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Topology (MAT411)

Lecture Notes

Preface

This series of lecture notes has been prepared for aiding students who took the BRAC University course **Topology (MAT411)** in Fall 2023 semester. These notes were typeset under the supervision of mathematician **Dr. Syed Hasibul Hassan Chowdhury**. The main goal of this typeset is to have an organized digital version of the notes, which is easier to share and handle. If you see any mistakes or typos, please send me an email at atonuroychowdhury@gmail.com

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References:

- *Topology*, by **James R. Munkres**

Contents

1 Topological Spaces

§1.1 Basic Definitions

Definition 1.1. Let X be a set. A **topology** on X is a collection $\mathcal{T}$ of subsets of X having the following properties:

1. $\varnothing$ and X are in $\mathcal{T}$.
2. For any subcollection $\{U_\alpha\}_{\alpha \in J}$ of $\mathcal{T}$, the union $\bigcup_{\alpha \in J} U_\alpha$ is in $\mathcal{T}$.
3. For any finite subcoluction $\{U_1, \ldots, U_n\}$ of $\mathcal{T}$, the intersection $\bigcap_{i=1}^{n} U_i$ is in $\mathcal{T}$.

A **topological space** $(X, \mathcal{T})$ is a set X with a given topology $\mathcal{T}$. A subset $U \subset X$ with $U \in \mathcal{T}$ is said to be an open set.

Example 1.1 (Two extreme examples)**.** Let X be a set. Following are 2 examples of topologies on X :

1. (Discrete topology) The discrete Topology on X, denoted by $\mathcal{T}_{\text{disc}}$ is the topology where all subsets $U \subset X$ are defined to be open. Hence, $\mathcal{T}_{\text{disc}} = \mathscr{P}(X)$, the power set of X. One can easily check that $\mathcal{T}_{\text{disc}}$ is indeed a topology.
2. (Indiscrete topology) The indiscrete topology on X, denoted by $\mathcal{T}_{\text{indis}}$ is the topology where only the subsets X and $\varnothing$ are defined to be open sets. In other words, $\mathcal{T}_{\text{indis}} = \{\varnothing, X\}$.

Definition 1.2 (Finite topological space)**.** If X is a finite set and $\mathcal{T}$ is a topology on X, we call $(X, \mathcal{T})$ a **finite topological space**.

Example 1.2. Let X be a 3-element set, $X = \{1, 2, 3\}$. Verify that the following are examples of finite topological spaces:

1. $\mathcal{T} = \{\varnothing, \{1, 2, 3\}\}$.
2. $\mathcal{T} = \{\varnothing, \{1, 2, 3\}, \{1, 2\}, \{2, 3\}, \{2\}\}$.
3. $\mathcal{T} = \{\varnothing, \{1, 2, 3\}, \{1\}, \{1, 2\}\}$.

Non-example: The collection $\mathcal{T} = \{\varnothing, \{1, 2, 3\}, \{1\}, \{2\}\}$ is not a topology on $X = \{1, 2, 3\}$, since it is not closed under union.

Definition 1.3. Let $\mathcal{T}$ and $\mathcal{T}'$ be 2 topologies on the same set X. If $\mathcal{T}' \supseteq \mathcal{T}$, we soy that $\mathcal{T}'$ is **finer** than $\mathcal{T}$, or $\mathcal{T}$ is **coarser** than $\mathcal{T}'$. If the containment above is proper, we say that $\mathcal{T}'$ is strictly finer than $\mathcal{T}$, or $\mathcal{T}$ is strictly coarser than $\mathcal{T}'$.

Example 1.3. In the context of Example 1.2, for the 3-element set $X = \{1, 2, 3\}$, consider the following 4 topologies:

1. $\mathcal{T} = \{\{1, 2, 3\}, \varnothing, \{1, 2\}, \{2\}, \{2, 3\}\}$.
2. $\mathcal{T}_1 = \{\{1, 2, 3\}, \varnothing\}$.
3. $\mathcal{T}_2 = \{\{1, 2, 3\}, \varnothing, \{2\}\}$
4. $\mathcal{T}_3 = \{\{1, 2, 3\}, \varnothing, \{1, 2\}\}$

Observe that $\mathcal{T}$ is strictly finer than all 3 of $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$. Also, one has $\mathcal{T}_1 \subset \mathcal{T}_3$, and $\mathcal{T}_1 \subset \mathcal{T}_2$, i.e. $\mathcal{T}_3$ is strictly finer than $\mathcal{T}_1$, and $\mathcal{T}_2$ is strict finer than $\mathcal{T}_1$.

§1.2 Review of Metric Space

Definition 1.4. A **metric** on a set X is a function $d : X \times X \to \mathbb{R}$ such that:

1. (Non-negativity) $d(x,y) \geq 0$ for any $x, y \in X$, and $d(x,y) = 0$ if and only if $x = y$.
2. (Symmetry) $d(x,y) = d(y,x)$, for any $x, y \in X$.
3. (Triangle inequality) $d(x,z) \leq d(x,y) + d(y,z)$ for any $x, y, z \in X$.

A **metric space** (X,d) is a set X equipped with a metric d.

Example 1.4. The real line $\mathbb{R}$ is a metric space, with distance function $d_{\text{Euc}}(x,y) = |y-x|$. More generally, in $\mathbb{R}^n$, one can define the Euclidean distance

$$d_{\text{Euc}}(\mathbf{x},\mathbf{y}) = \|\mathbf{y}-\mathbf{x}\| = \sqrt{(y_1-x_1)^2 + \cdots + (y_n-x_n)^2}, \tag{1.1}$$

for $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$. We call $(\mathbb{R}^n, d_{\text{Euc}})$ the Euclidean n-space.

Definition 1.5. Let (X,d) be a metric space. For each point $x \in X$ and each $\varepsilon < 0$, let

$$B_d(x,\varepsilon) = \{y \in X \mid d(x,y) < \varepsilon\}. \tag{1.2}$$

Then the set $B_d(x,\varepsilon)$ is called ε-ball around x in (X,d).

Definition 1.6 (Metric topology)**.** Let (X,d) be a metric space. The metric topology $\mathcal{T}_d$ on X is the collection of subsets $U \subset X$ such that for each $x \in U$, there exists $\varepsilon > 0$ such that $B_d(x,\varepsilon) \subset U$.

Lemma 1.1

The collection $\mathcal{T}_d$ is a topology on X.

Proof. Observe that $\varnothing$ is vacuously open in metric topology, i.e. $\varnothing \in \mathcal{T}_d$ since there is no element in $\varnothing$ to open the argument with. Also, the whole set $X \in \mathcal{T}_d$, i.e. the whole set X itself is open in the metric topology. This is so because for any $x \in X$, one can choose $B_d(x,1) = \{y \in X \mid d(x,y) < 1\} \subseteq X$ proving that X is open in the metric topology.

Next, let $\{U_\alpha\}_{\alpha \in J}$ be a subcollection of $\mathcal{T}_d$. Let $W = \bigcup_\alpha U_\alpha$. Consider $x \in W = \bigcup_\alpha U_\alpha$. Hence, there is some $\alpha_0 \in J$ such that $x \in U_{\alpha_0}$. Since $U_{\alpha_0} \in \mathcal{T}_d$, there exists $\varepsilon > 0$ such that

$$B_d(x,\varepsilon) \subset U_{\alpha_0} \subset \bigcup_{\alpha \in J} U_\alpha = W. \tag{1.3}$$

Hence, $\bigcup_{\alpha \in J} U_\alpha \in \mathcal{T}_d$.

Now, let $\{U_1, \ldots, U_n\}$ be a finite subcoluction of $\mathcal{T}_d$. Let $V = U_1 \cap \cdots \cap U_n$ and consider $x \in V$. Hence, $x \in U_i$ for each $i \in \{1, \ldots, n\}$. Since, each $U_i \in \mathcal{T}_d$, there exists $\varepsilon_i > 0$, such that $B_d(x,\varepsilon_i) \subset U_i$, for each $i \in \{1, \ldots, n\}$. Choose $\varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_n\} > 0$. Then one has $B_d(x,\varepsilon) \subset B_d(x,\varepsilon_i) \subset U_i$, for any i. Therefore,

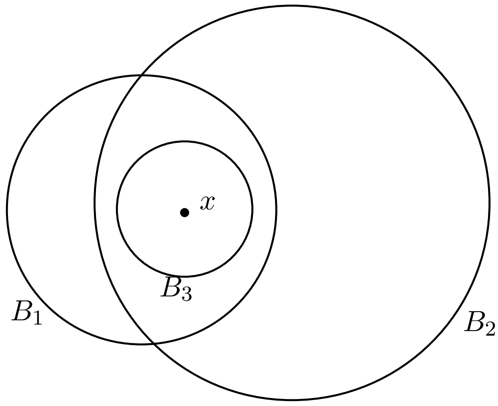
$$B_d(x,\varepsilon) \subset \bigcap_{i=1}^{n} U_i, \tag{1.4}$$

proving that $V = \bigcap_{i=1}^n U_i \in \mathcal{T}_d$. ■

§1.3 Basis for a Topology

Definition 1.7 (Basis)**.** Let X be a set. A **basis** for a topology on X is a collection $\mathscr{B}$ of subsets of X (called *basis elements*) such that

1. for each $x \in X$, there exists $B \in \mathscr{B}$ such that $x \in B \subset X$;
2. if $x \in B_1 \cap B_2$ with $B_1, B_2 \in \mathscr{B}$, then there exists $B_3 \in \mathscr{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.



Definition 1.8 (Topology generated by a basis)**.** Let $\mathscr{B}$ be a basis for a topology topology on a given set X. The topology $\mathcal{T}$ generated by $\mathscr{B}$ is the collection of subsets $U \subset X$ such that for each $x \in U$, there exists $B \in \mathscr{B}$ with $x \in B \subset U$. In other words, a subset $U \subset X$ is defined to be open in this topology if for each $x \in U$, there exists a basis element $B \subset U$ with $x \in B$.

Lemma 1.2

The collection $\mathcal{T}$ generated by a basis $\mathcal{B}$ as defined above is a topology on X.

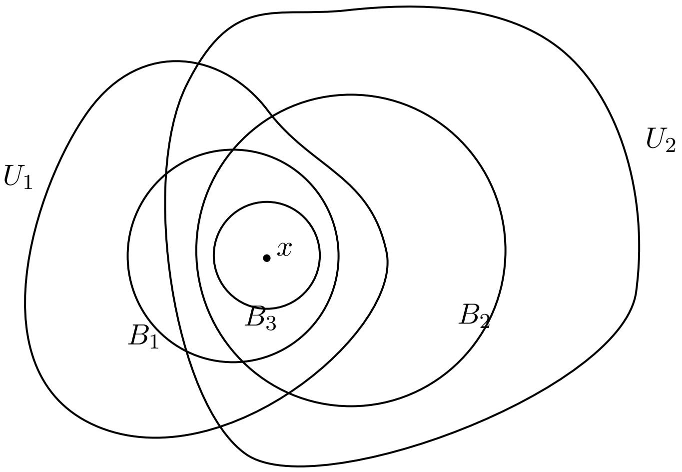
Proof. $\varnothing \in \mathcal{T}$ since there is no element in $\varnothing$ to verify the conditions, and hence $\varnothing$ is vacuously open. By the first condition of basis, for each $x \in X$, there exists $B \in \mathscr{B}$ such that $x \in B \subset X$. Therefore, from the definition of the topology generated by a basis, X is open, i.e. $X \in \mathcal{T}$.

Now, let $\{U_\alpha\}_{\alpha \in J}$ be a subcollection of $\mathcal{T}$. Also, let $\bigcup_{\alpha \in J} U_\alpha = W$. We need to show that $W \in \mathcal{T}$. Consider $x \in W = \bigcup_\alpha U_\alpha$. Hence, there is some $\alpha_0 \in J$ such that $x \in U_{\alpha_0}$. Since $U_{\alpha_0} \in \mathcal{T}$, there exists $B \in \mathscr{B}$ for which $x \in B \subset U_{\alpha_0}$ holds. In other words,

$$x \in B \subset U_{\alpha_0} \subset \bigcup_{\alpha \in J} U_\alpha = W. \tag{1.5}$$

Therefore, $W \in \mathcal{T}$.

Now, let $U_1, U_2 \in \mathcal{T}$. Given $x \in U_1 \cap U_2$, x is in both U_1 and U_2. Since $U_1, U_2 \in \mathcal{T}$, by the definition of topology generated by a basis, there exist basis elements $B_1, B_2 \in \mathscr{B}$ such that $x \in B_1 \subset U_1$ and $x \in B_2 \subset U_2$. Then we have $x \in B_1 \cap B_2$.



By the second condition for a basis, there is some $B_3 \in \mathscr{B}$ such that $x \in B_3 \subset B_1 \cap B_2$. Therefore,

$$x \in B_3 \subset B_1 \cap B_2 \subset U_1 \cap U_2. \tag{1.6}$$

So $U_1 \cap U_2 \in \mathcal{T}$. Now we use induction to prove that $V = \bigcap_{i=1}^{n} U_i \in \mathcal{T}$, where each $U_i \in \mathcal{T}$. The base case $n = 1$ is trivial. Now suppose that this is true for $n-1$, i.e. $\bigcap_{i=1}^{n-1} U_i \in \mathcal{T}$. We also have $U_n \in \mathcal{T}$. We have just proved that the intersection of two elements of $\mathcal{T}$ also belongs to $\mathcal{T}$. Therefore,

$$\left(\bigcap_{i=1}^{n-1} U_i\right) \cap U_n = \bigcap_{i=1}^{n} U_i \in \mathcal{T}. \tag{1.7}$$

Therefore, $\mathcal{T}$ is a topology on X. ■

Lemma 1.3

In any metric space (X, d), the collection of ε-balls

$$\mathscr{B} = \{B_d(x, \varepsilon) \mid x \in X, \varepsilon > 0\}$$

is a basis.

Proof. 1. For each $x \in X$, the 1-ball $B_d(x, 1) \in \mathscr{B}$.

2. Given $B_1 = B_d(x_1, \varepsilon_1)$ and $B_2 = B_d(x_2, \varepsilon_2)$, consider $x \in B_1 \cap B_2$. It is evident that

$$\varepsilon_1 - d(x, x_1) > 0 \text{ and } \varepsilon_2 - d(x, x_2) > 0. \tag{1.8}$$

Let $\varepsilon = \min\{\varepsilon_1 - d(x, x_1), \varepsilon_2 - d(x, x_2)\}$. Then $\varepsilon > 0$. Now we claim that $x \in B_d(x, \varepsilon) =: B_3 \subset B_1 \cap B_2$. Let $y \in B_3 = B_d(x, \varepsilon)$, so that $d(x, y) < \varepsilon$. Then

$$d(x, y) < \varepsilon \leq \varepsilon_1 - d(x, x_1).$$

By the triangle inequality,

$$d(x_1, y) \leq d(x, x_1) + d(x, y) < \varepsilon_1, \tag{1.9}$$

which implies that $y \in B_1 = B_d(x_1, \varepsilon_1)$. So $B_3 \subset B_1$. Similarly, $B_3 \subset B_2$. Therefore, $B_3 = B_d(x, \varepsilon) \subset B_1 \cap B_2$, as required. ■

Proposition 1.4

The metric topology $\mathcal{T}_d$ defined earlier on the metric space coincides with the topology $\mathcal{T}_d$ on (X, d) generated by the basis of ε-balls as in Lemma 1.3.

Proof. Suppose $U \in \mathcal{T}_d$. Hence, from the definition of metric topology, for each $y \in U$, there exists $\delta > 0$ such that $B_d(y,\delta) \subset U$. Since $B_d(y,\delta) \in \mathscr{B}$, and $y \in B_d(y,\delta) \subset U$, $U \in \mathcal{T}$, the topology on (X,d) generated by the basis $\mathscr{B}$. In other words, $\mathcal{T}_d \subset \mathcal{T}$.

Now conversely, suppose $U \in \mathcal{T}$. Hence, given $y \in U$, there is a basis element $B_d(x,\varepsilon) \in \mathscr{B}$ such that $y \in B_d(x,\varepsilon) \subset U$. Hence, $d(x,y) < \varepsilon$. Define $\delta = \varepsilon - d(x,y) > 0$. Then one immediately finds $B_d(y,\delta) \subset B_d(x,\varepsilon)$. Indeed, if $z \in B_d(y,\delta)$, then $d(y,z) < \delta = \varepsilon - d(x,y)$. By the triangle inequality,

$$d(x,z) \le d(x,y) + d(y,z) < \varepsilon. \tag{1.10}$$

Therefore, $z \in B_d(x,\varepsilon)$, proving that $y \in B_d(y,\delta) \subset B_d(x,\varepsilon) \subset U$. So we have proved that given $y \in U$, there exists $\delta > 0$ such that $B_d(y,\delta) \subset U$. In other words, $U \in \mathcal{T}_d$, so that $\mathcal{T} \subset \mathcal{T}_d$. Hence, $\mathcal{T} = \mathcal{T}_d$. ■

Example 1.5. Let $X = \mathbb{R}^2$, and $\mathscr{B}$ be the collection of all circular regions (interior of circles) in the plane. This is the collection of all ε-balls

$$B_{\text{Euc}}(\mathbf{x},\varepsilon) = \left\{\mathbf{y} \in \mathbb{R}^2 \mid d(\mathbf{x},\mathbf{y}) < \varepsilon\right\}$$

with respect to the Euclidean metric $d_{\text{Euc}}(\mathbf{x},\mathbf{y}) = \|\mathbf{x}-\mathbf{y}\| = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$, with $\mathbf{x} = (x_1,x_2), \mathbf{y} = (y_1,y_2) \in \mathbb{R}^2$. Indeed, $(\mathbb{R}^2, d_{\text{Euc}})$ is a metric space, and by means of Lemma 1.3 and Proposition 1.4, the collection

$$\mathscr{B} = \left\{B_{\text{Euc}}(\mathbf{x},\varepsilon) \mid \mathbf{x} \in \mathbb{R}^2,\ \varepsilon > 0\right\}$$

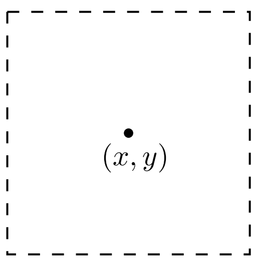
is a basis for the metric topology with respect to the Euclidean metric on $\mathbb{R}^2$.

Example 1.6. Let $X = \mathbb{R}^2$, but in contrast to Example 1.5, here choose $\mathscr{B}'$ to be the collection of all rectangular regions (interior of rectangles) in the plane $\mathbb{R}^2$. This is the collection of all sets of the form

$$(a,b) \times (c,d) \in \mathbb{R} \times \mathbb{R},$$

with $a < b$ and $c < d$. This is the open rectangular area bounded by the vertical lines $x = a$ and $x = b$, and horizontal lines $y = c$ and $y = d$. Let us verify that such a collection, indeed, satisfies the two conditions for a basis:

1. For each $(x,y) \in \mathbb{R}^2$, $(x,y) \in (x-1,x+1) \times (y-1,y+1)$, with $(x-1,x+1) \times (y-1,y+1) \in \mathscr{B}'$.



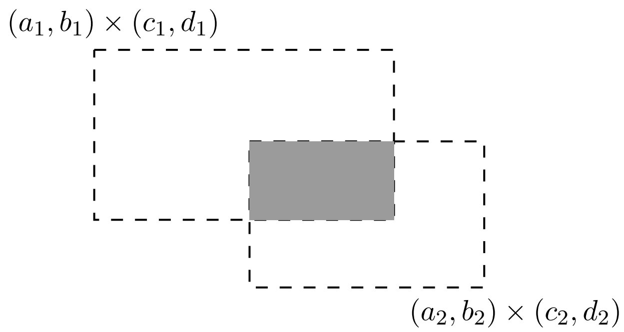
$$(x-1,x+1) \times (y-1,y+1)$$

2. Consider $B_1 = (a_1,b_1) \times (c_1,d_1)$ and $B_2 = (a_2,b_2) \times (c_2,d_2)$ to be two elements in $\mathscr{B}'$. Take $(x_0,y_0) \in B_1 \cap B_2$. Since $a_1 < x_0 < b_1$ and $a_2 < x_0 < b_2$, one has

$$a := \max\{a_1,a_2\} < x_0 < \min\{b_1,b_2\} =: b,$$

Similarly,

$$c := \max\{c_1,c_2\} < y_0 < \min\{d_1,d_2\} =: d.$$



Then $(x_0, y_0) \in (a, b) \times (c, d) =: B_3 = B_1 \cap B_2$, the shaded open rectangle in the diagram above. The diagram above is the case when $B_1 \cap B_2 \neq \varnothing$. The condition for this to happen is $a < b$ and $c < d$. Otherwise, the intersection is empty, and the second condition for basis is vacuously satisfied.

Proposition 1.5

Let $\mathscr{B}$ be a basis for a topology $\mathcal{T}$ on X, i.e. $\mathcal{T}$ is the topology on X generated by the basis $\mathscr{B}$. Then $\mathcal{T}$ equals the collection of all unions of elements of $\mathscr{B}$.

Proof. Let us first prove that $\mathcal{T}$ is contained in the collection of all unions of elements of $\mathscr{B}$. Let $U \in \mathcal{T}$. For each $x \in U$, there exists $B_x \in \mathscr{B}$ with $x \in B_x \subset U$. Then one easily has $U = \bigcup_{x \in U} B_x$. Indeed, since $x \in B_x \subset U$, taking union over all $x \in U$ gives us

$$\bigcup_{x \in U} \{x\} \subset \bigcup_{x \in U} B_x \subset U.$$

In other word,

$$U \subset \bigcup_{x \in U} B_x \subset U. \tag{1.11}$$

So $U = \bigcup_{x \in U} B_x$. Therefore, any open set on X in the topology $\mathcal{T}$ generated by a basis $\mathscr{B}$ is a union of basis elements from $\mathscr{B}$.

To prove the converse, i.e. any union of basis elements from $\mathscr{B}$ belongs to $\mathcal{T}$, note that every basis element B of $\mathscr{B}$ is open, i.e. it belongs to $\mathcal{T}$. This is because for each $x \in B$, there is a basis element, namely B itself, such that $x \in B \subset B$, proving that $B \in \mathcal{T}$, the topology generated by the basis $\mathscr{B}$. From the definition of topology, it follows that arbitrary union of basis elements from $\mathscr{B}$ will be in $\mathcal{T}$ as well. ∎

Example 1.7. If X is any set, the collection of all one-point subsets of X is a basis for the discrete topology $\mathcal{T}_{\text{dis}}$ on X. For example, if $X = \{a, b, c\}$, then

$$\mathcal{T}_{\text{dis}} = \{\{a, b, c\}, \varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}\} = \mathscr{P}(X);$$

and $\mathscr{B} = \{\{a\}, \{b\}, \{c\}\}$. Indeed, $\mathcal{T}_{\text{dis}}$ can be obtained from $\mathscr{B}$ by taking all possible unions. $\varnothing$ is understoor as the union of no basis elements at all.

Lemma 1.6 (Comparing topologies using bases)

Let $\mathscr{B}$ and $\mathscr{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$ on X, respectively. Then the following are equivalent:

1. $\mathcal{T}'$ is finer than $\mathcal{T}$.
2. For each $x \in X$ and any basis element $B \in \mathscr{B}$ containing x, there is a basis element $B' \in \mathscr{B}'$ such that $x \in B' \subset B$.

Proof. **(1⇒2)** Let $x \in X$ and $B \in \mathscr{B}$ with $x \in B$. We have seen in the proof of Proposition 1.5 that, $B \in \mathcal{T}$. By hypothesis, $\mathcal{T} \subset \mathcal{T}'$. Hence, $B \in \mathcal{T}'$. Since $\mathcal{T}'$ is the topology generated by $\mathscr{B}'$, there exists $B' \in \mathscr{B}'$ with $x \in B' \subset B$.

(2⇒1) Let $U \in \mathcal{T}$. Since $\mathcal{T}$ is generated by $\mathscr{B}$, for each $x \in U$, there exists some $B \in \mathscr{B}$ with $x \in B \subset U$. By hypothesis, there exists a $B' \in \mathscr{B}'$ with $x \in B' \subset B$. Therefore, $B' \in U$. We, therefore, have shown that for each $x \in U$, there exists $B' \in \mathscr{B}'$ with $x \in B' \subset U$. Hence, $U \in \mathcal{T}'$, the topology generated by $\mathscr{B}'$. Therefore, $\mathcal{T} \subset \mathcal{T}'$. ■

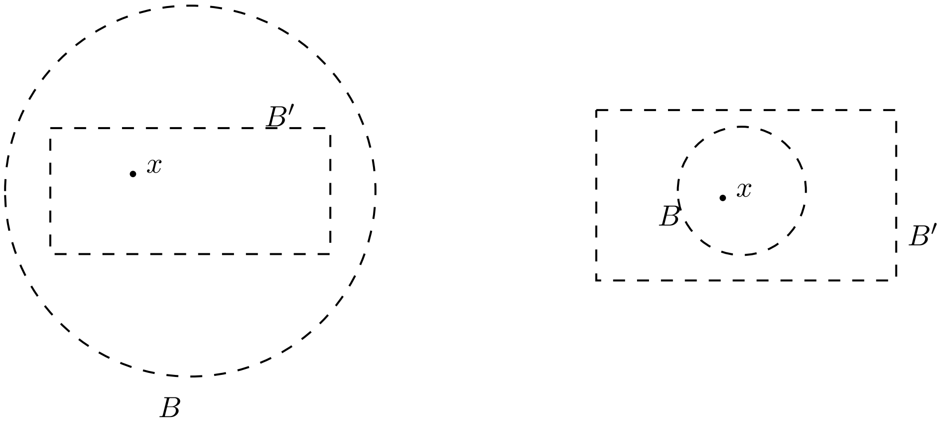
Corollary 1.7

Two bases $\mathscr{B}$ and $\mathscr{B}'$ for topologies on X generate the same topology if and only if

1. for each $x \in B \in \mathscr{B}$, there is a basis element $B' \in \mathscr{B}'$ such that $x \in B' \subset B$; and furthermore,
2. for each $x \in B' \in \mathscr{B}'$, there is a basis element $B \in \mathscr{B}$ such that $x \in B \subset B'$.

Proof. Let $\mathscr{B}$ and $\mathscr{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$ on X, respectively. By Lemma 1.6, $\mathcal{T} \subseteq \mathcal{T}'$ is equivalent to (1). By Lemma 1.6, $\mathcal{T}' \subseteq \mathcal{T}$ is equivalent to (2). ■

Example 1.8. The basis $\mathscr{B}$ of open circular regions in the plane $\mathbb{R}^2$ and the basis $\mathscr{B}'$ of open rectanglular regions generate the same topology on $\mathbb{R}^2$, namely the metric topology.



Example 1.9 (Three important topologies on $\mathbb{R}$)**.** Let $\mathscr{B}$ be the collection of all open intervals in thr real line $\mathbb{R}$:

$$(a, b) = \left(\frac{a+b}{2} - \frac{b-a}{2}, \frac{a+b}{2} + \frac{b-a}{2}\right) = B_{\text{Euc}}\left(\frac{a+b}{2}, \frac{b-a}{2}\right).$$

$\mathscr{B} = \{(a, b) \mid a, b \in \mathbb{R} \text{ with } a < b\}$. This collection $\mathscr{B}$ is a basis on $\mathbb{R}$, and the topology it generates is precisely the Euclidean metric topology on $\mathbb{R}$ by Proposition 1.4. This topology is also called the **standard topology** on $\mathbb{R}$.

Now, let $\mathscr{B}'$ denote the collection of all half-open intervals of the form

$$[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\},$$

for $a < b$. In other words, $\mathscr{B}' = \{[a, b) \mid a, b \in \mathbb{R} \text{ with } a < b\}$. The topology on $\mathbb{R}$ generated by the basis $\mathscr{B}'$ is called the **lower-limit topology**. When $\mathbb{R}$ is given the lower-limit topology, the resulting topological space is denoted by $\mathbb{R}_\ell$.

Now, let K denote the set of all numbers of the form $\frac{1}{n}$ for positive integers n. Also, let $\mathscr{B}''$ denote the collection of all open intervals (a, b) along with all sets of the form $(a, b) \setminus K$. The topology on $\mathbb{R}$ generated by $\mathscr{B}''$ is called the **K-topology** on $\mathbb{R}$. $\mathbb{R}$, equipped with the K-topology, is denoted by $\mathbb{R}_K$.

Lemma 1.8

The topologies $\mathbb{R}_\ell$ and $\mathbb{R}_K$ are strictly finer than the standard topology on $\mathbb{R}$.

Proof. Let $\mathcal{T}$, $\mathcal{T}'$ and $\mathcal{T}''$ be the topologies of $\mathbb{R}$, $\mathbb{R}_\ell$ and $\mathbb{R}_K$, respectively. Given a basis element $(a,b) \in \mathscr{B}$ generating $\mathcal{T}$, and $x \in (a,b)$, one finds $[x,b) \in \mathscr{B}'$ generating $\mathcal{T}'$ such that $x \in [x,b) \subset (a,b)$. This proves that $\mathcal{T}'$ is finer than $\mathcal{T}$ using Lemma 1.6.

On the other hand, choose $[x,y) \in \mathscr{B}'$ generating $\mathcal{T}'$. There exists no open interval $(a,b) \in \mathscr{B}$ such that $x \in (a,b) \subset [x,y)$ is satisfied. Hence, $\mathcal{T}'$ is strictly finer than $\mathcal{T}$.

Now, givena basis element $(a,b) \in \mathscr{B}$ generating $\mathcal{T}$, and $x \in (a,b)$, one finds $(a,b) \in \mathscr{B}''$ generating $\mathcal{T}''$ such that $x \in (a,b) \subset (a,b)$. This proves that $\mathcal{T}''$ is finer than $\mathcal{T}$ using Lemma 1.6.

On the other hand, observe that $B = (-2,2) \setminus K \in \mathscr{B}''$ generating $\mathcal{T}''$, and $0 \in B$. But there exists no open interval $(a,b) \in \mathscr{B}$ such that $0 \in (a,b) \subset B$. Hence, $\mathcal{T}''$ is strictly finer than $\mathcal{T}$. ■

§1.4 Subbasis

Definition 1.9 (Subbasis). A **subbasis** for a topology on X is a collection $\mathscr{S}$ of subsets of X, with union equal to X. One can form the collection $\mathscr{B}$ consisting of all finite intersections of elements of $\mathscr{S}$:

$$B = S_1 \cap \cdots \cap S_n,$$

with $S_1, \ldots, S_n \in \mathscr{S}$, for $n \geq 1$. By the topology $\mathcal{T}$ generated by $\mathscr{S}$, we mean the topology generated by the associated basis $\mathscr{B}$. One clearly has $\mathscr{S} \subset \mathscr{B} \subset \mathcal{T}$.

Lemma 1.9

Let $\mathscr{S}$ be a subbasis on X. The associated collection $\mathscr{B}$ is a basis for a topology.

Proof. There are 2 conditions of a basis to be fulfilled:

1. Since the union of all elements of $\mathscr{S}$ is X by the definition of a subbasis, each $x \in X$ lies in some $S \in \mathscr{S}$. Since S itself is a basis element, the first condition for basis is fulfilled.

2. Suppose $B_1 = S_1 \cap \cdots \cap S_m$ and $B_2 = S'_1 \cap \cdots \cap S'_n$, where $S_i, S'_i \in \mathscr{S}$. Then the intersection

$$B_1 \cap B_2 = (S_1 \cap \cdots \cap S_m) \cap (S'_1 \cap \cdots \cap S'_n) \tag{1.12}$$

is also a finite intersection of elements of $\mathscr{S}$, and hence $B_1 \cap B_2 \in \mathscr{B}$, fulfilling the second condition of basis.

■

Suppose $(X, \mathcal{T})$ is a topological space. We are given $\mathscr{C} \subset \mathcal{T}$, a subcollection of open subsets of X. How do we recognize if $\mathscr{C}$ is a basis for the topology $\mathcal{T}$ on X? The following lemma answers this question.

Lemma 1.10 (Recognition principle)

Let $(X, \mathcal{T})$ be a topological space. Suppose $\mathscr{C} \subset \mathcal{T}$ is a subcollection of open subsets of X, such that for each open $U \in \mathcal{T}$ and each $x \in U$, there exists $C \in \mathscr{C}$ with $x \in C \subset U$. Then $\mathscr{C}$ is a basis for the topology $\mathcal{T}$ on X.

Proof. Let us first check that $\mathscr{C}$ is a basis for *some* topology on X.

1. Observe that $X \in \mathcal{T}$. Hence, by hypothesis, for each $x \in X$, there exists $C \in \mathscr{C}$ with $x \in C \subset X$.

2. Suppose $C_1, C_2 \in \mathscr{C}$ with $x \in C_1 \cap C_2$. Since $\mathscr{C} \subset \mathcal{T}$, C_1 and C_2 are both open and so is their intersection $C_1 \cap C_2$, i.e. $C_1 \cap C_2 \in \mathcal{T}$. By hypothesis, there exists $C_3 \in \mathscr{C}$ such that $x \in C_3 \subset C_1 \cap C_2$. Thus the second condition for basis is also fulfilled.

Let us denote the topology on X generated by $\mathscr{C}$ with $\mathcal{T}'$. We are left to show that $\mathcal{T}' = \mathcal{T}$. Let $U \in \mathcal{T}$ and $x \in U$. By hypothesis, there exists $C \in \mathscr{C}$ with $x \in C \subset U$. Then, by definition of topology generated by a basis, $U \in \mathcal{T}'$. So $\mathcal{T} \subset \mathcal{T}'$.

Now, let $U \in \mathcal{T}'$. By Proposition 1.5, U is a union of elements of $\mathscr{C}$. Since $\mathscr{C} \subset \mathcal{T}$, each element of $\mathscr{C}$ is in $\mathcal{T}$. As $\mathcal{T}$ is a topology, it must be closed under arbitrary union. Hence, $U \in \mathcal{T}$. Therefore, $\mathcal{T}' \subset \mathcal{T}$. ■

§1.5 The Product Topology on $X \times Y$

Definition 1.10 (Product topology). Let X and Y be topological spaces. The **product topology** on $X \times Y$ is the topology generated by the basis

$$\mathscr{B} = \{U \times V \mid U \text{ open in } X, \text{ and } V \text{ open in Y} \}. \tag{1.13}$$

Lemma 1.11

The collection $\mathscr{B}$ as above is a basis for a topology on $X \times Y$.

Proof. Two conditions for basis need to be checked:

1. Observe that $X \times Y \in \mathscr{B}$.
2. Let $B_1 = U_1 \times V_1$ and $B_2 = U_2 \times V_2$ be two basis elements. Observe that

$$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2). \tag{1.14}$$

From the above set theoretic identity, one verifies that $B_1 \cap B_2 = B_3$, where $B_3 = U_3 \times V_3$ with $U_3 = U_1 \cap U_2$ being open in X and $V_3 = V_1 \cap V_2$ being open in Y. This proves that the second condition for basis is verified.

■

Lemma 1.11 gives us a basis for the product topology on $X \times Y$ in terms of open sets of X and Y. If we have information about the bases that generate the topologies on X and Y, then the following theorem gives us a basis generating the product topology on $X \times Y$.

Theorem 1.12

If $\mathscr{B}$ is a basis for the topology of X, $\mathscr{C}$ is a basis for the topology of Y, then the collection

$$\mathscr{D} = \{B \times C \mid B \in \mathscr{B} \text{ and } C \in \mathscr{C}\}$$

is a basis for the product topology of X.

Proof. We apply Recognition principle. We know from the definition of the product topology on $X \times Y$ that it is generated by the basis

$$\mathscr{B}_{\text{prod}} \{U \times V \mid U \subset X \text{ and } V \subset Y \text{ are open } \}. \tag{1.15}$$

Now, given an open set $W \subset X \times Y$ and $(x, y) \in W$, there exists a basis element $U \times V \in \mathscr{B}_{\text{prod}}$ with $(x, y) \in U \times V \subset W$. Now, since $U \subset X$ and $V \subset Y$ are open, there are basis elements $B \in \mathscr{B}$ and $C \in \mathscr{C}$ with $x \in B \subset U$ and $y \in C \subset V$. Therefore,

$$(x, y) \in B \times C \subset U \times V \subset W. \tag{1.16}$$

We, thus, have found that for an open set $W \subset X \times Y$ and any $(x, y) \in W$, there exists $B \times C \in \mathscr{D}$ such that $(x, y) \in B \times C \subset W$. So $\mathscr{D}$ is a basis for the product topology on $X \times Y$, by Recognition principle. ■

Definition 1.11. Let $\pi_1 : X \times Y \to X$ denote the projection onto the first component define by $\pi_1(x, y) = x$; and let $\pi_2 : X \times Y \to Y$ be the projection onto the second component defined by $\pi_2(x, y) = y$.

Observe that the preimage of $U \subset X$ under $\pi_1 : X \times Y \to X$ is $\pi_1^{-1}(U) U \times Y$; and the preimage of $V \subset Y$ under $\pi_2 : X \times Y \to Y$ is $\pi_2^{-1}(V) = X \times V$. Note the identity,

$$(U \times Y) \cap (X \times V) = U \times V. \tag{1.17}$$

Since each basis element B for the product topology on $X \times Y$ is of the form $U \times V$ with $U \subset X$ and $V \subset Y$ being open, the basis element $B = U \times V$ can be written as the intersection of $\pi_1^{-1}(U)$ and $\pi_2^{-1}(V)$. It follows that a basis element for the product topology can be written as intersection of subsets from the following collection $\mathscr{S}$:

$$\mathscr{S} = \{U \times Y \mid U \subset X \text{ open}\} \cup \{X \times V \mid V \subset Y \text{ open}\} \quad = \left\{\pi_1^{-1}(U) \mid U \subset X \text{ open}\right\} \cup \left\{\pi_2^{-1}(V) \mid V \subset Y \text{ open}\right\}$$

Theorem 1.13

The collection $\mathscr{S}$ as above is a subbasis for the product topology on $X \times Y$.

Proof. It is immediate that the collection $\mathscr{S}$ is a subbasis for a topology on $X \times Y$ as the union of all elements of $\mathscr{S}$ is equal to $X \times Y$. All that needs to be proved now is that the topology generated by this subbasis is equal to the product topology on $X \times Y$. Let $\mathcal{T}$ denote the product topology on $X \times Y$ and $\mathcal{T}'$ denote topology generated by $\mathscr{S}$. $\mathcal{T}'$ contains arbitrary unions of finite intersections of elements of $\mathscr{S}$. Since each element of $\mathscr{S}$ belongs to $\mathcal{T}$ as well, so do arbitrary unions of finite intersections of elements of $\mathscr{S}$ as $\mathcal{T}$ is a topology. Hence $\mathcal{T}' \subset \mathcal{T}$.

Conversely, since $U \times V$ is a generic basis element from $\mathscr{B}$ generating product topology on $X \times Y$ with U open in X and V open in Y, an arbitrary element W from $\mathcal{T}$ can be written as union of sets of the form $U \times V$. But

$$U \times V = \pi_1^{-1}(U) \cap \pi_2^{-1}(V),$$

with $\pi_1^{-1}(U), \pi_2^{-1}(V) \in \mathscr{S}$ and hence $\pi_1^{-1}(U) \cap \pi_2^{-1}(V) \in \mathcal{T}'$, the topology generated by the subbasis $\mathscr{S}$ so that an arbitrary union of sets of the form $\pi_1^{-1}(U) \cap \pi_2^{-1}(V)$ will also belong to $\mathcal{T}'$ leading to the fact that the arbitrary element $W \in \mathcal{T}$ also belongs to $\mathcal{T}'$. Hence, $\mathcal{T} \subset \mathcal{T}'$. Therefore, $\mathcal{T} = \mathcal{T}'$. ■

§1.6 Subspace Topology

Definition 1.12 (Subspace topology)**.** Let $(X, \mathcal{T})$ be a topological space. Ass, let $A \subset X$ be a subset. The collection

$$\mathcal{T}_A = \{A \cap U \mid U \in \mathcal{T}\} \tag{1.18}$$

of subsets of A is called the **subspace topology** on A. with this topology, $(A, \mathcal{T}_A)$ is called a subspace of $(X, \mathcal{T})$.

Lemma 1.14

The collection $\mathcal{T}_A$ as defined above is a topology on A.

Proof. Let us first note that $\varnothing$ can be written as $\varnothing = A \cap \varnothing$ with $\varnothing \in \mathcal{T}$ as $\mathcal{T}$ is a topology. Hence by definition (1.18), $\varnothing \in \mathcal{T}_A$. Also, notice that $A = A \cap X$ with $X \in \mathcal{T}$ again by the fact that $\mathcal{T}$ is a topology on X and $A \subseteq X$. Hence, by definition (1.18), $A \in \mathcal{T}_A$.

It remains to show that $\mathcal{T}_A$ is closed under arbitrary union and finite intersection. Let $\{A \cap U_\alpha\}_{\alpha \in J}$ be a subcollection of elements from $\mathcal{T}_A$ as defined in (1.18) associated with the subcollection $\{U_\alpha\}_{\alpha \in J}$ of open subsets from $\mathcal{T}$ indexed by J. Now, observe that

$$\bigcup_{\alpha \in J} (A \cap U_\alpha) = A \cap \left(\bigcup_{\alpha \in J} U_\alpha \right) \tag{1.19}$$

by distributive law in set theory. Hence, using the fact that $\bigcup_{\alpha \in J} U_\alpha \in \mathcal{T}$ holds, one deduces the fact that $\bigcup_{\alpha \in J} (A \cap U_\alpha) \in \mathcal{T}_A$. In other words, $\mathcal{T}_A$ is closed under arbitrary union of elements from it.

Now, let us choose a finite subcollection $\{A \cap U_1, \ldots, A \cap U_n\}$ of elements from $\mathcal{T}_A$ with $\{U_1, \ldots, U_n\} \subset \mathcal{T}$, being a finite subcollection of open subsets of X drawn from $\mathcal{T}$. Again, one observes that

$$\bigcap_{i=1}^{n} (A \cap U_i) = A \cap \left(\bigcap_{i=1}^{n} U_i \right), \tag{1.20}$$

with $\bigcap_{i=1}^{n} U_i \in \mathcal{T}$ as $\mathcal{T}$ a topology. Now, using definition (1.18), one concludes $\bigcap_{i=1}^{n} (A \cap U_i) \in \mathcal{T}_A$. In other words, $\mathcal{T}_A$ is closed under finite intersection of elements from it. ■

Remark 1.1. When $(A, \mathcal{T}_A)$ is a subspace of $(X, \mathcal{T})$, and $V \subset A \subset X$. one cant simply say that "V is open", as there may arise a potential ambiguity. One says "V is open in A" to indicate that $V \in \mathcal{T}_A$ while phrases "V is open in X" to imply that $V \in \mathcal{T}$. The former indicate that $V = A \cap U$ for some $U \in \mathcal{T}$.

Lemma 1.15

If $\mathscr{B}$ is a basis for topology $\mathcal{T}$ on X, and $A \subset X$, then the collection

$$\mathscr{B}_A = \{A \cap B \mid B \in \mathscr{B}\}$$

is a basis for the subspace topology $\mathcal{T}_A$ on A.

Proof. We apply Recognition principle for the collection $\mathscr{B}_A$ in the context of the topological space $(A, \mathcal{T}_A)$. Since each basis element $B \in \mathscr{B}$ is open in X, i.e. $B \in \mathcal{T}$, each subset $A \cap B \in \mathscr{B}_A$ is open in A, i.e. $A \cap B \in \mathcal{T}_A$. Additionally, given $x \in A \cap U \in \mathcal{T}_A$, with $U \in \mathcal{T}$, one has $x \in U \in \mathcal{T}$ so that ther exists $B \in \mathscr{B}$ with $x \in B \subset U$ as $\mathscr{B}$ is a basis for topology $\mathcal{T}$ on X. Also, since $x \in A \cap U$, $x \in A$ so that $x \in B \cap A \subset U \cap A$. Therefore, we have shown that given $A \cap U \in \mathcal{T}_A$ and $x \in A \cap U$, there exists $A \cap B \in \mathscr{B}_A$ such that $x \in A \cap B \subset A \cap U$. Therefore, the collection $\mathscr{B}_A$ meets the criteria as required by the Recognition principle to become a basis for the subspace topology $\mathscr{J}_A$ on A. ■

Example 1.10. Give $\mathbb{R}$ the standard Euclidean metric topology generated by the open intervals (a, b), and let $A = [0, 1)$. According to Lemma 1.15 above, the subspace topology $\mathcal{T}_A$ on A has a basis consisting of intersections $[0, 1) \cap (a, b)$. Observe that the basis consists of all intervals of the form $[0, b)$ and (a, b) with $0 < a < b \leq 1$.

Theorem 1.16

If A is a subspace of X and B is a subspace of Y, then the product topology on $A \times B$ is the same as the subspace topology on $A \times B$ as a subset of XXY.

Proof. The subspace topology $\mathcal{T}_A$ on A is given by the collection

$$\mathcal{T}_A = \{A \cap U \mid U \in \mathcal{T}\},$$

with $A \subset X$, a subset and $\mathcal{T}$ is the given topology on X. Similarly, the subspace toplogy $\mathcal{T}_B$ on a subset $B \subset Y$ is given by the collection

$$\mathcal{T}_B = \{B \cap V \mid V \in \mathcal{T}'\},$$

Where $\mathcal{T}'$ is the given topology on Y. Hence, by the definition of product topology, the product topology on $A \times B$ is generated by the basis

$$\mathscr{B}_{A\times B} = \{(A \cap U) \times (B \cap V) \mid U \in \mathcal{T},\ V \in \mathcal{T}'\}. \tag{1.21}$$

On the other hand, from the result provided in Lemma 1.15. the collection

$$\mathscr{B}'_{A\times B} = \{(A \times B) \cap (U \times V) \mid U \in \mathcal{T},\ V \in \mathcal{T}'\} \tag{1.22}$$

is a basis for the subspace topology $\mathcal{T}_{A\times B}$ on $A \times B$ as a subset of $X \times Y$. Note that here we use the fact that open subsets $U \times V$ of $X \times Y$, with U open in X and V open in Y, constitute a basis for the product topology on $X \times Y$. In view of the following set theoretic equality,

$$(A \cap U) \times (B \cap V) = (A \times B) \cap (U \times V) \tag{1.23}$$

one concludes that the 2 bases given, by (1.21) and (1.22) are actually the same, i.e. $\mathscr{B}_{A\times B} = \mathscr{B}'_{A\times B}$. Hence, the 2 topologies they generate are the same. ■

Definition 1.13. By an open subspace of X, we mean an open subset $A \subset X$ with the subspace topology.

Lemma 1.17

Let A be an open subspace of X. Then a subset $V \subset A$ is open if and only if it is open in X.

Proof. Suppose $V \subset A$ is open with respect to subspace topology on A, hence, $V = A \cap U$, for some U open in X. Since A is also open in X, $V = A \cap U$ must also be open in X. Now, let $V \subset A$ and V be open in X. Write $V = A \cap V$ with V open in X. From the definition of subspace topology, V is open in A with respect to subspace topology, as required. ■