **open sets in** A or **by open sets in** S.

- An **open cover of** A **in** S is a collection $\{U_\alpha\}_\alpha$ of open sets in S that covers A. In other words, $A \subseteq \bigcup_\alpha U_\alpha$ (Note that in this case $A = \bigcup_\alpha U_\alpha$ might not hold in general).
- An **open cover of** A **in** A is a collection $\{U_\alpha\}_\alpha$ of open sets in A in subsapce topology that covers A. In other words, $A \subseteq \bigcup_\alpha U_\alpha$ (Here, in fact, $A = \bigcup_\alpha U_\alpha$ as each $U_\alpha \subseteq A$).

Definition A.20 (Compact Set). Let S be a topological space and $A \subseteq S$. A is **compact** if and only if every open cover of A in A has finite subcover.

Proposition A.26

A subspace A of a topological space S is **compact** if and only if every **open cover of** A **in** S has a finite subcover.

Proof. ($\Rightarrow$) Assume A is compact and let $\{U_\alpha\}$ be an open cover of A in S. This means that $A \subseteq \bigcup\limits_\alpha U_\alpha$. Hence,

$$A \subseteq \left(\bigcup_\alpha U_\alpha\right) \bigcap A = \bigcup_\alpha \left(U_\alpha \bigcap A\right)$$

Now, $\{U_\alpha \cap A\}_\alpha$ is an open cover of A in A. Since A is compact, every open cover of A in A has a finite subcover. Let the finite sub-cover be $\{U_{\alpha_i} \cap A\}_{i=1}^n$. Thus,

$$A \subseteq \bigcup_{i=1}^n U_{\alpha_i}$$

which means that $\{U_{\alpha_i}\}_{i=1}^n$ is a finite sub-cover of the open cover $\{U_\alpha\}_\alpha$ of A in S.

($\Leftarrow$) Suppose every open cover of A in S has a finite subcover, and let $\{V_\alpha\}_\alpha$ be an open cover of A in A. Then each V_α is an open set of A in subspace topology. According to the definition of subspace topology, there is an open set U_α in S such that $V_\alpha = U_\alpha \cap A$. Now,

$$A \subseteq \bigcup_\alpha V_\alpha = \bigcup_\alpha (U_\alpha \cap A) = \left(\bigcup_\alpha U_\alpha\right) \cap A \subseteq \bigcup_\alpha U_\alpha$$

Therefore, $\{U_\alpha\}_\alpha$ is an open cover of A in S. By hypothesis, there are finitely many sets $\{U_{\alpha_i}\}_{i=1}^n$ such that $A \subseteq \bigcup\limits_{i=1}^n U_{\alpha_i}$. Hence,

$$A \subseteq \left(\bigcup_{i=1}^n U_{\alpha_i}\right) \cap A = \bigcup_{i=1}^n (U_{\alpha_i} \cap A) = \bigcup_{i=1}^n V_{\alpha_i}$$

So $\{V_{\alpha_i}\}_{i=1}^n$ is a finite subcover of $\{V_\alpha\}$ that covers A in A. Therefore, A is compact. ■

Proposition A.27

Every compact subset of K of a Hausdorff space S is closed.

Proof. We shall prove that $S \setminus K$ is open. Let's take $x \in S \setminus K$. We claim that there is a neighborhood U_x of x that is disjoint from K.

Since S is Hausdorff, for each $y \in K$, we can choose disjoint open sets U_y and V_y such that $U_y \ni x$ and $V_y \ni y$. The collection $\{V_y \ : \ y \in K\}$ is an open cover of K in S. Since K is compact, there exists a finite subcover $\{V_{y_i}\}_{i=1}^n$. That is $K \subseteq \bigcup_{i=1}^n V_{y_i}$. Since $U_{y_i} \cap V_{y_i}$ for every i, we have

$$\left(\bigcap_{i=1}^n U_{y_i}\right) \cap \left(\bigcup_{i=1}^n V_{y_i}\right) = \varnothing \implies U_x \cap K = \varnothing \text{ where } U_x = \bigcap_{i=1}^n U_{y_i}$$

U_x is the finite intersection of open sets, hence open. Also, every U_{y_i} contains x, hence their intersection U_x also contains x. So U_x is the desired open set that is disjoint from K, in other words $x \in U_x \subseteq S \setminus K$. As a result,

$$S \setminus K \subseteq \bigcup_{x \in S \setminus K} U_x \subseteq S \setminus K \implies S \setminus K = \bigcup_{x \in S \setminus K}$$

$S \setminus K$ is the union of open sets, hence open. Therefore K is closed. ■

Proposition A.28

The image of a compact set under a continuous map is compact.

Proof. Let $f : X \to Y$ be a continuous and K a compact subset of X. Suppose $\{U_\alpha\}$ is an open cover of $f(K)$ by open subsets of Y. Since, f is continuous, the inverse images of $f^{-1}(U_\alpha)$ are all open in X. Moreover,

$$K \subseteq f^{-1}(f(K)) \subseteq f^{-1}\left(\bigcup_\alpha U_\alpha\right) = \bigcup_\alpha f^{-1}(U_\alpha)$$

So $\{f^{-1}(U)_\alpha\}$ is an open cover of K in X. By Proposition A.26, there is a finite sub-collection $\{f^{-1}(U_{\alpha_i})\}_{i=1}^n$ such that

$$K \subseteq \bigcup_{i=1}^n f^{-1}(U_{\alpha_i}) = f^{-1}\left(\bigcup_{i=1}^n U_{\alpha_i}\right) \implies f(K) \subseteq f\left(f^{-1}\left(\bigcup_{i=1}^n U_{\alpha_i}\right)\right) \subseteq \bigcup_{i=1}^n U_{\alpha_i}$$

Thus $f(K)$ is compact. ■

Lemma A.29

A closed subset F of a compact topological space S is compact.

Proof. Let $\{U_\alpha\}_\alpha$ be an open cover of F in S. The collection $\{U_\alpha, S \setminus F\}$ is an open cover of S itself. By compactness of S, there is a finite sub-cover $\{U_{\alpha_i}, S \setminus F\}_{i=1}^n$ of S, that is,

$$F \subseteq S \subseteq \left(\bigcup_{i=1}^n U_{\alpha_i}\right) \cup (S \setminus F) \implies F \subseteq \bigcup_{i=1}^n U_{\alpha_i}$$

Therefore, $\{U_{\alpha_i}\}_{i=1}^n$ is a finite subcover of the open cover $\{U_\alpha\}$ of F in S. Hence, F is also compact. ■

Proposition A.30

A continuous map $f : X \to Y$ form a compact space X to a Hausdorff space Y is a closed map (a map that takes closed sets to closed sets).

Proof. Let $F \subseteq X$ be closed. Then F is compact by Lemma A.29. Since $f : X \to Y$ is a continuous map, by Proposition A.28, $f(F)$ is compact in Y. Since Y is Hausdorff, by Proposition A.27, $f(F)$ is closed in Y. Hence, f is a closed map. ■

Corollary A.31

A continuous bijection $f : X \to Y$ from a compact space X to a Hausdorff space is a homeomorphism.

Proof. We want to show thaat $f^{-1} : Y \to X$ is continuous. And in order to that it suffices to show that for every closed set F in X, $\left(f^{-1}\right)^{-1}(F) = f(F)$ is closed in Y. In other words, it suffices to show that f is a closed map. The corollary then follows from Proposition A.30. ■

Definition A.21 (Bounded Set)**.** A subset A of $\mathbb{R}^n$ is said to be bounded if it is contained in some open ball $B(p, r)$. otherwise, it is unbounded.

Theorem A.32 (Heine-Borel Theorem)

A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

Definition A.22 (Diameter of Set)**.** Let $A \subseteq X$ be a bounded subset of a metric space (X, d). The diameter of A is defined by

$$\operatorname{diam}(A) := \sup \left\{ d\left(a_1, a_2\right) \ : \ a_1, a_2 \in A \right\}$$

Lemma A.33 (Lebesgue Number Lemma)

Let (X, d) be a compact metric space. Given an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in J}$ of X, there exists a number $\delta > 0$ — called the Lebesgue number associated with the cover — such that for a given $A \subseteq X$ with $\operatorname{diam}(A) < \delta$, one must have $A \subseteq U_\alpha$ for some $\alpha \in J$.

Proof. Take $x \in X$. As $\mathcal{U}$ covers X, we can find $U_\alpha \in \mathcal{U}$ such that $x \in U_\alpha$. Since U_α is open and $x \in U_\alpha$, there exists $r_x > 0$ such that

$$B\left(x, r_x\right) \subseteq U_\alpha$$

We do this for every $x \in X$. So we get an open cover of X

$$X = \bigcup_{x \in X} B\left(x, \frac{r_x}{2}\right)$$

Since X is compact, there exists a finite subcover of this open cover. So

$$X = \bigcup_{i=1}^{n} B\left(x_i, \frac{r_{x_i}}{2}\right)$$

We define $\delta > 0$ in the following way:

$$\delta = \min\left\{ \frac{r_{x_i}}{2} \ : \ i = 1, 2, \ldots, n \right\}$$

We claim that this δ is our desired Lebesgue number of the open cover $\mathcal{U}$. Let $A \subseteq X$ with $\operatorname{diam}(A) < \delta$. Fix $a \in A$. Then there exists $j \in \{1, 2, \ldots, n\}$ such that

$$a \in B\left(x_j, \frac{r_{x_j}}{2}\right) \implies \boxed{d\left(x_j, a\right) < \frac{r_{x_j}}{2}}$$

By the construction of r_{x_j}, there exists $U_\beta \in \mathcal{U}$ such that $B\left(x_j, r_{x_j}\right) \subseteq U_\beta$. We claim that $A \subseteq U_\beta$. Take any $b \in A$.

$$d(a,b) \le \operatorname{diam}(A) < \delta \le \frac{r_{x_j}}{2} \implies \boxed{d(a,b) < \frac{r_{x_j}}{2}}$$

$$d(x_j,b) \le d(x_j,a) + d(a,b) < \frac{r_{x_j}}{2} + \frac{r_{x_j}}{2} = r_{x_j} \implies b \in B\left(x_j, r_{x_j}\right)$$

For every $b \in A$, we have $b \in B\left(x_j, r_{x_j}\right)$. Therefore, $A \subseteq B\left(x_j, r_{x_j}\right) \subseteq U_\beta$. ■

§A.9 Quotient Topology Continued

Let I be the closed interval $[0,1]$ in the standard topology of $\mathbb{R}^n$ and $I/\sim$ be the quotient space obtained from I by identifying the 2 points $\{0,1\}$ to a point. Denote by S^1 the unit circle in the complex plane. Define f by $f(x) = e^{2\pi i x}$.

Now the function $f : I \to S^1$ defined above assumes the same value at 0 and 1 and based on the discussion prior to Proposition A.25, f induces the map $\bar{f} : I/\!\sim \to S^1$.

Proposition A.34

The function $\bar{f} : I/\!\sim \to S^1$ is a homeomorphism.

Proof. The function $f : I \to S^1$ defined by $f(x) = e^{2\pi i x}$ is continuous (check!). Therefore, by Proposition A.25, $\bar{f} : I \to S^1$ is also continuous.

Note that $I = [0,1]$ in $\mathbb{R}$ is closed and bounded and hence by Heine-Borel Theorem, I is compact. Since the projection $\pi : I \to I/\!\sim$ is continuous, by Proposition A.28, the image of I under π, *i.e.*, $I/\!\sim$ is compact.

It should also be obvious that $\bar{f} : I/\!\sim \to S^1$ is a bijection. Since S^1 is a of the Hausdorff space $\mathbb{R}^2$, by Lemma A.16, S^1 is also Hausdorff. Hence, $\bar{f}$ is a continuous bijection from the compact space $I/\!\sim$ to the Hausdorff topological space S^1. Therefore, by Corollary A.31, $\bar{f} : I/\!\sim \to S^1$ is a homeomorphism. ■

Necessary Condition for a Hausdorff quotient

Even if S is a Hausdorff space, the quotient space $S/\!\sim$ may fail to be Hausdorff.

Proposition A.35

If the quotient space $S/\!\sim$ is Hausdorff, then the equivalence class $[p]$ of any point p in S is closed in S.

Proof. By Proposition A.15, every singleton set is closed in a Hausdorff topological space. Now, consider the canonical projection map $\pi : S \to S/\!\sim$. For a point $p \in S$, $\{\pi(p)\}$ is a singleton set in $S/\!\sim$.

Since, by hypothesis $S/\!\sim$ is Hausdorff, $\{\pi(p)\}$ must be closed in $S/\!\sim$ with respect to quotient topology. By continuity of π, $\pi^{-1}(\{\pi(p)\})$ is closed in S. But $\pi^{-1}(\{\pi(p)\}) = [p]$. Hence, $[p]$ is a closed set in S. ■

Remark A.1. In order to prove that a quotient space $S/\!\sim$ is not Hausdorff it is sufficient to prove that the equivalence class $[p]$ of some point $p \in S$ is not closed in S. We have the following example to elucidate this remark.

Example A.8. Define an equivalence relation $\sim$ on $\mathbb{R}$ by identifying the open interval $(0,\infty)$ to a point. The resulting quotient space $\mathbb{R}/\!\sim$ is not Hausdorff since the equivalence class $(0,\infty)$ is not a closed subset of $\mathbb{R}$.

§A.10 Open Equivalence Relations

> **Definition A.23.** An equivalence relation $\sim$ on a topological space S is said to be open if the underlying projection map $\pi : S \to S/\!\sim$ is open (maps open sets to open sets).

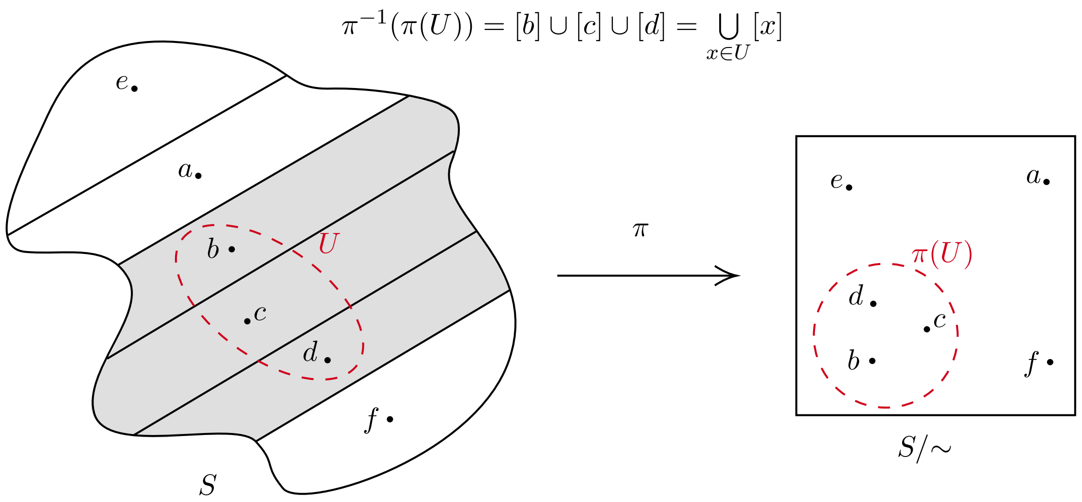


Figure A.3: Indeed $\pi^{-1}(\pi(U)) = \bigcup_{x \in U} [x]$

In other words, the equivalence relation $\sim$ on S is open if and only if for every open set $U \in S$, the set $\pi(U) \in S/\!\sim$ is open. Or equivalently, by definition of quotient topology,

$$\pi^{-1}(\pi(U)) = \bigcup_{x \in U} [x] \quad \text{is open in } S$$

$\bigcup_{x \in U} [x]$ denotes all points equivalent to some point of U (shaded region in Figure A.3).

Example A.9. The projection map onto a quotient space is, in general, not open. For example, let $\sim$ be the equivalence relation on the real line $\mathbb{R}$ that identifies the two points 1 and -1, and $\pi : \mathbb{R} \to \mathbb{R}/\!\sim$ the projection map.

The map π is open if and only if for every open set V in $\mathbb{R}$, its image $\pi(V)$ is open in $\mathbb{R}/\!\sim$, or equivalently $\pi^{-1}(\pi(V))$ is open in $\mathbb{R}$. Let V be the open interval $(-2, 0)$ in $\mathbb{R}$. Then,

$$\pi^{-1}(\pi(V)) = (-2, 0) \cup \{1\}, \qquad [\text{Since } \pi(1) \in \pi(V)]$$

which is not open in $\mathbb{R}$ and hence π is not an open map. In other words, the equivalence relation $\sim$ is not open.

> **Definition A.24** (Graph of Equivalence Relation)**.** Given an equivalence relation $\sim$ on S, let R be the subset of $S \times S$ that defines the relation $R = \{(x, y) \in S \times S \mid x \sim y\}$. We call R the **graph** of the equivalence relation $\sim$.

We have a necessary and sufficient condition for a quotient space to be Hausdorff if the underlying equivalence relation is an open equivalence relation.

> **Theorem A.36**
>
> Suppose $\sim$ is an open equivalence relation on a topological space S. Then the quotient space $S/\!\sim$ is Hausdorff if and only if the graph R of $\sim$ is closed in $S \times S$.

Proof. **(⇐)** Suppose R is closed in $S \times S$. Then $R^c = (S \times S) \setminus R$ is open. Therefore, for every $(x, y) \in R^c$, there exists basic open set $U \times V$ containing (x, y) such that $U \times V \subseteq R^c$. This is equivalent to saying, no element of U is equivalent to any element of V, and vice versa.

Since $\sim$ is an open equivalence relation, $\pi(U)$ and $\pi(V)$ are open sets containing $[x]$ and $[y]$, respectively. Since no element of U is equivalent to any element of V, $\pi(U)$ and $\pi(V)$ are disjoint. Therefore, for $[x] \neq [y]$, we have found their disjoint neighborhoods. Hence, $S/\sim$ is Hausdorff.

(⇒) Now suppose $S/\sim$ is Hausdorff. Take $[x] \neq [y]$ from $S/\sim$. Then there exist disjoint neighborhoods $A \ni [x]$ and $B \ni [y]$. A and B are open, so $U = \pi^{-1}(A)$ and $V = \pi^{-1}(B)$ are open in S.

$$\pi(U) = \pi\left(\pi^{-1}(A)\right) = A \quad \text{and} \ \pi(V) = \pi\left(\pi^{-1}(B)\right) = B\,.$$

So $\pi(U)$ and $\pi(V)$ are disjoint. In other words, no element of U is equivalent to any element of V. Therefore, $U \times V \subseteq R^c$. $[x] \in A$, and $U = \pi^{-1}(A)$, so $x \in U$. Similarly, $y \in V$. Therefore,

$$(x, y) \in U \times V \subseteq R^c\,.$$

So R^c is open, and hence R is closed. ■

If the equivalence relation $\sim$ is equality, *i.e.*, $x \sim y$ iff $x = y$, then the quotient space $S/\sim$ is S itself and the graph R of $\sim$ is simply the diagonal $\Delta = \{(x, x) \in S \times S\}$.

Corollary A.37

A topological space is Hausdorff if and only if the diagonal Δ is closed in $S \times S$.

Theorem A.38

Let $\sim$ be an open equivalence relation on a topological space S with projection $\pi : S \to S/\sim$. If $\mathcal{B} = \{B_\alpha\}$ is a basis for S, then its image $\{\pi(B_\alpha)\}$ under π is a basis for $S/\sim$.

Proof. Since π is open, $\{\pi(B_\alpha)\}$ is a collection of open sets in $S/\sim$. Let W be an open set in $S/\sim$ and $[x] \in W$ with $x \in S$. So $\pi(x) \in W$, *i.e.*, $x \in \pi^{-1}(W)$. Since $\pi^{-1}(W)$ is open in S, there is a basic open set $B \in \mathcal{B}$ such that, $x \in B \subseteq \pi^{-1}(W)$. Hence

$$[x] = \pi(x) \in \pi(B) \subseteq \pi(\pi^{-1}(W)) \subseteq W$$

Now, we have seen that given W open in $S/\sim$ and $[x] \in W$, there exists an open set $\pi(B)$ in the collection $\{\pi(B_\alpha)\}$ such that $[x] \in \pi(B) \subseteq W$. This proves that $\{\pi(B_\alpha)\}$ is a basis for $S/\sim$. ■

Corollary A.39

If $\sim$ is an open equivalence relation on a second-countable topological space, then the quotient space $S/\sim$ is second countable.

B Multivariable Calculus Review

§B.1 Differentiabiliy

Consider $f : \mathbb{R}^2 \to \mathbb{R}$ defined as follows:

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

For the piecewise defined function stated above, note that along the x-axis $y = 0$. So $f(x,0) = 0$ for every $x \in \mathbb{R}$. In other words, f is constant and identically 0 on the x-axis. Therefore,

$$\frac{\partial f}{\partial x}(x,y)\bigg|_{y=0} = 0\,.$$

Similarly, along the y-axis $x = 0$. So $f(0,y) = 0$ for every $y \in \mathbb{R}$. In other words, f is constant and identically 0 on the y-axis. Therefore,

$$\frac{\partial f}{\partial y}(x,y)\bigg|_{x=0} = 0\,.$$

Therefore, both the partial derivatives exist at $(0,0)$, and are equal to 0. We will now show that f is not even continuous at $(0,0)$. Consider the line $y = x$, and we shall evaluate the limit of $f(x,y)$ as $(x,y) \to (0,0)$ along this line.

$$\lim_{x\to 0} f(x,x) = \lim_{x\to 0} \frac{x \cdot x}{x^2 + x^2} = \frac{1}{2} \neq 0\,.$$

So we get,

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} f(x,y) &= 0\ ,\ \text{along } x\text{-axis;} \\
\lim_{(x,y)\to(0,0)} f(x,y) &= 0\ ,\ \text{along } y\text{-axis;} \\
\lim_{(x,y)\to(0,0)} f(x,y) &= \frac{1}{2}\ ,\ \text{along the line } y = x.
\end{aligned}$$

Therefore, f is not even continuous at $(0,0)$, let alone being differentiable. Therefore, mere existence of partial derivatives of order doesn't guarantee differentiability at a given point.

We will, first, consider functions whose domain is $U \subseteq \mathbb{R}^n$ and codomain is $\mathbb{R}$. If $f : U \to \mathbb{R}^n$ is such a function, then $f(\vec{x}) = f(x^1, x^2, \ldots, x^n)$ denotes its value at $\vec{x} \equiv (x^1, x^2, \ldots, x^n) \in U$. We also assume that the underlying domain of f is an open set $U \subseteq \mathbb{R}^n$. At each $\vec{a} \in U$, the partial derivative $\frac{\partial f}{\partial x^j}\Big|_{\vec{a}}$ of f with respect to x^j is the following limit, if it exists

$$\frac{\partial f}{\partial x^j}\bigg|_{\vec{x}=\vec{a}} = \lim_{h\to 0} \frac{f(a^1, \ldots, a^j + h, \ldots, a^n) - f(a^1, \ldots, a^j, \ldots, a^n)}{h}\,.$$

If $\frac{\partial f}{\partial x^j}$ is defined, that is, the limit above exists at each point of U for $1 \leq j \leq n$, this defines n functions on U. Should these functions be continuous on U for $1 \leq j \leq n$, f is said to be continuously differentiable on U, denoted by $f \in C^1(U)$.

We shall say that f is differentiable at $\vec{a} \in U$ if there is a homogenous linear expression $\sum\limits_{i=1}^{n} b_i\,(x^i - a^i)$ such that the inhomogenous expression $f(\vec{a}) + \sum\limits_{i=1}^{n} b_i\,(x^i - a^i)$ approximates $f(\vec{x})$ near $\vec{a}$ in the following sense:

$$\lim_{\vec{x}\to\vec{a}} \frac{f(\vec{x}) - f(\vec{a}) - \sum\limits_{i=1}^{n} b_i\,(x^i - a^i)}{\|\vec{x} - \vec{a}\|} = 0\,.$$

In other words, if there exist constants $b_1, b_2, \dots, b_n$ and a real valued function $r(\vec{x}, \vec{a})$ defined on a neghborhood V of $\vec{a} \in U$ such that the following two conditions hold:

$$f(\vec{x}) = f(\vec{a}) + \sum_{i=1}^{n} b_i \left(x^i - a^i\right) + \|\vec{x} - \vec{a}\| \, r(\vec{x}, \vec{a}) \quad \text{and} \quad \lim_{\vec{x} \to \vec{a}} r(\vec{x}, \vec{a}) = 0 \, .$$

b_i's are uniquely determined, and they are the partial derivatives at $\vec{a}$:

$$b_i = \left. \frac{\partial f}{\partial x^i} \right|_{\vec{x} = \vec{a}} \, .$$

In fact,

$$f(\vec{x}) = f(\vec{a}) + \sum_{i=1}^{n} \left. \frac{\partial f}{\partial x^i} \right|_{\vec{x} = \vec{a}} \left(x^i - a^i\right) + \|\vec{x} - \vec{a}\| \, r(\vec{x}, \vec{a}) \, .$$

Actually, existence of partial derivatives and their continuity guarantees differentiability at a given point $\vec{a} \in U \subseteq \mathbb{R}^n$.

§B.2 Chain Rule

By a differentiable curve in $\mathbb{R}^n$, we mean $f : (a, b) \to \mathbb{R}^n$, with $f(t) = (x^1(t), x^2(t), \dots, x^n(t))$, where the n coordinate functions $x^i(t)$ are all differentiable on (a, b). Recall that, for a function of one variable, differentiability is equivalent to existence of derivative.

Here, $(x^i(t))$ are real valued functions of one variable. And you must be familiar with the notion of C^r-differentiability of real valued functions of one variable. For example, $h(t) = t^{\frac{1}{3}}$ is not C^1, because its derivative does not exist at $t = 0$. Similarly, $k(t) = t^{\frac{4}{3}}$ is C^1, but not C^2.

Now, let's suppose $f : (a, b) \to \mathbb{R}^n$ is a C^r differentiable curve in the sense that all the n coordinate functions $x^i(t)$ are C^r differentiable. Take t_0 with $a < t_0 < b$, and $f : (a, b) \to U \subseteq \mathbb{R}^n$. Let g be a C^r-differentiable function from U to $\mathbb{R}$. In particular, $g : U \to \mathbb{R}$ is differentiable at $f(t_0) \in U$. Then $g \circ f : (a, b) \to \mathbb{R}$ is differentiable at t_0, and the derivative is given by:

$$\left. \frac{\mathrm{d}}{\mathrm{d}t} (g \circ f)(t) \right|_{t = t_0} = \sum_{i=1}^{n} \left. \frac{\partial g(f(t))}{\partial x^i} \right|_{f(t_0)} \cdot \left. \frac{\mathrm{d}x^i(t)}{\mathrm{d}t} \right|_{t = t_0} \, .$$

This result is known as the chain rule for real-valued functions.

Now, we can generalize this idea to functions on subsets U of $\mathbb{R}^n$, whose range is not in $\mathbb{R}$, but in $\mathbb{R}^n$. In other words, we consider $F : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^m$.

$$\vec{x} \equiv \left(x^1, x^2, \dots, x^n\right) \in U \; ; \; F(\vec{x}) = \left(F^1(\vec{x}), F^2(\vec{x}), \dots, F^m(\vec{x})\right) \, .$$

Now take a point $\vec{p} \in U$ with coordinate $(p^1, p^2, \dots, p^n)$. Then $F(\vec{p})$ is a point in V with coordinate $(F^1(\vec{p}), F^2(\vec{p}), \dots, F^m(\vec{p}))$. Now let $G : V \subseteq R^m \to R^l$. Write a point $\vec{y} \equiv (y^1, y^2, \dots, y^m) \in V \subseteq \mathbb{R}^m$. Then

$$G(\vec{y}) = \left(G^1(\vec{y}), G^2(\vec{y}), \dots, G^l(\vec{y})\right) \, .$$

In other words, $G^i : V \to \mathbb{R}$. Then we have $G^i \circ F : U \subseteq \mathbb{R}^n \to \mathbb{R}$. In this case, the chain rule is

$$\frac{\partial (G^i \circ F)}{\partial x^j}(\vec{p}) = \sum_{k=1}^{m} \frac{\partial G^i}{\partial y^k}(F(\vec{p})) \cdot \frac{\partial F^k}{\partial x^j}(\vec{p}) \, .$$

§B.3 Differential of a Map

Let $F : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^m$. Let $T_p\mathbb{R}^n$ denote the tangent space on $\mathbb{R}^n$ to the point $p \in \mathbb{R}^n$. (For convenience, we'll drop arrows in $\vec{p}$) The differential of F at p is a map $DF_p : T_p\mathbb{R}^n \to T_{F(p)}\mathbb{R}^m$. $T_p\mathbb{R}^n$

is clearly isomorphic to $\mathbb{R}^n$ as vector space. Hence, $DF_p : \mathbb{R}^n \to \mathbb{R}^m$. Let's try to see that DF_p is related to the Jacobian matrix of $F : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^m$.

$$\left\{ \left.\frac{\partial}{\partial x^1}\right|_p, \left.\frac{\partial}{\partial x^2}\right|_p, \ldots, \left.\frac{\partial}{\partial x^n}\right|_p \right\}$$

is a basis of $T_p\mathbb{R}^n$, which can be treated as $\mathbb{R}^n$ with origin at p. Similarly,

$$\left\{ \left.\frac{\partial}{\partial y^1}\right|_{F(p)}, \left.\frac{\partial}{\partial y^2}\right|_{F(p)}, \ldots, \left.\frac{\partial}{\partial y^m}\right|_{F(p)} \right\}$$

is a basis of $T_{F(p)}\mathbb{R}^m$, which can be treated as $\mathbb{R}^m$ with origin at $F(p)$.

Geometric tangent vectors like $\left.\frac{\partial}{\partial x^i}\right|_p$ or $\left.\frac{\partial}{\partial y^j}\right|_{F(p)}$ act on smooth functions of $\mathbb{R}^n$ or $\mathbb{R}^m$, respectively, and spit out real numbers.

$$\left.\frac{\partial}{\partial x^i}\right|_p f = \frac{\partial f}{\partial x^i}(p) \in \mathbb{R}\,.$$

Since DF_p is a linear map between two vector spaces, in order to express DF_p as a matrix, we need to find where the basis vectors are getting mapped. So we want to find $DF_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right)$. This is a vector in $T_{F(p)}\mathbb{R}^m$, and hence can be written as a linear combination of $\left.\frac{\partial}{\partial y^j}\right|_{F(p)}$'s. Now we wish to find the coefficients in the linear combination.

$DF_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right)$ acts on $f \in C^\infty(\mathbb{R}^m)$ and yields a real number.

$$DF_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) f := \left.\frac{\partial}{\partial x^i}\right|_p (f \circ F)\,.$$

This makes perfect sense as $f \circ F : U \subseteq \mathbb{R}^n \to \mathbb{R}$. By chain rule,

$$\left.\frac{\partial}{\partial x^i}\right|_p (f \circ F) = \frac{\partial(f \circ F)}{\partial x^i}(p) = \sum_{j=1}^m \left.\frac{\partial f}{\partial y^j}\right|_{F(p)} \left.\frac{\partial F^j}{\partial x^i}\right|_p\,.$$

$$\therefore DF_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) f = \sum_{j=1}^m \left.\frac{\partial F^j}{\partial x^i}\right|_p \left.\frac{\partial}{\partial y^j}\right|_{F(p)} f \implies \boxed{DF_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = \sum_{j=1}^m \frac{\partial F^j}{\partial x^i}(p) \cdot \left.\frac{\partial}{\partial y^j}\right|_{F(p)}}$$

Therefore, DF_p can be represented by the following $m \times n$ matrix:

$$\begin{bmatrix} \frac{\partial F^1}{\partial x^1}(p) & \frac{\partial F^1}{\partial x^2}(p) & \cdots & \frac{\partial F^1}{\partial x^n}(p) \\ \frac{\partial F^2}{\partial x^1}(p) & \frac{\partial F^2}{\partial x^2}(p) & \cdots & \frac{\partial F^2}{\partial x^n}(p) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F^m}{\partial x^1}(p) & \frac{\partial F^m}{\partial x^2}(p) & \cdots & \frac{\partial F^m}{\partial x^n}(p) \end{bmatrix}$$

F is differentiable at $p \in U \subseteq \mathbb{R}^n$ if all the entries in the $m \times n$ matrix DF exist and are continuous at p. If F is differentiable at every $p \in U$, we say that F is of class C^1. DF is called the total derivative in the language of multivariable calculus.

Similarly, if all the second order partial derivatives exist and are continuous at p, then we say F is twice differentiable at p. If F is twice differentiable at every $p \in U$, we say F is of class C^2. In a similar manner, we define maps of class C^r. If a map F is of class C^r for every $r \in \mathbb{N}$, we say F is **smooth** or **infinitely differentiable**, or F belongs in the class C^∞.

§B.4 Inverse Function Theorem

Definition B.1. Let U and V be open subsets of $\mathbb{R}^n$. A map $F : U \to V$ is said to be a $\boldsymbol{C^r}$**-diffeomorphism** if F is a homeomorphism, and both F and F^{-1} are of class C^r. When $r = \infty$, we just say F is a **diffeomorphism**.

Theorem B.1 (Inverse Function Theorem)

Let W be an open subset of $\mathbb{R}^n$ and $F : W \to \mathbb{R}^n$ a C^∞ mapping. If $p \in W$ and DF_p is nonsingular, then there exists a neighborhood U of p in W such that $V = F(U)$ is open and $F : U \to V$ is a diffomorphism. If $x \in U$, then

$$DF^{-1}_{F(x)} = (DF_x)^{-1}\ .$$

We are not going to prove it here. We will see an example now.

Example B.1. Let's consider the conversion of polar to rectangular coordinate. $F : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$F\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}.$$

Then the differential DF is

$$DF = \begin{bmatrix} \dfrac{\partial F^1}{\partial r} & \dfrac{\partial F^1}{\partial \theta} \\ \dfrac{\partial F^2}{\partial r} & \dfrac{\partial F^2}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

Hence, $\det DF = r$. So $DF_{(r,\theta)}$ is differentiable for $r \neq 0$. Choose $r = \sqrt{2}$ and $\theta = \frac{\pi}{4}$. Then

$$F\begin{pmatrix} \sqrt{2} \\ \frac{\pi}{4} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$$DF_{\left(\sqrt{2},\frac{\pi}{4}\right)} = \begin{bmatrix} \dfrac{1}{\sqrt{2}} & -1 \\ \dfrac{1}{\sqrt{2}} & 1 \end{bmatrix}.$$

By the Inverse Function Theorem, there is a local inverse

$$DF^{-1}_{(1,1)} = \left(DF_{\left(\sqrt{2},\frac{\pi}{4}\right)}\right)^{-1}.$$

Now, F^{-1} is given by

$$F^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sqrt{x^2+y^2} \\ \tan^{-1}\left(\frac{y}{x}\right) \end{pmatrix}.$$

Therefore,

$$DF^{-1} = \begin{bmatrix} \dfrac{2x}{2\sqrt{x^2+y^2}} & \dfrac{2y}{2\sqrt{x^2+y^2}} \\ \dfrac{-y}{x^2+y^2} & \dfrac{x}{x^2+y^2} \end{bmatrix} = \begin{bmatrix} \dfrac{x}{\sqrt{x^2+y^2}} & \dfrac{y}{\sqrt{x^2+y^2}} \\ \dfrac{-y}{x^2+y^2} & \dfrac{x}{x^2+y^2} \end{bmatrix}.$$

As a result,

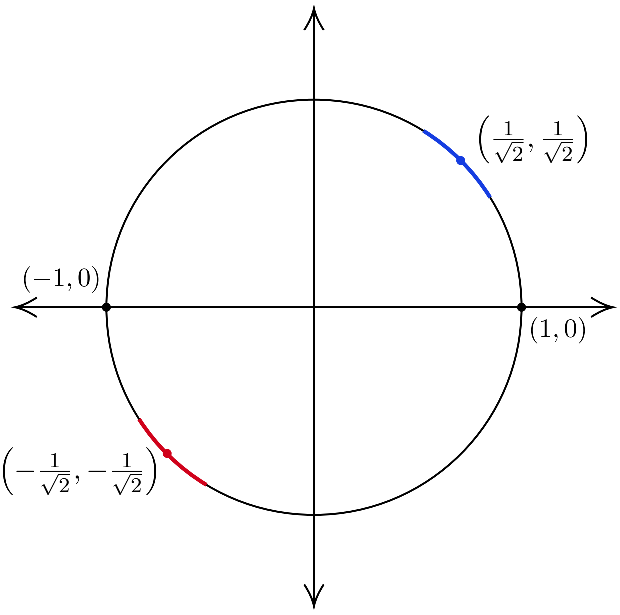
$$DF^{-1}_{(1,1)} = \begin{bmatrix} \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \\ \dfrac{-1}{2} & \dfrac{1}{2} \end{bmatrix}.$$

One can indeed check that

$$\begin{bmatrix} \dfrac{1}{\sqrt{2}} & -1 \\ \dfrac{1}{\sqrt{2}} & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \\ \dfrac{-1}{2} & \dfrac{1}{2} \end{bmatrix}.$$

§B.5 Implicit Function Theorem

Let us consider the equation of a unit circle in $\mathbb{R}^2$; $x^2+y^2=1$.



The graph of the unit circle above does not represent a function. Because, for a given value of x, there are 2 values for y that satisfy the equation. Choose a point, say $\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$, on the unit circle. Then one can consider an arc (colored blue in the figure above) containing $\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$ that indeed represents a function given by $y=\sqrt{1-x^2}$. Had we started with the point $\left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$, we could find an arc (colored red in the figure above) containing $\left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$ that represents a function given by $y=-\sqrt{1-x^2}$. The only problematic points are $(1,0)$ and $(-1,0)$. No matter how small an arc we choose about these points, it is not going to be represented by a function. Because, for those arcs, for a given x, there will be multiple values for y.

Now let us address the following 2-dimensional problem: Given an equation $F(x,y)=0$, which is not globally a functional relationship (in the unit circle example, $F(x,y)=x^2+y^2-1$), does there exist a point (x_0,y_0) satisfying $F(x_0,y_0)=0$ so that there exists a neighborhood of (x_0,y_0) where y can be written as $y=f(x)$ for some real valued function f of one variable? In other words, $F(x,f(x))=0$ should hold for all values of x in that neighborhood. In the unit circle example, this f was given by $f(x)=\sqrt{1-x^2}$ or $f(x)=-\sqrt{1-x^2}$, depending on the choice of the point (x_0,y_0) in the upper or lower semicircle, respectively. The Implicit Function Theorem guarantees the **local existence** of such a function provided the initial point (x_0,y_0) was chosen *appropriately*. In the unit circle example, $(1,0)$ and $(-1,0)$ were two *inappropriate* points. As required by the Implicit Function Theorem, one must have

$$\frac{\partial F}{\partial y}(x_0,y_0)\neq 0\,.$$

But in this case, for $F(x,y)=x^2+y^2-1$,

$$\frac{\partial F}{\partial y}=2y \implies \frac{\partial F}{\partial y}(1,0)=0=\frac{\partial F}{\partial y}(1,0)\ .$$

Therefore, in the light of Implicit Function Theorem, $(1,0)$ and $(-1,0)$ are not *appropriate* points on the unit circle around which we can construct a locally functional relationship. Now we state the most general form of Implicit Function Theorem.

Theorem B.2 (Implicit Function Theorem)

Let U be an open set in $\mathbb{R}^n\times\mathbb{R}^m$ and $F:U\to\mathbb{R}^m$ a C^∞ map. Write $(x,y)=(x^1,\ldots,x^n,y^1,\ldots,y^m)$

for a point in U. Suppose the matrix

$$\left[\frac{\partial F^i}{\partial y^j}(x_0, y_0)\right]_{1 \le i,j \le m}$$

is non-singular for a point $(x_0, y_0) \in U$ satisfying $F(x_0, y_0) = 0$. Then there exists a neighborhood $X \times Y$ of (x_0, y_0) in U and a unique C^∞ map $f : X \to Y$ such that in $X \times Y \subseteq U \subseteq \mathbb{R}^n \times \mathbb{R}^m$,

$$F(x, y) = 0 \iff y = f(x) \ .$$

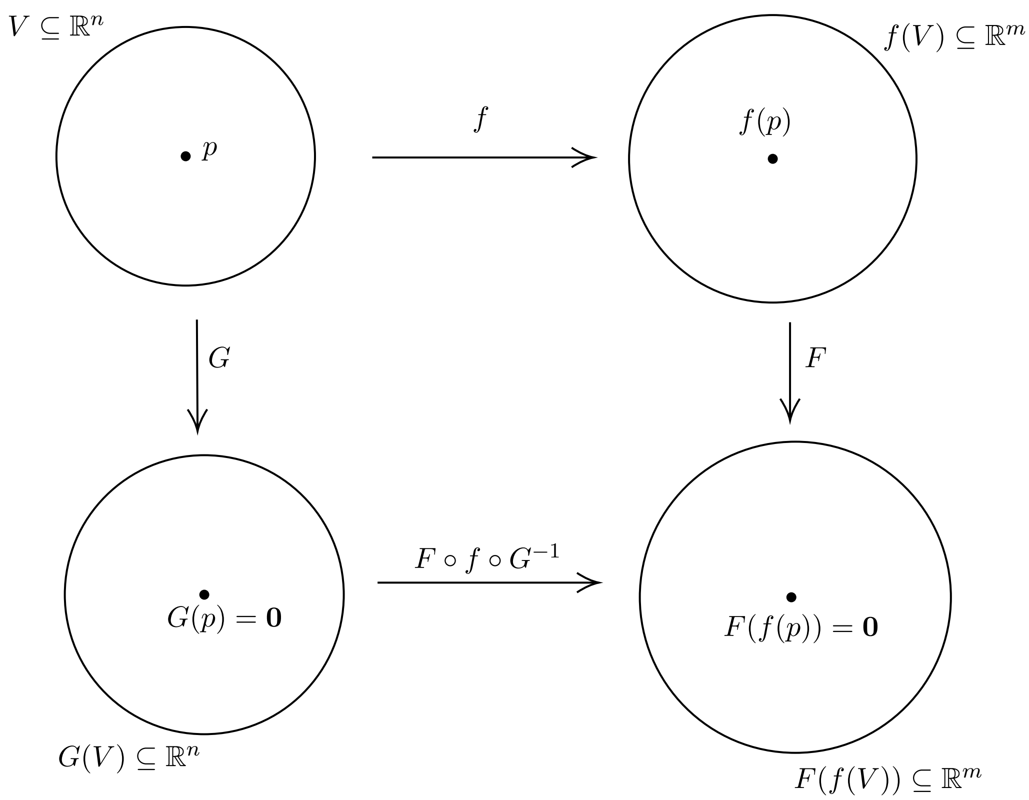
§B.6 Constant Rank Theorem

Definition B.2 (Rank of a Smooth Map at a Point)**.** Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be a C^∞ map. The **rank** of f at $p \in U$ is the rank of its Jacobian matrix $\left[\frac{\partial f^i}{\partial x^j}(p)\right]$.

Theorem B.3 (Constant Rank Theorem)

Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be a C^∞ map. Suppose f has a constant rank k in a neighborhood of $p \in U$. Then there are a diffeomorphism G of a neighborhood V of $p \in U$ sending p to $\mathbf{0} \in \mathbb{R}^n$, and a diffeomorphism F of a neighborhood W of $f(p) \in \mathbb{R}^m$ sending $f(p)$ to $\mathbf{0} \in \mathbb{R}^m$ such that

$$\left(F \circ f \circ G^{-1}\right)\left(x^1, \ldots, x^n\right) = \left(x^1, \ldots, x^k, 0, \ldots, 0\right) .$$



Remark B.1. Many textbooks include $f(V) \subseteq W$ in the statement of Constant Rank Theorem. Because if $f(V)$ is not a subset of W, we can always find a smaller V such that $f(V) \subseteq W$. Since G^{-1} is a map from $G(V)$ to V, we need to restrict f on V in order to form the composition $f \circ G^{-1}$. Then $f|_V \circ G^{-1}$ is a map from $G(V)$ to $f(V)$. Then we need to restrict F on $f(V)$ so that the composition $F \circ f \circ G^{-1}$ makes sense. We can do this because the domain W of F contains $f(V)$. Therefore, $F \circ f \circ G^{-1}$ is actually

$$F|_{f(V)} \circ f|_V \circ G^{-1} : G(V) \to F(f(V)) \ .$$

Oftentimes we just write $F \circ f \circ G^{-1}$ when what we actually mean is $F\big|_{f(V)} \circ f\big|_V \circ G^{-1}$.

C Review of Multilinear Algebra

§C.1 Dual Space

Let V and W be real vector spaces. We denote by $\mathrm{Hom}(V, W)$ the vector space of all linear maps $f : V \to W$. In particular, if we choose $W = \mathbb{R}$, we get the **dual space** V^*.

$$V^* = \mathrm{Hom}(V, \mathbb{R}).$$

The elements of V^* are called covectors on V. In the rest of the lecture, we will assume V to be a finite dimensional vector space. Let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be a basis for V. Then every $\mathbf{v} \in V$ is a unique linear combination

$$\mathbf{v} = \sum_{i=1}^{n} v^i \mathbf{e}_i, \tag{C.1}$$

with $v^i \in \mathbb{R}$. v^i's are called the coordinates of $\mathbf{v}$ relative to the basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$. Let $\widehat{\alpha}^i$ be the linear function on V that picks up the i-th coordinate of the vector, i.e.

$$\widehat{\alpha}^i(\mathbf{v}) = \widehat{\alpha}^i\left(\sum_{i=1}^{n} v^i \mathbf{e}_i\right) = v^i. \tag{C.2}$$

When $\mathbf{v}$ is one of the basis vectors,

$$\widehat{\alpha}^i(\mathbf{e}_j) = \delta^i{}_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{C.3}$$

Proposition C.1

The functions $\widehat{\alpha}^1, \ldots, \widehat{\alpha}^n$ form a basis for V^*.

Proof. Suppose $f \in V^*$. Then for any $\mathbf{v} = \sum_{i=1}^{n} v^i \mathbf{e}_i \in V$,

$$f(\mathbf{v}) = f\left(\sum_{i=1}^{n} v^i \mathbf{e}_i\right) = \sum_{i=1}^{n} v^i f(\mathbf{e}_i) = \sum_{i=1}^{n} f(\mathbf{e}_i)\,\widehat{\alpha}^i(\mathbf{v}).$$

Since this holds for any $\mathbf{v} \in V$,

$$f = \sum_{i=1}^{n} f(\mathbf{e}_i)\,\widehat{\alpha}^i. \tag{C.4}$$

Therefore, $\widehat{\alpha}^1, \ldots, \widehat{\alpha}^n$ span V^*. As for linear independence, suppose

$$\sum_{i=1}^{n} c_i \widehat{\alpha}^i = \mathbf{0}, \tag{C.5}$$

where $\mathbf{0}$ is the function that takes all of V to $0 \in \mathbb{R}$. If we evaluate (C.5) at $\mathbf{e}_j$, we get

$$0 = \sum_{i=1}^{n} c_i \widehat{\alpha}^i(\mathbf{e}_j) = \sum_{i=1}^{n} c_i \delta^i{}_j = c_j. \tag{C.6}$$

So $c_j = 0$, and this holds for each $j = 1, 2, \ldots, n$. Therefore, $\{\widehat{\alpha}^1, \ldots, \widehat{\alpha}^n\}$ is a linearly independent set that spans V^*, i.e. a basis. ■

Corollary C.2

The dual space V^* of a finite dimensional vector space has the same dimension as V.

The basis $\{\widehat{\alpha}^1, \ldots, \widehat{\alpha}^n\}$ for V^* is said to be dual to the basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ for V.

§C.2 Permutations

Fix a positive integer k. A permutation of the set $A = \{1, 2, \ldots, k\}$ is a bijection $\sigma : A \to A$. The product of two permutations τ and σ is the composition $\tau \circ \sigma : A \to A$. The **cyclic permutation** $(a_1\ a_2\ \cdots\ a_r)$ is the permutation σ such that

$$\sigma(a_1) = a_2,\ \sigma(a_2) = a_3,\ \cdots,\ \sigma(a_{r-1}) = a_r,\ \text{and}\ \sigma(a_r) = 1,$$

leaving all other elements unchanged, i.e. $\sigma(j) = j$ if j is not one of the a_i's. A cyclic permutation $(a_1\ a_2\ \cdots\ a_r)$ is also called a **cycle** of length r or an r-cycle. A **transposition** is a permutation of the form $(a\ b)$ that interchanges a and b, leaving all other elements of A fixed.

A permutation $\sigma : A \to A$ can be described by

$$\begin{bmatrix} 1 & 2 & \cdots & k \\ \sigma(1) & \sigma(2) & \cdots & \sigma(k) \end{bmatrix}.$$

We also write it as

$$[1\ 2\ \cdots\ k] \overset{\sigma}{\mapsto} [\sigma(1)\ \sigma(2)\ \cdots\ \sigma(k)]\,.$$

Example C.1. Suppose $\sigma : \{1,2,3,4,5\} \to \{1,2,3,4,5\}$ is the permutation given by

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 5 & 1 & 3 \end{bmatrix}.$$

In other words, $\sigma(1) = 2$, $\sigma(2) = 4$, $\sigma(3) = 5$, $\sigma(4) = 1$, and $\sigma(5) = 3$.

$$[1\ 2\ 3\ 4\ 5] \overset{\sigma}{\mapsto} [2\ 4\ 5\ 1\ 3]\,.$$

Observe that the cyclic permutation $\sigma' = (1\ 2\ 4)$ acts as $\sigma'(1) = 2$, $\sigma'(2) = 4$ and $\sigma'(4) = 1$, keeping 3 and 5 unchanged, i.e. $\sigma'(3) = 3$ and $\sigma'(5) = 5$.

$$[1\ 2\ 3\ 4\ 5] \xmapsto{(1\ 2\ 4)} [2\ 4\ 3\ 1\ 5]\,.$$

Now the transposition $\sigma'' = (3\ 5)$ acts as $\sigma''(3) = 5$ and $\sigma''(5) = 3$, keeping $1, 2, 4$ unchanged. Therefore,

$$\begin{tikzcd} {[1\ 2\ 3\ 4\ 5]} \arrow[r, mapsto, "(1\ 2\ 4)"] \arrow[rr, bend right, mapsto, "(3\ 5)(1\ 2\ 4)"'] & {[2\ 4\ 3\ 1\ 5]} \arrow[r, mapsto, "(3\ 5)"] & {[2\ 4\ 5\ 1\ 3]} \end{tikzcd}$$

so that $\sigma = (3\ 5)(1\ 2\ 4)$.

Let S_k be the group of permutations of the set $\{1, 2, \ldots, k\}$. The order of this group is $k!$. A permutation is *even* or *odd* depending on whether it is the product of an even or an odd number of transpositions. The sign of a permutation σ is 1 if the permutation is even, and -1 otherwise. It is denoted by $\operatorname{sgn} \sigma$. For example, in Example C.1, $\sigma = (3\ 5)(1\ 2\ 4)$. Note that we can write $(1\ 2\ 4)$ as a product of two transpositions:

$$\begin{tikzcd} {[1\ 2\ 3\ 4\ 5]} \arrow[r, mapsto, "(1\ 2)"] \arrow[rr, bend right, mapsto, "(1\ 4)(1\ 2){=}(1\ 2\ 4)"'] & {[2\ 1\ 3\ 4\ 5]} \arrow[r, mapsto, "(1\ 4)"] & {[2\ 4\ 3\ 1\ 5]} \end{tikzcd}$$

In other words, $\sigma = (3\ 5)(1\ 4)(1\ 2)$. Hence, $\operatorname{sgn} \sigma = -1$. One can easily check that

$$\operatorname{sgn}(\sigma\tau) = (\operatorname{sgn} \sigma)(\operatorname{sgn} \tau)\,. \tag{C.7}$$

So $\operatorname{sgn} : S_k \to \{1, -1\}$ is a group homomorphism.

Example C.2. Observe that the 5-cycle $(1\ 2\ 3\ 4\ 5)$ can be written as

$$(1\ 2\ 3\ 4\ 5) = (1\ 5)\,(1\ 4)\,(1\ 3)\,(1\ 2)\,.$$

Indeed,

$$\begin{tikzcd}
{[1\ 2\ 3\ 4\ 5]} \arrow[r, mapsto, "(1\ 2)"] \arrow[rrrr, bend right, "(1\ 2\ 3\ 4\ 5)=(1\ 5)(1\ 4)(1\ 3)(1\ 2)"'] & {[2\ 1\ 3\ 4\ 5]} \arrow[r, mapsto, "(1\ 3)"] & {[2\ 3\ 1\ 4\ 5]} \arrow[r, mapsto, "(1\ 4)"] & {[2\ 3\ 4\ 1\ 5]} \arrow[r, mapsto, "(1\ 5)"] & {[2\ 3\ 4\ 5\ 1]}
\end{tikzcd}$$

Therefore, $\operatorname{sgn}(1\ 2\ 3\ 4\ 5) = 1$.

An **inversion** in a permutation σ is an ordered pair $(\sigma(i), \sigma(j))$ such that $i < j$ but $\sigma(i) > \sigma(j)$. In Example C.1, $\sigma(1) = 2$, $\sigma(2) = 4$, $\sigma(3) = 5$, $\sigma(4) = 1$, and $\sigma(5) = 3$. So, the inversions in this permutation are

$$(\sigma(1), \sigma(4))\,,\ (\sigma(2), \sigma(4))\,,\ (\sigma(2), \sigma(5))\,,\ (\sigma(3), \sigma(4))\,,\ (\sigma(3), \sigma(5))\,.$$

Hence, there are 5 inversions associated with the permutation σ. There is an efficient way of determining the sign of a permutation.

Proposition C.3

A permutation is even if and only if it has an even number of inversions.

Proof. Let $\sigma \in S_k$ with n inversions. We shall prove that we can multiply σ by n transpositions and get the identity permutation. This will prove that $\operatorname{sgn}\sigma = (-1)^n$.

Suppose $\sigma(j_1) = 1$. Then for each $i < j_1$, $(\sigma(i), \sigma(j_1))$ is an inversion, and there are $j_1 - 1$ many of them. These are all the inversions with 1 in the second slot of the ordered pair of inversion. If we now multiply σ by the j_1-cycle

$$(\sigma(1)\ \ 1)\,(\sigma(2)\ \ 1)\cdots(\sigma(j_1 - 1)\ \ 1)$$

to the left of σ, the resulting permutation σ_1 would be

$$\begin{bmatrix} 1 & 2 & 3 & \cdots & j_1 & j_1 + 1 & \cdots & k \\ 1 & \sigma(1) & \sigma(2) & \cdots & \sigma(j_1 - 1) & \sigma(j_1 + 1) & \cdots & \sigma(k)\,. \end{bmatrix}$$

This permutation has no inversion with 1 in the second slot of the ordered pair of inversion. Suppose now that $\sigma(j_2) = 2$. Now observe that if $(\sigma_1(i), 2)$ is an inversion in σ_1, then either $(\sigma(i), 2)$ (if $i \geq j_1 + 1$) or $\sigma(i-1), 2$ (if $i \leq j_1 - 1$) is an inversion in σ. Therefore, the number of inversions in σ_1 ending in 2 is precisely the same as the number of inversions in σ ending in 2. So following a similar procedure as above, we can multiply σ_1 by i_2-many transpositions to the left (i_2 is the number of transpositions ending in 2) and get

$$\begin{bmatrix} 1 & 2 & 3 & \cdots & k \\ 1 & 2 & \sigma(1) & \cdots & \sigma(k)\,. \end{bmatrix}$$

We can continue these steps for each $j = 1, 2, \ldots, k$, and the number of transpositions required to move j to its natural position is the same as the number of inversions ending in j. In the end we achieve the identity permutation. Therefore, $\operatorname{sgn}\sigma = (-1)^n$, where n is the number of inversions. ∎

§C.3 Multilinear Functions

Definition C.1. Let V^k be the cartesian product of k-copies of a real vector space V.

$$V^k = \underbrace{V \times V \times \cdots \times V}_{k\text{-copies}}$$

A function $f : V^k \to \mathbb{R}$ is called k-linear if it is linear in each of its k arguments:

$$f(\ldots, a\mathbf{v} + b\mathbf{w}, \ldots) = a\, f(\ldots, \mathbf{v}, \ldots) + b\, f(\ldots, \mathbf{w}, \ldots), \tag{C.8}$$

for $a, b \in \mathbb{R}$ and $\mathbf{v}, \mathbf{w} \in V$.

Instead of 2-linear and 3-linear, it's customary to call "bilinear" and "trilinear", respectively. A k-linear function on V is called a **k-tensor** on V. We will denote the vector space of all k-tensors on V by $L_k(V)$. The vector addition and scalar multiplication of the real vector space $L_k(V)$ is the straightforward pointwise operation.

Example C.3. The dot product $f(\mathbf{v}, \mathbf{w}) = \mathbf{v} \cdot \mathbf{w}$ on $\mathbb{R}^n$ is bilinear: if $\mathbf{v} = \sum_{i=1}^n v^i \mathbf{e}_i$ and $\mathbf{w} = \sum_{i=1}^n w^i \mathbf{e}_i$, then

$$f(\mathbf{v}, \mathbf{w}) = \mathbf{v} \cdot \mathbf{w} = \sum_{i=1}^n v^i w^i.$$

Example C.4. The determinant

$$f(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n) = \det \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}$$

viewed as a function of the n column vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is n-linear.

Definition C.2 (Symmetric and alternating function)**.** A k-linear function $f : V^k \to \mathbb{R}$ is **symmetric** if

$$f\left(\mathbf{v}_{\sigma(1)}, \mathbf{v}_{\sigma(2)}, \ldots, \mathbf{v}_{\sigma(k)}\right) = f(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k), \tag{C.9}$$

for all permutations $\sigma \in S_k$. It is **alternating** if

$$f\left(\mathbf{v}_{\sigma(1)}, \mathbf{v}_{\sigma(2)}, \ldots, \mathbf{v}_{\sigma(k)}\right) = (\operatorname{sgn} \sigma)\, f(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k), \tag{C.10}$$

for all permutations $\sigma \in S_k$.

The dor product function on $\mathbb{R}^n$ in Theorem C.3 is symmetric, and the determinant function on $\mathbb{R}^n$ in Theorem C.4 is alternating.

We are especially interested in the vector space $A_k(V)$ of all alternating k-linear functions on a vector space V, for $k > 0$. The elements of $A_k(V)$ are called alternating k-tensors (also known as k-covectors). We define $A_0(V)$ to be $\mathbb{R}$. The elements of $A_0(V)$ are simply constants, which we call 0-covectors. The elements of $A_1(V)$ are simply covectors, i.e. the elements of V^*.

Permutation action on k-linear functions

If $f \in L_k(V)$ and $\sigma \in S_k$, define $\sigma f \in L_k(V)$ as follows:

$$(\sigma f)(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k) = f\left(\mathbf{v}_{\sigma(1)}, \mathbf{v}_{\sigma(2)}, \ldots, \mathbf{v}_{\sigma(k)}\right). \tag{C.11}$$

Thus, f is symmetric if and only if $f = \sigma f$ for all $\sigma \in S_k$; and f is alternating if and only if $\sigma f = (\operatorname{sgn} \sigma) f$ for all $\sigma \in S_k$. When $k = 1$, S_k only has the identity permutation. In that case, a 1-linear function or simply linear function on V is both symmetric and alternating. In particular,

$$A_1(V) = L_1(V) = V^*.$$

Lemma C.4

If $\sigma, \tau \in S_k$ and $f \in L_k(V)$, then $\tau(\sigma f) = (\tau\sigma) f$.

Proof. For any $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in V$,

$$\begin{aligned}
(\tau(\sigma f))(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k) &= (\sigma f)\left(\mathbf{v}_{\tau(1)}, \mathbf{v}_{\tau(2)}, \ldots, \mathbf{v}_{\tau(k)}\right) \\
&= (\sigma f)(\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k) && [\mathbf{w}_i = \mathbf{v}_{\tau(i)}] \\
&= f\left(\mathbf{w}_{\sigma(1)}, \mathbf{w}_{\sigma(2)}, \ldots, \mathbf{w}_{\sigma(k)}\right) \\
&= f\left(\mathbf{v}_{\tau(\sigma(1))}, \mathbf{v}_{\tau(\sigma(2))}, \ldots, \mathbf{v}_{\tau(\sigma(k))}\right) \\
&= ((\tau\sigma) f)(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k).
\end{aligned}$$

Therefore, $\tau(\sigma f) = (\tau\sigma) f$. ∎

Definition C.3. If G is a group and X is a set, a map

$$\begin{aligned} G \times X &\to X \\ (g, x) &\mapsto g \cdot x \end{aligned}$$

is called a **left action** of G on X if

(i) $e \cdot x = x$, where e is the identity element in G and x is any element in X; and

(ii) $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$, for all $g_1, g_2 \in G$ and $x \in X$.

Similarly, a **right action** of G on X is a map

$$\begin{aligned} X \times G &\to X \\ (x, g) &\mapsto x \cdot g \end{aligned}$$

such that

(i) $x \cdot e = x$, for all $x \in X$; and

(ii) $(x \cdot g_1) \cdot g_2 = x \cdot (g_1 g_2)$, for all $g_1, g_2 \in G$ and $x \in X$.

Symmetrizing and alternating operators

Given $f \in L_k(V)$, there is a way to make it a symmetric k-linear function $\mathcal{S}f$ from it:

$$(\mathcal{S}f)(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k) = \sum_{\sigma \in S_k} f\left(\mathbf{v}_{\sigma(1)}, \mathbf{v}_{\sigma(2)}, \ldots, \mathbf{v}_{\sigma(k)}\right). \tag{C.12}$$

In other words,

$$\mathcal{S}f = \sum_{\sigma \in S_k} \sigma f. \tag{C.13}$$

Similarly, there is a way to make an alternating k-linear function from f:

$$\mathcal{A}f = \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma)\, \sigma f. \tag{C.14}$$

Proposition C.5 (i) The k-linear function $\mathcal{S}f$ is symmetric.

(ii) The k-linear function $\mathcal{A}f$ is alternating.

Proof. (i) Let $\tau \in S_k$. Then

$$\tau(\mathcal{S}f) = \tau\left(\sum_{\sigma \in S_k} \sigma f\right). \tag{C.15}$$

The group action of S_k on $L_k(V)$ is distributive over the vector space addition. Therefore,

$$\tau(\mathcal{S}f) = \sum_{\sigma \in S_k} \tau(\sigma f) = \sum_{\sigma \in S_k} (\tau\sigma) f. \tag{C.16}$$

As σ varies over S_k, $\tau\sigma$ also varies over S_k. Therefore, $\sum_{\sigma \in S_k} (\tau\sigma) f = \mathcal{S}f$. In other words,

$$\tau(\mathcal{S}f) = \mathcal{S}f, \tag{C.17}$$

i.e. $\mathcal{S}f$ is symmetric.

(ii) Let $\tau \in S_k$. Then

$$\tau(\mathcal{A}f) = \tau\left(\sum_{\sigma \in S_k} (\operatorname{sgn}\sigma)\, \sigma f\right) = \sum_{\sigma \in S_k} (\operatorname{sgn}\sigma)\, \tau(\sigma f) = \sum_{\sigma \in S_k} (\operatorname{sgn}\sigma)\, (\tau\sigma) f. \tag{C.18}$$

Since $(\operatorname{sgn}\tau)^2 = 1$,

$$\begin{aligned}
\tau(\mathcal{A}f) &= \sum_{\sigma \in S_k} (\operatorname{sgn}\tau)^2 (\operatorname{sgn}\sigma)\, (\tau\sigma) f \\
&= (\operatorname{sgn}\tau) \sum_{\sigma \in S_k} (\operatorname{sgn}\tau)\, (\operatorname{sgn}\sigma)\, (\tau\sigma) f \\
&= (\operatorname{sgn}\tau) \sum_{\sigma \in S_k} (\operatorname{sgn}(\tau\sigma))\, (\tau\sigma) f.
\end{aligned} \tag{C.19}$$

As σ varies over S_k, $\tau\sigma$ also varies over S_k. Therefore, $\sum_{\sigma \in S_k} (\operatorname{sgn}(\tau\sigma))\, (\tau\sigma) f = \mathcal{A}f$. In other words,

$$\tau(\mathcal{A}f) = \mathcal{A}f, \tag{C.20}$$

i.e. $\mathcal{A}f$ is alternating.

■

Lemma C.6
If $f \in A_k(V)$, then $\mathcal{A}f = (k!)\, f$.

Proof. Since f is alternating,

$$\mathcal{A}f = \sum_{\sigma \in S_k} (\operatorname{sgn}\sigma)\, \sigma f = \sum_{\sigma \in S_k} (\operatorname{sgn}\sigma)^2 f = \sum_{\sigma \in S_k} f = (k!)\, f, \tag{C.21}$$

because the order of S_k is $k!$. ■

§C.4 Tensor Product and Wedge Product

Definition C.4 (Tensor Product)**.** Let f be a k-linear function and g an l-linear function on a vector space V. Their tensor product $f \otimes g$ is the $(k+l)$-linear function defined by

$$(f \otimes g)(\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}, \ldots, \mathbf{v}_{k+l}) = f(\mathbf{v}_1, \ldots, \mathbf{v}_k)\, g(\mathbf{v}_{k+1}, \ldots, \mathbf{v}_{k+l}). \tag{C.22}$$

$(k+l)$-linearity of $f \otimes g$ follows from k-linearity of f and l-linearity of g.

Lemma C.7 (Associativity of Tensor Product)
Let $f \in L_k(V)$, $g \in L_l(V)$ and $h \in L_m(V)$. Then

$$(f \otimes g) \otimes h = f \otimes (g \otimes h) .$$

Proof. For $\mathbf{v}_1, \ldots, \mathbf{v}_{k+l+m}$,

$$\begin{aligned} [(f \otimes g) \otimes h] (\mathbf{v}_1, \ldots, \mathbf{v}_{k+l+m}) &= (f \otimes g) (\mathbf{v}_1, \ldots, \mathbf{v}_{k+l}) \, h (\mathbf{v}_{k+l+1}, \ldots, \mathbf{v}_{k+l+m}) \\ &= f (\mathbf{v}_1, \ldots, \mathbf{v}_k) \, g (\mathbf{v}_{k+1}, \ldots, \mathbf{v}_{k+l}) \, h (\mathbf{v}_{k+l+1}, \ldots, \mathbf{v}_{k+l+m}) . \end{aligned} \tag{C.23}$$

$$\begin{aligned} [f \otimes (g \otimes h)] (\mathbf{v}_1, \ldots, \mathbf{v}_{k+l+m}) &= f (\mathbf{v}_1, \ldots, \mathbf{v}_k) \, (g \otimes h) (\mathbf{v}_{k+1}, \ldots, \mathbf{v}_{k+l+m}) \\ &= f (\mathbf{v}_1, \ldots, \mathbf{v}_k) \, g (\mathbf{v}_{k+1}, \ldots, \mathbf{v}_{k+l}) \, h (\mathbf{v}_{k+l+1}, \ldots, \mathbf{v}_{k+l+m}) . \end{aligned} \tag{C.24}$$

Therefore, $(f \otimes g) \otimes h = f \otimes (g \otimes h)$, i.e. tensor product is associative. ∎

Example C.5. Let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be the standard basis for $\mathbb{R}^n$, and $\{\widehat{\alpha}^1, \ldots, \widehat{\alpha}^n\}$ its dual basis. The Euclidean inner product on $\mathbb{R}^n$ is the bilinear function

$$\langle \ , \ \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$

defined by

$$\langle \ , \ \rangle (\mathbf{v}, \mathbf{w}) = \langle \mathbf{v}, \mathbf{w} \rangle = \sum_{i=1}^{n} v^i w^i ,$$

for $\mathbf{v} = \sum_{i=1}^n v^i \mathbf{e}_i$ and $\mathbf{w} = \sum_{i=1}^n w^i \mathbf{e}_i$. We can express $\langle \ , \ \rangle$ in terms of tensor product as follows:

$$\langle \ , \ \rangle (\mathbf{v}, \mathbf{w}) = \sum_{i=1}^{n} v^i w^i = \sum_{i=1}^{n} \widehat{\alpha}^i (\mathbf{v}) \, \widehat{\alpha}^i (\mathbf{w}) = \sum_{i=1}^{n} \left(\widehat{\alpha}^i \otimes \widehat{\alpha}^i \right) (\mathbf{v}, \mathbf{w}) .$$

Since $\mathbf{v}, \mathbf{w}$ are arbitrary,

$$\langle \ , \ \rangle = \sum_{i=1}^{n} \left(\widehat{\alpha}^i \otimes \widehat{\alpha}^i \right) . \tag{C.25}$$

If $f \in A_k(V)$ and $g \in A_l(V)$, then it's not true that $f \otimes g \in A_{k+l}(V)$, in general. We need to construct a product that is also alternating.

Definition C.5 (Wedge Product). For $f \in A_k(V)$ and $g \in A_l(V)$, the wedge product of f and g is defined as follows:

$$f \wedge g = \frac{1}{k!l!} \mathcal{A} (f \otimes g) . \tag{C.26}$$

Explicitly,

$$\begin{aligned} (f \wedge g) (\mathbf{v}_1, \ldots, \mathbf{v}_{k+l}) &= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma) \, \sigma (f \otimes g) (\mathbf{v}_1, \ldots, \mathbf{v}_{k+l}) \\ &= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma) (f \otimes g) \left(\mathbf{v}_{\sigma(1)}, \ldots, \mathbf{v}_{\sigma(k+l)} \right) \\ &= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma) \, f \left(\mathbf{v}_{\sigma(1)}, \ldots, \mathbf{v}_{\sigma(k)} \right) g \left(\mathbf{v}_{\sigma(k+1)}, \ldots, \mathbf{v}_{\sigma(k+l)} \right) . \end{aligned} \tag{C.27}$$

When $k = 0$, the element $f \in A_0(V)$ is simply a constant $c \in \mathbb{R}$ as discussed earlier. In this case, the wedge product $c \wedge g$ is just scalar multiplication as is evident from (C.27).

$$\begin{aligned}
(c \wedge g)(\mathbf{v}_1, \ldots, \mathbf{v}_l) &= \frac{1}{l!} \sum_{\sigma \in S_l} (\operatorname{sgn} \sigma)\, c\, g\left(\mathbf{v}_{\sigma(1)}, \ldots, \mathbf{v}_{\sigma(l)}\right) \\
&= \frac{1}{l!} \sum_{\sigma \in S_l} (\operatorname{sgn} \sigma)\, c\, (\operatorname{sgn} \sigma)\, g(\mathbf{v}_1, \ldots, \mathbf{v}_l) \\
&= \frac{1}{l!} \sum_{\sigma \in S_l} c\, g(\mathbf{v}_1, \ldots, \mathbf{v}_l) \\
&= \frac{1}{l!} l! c\, g(\mathbf{v}_1, \ldots, \mathbf{v}_l) \\
&= c\, g(\mathbf{v}_1, \ldots, \mathbf{v}_l).
\end{aligned}$$

Thus $c \wedge g = cg$, for $c \in \mathbb{R}$ and $g \in A_l(V)$.

Example C.6. For $f \in A_2(V)$ and $g \in A_1(V)$,

$$\begin{aligned}
\mathcal{A}(f \otimes g)(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = f(\mathbf{v}_1, \mathbf{v}_2)\, g(\mathbf{v}_3) - f(\mathbf{v}_1, \mathbf{v}_3)\, g(\mathbf{v}_2) - f(\mathbf{v}_2, \mathbf{v}_1)\, g(\mathbf{v}_3) \\
- f(\mathbf{v}_3, \mathbf{v}_2)\, g(\mathbf{v}_1) + f(\mathbf{v}_2, \mathbf{v}_3)\, g(\mathbf{v}_1) + f(\mathbf{v}_3, \mathbf{v}_1)\, g(\mathbf{v}_2).
\end{aligned}$$

Among these 6 terms, there are 3 pairs of equal terms due to the alternating nature of f.

$$\begin{aligned}
f(\mathbf{v}_1, \mathbf{v}_2)\, g(\mathbf{v}_3) &= -f(\mathbf{v}_2, \mathbf{v}_1)\, g(\mathbf{v}_3), \\
f(\mathbf{v}_3, \mathbf{v}_1)\, g(\mathbf{v}_2) &= -f(\mathbf{v}_1, \mathbf{v}_3)\, g(\mathbf{v}_2), \\
f(\mathbf{v}_2, \mathbf{v}_3)\, g(\mathbf{v}_1) &= -f(\mathbf{v}_3, \mathbf{v}_2)\, g(\mathbf{v}_1).
\end{aligned}$$

Therefore,

$$\mathcal{A}(f \otimes g)(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = 2f(\mathbf{v}_1, \mathbf{v}_2)\, g(\mathbf{v}_3) + 2f(\mathbf{v}_3, \mathbf{v}_1)\, g(\mathbf{v}_2) + 2f(\mathbf{v}_2, \mathbf{v}_3)\, g(\mathbf{v}_1). \tag{C.28}$$

Hence,

$$\begin{aligned}
(f \wedge g)(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) &= \frac{1}{2!1!} \mathcal{A}(f \otimes g)(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \\
&= f(\mathbf{v}_1, \mathbf{v}_2)\, g(\mathbf{v}_3) + f(\mathbf{v}_3, \mathbf{v}_1)\, g(\mathbf{v}_2) + f(\mathbf{v}_2, \mathbf{v}_3)\, g(\mathbf{v}_1).
\end{aligned} \tag{C.29}$$

Example C.7 (Wedge product of 2 covectors)**.** If $f, g \in A_1(V)$, and $\mathbf{v}_1, \mathbf{v}_2 \in V$, then

$$(f \wedge g)(\mathbf{v}_1, \mathbf{v}_2) = \frac{1}{1!1!} \mathcal{A}(f \otimes g)(\mathbf{v}_1, \mathbf{v}_2).$$

S_2 has 2 elements: the identity element e and $(1\ 2)$. Therefore,

$$(f \wedge g)(\mathbf{v}_1, \mathbf{v}_2) = \mathcal{A}(f \otimes g)(\mathbf{v}_1, \mathbf{v}_2) = f(\mathbf{v}_1)\, g(\mathbf{v}_2) - f(\mathbf{v}_2)\, g(\mathbf{v}_1).$$

Proposition C.8 (Anticommutativity of wedge product)

The wedge product is anticommutative: if $f \in A_k(V)$ and $g \in A_l(V)$, then

$$f \wedge g = (-1)^{kl} g \wedge f.$$

Proof. Define $\tau \in S_{k+l}$ to be the following permutation:

$$\begin{bmatrix} 1 & 2 & \cdots & l & l+1 & l+2 & \cdots & l+k \\ k+1 & k+2 & \cdots & k+l & 1 & 2 & \cdots & k \end{bmatrix}.$$

In other words,

$$\tau(i) = \begin{cases} k+i & \text{if } 1 \le i \le l, \\ i-l & \text{if } l+1 \le i \le l+k. \end{cases}$$

Then for any $\sigma \in S_{k+l}$,

$$\sigma(j) = \begin{cases} \sigma(\tau(l+j)) & \text{if } 1 \le j \le k, \\ \sigma(\tau(j-k)) & \text{if } k+1 \le j \le k+l. \end{cases} \tag{C.30}$$

Now, for any $\mathbf{v}_1, \ldots, \mathbf{v}_{k+l} \in V$,

$$\begin{aligned}
\mathcal{A}(f \otimes g)(\mathbf{v}_1, \ldots, \mathbf{v}_{k+l}) &= \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma) f\left(\mathbf{v}_{\sigma(1)}, \ldots, \mathbf{v}_{\sigma(k)}\right) g\left(\mathbf{v}_{\sigma(k+1)}, \ldots, \mathbf{v}_{\sigma(k+l)}\right) \\
&= \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma) f\left(\mathbf{v}_{\sigma(\tau(l+1))}, \ldots, \mathbf{v}_{\sigma(\tau(l+k))}\right) g\left(\mathbf{v}_{\sigma(\tau(1))}, \ldots, \mathbf{v}_{\sigma(\tau(l))}\right) \\
&= (\operatorname{sgn} \tau) \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma)(\operatorname{sgn} \tau)\, g\left(\mathbf{v}_{\sigma(\tau(1))}, \ldots, \mathbf{v}_{\sigma(\tau(l))}\right) f\left(\mathbf{v}_{\sigma(\tau(l+1))}, \ldots, \mathbf{v}_{\sigma(\tau(l+k))}\right) \\
&= (\operatorname{sgn} \tau) \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma\tau)\, g\left(\mathbf{v}_{\sigma(\tau(1))}, \ldots, \mathbf{v}_{\sigma(\tau(l))}\right) f\left(\mathbf{v}_{\sigma(\tau(l+1))}, \ldots, \mathbf{v}_{\sigma(\tau(l+k))}\right).
\end{aligned}$$

Again, as σ varies over S_{k+l}, $\sigma\tau$ also varies over S_{k+l}. Therefore,

$$\mathcal{A}(f \otimes g)(\mathbf{v}_1, \ldots, \mathbf{v}_{k+l}) = (\operatorname{sgn} \tau)\, \mathcal{A}(g \otimes f)(\mathbf{v}_1, \ldots, \mathbf{v}_{k+l}). \tag{C.31}$$

Now, let us evaluate the sign of the permutation τ. Let $(\tau(i), \tau(j))$ be an inversion of τ. Then it's not possible that $1 \le i < j \le l$, or $l+1 \le i < j \le l+k$; because if we have $1 \le i < j \le l$ or $l+1 \le i < j \le l+k$, then $\tau(i) < \tau(j)$. Therefore, i must be in between 1 and l (inclusive), and j must be in between $l+1$ and $l+k$ (inclusive). So there are l options for i, and k options for j. Therefore, τ has kl many inversions. So $\operatorname{sgn} \tau = (-1)^{kl}$. Using (C.31),

$$\mathcal{A}(f \otimes g) = (-1)^{kl} \mathcal{A}(g \otimes f). \tag{C.32}$$

Dividing by $k!l!$, we obtain

$$f \wedge g = (-1)^{kl} g \wedge f. \tag{C.33}$$

■

Corollary C.9

If f is a k-covector on V, i.e. $f \in A_k(V)$, and k is odd, then $f \wedge f = 0$.

Proof. By anticommutativity of wedge product,

$$f \wedge f = (-1)^{k^2} f \wedge f = -f \wedge f.$$

Therefore, $f \wedge f = 0$. ■

If f is a k-covector and g is an l-covector, i.e. $f \in A_k(V)$ and $g \in A_l(V)$, then we have defined their wedge product to be the $(k+l)$-covector

$$f \wedge g = \frac{1}{k!l!} \mathcal{A}(f \otimes g). \tag{C.34}$$

We have the following lemmas associated with the alternating operator $\mathcal{A}$.

Lemma C.10

Suppose $f \in L_k(V)$ and $g \in L_l(V)$. Then

(i) $\mathcal{A}(\mathcal{A}(f) \otimes g) = k!\mathcal{A}(f \otimes g)$.

(ii) $\mathcal{A}(f \otimes \mathcal{A}(g)) = l!\mathcal{A}(f \otimes g)$.

Proof. (i) By definition,

$$\begin{aligned}\mathcal{A}(\mathcal{A}(f) \otimes g) &= \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma)\, \sigma\, (\mathcal{A}(f) \otimes g) \\ &= \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma)\, \sigma \left[\sum_{\tau \in S_k} (\operatorname{sgn} \tau)\, (\tau f) \otimes g\right]. \end{aligned} \tag{C.35}$$

We can view $\tau \in S_k$ as a permutation in the following way: define $\tau' \in S_{k+l}$ as follows

$$\tau'(i) = \begin{cases} \tau(i) & \text{if } i \leq k, \\ i & \text{if } i > k. \end{cases} \tag{C.36}$$

Then for $\mathbf{v}_1, \ldots, \mathbf{v}_{k+l}$, we have

$$\begin{aligned}[(\tau f) \otimes g](\mathbf{v}_1, \ldots, \mathbf{v}_{k+l}) &= (\tau f)(\mathbf{v}_1, \ldots, \mathbf{v}_k)\, g(\mathbf{v}_{k+1}, \ldots, \mathbf{v}_{k+l}) \\ &= f\left(\mathbf{v}_{\tau(1)}, \ldots, \mathbf{v}_{\tau(k)}\right) g(\mathbf{v}_{k+1}, \ldots, \mathbf{v}_{k+l}) \\ &= f\left(\mathbf{v}_{\tau'(1)}, \ldots, \mathbf{v}_{\tau'(k)}\right) g\left(\mathbf{v}_{\tau'(k+1)}, \ldots, \mathbf{v}_{\tau'(k+l)}\right) \\ &= [\tau'(f \otimes g)](\mathbf{v}_1, \ldots, \mathbf{v}_{k+l}).\end{aligned}$$

Therefore, $(\tau f) \otimes g = \tau'(f \otimes g)$. Furthermore, $\operatorname{sgn} \tau = \operatorname{sgn} \tau'$ since the inversions $(\tau'(i), \tau'(j))$ occur only when $1 \leq i < j \leq k$, so that the τ and τ' has the same number of inversions.

Let us abuse notation a bit and denote by S_k the subgroup of permutations in S_{k+l} by keeping the last l arguments fixed. This subgroup of S_{k+l} is indeed isomorphic to S_k, so we will denote both these groups by S_k. Therefore, from (C.35),

$$\begin{aligned}\mathcal{A}(\mathcal{A}(f) \otimes g) &= \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma)\, \sigma \left[\sum_{\tau' \in S_k \subseteq S_{k+l}} (\operatorname{sgn} \tau')\, \tau'(f \otimes g)\right] \\ &= \sum_{\sigma \in S_{k+l}} \sum_{\tau' \in S_k \subseteq S_{k+l}} (\operatorname{sgn} \sigma)(\operatorname{sgn} \tau')\, \sigma\tau'(f \otimes g) \\ &= \sum_{\tau' \in S_k \subseteq S_{k+l}} \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma \operatorname{sgn} \tau')\, ((\sigma\tau')(f \otimes g)).\end{aligned}$$

For a fixed τ', as σ varies over S_{k+l}, $\sigma\tau'$ also varies over S_{k+l}. Therefore,

$$\mathcal{A}(\mathcal{A}(f) \otimes g) = \sum_{\tau' \in S_k \subseteq S_{k+l}} \mathcal{A}(f \otimes g) = k!\mathcal{A}(f \otimes g). \tag{C.37}$$

(ii) By (C.32),

$$\begin{aligned}\mathcal{A}(f \otimes \mathcal{A}(g)) &= \mathcal{A}\left((-1)^{kl}\mathcal{A}(g) \otimes f\right) \\ &= (-1)^{kl}\mathcal{A}(\mathcal{A}(g) \otimes f) \\ &= (-1)^{kl}\, l!\mathcal{A}(g \otimes f) \\ &= l!\mathcal{A}\left((-1)^{kl} g \otimes f\right) \\ &= l!\mathcal{A}(f \otimes g). \end{aligned} \tag{C.38}$$

■

Proposition C.11 (Associativity of wedge product)

Let V be a real vector space and f, g, h be alternating multilinear functions on V of degree k, l, m, respectively. Then

$$(f \wedge g) \wedge h = f \wedge (g \wedge h) .$$

Proof. Using the definition of wedge product,

$$\begin{aligned}
(f \wedge g) \wedge h &= \frac{1}{(k+l)!m!} \mathcal{A}\left[(f \wedge g) \otimes h\right] \\
&= \frac{1}{(k+l)!m!} \mathcal{A}\left[\frac{1}{k!l!} \mathcal{A}(f \otimes g) \otimes h\right] \\
&= \frac{1}{(k+l)!k!l!m!} \mathcal{A}\left[\mathcal{A}(f \otimes g) \otimes h\right] \\
&= \frac{(k+l)!}{(k+l)!k!l!m!} \mathcal{A}\left[(f \otimes g) \otimes h\right] \\
&= \frac{1}{k!l!m!} \mathcal{A}\left[(f \otimes g) \otimes h\right] .
\end{aligned}$$

On the other hand,

$$\begin{aligned}
f \wedge (g \wedge h) &= \frac{1}{k!\,(l+m)!} \mathcal{A}\left[f \otimes (g \wedge h)\right] \\
&= \frac{1}{k!\,(l+m)!} \mathcal{A}\left[f \otimes \left(\frac{1}{l!m!} \mathcal{A}(g \otimes h)\right)\right] \\
&= \frac{1}{k!\,(l+m)!l!m!} \mathcal{A}\left[f \otimes \mathcal{A}(g \otimes h)\right] \\
&= \frac{(l+m)!}{k!\,(l+m)!l!m!} \mathcal{A}\left[f \otimes (g \otimes h)\right] \\
&= \frac{1}{k!l!m!} \mathcal{A}\left[f \otimes (g \otimes h)\right] .
\end{aligned}$$

Since tensor product is associative (by Lemma C.7), we conclude that

$$(f \wedge g) \wedge h = f \wedge (g \wedge h) . \tag{C.39}$$

■

By associativity, we can omit the parenthesis and write univocally $f \wedge g \wedge h$ instead of $(f \wedge g) \wedge h$ or $f \wedge (g \wedge h)$.

Corollary C.12

Under the hypothesis of Proposition C.11,

$$f \wedge g \wedge h = \frac{1}{k!l!m!} \mathcal{A}\left[f \otimes g \otimes h\right] . \tag{C.40}$$

This easily generalizes to an arbitrary number of factors: if $f_i \in A_{d_i}(V)$ for $i = 1, 2, \ldots, r$, i.e. f_i is an alternating d_i-linear function on V, then

$$f_1 \wedge \cdots \wedge f_r = \frac{1}{d_1! \cdots d_r!} \mathcal{A}(f_1 \otimes \cdots \otimes f_r) . \tag{C.41}$$

Proposition C.13

Let $\widehat{\alpha}^1, \widehat{\alpha}^2, \ldots, \widehat{\alpha}^k$ be linear functions on a real vector space V (i.e. $\widehat{\alpha}^i : V \to \mathbb{R}$) and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in V$. Then

$$\begin{aligned}\left(\widehat{\alpha}^1 \wedge \cdots \wedge \widehat{\alpha}^k\right)(\mathbf{v}_1, \ldots, \mathbf{v}_k) &= \det\left[\widehat{\alpha}^i(\mathbf{v}_j)\right] \\ &= \det\begin{bmatrix} \widehat{\alpha}^1(\mathbf{v}_1) & \widehat{\alpha}^1(\mathbf{v}_2) & \cdots & \widehat{\alpha}^1(\mathbf{v}_k) \\ \widehat{\alpha}^2(\mathbf{v}_1) & \widehat{\alpha}^2(\mathbf{v}_2) & \cdots & \widehat{\alpha}^2(\mathbf{v}_k) \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{\alpha}^k(\mathbf{v}_1) & \widehat{\alpha}^k(\mathbf{v}_2) & \cdots & \widehat{\alpha}^k(\mathbf{v}_k) \end{bmatrix}.\end{aligned}$$

Proof. By C.41,

$$\left(\widehat{\alpha}^1 \wedge \cdots \wedge \widehat{\alpha}^k\right)(\mathbf{v}_1, \ldots, \mathbf{v}_k) = \mathcal{A}\left(\widehat{\alpha}^1 \otimes \cdots \otimes \widehat{\alpha}^k\right)(\mathbf{v}_1, \ldots, \mathbf{v}_k).$$

By the definition of the action of alternating operator,

$$\mathcal{A}\left(\widehat{\alpha}^1 \otimes \cdots \otimes \widehat{\alpha}^k\right)(\mathbf{v}_1, \ldots, \mathbf{v}_k) = \sum_{\sigma \in S_k} (\operatorname{sgn}\sigma)\, \widehat{\alpha}^1\left(\mathbf{v}_{\sigma(1)}\right) \cdots \widehat{\alpha}^k\left(\mathbf{v}_{\sigma(k)}\right). \tag{C.42}$$

By the definition of determinant of a $k \times k$ matrix $A = [a_{ij}]$,

$$\det A = \sum_{\sigma \in S_k} (\operatorname{sgn}\sigma)\, a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{k\sigma(k)}. \tag{C.43}$$

Using (C.43) in (C.42), we get

$$\mathcal{A}\left(\widehat{\alpha}^1 \otimes \cdots \otimes \widehat{\alpha}^k\right)(\mathbf{v}_1, \ldots, \mathbf{v}_k) = \det\left[\widehat{\alpha}^i(\mathbf{v}_j)\right]. \tag{C.44}$$

■

§C.5 A Basis for $A_k(V)$

Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be a basis for a real vector space V, and let $\{\widehat{\alpha}^1, \ldots, \widehat{\alpha}^n\}$ be the dual basis for V^*. Introduce the multi-index notation

$$I = (i_1, i_2, \ldots, i_k)$$

and write $\mathbf{e}_I$ for $(\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \ldots, \mathbf{e}_{i_k})$ and $\widehat{\alpha}^I$ for $\widehat{\alpha}^{i_1} \wedge \widehat{\alpha}^{i_2} \wedge \cdots \wedge \widehat{\alpha}^{i_k}$.

A k-linear function f on V is completely determined by its values on all k-tuples $(\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \ldots, \mathbf{e}_{i_k})$. If f is alternating, then f is completely determined by its values on all k-tuples $(\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \ldots, \mathbf{e}_{i_k})$ with

$$1 \le i_1 < i_2 < \cdots < i_k \le n.$$

In other words, it's sufficient to consider $\mathbf{e}_I$ with I in ascending order.

Lemma C.14

Suppose I and J are ascending multi-indices of length k. Then

$$\widehat{\alpha}^I(\mathbf{e}_J) = \delta^I{}_J := \begin{cases} 1 & \text{if } I = J, \\ 0 & \text{if } I \neq J. \end{cases}$$

Proof. Suppose $I = (i_1, \ldots, i_k)$ and $J = (j_1, \ldots, j_k)$. Using (C.42), we get

$$\begin{aligned}\widehat{\alpha}^I(\mathbf{e}_J) &= \left(\widehat{\alpha}^{i_1} \wedge \widehat{\alpha}^{i_2} \wedge \cdots \wedge \widehat{\alpha}^{i_k}\right)(\mathbf{e}_{j_1}, \mathbf{e}_{j_2}, \ldots, \mathbf{e}_{j_k}) \\ &= \sum_{\sigma \in S_k} (\operatorname{sgn}\sigma)\, \widehat{\alpha}^{i_1}\left(\mathbf{e}_{j_{\sigma(1)}}\right) \cdots \widehat{\alpha}^{i_k}\left(\mathbf{e}_{j_{\sigma(k)}}\right) \\ &= \sum_{\sigma \in S_k} (\operatorname{sgn}\sigma)\, \delta^{i_1}{}_{j_{\sigma(1)}} \cdots \delta^{i_k}{}_{j_{\sigma(k)}}. \end{aligned} \tag{C.45}$$

The terms in the sum (C.45) contribute $\operatorname{sgn}\sigma$ if and only if

$$(i_1, i_2, \ldots, i_k) = \left(j_{\sigma(1)}, j_{\sigma(2)}, \ldots, j_{\sigma(k)}\right);$$

otherwise they contribute 0 to the sum. Both I and J are ascending multi-indices. Permuting the elements of J no longer gives an ascending multi-index (unless the permutation σ is the identity permutation). Therefore, in (C.45), all the summands corresponding to σ being a non-identity permutation contribute 0.

$$\widehat{\alpha}^I(\mathbf{e}_J) = \sum_{\sigma \in S_k} (\operatorname{sgn}\sigma)\, \delta^{i_1}{}_{j_{\sigma(1)}} \cdots \delta^{i_k}{}_{j_{\sigma(k)}} = \delta^{i_1}{}_{j_1} \cdots \delta^{i_k}{}_{j_k} = \begin{cases} 1 & \text{if } I = J, \\ 0 & \text{if } I \neq J. \end{cases} \tag{C.46}$$

■

Proposition C.15

The alternating k-linear functions $\widehat{\alpha}^I$, $I = (i_1, \ldots, i_k)$, with $1 \leq i_1 < \cdots < i_k \leq n$ form a basis for the space $A_k(V)$ of alternating k-linear functions on V.

Proof. Let us first show linear independence. Suppose

$$\sum_I c_I \widehat{\alpha}^I = \mathbf{0}, \tag{C.47}$$

$c_I \in \mathbb{R}$ with I running over ascending multi-indices of length k. Applying $\mathbf{e}_J$ to both sides, we get

$$0 = \sum_I c_I \widehat{\alpha}^I(\mathbf{e}_J) = \sum_I c_I \delta^I{}_J = c_J. \tag{C.48}$$

Therefore, $\left\{\widehat{\alpha}^I \mid I \text{ is ascending multi-index of length } k\right\}$ is a linearly independent set. Now let us prove that this set spans $A_k(V)$. Let $f \in A_k(V)$. We claim that

$$f = \sum_I f(\mathbf{e}_I)\, \widehat{\alpha}^I. \tag{C.49}$$

Let $g = \sum_I f(\mathbf{e}_I)\, \widehat{\alpha}^I$. We need to prove that $f = g$. By k-linearity and alternating property, if two k-covectors agree on all $\mathbf{e}_J$ where J is an ascending multi-index, then they are equal. Now,

$$g(\mathbf{e}_J) = \sum_I f(\mathbf{e}_I)\, \widehat{\alpha}^I(\mathbf{e}_J) = \sum_I f(\mathbf{e}_I)\, \delta^I{}_J = f(\mathbf{e}_J). \tag{C.50}$$

Therefore, $f = g = \sum_I f(\mathbf{e}_I)\, \widehat{\alpha}^I$. ■

Corollary C.16

If the vector space V has dimension n, then the vector space $A_k(V)$ of k-covectors on V has dimension $\binom{n}{k}$.

Proof. An ascending multi-index $I = (i_1, i_2, \ldots, i_k)$, $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ is obtained by choosing a k-element subset of $\{1, 2, \ldots, n\}$. This can be done in $\binom{n}{k}$ ways. ■

Corollary C.17

If $k > \dim V$, then $A_k(V) = 0$.

Proof. If $k > \dim V = n$, then in the expression

$$\widehat{\alpha}^{i_1} \wedge \widehat{\alpha}^{i_2} \wedge \cdots \wedge \widehat{\alpha}^{i_k}$$

with each $i \in \{1, 2, \ldots, n\}$, there must be a repeated i_j's, say $\widehat{\alpha}^r$. Then $\widehat{\alpha}^r \wedge \widehat{\alpha}^r$ arises in the expression $\widehat{\alpha}^{i_1} \wedge \widehat{\alpha}^{i_2} \wedge \cdots \wedge \widehat{\alpha}^{i_k}$. But $\widehat{\alpha}^r \wedge \widehat{\alpha}^r = 0$ by Corollary C.9. Hence, $\widehat{\alpha}^{i_1} \wedge \widehat{\alpha}^{i_2} \wedge \cdots \wedge \widehat{\alpha}^{i_k} = 0$. Therefore, the basis set of $A_k(V)$ is empty, meaning $A_k(V) = 0$. ■

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Geometry is the science of correct reasoning on incorrect figures.

- George Pólya