



Inspiring Excellence

Category Theory (MAT434)

Lecture Notes

Preface

This series of lecture notes has been prepared for aiding students who took the BRAC University course **Category Theory (MAT434)** in Summer 2023 semester. These notes were typeset under the supervision of mathematician **Dr. Syed Hasibul Hassan Chowdhury**. The main goal of this typeset is to have an organized digital version of the notes, which is easier to share and handle. If you see any mistakes or typos, please send me an email at atonuroychowdhury@gmail.com

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References:

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- *Category Theory for Scientists*, by **David Spivak**.
- *Categories for the Working Mathematician*, by **Saunders Mac Lane**.
- *Basic Category Theory*, by **Tom Leinster**.

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1 Categories

§1.1 Definition of a Category

Category theory arises from the idea of a system of “functions” among some objects.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow g \circ f & \downarrow g \\ & & C \end{array}$$

A category consists of objects A, B, C, \dots and arrows $f : A \rightarrow B, g : B \rightarrow C, \dots$ that are closed under composition and satisfy certain conditions typical of composition of functions. Before formally defining what a category is, let us begin our discussion with the setting where the objects are sets and arrows are functions between sets.

Let f be a function from a set A to another set B . This is mathematically expressed as $f : A \rightarrow B$. Explicitly, it refers to the fact that f is defined for all of A , and all the values of f are contained in B . In other words, $\text{range}(f) \subseteq B$.

Now suppose we have another function $g : B \rightarrow C$. Then there is a unique function $g \circ f : A \rightarrow C$, given by

$$(g \circ f)(a) = g(f(a)), \quad \text{for } a \in A. \quad (1.1)$$

This unique function is called the composite of g and f , or g after f .

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow g \circ f & \downarrow g \\ & & C \end{array}$$

Now, this operation \circ of composition of functions is associative. In other words, the two arrows from A to D in the following diagram are the same:

$$\begin{array}{ccccc} & & (h \circ g) \circ f & & \\ & \nearrow & \text{---} & \searrow & \\ & & h \circ g & & \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & D \\ & \searrow & \text{---} & \nearrow & \\ & & g \circ f & & \\ & \nwarrow & \text{---} & \nearrow & \\ & & h \circ (g \circ f) & & \end{array}$$

Given $f : A \rightarrow B$, $g : B \rightarrow C$ and $h : C \rightarrow D$, one has unique compositions $h \circ g : B \rightarrow D$ and $g \circ f : A \rightarrow C$. These two composed functions can be further composed with f (from the left) and with h (from the right), respectively, to yield a unique function

$$(h \circ g) \circ f = h \circ (g \circ f), \quad (1.2)$$

from A to D as demanded by the associativity law. Using the definition of composition of functions, one verifies that this is indeed the case:

$$\begin{aligned} ((h \circ g) \circ f)(a) &= (h \circ g)(f(a)) = h(g(f(a))), \\ (h \circ (g \circ f))(a) &= h((g \circ f)(a)) = h(g(f(a))). \end{aligned}$$

Therefore, $(h \circ g) \circ f = h \circ (g \circ f)$.

Finally, note that for every set A , there is an identity function $\mathbb{1}_A : A \rightarrow A$ given by

$$\mathbb{1}_A(a) = a. \quad (1.3)$$

These identity functions act as units for composition, i.e. given $f : A \rightarrow B$, we have

$$\begin{aligned} (f \circ \mathbb{1}_A)(a) &= f(\mathbb{1}_A(a)) = f(a), \\ (\mathbb{1}_B \circ f)(a) &= \mathbb{1}_B(f(a)) = f(a), \end{aligned}$$

for each $a \in A$. Therefore,

$$f \circ \mathbb{1}_A = \mathbb{1}_B \circ f = f. \quad (1.4)$$

The equality above is equivalent to the following commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{\mathbb{1}_A} & A \\ & \searrow f \circ \mathbb{1}_A & \downarrow f \\ & & B \end{array} \quad \begin{array}{ccc} & & \searrow \mathbb{1}_B \circ f \\ & & B \\ & \xrightarrow{\mathbb{1}_B} & B \end{array}$$

We have the following abstract version of sets and functions between sets called a **category**.

Definition 1.1 (Category). A **category** \mathcal{C} consists of the following data:

- **Objects:** A, B, C, \dots The collection of objects of \mathcal{C} is denoted by $\text{Ob}(\mathcal{C})$.
- **Arrows:** f, g, h, \dots Given two objects A and B , the set of arrows from A to B is denoted by $\text{Hom}_{\mathcal{C}}(A, B)$.
- For each arrow f , there are given objects $\text{dom}(f)$, $\text{cod}(f)$ called the **domain** and **codomain** of f . We write $f : A \rightarrow B$ to indicate that $A = \text{dom}(f)$ and $B = \text{cod}(f)$.
- Given arrows $f : A \rightarrow B$ and $g : B \rightarrow C$, i.e. with $\text{cod}(f) = \text{dom}(g)$, there is a unique arrow $g \circ f : A \rightarrow C$, i.e. $g \circ f \in \text{Hom}_{\mathcal{C}}(A, C)$ called the **composite** of f and g . This fact can be rephrased as the following: given $A, B, C \in \text{Ob}(\mathcal{C})$, there is a function

$$\circ : \text{Hom}_{\mathcal{C}}(B, C) \times \text{Hom}_{\mathcal{C}}(A, B) \rightarrow \text{Hom}_{\mathcal{C}}(A, C), \quad (1.5)$$

with $(g, f) \mapsto g \circ f$. The well-definedness of \circ is synonymous to claiming that $g \circ f \in \text{Hom}_{\mathcal{C}}(A, C)$ is unique for given $g \in \text{Hom}_{\mathcal{C}}(B, C)$ and $f \in \text{Hom}_{\mathcal{C}}(A, B)$.

- For each $A \in \text{Ob}(\mathcal{C})$, there exists an unique arrow $\mathbb{1}_A \in \text{Hom}_{\mathcal{C}}(A, A)$.

The above data are required to satisfy the following laws:

- **Associativity:** For any $f \in \text{Hom}_{\mathcal{C}}(A, B)$, $g \in \text{Hom}_{\mathcal{C}}(B, C)$, $h \in \text{Hom}_{\mathcal{C}}(C, D)$ with $A, B, C, D \in \text{Ob}(\mathcal{C})$,

$$(h \circ g) \circ f = h \circ (g \circ f), \quad (1.6)$$

- **Unit:** For any $f \in \text{Hom}_{\mathcal{C}}(A, B)$ with $A, B \in \text{Ob}(\mathcal{C})$,

$$f \circ \mathbb{1}_A = \mathbb{1}_B \circ f = f. \quad (1.7)$$

Remark 1.1. Suppose we have the following commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow g & \downarrow h \\ & & C \end{array} \quad h \circ f = h \circ g$$

Commutativity of this diagram doesn't violate the uniqueness of the composition \circ . It just means that the map \circ in (1.5) is a many-to-one function.

§1.2 Examples of Categories

- Sets and functions between sets. This category is called **Sets**.
- Groups and group homomorphisms
- Vector spaces and linear mappings between them
- Graphs and graph isomorphisms
- The set of real numbers \mathbb{R} as an object, and continuous functions $f : \mathbb{R} \rightarrow \mathbb{R}$ as arrows
- Open subsets $U \subseteq \mathbb{R}$ and continuous functions $f : U \rightarrow V \subseteq \mathbb{R}$ defined on them
- Differentiable manifolds and smooth (C^∞) mappings
- Posets and monotone functions.

Let us discuss the last category at length.

Definition 1.2. A **partially ordered set** or **poset** is a set A equipped with a binary relation (a subset of $A \times A$) $a \leq_A b$ (in other words, $(a, b) \in R \subset A \times A$) such that the following conditions hold for all $a, b, c \in A$:

- (i) **Reflexivity:** $a \leq_A a$.
- (ii) **Transitivity:** if $a \leq_A b$ and $b \leq_A c$, then $a \leq_A c$.
- (iii) **Antisymmetry:** if $a \leq_A b$ and $b \leq_A a$, then $a = b$.

Remark 1.2. The antisymmetry condition tells us that if both $a \leq_A b$ and $b \leq_A a$ hold, then a and b cannot be distinct. Contrapositively, for distinct a and b in A , not both $a \leq_A b$ and $b \leq_A a$ hold true. Also, note that if (A, \leq_A) is a partially ordered set, there can be elements $a, b \in A$ such that neither (a, b) nor (b, a) is in R . If it happens that given any $a, b \in A$, either (a, b) or (b, a) is in R , i.e. either $a \leq_A b$ or $b \leq_A a$, then we call A a **totally ordered set**.

Example 1.1. (\mathbb{R}, \leq) , the set of real numbers with the usual ordering \leq is a totally ordered set.

Now we define an arrow from a poset (A, \leq_A) to another poset (B, \leq_B) to be a function $m : A \rightarrow B$ that is **monotone**, in the sense that for all $a, a' \in A$,

$$\text{whenever } a \leq_A a', \text{ one has } m(a) \leq_B m(a').$$

We need to verify that under this definition of arrows, we have a category. First of all, we must have $\mathbb{1}_A : A \rightarrow A$, defined by $\mathbb{1}_A(a) = a$ for each $a \in A$, to be monotone. Indeed, if $a \leq a'$ in A , then we automatically have $\mathbb{1}_A(a) \leq \mathbb{1}_A(a')$. Therefore, $\mathbb{1}_A$ is monotone.

Given monotone functions $f : A \rightarrow B$ between posets (A, \leq_A) and (B, \leq_B) , and $g : B \rightarrow C$ between posets (B, \leq_B) and (C, \leq_C) , we need to verify that the composition $g \circ f : A \rightarrow C$ is also monotone. Indeed, given $a \leq_A a'$, since f is monotone, we have

$$f(a) \leq_B f(a'). \quad (1.8)$$

Since g is monotone, this gives us

$$g(f(a)) \leq_C g(f(a')). \quad (1.9)$$

In other words, $(g \circ f)(a) \leq_C (g \circ f)(a')$ given $a \leq_A a'$. Therefore, $g \circ f : A \rightarrow C$ is monotone.

The category thus formed is called the category of posets and monotone functions, and is denoted by **Pos**.

Finite Categories

A finite category consists of finitely many objects and finitely many arrows between them.

- The category **1** looks as follows:



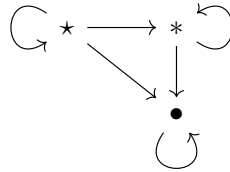
It has one object $*$ and its identity arrow.

- The category **2** looks as follows:



It has two objects $*$ and $*$, their identity arrows, and exactly one arrow $* \rightarrow *$.

- The category **3** looks as follows:



It has three objects $*$, $*$, \bullet , their respective

identity arrows, and the other arrows are $* \rightarrow *$, $* \rightarrow \bullet$, and $* \rightarrow \bullet$ (which is the composition of the previous two arrows).

- The category **0** looks as follows:

It has no objects or arrows.

§1.3 Functor

Definition 1.3 (Functor). A **functor** $F : \mathcal{C} \rightarrow \mathcal{D}$ between categories \mathcal{C} and \mathcal{D} is an assignment of $\text{Ob}(\mathcal{C})$ to $\text{Ob}(\mathcal{D})$ and a mapping of arrows in \mathcal{C} to arrows in \mathcal{D} , i.e. for any $A, B \in \text{Ob}(\mathcal{C})$, a mapping

$$\text{Hom}_{\mathcal{C}}(A, B) \rightarrow \text{Hom}_{\mathcal{D}}(F(A), F(B)),$$

where $F(A), F(B) \in \text{Ob}(\mathcal{D})$ are the assigned objects of \mathcal{D} under F . In other words, for given $A, B \in \text{Ob}(\mathcal{C})$ and an arrow $f : A \rightarrow B$, one has $F(A), F(B) \in \text{Ob}(\mathcal{D})$ and an arrow $F(f) : F(A) \rightarrow F(B)$ such that the following hold:

- (a) $F(1_A) = 1_{F(A)}$.
- (b) $F(g \circ f) = F(g) \circ F(f)$.

In other words, F preserves domains and codomains, identity arrows and composition.

$$\begin{array}{ccc}
 \mathcal{C} & & A \xrightarrow{f} B \\
 \downarrow F & & \searrow g \circ f \downarrow g \\
 \mathcal{D} & & C \\
 & & \downarrow F(g) \\
 & & F(C)
 \end{array}
 \quad
 \begin{array}{ccc}
 F(A) & \xrightarrow{F(f)} & F(B) \\
 \searrow F(g \circ f) = F(g) \circ F(f) & & \downarrow F(g) \\
 & & F(C)
 \end{array}$$

Now, one can see that functors compose in the expected way and that every category \mathcal{C} has a distinguished functor called the identity functor $\mathbb{1}_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}$. Thus we have a category, namely **Cat**, the category of all categories and functors between them.

Preorder

A **preorder** is a set P equipped with a binary relation \leq that is both reflexive and transitive: $a \leq a$; and if $a \leq b$ and $b \leq c$, then $a \leq c$ for any $a, b, c \in P$. Any preorder (P, \leq) can be regarded as a category by taking the objects of the category to be the elements of P and taking a unique arrow

$$a \rightarrow b \text{ if and only if } a \leq b \text{ in } (P, \leq). \quad (1.10)$$

Remark 1.3. Reflexivity and transitivity property ensures that the preorder (P, \leq) is indeed a category. Note that the above condition implies that there is at most one arrow from an object of (P, \leq) to another. In the other direction, any category with at most one arrow from an object to another determines a preorder simply by defining a binary relation \leq on the objects by (1.10).

Remark 1.4 (On the similarities between a poset and a preorder). A poset (P, \leq) is evidently a preorder with the additional condition of antisymmetry. Hence, a poset is also a category. Examples of poset include the power set $\mathcal{P}(X)$ of a given set X under the usual subset relation: $U \subseteq V$ between the subsets U, V of X .

There can be preorders that are not posets. For instance, $(\mathbb{Z}, |)$ is a preorder on the set of integers, where “ $|$ ” is the usual divides binary relation: given $a, b \in \mathbb{Z}$, we have $a | b$ (read a divides b) if and only if $b = ca$ for some $c \in \mathbb{Z}$. It is clearly reflexive and transitive. Note that $a | b$ and $b | a$ imply $a = \pm b$ which is not the same as $a = b$. Hence, “ $|$ ” is not antisymmetric. Therefore, $(\mathbb{Z}, |)$ is a preorder that is not a poset.

§1.4 Monoid

Definition 1.4 (Monoid). A **monoid** is a set M equipped with a binary operation $\cdot : M \times M \rightarrow M$ and a distinguished “unit” element $u \in M$ such that for each $x, y, z \in M$,

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z \text{ and } u \cdot x = x \cdot u = x. \quad (1.11)$$

Equivalently, a monoid is a category with just one object. The arrows of the category are the elements of the monoid. In particular, the identity arrow on the object is the unit element u . Composition of arrows is the binary operation $x \cdot y$ of the monoid.

For example, \mathbb{N} (we are adopting the convention that $0 \in \mathbb{N}$), \mathbb{Q} , \mathbb{R} with addition and 0 as the unit element. Also, \mathbb{N} , \mathbb{Q} , \mathbb{R} with multiplication and 1 as unit are monoids. For any set X , the set of functions from X to itself, written as

$$\text{Hom}_{\mathbf{Sets}}(X, X),$$

is a monoid under the operation of composition. Here **Sets** is the category of sets and functions between sets. More generally, for any object $C \in \text{Ob}(\mathcal{C})$ in a category \mathcal{C} , the set of arrows from C to itself, written as

$$\text{Hom}_{\mathcal{C}}(C, C),$$

is a monoid under the composition of arrows in \mathcal{C} .

Since monoids are structured sets (sets equipped with a binary operation fulfilling associativity, unitality etc.), there is a category **Mon** whose objects are monoids and arrows are functions that preserve the monoid structure, namely monoid homomorphisms. In detail, a **monoid homomorphism** from a monoid (M, \cdot_M) to a monoid (N, \cdot_N) is a function $f : M \rightarrow N$ such that for all $m, n \in M$,

$$h(m \cdot_M n) = h(m) \cdot_N h(n) \text{ and } h(u_M) = u_N. \quad (1.12)$$

Here u_M and u_N are unit elements of M and N , respectively.

§1.4.i Isomorphisms

Definition 1.5. In any category \mathcal{C} , an arrow $f : A \rightarrow B$ is called an **isomorphism** if there is an arrow $g : B \rightarrow A$ such that

$$g \circ f = \mathbb{1}_A \text{ and } f \circ g = \mathbb{1}_B. \quad (1.13)$$

Suppose there is another arrow $\tilde{g} : B \rightarrow A$ with

$$\tilde{g} \circ f = \mathbb{1}_A \text{ and } f \circ \tilde{g} = \mathbb{1}_B. \quad (1.14)$$

Then we have

$$g = g \circ \mathbb{1}_B = g \circ (f \circ \tilde{g}) = (g \circ f) \circ \tilde{g} = \mathbb{1}_A \circ \tilde{g} = \tilde{g}. \quad (1.15)$$

Hence, if an arrow $g : B \rightarrow A$ exists satisfying (1.13), then it is unique. Such unique arrow $g : B \rightarrow A$ is called the inverse of $f : A \rightarrow B$, and we write $g = f^{-1}$. When such an arrow $f : A \rightarrow B$ exists, we say that A is isomorphic to B , written $A \cong B$.

Definition 1.6 (Group). A **group** G is a monoid with an inverse g^{-1} for every element $g \in G$. Thus G is a category with one object in which every arrow is an isomorphism.

The natural numbers \mathbb{N} do not form a group either under addition or multiplication. But the integers \mathbb{Z} form a group under addition. So do the positive rationals \mathbb{Q}^+ under multiplication. For any set X , we have the group $\text{Aut}(X)$ of all the automorphisms of X , i.e. isomorphisms $f : X \rightarrow X$. A **group of permutations** is a subgroup $G \subseteq \text{Aut}(X)$ for some X . Thus the set G must satisfy the following:

1. The identity function $\mathbb{1}_X$ on X is in G .
2. If $g, g' \in G$, then $g \circ g' \in G$.
3. If $g \in G$, $g^{-1} \in G$.

We now have the following theorem due to Arthur Cayley.

Theorem 1.1 (Cayley's theorem)

Every group G is isomorphic to a group of permutations.

Sketch of proof. First, define the Cayley representation \overline{G} of G to be the following group of permutations on a set: the set is G itself, and for each $g \in G$, one has the permutation $\overline{g} : G \rightarrow G$ defined as

$$\overline{g}(h) = g \cdot h \text{ for each } h \in G. \quad (1.16)$$

Indeed, \overline{g} has an inverse $\overline{g}^{-1} = \overline{g^{-1}}$:

$$\overline{g}^{-1}(h) = g^{-1}h. \quad (1.17)$$

One, thus, verifies that $\overline{g} : G \rightarrow G$ is indeed an isomorphism, and hence a permutation on G .

Now define homomorphisms $i : G \rightarrow \overline{G}$ by $i(g) = \overline{g}$, and $j : \overline{G} \rightarrow G$ by $j(\overline{g}) = \overline{g}(u) = g$, with u being the identity element of the group G .

Observe that $i \circ j = \mathbb{1}_{\overline{G}}$ and $j \circ i = \mathbb{1}_G$. Indeed, for $g \in G$ and $\overline{g} \in \overline{G}$,

$$\begin{aligned} (j \circ i)(g) &= j(i(g)) = j(\overline{g}) = g, \\ (i \circ j)(\overline{g}) &= i(j(\overline{g})) = i(g) = \overline{g}, \end{aligned}$$

establishing that $i : G \rightarrow \overline{G}$ is an isomorphism. ■

Remark 1.5. There are two different types of isomorphisms involved in this proof. For each $g \in G$, one defines an isomorphism $\overline{g} : G \rightarrow G$. This is an isomorphism in the category **Sets**. Later, we defined an isomorphism $i : G \rightarrow \overline{G}$, which is an isomorphism in the category **Groups** of groups and group homomorphisms.

Remark 1.6. The group \overline{G} is the group of permutations (automorphisms) on the group G which is a subgroup of the automorphism group on G itself. This subgroup has the same unit element of that of the automorphism group on G , i.e. $\mathbb{1}_G$, the identity function on the group G . Note that this is also the unit of the group \overline{G} which is not the same as $\mathbb{1}_{\overline{G}}$. This identity function $\mathbb{1}_{\overline{G}}$ on \overline{G} was used to establish the required isomorphism in Cayley's theorem.

[Cayley's theorem](#) can be generalized to prove that any category not “too big” (which has the collection of objects to be a set) is isomorphic to a category in which the objects are sets and the arrows are functions between those sets. In other words, any not “too big” category is isomorphic to a subcategory of **Sets**.

Theorem 1.2 (Generalized Cayley's Theorem)

Every category \mathcal{C} with a set of arrows is isomorphic to one in which the objects are sets and the arrows are functions.

Sketch of proof. Define the Cayley representation $\overline{\mathcal{C}}$ of \mathcal{C} to be the following concrete category (i.e. a category where objects are sets and arrows are functions between them):

- **Objects:** objects are sets of the form

$$\overline{\mathcal{C}} = \{f : X \rightarrow C\} = \{f \in \text{Hom}_{\mathcal{C}}(X, C) \mid X \in \text{Ob}(\mathcal{C})\}. \quad (1.18)$$

In other words, the object $\overline{\mathcal{C}}$ is the set of functions whose codomains are C .

- **Arrows:** arrows are functions

$$\overline{g} : \overline{\mathcal{C}} \rightarrow \overline{\mathcal{C}} \quad (1.19)$$

for $g : C \rightarrow D$ in \mathcal{C} , defined for any $f : X \rightarrow C$ in $\overline{\mathcal{C}}$ by $\overline{g}(f) = g \circ f$.

$$\begin{array}{ccc}
X & & \\
f \downarrow & \searrow \bar{g}(f)=g \circ f & \\
C & \xrightarrow{g} & D
\end{array}$$

Then we define a functor $\mathcal{F} : \mathcal{C} \rightarrow \bar{\mathcal{C}}$ that takes the object $C \in \text{Ob}(\mathcal{C})$ to the object $\bar{C} \in \text{Ob}(\bar{\mathcal{C}})$, and takes the arrow $g : C \rightarrow D$ in \mathcal{C} to the arrow $\bar{g} : \bar{C} \rightarrow \bar{D}$ in $\bar{\mathcal{C}}$. In other words,

$$\mathcal{F}(C) = \bar{C} \text{ and } \mathcal{F}(g) = \bar{g}. \quad (1.20)$$

This is a functor since $\mathcal{F}(\mathbb{1}_C) = \mathbb{1}_{\bar{C}}$ and $\mathcal{F}(g \circ h) = \mathcal{F}(g) \circ \mathcal{F}(h)$ for composable arrows g, h in \mathcal{C} .

We then define another functor $\mathcal{G} : \bar{\mathcal{C}} \rightarrow \mathcal{C}$ that takes the object \bar{C} to the codomains of the functions that are in \bar{C} , i.e. $\mathcal{G}(\bar{C}) = C$, and takes the arrow $\bar{g} : \bar{C} \rightarrow \bar{D}$ to the arrow $\bar{g}(\mathbb{1}_{\mathcal{G}(\bar{C})}) = \bar{g}(\mathbb{1}_C) = g \circ \mathbb{1}_C = g : C \rightarrow D$ in \mathcal{C} . In other words,

$$\mathcal{G}(\bar{C}) = C \text{ and } \mathcal{G}(\bar{g}) = g. \quad (1.21)$$

This is a functor since $\mathcal{G}(\mathbb{1}_{\bar{C}}) = \mathbb{1}_C$ and $\mathcal{G}(\bar{g} \circ \bar{h}) = \mathcal{G}(\bar{g}) \circ \mathcal{G}(\bar{h})$ for composable arrows \bar{g}, \bar{h} in $\bar{\mathcal{C}}$. Then one can verify that

$$\mathcal{G} \circ \mathcal{F} = \mathbb{1}_{\mathcal{C}} \text{ and } \mathcal{F} \circ \mathcal{G} = \mathbb{1}_{\bar{\mathcal{C}}}. \quad (1.22)$$

Therefore, \mathcal{C} and $\bar{\mathcal{C}}$ are isomorphic. ■

§1.5 Construction on Categories

1. The product of 2 categories \mathcal{C} and \mathcal{D} , written as $\mathcal{C} \times \mathcal{D}$ has objects of the form (C, D) for $C \in \text{Ob}(\mathcal{C})$ and $D \in \text{Ob}(\mathcal{D})$, and arrows of the form

$$(f, g) : (C, D) \rightarrow (C', D'),$$

with $C, C' \in \text{Ob}(\mathcal{C})$, $D, D' \in \text{Ob}(\mathcal{D})$, $f \in \text{Hom}_{\mathcal{C}}(C, C')$ and $g \in \text{Hom}_{\mathcal{D}}(D, D')$.

Composition and units are defined componentwise, i.e.

$$(f', g') \circ (f, g) = (f' \circ f, g' \circ g) \text{ and } \mathbb{1}_{(C, D)} = (\mathbb{1}_C, \mathbb{1}_D), \quad (1.23)$$

with $C, C', C'' \in \text{Ob}(\mathcal{C})$ and $D, D', D'' \in \text{Ob}(\mathcal{D})$ and

$$\begin{array}{ccc}
\mathbb{1}_C \curvearrowright C & \xrightarrow{f} C' \xrightarrow{f'} C'' & \quad \quad \quad \mathbb{1}_D \curvearrowright D \xrightarrow{g} D' \xrightarrow{g'} D'' \\
& \searrow f' \circ f & \quad \quad \quad \searrow g' \circ g
\end{array}$$

Then in $\mathcal{C} \times \mathcal{D}$, we have

$$\mathbb{1}_{(C, D)} = (\mathbb{1}_C, \mathbb{1}_D) \curvearrowright (C, D) \xrightarrow{(f, g)} (C', D') \xrightarrow{(f', g')} (C'', D'')$$

$(f', g') \circ (f, g) = (f' \circ f, g' \circ g)$

Then there are two **projection functors**:

$$\mathcal{C} \xleftarrow{\pi_1} \mathcal{C} \times \mathcal{D} \xrightarrow{\pi_2} \mathcal{D}$$

Given $(C, D) \in \text{Ob}(\mathcal{C} \times \mathcal{D})$ and $(f, g) : (C, D) \rightarrow (C', D')$,

$$\pi_1(C, D) = C, \quad \pi_1(f, g) = f. \quad (1.24)$$

Similarly,

$$\pi_2(C, D) = D, \quad \pi_2(f, g) = g. \quad (1.25)$$

2. The opposite category \mathcal{C}^{op} has objects that are in a one-to-one correspondence with the objects of \mathcal{C} . Let $C^* \in \text{Ob}(\mathcal{C}^{\text{op}})$ be the object in \mathcal{C}^{op} that corresponds to $C \in \text{Ob}(\mathcal{C})$. Then an arrow $f : C \rightarrow D$ in \mathcal{C} corresponds to an arrow $f^* : D^* \rightarrow C^*$. With this notation, one can define composition and units in \mathcal{C}^{op} in terms of the corresponding operations in \mathcal{C} , namely

$$\mathbb{1}_{C^*} = (\mathbb{1}_C)^*. \quad (1.26)$$

For $f : C \rightarrow D$, $g : D \rightarrow E$ in \mathcal{C} , we have $f^* : D^* \rightarrow C^*$ and $g^* : E^* \rightarrow D^*$ in \mathcal{C}^{op} . Then their composition is defined as

$$f^* \circ g^* = (g \circ f)^*. \quad (1.27)$$

$$\begin{array}{ccc} \mathbb{1}_C \circlearrowleft C \xrightarrow{f} D \xrightarrow{g} E & \xleftrightarrow{\text{Duality}} & \begin{array}{c} \mathbb{1}_{C^*} = (\mathbb{1}_C)^* \\ \circlearrowleft C^* \xleftarrow{f^*} D^* \xleftarrow{g^*} E^* \\ \xleftarrow{f^* \circ g^* = (g \circ f)^*} \end{array} \end{array}$$

3. The slice category \mathcal{C}/C of a category \mathcal{C} over an object $C \in \text{Ob}(\mathcal{C})$ has

- **Objects:** all arrows f in \mathcal{C} such that $\text{cod}(f) = C$. In other words, all arrows $f \in \text{Hom}_{\mathcal{C}}(X, C)$ with some $X \in \text{Ob}(\mathcal{C})$.
- **Arrows:** an arrow a from $f : X \rightarrow C$ to $f' : X' \rightarrow C$ is precisely an arrow in $\text{Hom}_{\mathcal{C}}(X, X')$ such that $f' \circ a = f$. In other words, the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{a} & X' \\ & \searrow f & \swarrow f' \\ & C & \end{array}$$

Now suppose $f, g, h \in \text{Ob} \mathcal{C}/C$ and $a \in \text{Hom}_{\mathcal{C}/C}(f, g), b \in \text{Hom}_{\mathcal{C}/C}(g, h)$. Then there are objects $X, X', X'' \in \text{Ob}(\mathcal{C})$ such that the two triangles in the following diagram commute:

$$\begin{array}{ccccc} & & \text{boa} & & \\ & \searrow & \xrightarrow{\quad} & \searrow & \\ X & \xrightarrow{a} & X' & \xrightarrow{b} & X'' \\ & \searrow f & \downarrow g & \swarrow h & \\ & & C & & \end{array}$$

In other words, $g \circ a = f$ and $h \circ b = g$, so that one obtains

$$f = g \circ a = (h \circ b) \circ a = h \circ (b \circ a). \quad (1.28)$$

Therefore, we have the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\text{boa}} & X'' \\ & \searrow f & \swarrow h \\ & C & \end{array}$$

Hence, $b \circ a \in \text{Hom}_{\mathcal{C}/C}(f, h)$, using the definition of arrows in \mathcal{C}/C . For a given $f \in \text{Ob}(\mathcal{C}/C)$, $\mathbb{1}_f$ is precisely the identity arrow on $\text{dom}(f)$ in the category \mathcal{C} , which is evident from the following commutative diagram:

$$\begin{array}{ccc} \text{dom}(f) & \xrightarrow{\mathbb{1}_{\text{dom}(f)}} & \text{dom}(f) \\ & \searrow f & \swarrow f \\ & C & \end{array}$$

If $g : C \rightarrow D$ is any arrow in \mathcal{C} , then there is a functor called the **composition functor**:

$$g_* : \mathcal{C}/C \rightarrow \mathcal{C}/D,$$

defined on $\text{Ob}(\mathcal{C}/C)$ as

$$g_*(f) = g \circ f. \quad (1.29)$$

$$\begin{array}{ccc} X & & \\ f \downarrow & \searrow g \circ f & \\ C & \xrightarrow{g} & D \end{array}$$

Commutativity of the above diagram dictates that $g \circ f \in \text{Ob}(\mathcal{C}/D)$. Now suppose $f, f' \in \text{Ob}(\mathcal{C}/C)$, and consider $a \in \text{Hom}_{\mathcal{C}/C}(f, f')$ so that the following diagram commutes:

$$\begin{array}{ccccc} X & \xrightarrow{a} & X' & & \\ f \downarrow & & \searrow f' & \searrow g \circ f' & \\ & & C & \xrightarrow{g} & D \\ & \nearrow g \circ f & & & \end{array}$$

$$g \circ f = g \circ (f' \circ a) = (g \circ f') \circ a, \quad (1.30)$$

so the diagram indeed commutes. So we have the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{a} & X' \\ & \searrow g \circ f & \swarrow g \circ f' \\ & D & \end{array}$$

The commutativity of this diagram dictates that $g_*(a) = a$. In fact, the whole construction above is a functor $\mathcal{C}/(-) : \mathcal{C} \rightarrow \mathbf{Cat}$.

$$\boxed{\begin{array}{ccc} \mathcal{C} & & \\ C \xrightarrow{g} D & & \end{array}} \xrightarrow{\mathcal{C}/(-)} \boxed{\begin{array}{ccc} \mathbf{Cat} & & \\ \mathcal{C}/C \xrightarrow{g_*} \mathcal{C}/D & & \end{array}}$$

4. The coslice category C/\mathcal{C} of a category \mathcal{C} under an object $C \in \text{Ob}(\mathcal{C})$ has as objects all arrows f of \mathcal{C} such that $\text{dom}(f) = C$. An arrow in $\text{Hom}_{C/\mathcal{C}}(f, f')$ is an arrow $h \in \text{Hom}_{\mathcal{C}}(X, X')$ (where $X = \text{cod}(f)$ and $X' = \text{cod}(f')$) such that the diagram below commutes:

$$\begin{array}{ccc} & C & \\ f \swarrow & & \searrow f' \\ X & \xrightarrow{h} & X' \end{array}$$

In other words,

$$h \circ f = f'. \quad (1.31)$$

Question. How can the coslice category be defined in terms of the slice category and the opposite construction?

Example 1.2. The category \mathbf{Sets}_* of pointed sets consists of sets A with a distinguished element $a \in A$, and arrows $f : (A, a) \rightarrow (B, b)$ are functions $f : A \rightarrow B$ that preserves the distinguished elements $f(a) = b$. Now, \mathbf{Sets}_* is isomorphic to the coslice category $1/\mathbf{Sets}$ of sets under any singleton $1 = \{\star\}$.

$$\mathbf{Sets}_* \cong 1/\mathbf{Sets}. \quad (1.32)$$

Indeed, functions $\bar{a} : 1 \rightarrow A$ are uniquely determined by $\bar{a}(\star) = a \in A$, and are objects in $1/\mathbf{Sets}$. Now we define a functor $\mathcal{F} : \mathbf{Sets}_* \rightarrow 1/\mathbf{Sets}$ by

$$\mathcal{F}(A, a) = \bar{a} \text{ and } \mathcal{F}(f) = f. \quad (1.33)$$

Then we define $\mathcal{G} : 1/\mathbf{Sets} \rightarrow \mathbf{Sets}_*$ by

$$\mathcal{G}(\bar{a}) = (A, \bar{a}(\star)) \text{ and } \mathcal{G}(f) = f. \quad (1.34)$$

One can easily verify that \mathcal{G} and \mathcal{F} are functors, and

$$\mathcal{G} \circ \mathcal{F} = \mathbf{1}_{\mathbf{Sets}_*} \text{ and } \mathcal{F} \circ \mathcal{G} = \mathbf{1}_{1/\mathbf{Sets}}. \quad (1.35)$$

Therefore, $1/\mathbf{Sets}$ and \mathbf{Sets}_* are isomorphic categories.

2 Free Categories

§2.1 Free Monoid

Start with an “alphabet” A of “letters” a, b, c, \dots , i.e. a set

$$A = \{a, b, c, \dots\}. \quad (2.1)$$

A word over A is a finite sequence of letters:

`thisword, categoriesarefun, asdfghjkl, ...`

We write “-” for empty word. The **Kleene closure** of A is defined to be the set

$$A^* = \{\text{words over } A\}. \quad (2.2)$$

Define a binary operation $*$ on A^* by $w * w' = ww'$ for words $w, w' \in A^*$. Thus, the binary operation $*$ on A^* is just concatenation. The operation can easily be seen to be associative, and the empty word “-” is a unit. Therefore, A^* is a monoid—called the **free monoid** on the set A .

The number of letters in a word is called its **length**. The elements $a \in A$ can be regarded as words of length 1. One has a function $i : A \rightarrow A^*$ defined by $i(a) = a$, and called the “insertion of generators”. The elements of A generate the free monoid, in the sense that every $w \in A^*$ can be written as a $*$ products of elements of A , i.e.,

$$w = a_1 * a_2 * \dots * a_n,$$

for some $a_1, \dots, a_n \in A$.

A monoid M is **freely generated** by a subset A of M , if the following conditions hold:

- (a) Every element $m \in M$ can be written as a product of elements of A :

$$m = a_1 \cdot_M a_2 \cdot_M \dots \cdot_M a_n, \quad \text{where } a_i \in A.$$

- (b) No “nontrivial” relations hold in M . In other words, if

$$a_1 \cdot_M \dots \cdot_M a_n = a'_1 \cdot_M \dots \cdot_M a'_k,$$

for $a_i, a'_j \in A$, then this is required by the axioms of monoids.

The second condition of the definition of a free monoid is made more precise in the following way:

First, every monoid N has an underlying set $|N|$, and every monoid homomorphism $f : N \rightarrow M$ has an underlying function $|f| : |N| \rightarrow |M|$. This way, one has a functor **Mon** \rightarrow **Sets**. This functor is called the **forgetful functor**.

The free monoid $M(A)$ on a set A is by definition “the” monoid with the following universal mapping property or UMP:

Universal mapping property (UMP) of $M(A)$:

There is a function $i : A \rightarrow |M(A)|$; and given any monoid N and any function $f : A \rightarrow |N|$, there is a **unique** monoid homomorphism $\bar{f} : M(A) \rightarrow N$ such that $|\bar{f}| \circ i = f$, as indicated in the following diagram:

in **Mon** : $M(A) \xrightarrow{\bar{f}} N$

in **Sets** :

$$\begin{array}{ccc} |M(A)| & \xrightarrow{|\bar{f}|} & |N| \\ \uparrow i & \nearrow f & \\ A & & \end{array}$$

i is called the insertion of generators.

Proposition 2.1

A^* has the UMP of the free monoid on A .

Proof. Given any monoid N and any function $f : A \rightarrow |N|$, define $\bar{f} : A^* \rightarrow N$ by $\bar{f}(-) = u_N$, and

$$\bar{f}(a_1 a_2 \cdots a_n) = \bar{f}(a_1 * a_2 * \cdots * a_n) := f(a_1) \cdot_N \cdots \cdot_N f(a_n). \quad (2.3)$$

$\bar{f} : A^* \rightarrow N$ is clearly a monoid homomorphism, with $\bar{f}(a) = f(a)$, so that

$$(|\bar{f}| \circ i)(a) = f(a). \quad (2.4)$$

Therefore, the following diagram commutes:

$$\begin{array}{ccc} |A^*| & \xrightarrow{|\bar{f}|} & |N| \\ \uparrow i & \nearrow f & \\ A & & \end{array}$$

This proves the existence of $\bar{f} : A^* \rightarrow N$ with the required universal mapping property. Let us now prove the uniqueness. Suppose there is another monoid homomorphism $g : A^* \rightarrow N$ so that $g(a) = f(a)$, which in turn will give us the commutative diagram exhibiting UMP. Therefore, for all $a_1, \dots, a_n \in A$,

$$\begin{aligned} g(a_1 a_2 \cdots a_n) &= g(a_1 * a_2 * \cdots * a_n) \\ &= g(a_1) \cdot_N \cdots \cdot_N g(a_n) \\ &= f(a_1) \cdot_N \cdots \cdot_N f(a_n) \\ &= \bar{f}(a_1 a_2 \cdots a_n). \end{aligned}$$

Therefore, $g = \bar{f}$, proving the uniqueness of $\bar{f} : A^* \rightarrow N$. ■

Remark 2.1. Existence of a monoid homomorphism $\bar{f} : M(A) \rightarrow N$ implies that if there is an additional equality (sometimes referred to as “noise”) besides the ones imposed by associativity law and unitality law in $M(A)$, then the additional equality is transported to the monoid N . But N is **any** monoid to which the free monoid $M(A)$ is supposed to be mapped to via the monoid homomorphism $\bar{f} : M(A) \rightarrow N$. Hence, the existence of monoid homomorphism $\bar{f} : M(A) \rightarrow N$ is equivalent to the second condition of absence of any “noise” in $M(A)$.

Proposition 2.2

Given monoids M and N with functions $i : A \rightarrow |M|$ and $j : A \rightarrow |N|$, each with the UMP of the

free monoid on A , there is a unique monoid isomorphism $h : M \xrightarrow{\cong} N$ such that

$$|h| \circ i = j \text{ and } |h^{-1}| \circ j = i.$$

Proof. From $j : A \rightarrow |N|$ and the UMP of M , one has $\bar{j} : M \rightarrow N$ with $|\bar{j}| \circ i = j$.

$$\text{in } \mathbf{Mon} : \quad M \xrightarrow{\bar{j}} N$$

$$\text{in } \mathbf{Sets} : \quad \begin{array}{ccc} |M| & \xrightarrow{|\bar{j}|} & |N| \\ i \uparrow & \nearrow j & \\ A & & \end{array}$$

From $i : A \rightarrow |M|$ and the UMP of N , one has $\bar{i} : N \rightarrow M$ with $|\bar{i}| \circ j = i$.

$$\text{in } \mathbf{Mon} : \quad N \xrightarrow{\bar{i}} M$$

$$\text{in } \mathbf{Sets} : \quad \begin{array}{ccc} |N| & \xrightarrow{|\bar{i}|} & |M| \\ j \uparrow & \nearrow i & \\ A & & \end{array}$$

Combining these two, we get the following commutative diagram:

$$\begin{array}{ccc} & \xrightarrow{\bar{i} \circ \bar{j}} & \\ \text{in } \mathbf{Mon} : & M \xrightarrow{\bar{j}} N \xrightarrow{\bar{i}} M & \\ & \searrow \bar{j} & \nearrow \bar{i} \\ \text{in } \mathbf{Sets} : & \begin{array}{ccccc} |M| & \xrightarrow{|\bar{j}|} & |N| & \xrightarrow{|\bar{i}|} & |M| \\ & \nwarrow i & \uparrow j & \nearrow i & \\ & & A & & \end{array} & \end{array}$$

From $i : A \rightarrow |M|$ and the UMP of M , we have the existence of a unique homomorphism $f : M \rightarrow M$ such that $|f| \circ i = i$. From the above commutative diagram, we get that $f = \bar{i} \circ \bar{j}$ satisfies $|f| \circ i = i$. Furthermore, $f = 1_M : M \rightarrow M$ also satisfies $|f| \circ i = i$. Therefore,

$$\bar{i} \circ \bar{j} = 1_M. \quad (2.5)$$

Similarly, exchanging M and N , we get

$$\bar{j} \circ \bar{i} = 1_N. \quad (2.6)$$

Now, $\bar{j} : M \rightarrow N$ is the required monoid isomorphism h , i.e. $\bar{j} = h$ and $\bar{i} = h^{-1}$, so that we have $|h| \circ i = j$ and $|h^{-1}| \circ j = i$. \blacksquare

In light of [Proposition 2.1](#) and [Proposition 2.2](#), we can say that if $M(A)$ has the UMP of a free monoid on A , then $M(A)$ is isomorphic to A^* .

§2.2 Free Category

Just as a monoid has an underlying set, a category has an underlying graph. A directed graph consists of vertices and edges, each of which has a “source” and a “target” vertex. [Figure 2.1](#) is an example of a graph.



Figure 2.1: A graph

Definition 2.1. A **(directed) graph** consists of two sets: a set E of edges, and a set V of vertices, and two functions $s : E \rightarrow V$ (called source) and $t : E \rightarrow V$ (called target). We denote a directed graph G by a quadruple (V, E, s, t) . A **path** in a graph G is a finite sequence of edges e_1, \dots, e_n such that $t(e_i) = s(e_{i+1})$ for each $i = 1, \dots, n-1$.

Suppose we have a path e_1, \dots, e_n in G .

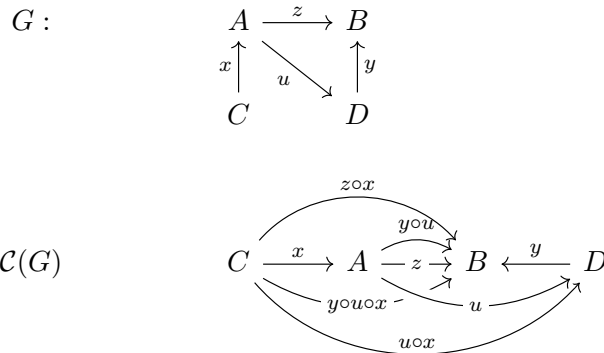
$$v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} v_2 \xrightarrow{e_3} \dots \xrightarrow{e_n} v_n$$

Put $\text{dom}(e_n \cdots e_1) = s(e_1)$ and $\text{cod}(e_n \cdots e_1) = t(e_n)$, and define composition by concatenation:

$$e_n \cdots e_1 \circ e'_m \cdots e'_1 = e_n \cdots e_1 e'_m \cdots e'_1, \quad (2.7)$$

where $\text{dom}(e_n \cdots e_1) = \text{cod}(e'_m \cdots e'_1)$.

For each vertex v , we have an “empty path” denoted by 1_v which is to be the identity arrow at v . With all of these terminologies at our disposal, we see that every graph G generates a category $\mathcal{C}(G)$ called the **free category** on G . It is defined by taking vertices of G as objects and paths in G as arrows. For example, take the graph given in Figure 2.1 with 4 vertices A, B, C, D . The corresponding free category on G is given by:



Definition 2.2 (Graph Homomorphism). Let $G \equiv (V, E, s, t)$ and $G' \equiv (V', E', s', t')$ be two graphs. A **graph homomorphism** f from G to G' , denoted by $f : G \rightarrow G'$ consists of two functions $f_0 : V \rightarrow V'$ and $f_1 : E \rightarrow E'$ such that the following diagrams commute:

$$\begin{array}{ccc} E & \xrightarrow{f_1} & E' \\ s \downarrow & & \downarrow s' \\ V & \xrightarrow{f_0} & V' \end{array} \quad \begin{array}{ccc} E & \xrightarrow{f_1} & E' \\ t \downarrow & & \downarrow t' \\ V & \xrightarrow{f_0} & V' \end{array}$$

Remark 2.2. Note that if G has only one vertex, then $\mathcal{C}(G)$ is just the free monoid on the set of edges of G . If, on the other hand, G has only vertices with no edges, then $\mathcal{C}(G)$ is the discrete category on the set of vertices of G .

Let us now see that $\mathcal{C}(G)$ has a UMP (universal mapping property). Define a “forgetful functor” $\mathcal{U} : \mathbf{Cat} \rightarrow \mathbf{Graphs}$ in the following way: the underlying graph of a (small) category has the collection of arrows as the set of edges E and the collection of objects as the set of vertices V , with $s = \text{dom}$ and $t = \text{cod}$.

Also, observe that we can describe a category \mathcal{C} with a diagram as below:

$$C_2 \xrightarrow{\circ} C_1 \begin{array}{c} \xrightarrow{\text{cod}} \\ \xleftarrow{i} \\ \xrightarrow{\text{dom}} \end{array} C_0,$$

where C_0 is the collection of objects of \mathcal{C} , C_1 is the collection of arrows, i is the identity arrow operation, and C_2 is the collection

$$C_2 = \{(f, g) \in C_1 \times C_1 \mid \text{cod } f = \text{dom } g\}.$$

Then a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ from \mathcal{C} to another category \mathcal{D} (with D_2, D_1, D_0 as given above) is a pair of assignments $F_0 : C_0 \rightarrow D_0$ and $F_1 : C_1 \rightarrow D_1$ such that each similarly labeled square in the following diagram commutes:

$$\begin{array}{ccccc} C_2 & \xrightarrow{\circ} & C_1 & \begin{array}{c} \xrightarrow{\text{cod}} \\ \xleftarrow{i} \\ \xrightarrow{\text{dom}} \end{array} & C_0 \\ F_2 \downarrow & & F_1 \downarrow & & \downarrow F_0 \\ D_2 & \xrightarrow{\circ} & D_1 & \begin{array}{c} \xrightarrow{\text{cod}} \\ \xleftarrow{i} \\ \xrightarrow{\text{dom}} \end{array} & D_0 \end{array}$$

where $F_2(f, g) = (F_1(f), F_1(g))$. Commutativity of the first square tells us that

$$F_1(g \circ f) = F_1(g) \circ F_1(f). \quad (2.8)$$

Commutativity of the second square is reminiscent of graph homomorphism if one removes the identity arrow operation. Note that the underlying graph of a category \mathcal{C} can be depicted as

$$C_1 \begin{array}{c} \xrightarrow{\text{cod}} \\ \xleftarrow{i} \\ \xrightarrow{\text{dom}} \end{array} C_0.$$

Therefore, at the level of objects, the forgetful functor

$$\mathcal{U} : \mathbf{Cat} \rightarrow \mathbf{Graphs} \quad (2.9)$$

sends the category (object of \mathbf{Cat})

$$C_2 \xrightarrow{\circ} C_1 \begin{array}{c} \xrightarrow{\text{cod}} \\ \xleftarrow{i} \\ \xrightarrow{\text{dom}} \end{array} C_0$$

to the underlying graph (object of \mathbf{Graphs})

$$C_1 \begin{array}{c} \xrightarrow{\text{cod}} \\ \xleftarrow{i} \\ \xrightarrow{\text{dom}} \end{array} C_0.$$

And functors (arrows of \mathbf{Cat}) under \mathcal{U} are sent to graph homomorphisms (arrows of \mathbf{Graphs}), i.e.

$$\begin{array}{ccccc} C_2 & \xrightarrow{\circ} & C_1 & \begin{array}{c} \xrightarrow{\text{cod}} \\ \xleftarrow{i} \\ \xrightarrow{\text{dom}} \end{array} & C_0 \\ F_2 \downarrow & & F_1 \downarrow & & \downarrow F_0 \\ D_2 & \xrightarrow{\circ} & D_1 & \begin{array}{c} \xrightarrow{\text{cod}} \\ \xleftarrow{i} \\ \xrightarrow{\text{dom}} \end{array} & D_0 \end{array}$$

is sent to

$$\begin{array}{ccc}
C_1 & \xrightleftharpoons[\text{dom}]{\text{cod}} & C_0 \\
F_1 \downarrow & & \downarrow F_0 \\
D_1 & \xrightleftharpoons[\text{dom}]{\text{cod}} & D_0
\end{array}$$

under \mathcal{U} . Given a category \mathcal{C} , its underlying graph is denoted by $|\mathcal{C}| = \mathcal{U}(\mathcal{C})$, where \mathcal{U} is the forgetful functor $\mathcal{U} : \mathbf{Cat} \rightarrow \mathbf{Graphs}$. The **free category** $\mathcal{C}(G)$ on a graph G has the following universal mapping property (UMP).

Universal mapping property (UMP) of $\mathcal{C}(G)$:

There is a graph homomorphism $i : G \rightarrow |\mathcal{C}(G)|$; and given any category \mathcal{D} and any graph homomorphism $h : G \rightarrow |\mathcal{D}|$, there is a **unique** functor $\bar{h} : \mathcal{C}(G) \rightarrow \mathcal{D}$ such that $|\bar{h}| \circ i = h$, as indicated in the following diagram:

$$\begin{array}{ccc}
\text{in } \mathbf{Cat} : & & \mathcal{C}(G) \xrightarrow{\bar{h}} \mathcal{D} \\
\\
\text{in } \mathbf{Graphs} : & & \begin{array}{ccc} |\mathcal{C}(G)| & \xrightarrow{|\bar{h}|} & |\mathcal{D}| \\ \uparrow i & \nearrow h & \\ G & & \end{array}
\end{array}$$

The free category on a graph with just one vertex is just a free monoid on the set of edges. The free category on a graph with two vertices and one edge between them is the finite category **2**. On the other hand, the free category on a graph of the form

$$A \xrightleftharpoons[f]{e} B$$

has (in addition to the identity arrows) the infinitely many arrows:

$$e, f, ef, fe, efe, fef, efef, fefe, \dots \quad (2.10)$$

3 Abstract Characterizations

§3.1 Epis and Monos

In **Sets**, a function $f : A \rightarrow B$ is called

- **injective** if $f(a) = f(a')$ implies $a = a'$ for all $a, a' \in A$,
- **surjective** if for each $b \in B$, there exists $a \in A$ such that $f(a) = b$.

We have the following abstract characterizations of these properties:

Definition 3.1 (Monomorphism and Epimorphism). In any category \mathcal{C} , an arrow $f : A \rightarrow B$ is called a **monomorphism** if given any $g, h : C \rightarrow A$, $f \circ g = f \circ h$ implies $g = h$.

$$C \xrightarrow[g]{g} A \xrightarrow{f} B$$

An arrow $f : A \rightarrow B$ is called an **epimorphism** if given any $i, j : B \rightarrow D$, $i \circ f = j \circ f$ implies $i = j$.

$$A \xrightarrow{f} B \xrightarrow[i]{i} D$$

We often write $f : A \rightarrowtail B$ if f is a monomorphism and $f : A \twoheadrightarrow B$ if f is an epimorphism.

Proposition 3.1

A function $f : A \rightarrow B$ between sets is a monomorphism if and only if it is injective.

Proof. (\Rightarrow) Suppose $f : A \rightarrowtail B$. We need to show that f is injective. Suppose $f(a) = f(a')$ for some $a, a' \in A$. Consider functions $g, h : \{*\} \rightarrow A$ as $g(*) = a$ and $h(*) = a'$. Then

$$(f \circ g)(*) = f(g(*)) = f(a) = f(a') = f(h(*)) = (f \circ h)(*). \quad (3.1)$$

So $f \circ g = f \circ h$. Since f is a monomorphism, we must have $g = h$. Therefore, $a = a'$, proving the injectivity of f .

(\Leftarrow) Conversely, let $f : A \rightarrow B$ be injective. Let $g, h : C \rightarrow A$ such that $f \circ g = f \circ h$. Then for any $c \in C$,

$$f(g(c)) = (f \circ g)(c) = (f \circ h)(c) = f(h(c)). \quad (3.2)$$

So $f(g(c)) = f(h(c))$. Since f is injective, this implies $g(c) = h(c)$. This is true for any $c \in C$. Therefore, $g = h$. In other words, $f \circ g = f \circ h$ implies $g = h$. Hence, f is a monomorphism. ■

Proposition 3.2

A function $f : A \rightarrow B$ between sets is an epimorphism if and only if it is surjective.

Proof. (\Rightarrow) Suppose $f : A \twoheadrightarrow B$. We need to show that f is surjective. We construct two functions $g_1, g_2 : B \rightarrow \{0, 1\}$ as follows:

$$g_1(b) = 0 \text{ and } g_2(b) = \begin{cases} 0 & \text{if } b \in \text{im } f \\ 1 & \text{otherwise.} \end{cases} \quad (3.3)$$

$$A \xrightarrow{f} B \xrightarrow[g_2]{g_1} \{0, 1\}$$

Then for each $a \in A$, $g_1(f(a)) = g_2(f(a)) = 0$. So $g_1 \circ f = g_2 \circ f$. Since f is an epimorphism, this gives us $g_1 = g_2$. In other words,

$$g_2(b) = 0 \text{ for each } b \in B. \quad (3.4)$$

Therefore, all $b \in B$ are in $\text{im } f$, proving the surjectivity of f .

(\Leftarrow) Conversely, let $f : A \rightarrow B$ be surjective. Let $g_1, g_2 : B \rightarrow C$ such that $g_1 \circ f = g_2 \circ f$.

$$A \xrightarrow{f} B \xrightarrow[g_2]{g_1} \{0, 1\}$$

Since f is surjective, for any $b \in B$, there exists $a \in A$ such that $f(a) = b$. Then we have

$$g_1(b) = g_1(f(a)) = (g_1 \circ f)(a) = (g_2 \circ f)(a) = g_2(f(a)) = g_2(b). \quad (3.5)$$

So $g_1(b) = g_2(b)$, and this is true for any $b \in B$. Therefore, $g_1 = g_2$ and hence f is an epimorphism. ■

Monomorphisms are often abbreviated as monos or monic, epimorphisms are often abbreviated as epis or epic.

Proposition 3.3

A monoid homomorphism $h : M \rightarrow N$ is monic if and only if the underlying function $|h| : |M| \rightarrow |N|$ is monic (i.e. injective by Proposition 3.1).

Proof. (\Rightarrow) Suppose h is monic. We need to show that $|h|$ is injective. Let $|h|(m_1) = |h|(m_2)$ for $m_1, m_2 \in |M|$. Take two functions $x, y : 1 \rightarrow |M|$, where $1 = \{*\}$ is any singleton set, defined by $x(*) = m_1$ and $y(*) = m_2$. Let $M(1)$ be the free monoid generated by 1. By the UMP of a free monoid, there are unique monoid homomorphisms $\bar{x} : M(1) \rightarrow M$ and $\bar{y} : M(1) \rightarrow M$ such that the following diagrams commute:

$$\begin{array}{ccc} |M(1)| & \xrightarrow{|\bar{x}|} & |M| \\ & \swarrow i \quad \searrow x & \\ & 1 & \end{array} \quad \begin{array}{ccc} |M(1)| & \xrightarrow{|\bar{y}|} & |M| \\ & \swarrow i \quad \searrow y & \\ & 1 & \end{array}$$

In other words, $|\bar{x}| \circ i = x$ and $|\bar{y}| \circ i = y$. Furthermore, by the UMP of $M(1)$ for the functions $|h| \circ x, |h| \circ y : 1 \rightarrow N$, there are unique monoid homomorphisms $f : M(1) \rightarrow N$ and $g : M(1) \rightarrow N$ such that the following diagrams commute:

$$\begin{array}{ccc} |M(1)| & \xrightarrow{|f|} & |N| \\ & \swarrow i \quad \searrow |h| \circ x & \\ & 1 & \end{array} \quad \begin{array}{ccc} |M(1)| & \xrightarrow{|g|} & |N| \\ & \swarrow i \quad \searrow |h| \circ y & \\ & 1 & \end{array}$$

But since the following diagrams commute,

$$\begin{array}{ccccc} & & |h| \circ |\bar{x}| = |h \circ \bar{x}| & & \\ & \nearrow & & \searrow & \\ |M(1)| & \xrightarrow{|\bar{x}|} & |M| & \xrightarrow{|h|} & |N| \\ & \swarrow i \quad \searrow x & & \nearrow |h| \circ x & \\ & 1 & & & \end{array} \quad \begin{array}{ccccc} & & |h| \circ |\bar{y}| = |h \circ \bar{y}| & & \\ & \nearrow & & \searrow & \\ |M(1)| & \xrightarrow{|\bar{y}|} & |M| & \xrightarrow{|h|} & |N| \\ & \swarrow i \quad \searrow y & & \nearrow |h| \circ y & \\ & 1 & & & \end{array}$$

the uniqueness of f and g guarantees that

$$f = h \circ \bar{x} \text{ and } g = h \circ \bar{y}. \quad (3.6)$$

However, $|h| \circ x$ and $|h| \circ y$ are equal, since

$$(|h| \circ x)(*) = |h|(m_1) = |h|(m_2) = (|h| \circ y)(*). \quad (3.7)$$

Therefore, $f = g$, and hence,

$$h \circ \bar{x} = h \circ \bar{y}. \quad (3.8)$$

Since h is monic, $\bar{x} = \bar{y}$. Then we have

$$x = |\bar{x}| \circ i = |\bar{y}| \circ i = y. \quad (3.9)$$

Hence, $m_1 = x(*) = y(*) = m_2$, and thus $|h|$ is injective.

(\Leftarrow) Conversely, suppose $|h| : |M| \rightarrow |N|$ is monic. Let $f, g : X \rightarrow M$ be monoid homomorphisms such that $h \circ f = h \circ g$.

$$X \begin{array}{c} \xrightarrow{g} \\ \xrightarrow{f} \end{array} M \xrightarrow{h} N.$$

In **Sets**, we then have $|h| \circ |f| = |h| \circ |g|$.

$$|X| \begin{array}{c} \xrightarrow{|g|} \\ \xrightarrow{|f|} \end{array} |M| \xrightarrow{|h|} |N|.$$

Since $|h|$ is monic, we must have $|f| = |g|$. Therefore, $f = g$, and hence $h : M \rightarrow N$ is monic. \blacksquare

Example 3.1. In the category **Mon** of monoids and monoid homomorphisms, there is a monic homomorphism $\bar{\mathbb{N}} \rightarrow \bar{\mathbb{Z}}$, where $\bar{\mathbb{N}} = (\mathbb{N}, +, 0)$ is the additive monoid of natural numbers, and $\bar{\mathbb{Z}} = (\mathbb{Z}, +, 0)$ is the additive monoid of integers. This map given by the inclusion $\mathbb{N} \subset \mathbb{Z}$ of sets is monic in **Mon**, by [Proposition 3.3](#), since it is an injective homomorphism. We shall now show that this is also epic in **Mon**.

$$\bar{\mathbb{N}} \xrightarrow{i} \bar{\mathbb{Z}} \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} \bar{M}$$

Given any monoid $\bar{M} = (M, *, u)$, with the underlying set M , let $f, g : \bar{\mathbb{Z}} \rightarrow \bar{M}$ be monoid homomorphisms such that $f \circ i = g \circ i$. To prove that i is epic, we need to show that $f = g$.

Since $i : \bar{\mathbb{N}} \rightarrow \bar{\mathbb{Z}}$ is the inclusion, $f \circ i = g \circ i$ gives us that $f|_{\bar{\mathbb{N}}} = g|_{\bar{\mathbb{N}}}$, i.e. $f(n) = g(n)$ for each $n \in \bar{\mathbb{N}}$. We need to show that $f(-n) = g(-n)$.

$$\begin{aligned} f(-n) &= f(-n) * u = f(-n) * g(0) = f(-n) * g(n + (-n)) \\ &= f(-n) * [g(n) * g(-n)] = [f(-n) * g(n)] * g(-n) \\ &= [f(-n) * f(n)] * g(-n) = f(-n + n) * g(-n) \\ &= u * g(-n) = g(-n). \end{aligned} \quad (3.10)$$

Therefore, $f = g$, and hence i is epic.

Proposition 3.4

Every isomorphism is both monic and epic.

Proof. Let $m : B \rightarrow C$ be an isomorphism. Then there exists $e : C \rightarrow B$ such that $m \circ e = \mathbb{1}_C$ and $e \circ m = \mathbb{1}_B$. So the two triangles in the following diagram commute:

$$\begin{array}{ccccc}
A & \xrightarrow[x]{y} & B & \xrightarrow{m} & C \\
& & \searrow e \circ m = \mathbb{1}_B & \downarrow e & \searrow m \circ e = \mathbb{1}_C \\
& & & B & \xrightarrow{m} C \xrightarrow[f]{g} C
\end{array}$$

Given any arrows $x, y : A \rightarrow B$ such that $m \circ x = m \circ y$, we have

$$\begin{aligned}
m \circ x = m \circ y &\implies e \circ (m \circ x) = e \circ (m \circ y) \\
&\implies (e \circ m) \circ x = (e \circ m) \circ y \\
&\implies \mathbb{1}_B \circ x = \mathbb{1}_B \circ y \\
&\implies x = y.
\end{aligned} \tag{3.11}$$

Therefore, m is monic. Now, given arrows $f, g : C \rightarrow D$ with $f \circ m = g \circ m$, we have

$$\begin{aligned}
f \circ m = g \circ m &\implies (f \circ m) \circ e = (g \circ m) \circ e \\
&\implies f \circ (m \circ e) = g \circ (m \circ e) \\
&\implies f \circ \mathbb{1}_C = g \circ \mathbb{1}_C \\
&\implies f = g.
\end{aligned} \tag{3.12}$$

Therefore, m is epic. ■

Remark 3.1. In **Sets**, the converse of [Proposition 3.4](#) also holds: every monic and epic is an isomorphism. But this is not true, in general. The counterexample is provided in the context of the category **Mon** in [Example 3.1](#). There we saw that the inclusion homomorphism $i : (\mathbb{N}, +, 0) \rightarrow (\mathbb{Z}, +, 0)$ is both monic and epic. But this arrow in **Mon** is not an iso, i.e. its inverse does not exist. In particular, there does not exist an arrow $j : \mathbb{Z} \rightarrow \mathbb{N}$ such that $i \circ j = \mathbb{1}_{\mathbb{Z}}$ and $j \circ i = \mathbb{1}_{\mathbb{N}}$.

Proposition 3.5

With regard to a commutative triangle,

$$\begin{array}{ccc}
A & \xrightarrow{f} & B \\
& \searrow h & \downarrow g \\
& & C
\end{array}$$

in any category \mathcal{C} ,

- (a) if f and g are isos (resp. monos, resp. epis), so is h ;
- (b) if h is monic, so is f ;
- (c) if h is epic, so is g ;

Proof. (a) Suppose f and g are isos. Then there are arrows $f^{-1} : B \rightarrow A$ and $g^{-1} : C \rightarrow B$ such that $f \circ f^{-1} = \mathbb{1}_A$, $f^{-1} \circ f = \mathbb{1}_B$, $g \circ g^{-1} = \mathbb{1}_C$, $g^{-1} \circ g = \mathbb{1}_B$. Since the triangle above is commutative, $h = g \circ f$. We define $k : C \rightarrow A$ as $k = f^{-1} \circ g^{-1}$. Then

$$h \circ k = (g \circ f) \circ (f^{-1} \circ g^{-1}) = g \circ \mathbb{1}_A \circ g^{-1} = \mathbb{1}_C. \tag{3.13}$$

$$k \circ h = (f^{-1} \circ g^{-1}) \circ (g \circ f) = f^{-1} \circ \mathbb{1}_B \circ f = \mathbb{1}_A. \tag{3.14}$$

Therefore, h is an isomorphism, and $k = h^{-1}$.

Suppose f and g are monos. We want to show that $h : A \rightarrow C$ is monic. Suppose there are arrows $x, y : X \rightrightarrows A$ such that $h \circ x = h \circ y$.

$$\begin{array}{ccccc}
 & & B & & \\
 & f \nearrow & & \searrow g & \\
 X & \xrightarrow[x]{y} & A & \xrightarrow{h=g \circ f} & C
 \end{array}$$

$$h \circ x = h \circ y \implies (g \circ f) \circ x = (g \circ f) \circ y \implies g \circ (f \circ x) = g \circ (f \circ y). \quad (3.15)$$

Since g is monic, we must have $f \circ x = f \circ y$. Again, since f is monic, we must have $x = y$. So $h \circ x = h \circ y$ implies $x = y$. Therefore, h is monic.

Suppose f and g are epis. We want to show that $h : A \rightarrow C$ is epic. Suppose there are arrows $x, y : C \rightrightarrows Y$ such that $x \circ h = y \circ h$.

$$\begin{array}{ccccc}
 & & B & & \\
 & f \nearrow & & \searrow g & \\
 A & \xrightarrow{h=g \circ f} & C & \xrightarrow[x]{y} & Y
 \end{array}$$

$$x \circ h = y \circ h \implies x \circ (g \circ f) = y \circ x \circ (g \circ f) \implies (x \circ g) \circ f = (y \circ g) \circ f. \quad (3.16)$$

Since f is epic, we must have $x \circ g = y \circ g$. Again, since g is epic, we must have $x = y$. So $x \circ h = y \circ h$ implies $x = y$. Therefore, h is epic.

- (b) We need to show that $f : A \rightarrow B$ is monic. Suppose there are arrows $x, y : X \rightrightarrows A$ such that $f \circ x = f \circ y$.

$$\begin{array}{ccccc}
 & & B & & \\
 & f \nearrow & & \searrow g & \\
 X & \xrightarrow[x]{y} & A & \xrightarrow{h=g \circ f} & C
 \end{array}$$

$$f \circ x = f \circ y \implies g \circ f \circ x = g \circ f \circ y \implies h \circ x = h \circ y. \quad (3.17)$$

Since h is monic, $h \circ x = h \circ y$ implies $x = y$. Therefore, since $f \circ x = f \circ y$ forces $x = y$, f is monic.

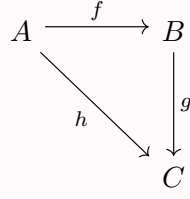
- (c) We need to show that $g : B \rightarrow C$ is epic. Suppose there are arrows $x, y : C \rightrightarrows Y$ such that $x \circ g = y \circ g$.

$$\begin{array}{ccccc}
 & & B & & \\
 & f \nearrow & & \searrow g & \\
 A & \xrightarrow{h=g \circ f} & C & \xrightarrow[x]{y} & Y
 \end{array}$$

$$x \circ g = y \circ g \implies x \circ g \circ f = y \circ g \circ f \implies x \circ h = y \circ h. \quad (3.18)$$

Since h is epic, $x \circ h = y \circ h$ implies $x = y$. Therefore, since $x \circ g = y \circ g$ forces $x = y$, g is epic. ■

Remark 3.2. In the commutative triangle of [Proposition 3.5](#), we have $h = g \circ f$.



If h is monic, then so is f as proven in [Proposition 3.5](#). However, h being monic does not necessarily imply g is monic. Similarly, h being epic also does not imply f is epic. The following is such an example.

Example 3.2. Let us consider the category **Sets**. Take $A = C = \{0\}$ and $B = \{0, 1\}$. Define $f : A \rightarrow B$ by $f(0) = 0$, and $g : B \rightarrow C$ by $g(x) = 0$ for $x \in \{0, 1\}$.

Then $g \circ f : A \rightarrow C$ is

$$(g \circ f)(0) = g(0) = 0. \quad (3.19)$$

So $h = g \circ f$ is an isomorphism. In particular, h is monic. However, $g : B \rightarrow C$ is not monic (i.e. injective since we are in **Sets**) because $g(0) = g(1)$. Similarly, h is epic as well, but f is not.

§3.2 Sections and Retractions

We have seen in [Proposition 3.4](#) that any iso is both monic and epic. More generally, if an arrow $f : A \rightarrow B$ has a left inverse $g : B \rightarrow A$, i.e. $g \circ f = \mathbb{1}_A$, then f must be monic and g epic.

$$\begin{array}{ccccc} C & \xrightarrow{x} & A & \xrightarrow{f} & B \\ & \searrow y & & \searrow g & \\ & & A & \xrightarrow[p]{q} & D \end{array}$$

$g \circ f = \mathbb{1}_A$

Given arrows $x, y : C \rightarrow A$ with $f \circ x = f \circ y$,

$$\begin{aligned} f \circ x = f \circ y &\implies g \circ (f \circ x) = g \circ (f \circ y) \\ &\implies (g \circ f) \circ x = (g \circ f) \circ y \\ &\implies \mathbb{1}_A \circ x = \mathbb{1}_A \circ y \\ &\implies x = y. \end{aligned} \quad (3.20)$$

Therefore, $f : A \rightarrow B$ is monic. Now, given $p, q : A \rightarrow D$ with $p \circ g = q \circ g$,

$$\begin{aligned} p \circ g = q \circ g &\implies (p \circ g) \circ f = (q \circ g) \circ f \\ &\implies p \circ (g \circ f) = q \circ (g \circ f) \\ &\implies p \circ \mathbb{1}_A = q \circ \mathbb{1}_A \\ &\implies p = q. \end{aligned} \quad (3.21)$$

Therefore, $g : B \rightarrow A$ is epic.

Definition 3.2. A **split mono** is an arrow with a left inverse. A **split epi** is an arrow with a right inverse. Given arrows $e : X \rightarrow A$ and $s : A \rightarrow X$ with $e \circ s = \mathbb{1}_A$, the arrow s is called a **section** or **splitting** of e , and e is called a **retraction** of s . The object A is called a **retract** of X . Therefore, section is monic and retraction is epic.

Remark 3.3. Since functors preserve identities, they also preserve split epis and split monos. Functors do not preserve epi, in general. Counterexample is the forgetful functor $\mathcal{U} : \mathbf{Mon} \rightarrow \mathbf{Sets}$, which does not preserve the epi $i : \bar{\mathbb{N}} \rightarrow \bar{\mathbb{Z}}$, i.e. $\mathcal{U}(i) : \mathbb{N} \rightarrow \mathbb{Z}$ is not an epi in \mathbf{Sets} as it's not surjective.

In \mathbf{Sets} , every mono splits except those of the form $\emptyset \rightarrow A$. The condition that every epi splits is the categorical version of the axiom of choice. Let us see this fact in detail. Consider an epi $e : E \twoheadrightarrow X$. We then have the following family of nonempty sets

$$E_x = e^{-1}(\{x\}) \quad \text{for } x \in X.$$

Each E_x is nonempty since e is surjective. Now, for each $x \in X$, one chooses an element $s(x)$ from $E_x \subset E$, and thus define the function $s : X \rightarrow E$. By construction, $e(s(x)) = x$ for each $x \in X$, i.e. $e \circ s = 1_X$. One, thus finds that the choice function $s : X \rightarrow E$ associated with the family $(E_x)_{x \in X}$ is exactly a splitting of e .

Let us do the reverse construction now. Given a family of nonempty subsets $(E_x)_{x \in X}$, take

$$E = \{(x, y) \mid x \in X, y \in E_x\} = \bigcup_{x \in X} (\{x\} \times E_x), \quad (3.22)$$

and define the epi $e : E \twoheadrightarrow X$ by $(x, y) \mapsto x$. A splitting $s : X \rightarrow E$ hence determines a choice function for the family $(E_x)_{x \in X}$,

Remark 3.4. A notion related to the existence of “choice function” is that of being projective. An object P is said to be **projective** if for any $e : E \twoheadrightarrow X$ and arrow $f : P \rightarrow X$, there is some (not necessarily unique) arrow $\bar{f} : P \rightarrow E$ such that $e \circ \bar{f} = f$, as indicated in the following commutative diagram:

$$\begin{array}{ccc} & & E \\ & \nearrow \bar{f} & \downarrow e \\ P & \xrightarrow{f} & X \end{array}$$

One says that f **lifts across** e .

Let P be projective and let $e : E \twoheadrightarrow P$ be an epi. Since P is projective, given $1_P : P \rightarrow P$, there exists $s : P \rightarrow E$ such that the following diagram commutes:

$$\begin{array}{ccc} & & E \\ & \nearrow s & \downarrow e \\ P & \xrightarrow{1_P} & P \end{array}$$

yielding $s \circ e = 1_P$. In other words, the epi $e : E \twoheadrightarrow P$ splits.

§3.3 Initial and Terminal Objects

Definition 3.3. In any category \mathcal{C} , an object

- 0 is **initial** if for any object C , there is a unique arrow $0 \rightarrow C$,
- 1 is **terminal** if for any object C , there is a unique arrow $C \rightarrow 1$.

Proposition 3.6

Initial and terminal objects are unique up to isomorphism.

Proof. Suppose both 0 and $0'$ are initial objects in some category \mathcal{C} . Since 0 is initial, there is a unique arrow $u : 0 \rightarrow 0'$. Again, since $0'$ is initial, there is a unique arrow $v : 0' \rightarrow 0$. Then we can form the composition $v \circ u : 0 \rightarrow 0$. But since 0 is an initial object, there is a unique arrow $0 \rightarrow 0$, which has to be the identity arrow $\mathbb{1}_0$. Therefore, $v \circ u = \mathbb{1}_0$.

$$\begin{array}{ccc}
 0 & \xrightarrow{u} & 0' \\
 & \searrow & \downarrow v \\
 & & 0 \\
 & \swarrow & \uparrow u \\
 & & 0'
 \end{array}
 \quad
 \begin{array}{l}
 v \circ u = \mathbb{1}_0 \\
 u \circ v = \mathbb{1}_{0'}
 \end{array}$$

Similarly, $u \circ v : 0' \rightarrow 0'$ is an arrow from $0'$ to itself. But since $0'$ is an initial object, there is a unique arrow $0' \rightarrow 0'$, which has to be the identity arrow $\mathbb{1}_{0'}$. Therefore, $u \circ v = \mathbb{1}_{0'}$. So 0 and $0'$ are isomorphic via a unique isomorphism $u : 0 \rightarrow 0'$.

Let 1 and $1'$ be terminal objects of \mathcal{C} . Since $1'$ is terminal, there is a unique arrow $u : 1 \rightarrow 1'$. Again, since 1 is terminal, there is a unique arrow $v : 1' \rightarrow 1$. Then we can form the composition $v \circ u : 1 \rightarrow 1$. But since 1 is a terminal object, there is a unique arrow $1 \rightarrow 1$, which has to be the identity arrow $\mathbb{1}_1$. Therefore, $v \circ u = \mathbb{1}_1$.

$$\begin{array}{ccc}
 1 & \xrightarrow{u} & 1' \\
 & \searrow & \downarrow v \\
 & & 1 \\
 & \swarrow & \uparrow u \\
 & & 1'
 \end{array}
 \quad
 \begin{array}{l}
 v \circ u = \mathbb{1}_1 \\
 u \circ v = \mathbb{1}_{1'}
 \end{array}$$

Similarly, $u \circ v : 1' \rightarrow 1'$ is an arrow from $1'$ to itself. But since $1'$ is a terminal object, there is a unique arrow $1' \rightarrow 1'$, which has to be the identity arrow $\mathbb{1}_{1'}$. Therefore, $u \circ v = \mathbb{1}_{1'}$. So 1 and $1'$ are isomorphic via a unique isomorphism $u : 1 \rightarrow 1'$. \blacksquare

Example 3.3. 1. In **Sets**, the empty set \emptyset is initial, and any singleton set is terminal. Indeed, for any set B , there is a unique function from \emptyset to B , called the empty function. When $B = \emptyset$, the empty function from \emptyset to \emptyset is the required identity arrow on the object $\emptyset \in \text{Ob}(\mathbf{Sets})$.

There is also a unique function from any set B to a singleton set $\{*\}$, $f : B \rightarrow \{*\}$, given by $f(b) = *$ for every $b \in B$. It is unique in the sense that there can't be any other function from the same set B to the singleton set $\{*\}$. In other words, $\text{Hom}_{\mathbf{Sets}}(B, \{*\})$ for a given set $B \in \text{Ob}(\mathbf{Sets})$ is a singleton set.

2. In $\mathbf{Vect}_{\mathbb{K}}$, the category of vector spaces over the field \mathbb{K} and linear transformations, the 0-dimensional vector space $\{\mathbf{0}\}$ consisting of the zero-vector (additive identity) only is both the initial and terminal objects.

Given a \mathbb{K} -vector space V , there is only one linear transformation from $\{\mathbf{0}\}$ to V , namely the one that takes $\mathbf{0}$ to $\mathbf{0}_V$, the additive identity of the \mathbb{K} -vector space V . Also, given a \mathbb{K} -vector space V , there is only one linear transformation from V to $\{\mathbf{0}\}$ that takes all of V to $\mathbf{0}$. In other words, for a given $V \in \text{Ob}(\mathbf{Vect}_{\mathbb{K}})$, both $\text{Hom}_{\mathbf{Vect}_{\mathbb{K}}}(\{\mathbf{0}\}, V)$ and $\text{Hom}_{\mathbf{Vect}_{\mathbb{K}}}(V, \{\mathbf{0}\})$ are singleton sets.

§3.3.i The Category of Boolean Algebras

Definition 3.4 (Boolean algebra). A **Boolean algebra** is a poset (B, \leq) together with distinguished elements $0, 1$, binary operations $a \vee b$ (read a join b) and $a \wedge b$ (read a meet b) and a unary operation $\neg b$ (read b complement) such that the following conditions are satisfied for all $a, b, c \in B$:

1. $0 \leq a$ and $a \leq 1$.
2. $a \leq c$ and $b \leq c$ if and only if $a \vee b \leq c$.
3. $c \leq a$ and $c \leq b$ if and only if $c \leq a \wedge b$.
4. $a \leq \neg b$ if and only if $a \wedge b = 0$.
5. $\neg(\neg a) = a$.

A typical example of a Boolean algebra is the power set $\mathcal{P}(X)$ of a set X , ordered by inclusion $A \subseteq B$, with the distinguished elements $0 = \emptyset$ and $1 = X$. The binary operations join and meet are given by union and intersection of subsets, respectively, and the unary operation complementation is $\neg A = X \setminus A$.

The union and intersection of sets satisfy $A \cap B \subseteq A \subseteq A \cup B$ and $A \cap B \subseteq B \subseteq A \cup B$. The same is satisfied in any Boolean algebra.

Proposition 3.7

In a Boolean algebra B , $a \wedge b \leq a \leq a \vee b$ and $a \wedge b \leq b \leq a \vee b$ for any $a, b \in B$.

Proof. Since (B, \leq) is a poset, by the reflexivity property, $a \vee b \leq a \vee b$. Then using property 2 from the definition of Boolean algebra, we get

$$a \leq a \vee b \text{ and } b \leq a \vee b. \quad (3.23)$$

Again, $a \wedge b \leq a \wedge b$ by reflexivity. Then using property 3 from the definition of Boolean algebra, we get

$$a \wedge b \leq a \text{ and } a \wedge b \leq b. \quad (3.24)$$

Therefore, combining (3.23) and (3.24), we get $a \wedge b \leq a \leq a \vee b$ and $a \wedge b \leq b \leq a \vee b$. ■

A special case of Boolean algebras is the 2-element Boolean algebra $\mathbf{2} = \{0, 1\}$, considered as $\mathcal{P}(\{*\})$, the power set of the singleton set $\{*\}$ which consists of only the empty set \emptyset and $\{*\}$.

The category of Boolean algebras is denoted by **BA**, where objects are Boolean algebras and arrows are Boolean homomorphisms, that is functions $h : B \rightarrow B'$ that, in addition to being monotone, satisfy

- $h(0_B) = 0_{B'}$, $h(1_B) = 1_{B'}$;
- $h(a \vee_B b) = h(a) \vee_{B'} h(b)$;
- $h(a \wedge_B b) = h(a) \wedge_{B'} h(b)$;
- $h(\neg_B a) = \neg_{B'} h(a)$,

for any $a, b \in B$. In this category, $\mathbf{2}$ is the initial object. There is this 1-element Boolean algebra $\mathbf{1} = \{0\}$, which is regarded as the power set of \emptyset , i.e. $\mathbf{1} \equiv \mathcal{P}(\emptyset) = \{\emptyset\}$. In this situation, the distinguished elements 0 and 1 coincide. $\mathbf{1}$ acts as the terminal object of **BA**. In other words, given a Boolean algebra B , there is exactly one boolean homomorphism from $h : \mathbf{2} \rightarrow B$, given by

$$h(0) = 0_B \text{ and } h(1) = 1_B.$$

Furthermore, there is exactly one boolean homomorphism $f : B \rightarrow \mathbf{1}$ given by

$$f(b) = 0$$

for any $b \in B$.

Definition 3.5 (Filter). A **filter** in a Boolean algebra B is a nonempty subset $F \subseteq B$ that is closed upward and under meets, i.e.

- $a \in F$ and $a \leq b$ implies $b \in F$,
- $a, b \in F$ implies $a \wedge b \in F$.

Definition 3.6 (Ultrafilter). A filter F is called **maximal** if $F \subset F'$ implies $F' = B$, i.e. the only strictly larger filter is the “improper” filter B itself. A maximal filter is called an **ultrafilter**.

One can easily verify that a filter F is an ultrafilter if and only if for every element $b \in B$, either $b \in F$ or $\neg b \in F$ and not both.

§3.4 Generalized Elements

In **Sets**, we have the notion of an element of a set. However, in a general category, we don’t have the notion of an element of an object, because the objects need not be sets. That’s why we need to generalize the concept of an element to any category. One idea can be that for a given object X , we can consider all the arrows from the terminal object to X . This idea works in **Sets**, since any set X is determined uniquely by the set of all the functions from $1 = \{*\}$ to X . In other words,

$$X \cong \text{Hom}_{\mathbf{Sets}}(1, X). \quad (3.25)$$

Any $x \in X$ uniquely corresponds to a function $\bar{x} : 1 \rightarrow X$ given by $\bar{x}(*) = x$. Conversely, any function $f : 1 \rightarrow X$ uniquely corresponds to an element $f(*)$ of X . Thus, we get the one-to-one correspondence (i.e. an isomorphism in **Sets**) $X \cong \text{Hom}_{\mathbf{Sets}}(1, X)$.

The arrows $1 \rightarrow X$ are called **global elements** of X . However, this idea of taking all the arrows from the terminal object does not always work in other categories. For instance, in $\mathbf{Vect}_{\mathbb{K}}$, the terminal object is $\{0\}$, which is also the initial object. So given a \mathbb{K} -vector space V , there is only one linear transformation from $\{0\}$ to V , $x : \{0\} \rightarrow V$, given by

$$x(0) = 0_V.$$

Given two \mathbb{K} -vector spaces V and W , and two linear transformations $f, g : V \rightarrow W$, we always have $f \circ x = g \circ x$, since

$$\begin{aligned} (f \circ x)(0) &= f(x(0)) = f(0_V) = 0_W, \\ (g \circ x)(0) &= g(x(0)) = g(0_V) = 0_W. \end{aligned}$$

No matter how different f and g are, composing them with x yields the same arrow. This is not what we want. Ideally, we want generalized elements to have the following property:

$$f = g \text{ if and only if } f \circ x = g \circ x \text{ for every generalized element } x \text{ of } \text{dom } f.$$

The motivation for requiring this condition comes from sets and functions. We call two functions $f, g : X \rightarrow Y$ the same if and only if $f(x) = g(x)$ for every $x \in X$.

For this, we draw motivations from the proof of [Generalized Cayley’s Theorem](#). In the sketch of the proof, we showed that a “not too big” category is isomorphic to a concrete category (i.e. a category where objects are sets and arrows are functions between them). We’ve seen that under the isomorphism, the object C of the “not too big” category \mathcal{C} corresponds to the set of arrows whose codomains are C . Keeping this in mind, we define the generalized elements of C to be the arrows whose codomains are C .

Definition 3.7 (Generalized Element). Let \mathcal{C} be a category, and $A \in \text{Ob}(\mathcal{C})$. The arrows whose codomains are A are called **generalized elements** of A . In other words, any arbitrary arrow $f : X \rightarrow A$ (with arbitrary domain X) is a generalized element of A .

Then these generalized elements satisfy the required property that $f = g$ if and only if $f \circ x = g \circ x$ for every generalized element x of $\text{dom } f (= \text{dom } g)$. Let $f, g : A \rightarrow B$ be two arrows. If $f = g$, then obviously $f \circ x = g \circ x$. Conversely, suppose $f \circ x = g \circ x$ for every generalized element x of A . Then if we choose $x = \mathbb{1}_A$ (which is an arrow with codomain A , so it's a generalized element of A), we get

$$f \circ \mathbb{1}_A = g \circ \mathbb{1}_A \implies f = g. \quad (3.26)$$

Furthermore, we defined monomorphisms (generalizations of injective functions) as follows: $f : A \rightarrow B$ is a monomorphism if for any $x, y : C \rightarrow A$, $f \circ x = f \circ y$ implies $x = y$.

$$C \xrightarrow[x]{y} A \xrightarrow{f} B$$

In other words, $f : A \rightarrow B$ is a monomorphism if it is “injective on generalized elements of A ”.

§3.5 Products

Let us begin by considering products of sets. Given sets A and B , the cartesian product of A and B is the set of ordered pairs

$$A \times B = \{(a, b) \mid a \in A, b \in B\}.$$

Observe that there are 2 coordinate projections

$$A \xleftarrow{\pi_1} A \times B \xrightarrow{\pi_2} B$$

with $\pi_1(a, b) = a$ and $\pi_2(a, b) = b$. Indeed, given any element $c \in A \times B$, one has $c = (\pi_1 c, \pi_2 c)$. This situation is captured by the following commutative diagram:

$$\begin{array}{ccccc} & & 1 & & \\ & \swarrow a & \downarrow (a,b) & \searrow b & \\ A & \xleftarrow{\pi_1} & A \times B & \xrightarrow{\pi_2} & B \end{array}$$

$$(a, b)(*) = (a(*), b(*)).$$

$$(\pi_1 \circ (a, b))(*) = \pi_1(a(*), b(*)) = a(*).$$

$$(\pi_2 \circ (a, b))(*) = \pi_2(a(*), b(*)) = b(*).$$

Here, $a \in \text{Hom}_{\mathbf{Sets}}(1, A)$, $b \in \text{Hom}_{\mathbf{Sets}}(1, B)$ are global elements or constants of A and B , respectively. Replacing global elements by generalized elements, one has the following definition.

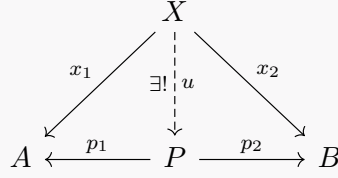
Definition 3.8 (Product). In any category \mathcal{C} , a product diagram for the objects A and B consists of an object P and arrows

$$A \xleftarrow{p_1} P \xrightarrow{p_2} B$$

satisfying the following UMP: given any diagram of the form

$$A \xleftarrow{x_1} X \xrightarrow{x_2} B$$

there exists a unique morphism $u : X \rightarrow P$ such that the following diagram commutes:



i.e. $x_1 = p_1 \circ u$ and $x_2 = p_2 \circ u$.

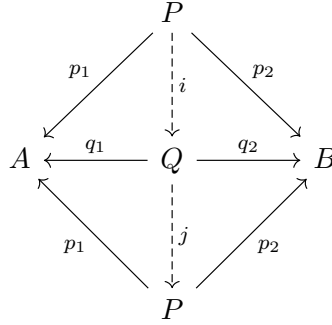
Proposition 3.8

Products are unique up to isomorphism.

Proof. Suppose one has two product diagrams as below:

$$\begin{array}{ccccc} A & \xleftarrow{p_1} & P & \xrightarrow{p_2} & B \\ A & \xleftarrow{q_1} & Q & \xrightarrow{q_2} & B \end{array}$$

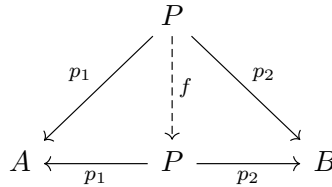
where P, Q are both taken to be products of A and B . Since Q is a product, there is a unique $i : P \rightarrow Q$ such that $q_1 \circ i = p_1$ and $q_2 \circ i = p_2$. Again, since P is a product, there is a unique $j : Q \rightarrow P$ such that $p_1 \circ j = q_1$ and $p_2 \circ j = q_2$. In other words, the following diagram commutes:



Then we have

$$p_1 = q_1 \circ i = p_1 \circ (j \circ i) \text{ and } p_2 = q_2 \circ i = p_2 \circ (j \circ i). \quad (3.27)$$

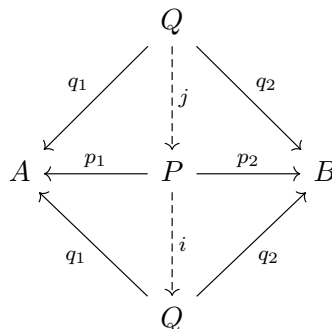
Since P is a product, there is a unique $f : P \rightarrow P$ such that the following diagram commutes:



$f = \mathbb{1}_P$ makes this diagram commutative, since $p_1 = p_1 \circ \mathbb{1}_P$ and $p_2 = p_2 \circ \mathbb{1}_P$. Therefore, the uniqueness of f guarantees that $f = \mathbb{1}_P$. Furthermore, (3.27) tells us that $f = j \circ i$ also makes the above diagram commutative. Therefore,

$$f = \mathbb{1}_P = j \circ i. \quad (3.28)$$

Similarly, using the following commutative diagram,



we have

$$q_1 = p_1 \circ j = q_1 \circ (i \circ j) \text{ and } q_2 = p_2 \circ i = q_2 \circ (i \circ j). \quad (3.29)$$

Since Q is a product, there is a unique $g : Q \rightarrow Q$ such that the following diagram commutes:

$$\begin{array}{ccccc} & & Q & & \\ & \swarrow q_1 & \downarrow g & \searrow q_2 & \\ A & \xleftarrow{q_1} & Q & \xrightarrow{q_2} & B \end{array}$$

$g = \mathbb{1}_Q$ makes this diagram commutative, since $q_1 = q_1 \circ \mathbb{1}_Q$ and $q_2 = q_2 \circ \mathbb{1}_Q$. Therefore, the uniqueness of g guarantees that $g = \mathbb{1}_Q$. Furthermore, (3.29) tells us that $g = i \circ j$ also makes the above diagram commutative. Therefore,

$$g = \mathbb{1}_Q = i \circ j. \quad (3.30)$$

Therefore, we have $i \circ j = \mathbb{1}_Q$ and $j \circ i = \mathbb{1}_P$. Hence, i is an isomorphism and $P \cong Q$. ■

If A and B have a product, we write

$$A \xleftarrow{p_1} A \times B \xrightarrow{p_2} B$$

for one such product. Then given X, x_1, x_2 as below

$$A \xleftarrow{x_1} X \xrightarrow{x_2} B$$

there is a unique arrow $u : X \rightarrow A \times B$ such that $x_i = p_i \circ u$ for $i = 1, 2$. We denote $u = \langle x_1, x_2 \rangle$. Hence, $p_1 \circ \langle x_1, x_2 \rangle = x_1$ and $p_2 \circ \langle x_1, x_2 \rangle = x_2$. Now, however, a pair of objects may have different products in a category \mathcal{C} . For example, given a product $A \times B, p_1, p_2$ and an isomorphism $h : A \times B \rightarrow Q$, one finds that $Q, p_1 \circ h^{-1}, p_2 \circ h^{-1}$ is also a product of A and B .

$$\begin{array}{ccccc} & & X & & \\ & \swarrow x_1 & \downarrow u & \searrow x_2 & \\ & & Q & & \\ & \swarrow p_1 \circ h^{-1} & \downarrow h^{-1} & \searrow p_2 \circ h^{-1} & \\ A & \xleftarrow{p_1} & A \times B & \xrightarrow{p_2} & B \end{array}$$

Now, given a diagram $A \xleftarrow{x_1} X \xrightarrow{x_2} B$, there exists unique $\tilde{u} : X \rightarrow A \times B$ such that $p_1 \circ \tilde{u} = x_1$ and $p_2 \circ \tilde{u} = x_2$. Define $u : X \rightarrow Q$ as $u = h \circ \tilde{u}$. Then $h^{-1} \circ u = \tilde{u}$. So

$$x_1 = p_1 \circ \tilde{u} = (p_1 \circ h^{-1}) \circ u \text{ and } x_2 = p_2 \circ \tilde{u} = (p_2 \circ h^{-1}) \circ u,$$

and the diagram above commutes. Furthermore, u is unique. Indeed, if there is another $v : X \rightarrow Q$ such that $x_1 = (p_1 \circ h^{-1}) \circ v$ and $x_2 = (p_2 \circ h^{-1}) \circ v$, then by the uniqueness of \tilde{u} ,

$$\tilde{u} = h^{-1} \circ v \implies v = h \circ \tilde{u} = u.$$

So u is unique. This proves that $Q, p_1 \circ h^{-1}, p_2 \circ h^{-1}$ fulfills the universal property of a product diagram.

Example 3.4. 1. Products of “structured sets” like monoids or groups can often be constructed as products of the underlying sets with componentwise operations: if G and H are groups, then $G \times H$ can be constructed by taking the underlying set of $G \times H$ to be the set $\{\langle g, h \rangle \mid g \in G, h \in H\}$ and define the binary operation in $G \times H$ by

$$\langle g, h \rangle *_{G \times H} \langle g', h' \rangle = \langle g *_G g', h *_H h' \rangle,$$

where the group G is given by the triple $(G, *_G, u_G)$ and the group H is given by the triple $(H, *_H, u_H)$. The unit of $G \times H$ is given by

$$u_{G \times H} = \langle u_G, u_H \rangle.$$

The inverse of $\langle g, h \rangle$ in $G \times H$ is

$$\langle g, h \rangle^{-1} = \langle g^{-1}, h^{-1} \rangle.$$

The projection homomorphism $G \times H \rightarrow G$ (or H) are the evident ones

$$\langle g, h \rangle \mapsto g \text{ (or } h \text{)}.$$

2. Let (P, \leq) be a poset and consider a product $p \times q$ of elements $p, q \in P$. We must have projections $p \times q \rightarrow p$ and $p \times q \rightarrow q$. In a poset, these arrows indicate

$$p \times q \leq p \text{ and } p \times q \leq q$$

. Furthermore, we need $p \times q$ to fulfill the UMP of product: if for a given element $x \in P$, $x \rightarrow p$ and $x \rightarrow q$ (i.e. $x \leq p$ and $x \leq q$ in the poset), then there exists a unique arrow $x \rightarrow p \times q$ (i.e. $x \leq p \times q$). The product $p \times q$ is easily recognized to be the greatest lower bound $p \wedge q$.

Remark 3.5. Note that the projection arrows $p_1 : A \times B \rightarrow A$ and $p_2 : A \times B \rightarrow B$ need not be epic. In **Sets**, let A be a non-empty set. Consider the product of \emptyset and A . Then the projection

$$p_2 : \emptyset \times A \rightarrow A$$

is not surjective, since the domain is empty set but the codomain is nonempty. Therefore, the projection p_2 is not an epimorphism in **Sets**.

§3.5.i Product Topological Space

Let us show that the product of two topological spaces X and Y is a product $X \times Y$ in **Top**, the category of topological spaces and continuous maps. Let $\mathcal{O}(X)$ be the collection of open sets in X . Suppose we have two topological space X and Y and the product space $X \times Y$ along with the projections

$$X \xleftarrow{p_1} X \times Y \xrightarrow{p_2} Y$$

Recall that $\mathcal{O}(X \times Y)$ is generated by basic open sets of the form $U \times V$, where $U \in \mathcal{O}(X)$ and $V \in \mathcal{O}(Y)$, so that every $W \in \mathcal{O}(X \times Y)$ is a union of such basic open sets. We have the following observations:

- Clearly $p_1 : X \times Y \rightarrow X$ and $p_2 : X \times Y \rightarrow Y$ are continuous: for $U \in \mathcal{O}(X)$ and $V \in \mathcal{O}(Y)$,

$$\begin{aligned} p_1^{-1}(U) &= U \times Y \in \mathcal{O}(X \times Y), \\ p_2^{-1}(V) &= X \times V \in \mathcal{O}(X \times Y). \end{aligned}$$

So $p_1 : X \times Y \rightarrow X$ and $p_2 : X \times Y \rightarrow Y$ are continuous.

- Given any continuous $f_1 : Z \rightarrow X$ and $f_2 : Z \rightarrow Y$, let $f : Z \rightarrow X \times Y$ be defined as

$$f(z) = (f_1(z), f_2(z)).$$

We need to verify that f is continuous. Given $W \in \mathcal{O}(X \times Y)$, W can be expressed as

$$W = \bigcup_{\alpha} (U_{\alpha} \times V_{\alpha}),$$

for $U_{\alpha} \in \mathcal{O}(X)$ and $V_{\alpha} \in \mathcal{O}(Y)$. Then

$$f^{-1}(W) = \bigcup_{\alpha} f^{-1}(U_{\alpha} \times V_{\alpha}).$$

Therefore, it suffices to show that $f^{-1}(U \times V) \in \mathcal{O}(Z)$ for $U \in \mathcal{O}(X)$ and $V \in \mathcal{O}(Y)$.

$$\begin{aligned} f^{-1}(U \times V) &= f^{-1}((U \times Y) \cap (X \times V)) \\ &= f^{-1}(U \times Y) \cap f^{-1}(X \times V) \\ &= f^{-1}(p_1^{-1}(U)) \cap f^{-1}(p_2^{-1}(V)) \\ &= (p_1 \circ f)^{-1}(U) \cap (p_2 \circ f)^{-1}(V) \\ &= f_1^{-1}(U) \cap f_2^{-1}(V). \end{aligned}$$

Both $f_1^{-1}(U)$ and $f_2^{-1}(V)$ are open in Z since f_1, f_2 are continuous. Hence, f is continuous.

- Therefore, given $f_1 : Z \rightarrow X$ and $f_2 : Z \rightarrow Y$ continuous, there exists a continuous $f : Z \rightarrow X \times Y$ such that $p_1 \circ f = f_1$ and $p_2 \circ f = f_2$. Now, suppose there exists another continuous $g : Z \rightarrow X \times Y$ such that $p_1 \circ g = f_1$ and $p_2 \circ g = f_2$. Let $g(z) = (g_1(z), g_2(z))$. Then

$$f_1(z) = (p_1 \circ g)(z) = p_1(g_1(z), g_2(z)) = g_1(z), \quad (3.31)$$

for every $z \in Z$. Similarly, $f_2(z) = g_2(z)$. Therefore, $g = f$, proving the uniqueness of f .

So $X \times Y$ is a product of X and Y in **Top**.

§3.5.ii Category with Products

Let \mathcal{C} be a category that has a product diagram for every pair of objects. Such a category is said to have a binary product. Suppose we have objects and arrows as indicated below:

$$\begin{array}{ccccc} A & \xleftarrow{p_1} & A \times A' & \xrightarrow{p_2} & A' \\ f \downarrow & \swarrow f \circ p_1 & \vdots f \times f' & \searrow f' \circ p_2 & \downarrow f' \\ B & \xleftarrow{q_1} & B \times B' & \xrightarrow{q_2} & B' \end{array} \quad (3.32)$$

Then there exists a unique arrow from the product $A \times A'$ to $B \times B'$ as follows:

$$f \times f' = \langle f \circ p_1, f' \circ p_2 \rangle. \quad (3.33)$$

In other words, $f \times f'$ is the unique arrow from $A \times A'$ to $B \times B'$ which makes the above diagram (3.32) commutative. One can, therefore, construct a functor $\times : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}^1$ as follows:

$$\times (A, A') = A \times A' \text{ and } \times (f, f') = f \times f'. \quad (3.34)$$

Let us now verify that \times is indeed a functor. For an object (A, A') in $\mathcal{C} \times \mathcal{C}$, let $\mathbb{1}_{(A, A')} = (\mathbb{1}_A, \mathbb{1}_{A'})$ be its identity arrow. We need to show that $\times(\mathbb{1}_{(A, A')}) = \mathbb{1}_A \times \mathbb{1}_{A'} = \mathbb{1}_{A \times A'}$.

¹ $\mathcal{C} \times \mathcal{C}$ is the product category as discussed in Section 1.5. An object in $\mathcal{C} \times \mathcal{C}$ is of the form (A, A') with $A, A' \in \text{Ob}(\mathcal{C})$. Given arrows $f : A \rightarrow B$ and $f' : A' \rightarrow B'$ in \mathcal{C} , one has $(f, f') : (A, A') \rightarrow (B, B')$ as an arrow in $\mathcal{C} \times \mathcal{C}$ between objects (A, A') and (B, B') .

$$\begin{array}{ccccc}
A & \xleftarrow{p_1} & A \times A' & \xrightarrow{p_2} & A' \\
\downarrow \mathbb{1}_A & \swarrow \mathbb{1}_A \circ p_1 & \downarrow \mathbb{1}_A \times \mathbb{1}_{A'} & \searrow \mathbb{1}_{A'} \circ p_2 & \downarrow \mathbb{1}_{A'} \\
A & \xleftarrow{p_1} & A \times A' & \xrightarrow{p_2} & A'
\end{array}$$

$\mathbb{1}_A \times \mathbb{1}_{A'}$ is the **unique** arrow $A \times A' \rightarrow A \times A'$ that makes this diagram commute. If we choose $\mathbb{1}_{A \times A'}$ in place of $\mathbb{1}_A \times \mathbb{1}_{A'}$, then also the diagram commutes:

$$\begin{array}{ccccc}
A & \xleftarrow{p_1} & A \times A' & \xrightarrow{p_2} & A' \\
\downarrow \mathbb{1}_A & \swarrow \mathbb{1}_A \circ p_1 & \downarrow \mathbb{1}_{A \times A'} & \searrow \mathbb{1}_{A'} \circ p_2 & \downarrow \mathbb{1}_{A'} \\
A & \xleftarrow{p_1} & A \times A' & \xrightarrow{p_2} & A'
\end{array}$$

since

$$p_1 \circ \mathbb{1}_{A \times A'} = p_1 = \mathbb{1}_A \circ p_1 \text{ and } p_2 \circ \mathbb{1}_{A \times A'} = p_2 = \mathbb{1}_{A'} \circ p_2.$$

Therefore, by the uniqueness, $\mathbb{1}_A \times \mathbb{1}_{A'} = \mathbb{1}_{A \times A'}$. Hence,

$$\times (\mathbb{1}_{(A, A')}) = \times (\mathbb{1}_A, \mathbb{1}_{A'}) = \mathbb{1}_A \times \mathbb{1}_{A'} = \mathbb{1}_{A \times A'}. \quad (3.35)$$

Let $(f, f') : (A, A') \rightarrow (B, B')$ and $(g, g') : (B, B') \rightarrow (C, C')$ be arrows in $\mathcal{C} \times \mathcal{C}$. Then $(g, g') \circ (f, f') = (g \circ f, g' \circ f')$. Consider the following diagram:

$$\begin{array}{ccccc}
A & \xleftarrow{p_1} & A \times A' & \xrightarrow{p_2} & A' \\
\downarrow f & \swarrow f \circ p_1 & \downarrow f \times f' & \searrow f' \circ p_2 & \downarrow f' \\
B & \xleftarrow{q_1} & B \times B' & \xrightarrow{q_2} & B' \\
\downarrow g & \swarrow g \circ q_1 & \downarrow g \times g' & \searrow g' \circ q_2 & \downarrow g' \\
C & \xleftarrow{r_1} & C \times C' & \xrightarrow{r_2} & C'
\end{array}$$

In this diagram, $f \times f' : A \times A' \rightarrow B \times B'$ and $g \times g' : B \times B' \rightarrow C \times C'$ are the **unique** maps such that the diagram above commutes. In other words,

$$q_1 \circ f \times f' = f \circ p_1, \quad q_2 \circ f \times f' = f' \circ p_2, \quad \text{and} \quad (3.36)$$

$$r_1 \circ g \times g' = g \circ q_1, \quad r_2 \circ g \times g' = g' \circ q_2. \quad (3.37)$$

$g \circ f : A \rightarrow C$ and $g' \circ f' : A' \rightarrow C'$. Then $(g \circ f) \times (g' \circ f')$ is the **unique** map from $A \times A'$ to $C \times C'$ such that the following diagram commutes:

$$\begin{array}{ccccc}
A & \xleftarrow{p_1} & A \times A' & \xrightarrow{p_2} & A' \\
\downarrow g \circ f & \swarrow g \circ f \circ p_1 & \downarrow (g \circ f) \times (g' \circ f') & \searrow g' \circ f' \circ p_2 & \downarrow g' \circ f' \\
C & \xleftarrow{r_1} & C \times C' & \xrightarrow{r_2} & C'
\end{array}$$

In other words,

$$r_1 \circ (g \circ f) \times (g' \circ f') = g \circ f \circ p_1 \text{ and } r_2 \circ (g \circ f) \times (g' \circ f') = g' \circ f' \circ p_2. \quad (3.38)$$

If we take $(g \times g') \circ (f \times f')$ in place of $(g \circ f) \times (g' \circ f')$, then also the diagram commutes, since

$$r_1 \circ (g \times g') \circ (f \times f') = g \circ q_1 \circ (f \times f') = g \circ f \circ p_1, \quad (3.39)$$

$$r_2 \circ (g \times g') \circ (f \times f') = g' \circ q_2 \circ (f \times f') = g' \circ f' \circ p_2. \quad (3.40)$$

$$\begin{array}{ccccc}
A & \xleftarrow{p_1} & A \times A' & \xrightarrow{p_2} & A' \\
\downarrow g \circ f & \swarrow g \circ f \circ p_1 & \downarrow (g \times g') \circ (f \times f') & \searrow g' \circ f' \circ p_2 & \downarrow g' \circ f' \\
C & \xleftarrow{r_1} & C \times C' & \xrightarrow{r_2} & C'
\end{array}$$

Therefore, by the uniqueness, we have $(g \times g') \circ (f \times f') = (g \circ f) \times (g' \circ f')$. Then

$$\times ((g, g') \circ (f, f')) = \times (g \circ f, g' \circ f') = (g \circ f) \times (g' \circ f') = (g \times g') \circ (f \times f') = \times (g, g') \circ \times (f, f'). \quad (3.41)$$

Therefore, $\times : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ is indeed a functor.

§3.6 Hom-sets

We are dealing with locally small categories, where given any pair of objects A and B , the collection $\text{Hom}_{\mathcal{C}}(A, B)$ of all the arrows from A to B forms a set. We call such a set of arrows **Hom-set**. Now, fix $A \in \text{Ob}(\mathcal{C})$ once and for all. Then we consider a functor $\text{Hom}_{\mathcal{C}}(A, -) : \mathcal{C} \rightarrow \mathbf{Sets}$ as follows:

$$\begin{array}{ccc}
\mathcal{C} & & \mathbf{Sets} \\
\boxed{\begin{array}{c} C \\ \downarrow g \\ D \end{array}} & \xrightarrow{\text{Hom}_{\mathcal{C}}(A, -)} & \boxed{\begin{array}{c} \text{Hom}_{\mathcal{C}}(A, C) \\ \downarrow \text{Hom}_{\mathcal{C}}(A, g) \\ \text{Hom}_{\mathcal{C}}(A, D) \end{array}}
\end{array}$$

with the function $\text{Hom}_{\mathcal{C}}(A, g) : \text{Hom}_{\mathcal{C}}(A, C) \rightarrow \text{Hom}_{\mathcal{C}}(A, D)$ defined by

$$\text{Hom}_{\mathcal{C}}(A, g)(f) = g \circ f,$$

for each $f \in \text{Hom}_{\mathcal{C}}(A, C)$. $\text{Hom}_{\mathcal{C}}(A, -)$ is called the **representable functor** of A . Let us now show that $\text{Hom}_{\mathcal{C}}(A, \mathbb{1}_X) = \mathbb{1}_{\text{Hom}_{\mathcal{C}}(A, X)}$.

$$\begin{array}{ccc}
\mathcal{C} & & \mathbf{Sets} \\
\boxed{\begin{array}{c} X \\ \downarrow \mathbb{1}_X \\ X \end{array}} & \xrightarrow{\text{Hom}_{\mathcal{C}}(A, -)} & \boxed{\begin{array}{c} \text{Hom}_{\mathcal{C}}(A, X) \\ \downarrow \text{Hom}_{\mathcal{C}}(A, \mathbb{1}_X) \\ \text{Hom}_{\mathcal{C}}(A, X) \end{array}}
\end{array}$$

Indeed, let $x \in \text{Hom}_{\mathcal{C}}(A, X)$. Then

$$\text{Hom}_{\mathcal{C}}(A, \mathbb{1}_X)(x) = \mathbb{1}_X \circ x = x = \mathbb{1}_{\text{Hom}_{\mathcal{C}}(A, X)}(x), \quad (3.42)$$

for any $x \in \text{Hom}_{\mathcal{C}}(A, X)$. Therefore, $\text{Hom}_{\mathcal{C}}(A, \mathbb{1}_X) = \mathbb{1}_{\text{Hom}_{\mathcal{C}}(A, X)}$.

Now, take two composable arrows $g : Y \rightarrow Z$ and $f : X \rightarrow Y$ in \mathcal{C} . We want to show that

$$\text{Hom}_{\mathcal{C}}(A, g \circ f) = \text{Hom}_{\mathcal{C}}(A, g) \circ \text{Hom}_{\mathcal{C}}(A, f) : \text{Hom}_{\mathcal{C}}(A, X) \rightarrow \text{Hom}_{\mathcal{C}}(A, Z).$$

Let's take any $x \in \text{Hom}_{\mathcal{C}}(A, X)$. Then,

$$\begin{aligned}
\text{Hom}_{\mathcal{C}}(A, g \circ f)(x) &= g \circ f \circ x = g \circ (\text{Hom}_{\mathcal{C}}(A, f)(x)) \\
&= \text{Hom}_{\mathcal{C}}(A, g)(\text{Hom}_{\mathcal{C}}(A, f)(x)) \\
&= [\text{Hom}_{\mathcal{C}}(A, g) \circ \text{Hom}_{\mathcal{C}}(A, f)](x), \quad (3.43)
\end{aligned}$$

which is true for any $x \in \text{Hom}_{\mathcal{C}}(A, X)$. Therefore,

$$\text{Hom}_{\mathcal{C}}(A, g \circ f) = \text{Hom}_{\mathcal{C}}(A, g) \circ \text{Hom}_{\mathcal{C}}(A, f). \quad (3.44)$$

Hence, $\text{Hom}_{\mathcal{C}}(A, -) : \mathcal{C} \rightarrow \mathbf{Sets}$ is indeed a functor.

We will now present an alternative characterization of product using Hom-sets. For any object P , a pair of arrows $p_1 : P \rightarrow A$ and $p_2 : P \rightarrow B$ determine an element (p_1, p_2) of the set

$$\text{Hom}_{\mathcal{C}}(P, A) \times \text{Hom}_{\mathcal{C}}(P, B).$$

Given any arrow $x : X \rightarrow P$, composing with p_1 and p_2 gives a pair of arrows $x_1 = p_1 \circ x : X \rightarrow A$ and $x_2 = p_2 \circ x : X \rightarrow B$ as indicated in the following diagram:

$$\begin{array}{ccccc} & & X & & \\ & \swarrow & \downarrow x & \searrow & \\ x_1 = p_1 \circ x & & & & x_2 = p_2 \circ x \\ & \swarrow & \downarrow & \searrow & \\ A & \xleftarrow{p_1} & P & \xrightarrow{p_2} & B \end{array}$$

In this way, one has a function

$$\vartheta_X \equiv (\text{Hom}_{\mathcal{C}}(X, p_1), \text{Hom}_{\mathcal{C}}(X, p_2)) : \text{Hom}_{\mathcal{C}}(X, P) \rightarrow \text{Hom}_{\mathcal{C}}(X, A) \times \text{Hom}_{\mathcal{C}}(X, B).$$

Given any $x : X \rightarrow P$, ϑ maps it to

$$(\text{Hom}_{\mathcal{C}}(X, p_1)(x), \text{Hom}_{\mathcal{C}}(X, p_2)(x)) = (p_1 \circ x, p_2 \circ x).$$

In other words,

$$\vartheta(x) = (p_1 \circ x, p_2 \circ x) = (x_1, x_2). \quad (3.45)$$

Proposition 3.9

A diagram of the form $A \xleftarrow{p_1} P \xrightarrow{p_2} B$ is a product diagram if and only if for every object X , the canonical function ϑ_X defined by (3.45) is an isomorphism,

$$\vartheta_X : \text{Hom}_{\mathcal{C}}(X, P) \xrightarrow{\cong} \text{Hom}_{\mathcal{C}}(X, A) \times \text{Hom}_{\mathcal{C}}(X, B).$$

Proof. (\Rightarrow) Suppose P is a product. Then given any object X of \mathcal{C} and a pair of arrows $x_1 : X \rightarrow A$ and $x_2 : X \rightarrow B$, there is a unique arrow $x : X \rightarrow P$ such that the following diagram commutes:

$$\begin{array}{ccccc} & & X & & \\ & \swarrow & \downarrow x & \searrow & \\ x_1 & & & & x_2 \\ & \swarrow & \downarrow & \searrow & \\ A & \xleftarrow{p_1} & P & \xrightarrow{p_2} & B \end{array}$$

In other words, $p_1 \circ x = x_1$ and $p_2 \circ x = x_2$. Therefore, for any object X , and $x_1 \in \text{Hom}_{\mathcal{C}}(X, A)$, $x_2 \in \text{Hom}_{\mathcal{C}}(X, B)$, there is a unique $x \in \text{Hom}_{\mathcal{C}}(X, P)$ which makes the above diagram commutative. We, therefore, have a function

$$\wp_X : \text{Hom}_{\mathcal{C}}(X, A) \times \text{Hom}_{\mathcal{C}}(X, B) \rightarrow \text{Hom}_{\mathcal{C}}(X, P)$$

defined by

$$\wp_X(x_1, x_2) = x, \quad (3.46)$$

where x is the **unique** arrow from X to P such that $p_1 \circ x = x_1$ and $p_2 \circ x = x_2$. In particular,

$$\wp_X(p_1 \circ x, p_2 \circ x) = x. \quad (3.47)$$

Then for any $(x_1, x_2) \in \text{Hom}_{\mathcal{C}}(X, A) \times \text{Hom}_{\mathcal{C}}(X, B)$,

$$(\vartheta_X \circ \wp_X)(x_1, x_2) = \vartheta_X(x) = (p_1 \circ x, p_2 \circ x) = (x_1, x_2),$$

and for any $x \in \text{Hom}_{\mathcal{C}}(X, P)$,

$$(\wp_X \circ \vartheta_X)(x) = \wp_X(p_1 \circ x, p_2 \circ x) = x.$$

Therefore,

$$\vartheta_X \circ \wp_X = \mathbb{1}_{\text{Hom}_{\mathcal{C}}(X, A) \times \text{Hom}_{\mathcal{C}}(X, B)} \quad \text{and} \quad \wp_X \circ \vartheta_X = \mathbb{1}_{\text{Hom}_{\mathcal{C}}(X, P)}. \quad (3.48)$$

Therefore, ϑ_X is an isomorphism for any object X .

(\Leftarrow) Conversely, suppose, for every object X , the canonical function

$$\vartheta_X : \text{Hom}_{\mathcal{C}}(X, P) \xrightarrow{\cong} \text{Hom}_{\mathcal{C}}(X, A) \times \text{Hom}_{\mathcal{C}}(X, B)$$

is an isomorphism. Given any $x_1 \in \text{Hom}_{\mathcal{C}}(X, A)$ and $x_2 \in \text{Hom}_{\mathcal{C}}(X, B)$, since ϑ_X is an isomorphism (i.e. a bijection since we are in **Sets**), there exists a **unique** $x \in \text{Hom}_{\mathcal{C}}(X, P)$ such that $\vartheta_X(x) = (x_1, x_2)$. Using (3.45), we get

$$(x_1, x_2) = \vartheta_X(x) = (p_1 \circ x, p_2 \circ x). \quad (3.49)$$

Therefore, given any $A \xleftarrow{x_1} X \xrightarrow{x_2} B$, there is a unique $x : X \rightarrow P$ such that $p_1 \circ x = x_1$ and $p_2 \circ x = x_2$. In other words, the following diagram commutes:

$$\begin{array}{ccccc} & & X & & \\ & \swarrow x_1 & \vdots x & \searrow x_2 & \\ A & \xleftarrow{p_1} & P & \xrightarrow{p_2} & B \end{array}$$

Therefore, P is a product of A and B . ■

Definition 3.9. Let \mathcal{C} and \mathcal{D} be categories with binary products. A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is said to **preserve binary products** if it takes every product diagram

$$A \xleftarrow{p_1} A \times B \xrightarrow{p_2} B \quad \text{in } \mathcal{C}$$

to a product diagram

$$F(A) \xleftarrow{F(p_1)} F(A \times B) \xrightarrow{F(p_2)} F(B) \quad \text{in } \mathcal{D}.$$

The latter is a product diagram in \mathcal{D} if and only if $F(A \times B) \cong F(A) \times F(B)$. In other words, if and only if the canonical arrow

$$\langle F(p_1), F(p_2) \rangle : F(A \times B) \xrightarrow{\cong} F(A) \times F(B)$$

is an isomorphism. Note that $\langle F(p_1), F(p_2) \rangle$ is the **unique** arrow from $F(A \times B)$ to $F(A) \times F(B)$ such that the diagram below commutes:

$$\begin{array}{ccccc} & & F(A \times B) & & \\ & \swarrow F(p_1) & \vdots \langle F(p_1), F(p_2) \rangle & \searrow F(p_2) & \\ F(A) & \xleftarrow{\tilde{p}_1} & F(A) \times F(B) & \xrightarrow{\tilde{p}_2} & F(B) \end{array}$$

Therefore, if \mathcal{C} and \mathcal{D} are categories with binary products, then a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ preserves binary products if and only if $F(A \times B) \cong F(A) \times F(B)$ in \mathcal{D} for any $A, B \in \text{Ob}(\mathcal{C})$. For example, the forgetful functor $\mathcal{U} : \mathbf{Mon} \rightarrow \mathbf{Sets}$ preserves binary products.

Corollary 3.10

For any object X in a category \mathcal{C} with products, the covariant representable functor

$$\text{Hom}_{\mathcal{C}}(X, -) : \mathcal{C} \rightarrow \mathbf{Sets}$$

preserves products.

Proof. For $A, B \in \text{Ob}(\mathcal{C})$, [Proposition 3.9](#) says that there is a canonical isomorphism

$$\text{Hom}_{\mathcal{C}}(X, A \times B) \cong \text{Hom}_{\mathcal{C}}(X, A) \times \text{Hom}_{\mathcal{C}}(X, B),$$

as $A \times B$ is a product of A and B . In other words,

$$\text{Hom}_{\mathcal{C}}(X, -)(A \times B) \cong \text{Hom}_{\mathcal{C}}(X, -)(A) \times \text{Hom}_{\mathcal{C}}(X, -)(B). \quad (3.50)$$

Hence, $\text{Hom}_{\mathcal{C}}(X, -) : \mathcal{C} \rightarrow \mathbf{Sets}$ is indeed a binary product preserving functor. ■

§3.7 Associativity of Product**Theorem 3.11**

Product is associative up to isomorphism, i.e. if A , B , and C are objects in a category with products, then $(A \times B) \times C \cong A \times (B \times C)$.

Proof. Let the following be product diagrams:

$$\begin{array}{ccccc} A & \xleftarrow{p_1} & A \times B & \xrightarrow{p_2} & B \\ B & \xleftarrow{q_1} & B \times C & \xrightarrow{q_2} & C \\ A \times B & \xleftarrow{r_1} & (A \times B) \times C & \xrightarrow{r_2} & C \\ A & \xleftarrow{s_1} & A \times (B \times C) & \xrightarrow{s_2} & B \times C \end{array}$$

Consider the following composition:

$$(A \times B) \times C \xrightarrow{r_1} A \times B \xrightarrow{p_2} B.$$

Given $p_2 \circ r_1 : (A \times B) \times C \rightarrow B$ and $r_2 : (A \times B) \times C \rightarrow C$, there exists a **unique** $f : (A \times B) \times C \rightarrow B \times C$ such that the following diagram commutes:

$$\begin{array}{ccccc} & & (A \times B) \times C & & \\ & \swarrow p_2 \circ r_1 & \downarrow f & \searrow r_2 & \\ B & \xleftarrow{q_1} & B \times C & \xrightarrow{q_2} & C \end{array}$$

In other words,

$$q_1 \circ f = p_2 \circ r_1, \quad (3.51)$$

$$q_2 \circ f = r_2. \quad (3.52)$$

Then consider the following composition:

$$(A \times B) \times C \xrightarrow{r_1} A \times B \xrightarrow{p_1} A.$$

Given $p_1 \circ r_1 : (A \times B) \times C \rightarrow A$ and $f : (A \times B) \times C \rightarrow (B \times C)$, there exists a **unique** $g : (A \times B) \times C \rightarrow A \times (B \times C)$ such that the following diagram commutes:

$$\begin{array}{ccccc}
& (A \times B) \times C & & & \\
& \swarrow p_1 \circ r_1 & \downarrow g & \searrow f & \\
A & \xleftarrow{s_1} & A \times (B \times C) & \xrightarrow{s_2} & B \times C
\end{array}$$

In other words,

$$s_1 \circ g = p_1 \circ r_1, \quad (3.53)$$

$$s_2 \circ g = f. \quad (3.54)$$

Now, we consider the following composition:

$$A \times (B \times C) \xrightarrow{s_2} B \times C \xrightarrow{q_1} B.$$

Given $s_1 : A \times (B \times C) \rightarrow A$ and $q_1 \circ s_2 : A \times (B \times C) \rightarrow B$, there exists a **unique** $h : A \times (B \times C) \rightarrow A \times B$ such that the following diagram commutes:

$$\begin{array}{ccccc}
& A \times (B \times C) & & & \\
& \swarrow s_1 & \downarrow h & \searrow q_1 \circ s_2 & \\
A & \xleftarrow{p_1} & A \times B & \xrightarrow{p_2} & B
\end{array}$$

In other words,

$$p_1 \circ h = s_1, \quad (3.55)$$

$$p_2 \circ h = q_1 \circ s_2. \quad (3.56)$$

Then we consider the following composition:

$$A \times (B \times C) \xrightarrow{s_2} B \times C \xrightarrow{q_2} C.$$

Given $h : A \times (B \times C) \rightarrow A \times B$ and $q_2 \circ s_2 : A \times (B \times C) \rightarrow C$, there exists a **unique** $i : A \times (B \times C) \rightarrow (A \times B) \times C$ such that the following diagram commutes:

$$\begin{array}{ccccc}
& A \times (B \times C) & & & \\
& \swarrow h & \downarrow i & \searrow q_2 \circ s_2 & \\
A \times B & \xleftarrow{r_1} & (A \times B) \times C & \xrightarrow{r_2} & C
\end{array}$$

In other words,

$$r_1 \circ i = h, \quad (3.57)$$

$$r_2 \circ i = q_2 \circ s_2. \quad (3.58)$$

We claim that $g : (A \times B) \times C \rightarrow A \times (B \times C)$ is our desired isomorphism, with inverse $i : A \times (B \times C) \rightarrow (A \times B) \times C$. For that purpose, we need to show that $g \circ i = \mathbb{1}_{A \times (B \times C)}$ and $i \circ g = \mathbb{1}_{(A \times B) \times C}$.

Now, consider the following diagram:

$$\begin{array}{ccccc}
& \overset{g \circ i}{\curvearrowright} & & & \\
& A \times (B \times C) & & & \\
& \swarrow q_1 \circ s_2 & \downarrow s_2 & \searrow q_1 \circ s_2 & \\
B & \xleftarrow{q_1} & B \times C & \xrightarrow{q_2} & C
\end{array}$$

$s_2 \circ g \circ i$

Using (3.54), (3.51), (3.57) and (3.56), we get

$$q_1 \circ s_2 \circ g \circ i = q_1 \circ f \circ i = p_2 \circ r_1 \circ i = p_2 \circ h = q_1 \circ s_2. \quad (3.59)$$

Furthermore, using (3.54), (3.52) and (3.58),

$$q_2 \circ s_2 \circ g \circ i = q_2 \circ f \circ i = r_2 \circ i = q_2 \circ s_2. \quad (3.60)$$

So there are two arrows $s_2, s_2 \circ g \circ i : A \times (B \times C) \rightarrow B \times C$ and both of them makes the above diagram commute. Therefore, by the universal property of the product $B \times C$,

$$s_2 = s_2 \circ g \circ i. \quad (3.61)$$

Now, let's consider the following diagram:

$$\begin{array}{ccccc} & & A \times (B \times C) & & \\ & \swarrow s_1 & \downarrow \mathbb{1}_{A \times (B \times C)} \quad g \circ i & \searrow s_2 & \\ A & \xleftarrow{s_1} & A \times (B \times C) & \xrightarrow{s_2} & B \times C \end{array}$$

Here, (3.53), (3.57) and (3.55) gives us

$$s_1 \circ g \circ i = p_1 \circ r_1 \circ i = p_1 \circ h = s_1 = s_1 \circ \mathbb{1}_{A \times (B \times C)}. \quad (3.62)$$

Also, from (3.61),

$$s_2 \circ g \circ i = s_2 = s_2 \circ \mathbb{1}_{A \times (B \times C)}. \quad (3.63)$$

So there are two arrows $\mathbb{1}_{A \times (B \times C)}, g \circ i : A \times (B \times C) \rightarrow A \times (B \times C)$ and both of them makes the above diagram commute. Therefore, by the universal property of the product $A \times (B \times C)$,

$$g \circ i = \mathbb{1}_{A \times (B \times C)}. \quad (3.64)$$

Similarly, consider the following diagram:

$$\begin{array}{ccccc} & & (A \times B) \times C & & \\ & \swarrow p_1 \circ r_1 & \downarrow r_1 \quad r_1 \circ i \circ g & \searrow p_2 \circ r_1 & \\ A & \xleftarrow{p_1} & A \times B & \xrightarrow{p_2} & B \end{array}$$

Using (3.57), (3.55) and (3.53),

$$p_1 \circ r_1 \circ i \circ g = p_1 \circ h \circ g = s_1 \circ g = p_1 \circ r_1. \quad (3.65)$$

Furthermore, by (3.57), (3.56), (3.54) and (3.51),

$$p_2 \circ r_1 \circ i \circ g = p_2 \circ h \circ g = q_1 \circ s_2 \circ g = q_1 \circ f = p_2 \circ r_1. \quad (3.66)$$

So there are two arrows $r_1, r_1 \circ i \circ g : (A \times B) \times C \rightarrow A \times B$ and both of them makes the above diagram commute. Therefore, by the universal property of the product $A \times B$,

$$r_1 = r_1 \circ i \circ g. \quad (3.67)$$

Now, let's consider the following diagram:

$$\begin{array}{ccccc}
& & (A \times B) \times C & & \\
& \swarrow r_1 & \downarrow \mathbb{1}_{(A \times B) \times C} & \searrow r_2 & \\
A \times B & \xleftarrow{r_1} & (A \times B) \times C & \xrightarrow{r_2} & C
\end{array}$$

$i \circ g$ is indicated on the vertical arrow from $(A \times B) \times C$ to $(A \times B) \times C$.

From (3.67),

$$r_1 \circ i \circ g = r_1 = r_1 \circ \mathbb{1}_{(A \times B) \times C}. \quad (3.68)$$

Also, (3.58), (3.54) and (3.52) gives us

$$r_2 \circ i \circ g = q_2 \circ s_2 \circ g = q_2 \circ f = r_2 = r_2 \circ \mathbb{1}_{(A \times B) \times C}. \quad (3.69)$$

So there are two arrows $\mathbb{1}_{(A \times B) \times C}, i \circ g : (A \times B) \times C \rightarrow (A \times B) \times C$ and both of them makes the above diagram commute. Therefore, by the universal property of the product $(A \times B) \times C$,

$$i \circ g = \mathbb{1}_{(A \times B) \times C}. \quad (3.70)$$

Therefore, from (3.64) and (3.70), we can conclude that $g : (A \times B) \times C \rightarrow A \times (B \times C)$ is an isomorphism, with inverse $i : A \times (B \times C) \rightarrow (A \times B) \times C$. ■

Furthermore, this isomorphism is “natural”. To make this notion precise, we need to define what a natural transformation is.

Definition 3.10 (Natural Transformation). Let F and G be two functors between the categories \mathcal{C} and \mathcal{D} . Then a **natural transformation** $\eta : F \Rightarrow G$ is a family of arrows in \mathcal{D} that satisfies the following requirements:

1. The natural transformation η must associate to each object $X \in \text{Ob}(\mathcal{C})$ an arrow $\eta_X : F(X) \rightarrow G(X)$ in \mathcal{D} . This arrow η_X is called the component of η at X .
2. Components must be such that for every arrow $f : X \rightarrow Y$ in \mathcal{C} , one must have $\eta_Y \circ F(f) = G(f) \circ \eta_X$. In other words, the diagram below commutes:

$$\begin{array}{ccc}
F(X) & \xrightarrow{F(f)} & F(Y) \\
\eta_X \downarrow & & \downarrow \eta_Y \\
G(X) & \xrightarrow{G(f)} & G(Y)
\end{array}$$

If $F, G : \mathcal{C} \rightarrow \mathcal{D}$ are functors, and $\eta : F \Rightarrow G$ is a functor, it is denoted as follows

$$\begin{array}{ccc}
& F & \\
\mathcal{C} & \begin{array}{c} \downarrow \eta \\ \downarrow G \end{array} & \mathcal{D}
\end{array}$$

If for any object $X \in \text{Ob}(\mathcal{C})$, the arrow $\eta_X : F(X) \rightarrow G(X)$ is an isomorphism in \mathcal{D} , then $\eta : F \Rightarrow G$ is said to be a **natural isomorphism**. Two functors F and G between two categories are called naturally isomorphic if there exists a natural isomorphism between them.

When we say the isomorphism $g : (A \times B) \times C \xrightarrow{\cong} A \times (B \times C)$ is natural, we mean that there is a natural isomorphism whose components are precisely these isomorphisms $g : (A \times B) \times C \xrightarrow{\cong} A \times (B \times C)$. To prove it, let us first define the product of three categories.

Definition 3.11. Let \mathcal{C} , \mathcal{D} , \mathcal{E} be categories. Then we define the product category $\mathcal{C} \times \mathcal{D} \times \mathcal{E}$ as follows:

- The objects of $\mathcal{C} \times \mathcal{D} \times \mathcal{E}$ are ordered triples (C, D, E) , where $C \in \text{Ob}(\mathcal{C})$, $D \in \text{Ob}(\mathcal{D})$, $E \in \text{Ob}(\mathcal{E})$.
- Given two objects $(C, D, E), (C', D', E') \in \text{Ob}(\mathcal{C} \times \mathcal{D} \times \mathcal{E})$, an arrow from (C, D, E) to (C', D', E') is an ordered triple (f, g, h) , where $f \in \text{Hom}_{\mathcal{C}}(C, C')$, $g \in \text{Hom}_{\mathcal{D}}(D, D')$, $h \in \text{Hom}_{\mathcal{E}}(E, E')$.
- Given an object $(C, D, E) \in \text{Ob}(\mathcal{C} \times \mathcal{D} \times \mathcal{E})$, its identity arrow is $(1_C, 1_D, 1_E)$, i.e.

$$1_{(C,D,E)} = (1_C, 1_D, 1_E).$$

- Given two composable arrows $(f, g, h) : (C, D, E) \rightarrow (C', D', E')$ and $(f', g', h') : (C', D', E') \rightarrow (C'', D'', E'')$, their composition is defined as

$$(f', g', h') \circ (f, g, h) = (f' \circ f, g' \circ g, h' \circ h).$$

One can easily verify that $\mathcal{C} \times \mathcal{D} \times \mathcal{E}$ is indeed a category. Now, suppose \mathcal{C} is a category with products. Then we define two functors from $\mathcal{C} \times \mathcal{C} \times \mathcal{C}$ to \mathcal{C} .

$$\begin{aligned} F : \mathcal{C} \times \mathcal{C} \times \mathcal{C} &\rightarrow \mathcal{C}, \\ (A, B, C) &\mapsto (A \times B) \times C, \\ (f, g, h) &\mapsto (f \times g) \times h. \end{aligned} \tag{3.71}$$

$$\begin{aligned} G : \mathcal{C} \times \mathcal{C} \times \mathcal{C} &\rightarrow \mathcal{C}, \\ (A, B, C) &\mapsto A \times (B \times C), \\ (f, g, h) &\mapsto f \times (g \times h). \end{aligned} \tag{3.72}$$

One can easily check that F and G are functors. Now we define the natural transformation α as follows:

$$\begin{array}{ccc} & F & \\ \curvearrowright & \downarrow \alpha & \curvearrowleft \\ \mathcal{C} \times \mathcal{C} \times \mathcal{C} & & \mathcal{C} \\ \curvearrowleft & \downarrow G & \end{array}$$

Given $(A, B, C) \in \text{Ob}(\mathcal{C} \times \mathcal{C} \times \mathcal{C})$, $F(A, B, C) = (A \times B) \times C$, and $G(A, B, C) = A \times (B \times C)$. So the component of α at (A, B, C) is an arrow from $(A \times B) \times C$ to $A \times (B \times C)$ in \mathcal{C} , which we define to be the isomorphism $g : (A \times B) \times C \rightarrow A \times (B \times C)$ defined in the course of proof of [Theorem 3.11](#).

$$\alpha_{(A,B,C)} = g : (A \times B) \times C \xrightarrow{\cong} A \times (B \times C). \tag{3.73}$$

Theorem 3.12

α defined as above is a natural isomorphism.

§3.8 Product and Terminal Object

Proposition 3.13

Let \mathcal{C} be a category with products, and 1 is a terminal object of \mathcal{C} . Then for any $X \in \text{Ob}(\mathcal{C})$, $X \times 1 \cong X$.

Proof. Let the following be a product diagram:

$$X \xleftarrow{p_1} X \times 1 \xrightarrow{p_2} 1$$

Since 1 is a terminal object, there is a **unique** arrow $u : X \rightarrow 1$. Given $\mathbb{1}_X : X \rightarrow X$ and $u : X \rightarrow 1$, there is a **unique** arrow $\langle \mathbb{1}_X, u \rangle : X \rightarrow X \times 1$ such that the following diagram commutes:

$$\begin{array}{ccccc} & & X & & \\ & \swarrow \mathbb{1}_X & \downarrow \langle \mathbb{1}_X, u \rangle & \searrow u & \\ X & \xleftarrow{p_1} & X \times 1 & \xrightarrow{p_2} & 1 \end{array}$$

In other words,

$$p_1 \circ \langle \mathbb{1}_X, u \rangle = \mathbb{1}_X \text{ and } p_2 \circ \langle \mathbb{1}_X, u \rangle = u. \quad (3.74)$$

We claim that $p_1 : X \times 1 \rightarrow X$ is an isomorphism, with inverse $\langle \mathbb{1}_X, u \rangle$. Clearly, we have $p_1 \circ \langle \mathbb{1}_X, u \rangle = \mathbb{1}_X$. We need to show that $\langle \mathbb{1}_X, u \rangle \circ p_1 = \mathbb{1}_{X \times 1}$.

$p_2 \circ \langle \mathbb{1}_X, u \rangle \circ p_1$ is an arrow from $X \times 1$ to 1 . Since 1 is a terminal object, there is exactly one arrow from $X \times 1$ to 1 . So $p_2 \circ \langle \mathbb{1}_X, u \rangle \circ p_1 = p_2$. Therefore,

$$p_1 \circ \langle \mathbb{1}_X, u \rangle \circ p_1 = p_1 \text{ and } p_2 \circ \langle \mathbb{1}_X, u \rangle \circ p_1 = p_2. \quad (3.75)$$

So the following diagrams commute:

$$\begin{array}{ccc} & X \times 1 & \\ p_1 \swarrow & \downarrow \langle \mathbb{1}_X, u \rangle \circ p_1 & \searrow p_2 \\ X & \xleftarrow{p_1} X \times 1 \xrightarrow{p_2} & 1 \end{array} \quad \begin{array}{ccc} & X \times 1 & \\ p_1 \swarrow & \downarrow \mathbb{1}_{X \times 1} & \searrow p_2 \\ X & \xleftarrow{p_1} X \times 1 \xrightarrow{p_2} & 1 \end{array}$$

Therefore, by the universal property of products, we must have $\langle \mathbb{1}_X, u \rangle \circ p_1 = \mathbb{1}_{X \times 1}$. Therefore, $p_1 : X \times 1 \rightarrow X$ is an isomorphism, with inverse $\langle \mathbb{1}_X, u \rangle : X \rightarrow X \times 1$, i.e. $X \times 1 \cong X$. \blacksquare

However, the converse of [Proposition 3.13](#) is not, in general, true. If A is an object in a category \mathcal{C} with products such that $X \times A \cong A$ for every object $X \in \text{Ob}(\mathcal{C})$, then A need not be a terminal object. The following is a counterexample.

Example 3.5. Let $\mathbb{Z}^{\mathbb{N}}$ denote the set of all integer sequences, which is an object of the category **Groups**. Let us consider the subcategory \mathcal{C} of **Groups** consisting of only one object $\mathbb{Z}^{\mathbb{N}}$, and all the arrows from $\mathbb{Z}^{\mathbb{N}}$ to itself in **Groups** (often referred to as the *full subcategory* of **Groups**, with the object $\mathbb{Z}^{\mathbb{N}}$). Then in this category \mathcal{C} ,

$$\mathbb{Z}^{\mathbb{N}} \times \mathbb{Z}^{\mathbb{N}} = \mathbb{Z}^{\mathbb{N}}.$$

The product diagram is as follows:

$$\mathbb{Z}^{\mathbb{N}} \xleftarrow{\pi_1} \mathbb{Z}^{\mathbb{N}} \xrightarrow{\pi_2} \mathbb{Z}^{\mathbb{N}}$$

The homomorphisms π_1 and π_2 are defined as follows:

$$\pi_1(a_0, a_1, a_2, a_3, a_4, \dots) = (a_1, a_3, a_5, \dots),$$

$$\pi_2(a_0, a_1, a_2, a_3, a_4, \dots) = (a_0, a_2, a_4, a_6, \dots).$$

This is easily seen to satisfy the UMP of product. Now, in the category \mathcal{C} , there is only one object. So $X \times A \cong A$ for every object $X \in \text{Ob}(\mathcal{C})$ is clearly satisfied. But $\mathbb{Z}^{\mathbb{N}}$ is **not** a terminal object in this category, since there are more than one arrows from $\mathbb{Z}^{\mathbb{N}}$ to itself in \mathcal{C} .

The isomorphism we showed in [Proposition 3.13](#) is not just any isomorphism. This isomorphism is actually “natural”, meaning there is a natural isomorphism whose components are the isomorphisms we saw in [Proposition 3.13](#).

$$\begin{array}{ccc} & \times(-,1) & \\ \curvearrowright & \downarrow \eta & \curvearrowleft \\ \mathcal{C} & & \mathcal{C} \\ & \downarrow \mathbb{1}_{\mathcal{C}} & \end{array}$$

The components of η are given by

$$\eta_X = p_1 : X \times 1 \xrightarrow{\cong} X, \quad (3.76)$$

where $p_1 : X \times 1 \rightarrow X$ is the projection on X arrow. To show that η is a natural isomorphism, we just need to show the commutativity of the following square:

$$\begin{array}{ccc} X \times 1 & \xrightarrow{f \times \mathbb{1}_1} & X' \times 1 \\ \eta_X = p_1 \downarrow & & \downarrow \eta_{X'} = p'_1 \\ X & \xrightarrow{f} & X' \end{array}$$

But this square obviously commutes, since $f \times \mathbb{1}_1$ is the **unique** arrow $X \times 1 \rightarrow X' \times 1$ such that the following diagram commutes:

$$\begin{array}{ccccc} X & \xleftarrow{p_1} & X \times 1 & \xrightarrow{p_2} & 1 \\ \downarrow f & & \downarrow f \times \mathbb{1}_1 & & \downarrow \mathbb{1}_1 \\ X' & \xleftarrow{p'_1} & X' \times 1 & \xrightarrow{p'_2} & 1 \end{array}$$

The naturality square is precisely the left hand square of this diagram. So naturality holds. Therefore, the isomorphism $X \times 1 \cong X$ is **natural**.

Now, the converse of [Proposition 3.13](#) holds if we further assume that the isomorphism is natural.

Theorem 3.14

Let \mathcal{C} be a category with products, and $A \in \text{Ob}(\mathcal{C})$ such that $\eta : \times(-, A) \Rightarrow \mathbb{1}_{\mathcal{C}}$ is a natural isomorphism. Then A is a terminal object of \mathcal{C} .

Proof. $\eta_X : X \times A \rightarrow X$ is an isomorphism for every $X \in \text{Ob}(\mathcal{C})$. Suppose the following is a product diagram:

$$X \xleftarrow{p_1} X \times A \xrightarrow{p_2} A$$

Let us define a transformation $\alpha : \mathbb{1}_{\mathcal{C}} \Rightarrow \mathbb{1}_{\mathcal{C}}$, which we are going to show to be a natural transformation.

$$\begin{array}{ccc} & \mathbb{1}_{\mathcal{C}} & \\ \curvearrowright & \downarrow \alpha & \curvearrowleft \\ \mathcal{C} & & \mathcal{C} \\ & \downarrow \mathbb{1}_{\mathcal{C}} & \end{array}$$

The components of α are given by

$$\alpha_X = p_1 \circ \eta_X^{-1} : X \rightarrow X, \quad (3.77)$$

where p_1 is the projection $X \times A \rightarrow X$. In order to show that α is a natural transformation, we need to show the commutativity of the following square given any arrow $g : X \rightarrow X'$:

$$\begin{array}{ccc} X & \xrightarrow{g} & X' \\ \alpha_X = p_1 \circ \eta_X^{-1} \downarrow & & \downarrow \alpha_{X'} = p'_1 \circ \eta_{X'}^{-1} \\ X & \xrightarrow{g} & X' \end{array} \quad (3.78)$$

$g \times \mathbb{1}$ is the **unique** arrow $X \times A \rightarrow X' \times A$ such that the diagram below commutes:

$$\begin{array}{ccccc} X & \xleftarrow{p_1} & X \times A & \xrightarrow{p_2} & A \\ g \downarrow & & \downarrow g \times \mathbb{1}_A & & \downarrow \mathbb{1}_A \\ X' & \xleftarrow{p'_1} & X' \times A & \xrightarrow{p'_2} & A \end{array}$$

In particular,

$$g \circ p_1 = p'_1 \circ (g \times \mathbb{1}_A). \quad (3.79)$$

The naturality of η guarantees the commutativity of the following square:

$$\begin{array}{ccc} X \times A & \xrightarrow{g \times \mathbb{1}_A} & X' \times A \\ \eta_X \downarrow & & \downarrow \eta_{X'} \\ X & \xrightarrow{g} & X' \end{array}$$

In other words,

$$\eta_{X'} \circ (g \times \mathbb{1}_A) = g \circ \eta_X. \quad (3.80)$$

Composing by $\eta_{X'}^{-1}$ on the left, and η_X^{-1} on the right, we get

$$(g \times \mathbb{1}_A) \circ \eta_X^{-1} = \eta_{X'}^{-1} \circ g. \quad (3.81)$$

Therefore, using (3.79) and (3.81),

$$\alpha_{X'} \circ g = p'_1 \circ \eta_{X'}^{-1} \circ g = p'_1 \circ (g \times \mathbb{1}_A) \circ \eta_X^{-1} = g \circ p_1 \circ \eta_X^{-1} = g \circ \alpha_X. \quad (3.82)$$

So (3.78) commutes, and hence $\alpha : \mathbb{1}_C \rightarrow \mathbb{1}_C$ is a natural transformation. Therefore, for any arrow $u : X \rightarrow X$, by the naturality of α , the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{u} & X \\ \alpha_X = p_1 \circ \eta_X^{-1} \downarrow & & \downarrow \alpha_X = p_1 \circ \eta_X^{-1} \\ X & \xrightarrow{u} & X \end{array}$$

In other words, for any $u : X \rightarrow X$,

$$p_1 \circ \eta_X^{-1} \circ u = u \circ p_1 \circ \eta_X^{-1}. \quad (3.83)$$

In order to prove that A is a terminal object, we need to show that there exists a **unique** arrow $X \rightarrow A$. Clearly, there is an arrow $X \rightarrow A$; for instance $p_2 \circ \eta_X^{-1} : X \rightarrow A$. We now need to show that the arrow $X \rightarrow A$ is unique. Let f be **any** arrow $X \rightarrow A$.

$$\begin{array}{ccccc}
& & X & & \\
& \swarrow \mathbb{1}_X & \downarrow \langle \mathbb{1}_X, f \rangle & \searrow f & \\
X & \xleftarrow{p_1} & X \times A & \xrightarrow{p_2} & A
\end{array}$$

Choose the arrow $u : X \rightarrow X$ to be $u = \eta_X \circ \langle \mathbb{1}_X, f \rangle : X \rightarrow X$. Using (3.83), we have

$$\eta_X \circ \langle \mathbb{1}_X, f \rangle \circ p_1 \circ \eta_X^{-1} = p_1 \circ \eta_X^{-1} \circ \eta_X \circ \langle \mathbb{1}_X, f \rangle = p_1 \circ \langle \mathbb{1}_X, f \rangle = \mathbb{1}_X. \quad (3.84)$$

So, $\eta_X \circ \langle \mathbb{1}_X, f \rangle \circ p_1 \circ \eta_X^{-1} = \mathbb{1}_X$. Now, composing by η_X on the right, and η_X^{-1} on the left, we get

$$\langle \mathbb{1}_X, f \rangle \circ p_1 = \eta_X^{-1} \circ \eta_X = \mathbb{1}_{X \times A}. \quad (3.85)$$

Therefore, $p_1 : X \times A \rightarrow X$ is an isomorphism, with the inverse $\langle \mathbb{1}_X, f \rangle$. As a result,

$$f = p_2 \circ \langle \mathbb{1}_X, f \rangle = p_2 \circ p_1^{-1}. \quad (3.86)$$

Hence, given **any** arrow $f : X \rightarrow A$, f must be equal to $p_2 \circ p_1^{-1}$. Therefore, there is a unique arrow $X \rightarrow A$, and hence A is a terminal object. \blacksquare

4 Duality

§4.1 Duality Principle

The **Elementary Theory of an Abstract Category (ETAC)** consists of certain statements which involves letters A, B, C, \dots for objects and f, g, h, \dots for arrows. For example, the statement $f : A \rightarrow B$ can be phrased as “ A is the domain of f , and B is the codomain of f .” A sentence is a statement with all variables quantified (quantifiers are “for all A ”, “for all f ”, “there exists an A ”, “there exists an f ” etc.), and none being free. For example,

For all f , there exists A and B such that $f : A \rightarrow B$

is a sentence. Axioms of ETAC are examples of sentences that are true in every category.

To each category \mathcal{C} , we also associate the **opposite category** \mathcal{C}^{op} . The objects of \mathcal{C}^{op} are the same as those of \mathcal{C} , i.e. given $A \in \text{Ob}(\mathcal{C})$, one also has $A \in \text{Ob}(\mathcal{C}^{\text{op}})$.

$$\text{Ob}(\mathcal{C}) = \text{Ob}(\mathcal{C}^{\text{op}}).$$

The arrows of \mathcal{C}^{op} are arrows f^{op} in the reverse direction, and hence are in 1-1 correspondence

$$f \mapsto f^{\text{op}}$$

with the arrows f of \mathcal{C} . In other words, if $f : A \rightarrow B$ is an arrow in \mathcal{C} , then the corresponding arrow in \mathcal{C}^{op} is given by $f^{\text{op}} : B \rightarrow A$ so that

$$\text{dom}(f^{\text{op}}) = \text{cod}(f) \text{ and } \text{cod}(f^{\text{op}}) = \text{dom}(f).$$

Now, given arrows

$$\begin{array}{c} A \xrightarrow{f} B \xrightarrow{g} C \\ \quad \quad \quad \curvearrowright \\ \quad \quad \quad g \circ f \end{array}$$

in \mathcal{C} , one has

$$\begin{array}{c} A \xleftarrow{f^{\text{op}}} B \xleftarrow{g^{\text{op}}} C \\ \quad \quad \quad \curvearrowleft \\ \quad \quad \quad f^{\text{op}} \circ g^{\text{op}} = (g \circ f)^{\text{op}} \end{array}$$

in \mathcal{C}^{op} so that $f^{\text{op}} \circ g^{\text{op}} = (g \circ f)^{\text{op}}$ is defined in \mathcal{C}^{op} exactly when the composite $g \circ f$ is defined in \mathcal{C} , i.e. (g, f) is a composable pair (g after f) in \mathcal{C} if and only if $(f^{\text{op}}, g^{\text{op}})$ is a composable pair (f^{op} after g^{op}) in \mathcal{C}^{op} , and $f^{\text{op}} \circ g^{\text{op}}$ is precisely the opposite arrow of the composite arrow $g \circ f$ in \mathcal{C} .

Now, the dual of any statement Σ of ETAC is formed by making the following replacements throughout in Σ : “domain” by “codomain”, “codomain” by “domain”, “ h is the composite of f with g ” by “ h is the composite of g with f ”. As a result, arrows and composites are reversed in the dual statement Σ^* . While forming the dual sentence, logic / quantifiers (and, or, for all, there exists etc.) remain unchanged. Some examples of statements Σ and their dual statements Σ^* are listed below:

Statement Σ	Dual Statement Σ^*
$f : A \rightarrow B$	$f : B \rightarrow A$
$A = \text{dom}(f)$	$A = \text{cod}(f)$
$i = \mathbb{1}_A$	$i = \mathbb{1}_A$
$h = g \circ f$	$h = f \circ g$
T is a terminal object	T is an initial object

A sentence Σ is true in \mathcal{C}^{op} if and only if its dual statement Σ^* is true in \mathcal{C} (arrows in \mathcal{C}^{op} are read without the op prefix).

Remark 4.1. Note that the dual of the dual is the original statement ($\Sigma^{**} = \Sigma$). There could be broadly two types of statements or sentences. A statement can be consequences of the axioms of ETAC, for example, “a terminal object of a category, if it exists, is unique up to isomorphism.” The other type of statements do not follow from the axioms of ETAC. For example, “ $a = \text{dom } f$ ”, or $h = g \circ f$. For the latter type of statements Σ in \mathcal{C} , the dual statements Σ^* refer to opposite category \mathcal{C}^{op} . For the former type of statements, i.e. for the statements that are consequences of the axioms of ETAC, Σ and its dual Σ^* refer to the category \mathcal{C} . This fact is captured by the **duality principle**.

Duality Principle:

If a statement Σ of ETAC is a consequence of the axioms, so is the dual statement Σ^* .

For example, take Σ to be “a terminal object, if it exists, is unique up to isomorphism.” We have the dual Σ^* that reads “an initial object, if it exists, is unique up to isomorphism.” This dual statement Σ^* applies to the same category \mathcal{C} .

§4.2 Coproducts

Definition 4.1. A diagram $A \xrightarrow{q_1} Q \xleftarrow{q_2} B$ is called a coproduct of A and B if for any object Z and $A \xrightarrow{z_1} Z \xleftarrow{z_2} B$, there is a **unique** arrow $u : Q \rightarrow Z$ such that the following diagram commutes:

$$\begin{array}{ccccc} & & Z & & \\ & \nearrow z_1 & \uparrow \exists! u & \nwarrow z_2 & \\ A & \xrightarrow{q_1} & Q & \xleftarrow{q_2} & B \end{array}$$

i.e. $u \circ q_1 = z_1$ and $u \circ q_2 = z_2$.

We usually write $A \xrightarrow{i_1} A + B \xleftarrow{i_2} B$ for the coproduct, and $[f, g]$ for the uniquely determined arrow $[f, g] : A + B \rightarrow Z$ as in the following diagram:

$$\begin{array}{ccccc} & & Z & & \\ & \nearrow f & \uparrow [f, g] & \nwarrow g & \\ A & \xrightarrow{i_1} & A + B & \xleftarrow{i_2} & B \end{array}$$

Remark 4.2. Similar to Remark 3.5, the “co-projections” $i_1 : A \rightarrow A + B$ and $i_2 : B \rightarrow A + B$ need not be monomorphisms. A coproduct of two objects is a product of the same two objects in the opposite category, as can be seen by reversing the arrows above and using the commutative diagram of a product. Therefore, given a nonempty set A , the product $\emptyset \times A$ in **Sets** is the coproduct $\emptyset + A$ in **Sets**^{op}. In other words,

$$\emptyset +_{\mathbf{Sets}^{\text{op}}} A = \emptyset \times_{\mathbf{Sets}} A. \quad (4.1)$$

Then the coprojection

$$i_2 : A \rightarrow \emptyset +_{\mathbf{Sets}^{\text{op}}} A$$

is not a monomorphism in **Sets**^{op}, since the opposite arrow $p_2 : \emptyset \times_{\mathbf{Sets}} A \rightarrow A$ is not an epimorphism.

The ‘co-projections’ $i_1 : A \rightarrow A + B$ and $i_2 : B \rightarrow A + B$ are often called coproduct injections.

Example 4.1. In **Sets**, the coproduct $A + B$ of two sets is their disjoint union which can be constructed as

$$A + B = \{(a, 1) \mid a \in A\} \cup \{(b, 2) \mid b \in B\} = (A \times \{1\}) \cup (B \times \{2\}), \quad (4.2)$$

with evident coproduct injections $i_1(a) = (a, 1)$ and $i_2(b) = (b, 2)$. Given any functions $f : A \rightarrow Z$ and $g : B \rightarrow Z$,

we define $[f, g] : A + B \rightarrow Z$ as follows:

$$[f, g](x, \delta) = \begin{cases} f(x) & \text{if } \delta = 1, \\ g(x) & \text{if } \delta = 2. \end{cases} \quad (4.3)$$

Uniqueness of $[f, g]$ defined in (4.3) follows by noting that if for $h : A + B \rightarrow Z$ the above diagram commutes, then one must have $h \circ i_1 = f$ and $h \circ i_2 = g$. This leads to

$$f(a) = (h \circ i_1)(a) = h(i_1(a)) = h(a, 1), \text{ and} \quad (4.4)$$

$$g(b) = (h \circ i_2)(b) = h(i_2(b)) = h(b, 2). \quad (4.5)$$

Therefore, $h(x, \delta) = [f, g](x, \delta)$, for any $(x, \delta) \in A + B$. So $[f, g] = h$ proving the uniqueness of $[f, g]$.

In **Sets**, if A is a finite set with cardinality n , then

$$A \cong \underbrace{1 + 1 + \cdots + 1}_{n\text{-times}}, \quad (4.6)$$

where 1 stands for a singleton set. Indeed, if $A = \{a_1, a_2, \dots, a_n\}$, then

$$\{a_1\} + \{a_2\} + \cdots + \{a_n\} = \{(a_1, 1), (a_2, 2), \dots, (a_n, n)\} \cong \{a_1, a_2, \dots, a_n\}. \quad (4.7)$$

Proposition 4.1

Coproducts are unique up to isomorphism.

We don't need to prove it again. We can just use the fact that products are unique up to isomorphism (Proposition 3.8), and then invoke duality principle.

Theorem 4.2

Coproducts are associative up to isomorphism, i.e. if A , B , and C are objects in a category with coproducts, then $(A + B) + C \cong A + (B + C)$.

This follows from Theorem 3.11 and duality principle.

Example 4.2. If $M(A)$ and $M(B)$ are free monoids on sets A and B , then in the category **Mon**, we can construct their coproduct as

$$M(A) + M(B) \cong M(A + B).$$

In other words, the coproduct of the free monoids on A and B is the free monoid on the coproduct $A + B$ of A and B in **Sets**.

We shall prove that $M(A + B)$ satisfies the UMP of coproduct of $M(A)$ and $M(B)$. Let $\mathcal{U} : \mathbf{Mon} \rightarrow \mathbf{Sets}$ be the forgetful functor, and let

$$\begin{aligned}\eta_A &: A \rightarrow \mathcal{U}(M(A)), \\ \eta_B &: B \rightarrow \mathcal{U}(M(B)), \\ \eta_{A+B} &: A + B \rightarrow \mathcal{U}(M(A + B))\end{aligned}$$

be the respective insertions of generators. Let

$$A \xrightarrow{i_1} A + B \xleftarrow{i_2} B$$

be the coproduct diagram for A and B . Consider the following diagram in **Sets**:

$$\begin{array}{ccccc}\mathcal{U}(M(A)) & \xrightarrow{\mathcal{U}(j_1)} & \mathcal{U}(M(A + B)) & \xleftarrow{\mathcal{U}(j_2)} & \mathcal{U}(M(B)) \\ \uparrow \eta_A & & \uparrow \eta_{A+B} & & \uparrow \eta_B \\ A & \xrightarrow{i_1} & A + B & \xleftarrow{i_2} & B\end{array}$$

Given the monoid $M(A + B)$ and the function $\eta_{A+B} \circ i_1 : A \rightarrow \mathcal{U}(M(A + B))$, by the UMP of free monoid on the set A , there is a **unique** monoid homomorphism $j_1 : M(A) \rightarrow M(A + B)$ such that

$$\eta_{A+B} \circ i_1 = \mathcal{U}(j_1) \circ \eta_A. \quad (4.8)$$

Similarly, using the UMP of free monoid on the set B and the function $\eta_{A+B} \circ i_2 : B \rightarrow \mathcal{U}(M(A + B))$, there is a **unique** monoid homomorphism $j_2 : M(B) \rightarrow M(A + B)$ such that

$$\eta_{A+B} \circ i_2 = \mathcal{U}(j_2) \circ \eta_B. \quad (4.9)$$

In other words, the diagram above commutes in **Sets**. We now claim that

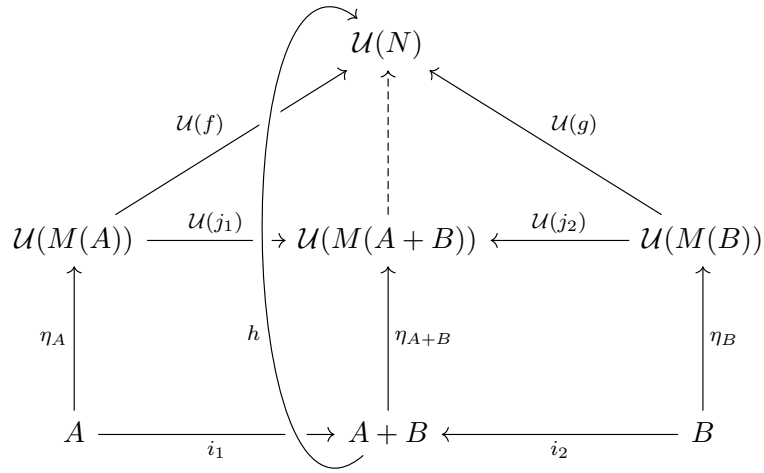
$$M(A) \xrightarrow{j_1} M(A + B) \xleftarrow{j_2} M(B)$$

is a coproduct diagram. Let N be any monoid and let $f : M(A) \rightarrow N$ and $g : M(B) \rightarrow N$ be monoid homomorphisms.

$$\begin{array}{ccccc}\mathcal{U}(N) & & & & \\ \uparrow \mathcal{U}(f) & & \uparrow h & & \uparrow \mathcal{U}(g) \\ \mathcal{U}(M(A)) & & \mathcal{U}(M(A + B)) & & \mathcal{U}(M(B)) \\ \uparrow \eta_A & & \uparrow \eta_{A+B} & & \uparrow \eta_B \\ A & \xrightarrow{i_1} & A + B & \xleftarrow{i_2} & B\end{array}$$

Since $A + B$ is the coproduct of A and B in **Sets**, there exists a **unique** $h : A + B \rightarrow \mathcal{U}(N)$ such that

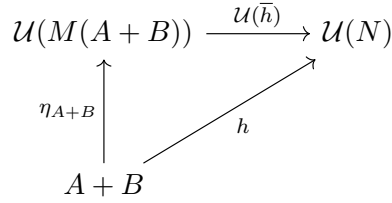
$$h \circ i_1 = \mathcal{U}(f) \circ \eta_A \text{ and } h \circ i_2 = \mathcal{U}(g) \circ \eta_B. \quad (4.10)$$



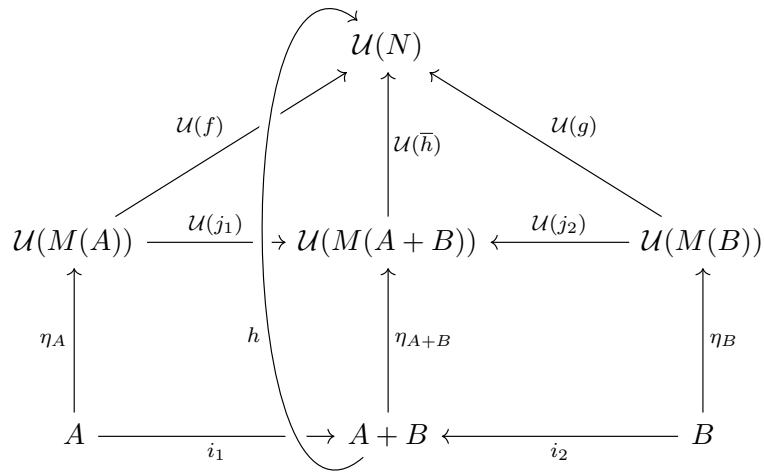
Now, given the monoid N and the function $h : A + B \rightarrow \mathcal{U}(N)$, by the UMP of free monoid on the set $A + B$, there exists a **unique** monoid homomorphism $\bar{h} : M(A + B) \rightarrow N$ such that

$$h = \mathcal{U}(\bar{h}) \circ \eta_{A+B}. \quad (4.11)$$

In other words, the following diagram commutes



So we get the following diagram:



In this diagram, using (4.11) and (4.10), we have

$$\mathcal{U}(\bar{h}) \circ \eta_{A+B} \circ i_1 = h \circ i_1 = \mathcal{U}(f) \circ \eta_A. \quad (4.12)$$

Again, using (4.8),

$$\mathcal{U}(\bar{h}) \circ \eta_{A+B} \circ i_1 = \mathcal{U}(\bar{h}) \circ \mathcal{U}(j_1) \circ \eta_A = \mathcal{U}(\bar{h} \circ j_1) \circ \eta_A. \quad (4.13)$$

However, for the function $\mathcal{U}(\bar{h}) \circ \eta_{A+B} \circ i_1 : A \rightarrow \mathcal{U}(N)$, using the UMP of free monoid on the set A , there exists a **unique** monoid homomorphism $u : M(A) \rightarrow N$ such that

$$\mathcal{U}(\bar{h}) \circ \eta_{A+B} \circ i_1 = \mathcal{U}(u) \circ \eta_A. \quad (4.14)$$

In other words, there is a **unique** monoid homomorphism $u : M(A) \rightarrow N$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{U}(M(A)) & \xrightarrow{\mathcal{U}(u)} & \mathcal{U}(N) \\ \eta_A \uparrow & \nearrow \mathcal{U}(\bar{h}) \circ \eta_{A+B} \circ i_1 & \\ A & & \end{array}$$

But from (4.12) and (4.13), we get that taking $u = f$ and $u = \bar{h} \circ j_1$ makes the above diagram commute. Therefore, we must have

$$f = \bar{h} \circ j_1. \quad (4.15)$$

Similarly,

$$g = \bar{h} \circ j_2. \quad (4.16)$$

Therefore, given the following diagram commutes:

$$\begin{array}{ccccc} & & N & & \\ & \nearrow f & \uparrow \bar{h} & \nwarrow g & \\ M(A) & \xrightarrow{j_1} & M(A+B) & \xleftarrow{j_2} & M(B) \end{array}$$

Furthermore, we need to show that \bar{h} is unique. Suppose there exists another monoid homomorphism $v : M(A+B) \rightarrow N$ such that $f = v \circ j_1$ and $g = v \circ j_2$. Then using (4.8), we get

$$\mathcal{U}(f) \circ \eta_A = \mathcal{U}(v) \circ \mathcal{U}(j_1) \circ \eta_A = \mathcal{U}(v) \circ \eta_{A+B} \circ i_1. \quad (4.17)$$

Similarly, using (4.9),

$$\mathcal{U}(g) \circ \eta_B = \mathcal{U}(v) \circ \mathcal{U}(j_2) \circ \eta_B = \mathcal{U}(v) \circ \eta_{A+B} \circ i_2. \quad (4.18)$$

However, $h : A+B \rightarrow \mathcal{U}(N)$ is the **unique** map such that

$$h \circ i_1 = \mathcal{U}(f) \circ \eta_A \text{ and } h \circ i_2 = \mathcal{U}(g) \circ \eta_B.$$

The uniqueness of h along with (4.17) and (4.18) implies that

$$h = \mathcal{U}(v) \circ \eta_{A+B}. \quad (4.19)$$

On the other hand, \bar{h} is the **unique** monoid homomorphism $M(A+B) \rightarrow N$ such that $h = \mathcal{U}(\bar{h}) \circ \eta_{A+B}$. Therefore, from (4.19), we must have $\bar{h} = v$. Hence, \bar{h} is the **unique** monoid homomorphism $M(A+B) \rightarrow N$ such that $f = \bar{h} \circ j_1$ and $g = \bar{h} \circ j_2$. So $M(A+B)$ satisfies the UMP of coproduct of $M(A)$ and $M(B)$. Therefore,

$$M(A) + M(B) \cong M(A+B). \quad (4.20)$$

Example 4.3. Two monoids A, B have a coproduct of the form

$$A + B = M(|A| + |B|) / \sim .$$

Here, $|A| + |B|$ is the disjoint union of the underlying sets $|A|$ and $|B|$ of the monoids A and B , respectively. One then forms the free monoid $M(|A| + |B|)$ on the set $|A| + |B|$, which is the collection of words over $|A| + |B|$. Now given two words $v, w \in M(|A| + |B|)$, one declares them to be equivalent as follows:

$$\begin{aligned} \cdots x u_A y \cdots &\sim \cdots x y \cdots , \\ \cdots x u_B y \cdots &\sim \cdots x y \cdots , \\ \cdots a \cdot_A a' \cdots &\sim \cdots a a' \cdots , \\ \cdots b \cdot_B b' \cdots &\sim \cdots b b' \cdots . \end{aligned} \tag{4.21}$$

One, thus, forms quotient of the free monoid $M(|A| + |B|)$ subjected to the equivalence relation \sim provided in (4.21). Two equivalence classes $[x \dots y], [x' \dots y'] \in M(|A| + |B|) / \sim$ have the straightforward multiplication:

$$[x \dots y] \cdot [x' \dots y'] = [x \dots y x' \dots y'] . \tag{4.22}$$

The unit in $M(|A| + |B|) / \sim$ is the equivalence class $[-]$ consisting of the empty word. Clearly,

$$[-] = [u_A] = [u_B] .$$

The collection of all the equivalence classes thus formed is defined as the coproduct of A and B :

$$A + B = M(|A| + |B|) / \sim . \tag{4.23}$$

The coproduct injections $i_A : A \rightarrow A + B$ and $i_B : B \rightarrow A + B$ are simply

$$i_A(a) = [a] , \quad i_B(b) = [b] , \tag{4.24}$$

which can easily be verified to be monoid homomorphism. The UMP of $A + B$ should ensure the existence of a **unique** monoid homomorphism $[f, g] : A + B \rightarrow N$ for a given monoid N and monoid homomorphisms $f : A \rightarrow N$ and $g : B \rightarrow N$ such that the following diagram commutes:

$$\begin{array}{ccccc} & & N & & \\ & \nearrow f & \uparrow [f, g] & \nwarrow g & \\ A & \xrightarrow{i_A} & A + B & \xleftarrow{i_B} & B \end{array}$$

Let us now construct $[f, g]$. From the given monoid homomorphisms $f : A \rightarrow N$ and $g : B \rightarrow N$, one obtains the functions $|f| : |A| \rightarrow |N|$ and $|g| : |B| \rightarrow |N|$. Now, using the UMP of the coproduct $|A| + |B|$, one knows that there exists a **unique** function $h = [|f|, |g|] : |A| + |B| \rightarrow |N|$ such that the following diagram commutes:

$$\begin{array}{ccccc} & & |N| & & \\ & \nearrow |f| & \uparrow h & \nwarrow |g| & \\ |A| & \xrightarrow{i_1} & |A| + |B| & \xleftarrow{i_2} & |B| \end{array}$$

where $i_1 : |A| \rightarrow |A| + |B|$ and $i_2 : |B| \rightarrow |A| + |B|$ are coproduct injections of the coproduct $|A| + |B|$ in **Sets**. Now, consider the free monoid $M(|A| + |B|)$ on the set $|A| + |B|$. Invoking the UMP of free monoid, given the monoid N and the function $h : |A| + |B| \rightarrow |N|$, there exists a **unique** monoid homomorphism $\bar{h} : M(|A| + |B|) \rightarrow N$ such that the following diagram commutes in **Sets**:

$$\begin{array}{ccc}
M(|A| + |B|) & \xrightarrow{\bar{h}} & |N| \\
\uparrow i & \nearrow h & \\
|A| + |B| & &
\end{array}$$

where $i : |A| + |B| \rightarrow M(|A| + |B|)$ is the insertion of generators. We now need to verify that \bar{h} “respects the equivalence relation \sim ”, i.e. $v \sim w$ implies $\bar{h}(v) = \bar{h}(w)$.

Given $a \in A$,

$$h(a) = h(i_1(a)) = f(a). \quad (4.25)$$

Since $i(a) = a$, so

$$\bar{h}(a) = \bar{h}(i(a)) = h(a) = f(a). \quad (4.26)$$

Similarly, for $b \in B$,

$$\bar{h}(b) = g(b). \quad (4.27)$$

So $\bar{h}(u_A) = \bar{h}(u_B) = u_N$. As a result,

$$\bar{h}(\cdots x u_A y \cdots) = \cdots \bar{h}(x) \bar{h}(u_A) \bar{h}(y) \cdots = \cdots \bar{h}(x) \bar{h}(y) \cdots = \bar{h}(\cdots x y \cdots).$$

Similarly,

$$\bar{h}(\cdots x u_B y \cdots) = \cdots \bar{h}(x) \bar{h}(u_B) \bar{h}(y) \cdots = \cdots \bar{h}(x) \bar{h}(y) \cdots = \bar{h}(\cdots x y \cdots).$$

Since f is a homomorphism,

$$\begin{aligned}
\bar{h}(\cdots a \cdot_A a' \cdots) &= \cdots \bar{h}(a \cdot_A a') \cdots = \cdots f(a \cdot_A a') \cdots \\
&= \cdots f(a) f(a') \cdots = \cdots \bar{h}(a) \bar{h}(a') \cdots \\
&= \bar{h}(\cdots a a' \cdots).
\end{aligned}$$

Similarly, using the fact that g is a homomorphism,

$$\begin{aligned}
\bar{h}(\cdots b \cdot_B b' \cdots) &= \cdots \bar{h}(b \cdot_B b') \cdots = \cdots g(b \cdot_B b') \cdots \\
&= \cdots g(b) g(b') \cdots = \cdots \bar{h}(b) \bar{h}(b') \cdots \\
&= \bar{h}(\cdots b b' \cdots).
\end{aligned}$$

Therefore, $v \sim w$ implies $\bar{h}(v) = \bar{h}(w)$. So we have a monoid homomorphism $\tilde{h} : M(|A| + |B|) / \sim \rightarrow N$ defined by

$$\tilde{h}[w] = \bar{h}(w). \quad (4.28)$$

This is well-defined since \bar{h} is constant on the equivalence classes. Therefore, the following diagram commutes:

$$\begin{array}{ccc}
M(|A| + |B|) & \xrightarrow{\pi} & M(|A| + |B|) / \sim \\
& \searrow \bar{h} & \downarrow \tilde{h} \\
& & N
\end{array}$$

Now, this $\tilde{h} : M(|A| + |B|) / \sim \rightarrow N$ is our desired $[f, g]$. Indeed,

$$(\tilde{h} \circ i_A)(a) = \tilde{h}([a]) = \bar{h}(a) = f(a). \quad (4.29)$$

$$(\tilde{h} \circ i_B)(b) = \tilde{h}([b]) = \bar{h}(b) = g(b). \quad (4.30)$$

Therefore, $\tilde{h} \circ i_A = f$ and $\tilde{h} \circ i_B = g$. In other words, the following diagram commutes:

$$\begin{array}{ccccc}
& & N & & \\
& \nearrow f & \uparrow \tilde{h} & \nwarrow g & \\
A & \xrightarrow{i_A} & M(|A| + |B|) / \sim & \xleftarrow{i_B} & B
\end{array}$$

Furthermore, \tilde{h} is unique. If there exists another monoid homomorphism $u : M(|A| + |B|) / \sim \rightarrow N$ such that $u \circ i_A = f$ and $u \circ i_B = g$, then we must have

$$\tilde{h}([a]) = f(a) = (u \circ i_A)(a) = u([a]), \quad (4.31)$$

and similarly,

$$\tilde{h}([b]) = g(b) = (u \circ i_B)(b) = u([b]), \quad (4.32)$$

for any $a \in A$ and $b \in B$. So u agrees with \tilde{h} on the equivalence classes of elements of A and B . Since any element of $M(|A| + |B|) / \sim$ can be written as a finite product of equivalence classes of elements of A and B , we can conclude that u and \tilde{h} agrees everywhere in $M(|A| + |B|) / \sim$. Therefore, \tilde{h} is indeed the **unique** arrow in **Mon** such that $\tilde{h} \circ i_A = f$ and $\tilde{h} \circ i_B = g$. Therefore, $M(|A| + |B|) / \sim = A + B$, and $\tilde{h} = [f, g]$.

Abuse of Notation. In [Example 4.3](#), we did not distinguish between $a \in |A|$ and $(a, 1) \in |A| + |B|$; and between $b \in B$ and $(b, 2) \in |A| + |B|$. This can be justified by assuming without loss of generality that $|A|$ and $|B|$ are disjoint. Indeed, if they are not disjoint, we can just replace them by $|A| \times \{1\}$ and $|B| \times \{2\}$.

Example 4.4. Coproduct in groups is called the free product. Suppose A and B are groups. The free product $A * B$ consists of words of the form

$$a_1 b_1 a_2 b_2 \cdots,$$

where $a_i \in A$ and $b_j \in B$. For example,

$$(a_1 b_1 a_2 b_2) = (a_1 b_1) * (a_2 b_2) \text{ and } (a_2 b_2 a_1 b_1) = (a_2 b_2) * (a_1 b_1)$$

are both words in $A * B$, but they are unequal highlighting the non-abelian nature of the free product of $A * B$, even when A and B are both abelian groups. When A and B are both abelian groups, one defines their direct sum as the quotient

$$A \oplus B = A * B / \sim,$$

where the equivalence relation \sim is defined as

$$(a_1 b_1 a_2 b_2 \cdots a_n b_n \cdots) \sim (a_1 a_2 \cdots a_n \cdots b_1 b_2 \cdots b_n \cdots). \quad (4.33)$$

On the RHS, all the a_i 's are flushed to the left. Now, since A and B are both abelian, so is $A \oplus B$:

$$(a_1 b_1) * (a_2 b_2) = a_1 b_1 a_2 b_2 = a_1 a_2 b_1 b_2 = a_2 a_1 b_2 b_1 = a_2 b_2 a_1 b_1 = (a_2 b_2) * (a_1 b_1).$$

Thus, using (4.33), one can identify a word in $A \oplus B$ by a pair

$$(a_1 + a_2 + \cdots + a_n + \cdots, b_1 + b_2 + \cdots + b_n + \cdots) \equiv (a, b)$$

with $a \in A$ and $b \in B$. But

$$|A \times B| = \{(a, b) \mid a \in A, b \in B\}.$$

Given two abelian groups A and B , their coproduct is defined as their direct sum, which has their product set as the underlying set as we have just seen. Hence, $|A + B| = |A \times B|$, for A, B abelian groups. Then the coproduct injections $i_A : A \rightarrow A + B$ and $i_B : B \rightarrow A + B$ are defined as

$$i_A(a) = (a, 0_B) \text{ and } i_B(b) = (0_A, b). \quad (4.34)$$

Given any group homomorphisms $A \xrightarrow{f} X \xleftarrow{g} B$ for any abelian group X , we construct $[f, g] : A + B \rightarrow X$ as follows:

$$[f, g](a, b) = f(a) +_X g(b). \quad (4.35)$$

Then,

$$\begin{aligned} ([f, g] \circ i_A)(a) &= [f, g](a, 0_B) = f(a) + g(0_B) = f(a), \\ ([f, g] \circ i_B)(b) &= [f, g](0_A, b) = f(0_A) + g(b) = g(b). \end{aligned}$$

Therefore, $[f, g] \circ i_A = f$ and $[f, g] \circ i_B = g$. In other words, the following diagram commutes:

$$\begin{array}{ccccc} & & X & & \\ & \nearrow f & \uparrow [f, g] & \nwarrow g & \\ A & \xrightarrow{i_A} & A + B & \xleftarrow{i_B} & B \end{array}$$

Furthermore, $[f, g]$ is the **unique** homomorphism $A + B \rightarrow X$ such that the diagram above commutes. If there exists another homomorphism $h : A + B \rightarrow X$ such that $h \circ i_A = f$ and $h \circ i_B = g$, then for any $a \in A$ and $b \in B$,

$$h(a, 0_B) = h(i_A(a)) = f(a) \text{ and } h(0_A, b) = h(i_B(b)) = g(b).$$

Therefore,

$$h(a, b) = h(a, 0_B) + h(0_A, b) = f(a) + g(b) = [f, g](a, b). \quad (4.36)$$

So $[f, g] = h$, and hence $[f, g]$ is unique. Therefore, $A + B$ is indeed the coproduct of A and B .

We have just seen that $|A + B| = |A \times B|$ for A, B abelian groups. In fact, a stronger result holds in the category **Ab** of abelian groups.

Proposition 4.3

In the category **Ab** of abelian groups, there is a canonical isomorphism between the binary product and coproduct, i.e. $A + B \cong A \times B$.

Proof. The goal is to define an arrow $\vartheta : A + B \rightarrow A \times B$. In order to do so, one needs first an arrow $A \rightarrow A \times B$ (which is determined by an arrow $A \rightarrow A$ and an arrow $A \rightarrow B$) and another arrow $B \rightarrow A \times B$ (which is determined by an arrow $B \rightarrow A$ and an arrow $B \rightarrow B$). Therefore, we need 4 arrows altogether, which we choose as follows:

$$1_A : A \rightarrow A, \quad 0_B : A \rightarrow B, \quad 0_A : B \rightarrow A, \quad 1_B : B \rightarrow B,$$

where $0_B : A \rightarrow B$ maps all of A to the identity $0_B \in B$, and $0_A : B \rightarrow A$ maps all of B to the identity $0_A \in A$.

$$\begin{array}{c} \begin{array}{ccccc} & & A & & \\ & \swarrow 1_A & \downarrow \langle 1_A, 0_B \rangle & \searrow 0_B & \\ A & \xleftarrow{p_1} & A \times B & \xrightarrow{p_2} & B \end{array} & \begin{array}{ccccc} & & B & & \\ & \swarrow 0_A & \downarrow \langle 0_A, 1_B \rangle & \searrow 1_B & \\ A & \xleftarrow{p_1} & A \times B & \xrightarrow{p_2} & B \end{array} \\ \\ \begin{array}{ccccc} & & A \times B & & \\ & \swarrow \langle 0_A, 1_B \rangle & \uparrow [\langle 1_A, 0_B \rangle, \langle 0_A, 1_B \rangle] & \nwarrow \langle 1_A, 0_B \rangle & \\ A & \xrightarrow{i_A} & A + B & \xleftarrow{i_B} & B \end{array} \end{array}$$

Using the commutative diagrams above, we define

$$\vartheta = [\langle \mathbf{1}_A, \mathbf{0}_B \rangle, \langle \mathbf{0}_A, \mathbf{1}_B \rangle] : A + B \rightarrow A \times B. \quad (4.37)$$

Given $(a, b) \in A + B$, one obtains

$$\begin{aligned} \vartheta(a, b) &= [\langle \mathbf{1}_A, \mathbf{0}_B \rangle, \langle \mathbf{0}_A, \mathbf{1}_B \rangle](a, b) \\ &= \langle \mathbf{1}_A, \mathbf{0}_B \rangle(a) + \langle \mathbf{0}_A, \mathbf{1}_B \rangle(b) \\ &= (\mathbf{1}_A(a), \mathbf{0}_B(a)) + (\mathbf{0}_A(b), \mathbf{1}_B(b)) \\ &= (a, 0_B) + (0_A, b) \\ &= (a, b). \end{aligned} \quad (4.38)$$

Therefore, ϑ is a bijective group homomorphism, and hence an isomorphism. \blacksquare

§4.3 Equalizers

Definition 4.2 (Equalizer). In any category \mathcal{C} , given parallel arrows $A \begin{smallmatrix} \xrightarrow{f} \\ \xrightarrow{g} \end{smallmatrix} B$, an **equalizer** of f and g consists of an object E and an arrow $e : E \rightarrow A$ *universal* such that $f \circ e = g \circ e$, i.e. the following diagram commutes:

$$E \xrightarrow{e} A \begin{smallmatrix} \xrightarrow{f} \\ \xrightarrow{g} \end{smallmatrix} B$$

Here $e : E \rightarrow A$ is universal in the sense that given $z : Z \rightarrow A$ with $f \circ z = g \circ z$, there exists a unique $u : Z \rightarrow E$ satisfying $e \circ u = z$. This is encapsulated in the following commutative diagram:

$$\begin{array}{ccccc} E & \xrightarrow{e} & A & \begin{smallmatrix} \xrightarrow{f} \\ \xrightarrow{g} \end{smallmatrix} & B \\ \uparrow \scriptstyle \exists! u & \nearrow z & & & \\ Z & & & & \end{array}$$

Example 4.5. Suppose one has continuous functions $f, g : \mathbb{R}^2 \rightarrow \mathbb{R}$ in **Top** defined by

$$f(x, y) = x^2 + y^2 \text{ and } g(x, y) = 1. \quad (4.39)$$

The equalizer of f and g is the subspace

$$S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\},$$

along with the inclusion

$$i : S^1 \hookrightarrow \mathbb{R}^2.$$

Given any generalized element $z : Z \rightarrow \mathbb{R}^2$ of \mathbb{R}^2 , we can write

$$z(\alpha) = (z_1(\alpha), z_2(\alpha)),$$

for $\alpha \in Z$. So $z = \langle z_1, z_2 \rangle$. If $f \circ z = g \circ z$, then for any $\alpha \in Z$,

$$1 = g(z(\alpha)) = f(z(\alpha)) = z_1(\alpha)^2 + z_2(\alpha)^2. \quad (4.40)$$

So $(z_1(\alpha), z_2(\alpha)) = z(\alpha) \in S^1$ for every $\alpha \in Z$. Therefore, there is a factorization $z = i \circ \bar{z}$ through some $\bar{z} : Z \rightarrow S^1$. \bar{z} is nothing but restricting the codomain of $z : Z \rightarrow \mathbb{R}^2$ to S^1 . Then the following diagram commutes:

$$\begin{array}{ccccc}
S^1 & \xhookrightarrow{e} & \mathbb{R}^2 & \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} & \mathbb{R} \\
\uparrow \bar{z} & & \nearrow z & & \\
Z & & & &
\end{array}$$

Since the inclusion i is monic, such $\bar{z} : Z \rightarrow S^1$ is unique. Hence, the unit sphere S^1 along with the inclusion $i : S^1 \hookrightarrow \mathbb{R}^2$ is indeed the equalizer of f and g in **Top**.

Example 4.6. Similarly, in **Sets**, given any functions $f, g : A \rightarrow B$, their equalizer is the inclusion of the following subset

$$E = \{x \in A \mid f(x) = g(x)\} \quad (4.41)$$

into A . Then

$$E \xhookrightarrow{e} A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} B$$

is a commutative diagram. Furthermore, given any set Z and a function $z : Z \rightarrow A$ such that $f \circ z = g \circ z$, we have

$$f(z(\alpha)) = g(z(\alpha)),$$

for any $\alpha \in Z$. So $z(\alpha) \in E$. So we can define a function $\bar{z} : Z \rightarrow E$ by just restricting the codomain of z , then the following diagram commutes:

$$\begin{array}{ccccc}
E & \xhookrightarrow{e} & A & \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} & B \\
\uparrow \bar{z} & & \nearrow z & & \\
Z & & & &
\end{array}$$

Furthermore, \bar{z} is unique since the inclusion $E \hookrightarrow A$ is injective. Therefore, $e : E \hookrightarrow A$ is the equalizer of f and g .

Furthermore, every subset $U \subseteq A$ is of this “equational form”, i.e. every subset is an equalizer for some pair of functions. Let us undertake this canonical construction. First, let us put

$$\mathbf{2} = \{0, 1\},$$

thinking of it as the set of “truth values” (or the 2-element Boolean algebra). Then consider the characteristic function $\chi_U : A \rightarrow \mathbf{2}$ defined by

$$\chi_U(x) = \begin{cases} 1 & \text{if } x \in U, \\ 0 & \text{if } x \notin U. \end{cases} \quad (4.42)$$

As a result,

$$U = \{x \in A \mid \chi_U(x) = 1\}.$$

We can also define a constant function $\top : A \rightarrow \mathbf{2}$ that takes all of A to $1 \in \mathbf{2}$, i.e.

$$\top(x) = 1, \quad (4.43)$$

for any $x \in A$. Then the following is a commutative diagram:

$$U \xhookrightarrow{i} A \begin{array}{c} \xrightarrow{\top} \\ \xrightarrow{\chi_U} \end{array} \mathbf{2}$$

In other words, $\top \circ i = \chi_U \circ i$.

On the other hand, let $\varphi : A \rightarrow \mathbf{2}$ be given. Form the variety (i.e. “equational subset”)

$$V_\varphi = \{x \in A \mid \varphi(x) = 1\} = \varphi^{-1}(\{1\}). \quad (4.44)$$

Now one can easily verify that the inclusion $i : V_\varphi \hookrightarrow A$ is an equalizer of the parallel arrows

$A \begin{smallmatrix} \top \\ \varphi \end{smallmatrix} \rightrightarrows \mathbf{2}$, i.e. one can verify that the following diagram is commutative:

$$V_\varphi \xrightarrow{i} A \begin{smallmatrix} \top \\ \varphi \end{smallmatrix} \rightrightarrows \mathbf{2}$$

Indeed, for any $x \in V_\varphi$,

$$(\varphi \circ i)(x) = \varphi(x) = 1 \text{ and } (\top \circ i)(x) = 1.$$

Let us now show that indeed $\mathcal{P}(A) \cong \text{Hom}_{\mathbf{Sets}}(A, \mathbf{2})$. Let $\vartheta : \mathcal{P}(A) \rightarrow \text{Hom}_{\mathbf{Sets}}(A, \mathbf{2})$ be given by

$$\vartheta(U) = \chi_U, \quad (4.45)$$

for any $U \subseteq A$; and $\lambda : \text{Hom}_{\mathbf{Sets}}(A, \mathbf{2}) \rightarrow \mathcal{P}(A)$ be given by

$$\lambda(\varphi) = V_\varphi = \varphi^{-1}(\{1\}), \quad (4.46)$$

for any function $\varphi : A \rightarrow \mathbf{2}$. Then from (4.45) and (4.46), we get

$$(\vartheta \circ \lambda)(\varphi) = \chi_{\lambda(\varphi)} = \chi_{V_\varphi}.$$

$$\chi_{V_\varphi}(x) = \begin{cases} 1 & \text{if } x \in V_\varphi, \\ 0 & \text{if } x \notin V_\varphi \end{cases} = \begin{cases} 1 & \text{if } \varphi(x) = 1, \\ 0 & \text{if } \varphi(x) = 0 \end{cases} = \varphi(x).$$

Therefore, $\chi_{V_\varphi} = \varphi$, and hence

$$\vartheta \circ \lambda = \mathbb{1}_{\text{Hom}_{\mathbf{Sets}}(A, \mathbf{2})}. \quad (4.47)$$

On the other hand,

$$(\lambda \circ \vartheta)(U) = V_{\vartheta(U)} = V_{\chi_U}.$$

$$V_{\chi_U} = \{x \in A \mid \chi_U(x) = 1\} = \{x \in A \mid x \in U\} = U.$$

Therefore, $V_{\chi_U} = U$, and hence

$$\lambda \circ \vartheta = \mathbb{1}_{\mathcal{P}(A)}. \quad (4.48)$$

From (4.47) and (4.48), one can conclude that

$$\vartheta : \mathcal{P}(A) \rightarrow \text{Hom}_{\mathbf{Sets}}(A, \mathbf{2})$$

is an isomorphism, i.e. $\mathcal{P}(A) \cong \text{Hom}_{\mathbf{Sets}}(A, \mathbf{2})$.

Example 4.7. Let G and H be two groups with $h : G \rightarrow H$ being a group homomorphism. The kernel of h is the subgroup of G defined by

$$\text{Ker } h = \{g \in G \mid h(g) = e_H\},$$

where e_H is the identity element of the group H . Then $\iota : \text{Ker } h \hookrightarrow G$ is an equalizer of h and u , where $u : G \rightarrow H$ is the trivial group homomorphism, i.e. the homomorphism that maps all of G to the identity element e_H of H .

$$\text{Ker } h = \{g \in G \mid h(g) = e_H\} \xrightarrow{\iota} G \begin{smallmatrix} h \\ u \end{smallmatrix} \rightrightarrows H$$

Indeed, $h \circ \iota = u \circ \iota$. Suppose there is another group A and a homomorphism $a : A \rightarrow G$ such that $h \circ a = u \circ a$.

$$\begin{array}{ccccc}
 \text{Ker } h & \xrightarrow{\iota} & G & \xrightarrow[h]{u} & H \\
 \uparrow u & \nearrow a & & & \\
 A & & & &
 \end{array}$$

Then for any $x \in A$, $h(a(x)) = u(a(x)) = e_H$, so $a(x) \in \text{Ker } h$. So we can define a homomorphism $u : A \rightarrow \text{Ker } h$, which is formed by restricting the codomain of a to $\text{Ker } h$. For this u , we have

$$\iota \circ u = a.$$

Furthermore, u is unique since the inclusion $\iota : \text{Ker } h \rightarrow G$ is monic. Therefore, $\iota : \text{Ker } h \rightarrow G$ is an equalizer of $h, u : G \rightarrow H$.

Proposition 4.4

In any category, if $e : E \rightarrow A$ is an equalizer of some pair of arrows, then e is monic.

Proof. Suppose $e : E \rightarrow A$ is an equalizer of the parallel arrows $f, g : A \rightarrow B$. Then $f \circ e = g \circ e$. Suppose $x, y : Z \rightarrow E$ such that $e \circ x = e \circ y$.

$$\begin{array}{ccccc}
 E & \xrightarrow{e} & A & \xrightarrow[f]{g} & B \\
 \uparrow x & \nearrow z & & & \\
 \uparrow y & & & & \\
 Z & & & &
 \end{array}$$

We want to show that $x = y$. Put $z = e \circ x = e \circ y$. Then

$$f \circ z = f \circ e \circ x = g \circ e \circ x = g \circ z. \quad (4.49)$$

Then from the definition of equalizer, there is a **unique** arrow $u : Z \rightarrow E$ such that $e \circ u = z$. But $z = e \circ x = e \circ y$. Therefore, from the uniqueness of u , we have

$$u = x = y. \quad (4.50)$$

Therefore, $e : E \rightarrow A$ is monic. ■

Proposition 4.5

Equalizers are unique up to isomorphism. In other words, if $e_1 : E_1 \rightarrow A$ and $e_2 : E_2 \rightarrow A$ are equalizers of parallel arrows $f, g : A \rightarrow B$, then there is an isomorphism $i : E_1 \rightarrow E_2$ such that $e_1 = e_2 \circ i$ and $e_2 = e_1 \circ i^{-1}$.

Proof. Since $e_1 : E_1 \rightarrow A$ is an equalizer, then $f \circ e_1 = g \circ e_1$. $e_2 : E_2 \rightarrow A$ is also an equalizer, so $f \circ e_2 = g \circ e_2$. But since E_1 is an equalizer, by the UMP of equalizer, there exists a **unique** arrow $j : E_2 \rightarrow E_1$ such that the following diagram commutes:

$$\begin{array}{ccccc}
 E_1 & \xrightarrow{e_1} & A & \xrightarrow[f]{g} & B \\
 \uparrow j & \nearrow e_2 & & & \\
 E_2 & & & &
 \end{array}$$

In other words,

$$e_1 \circ j = e_2. \quad (4.51)$$

Similarly, since E_2 is an equalizer, by the UMP of equalizer, there exists a **unique** arrow $i : E_1 \rightarrow E_2$ such that the following diagram commutes:

$$\begin{array}{ccccc}
 E_2 & \xrightarrow{e_2} & A & \xrightleftharpoons[g]{f} & B \\
 \uparrow i & \nearrow e_1 & & & \\
 E_1 & & & &
 \end{array}$$

In other words,

$$e_2 \circ i = e_1. \quad (4.52)$$

We claim that this $i : E_1 \rightarrow E_2$ is our desired isomorphism with inverse $j : E_2 \rightarrow E_1$. From (4.51) and (4.52), we get $e_1 = e_2 \circ i = e_1 \circ j \circ i$. So we have the following commutative diagram:

$$\begin{array}{ccccc}
 E_1 & \xrightarrow{e_1} & A & \xrightleftharpoons[g]{f} & B \\
 \uparrow j \circ i & \nearrow e_1 & & & \\
 E_1 & & & &
 \end{array}$$

Since $e_1 : E_1 \rightarrow A$ is an equalizer, and $f \circ e_1 = g \circ e_1$, there exists a **unique** arrow $u : E_1 \rightarrow E_1$ such that $e_1 \circ u = e_1$. But the above diagram shows that there are two such arrows $E_1 \rightarrow E_1$, $j \circ i$ and 1_{E_1} . Therefore, by the uniqueness, we must have

$$j \circ i = 1_{E_1}. \quad (4.53)$$

Similarly, again using (4.51) and (4.52), we get $e_2 = e_1 \circ j = e_2 \circ i \circ j$. So we have the following commutative diagram:

$$\begin{array}{ccccc}
 E_2 & \xrightarrow{e_2} & A & \xrightleftharpoons[g]{f} & B \\
 \uparrow i \circ j & \nearrow e_2 & & & \\
 E_2 & & & &
 \end{array}$$

Since $e_2 : E_2 \rightarrow A$ is an equalizer, and $f \circ e_2 = g \circ e_2$, there exists a **unique** arrow $v : E_2 \rightarrow E_2$ such that $e_2 \circ v = e_2$. But the above diagram shows that there are two such arrows $E_2 \rightarrow E_2$, $i \circ j$ and 1_{E_2} . Therefore, by the uniqueness, we must have

$$i \circ j = 1_{E_2}. \quad (4.54)$$

Combining (4.53) and (4.54), we conclude that $i : E_1 \rightarrow E_2$ is an isomorphism and $j = i^{-1}$. Furthermore, $e_1 = e_2 \circ i$ and $e_2 = e_1 \circ j = e_1 \circ i^{-1}$. ■

§4.4 Coequalizer

Definition 4.3 (Coequalizer). For any parallel arrows $f, g : A \rightarrow B$ in a category \mathcal{C} , a **coequalizer** consists of an object Q and an arrow $q : B \rightarrow Q$, *universal* with the property $q \circ f = q \circ g$ as depicted in the following diagram:

$$\begin{array}{ccccc}
 A & \xrightleftharpoons[g]{f} & B & \xrightarrow{q} & Q \\
 & & \searrow z & \downarrow \exists! u & \\
 & & & Z &
 \end{array}$$

That is, given any object Z and arrow $z : B \rightarrow Z$ with $z \circ f = z \circ g$, then there exists a **unique**

$u : Q \rightarrow Z$ such that $u \circ q = z$.

Consider the statement given in [Proposition 4.4](#): “if $e : E \rightarrow A$ is an equalizer of some pair of arrows, then e is monic.” This statement follows from the axioms of ETAC, and hence its dual is also a valid statement in the same category:

Proposition 4.6

In any category, if $q : B \rightarrow Q$ is a coequalizer of some pair of arrows, then q is epic.

The statement of [Proposition 4.5](#) —equalizers are unique up to isomorphism— also follows from the axioms of ETAC. So its dual also holds in the same category.

Proposition 4.7

Coequalizers are unique up to isomorphism. In other words, if $q_1 : B \rightarrow Q_1$ and $q_2 : B \rightarrow Q_2$ are equalizers of parallel arrows $f, g : A \rightarrow B$, then there is an isomorphism $i : Q_1 \rightarrow Q_2$ such that $q_2 = i \circ q_1$ and $q_1 = i^{-1} \circ q_2$.

Example 4.8. Let $R \subseteq X \times X$ be an equivalence relation on a set X , and consider the diagram

$$R \begin{array}{c} \xrightarrow{r_1} \\ \xrightarrow{r_2} \end{array} X$$

where the r_i ’s are the two projections of the inclusion $i : R \hookrightarrow X \times X$, i.e. $r_1 = \pi_1 \circ i$ and $r_2 = \pi_2 \circ i$, with $\pi_1, \pi_2 : X \times X \rightarrow X$ are defined as

$$\pi_1(x_1, x_2) = x_1 \text{ and } \pi_2(x_1, x_2) = x_2.$$

$$\begin{array}{ccccc} & & R & & \\ & \swarrow r_1 & \downarrow i & \searrow r_2 & \\ X & \xleftarrow{\pi_2} & X \times X & \xrightarrow{\pi_1} & X \end{array}$$

Let X/R be the set of all the equivalence classes of X under the equivalence relation R . The quotient projection $\pi : X \rightarrow X/R$ is defined by $x \mapsto [x]$. Then for $(x_1, x_2) \in R$, we have

$$(\pi \circ r_1)(x_1, x_2) = \pi(x_1) = [x_1] \text{ and } (\pi \circ r_2)(x_1, x_2) = \pi(x_2) = [x_2].$$

Since $(x_1, x_2) \in R$, they belong to the same equivalence class, so $[x_1] = [x_2]$. Therefore,

$$\pi \circ r_1 = \pi \circ r_2. \tag{4.55}$$

Now, suppose in the following diagram we have $f : X \rightarrow Z$ such that $f \circ r_1 = f \circ r_2$.

$$\begin{array}{ccccc} R & \begin{array}{c} \xrightarrow{r_1} \\ \xrightarrow{r_2} \end{array} & X & \xrightarrow{\pi} & X/R \\ & & \searrow f & & \downarrow \bar{f} \\ & & & & Z \end{array}$$

$f \circ r_1 = f \circ r_2$ means that for any $(x_1, x_2) \in R$,

$$\begin{aligned} (f \circ r_1)(x_1, x_2) &= (f \circ r_2)(x_1, x_2) \\ \implies f(r_1(x_1, x_2)) &= f(r_2(x_1, x_2)) \\ \implies f(x_1) &= f(x_2). \end{aligned} \tag{4.56}$$

In other words, f “respects the equivalence relation R ” in the sense that f is constant on the equivalence classes. So we can define a function $\bar{f} : X/R \rightarrow Z$ by

$$\bar{f}[x] = f(x). \quad (4.57)$$

This is well-defined since f is constant on the equivalence classes. Therefore, we have $\bar{f}(\pi(x)) = f(x)$, i.e.

$$\bar{f} \circ \pi = f. \quad (4.58)$$

Furthermore, \bar{f} is unique since π is surjective.

$$\begin{array}{ccccc} R & \xrightarrow[r_2]{r_1} & X & \xrightarrow{\pi} & X/R \\ & & \searrow f & & \downarrow \bar{f} \\ & & & & Z \end{array}$$

To summarize, given two parallel arrows $r_1, r_2 : R \rightarrow X$, we have an arrow $\pi : X \rightarrow X/R$ satisfying $\pi \circ r_1 = \pi \circ r_2$. Furthermore, given any arrow $f : X \rightarrow Z$ such that $f \circ r_1 = f \circ r_2$, then there exists a **unique** arrow $\bar{f} : X/R \rightarrow Z$ such that $\bar{f} \circ \pi = f$. Hence, the object X/R along with the arrow $\pi : X \rightarrow X/R$ is the required coequalizer of $R \xrightarrow[r_2]{r_1} X$.

Example 4.9. Let $\alpha, \beta : A \rightarrow B$ be two parallel arrows in **Sets**. Let R be the minimal equivalence relation on B containing all the elements of the form $(\alpha(a), \beta(a))$ for all $a \in A$. In other words, R is the intersection of all the equivalence relations on B containing all the elements of the form $(\alpha(a), \beta(a))$.

$$R = \bigcap \{E \mid E \subseteq B \times B \text{ is an equivalence relation, and } (\alpha(a), \beta(a)) \in E \text{ for all } a \in A\}.$$

Then we form the set of equivalence classes of B under this equivalence relation R . The set of equivalence classes is B/R . Then we claim that the coequalizer of $\alpha, \beta : A \rightarrow B$ is

$$\gamma : B \rightarrow B/R,$$

the canonical quotient map that takes each $b \in B$ to its equivalence class $[b] \in B/R$.

$$\begin{array}{ccccc} A & \xrightarrow[\beta]{\alpha} & B & \xrightarrow{\gamma} & B/R \\ & & \searrow \varphi & & \downarrow \psi \\ & & & & C \end{array}$$

Indeed, $\gamma \circ \alpha = \gamma \circ \beta$, since for each $a \in A$, we have $(\alpha(a), \beta(a)) \in R$. Therefore, $[\alpha(a)] = [\beta(a)]$. In other words,

$$(\gamma \circ \alpha)(a) = (\gamma \circ \beta)(a),$$

for each $a \in A$. Therefore,

$$\gamma \circ \alpha = \gamma \circ \beta. \quad (4.59)$$

Now, suppose there is another arrow $\varphi : B \rightarrow C$ satisfying $\varphi \circ \alpha = \varphi \circ \beta$. We define another equivalence relation R' on B as follows:

$$(b_1, b_2) \in R' \iff \varphi(b_1) = \varphi(b_2). \quad (4.60)$$

Then for each $a \in A$, we have

$$(\varphi \circ \alpha)(a) = (\varphi \circ \beta)(a) \implies \varphi(\alpha(a)) = \varphi(\beta(a)).$$

Therefore, $(\alpha(a), \beta(a)) \in R'$ for each $a \in A$. Then by the minimality of R , we must have $R \subseteq R'$. In other words, if $(b_1, b_2) \in R$, or equivalently $[b_1] = [b_2]$ in B/R , then $\varphi(b_1) = \varphi(b_2)$. So we can define a well-defined function $\psi : B/R \rightarrow C$ by

$$\psi([b]) = \varphi(b). \quad (4.61)$$

Then we have

$$\varphi(b) = \psi(\gamma(b)) = (\psi \circ \gamma)(b),$$

for each $b \in B$. Therefore,

$$\varphi = \psi \circ \gamma.$$

Furthermore, this ψ is unique. Indeed, if there exists another $\psi' : B/R \rightarrow C$ satisfying $\varphi = \psi' \circ \gamma$, then we must have

$$\varphi(b) = (\psi' \circ \gamma)(b) = \psi'(\gamma(b)) = \psi'([b]),$$

for each $b \in B$. Therefore, $\psi' = \psi$, so $\psi : B/R \rightarrow C$ is unique. Hence, $\gamma : B \rightarrow B/R$ is a coequalizer of $\alpha, \beta : A \rightarrow B$ in **Sets**.

Example 4.10. Let $f, g : X \rightarrow Y$ be given arrows in **Top**. Then $f, g : X \rightarrow Y$ are continuous maps between the topological spaces X and Y , whose underlying sets are $|X|$ and $|Y|$, respectively. Then $|f|, |g| : |X| \rightarrow |Y|$ are parallel arrows in **Sets**. As above, we can form the coequalizer of $|f|$ and $|g|$ in **Sets**. Let $|q| : |Y| \rightarrow |Q|$ be the coequalizer of $|f|$ and $|g|$ in **Sets**. Then we topologize the set $|Q|$ as follows:

$$V \subseteq |Q| \text{ is open} \iff |q|^{-1}(V) \subseteq |Y| \text{ is open.} \quad (4.62)$$

Let us verify that this indeed forms a topology:

- \emptyset is open, since $|q|^{-1}(\emptyset) = \emptyset$ is open in Y .
- $|Q|$ is open, since $|q|^{-1}(|Q|) = |Y|$ is open in Y .
- Let $\{U_i\}_{i \in I}$ be a collection of open sets in $|Q|$. Then

$$|q|^{-1}\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} |q|^{-1}(U_i). \quad (4.63)$$

Since U_i is open, $|q|^{-1}(U_i)$ is open in Y . Therefore, as an union of open sets, $|q|^{-1}(\bigcup_{i \in I} U_i)$ is also open in Y . Hence, $\bigcup_{i \in I} U_i$ is open in $|Q|$.

- Let U_1 and U_2 be open in $|Q|$. Then

$$|q|^{-1}(U_1 \cap U_2) = |q|^{-1}(U_1) \cap |q|^{-1}(U_2). \quad (4.64)$$

Since U_1, U_2 are open, $|q|^{-1}(U_1), |q|^{-1}(U_2)$ are also open in Y . Therefore, as an intersection of finitely many open sets, $|q|^{-1}(U_1 \cap U_2)$ is also open in Y . Hence, $U_1 \cap U_2$ is open in $|Q|$.

Let Q be the topological space whose underlying set is $|Q|$, and the topology is defined as in (4.62). Then the function $q : Y \rightarrow Q$ is continuous, since the preimage of every open set in Q is also open in Y . We claim that $q : Y \rightarrow Q$ is a coequalizer of $f, g : X \rightarrow Y$.

$$\begin{array}{ccccc} X & \xrightarrow[f]{g} & Y & \xrightarrow{q} & Q \\ & & & \searrow z & \downarrow u \\ & & & & Z \end{array}$$

Since $|q|$ is a coequalizer of $|f|, |g| : |X| \rightarrow |Y|$ in **Sets**, then

$$|q| \circ |f| = |q| \circ |g| \implies |q \circ f| = |q \circ g| \implies q \circ f = q \circ g. \quad (4.65)$$

Now suppose there is another continuous map $z : Y \rightarrow Z$ such that $z \circ f = z \circ g$. Then $|z| \circ |f| = |z| \circ |g|$ in **Sets**. Since $|q|$ is a coequalizer of $|f|, |g| : |X| \rightarrow |Y|$, there is a **unique** function $|u| : |Q| \rightarrow |Z|$ such that $|u| \circ |q| = |z|$. Now we need to show that $|u|$ defines a continuous map $u : Q \rightarrow Z$. Let $U \subseteq |Z|$ be open. We need to show that $|u|^{-1}(U)$ is open in Q .

$$|q|^{-1}(|u|^{-1}(U)) = |z|^{-1}(U). \quad (4.66)$$

Then $|z|^{-1}(U)$ is open in Y since z is continuous. Therefore, $|q|^{-1}(|u|^{-1}(U))$ is open in Y . Then by (4.62), $|u|^{-1}(U)$ is open in Q . Hence, u is open.

Therefore, there exists a **unique** continuous map $u : Q \rightarrow Z$ such that $u \circ q = z$. So $q : Y \rightarrow Q$ is indeed a coequalizer of $f, g : X \rightarrow Y$ in **Top**.

5 Groups and Categories

§5.1 Groups in a Category

A group arises as an abstraction of the automorphisms of an object. Recall the definition of a concrete category where the objects are some “structured” sets, and the arrows are “structure preserving” maps. In a specific and concrete case, a group G may thus consist of certain arrows for a given object X in a category \mathcal{C} :

$$G \subseteq \text{Hom}_{\mathcal{C}}(X, X).$$

One requires all these arrows to be isomorphisms. But the abstract group concept can also be described directly as an object in a category, equipped with a certain structure.

Definition 5.1. Let \mathcal{C} be a category admitting binary products and a final object 1 . A **group** in \mathcal{C} consists of objects and arrows as given below

$$\begin{array}{ccccc} G \times G & \xrightarrow{m} & G & \xleftarrow{i} & G \\ & & \uparrow u & & \\ & & 1 & & \end{array}$$

satisfying the following conditions:

1. m is associative, i.e. the following diagram commutes:

$$\begin{array}{ccc} (G \times G) \times G & \xrightarrow[\cong]{\mathcal{I}} & G \times (G \times G) \\ m \times \mathbb{1}_G \downarrow & & \downarrow \mathbb{1}_G \times m \\ G \times G & & G \times G \\ & \searrow m & \swarrow m \\ & G & \end{array} \quad (5.1)$$

where $\mathcal{I} : (G \times G) \times G \rightarrow G \times (G \times G)$ is the canonical associativity isomorphism for product (the same isomorphism we proved in [Theorem 3.11](#)). In other words,

$$m \circ (m \times \mathbb{1}_G) = m \circ (\mathbb{1}_G \times m) \circ \mathcal{I}. \quad (5.2)$$

2. $u : 1 \rightarrow G$ is a unit for m , i.e. both triangles below commute:

$$\begin{array}{ccc} G & \xrightarrow{\langle u!, \mathbb{1}_G \rangle} & G \times G \\ \langle \mathbb{1}_G, u! \rangle \downarrow & \searrow \mathbb{1}_G & \downarrow m \\ G \times G & \xrightarrow{m} & G \end{array} \quad (5.3)$$

Here $u!$ is the “constant arrow” $u! : G \rightarrow G$ defined by the following composition:

$$G \xrightarrow{!} 1 \xrightarrow{u} G.$$

So

$$m \circ \langle u!, \mathbb{1}_G \rangle = \mathbb{1}_G = m \circ \langle \mathbb{1}_G, u! \rangle. \quad (5.4)$$

3. $i : G \rightarrow G$ is an inverse with respect to m , i.e. the both squares of the following diagram commute:

$$\begin{array}{ccccc} G \times G & \xleftarrow{\langle \mathbb{1}_G, \mathbb{1}_G \rangle} & G & \xrightarrow{\langle \mathbb{1}_G, \mathbb{1}_G \rangle} & G \times G \\ \downarrow \mathbb{1}_G \times i & & \downarrow u! & & \downarrow i \times \mathbb{1}_G \\ G \times G & \xrightarrow{m} & G & \xleftarrow{m} & G \times G \end{array} \quad (5.5)$$

In other words,

$$m \circ (\mathbb{1}_G \times i) \circ \langle \mathbb{1}_G, \mathbb{1}_G \rangle = u! = m \circ (i \times \mathbb{1}_G) \circ \langle \mathbb{1}_G, \mathbb{1}_G \rangle. \quad (5.6)$$

These commutative diagrams don't seem to manifest the usual group axioms (associativity, unit element, inverse), but they actually do! Let us see them in action.

1. **Associativity:** When G is a (structured) set and x, y, z are arbitrary elements of the underlying set, then the commutativity of (5.1) yields:

$$\begin{aligned} & m \circ (m \times \mathbb{1}_G) ((x, y), z) = (m \circ (\mathbb{1}_G \circ m) \circ \mathcal{I}) ((x, y), z) \\ \implies & m(x * y, z) = (m \circ (\mathbb{1}_G \circ m)) (x, (y, z)) \\ \implies & m(x * y, z) = m(x, y * z) \\ \implies & (x * y) * z = x * (y * z), \end{aligned}$$

which is precisely the associativity equation of the binary operation $*$ (here we have written $m(a, b) = a * b$).

2. **Unit element:** When G is a (structured) set, the terminal object 1 is a singleton set. So $u! : G \xrightarrow{!} 1 \xrightarrow{u} G$ is a constant function. Let $e = u!(x)$. Then the commutativity of (5.3) yields:

$$\begin{aligned} & (m \circ \langle u!, \mathbb{1}_G \rangle) (x) = \mathbb{1}_G(x) = (m \circ \langle \mathbb{1}_G, u! \rangle) (x) \\ \implies & m(u!(x), x) = x = m(x, u!(x)) \\ \implies & e * x = x = x * e. \end{aligned}$$

This suggests that when G is a (structured) set, the element $e = u!(x)$ is the unit of the binary operation $*$.

3. **Inverse:** When G is a (structured) set, the commutativity of (5.5) yields:

$$\begin{aligned} & (m \circ (\mathbb{1}_G \times i) \circ \langle \mathbb{1}_G, \mathbb{1}_G \rangle) (x) = u!(x) = (m \circ (i \times \mathbb{1}_G) \circ \langle \mathbb{1}_G, \mathbb{1}_G \rangle) (x) \\ \implies & (m \circ (\mathbb{1}_G \times i)) (x, x) = e = (m \circ (i \times \mathbb{1}_G)) (x, x) \\ \implies & m(x, i(x)) = e = m(i(x), x) \\ \implies & x * i(x) = e = i(x) * x. \end{aligned}$$

Since $e = u!(x)$ is the identity of the binary operation $*$, this equation suggests that $i(x)$ is the inverse element of x .

Definition 5.2 (Group Homomorphism). A homomorphism $h : G \rightarrow H$ of groups in \mathcal{C} consists of an arrow $h : G \rightarrow H$ in \mathcal{C} such that

1. h preserves m , i.e. the following diagram commutes:

$$\begin{array}{ccc} G \times G & \xrightarrow{h \times h} & H \times H \\ m_G \downarrow & & \downarrow m_H \\ G & \xrightarrow{h} & H \end{array} \quad (5.7)$$

In other words,

$$h \circ m_G = m_H \circ (h \times h). \quad (5.8)$$

2. h preserves u , i.e. the following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{h} & H \\ u_G \uparrow & \nearrow u_H & \\ 1 & & \end{array} \quad (5.9)$$

In other words,

$$h \circ u_G = u_H. \quad (5.10)$$

3. h preserves i , i.e. the following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{h} & H \\ i_G \downarrow & & \downarrow i_H \\ G & \xrightarrow{h} & H \end{array} \quad (5.11)$$

In other words,

$$h \circ i_G = i_H \circ h. \quad (5.12)$$

This definition of homomorphism complies with the usual definition of homomorphism we see while studying group theory. Suppose G and H are now (structured) sets, and take any $x, y \in G$. Commutativity of (5.7) implies

$$\begin{aligned} (h \circ m_G)(x, y) &= (m_H \circ (h \times h))(x, y) \\ \implies h(x *_G y) &= m_H(h(x), h(y)) \\ \implies h(x *_G y) &= h(x) *_H h(y). \end{aligned}$$

When G and H are (structured) sets, the terminal object 1 is a singleton set $\{*\}$. Commutativity of (5.9) gives us

$$\begin{aligned} (h \circ u_G)(*) &= u_H(*) \\ \implies h(e_G) &= e_H, \end{aligned}$$

where e_G and e_H are identities of $*_G$ and $*_H$, respectively. Furthermore, commutativity of (5.11) yields

$$\begin{aligned} (h \circ i_G)(x) &= (i_H \circ h)(x) \\ \implies h(i_G(x)) &= i_H(h(x)) \\ \implies h(x^{-1}) &= h(x)^{-1}. \end{aligned}$$

With the evident identities u , inverses i and multiplications m we thus have a category of groups in \mathcal{C} , denoted by

$$\text{Group}(\mathcal{C}).$$

With this construction,

- A **group** in the usual sense is a group in the category **Sets**.
- A **topological group** is a group in the category **Top**.
- A **Lie group** is a group in the category **Man** of smooth manifolds and smooth maps.
- A **partially ordered group** is a group in the category **Pos**.

Example 5.1. In this example, we focus on $\text{Group}(\mathbf{Groups})$. So it refers to the diagram

$$\begin{array}{ccccc} G \times G & \xrightarrow{m} & G & \xleftarrow{i} & G \\ & & \uparrow u & & \\ & & 1 & & \end{array}$$

in the category of groups subjected to the properties mentioned earlier. The terminal object $1 \in \text{Ob}(\mathbf{Groups})$ is the trivial group consisting of the identity element only. Here, G itself is a group. Let us denote the group operation on G by \circ_G . \circ_G induces $\circ_{G \times G}$ on $G \times G$ in the following way:

$$(g_1, g_2) \circ_{G \times G} (h_1, h_2) = (g_1 \circ_G h_1, g_2 \circ_G h_2). \quad (5.13)$$

Denote $m : G \times G \rightarrow G$ by $m(g_1, g_2) = g_1 \star g_2$. Since m is an arrow in **Groups**, it is a group homomorphism. So we must have

$$\begin{aligned} m((g_1, g_2) \circ_{G \times G} (h_1, h_2)) &= m(g_1, g_2) \circ_G m(h_1, h_2) \\ \implies m(g_1 \circ_G h_1, g_2 \circ_G h_2) &= (g_1 \star g_2) \circ_G (h_1 \star h_2) \\ \implies (g_1 \circ_G h_1) \star (g_2 \circ_G h_2) &= (g_1 \star g_2) \circ_G (h_1 \star h_2). \end{aligned} \quad (5.14)$$

Write 1° for the unit of G with respect to \circ_G , and 1^\star for the unit of G with respect to \star , i.e.

$$g \circ_G 1^\circ = g = 1^\circ \circ_G g \text{ and } g \star 1^\star = g = 1^\star \star g. \quad (5.15)$$

Proposition 5.1 (Eckmann-Hilton argument)

Given any set G equipped with two binary operations $\circ, \star : G \times G \rightarrow G$ with units 1° and 1^\star , respectively, and satisfying (5.14), the following hold:

1. $1^\circ = 1^\star$,
2. $\circ = \star$,
3. The operation $\circ = \star$ is commutative.

Proof. First one has

$$\begin{aligned} 1^\circ &= 1^\circ \circ 1^\circ && [1^\circ \text{ is the unit with respect to } \circ] \\ &= (1^\circ \star 1^\star) \circ (1^\star \star 1^\circ) && [1^\star \text{ is the unit with respect to } \star] \\ &= (1^\circ \circ 1^\star) \star (1^\star \circ 1^\circ) && [\text{Applying (5.14)}] \\ &= 1^\star \star 1^\star && [1^\circ \text{ is the unit with respect to } \circ] \\ &= 1^\star. \end{aligned}$$

Let us write $1^\circ = 1 = 1^\star$, and observe that for any $x, y \in G$,

$$\begin{aligned} x \circ y &= (x \star 1) \circ (1 \star y) && [1 = 1^\star \text{ is the unit with respect to } \star] \\ &= (x \circ 1) \star (1 \circ y) && [\text{Applying (5.14)}] \\ &= x \star y. && [1 = 1^\circ \text{ is the unit with respect to } \circ] \end{aligned}$$

Thus for $x, y \in G$, let us write $x \circ y = x \cdot y = x \star y$. Finally, we have

$$\begin{aligned} x \cdot y &= (1 \cdot x) \cdot (y \cdot 1) && [1 \text{ is the unit with respect to } \circ = \star = \cdot] \\ &= (1 \cdot y) \cdot (x \cdot 1) && [\text{Applying (5.14)}] \\ &= y \cdot x. && [1 \text{ is the unit with respect to } \cdot] \end{aligned}$$

Therefore, \cdot is commutative. ■

Corollary 5.2

The groups in the category **Groups** are exactly the abelian groups.

Proof. In Proposition 5.1, we have shown that a group in the category **Groups** of groups is abelian. In other words, **Group**(**Groups**) is a subcategory of **Ab**. One now has to prove the converse, i.e. any abelian group G admits homomorphic group multiplication $m : G \times G \rightarrow G$ and homomorphic inverse $i : G \rightarrow G$. In other words, for any $g, h \in G \times G$ and $x, y \in G$,

$$m(g \circ_{G \times G} h) = m(g) \circ_G m(h), \text{ and } i(x \circ_G y) = i(x) \circ_G i(y).$$

This is left as an exercise. ■

§5.2 Monoidal Categories

A functor from the product category $\mathcal{C} \times \mathcal{D}$ to a category \mathcal{E} is called a bifunctor. We denote such a functor by T , so that

$$T : \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{E}$$

is a functor. Given objects $A \in \text{Ob}(\mathcal{C})$ and $B \in \text{Ob}(\mathcal{D})$, some object $C \in \text{Ob}(\mathcal{E})$ is assigned, i.e.

$$T(A, B) = C.$$

To each pair of arrows $\alpha : A \rightarrow A'$ in \mathcal{C} , and $\beta : B \rightarrow B'$ in \mathcal{D} , an arrow

$$T(\alpha, \beta) : T(A, B) \rightarrow T(A', B')$$

in \mathcal{E} is assigned, such that the following conditions are satisfied:

1. Given objects $A \in \text{Ob}(\mathcal{C})$ and $B \in \text{Ob}(\mathcal{D})$

$$T(1_A, 1_B) = 1_{T(A, B)}. \quad (5.16)$$

2. If $A \xrightarrow{\alpha} A' \xrightarrow{\alpha'} A''$ in \mathcal{C} and $B \xrightarrow{\beta} B' \xrightarrow{\beta'} B''$ in \mathcal{D} , then in the product category $\mathcal{C} \times \mathcal{D}$, we have

$$\begin{array}{ccccc} (A, B) & \xrightarrow{(\alpha, \beta)} & (A', B') & \xrightarrow{(\alpha', \beta')} & (A'', B'') \\ & \searrow & \text{curved arrow} & \nearrow & \\ & & (\alpha', \beta') \circ (\alpha, \beta) = (\alpha' \circ \alpha, \beta' \circ \beta) & & \end{array}$$

Since $T : \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{E}$ is considered a functor, it has to respect the composition of arrows, i.e.

$$T(\alpha' \circ \alpha, \beta' \circ \beta) = T(\alpha', \beta') \circ T(\alpha, \beta). \quad (5.17)$$

Example 5.2. $\text{Hom}(-, -) : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathbf{Sets}$ is an example of a bifunctor. Suppose A and B are objects of \mathcal{C} , and hence objects of \mathcal{C}^{op} . Then this functor takes (A, B) to $\text{Hom}_{\mathcal{C}}(A, B)$, i.e.

$$\text{Hom}(A, B) = \text{Hom}_{\mathcal{C}}(A, B). \quad (5.18)$$

Let $\alpha^{\text{op}} : A \rightarrow A'$ be an arrow in \mathcal{C}^{op} (the corresponding arrow in \mathcal{C} reads $\alpha : A' \rightarrow A$). Suppose $\beta : B \rightarrow B'$ is another arrow in \mathcal{C} . Using these data, let us now construct an arrow $\text{Hom}(\alpha^{\text{op}}, \beta) : \text{Hom}_{\mathcal{C}}(A, B) \rightarrow \text{Hom}_{\mathcal{C}}(A', B')$ in the category \mathbf{Sets} as follows:

$$\text{Hom}(\alpha^{\text{op}}, \beta)(\varphi) = \beta \circ \varphi \circ \alpha, \quad (5.19)$$

for $\varphi \in \text{Hom}_{\mathcal{C}}(A, B)$. Let us now verify that the definition in (5.19) satisfies both (5.16) and (5.17).

$$\text{Hom}(\mathbb{1}_A^{\text{op}}, \mathbb{1}_B)(\varphi) = \mathbb{1}_B \circ \varphi \circ \mathbb{1}_A = \varphi,$$

for any $\varphi \in \text{Hom}_{\mathcal{C}}(A, B)$. Therefore,

$$\text{Hom}(\mathbb{1}_A^{\text{op}}, \mathbb{1}_B) = \mathbb{1}_{\text{Hom}_{\mathcal{C}}(A, B)}. \quad (5.20)$$

So (5.16) is satisfied. Now, let $A \xrightarrow{\alpha_1^{\text{op}}} A' \xrightarrow{\alpha_2^{\text{op}}} A''$ be arrows in \mathcal{C}^{op} and $B \xrightarrow{\beta_1} B' \xrightarrow{\beta_2} B''$ be arrows in \mathcal{C} .

$$\begin{array}{c} \boxed{\begin{array}{ccc} & \mathcal{C}^{\text{op}} \times \mathcal{C} & \\ (A, B) & \xrightarrow{(\alpha_1^{\text{op}}, \beta_1)} (A', B') & \xrightarrow{(\alpha_2^{\text{op}}, \beta_2)} (A'', B'') \\ & \searrow & \nearrow \\ & (\alpha_2^{\text{op}}, \beta_2) \circ (\alpha_1^{\text{op}}, \beta_1) = (\alpha_2^{\text{op}} \circ \alpha_1^{\text{op}}, \beta_2 \circ \beta_1) & \end{array}} \\ \downarrow \text{Hom}(-, -) \\ \boxed{\begin{array}{ccc} & \mathbf{Sets} & \\ \text{Hom}_{\mathcal{C}}(A, B) & \xrightarrow{\text{Hom}(\alpha_1^{\text{op}}, \beta_1)} \text{Hom}_{\mathcal{C}}(A', B') & \xrightarrow{\text{Hom}(\alpha_2^{\text{op}}, \beta_2)} \text{Hom}_{\mathcal{C}}(A'', B'') \\ & \searrow & \nearrow \\ & \text{Hom}(\alpha_2^{\text{op}} \circ \alpha_1^{\text{op}}, \beta_2 \circ \beta_1) & \end{array}} \end{array}$$

Now we need to verify equation (5.17) for the functor $\text{Hom}(-, -) : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathbf{Sets}$ which entails the commutativity of the diagram in \mathbf{Sets} given above.

$$\begin{aligned} \text{Hom}(\alpha_2^{\text{op}} \circ \alpha_1^{\text{op}}, \beta_2 \circ \beta_1)(\varphi) &= \text{Hom}((\alpha_1 \circ \alpha_2)^{\text{op}}, \beta_2 \circ \beta_1)(\varphi) \\ &= \beta_2 \circ \beta_1 \circ \varphi \circ \alpha_1 \circ \alpha_2. \end{aligned}$$

On the other hand,

$$\begin{aligned} (\text{Hom}(\alpha_2^{\text{op}}, \beta_2) \circ \text{Hom}(\alpha_1^{\text{op}}, \beta_1))(\varphi) &= \text{Hom}(\alpha_2^{\text{op}}, \beta_2)(\beta_1 \circ \varphi \circ \alpha_1) \\ &= \beta_2 \circ \beta_1 \circ \varphi \circ \alpha_1 \circ \alpha_2. \end{aligned}$$

Therefore,

$$\text{Hom}(\alpha_2^{\text{op}} \circ \alpha_1^{\text{op}}, \beta_2 \circ \beta_1) = \text{Hom}(\alpha_2^{\text{op}}, \beta_2) \circ \text{Hom}(\alpha_1^{\text{op}}, \beta_1). \quad (5.21)$$

Therefore, $\text{Hom}(-, -) : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathbf{Sets}$ is indeed a bifunctor.

Definition 5.3 (Monoidal Category). A **monoidal category** $\mathcal{B} = \langle \mathcal{B}, \square, e, \alpha, \lambda, \rho \rangle$ is a category \mathcal{B} equipped with a bifunctor $\square : \mathcal{B} \times \mathcal{B} \rightarrow \mathcal{B}$, an object $e \in \text{Ob}(\mathcal{B})$ and three natural isomorphisms α, λ, ρ with the following properties:

- Let $\mathcal{I} : \mathcal{B} \times \mathcal{B} \times \mathcal{B} \rightarrow (\mathcal{B} \times \mathcal{B}) \times \mathcal{B}$ and $\mathcal{I}' : \mathcal{B} \times \mathcal{B} \times \mathcal{B} \rightarrow \mathcal{B} \times (\mathcal{B} \times \mathcal{B})$ be the canonical isomorphisms. Then $\alpha : \square \circ (\square \times \mathbf{1}_{\mathcal{B}}) \circ \mathcal{I} \Rightarrow \square \circ (\mathbf{1}_{\mathcal{B}} \times \square) \circ \mathcal{I}'$ is a natural isomorphism.

$$\begin{array}{ccc}
 & \square \circ (\square \times \mathbf{1}_{\mathcal{B}}) \circ \mathcal{I} & \\
 & \downarrow \alpha & \\
 \mathcal{B} \times \mathcal{B} \times \mathcal{B} & & \mathcal{B} \\
 & \downarrow \square \circ (\mathbf{1}_{\mathcal{B}} \times \square) \circ \mathcal{I}' &
 \end{array}$$

That is, given arrows $a \xrightarrow{f} a'$, $b \xrightarrow{g} b'$, $c \xrightarrow{h} c'$ in \mathcal{B} , the following diagram commutes:

$$\begin{array}{ccc}
 (a \square b) \square c & \xrightarrow{(f \square g) \square h} & (a' \square b') \square c' \\
 \downarrow \alpha_{a,b,c} & & \downarrow \alpha_{a',b',c'} \\
 a \square (b \square c) & \xrightarrow{f \square (g \square h)} & a' \square (b' \square c')
 \end{array}$$

Furthermore, the components $\alpha_{a,b,c} : (a \square b) \square c \rightarrow a \square (b \square c)$ are all isomorphisms.

- The bifunctor $\square : \mathcal{B} \times \mathcal{B} \rightarrow \mathcal{B}$ induces a functor in the second component by keeping the first component e , i.e. $\square(e, -) : \mathcal{B} \rightarrow \mathcal{B}$ is a functor. $\lambda : \square(e, -) \Rightarrow \mathbf{1}_{\mathcal{B}}$ is a natural isomorphism.

$$\begin{array}{ccc}
 & \square(e, -) & \\
 & \downarrow \lambda & \\
 \mathcal{B} & & \mathcal{B} \\
 & \downarrow \mathbf{1}_{\mathcal{B}} &
 \end{array}$$

That is, given an arrow $a \xrightarrow{f} b$ in \mathcal{B} , the following diagram commutes:

$$\begin{array}{ccc}
 e \square a & \xrightarrow{\mathbf{1}_e \square f} & e \square b \\
 \downarrow \lambda_a & & \downarrow \lambda_b \\
 a & \xrightarrow{f} & b
 \end{array}$$

Furthermore, the components $\lambda_a : e \square a \rightarrow a$ are all isomorphisms.

- In the same manner, $\rho : \square(-, e) \Rightarrow \mathbf{1}_{\mathcal{B}}$ is a natural isomorphism.

$$\begin{array}{ccc}
 & \square(-, e) & \\
 & \downarrow \rho & \\
 \mathcal{B} & & \mathcal{B} \\
 & \downarrow \mathbf{1}_{\mathcal{B}} &
 \end{array}$$

That is, given an arrow $a \xrightarrow{f} b$ in \mathcal{B} , the following diagram commutes:

$$\begin{array}{ccc}
 a \square e & \xrightarrow{f \square \mathbf{1}_e} & b \square e \\
 \downarrow \rho_a & & \downarrow \rho_b \\
 a & \xrightarrow{f} & b
 \end{array}$$

Furthermore, the components $\rho : a \square e \rightarrow a$ are all isomorphisms.

This data is required to satisfy the following conditions:

- For all $a, b, c, d \in \text{Ob}(\mathcal{B})$, the following diagram commutes:

$$\begin{array}{ccc}
 & (a \square b) \square (c \square d) & \\
 \alpha_{a \square b, c, d} \nearrow & & \searrow \alpha_{a, b, c \square d} \\
 ((a \square b) \square c) \square d & & a \square (b \square (c \square d)) \\
 \alpha_{a, b, c} \square \mathbb{1}_d \downarrow & & \uparrow \mathbb{1}_a \square \alpha_{b, c, d} \\
 (a \square (b \square c)) \square d & \xrightarrow{\alpha_{a, b \square c, d}} & a \square ((b \square c) \square d)
 \end{array}$$

- For all $a, c \in \text{Ob}(\mathcal{B})$, the following diagram commutes:

$$\begin{array}{ccc}
 (a \square e) \square c & \xrightarrow{\alpha_{a, e, c}} & a \square (e \square c) \\
 \rho_a \square \mathbb{1}_c \searrow & & \downarrow \mathbb{1}_a \square \lambda_c \\
 & & a \square c
 \end{array}$$

- $\lambda_e = \rho_e$.

$\mathcal{B} = \langle \mathcal{B}, \square, e \rangle$ is called a strict monoidal category when all 3 natural isomorphisms α, λ, ρ are identity natural transformation. For example, any category with finite products can be regarded as a monoidal category with the product as the monoidal product and the terminal object as the unit.

§5.3 The Category of Groups

Let us now focus on the category of groups, i.e. $\text{Group}(\mathbf{Sets})$. Let G and H be two groups with $h : G \rightarrow H$ being a group homomorphism. The kernel of h is defined by the equalizer

$$\text{Ker } h = \{g \in G \mid h(g) = e_H\} \xrightarrow{\iota} G \xrightarrow[u_H!]{h} H$$

where, again, we write $u_H! : G \rightarrow H$ as a composition:

$$u_H! : G \xrightarrow{!} 1 \xrightarrow{u_H} H,$$

i.e. $u_H! = u_H \circ !$, and e_H is the identity element of H . Also, $\iota : \text{Ker } h \rightarrow G$ is the inclusion. We have seen in [Example 4.7](#) that this specification makes $\iota : \text{Ker } h \rightarrow G$ an equalizer of h and $u_H!$.

Observe that $\text{Ker } h$ is a normal subgroup of G . Now, if $N \subseteq G$ is a normal subgroup of G , then the inclusion $i : N \hookrightarrow G$ is a monomorphism. We can now construct the coequalizer

$$N \xrightarrow[u_G!]{i} G \xrightarrow{\pi} G/N$$

π is defined by $\pi(g) = [g] = gN$. For $n \in N$,

$$\pi(n) = nN = N = [e_G],$$

where e_G is the identity element of G . So we have $\pi(i(n)) = \pi(u_G!(n))$ for each $n \in N$. Therefore,

$$\pi \circ i = \pi \circ u_G!.$$

It remains to show that π is universal. If $f : G \rightarrow H$ satisfies $f \circ i = f \circ u_G!$, in the diagram below,

$$\begin{array}{ccccc}
 N & \xrightarrow[u_G!]{i} & G & \xrightarrow{\pi} & G/N \\
 & & \searrow f & & \downarrow \bar{f} \\
 & & & & H
 \end{array}$$

then $f(n) = e_H$ for each $n \in N$. Then there is a **unique** homomorphism $\bar{f} : G/N \rightarrow H$ such that $\bar{f} \circ \pi = f$, i.e. the triangle in the diagram above commutes. \bar{f} is defined as follows:

$$\bar{f}([g]) = \bar{f}(gN) = f(g). \quad (5.22)$$

To check the well-definedness of \bar{f} , consider $g, h \in G$ such that $[g] = [h]$. Then $gN = hN$, so $h^{-1}g \in N$. Therefore,

$$f(g) = f(h \cdot_G h^{-1}g) = f(h) f(h^{-1}g) = f(h) \cdot_H e_H = f(h).$$

So $\bar{f}([g]) = \bar{f}([h])$, and hence \bar{f} is well-defined. The uniqueness of \bar{f} follows from the fact that π is an epimorphism. Indeed, if there is another arrow $\bar{f}' : G/N \rightarrow H$ such that $\bar{f}' \circ \pi = f$, then we have the following commutative diagram.

$$\begin{array}{ccc}
 G & \xrightarrow{\pi} & G/N \\
 & \searrow f & \downarrow \bar{f} \\
 & & H
 \end{array}$$

Then $\bar{f}' \circ \pi = \bar{f} \circ \pi$. Since π is an epi, $\bar{f} = \bar{f}'$, so \bar{f} is unique.

Theorem 5.3

Every group homomorphism $h : G \rightarrow H$ has a kernel $\text{Ker } h = h^{-1}(u_H)$ (e_H is the unit element of H), which is a normal subgroup of G , with the property that for any normal subgroup $N \subseteq G$,

$$N \subseteq \text{Ker } h \iff \exists! \bar{h} : G/N \rightarrow H$$

such that $\bar{h} \circ \pi = h$, as indicated in the following diagram:

$$\begin{array}{ccc}
 G & \xrightarrow{h} & H \\
 \downarrow \pi & \nearrow \bar{h} & \\
 G/N & &
 \end{array}$$

Proof. (\Rightarrow) Suppose $N \subseteq \text{Ker } h$. Consider the following diagram:

$$\begin{array}{ccccc}
 N & \xrightarrow[u_G!]{i} & G & \xrightarrow{\pi} & G/N \\
 & & \searrow h & & \downarrow \bar{h} \\
 & & & & H
 \end{array}$$

Since $N \subseteq \text{Ker } h$, one has for each $n \in N$,

$$h(i(n)) = h(n) = e_H = h(e_G) = h(u_G!(n)).$$

Therefore,

$$h \circ i = h \circ u_G!. \quad (5.23)$$

We have seen earlier that $\pi : G \rightarrow G/N$ is a coequalizer of the parallel arrows $i, u_G! : N \rightarrow G$. Hence, there is a **unique** homomorphism $\bar{h} : G/N \rightarrow H$ such that $\bar{h} \circ \pi = h$.

(\Leftarrow) Conversely, suppose there is a unique homomorphism $\bar{h} : G/N \rightarrow H$ such that $\bar{h} \circ \pi = h$. Then for any $n \in N$,

$$h(n) = \bar{h}(\pi(n)) = \bar{h}([n]) = \bar{h}([e_G]) = e_H. \quad (5.24)$$

Therefore, $n \in \text{Ker } h$, proving that $N \subseteq \text{Ker } h$. ■

Corollary 5.4

Every group homomorphism $h : G \rightarrow H$ factors as a quotient followed by an injective homomorphism:

$$\begin{array}{ccc} G & \xrightarrow{h} & H \\ \pi \downarrow & \nearrow \bar{h} & \\ G/\text{Ker } h & & \end{array}$$

In other words, $h = \bar{h} \circ \pi$, where \bar{h} is injective. Thus $\bar{h} : G/\text{Ker } h \xrightarrow{\cong} \text{im}(h) \subseteq H$ is an isomorphism to the subgroup $\text{im}(h)$.

Proof. Let us take $N = \text{Ker } h$ in [Theorem 5.3](#). Then there exists a **unique** homomorphism $\bar{h} : G/\text{Ker } h \rightarrow H$ such that $\bar{h} \circ \pi = h$. So, the following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{h} & H \\ \pi \downarrow & \nearrow \bar{h} & \\ G/\text{Ker } h & & \end{array}$$

Now we need to show that \bar{h} is injective. Choose $[x], [y] \in G/\text{Ker } h$ such that $\bar{h}[x] = \bar{h}[y]$. Then we have

$$\begin{aligned} \bar{h}([x]) = \bar{h}([y]) &\implies \bar{h}(\pi(x)) = \bar{h}(\pi(y)) \\ &\implies h(x) = h(y) \\ &\implies h(x) \cdot (h(y))^{-1} = e_H \\ &\implies h(x) \cdot h(y^{-1}) = e_H \\ &\implies h(xy^{-1}) = e_H \\ &\implies xy^{-1} \in \text{Ker } h \\ &\implies [x] = [y]. \end{aligned}$$

Therefore, $\bar{h} : G/\text{Ker } h \rightarrow H$ is injective. ■

§5.4 Groups as Categories

A group is a category with one object and every arrow from the object to itself is an isomorphism. If G and H are groups, regarded as categories, then we can consider arbitrary functors between them

$$f : G \rightarrow H.$$

It then follows immediately that a functor between groups is exactly the same thing as a group homomorphism.

Consider a functor $R : G \rightarrow \mathbf{Vect}_{\mathbb{K}}$ from a group G to the category of finite dimensional \mathbb{K} -vector spaces. Since G is a one object category, we denote this object by $*$. Hence, $R(*) =: V$ is a finite dimensional \mathbb{K} -vector space. In the category G , an arrow $g : * \rightarrow *$ is an isomorphism, so that $R(g) \in \mathrm{GL}(V)$ is an invertible linear operator on V . This is precisely a **linear representation** of G .

We now want to generalize the notions of kernel of a homomorphism and quotient group by a normal subgroup, from groups to arbitrary categories.

Definition 5.4. A congruence on a category \mathcal{C} is an equivalence relation $f \sim g$ on arrows such that

1. $f \sim g$ implies $\mathrm{dom}(f) = \mathrm{dom}(g)$ and $\mathrm{cod}(f) = \mathrm{cod}(g)$.

$$X \xrightarrow[g]{f} Y$$

2. $f \sim g$ implies $b \circ f \circ a \sim b \circ g \circ a$ for any arrows $a : A \rightarrow X$ and $b : Y \rightarrow B$, where $X = \mathrm{dom}(f) = \mathrm{dom}(g)$ and $Y = \mathrm{cod}(f) = \mathrm{cod}(g)$.

$$A \xrightarrow{a} X \xrightarrow[g]{f} Y \xrightarrow{b} B$$

Remark 5.1. We can also rephrase the definition of congruence as follows: a congruence is a collection of equivalence relations $R_{X,Y}$ for $X, Y \in \mathrm{Ob}(\mathcal{C})$ such that

1. $R_{X,Y} \subseteq \mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(X, Y)$ is an equivalence relation (i.e. it's reflexive, symmetric, transitive)
2. Given $(f, g) \in R_{X,Y}$, one has

$$(b \circ f \circ a, b \circ g \circ a) \in R_{\mathrm{dom}(a), \mathrm{cod}(b)} \quad (5.25)$$

for any arrow a, b with $\mathrm{cod}(a) = X$ and $\mathrm{dom}(b) = Y$.

Such equivalence relations clearly exist, as we can just take $R_{X,Y} = \mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(X, Y)$. But such equivalence relations are not unique. So **any** equivalence relations $R_{X,Y}$ on $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ satisfying (5.25) will be called a congruence.

Definition 5.5 (Congruence Category). Let \sim be a congruence on the category \mathcal{C} , and define the **congruence category** \mathcal{C}^{\sim} by

$$\begin{aligned} \mathrm{Ob}(\mathcal{C}^{\sim}) &= \mathrm{Ob}(\mathcal{C}), \\ \mathrm{Hom}_{\mathcal{C}^{\sim}}(X, Y) &= \{(f, g) \mid f, g \in \mathrm{Hom}_{\mathcal{C}}(X, Y) \text{ and } f \sim g\}, \\ \tilde{1}_X &= (1_X, 1_X), \\ (f', g') \circ (f, g) &= (f' \circ f, g' \circ g). \end{aligned}$$

Note that, in this definition, $\mathrm{Hom}_{\mathcal{C}^{\sim}}(X, Y)$ is nothing but the $R_{X,Y}$ mentioned in Remark 5.1. We need to check the well-definedness of the definition of composition.

Let $(f, g) \in \mathrm{Hom}_{\mathcal{C}^{\sim}}(X, Y)$ and $(f', g') \in \mathrm{Hom}_{\mathcal{C}^{\sim}}(Y, Z)$. We need to show that $(f' \circ f, g' \circ g) \in \mathrm{Hom}_{\mathcal{C}^{\sim}}(X, Z)$, i.e. $f' \circ f \sim g' \circ g$.

$$X \xrightarrow{1_X} X \xrightarrow[g]{f} Y \xrightarrow[g']{f'} Z \xrightarrow{1_Z} Z$$

$(f, g) \in \text{Hom}_{\mathcal{C}^\sim}(X, Y)$ and $(f', g') \in \text{Hom}_{\mathcal{C}^\sim}(Y, Z)$ implies $f \sim g$ and $f' \sim g'$. Therefore,

$$\begin{aligned}
 f' \circ f &= \mathbb{1}_Z \circ f' \circ (f \circ \mathbb{1}_X) \\
 &\sim \mathbb{1}_Z \circ g' \circ (f \circ \mathbb{1}_X) && [\text{since } f' \sim g'] \\
 &= g' \circ f \circ \mathbb{1}_X \\
 &\sim g' \circ g \circ \mathbb{1}_X && [\text{since } f \sim g] \\
 &= g' \circ g.
 \end{aligned} \tag{5.26}$$

Therefore, $f' \circ f \sim g' \circ g$, so $(f' \circ f, g' \circ g) \in \text{Hom}_{\mathcal{C}^\sim}(X, Z)$ and the composition is well-defined.

$$\begin{array}{ccccc}
 X & \xrightarrow{(f,g)} & Y & \xrightarrow{(f',g')} & Z \\
 & \searrow & & \nearrow & \\
 & & & & (f',g') \circ (f,g) = (f' \circ f, g' \circ g)
 \end{array}$$

There are two obvious projection functors:

$$\mathcal{C}^\sim \begin{array}{c} \xrightarrow{p_1} \\ \xrightarrow{p_2} \end{array} \mathcal{C}$$

They leave the objects unchanged, i.e. $p_i(X) = X$ for $X \in \text{Ob}(\mathcal{C}^\sim) = \text{Ob}(\mathcal{C})$, $i = 1, 2$, and

$$p_1(f, g) = f \text{ and } p_2(f, g) = g. \tag{5.27}$$

Clearly, for $i = 1, 2$,

$$p_i(\tilde{\mathbb{1}}_X) = p_i(\mathbb{1}_X, \mathbb{1}_X) = \mathbb{1}_X. \tag{5.28}$$

Furthermore,

$$\begin{aligned}
 p_1((f', g') \circ (f, g)) &= p_1(f' \circ f, g' \circ g) \\
 &= f' \circ f \\
 &= p_1(f', g') \circ p_1(f, g).
 \end{aligned} \tag{5.29}$$

$$\begin{aligned}
 p_2((f', g') \circ (f, g)) &= p_2(f' \circ f, g' \circ g) \\
 &= g' \circ g \\
 &= p_2(f', g') \circ p_2(f, g).
 \end{aligned} \tag{5.30}$$

Therefore, p_1 and p_2 are indeed functors. We now build the quotient category \mathcal{C}/\sim as follows:

$$\begin{aligned}
 \text{Ob}(\mathcal{C}/\sim) &= \text{Ob}(\mathcal{C}) \\
 \text{Hom}_{\mathcal{C}/\sim}(X, Y) &= \text{Hom}_{\mathcal{C}}(X, Y) / \sim.
 \end{aligned}$$

The arrows of \mathcal{C}/\sim have the form $[f]$, where f is an arrow in \mathcal{C} , and we put

$$[\mathbb{1}_X] = \mathbb{1}_{[X]}, \tag{5.31}$$

for $X \in \text{Ob}(\mathcal{C})$, where the equivalence class $[X]$ consists of X only. Furthermore,

$$[g] \circ [f] = [g \circ f], \tag{5.32}$$

for $\text{cod}(f) = \text{dom}(g)$ in \mathcal{C} . We need to verify the well-definedness of this definition. Suppose $[f] = [f']$ and $[g] = [g']$. Then $f \sim f'$ and $g \sim g'$. Therefore, using (5.26), we get $g \circ f \sim g' \circ f'$. So $[g \circ f] = [g' \circ f']$ and the composition in \mathcal{C}/\sim is well-defined.

Consider the quotient functor $\pi : \mathcal{C} \rightarrow \mathcal{C}/\sim$ by

$$\pi(X) = [X] \text{ and } \pi(f) = [f], \tag{5.33}$$

for objects X and arrows f . Then the following is a coequalizer diagram in **Cat**:

$$\mathcal{C}^\sim \begin{array}{c} \xrightarrow{p_1} \\ \xrightarrow{p_2} \end{array} \mathcal{C} \xrightarrow{\pi} \mathcal{C}/\sim$$

The arrows p_1, p_2, π do not change the objects. So $\pi \circ p_1 = \pi \circ p_2$ clearly holds on objects. Let (f, g) be an arrow in \mathcal{C}^\sim . Then $f \sim g$, so that $[f] = [g]$. Now,

$$\begin{aligned} (\pi \circ p_1)(f, g) &= \pi(f) = [f], \\ (\pi \circ p_2)(f, g) &= \pi(g) = [g]. \end{aligned}$$

Therefore, $\pi \circ p_1 = \pi \circ p_2$ holds at the level of arrows as well. Hence,

$$\pi \circ p_1 = \pi \circ p_2. \quad (5.34)$$

Now, suppose there is another functor $F : \mathcal{C} \rightarrow \mathcal{D}$ such that $F \circ p_1 = F \circ p_2$.

$$\begin{array}{ccc} \mathcal{C}^\sim & \begin{array}{c} \xrightarrow{p_1} \\ \xrightarrow{p_2} \end{array} & \mathcal{C} \xrightarrow{\pi} \mathcal{C}/\sim \\ & & \searrow F \quad \downarrow \overline{F} \\ & & \mathcal{D} \end{array} \quad (5.35)$$

Since $F \circ p_1 = F \circ p_2$, given any arrow (f, g) in \mathcal{C}^\sim , i.e. $f \sim g$, we have

$$(F \circ p_1)(f, g) = (F \circ p_2)(f, g) \implies F(f) = F(g). \quad (5.36)$$

Now we define the functor $\overline{F} : \mathcal{C}/\sim \rightarrow \mathcal{D}$ as follows: given any object $[X] \in \text{Ob}(\mathcal{C}/\sim)$, we define

$$\overline{F}([X]) = F(X) \in \text{Ob}(\mathcal{D}); \quad (5.37)$$

and given an arrow $[f] \in \text{Hom}_{\mathcal{C}/\sim}(X, Y)$,

$$\overline{F}([f]) = F(f). \quad (5.38)$$

This definition is well-defined, since for $[f] = [g]$, i.e. $f \sim g$, we have $F(f) = F(g)$.

Now, given any $X \in \text{Ob}(\mathcal{C})$, we have

$$(\overline{F} \circ \pi)(X) = \overline{F}([X]) = F(X).$$

Furthermore, given any arrow $f \in \text{Hom}_{\mathcal{C}}(X, Y)$,

$$(\overline{F} \circ \pi)(f) = \overline{F}([f]) = F(f).$$

Therefore, we have

$$\overline{F} \circ \pi = F. \quad (5.39)$$

Therefore, there exists a functor $\overline{F} : \mathcal{C}/\sim \rightarrow \mathcal{D}$ such that the triangle of (5.35) commutes. Now we need to show the uniqueness of \overline{F} . Suppose there is another functor $G : \mathcal{C}/\sim \rightarrow \mathcal{D}$ such that $G \circ \pi = F$.

Given any object $[X] \in \text{Ob}(\mathcal{C}/\sim)$,

$$G([X]) = G(\pi(X)) = F(X) = \overline{F}([X]). \quad (5.40)$$

Furthermore, given an arrow $[f] \in \text{Hom}_{\mathcal{C}/\sim}(X, Y)$,

$$G([f]) = G(\pi(f)) = F(f) = \overline{F}([f]). \quad (5.41)$$

Therefore, $G = \overline{F}$, and hence \overline{F} is unique. In summary, if there is a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ such that $F \circ p_1 = F \circ p_2$, then there is a **unique** functor $\overline{F} : \mathcal{C}/\sim \rightarrow \mathcal{D}$ such that $\overline{F} \circ \pi = F$, i.e. the diagram (5.35) commutes. So $\pi : \mathcal{C} \rightarrow \mathcal{C}/\sim$ is indeed a coequalizer of p_1 and p_2 in **Cat**.

Let us now see how we can use the notion of coequalizer to prove analogous homomorphism theorems for categories. Suppose one has categories \mathcal{C} and \mathcal{D} and a functor $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$. Then \mathcal{F} defines a congruence $\sim_{\mathcal{F}}$ on \mathcal{C} by setting

$$f \sim_{\mathcal{F}} g \iff \text{dom}(f) = \text{dom}(g), \text{cod}(f) = \text{cod}(g), \text{ and } \mathcal{F}(f) = \mathcal{F}(g). \quad (5.42)$$

Suppose $f \sim_{\mathcal{F}} g$ with $\text{dom}(f) = \text{dom}(g) = X$ and $\text{cod}(f) = \text{cod}(g) = Y$. Then for any arrow $a : A \rightarrow X$ and $b : Y \rightarrow B$,

$$\mathcal{F}(b \circ f \circ a) = \mathcal{F}(b) \circ \mathcal{F}(f) \circ \mathcal{F}(a) = \mathcal{F}(b) \circ \mathcal{F}(g) \circ \mathcal{F}(a) = \mathcal{F}(b \circ g \circ a).$$

Therefore, $b \circ f \circ a \sim_{\mathcal{F}} b \circ g \circ a$. So $\sim_{\mathcal{F}}$ is indeed a congruence. Then we can form the congruence category $\mathcal{C}^{\sim_{\mathcal{F}}}$, which we call the **kernel** category of the functor \mathcal{F} .

$$\text{Ker } \mathcal{F} = \mathcal{C}^{\sim_{\mathcal{F}}} \rightrightarrows_{p_2}^{p_1} \mathcal{C}$$

The functors $p_1, p_2 : \mathcal{C}^{\sim_{\mathcal{F}}} \rightarrow \mathcal{C}$ are defined as (5.27). One then obtains the quotient category as the coequalizer

$$\text{Ker } \mathcal{F} = \mathcal{C}^{\sim_{\mathcal{F}}} \rightrightarrows_{p_2}^{p_1} \mathcal{C} \xrightarrow{\pi} \mathcal{C} / \sim_{\mathcal{F}}$$

where π is defined as (5.33). The quotient category $\mathcal{C} / \sim_{\mathcal{F}}$ then has the following UMP.

Theorem 5.5

Every functor $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$ has a kernel category $\text{Ker } \mathcal{F} = \mathcal{C}^{\sim_{\mathcal{F}}}$, determined by a congruence $\sim_{\mathcal{F}}$ on \mathcal{C} such that given any congruence \sim on \mathcal{C} , one has

$$f \sim g \implies f \sim_{\mathcal{F}} g$$

if and only if there is a unique factorization $\tilde{\mathcal{F}} : \mathcal{C} / \sim \rightarrow \mathcal{D}$ as indicated in the following commutative diagram:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\mathcal{F}} & \mathcal{D} \\ \pi \downarrow & \nearrow \tilde{\mathcal{F}} & \\ \mathcal{C} / \sim & & \end{array}$$

In other words, $\mathcal{F} = \tilde{\mathcal{F}} \circ \pi$.

Proof. (\implies) Suppose that $f \sim g \implies f \sim_{\mathcal{F}} g$. Therefore, $f \sim g$ implies $\mathcal{F}(f) = \mathcal{F}(g)$. Consider the following coequalizer diagram:

$$\begin{array}{ccccc} \mathcal{C}^{\sim} & \rightrightarrows_{p_2}^{p_1} & \mathcal{C} & \xrightarrow{\pi} & \mathcal{C} / \sim \\ & & \searrow \mathcal{F} & & \downarrow \tilde{\mathcal{F}} \\ & & & & \mathcal{D} \end{array}$$

Given an arrow (f, g) in \mathcal{C}^{\sim} , we have $f \sim g$. So

$$(\mathcal{F} \circ p_1)(f, g) = \mathcal{F}(f) = \mathcal{F}(g) = (\mathcal{F} \circ p_2)(f, g). \quad (5.43)$$

Therefore, $\mathcal{F} \circ p_1 = \mathcal{F} \circ p_2$. Since $\pi : \mathcal{C} \rightarrow \mathcal{C} / \sim$ is a coequalizer of p_1 and p_2 in **Cat**, and $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$ is another functor such that $\mathcal{F} \circ p_1 = \mathcal{F} \circ p_2$, there exists a **unique** arrow $\tilde{\mathcal{F}} : \mathcal{C} / \sim \rightarrow \mathcal{D}$ such that the diagram above commutes, i.e. $\mathcal{F} = \tilde{\mathcal{F}} \circ \pi$.

(\Leftarrow) Conversely, suppose there is a unique functor $\tilde{\mathcal{F}} : \mathcal{C}/\sim \rightarrow \mathcal{D}$ such that $\mathcal{F} = \tilde{\mathcal{F}} \circ \pi$. Let $f \sim g$. Then $[f] = [g]$ in \mathcal{C}/\sim . Then we have

$$\begin{aligned} [f] = [g] &\implies \pi(f) = \pi(g) \\ &\implies \tilde{\mathcal{F}}(\pi(f)) = \tilde{\mathcal{F}}(\pi(g)) \\ &\implies \mathcal{F}(f) = \mathcal{F}(g). \end{aligned} \tag{5.44}$$

Therefore, $f \sim_{\mathcal{F}} g$. ■

Corollary 5.6

Every functor $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$ factors as $\mathcal{F} = \tilde{\mathcal{F}} \circ \pi$, where π is bijective on objects and surjective on Hom-sets (i.e. π is a *full* functor), and $\tilde{\mathcal{F}}$ is injective on Hom-sets (i.e. $\tilde{\mathcal{F}}$ is a *faithful* functor):

$$\tilde{\mathcal{F}}_{A,B} : \text{Hom}_{\mathcal{C}/\sim_{\mathcal{F}}}(A, B) \rightarrow \text{Hom}_{\mathcal{D}}(\mathcal{F}(A), \mathcal{F}(B)), \forall A, B \in \text{Ob}(\mathcal{C}).$$

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\mathcal{F}} & \mathcal{D} \\ \pi \downarrow & \nearrow \tilde{\mathcal{F}} & \\ \mathcal{C}/\sim_{\mathcal{F}} & & \end{array}$$

Proof. Let us take $\sim = \sim_{\mathcal{F}}$ in [Theorem 5.5](#). Then there exists a **unique** functor $\tilde{\mathcal{F}} : \mathcal{C}/\sim \rightarrow \mathcal{D}$ such that $\mathcal{F} = \tilde{\mathcal{F}} \circ \pi$. So, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\mathcal{F}} & \mathcal{D} \\ \pi \downarrow & \nearrow \tilde{\mathcal{F}} & \\ \mathcal{C}/\sim_{\mathcal{F}} & & \end{array}$$

Now, clearly π is bijective on objects since the objects of $\mathcal{C}/\sim_{\mathcal{F}}$ are equivalence classes $[X]$ containing only $X \in \text{Ob}(\mathcal{C})$. Furthermore, π is surjective on Hom-sets, since given any $[f] \in \text{Hom}_{\mathcal{C}/\sim_{\mathcal{F}}}(X, Y)$, $[f] = \pi(f)$. We are only left to show that $\tilde{\mathcal{F}}$ is injective on Hom-sets.

Take $[f], [g] \in \text{Hom}_{\mathcal{C}/\sim_{\mathcal{F}}}(A, B)$ such that $\tilde{\mathcal{F}}([f]) = \tilde{\mathcal{F}}([g])$. Then we have

$$\tilde{\mathcal{F}}([f]) = \tilde{\mathcal{F}}(\pi(f)) = \mathcal{F}(f).$$

A similar computation reveals that $\tilde{\mathcal{F}}([g]) = \mathcal{F}(g)$. Therefore,

$$\begin{aligned} \tilde{\mathcal{F}}([f]) = \tilde{\mathcal{F}}([g]) &\implies \mathcal{F}(f) = \mathcal{F}(g) \\ &\implies f \sim_{\mathcal{F}} g \\ &\implies [f] = [g]. \end{aligned} \tag{5.45}$$

Therefore, $\tilde{\mathcal{F}}$ is injective on Hom-sets, i.e. faithful. ■

6 Subobjects and Pullbacks

§6.1 Subobjects

Every subset $U \subseteq X$ of a set X occurs as equalizer, and the equalizers are always monomorphisms.

$$U \xhookrightarrow{i} X \xrightarrow[\chi_U]{\top} \mathbf{2},$$

where $\mathbf{2} = \{0, 1\}$, $\top : A \rightarrow \mathbf{2}$ that takes all of A to $1 \in \mathbf{2}$, and

$$\chi_U(x) = \begin{cases} 1 & \text{if } x \in U, \\ 0 & \text{if } x \notin U. \end{cases}$$

Since every subset $U \subseteq X$ occurs as equalizer and equalizers are monic, it is natural to treat monos as generalized subsets. For instance, monos in **Groups** can be regarded as *subgroups*, and monos in **Top** can be regarded as *subspaces*. Given a monomorphism $m : M \rightarrowtail X$ in a category \mathcal{C} of structured sets of some sort (call it “gadget”), the image subset

$$\{m(y) \mid y \in M\} = m(M) \subseteq X$$

is a “sub-gadget” of X to which M is isomorphic to via m .

$$m : M \xrightarrow{\cong} m(M) \subseteq X.$$

More generally, we can think of the mono $m : M \rightarrowtail X$ itself as determining a “part” of X , even in categories where the arrows are not necessarily functions.

Definition 6.1 (Subobject). A **subobject** of an object X in a category \mathcal{C} is a monomorphism

$$m : M \rightarrowtail X.$$

Given subobjects m and m' of X , a morphism $f : m \rightarrow m'$ is an arrow in \mathcal{C}/X . If $m : M \rightarrowtail X$ and $m' : M' \rightarrowtail X$ are subobjects of X , then an arrow $f : m \rightarrow m'$ is an arrow $f : M \rightarrow M'$ such that $m' \circ f = m$, i.e. the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{f} & M' \\ & \searrow m & \downarrow m' \\ & & X \end{array}$$

Thus we have the category $\text{Sub}_{\mathcal{C}}(X)$ of subobjects of X in \mathcal{C} . The objects of $\text{Sub}_{\mathcal{C}}(X)$ are monomorphisms of \mathcal{C} with codomain X . The objects of \mathcal{C}/X are, on the other hand, all arrows of \mathcal{C} with codomain X .

In the construction of $\text{Sub}_{\mathcal{C}}(X)$, since m' is monic, given the following commutative diagram,

$$\begin{array}{ccc} M & \xrightarrow{f} & M' \\ & \searrow m & \downarrow m' \\ & & X \end{array}$$

there is at most one $f : M \rightarrow M'$. In other words, there is at most one element in the hom-set $\text{Hom}_{\text{Sub}_{\mathcal{C}}(X)}(m, m')$. Indeed, if there are $f, f' : m \rightarrow m'$ in $\text{Sub}_{\mathcal{C}}(X)$, $f, f' : M \rightarrow M'$ are arrows in \mathcal{C} such that $m' \circ f = m' \circ f' = m$.

$$\begin{array}{ccc} M & \xrightleftharpoons[f']{f} & M' \\ & \searrow m & \downarrow m' \\ & & X \end{array}$$

Since $m' \circ f = m' \circ f'$, and m' is monic, we have $f = f'$. Therefore, there is at most one element in the hom-set $\text{Hom}_{\text{Sub}_{\mathcal{C}}(X)}(m, m')$, so that $\text{Sub}_{\mathcal{C}}(X)$ is a **preorder category**. We define the relation of inclusion of subobjects by

$$m \subseteq m' \iff \text{there exists some } f : m \rightarrow m'. \quad (6.1)$$

Finally, we say that m and m' are equivalent, written $m \equiv m'$ if and only if they are isomorphic as subobjects, i.e. $m \subseteq m'$ and $m' \subseteq m$. This holds if and only if f and f' making both triangles below commute:

$$\begin{array}{ccc} M & \xrightleftharpoons[f']{f} & M' \\ & \searrow m & \downarrow m' \\ & & X \end{array}$$

In other words,

$$m' \circ f = m \text{ and } m \circ f' = m'. \quad (6.2)$$

Using (6.2), we get

$$m \circ \mathbb{1}_M = m = m' \circ f = m \circ f' \circ f. \quad (6.3)$$

So $m \circ \mathbb{1}_M = m \circ (f' \circ f)$.

$$M \xrightleftharpoons[\mathbb{1}_M]{f' \circ f} M \xrightarrow{m} X$$

Since m is a monomorphism,

$$f' \circ f = \mathbb{1}_M. \quad (6.4)$$

Similarly,

$$f \circ f' = \mathbb{1}_{M'}. \quad (6.5)$$

Hence, $M \cong M'$ via f . Thus, we can see that equivalent subobjects ($m \equiv m'$) have isomorphic domains ($M \cong M'$).

Abuse of Notation. We sometimes abuse notation and language by calling M the subobject when the monomorphism $m : M \rightarrowtail X$ is clear.

It is often convenient to pass from the preorder $\text{Sub}_{\mathcal{C}}(X)$ to the poset $\text{Sub}_{\mathcal{C}}(X) / \equiv$ by factoring out the equivalence relation \equiv . Notice that two distinct monos m and m' in $\text{Sub}_{\mathcal{C}}(X)$ can be equivalent, i.e. $m \equiv m'$ so that both $m \subseteq m'$ and $m' \subseteq m$ hold with m and m' being distinct. That's why $\text{Sub}_{\mathcal{C}}(X)$ is just a preorder category and not a poset category. As soon as we construct $\text{Sub}_{\mathcal{C}}(X) / \equiv$, distinct m and m' in $\text{Sub}_{\mathcal{C}}(X)$ with $m \equiv m'$ coincide, i.e. in $\text{Sub}_{\mathcal{C}}(X)$, $[m] = [m']$.

Now, in $\text{Sub}_{\mathcal{C}}(X) / \equiv$, for distinct equivalence classes $[m] \neq [m']$ one will have either $[m] \subseteq [m']$ or $[m'] \subseteq [m]$ (or neither) and noth both. Hence, \subseteq is antisymmetric in $\text{Sub}_{\mathcal{C}}(X) / \equiv$ so that $\text{Sub}_{\mathcal{C}}(X) / \equiv$ is a **poset category** with respect to the inclusion \subseteq .

Proposition 6.1

In **Sets**, under this notion of subobject, one then has the isomorphism

$$\text{Sub}_{\mathbf{Sets}}(X) / \equiv \cong \mathcal{P}(X).$$

Suppose m, m' are subobjects of $X \in \text{Ob}(\mathcal{C})$, i.e. $m : M \rightarrowtail X$ and $m' : M' \rightarrowtail X$ are monic. Also, let $f : m \rightarrow m'$ be an arrow in $\text{Sub}_{\mathcal{C}}(X)$. Then the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{f} & M' \\ & \searrow m & \downarrow m' \\ & & X \end{array}$$

By [Proposition 3.5](#), since m is monic, so is f . Hence, $f \in \text{Sub}_{\mathcal{C}}(M')$.

Given a monic arrow $m : M \rightarrowtail X$ in \mathcal{C} , one can construct a functor called **composition functor** $m_* : \text{Sub}_{\mathcal{C}}(M) \rightarrow \text{Sub}_{\mathcal{C}}(X)$ by

$$m_*(f) = m \circ f, \tag{6.6}$$

at the level of objects; and at the level of arrows, m_* is defined trivially:

$$m_*(a) = a, \tag{6.7}$$

as can be seen by looking at the following commutative diagram.

$$\begin{array}{ccc} N & \xrightarrow{a} & P \\ & \searrow f & \swarrow g \\ & M & \\ m_*(f)=m \circ f \swarrow & \downarrow m & \searrow m_*(g)=m \circ g \\ & X & \end{array}$$

If $f : N \rightarrowtail M$ is monic, then $m \circ f : N \rightarrow X$ is also monic by [Proposition 3.5](#). Similarly, $m \circ g : P \rightarrow X$ is a monomorphism. Now, $a : f \rightarrow g$ is an arrow in $\text{Sub}_{\mathcal{C}}(M)$, i.e. $a : N \rightarrow P$ is an arrow in \mathcal{C} such that $f = g \circ a$. $m_*(a)$ is supposed to be an arrow from $m_*(f) = m \circ f$ to $m_*(g) = m \circ g$, i.e. $m_*(a) : N \rightarrow P$ is an arrow in \mathcal{C} such that

$$m \circ f = m \circ g \circ m_*(a).$$

Choosing $m_*(a) = a$ clearly satisfies our requirement, since $f = g \circ a$.

§6.2 Pullback

Definition 6.2 (Pullback). In any category \mathcal{C} , given arrows f, g with $\text{cod } f = \text{cod } g$,

$$\begin{array}{ccc} & B & \\ & \downarrow g & \\ A & \xrightarrow{f} & C \end{array}$$

the pullback of f and g consists of arrows

$$\begin{array}{ccc}
 P & \xrightarrow{p_2} & B \\
 \downarrow p_1 & & \\
 A & &
 \end{array}$$

such that $f \circ p_1 = g \circ p_2$ with a certain universal property.

$$\begin{array}{ccc}
 P & \xrightarrow{p_2} & B \\
 \downarrow p_1 & & \downarrow g \\
 A & \xrightarrow{f} & C
 \end{array}$$

The universal property is as follows: given any $z_1 : Z \rightarrow A$ and $z_2 : Z \rightarrow B$ with $f \circ z_1 = g \circ z_2$, there is a **unique** arrow $u : Z \rightarrow P$ such that $z_1 = p_1 \circ u$ and $z_2 = p_2 \circ u$. In other words, the diagram below commutes:

$$\begin{array}{ccccc}
 Z & & & & \\
 \downarrow z_1 & \searrow u & & \searrow z_2 & \\
 & P & \xrightarrow{p_2} & B & \\
 & \downarrow p_1 & & \downarrow g & \\
 & A & \xrightarrow{f} & C &
 \end{array}$$

There is a product-style notation that is often used:

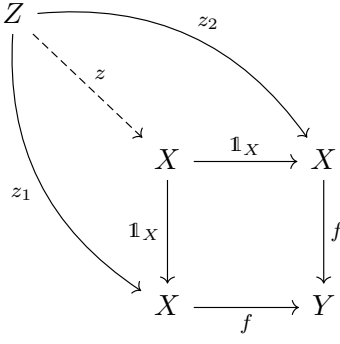
$$\begin{array}{ccccc}
 Z & & & & \\
 \downarrow z_1 & \searrow (z_1, z_2) & & \searrow z_2 & \\
 & A \times_C B & \xrightarrow{p_2} & B & \\
 & \downarrow p_1 & & \downarrow g & \\
 & A & \xrightarrow{f} & C &
 \end{array}$$

Pullbacks are clearly unique up to isomorphism as they are given by a universal mapping property.

Example 6.1. If $f : X \rightarrow Y$ is a monomorphism, then the following is a pullback diagram:

$$\begin{array}{ccc}
 X & \xrightarrow{1_X} & X \\
 \downarrow 1_X & & \downarrow f \\
 X & \xrightarrow{f} & Y
 \end{array} \tag{6.8}$$

Clearly, $f \circ 1_X = f \circ 1_X$. Now, suppose there is another object Z with arrows $z_1 : Z \rightarrow X$ and $z_2 : Z \rightarrow X$ such that $f \circ z_1 = f \circ z_2$.



Since f is monic, we must have $z_1 = z_2$. Therefore, the unique arrow $z : Z \rightarrow X$ is equal to $z_1 = z_2$. It clearly makes the above diagram commutative. Furthermore, it is unique. If there is another $z' : Z \rightarrow X$ commuting the above diagram, then we would have

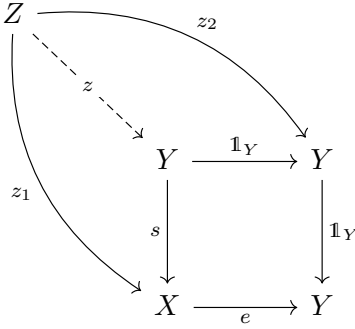
$$z = z_1 = 1_X \circ z' = z'. \quad (6.9)$$

Therefore, (6.8) is a pullback diagram.

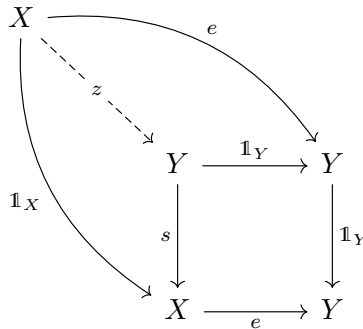
Example 6.2. If $e : X \rightarrow Y$ is a split epi with right inverse $s : Y \rightarrow X$, i.e. $e \circ s = 1_Y$, and e is **not** an isomorphism, then the following is **not** a pullback diagram:

$$\begin{array}{ccc} Y & \xrightarrow{1_Y} & Y \\ s \downarrow & & \downarrow 1_Y \\ X & \xrightarrow{e} & Y \end{array} \quad (6.10)$$

If it were a pullback diagram, then for any object Z with arrows $z_1 : Z \rightarrow X$ and $z_2 : Z \rightarrow Y$ such that $e \circ z_1 = 1_Y \circ z_2$, there would've existed a unique $z : Z \rightarrow Y$ such that $s \circ z = z_1$ and $1_Y \circ z = z_2$.



Let us now choose $Z = X$, $z_1 = 1_X$ and $z_2 = e$. It clearly satisfies $e \circ z_1 = 1_Y \circ z_2$. So there is a unique $z : X \rightarrow Y$ such that the following diagram commutes.



In other words,

$$s \circ z = z_1 = 1_X \quad \text{and} \quad 1_Y \circ z = e. \quad (6.11)$$

The second equation $\mathbb{1}_Y \circ z = e$ gives us that $z = e$. Plugging it into the first equation, we get

$$s \circ e = \mathbb{1}_X. \quad (6.12)$$

This clearly violates the hypothesis that e is not an isomorphism. Therefore, (6.10) is **not** a pullback diagram.

Proposition 6.2

In a category with products and equalizers, given a corner of arrows

$$\begin{array}{ccc} & B & \\ & \downarrow g & \\ A & \xrightarrow{f} & C \end{array}$$

Consider the diagram

$$\begin{array}{ccccc} E & \xrightarrow{p_2} & B & & \\ & \searrow e & \nearrow \pi_2 & & \\ & A \times B & & & \\ & \swarrow \pi_1 & & & \\ A & \xrightarrow{f} & C & & \\ & \uparrow p_1 & & & \\ & & & & \end{array}$$

in which e is an equalizer of $f \circ \pi_1$ and $g \circ \pi_2$ and $\pi_1 \circ e = p_1$, $\pi_2 \circ e = p_2$. Then E, p_1, p_2 is a pullback of f and g . Conversely, if E, p_1, p_2 is a pullback of f and g are given as such a pullback, then the arrow

$$e = \langle p_1, p_2 \rangle : E \rightarrow A \times B$$

is an equalizer of $f \circ \pi_1$ and $g \circ \pi_2$.

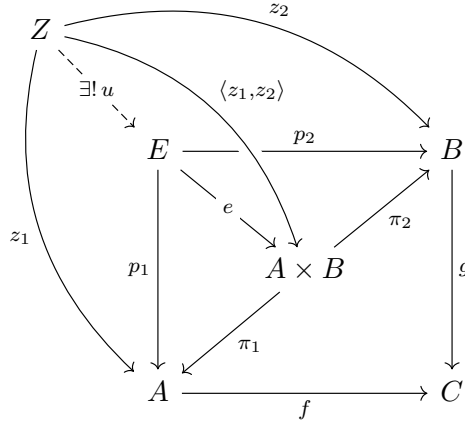
Proof. Suppose that $e : E \rightarrow A \times B$ is an equalizer diagram of the parallel arrows $f \circ \pi_1, g \circ \pi_2 : A \times B \rightarrow C$.

$$E \xrightarrow{e} A \times B \xrightarrow[f \circ \pi_1]{g \circ \pi_2} C$$

Also, $\pi_1 \circ e = p_1$ and $\pi_2 \circ e = p_2$. We need to show that

$$\begin{array}{ccc} E & \xrightarrow{p_2} & B \\ \downarrow p_1 & & \\ A & & \end{array}$$

satisfies the universal mapping property of pullback. In other words, we need to show that $f \circ p_1 = g \circ p_2$ holds, and given any object Z and arrows $z_1 : Z \rightarrow A$ and $z_2 : Z \rightarrow B$ satisfying $f \circ z_1 = g \circ z_2$, there exists a **unique** $u : Z \rightarrow E$ such that $p_1 \circ u = z_1$ and $p_2 \circ u = z_2$.



Since $e : E \rightarrow A \times B$ is an equalizer of $f \circ \pi_1, g \circ \pi_2 : A \times B \rightarrow C$, we have

$$f \circ \pi_1 \circ e = g \circ \pi_2 \circ e. \quad (6.13)$$

We have $\pi_1 \circ e = p_1$ and $\pi_2 \circ e = p_2$. Therefore,

$$f \circ p_1 = g \circ p_2. \quad (6.14)$$

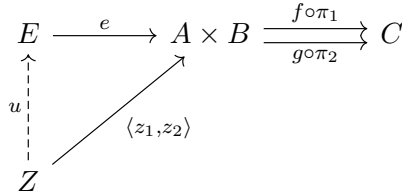
Since $A \xleftarrow{\pi_1} A \times B \xrightarrow{\pi_2} B$ is a product diagram, for $A \xleftarrow{z_1} A \times B \xrightarrow{z_2} B$, there exists a **unique** $\langle z_1, z_2 \rangle : Z \rightarrow A \times B$ such that

$$\pi_1 \circ \langle z_1, z_2 \rangle = z_1 \text{ and } \pi_2 \circ \langle z_1, z_2 \rangle = z_2. \quad (6.15)$$

According to the hypothesis, $f \circ z_1 = g \circ z_2$. Therefore,

$$(f \circ \pi_1) \circ \langle z_1, z_2 \rangle = f \circ z_1 = g \circ z_2 = (g \circ \pi_2) \circ \langle z_1, z_2 \rangle. \quad (6.16)$$

Therefore, $(f \circ \pi_1) \circ \langle z_1, z_2 \rangle = (g \circ \pi_2) \circ \langle z_1, z_2 \rangle$.



By the UMP of equalizer, there is a **unique** $u : Z \rightarrow E$ such that

$$e \circ u = \langle z_1, z_2 \rangle. \quad (6.17)$$

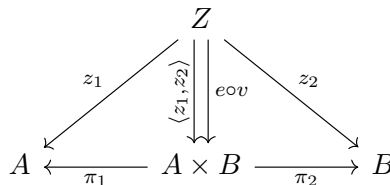
Now,

$$p_1 \circ u = \pi_1 \circ e \circ u = \pi_1 \circ \langle z_1, z_2 \rangle = z_1, \quad (6.18)$$

$$p_2 \circ u = \pi_2 \circ e \circ u = \pi_2 \circ \langle z_1, z_2 \rangle = z_2. \quad (6.19)$$

Therefore, there exists an arrow $u : Z \rightarrow E$ such that $p_1 \circ u = z_1$ and $p_2 \circ u = z_2$. Now we are left to show the uniqueness of u . Suppose there is another arrow $v : Z \rightarrow E$ satisfying $p_1 \circ v = z_1$ and $p_2 \circ v = z_2$. Then we have

$$z_1 = p_1 \circ v = \pi_1 \circ e \circ v \text{ and } z_2 = p_2 \circ v = \pi_2 \circ e \circ v. \quad (6.20)$$



Therefore, $e \circ v = \langle z_1, z_2 \rangle$ by the UMP of the product $A \times B$. Then again, by (6.17), $e \circ u = \langle z_1, z_2 \rangle$. As a result,

$$e \circ v = e \circ u. \quad (6.21)$$

$$Z \begin{array}{c} \xrightarrow{u} \\ \xrightarrow{v} \end{array} E \xrightarrow{e} A \times B$$

But since e is an equalizer, it is monic. Therefore, $u = v$, and hence u is unique. So

$$\begin{array}{ccc} E & \xrightarrow{p_2} & B \\ p_1 \downarrow & & \\ A & & \end{array}$$

is a pullback of f and g .

The converse is left as an exercise. ■

Corollary 6.3

If a category \mathcal{C} has binary products and equalizers, then it has pullbacks.

Lemma 6.4 (Two pullbacks lemma)

Consider the commutative diagram below in a category \mathcal{C} with pullbacks:

$$\begin{array}{ccccc} F & \xrightarrow{f'} & E & \xrightarrow{g'} & D \\ h'' \downarrow & & \downarrow h' & & \downarrow h \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C \end{array}$$

1. If the two squares are pullbacks, then so is the outer rectangle. Thus

$$A \times_B (B \times_C D) \cong A \times_C D.$$

2. If the right square and the outer rectangle are pullbacks, so is the left square.

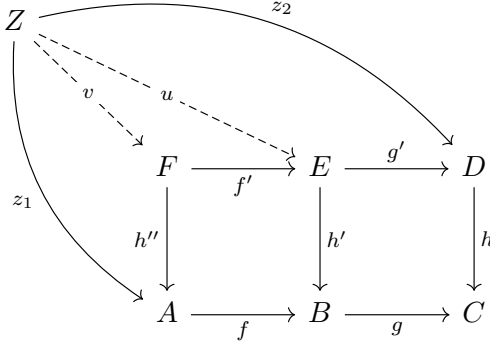
Proof. 1. Suppose both the squares are pullbacks. One needs to show that the outer rectangle is a pullback. The outer rectangle is

$$\begin{array}{ccc} F & \xrightarrow{g' \circ f'} & D \\ h'' \downarrow & & \downarrow h \\ A & \xrightarrow{g \circ f} & C \end{array}$$

Clearly this square is commutative, since

$$g \circ f \circ h'' = g \circ h' \circ f' = h \circ g' \circ f'. \quad (6.22)$$

Now suppose there is an object Z and arrows $z_1 : Z \rightarrow A$ and $z_2 : Z \rightarrow D$ such that $g \circ f \circ z_1 = h \circ z_2$. We need to show the existence of a **unique** $v : Z \rightarrow F$ such that $h'' \circ v = z_1$ and $g' \circ f' \circ v = z_2$.



$g \circ f \circ z_1 = h \circ z_2$ gives us that

$$g \circ (f \circ z_1) = h \circ z_2. \quad (6.23)$$

Since the right square is a pullback for g and h , there is a **unique** $u : Z \rightarrow E$ such that

$$f \circ z_1 = h' \circ u \text{ and } z_2 = g' \circ u. \quad (6.24)$$

Now, $f \circ z_1 = h' \circ u$, and the left square is a pullback of f and h' . Therefore, there exists a **unique** $v : Z \rightarrow F$ such that

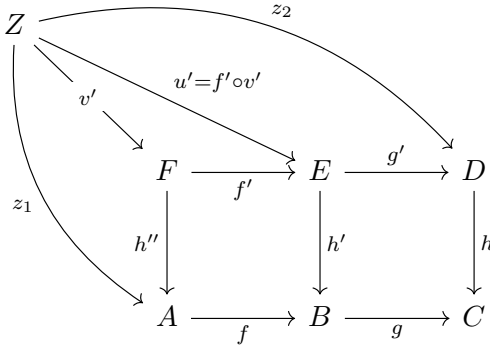
$$z_1 = h'' \circ v \text{ and } u = f' \circ v. \quad (6.25)$$

Now, we have $h'' \circ v = z_1$, and

$$g' \circ f' \circ v = g' \circ u = z_2. \quad (6.26)$$

Therefore, there **exists** $v : Z \rightarrow F$ such that $h'' \circ v = z_1$ and $g' \circ f' \circ v = z_2$. We need to show the uniqueness of v . Suppose there is another $v' : Z \rightarrow F$ such that

$$h'' \circ v' = z_1 \text{ and } g' \circ f' \circ v' = z_2. \quad (6.27)$$



Now, suppose $u' = f' \circ v'$. Then,

$$g' \circ u' = g' \circ f' \circ v' = z_2, \text{ and} \quad (6.28)$$

$$h' \circ u' = h' \circ f' \circ v' = f \circ h'' \circ v' = f \circ z_1. \quad (6.29)$$

But $u : Z \rightarrow E$ is the **unique** arrow such that $f \circ z_1 = h' \circ u$ and $z_2 = g' \circ u$. Therefore, u and u' must coincide.

$$u = u' = f' \circ v'. \quad (6.30)$$

Furthermore, $v : Z \rightarrow F$ is the **unique** arrow such that $z_1 = h'' \circ v$ and $u = f' \circ v$. By (6.27) and (6.30), $v' : Z \rightarrow F$ also satisfies $h'' \circ v' = z_1$ and $f' \circ v' = u$. Therefore, $v = v'$, proving the uniqueness of v . Therefore, the outer rectangle is also a pullback diagram.

2. Left as an exercise. ■

Remark 6.1. In practice, we often see a slightly different form of [Two pullbacks lemma](#). In the following commutative diagram,

$$\begin{array}{ccc}
 F & \xrightarrow{h''} & A \\
 f' \downarrow & & \downarrow f \\
 E & \xrightarrow{h'} & B \\
 g' \downarrow & & \downarrow g \\
 D & \xrightarrow{h} & C
 \end{array}$$

1. if the two squares are pullbacks, then so is the outer rectangle.
2. if the bottom square and the outer rectangle is pullback, so is the top square.

This is equivalent to the version we just proved. This diagram can be obtained by rotating our diagram 90 degrees clockwise, and then reflecting about the diagonal.

Remark 6.2. In the [Two pullbacks lemma](#), if the left square and the outer rectangle are pullbacks, then the right square need not be a pullback. The following is a counterexample: let $e : X \rightarrow Y$ be a split epi with right inverse $s : Y \rightarrow X$, i.e. $e \circ s = \mathbb{1}_Y$. Furthermore, suppose e is **not** an isomorphism. Then consider the following commutative diagram:

$$\begin{array}{ccccc}
 Y & \xrightarrow{\mathbb{1}_Y} & Y & \xrightarrow{\mathbb{1}_Y} & Y \\
 \mathbb{1}_Y \downarrow & & \downarrow s & & \downarrow \mathbb{1}_Y \\
 Y & \xrightarrow{s} & X & \xrightarrow{e} & Y
 \end{array}$$

This diagram commutes, since $e \circ s = \mathbb{1}_Y$. Furthermore, s has a left inverse, so it is monic. Therefore, the left square is a pullback diagram, by [Example 6.1](#). Also, the outer rectangle is

$$\begin{array}{ccc}
 Y & \xrightarrow{\mathbb{1}_Y} & Y \\
 \mathbb{1}_Y \downarrow & & \downarrow \mathbb{1}_Y \\
 Y & \xrightarrow{e \circ s = \mathbb{1}_Y} & Y
 \end{array}$$

This is also a pullback diagram by [Example 6.1](#), since $\mathbb{1}_Y$ is monic. However, we assumed e is **not** an isomorphism. So by [Example 6.2](#), the right square is **not** a pullback diagram. Therefore, the right square can fail to be a pullback even if the left square and the outer rectangle are pullbacks.

Corollary 6.5

The pullback of a commutative triangle is a commutative triangle. Specifically, given a commutative triangle as on the right end of the following “prism diagram”:

$$\begin{array}{ccccc}
 A' & \xrightarrow{h_\alpha} & A & & \\
 \downarrow \alpha' & \searrow \gamma' & \downarrow \alpha & \searrow \gamma & \\
 & B' & \xrightarrow{h_\beta} & B & \\
 & \swarrow \beta' & \downarrow \beta & & \\
 C' & \xrightarrow{h} & C & &
 \end{array} \tag{6.31}$$

for any $h : C' \rightarrow C$, if one forms the pullbacks α' and β' as on the left end, i.e. if $C' \xleftarrow{\alpha'} A' \xrightarrow{h_\alpha} A$ is a pullback of $C' \xrightarrow{h} C \xleftarrow{\alpha} A$ and $C' \xleftarrow{\beta'} B' \xrightarrow{h_\beta} B$ is a pullback of $C' \xrightarrow{h} C \xleftarrow{\beta} B$, then there exists a unique $\gamma' : A' \rightarrow C'$ making the left end a commutative triangle, and the upper face a commutative rectangle, and indeed a pullback .

Proof. Since $C' \xleftarrow{\alpha'} A' \xrightarrow{h_\alpha} A$ is a pullback of $C' \xrightarrow{h} C \xleftarrow{\alpha} A$, we have the following commutative pullback square:

$$\begin{array}{ccc}
 A' & \xrightarrow{h_\alpha} & A \\
 \downarrow \alpha' & & \downarrow \alpha \\
 C' & \xrightarrow{h} & C
 \end{array}$$

Therefore,

$$h \circ \alpha' = \alpha \circ h_\alpha = \beta \circ \gamma \circ h_\alpha. \tag{6.32}$$

Furthermore, $C' \xleftarrow{\beta'} B' \xrightarrow{h_\beta} B$ is a pullback of $C' \xrightarrow{h} C \xleftarrow{\beta} B$, and $\alpha' : A' \rightarrow C'$ and $\gamma \circ h_\alpha : A' \rightarrow C$ are arrows such that $h \circ \alpha' = \beta \circ \gamma \circ h_\alpha$.

$$\begin{array}{ccccc}
 A' & & & & \\
 \downarrow \alpha' & \searrow \gamma' & & \searrow \gamma \circ h_\alpha & \\
 & B' & \xrightarrow{h_\beta} & B & \\
 & \downarrow \beta' & & \downarrow \beta & \\
 & C' & \xrightarrow{h} & C &
 \end{array}$$

Therefore, there exists a **unique** $\gamma' : A' \rightarrow B'$ such that

$$\alpha' = \beta' \circ \gamma' \text{ and } \gamma \circ h_\alpha = h_\beta \circ \gamma'. \tag{6.33}$$

So γ' is the unique arrow making the left end triangle and the upper face of the prism diagram commutative. Now, the prism diagram is equivalent to the following commutative diagram:

$$\begin{array}{ccc}
A' & \xrightarrow{h_\alpha} & A \\
\downarrow \gamma' & & \downarrow \gamma \\
B' & \xrightarrow{h_\beta} & B \\
\downarrow \beta' & & \downarrow \beta \\
C' & \xrightarrow{h} & C
\end{array}
\quad
\begin{array}{l}
\alpha' = \beta' \circ \gamma' \\
\alpha = \beta \circ \gamma
\end{array}$$

The outer rectangle and the bottom square are pullbacks. Therefore, by [Two pullbacks lemma](#), the top square is also a pullback. \blacksquare

Proposition 6.6

Pullback is a functor, i.e. for fixed $h : C' \rightarrow C$ in a category \mathcal{C} with pullbacks, there is a functor $h^* : \mathcal{C}/C \rightarrow \mathcal{C}/C'$ defined by

$$(A \xrightarrow{\alpha} C) \mapsto (C' \times_C A \xrightarrow{\alpha'} C'),$$

where α' is the pullback of α along h . The effect of the functor h^* on an arrow $\gamma : \alpha \rightarrow \beta$ in \mathcal{C}/C is given by the pullback diagram of a commutative triangle (6.31): $h^*(\gamma) = \gamma'$, whose unique existence is guaranteed by [Corollary 6.5](#).

Proof. Given an object $\alpha \in \text{Ob}(\mathcal{C}/C)$, we must check that $h^*(\mathbb{1}_\alpha) = \mathbb{1}_{h^*(\alpha)}$. Given $\alpha : A \rightarrow C$ in \mathcal{C} , it is an object in \mathcal{C}/C . The arrow $\mathbb{1}_\alpha$ in \mathcal{C}/C is the arrow $\mathbb{1}_A$ in \mathcal{C} . Then the following is a commutative triangle in \mathcal{C} .

$$\begin{array}{ccc}
A & & \\
\downarrow \alpha & \searrow \mathbb{1}_A & \\
& & A \\
& \swarrow \alpha & \\
C & &
\end{array}$$

Consider the pullback of this commutative triangle along $h : C' \rightarrow C$.

$$\begin{array}{ccccc}
C' \times_C A = A' & \xrightarrow{h_\alpha} & A & & \\
\downarrow \alpha' & \searrow h^*(\mathbb{1}_\alpha) & \downarrow \alpha & \searrow \mathbb{1}_A & \\
& & A' & \xrightarrow{h_\alpha} & A \\
& \swarrow \alpha' & \downarrow \alpha & \swarrow \alpha & \\
C' & \xrightarrow{h} & C & &
\end{array}$$

By [Corollary 6.5](#) $h^*(\mathbb{1}_\alpha)$ is the unique arrow $A' \rightarrow A'$ such that the left end triangle and the upper face of the prism are commutative. In other words,

$$\alpha' = \alpha' \circ h^*(\mathbb{1}_\alpha) \text{ and } h_\alpha \circ h^*(\mathbb{1}_\alpha) = \mathbb{1}_A \circ h_\alpha. \quad (6.34)$$

One can easily check that $\mathbb{1}_{A'}$ clearly satisfies this requirement.

$$\begin{array}{ccc}
A' & \xrightarrow{\mathbb{1}_{A'}} & A' \\
\alpha' \downarrow & \swarrow \alpha' & \\
C' & &
\end{array}
\qquad
\begin{array}{ccc}
A' & \xrightarrow{h_\alpha} & A \\
\mathbb{1}_{A'} \downarrow & & \downarrow \mathbb{1}_A \\
A' & \xrightarrow{h_\alpha} & A
\end{array}$$

Indeed, $\alpha' = \alpha' \circ \mathbb{1}_{A'}$, and

$$h_\alpha \circ \mathbb{1}_{A'} = h_\alpha = \mathbb{1}_A \circ h_\alpha. \quad (6.35)$$

Therefore, by the uniqueness of $h^*(\mathbb{1}_\alpha)$, $h^*(\mathbb{1}_\alpha) = \mathbb{1}_{A'}$, which is the identity arrow of $\alpha' = h^*(\alpha)$ in the slice category \mathcal{C}/C' . So

$$h^*(\mathbb{1}_\alpha) = \mathbb{1}_{\alpha'} = \mathbb{1}_{h^*(\alpha)}. \quad (6.36)$$

Let $\gamma_1 : \alpha_1 \rightarrow \alpha_2, \gamma_2 : \alpha_2 \rightarrow \alpha_3$ be arrows in \mathcal{C}/C . We have to verify that $h^*(\gamma_2 \circ \gamma_1) = h^*(\gamma_2) \circ h^*(\gamma_1)$. In other words, we need to check that $(\gamma_2 \circ \gamma_1)' = \gamma_2' \circ \gamma_1'$.

Suppose $\alpha_1 : A \rightarrow C, \alpha_2 : B \rightarrow C, \alpha_3 : D \rightarrow C$ in \mathcal{C} . Then the following is a commutative diagram in \mathcal{C} .

$$\begin{array}{ccc}
A & & \\
\alpha_1 \downarrow & \searrow \gamma_1 & \\
& B & \\
& \swarrow \alpha_2 & \downarrow \gamma_2 \\
C & & D \\
& \swarrow \alpha_3 &
\end{array}$$

γ_1' and γ_2' are the **unique** arrows such that the left triangles and upper faces of the following prism diagrams commute.

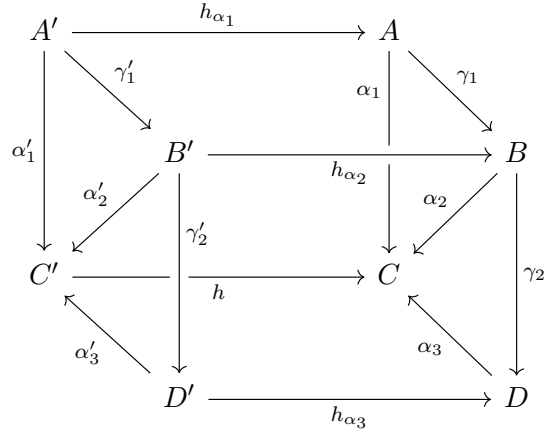
$$\begin{array}{ccc}
A' & \xrightarrow{h_{\alpha_1}} & A \\
\alpha_1' \downarrow & \searrow \gamma_1' & \downarrow \alpha_1 \\
& B' & \xrightarrow{h_{\alpha_2}} B \\
& \swarrow \alpha_2' & \downarrow \alpha_2 \\
C' & \xrightarrow{h} & C
\end{array}
\qquad
\begin{array}{ccc}
B' & \xrightarrow{h_{\alpha_2}} & B \\
\alpha_2' \downarrow & \searrow \gamma_2' & \downarrow \alpha_2 \\
& D' & \xrightarrow{h_{\alpha_3}} D \\
& \swarrow \alpha_3' & \downarrow \alpha_3 \\
C' & \xrightarrow{h} & C
\end{array}$$

In other words,

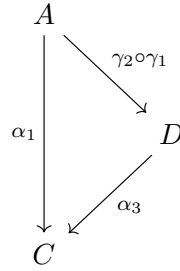
$$\alpha_1' = \alpha_2' \circ \gamma_1' \text{ and } \gamma_1 \circ h_{\alpha_1} = h_{\alpha_2} \circ \gamma_1'; \quad (6.37)$$

$$\alpha_2' = \alpha_3' \circ \gamma_2' \text{ and } \gamma_2 \circ h_{\alpha_2} = h_{\alpha_3} \circ \gamma_2'. \quad (6.38)$$

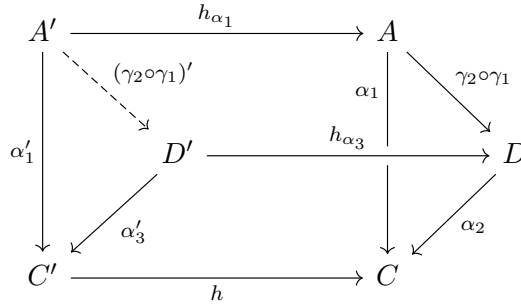
Combining these two prism diagrams, we get a commutative “cubic diagram”:



Now consider the following commutative triangle and its pullback:



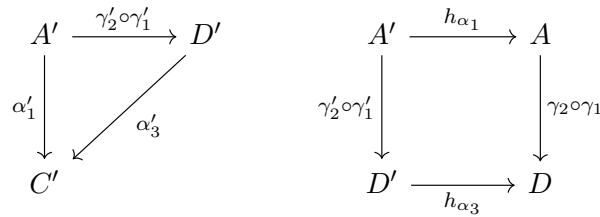
$(\gamma_2 \circ \gamma_1)'$ is the **unique** arrow such that left triangle and upper face of the following prism diagram commute:



In other words,

$$\alpha'_1 = \alpha'_3 \circ (\gamma_2 \circ \gamma_1)' \text{ and } (\gamma_2 \circ \gamma_1) \circ h_{\alpha_1} = h_{\alpha_3} \circ (\gamma_2 \circ \gamma_1)'. \quad (6.39)$$

One can easily check that $\gamma'_2 \circ \gamma'_1$ clearly satisfies this requirement.



Using (6.37) and (6.38), we get

$$\alpha'_3 \circ (\gamma'_2 \circ \gamma'_1) = (\alpha'_3 \circ \gamma'_2) \circ \gamma'_1 = \alpha'_2 \circ \gamma'_1 = \alpha'_1. \quad (6.40)$$

Again, applying (6.37) and (6.38), we get

$$\begin{aligned} h_{\alpha_3} \circ (\gamma'_2 \circ \gamma'_1) &= (h_{\alpha_3} \circ \gamma'_2) \circ \gamma'_1 = (\gamma_2 \circ h_{\alpha_2}) \circ \gamma'_1 \\ &= \gamma_2 \circ (h_{\alpha_2} \circ \gamma'_1) = \gamma_2 \circ (\gamma_1 \circ h_{\alpha_1}) \\ &= (\gamma_2 \circ \gamma_1) \circ h_{\alpha_1}. \end{aligned} \quad (6.41)$$

Therefore, $\gamma'_2 \circ \gamma'_1$ satisfies (6.39). Since $(\gamma_2 \circ \gamma_1)'$ is **unique**, we must have

$$(\gamma_2 \circ \gamma_1)' = \gamma'_2 \circ \gamma'_1. \quad (6.42)$$

Hence, h^* is indeed a functor. ■