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Differential Geometry II (MAT401)

Lecture Notes

Preface

This series of lecture notes has been prepared for aiding students who took the BRAC University course **Differential Geometry II (MAT401)** in Fall 2023 semester. These notes were typeset under the supervision of mathematician **Dr. Syed Hasibul Hassan Chowdhury**. The main goal of this typeset is to have an organized digital version of the notes, which is easier to share and handle. If you see any mistakes or typos, please send me an email at atonuroychowdhury@gmail.com

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References:

- *An Introduction to Manifolds*, by **Loring W. Tu**
- *An Introduction to Differentiable Manifolds and Riemannian Geometry*, by **William Boothby**
- *Introduction to Smooth Manifolds*, by **John M. Lee**
- *Lectures on Differential geometry*, by **S.S Chern, W.H. Chen and K.S. Lam**
- *Geometry of Differential Forms*, by **Shigeyuki Morita**
- *From Calculus to Cohomology: De Rham Cohomology and Characteristic Classes*, by **Ib Madsen and Jxrgen Tornehave**.

Contents

1 Review of Multilinear Algebra

§1.1 Dual Space

Let V and W be real vector spaces. We denote by $\mathrm{Hom}\,(V, W)$ the vector space of all linear maps $f : V \to W$. In particular, if we choose $W = \mathbb{R}$, we get the **dual space** V^*.

$$V^* = \mathrm{Hom}\,(V, \mathbb{R})\,.$$

The elements of V^* are called covectors on V. In the rest of the lecture, we will assume V to be a finite dimensional vector space. Let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be a basis for V. Then every $\mathbf{v} \in V$ is a unique linear combination

$$\mathbf{v} = \sum_{i=1}^{n} v^i \mathbf{e}_i, \tag{1.1}$$

with $v^i \in \mathbb{R}$. v^i's are called the coordinates of $\mathbf{v}$ relative to the basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$. Let $\widehat{\alpha}^i$ be the linear function on V that picks up the i-th coordinate of the vector, i.e.

$$\widehat{\alpha}^i\,(\mathbf{v}) = \widehat{\alpha}^i \left(\sum_{i=1}^{n} v^i \mathbf{e}_i \right) = v^i. \tag{1.2}$$

When $\mathbf{v}$ is one of the basis vectors,

$$\widehat{\alpha}^i\,(\mathbf{e}_j) = \delta^i{}_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{1.3}$$

Proposition 1.1

The functions $\widehat{\alpha}^1, \ldots, \widehat{\alpha}^n$ form a basis for V^*.

Proof. Suppose $f \in V^*$. Then for any $\mathbf{v} = \sum_{i=1}^{n} v^i \mathbf{e}_i \in V$,

$$f\,(\mathbf{v}) = f \left(\sum_{i=1}^{n} v^i \mathbf{e}_i \right) = \sum_{i=1}^{n} v^i f\,(\mathbf{e}_i) = \sum_{i=1}^{n} f\,(\mathbf{e}_i)\,\widehat{\alpha}^i\,(\mathbf{v})\,.$$

Since this holds for any $\mathbf{v} \in V$,

$$f = \sum_{i=1}^{n} f\,(\mathbf{e}_i)\,\widehat{\alpha}^i. \tag{1.4}$$

Therefore, $\widehat{\alpha}^1, \ldots, \widehat{\alpha}^n$ span V^*. As for linear independence, suppose

$$\sum_{i=1}^{n} c_i \widehat{\alpha}^i = \mathbf{0}, \tag{1.5}$$

where $\mathbf{0}$ is the function that takes all of V to $0 \in \mathbb{R}$. If we evaluate (1.5) at $\mathbf{e}_j$, we get

$$0 = \sum_{i=1}^{n} c_i \widehat{\alpha}^i\,(\mathbf{e}_j) = \sum_{i=1}^{n} c_i \delta^i{}_j = c_j. \tag{1.6}$$

So $c_j = 0$, and this holds for each $j = 1, 2, \ldots, n$. Therefore, $\{\widehat{\alpha}^1, \ldots, \widehat{\alpha}^n\}$ is a linearly independent set that spans V^*, i.e. a basis. ■

Corollary 1.2

The dual space V^* of a finite dimensional vector space has the same dimension as V.

The basis $\{\widehat{\alpha}^1, \ldots, \widehat{\alpha}^n\}$ for V^* is said to be dual to the basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ for V.

§1.2 Permutations

Fix a positive integer k. A permutation of the set $A = \{1, 2, \ldots, k\}$ is a bijection $\sigma : A \to A$. The product of two permutations τ and σ is the composition $\tau \circ \sigma : A \to A$. The **cyclic permutation** $(a_1\ a_2\ \cdots\ a_r)$ is the permutation σ such that

$$\sigma(a_1) = a_2,\ \sigma(a_2) = a_3,\ \cdots,\ \sigma(a_{r-1}) = a_r,\ \text{and } \sigma(a_r) = 1,$$

leaving all other elements unchanged, i.e. $\sigma(j) = j$ if j is not one of the a_i's. A cyclic permutation $(a_1\ a_2\ \cdots\ a_r)$ is also called a **cycle** of length r or an r-cycle. A **transposition** is a permutation of the form $(a\ b)$ that interchanges a and b, leaving all other elements of A fixed.

A permutation $\sigma : A \to A$ can be described by

$$\begin{bmatrix} 1 & 2 & \cdots & k \\ \sigma(1) & \sigma(2) & \cdots & \sigma(k) \end{bmatrix}.$$

We also write it as

$$[1\ 2\ \cdots\ k] \overset{\sigma}{\mapsto} [\sigma(1)\ \sigma(2)\ \cdots\ \sigma(k)].$$

Example 1.1. Suppose $\sigma : \{1, 2, 3, 4, 5\} \to \{1, 2, 3, 4, 5\}$ is the permutation given by

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 5 & 1 & 3 \end{bmatrix}.$$

In other words, $\sigma(1) = 2$, $\sigma(2) = 4$, $\sigma(3) = 5$, $\sigma(4) = 1$, and $\sigma(5) = 3$.

$$[1\ 2\ 3\ 4\ 5] \overset{\sigma}{\mapsto} [2\ 4\ 5\ 1\ 3].$$

Observe that the cyclic permutation $\sigma' = (1\ 2\ 4)$ acts as $\sigma'(1) = 2$, $\sigma'(2) = 4$ and $\sigma'(4) = 1$, keeping 3 and 5 unchanged, i.e. $\sigma'(3) = 3$ and $\sigma'(5) = 5$.

$$[1\ 2\ 3\ 4\ 5] \xmapsto{(1\ 2\ 4)} [2\ 4\ 3\ 1\ 5].$$

Now the transposition $\sigma'' = (3\ 5)$ acts as $\sigma''(3) = 5$ and $\sigma''(5) = 3$, keeping $1, 2, 4$ unchanged. Therefore,

$$\begin{tikzcd} {[1\ 2\ 3\ 4\ 5]} \arrow[r, mapsto, "(1\ 2\ 4)"] \arrow[rr, bend right, "(3\ 5)(1\ 2\ 4)"'] & {[2\ 4\ 3\ 1\ 5]} \arrow[r, mapsto, "(3\ 5)"] & {[2\ 4\ 5\ 1\ 3]} \end{tikzcd}$$

so that $\sigma = (3\ 5)(1\ 2\ 4)$.

Let S_k be the group of permutations of the set $\{1, 2, \ldots, k\}$. The order of this group is $k!$. A permutation is *even* or *odd* depending on whether it is the product of an even or an odd number of transpositions. The sign of a permutation σ is 1 if the permutation is even, and -1 otherwise. It is denoted by $\operatorname{sgn}\sigma$. For example, in Example 1.1, $\sigma = (3\ 5)(1\ 2\ 4)$. Note that we can write $(1\ 2\ 4)$ as a product of two transpositions:

$$\begin{tikzcd} {[1\ 2\ 3\ 4\ 5]} \arrow[r, mapsto, "(1\ 2)"] \arrow[rr, bend right, "(1\ 4)(1\ 2)=(1\ 2\ 4)"'] & {[2\ 1\ 3\ 4\ 5]} \arrow[r, mapsto, "(1\ 4)"] & {[2\ 4\ 3\ 1\ 5]} \end{tikzcd}$$

In other words, $\sigma = (3\ 5)(1\ 4)(1\ 2)$. Hence, $\operatorname{sgn}\sigma = -1$. One can easily check that

$$\operatorname{sgn}(\sigma\tau) = (\operatorname{sgn}\sigma)(\operatorname{sgn}\tau). \tag{1.7}$$

So $\operatorname{sgn} : S_k \to \{1, -1\}$ is a group homomorphism.

Example 1.2. Observe that the 5-cycle $(1\ 2\ 3\ 4\ 5)$ can be written as

$$(1\ 2\ 3\ 4\ 5) = (1\ 5)\,(1\ 4)\,(1\ 3)\,(1\ 2)\,.$$

Indeed,

$$\begin{tikzcd}
{[1\ 2\ 3\ 4\ 5]} \arrow[r, mapsto, "(1\ 2)"] \arrow[rrrr, bend right, "(1\ 2\ 3\ 4\ 5)=(1\ 5)(1\ 4)(1\ 3)(1\ 2)"'] & {[2\ 1\ 3\ 4\ 5]} \arrow[r, mapsto, "(1\ 3)"] & {[2\ 3\ 1\ 4\ 5]} \arrow[r, mapsto, "(1\ 4)"] & {[2\ 3\ 4\ 1\ 5]} \arrow[r, mapsto, "(1\ 5)"] & {[2\ 3\ 4\ 5\ 1]}
\end{tikzcd}$$

Therefore, $\operatorname{sgn}(1\ 2\ 3\ 4\ 5) = 1$.

An **inversion** in a permutation σ is an ordered pair $(\sigma(i), \sigma(j))$ such that $i < j$ but $\sigma(i) > \sigma(j)$. In Example 1.1, $\sigma(1) = 2$, $\sigma(2) = 4$, $\sigma(3) = 5$, $\sigma(4) = 1$, and $\sigma(5) = 3$. So, the inversions in this permutation are

$$(\sigma(1), \sigma(4))\,,\ (\sigma(2), \sigma(4))\,,\ (\sigma(2), \sigma(5))\,,\ (\sigma(3), \sigma(4))\,,\ (\sigma(3), \sigma(5))\,.$$

Hence, there are 5 inversions associated with the permutation σ. There is an efficient way of determining the sign of a permutation.

Proposition 1.3

A permutation is even if and only if it has an even number of inversions.

Proof. Let $\sigma \in S_k$ with n inversions. We shall prove that we can multiply σ by n transpositions and get the identity permutation. This will prove that $\operatorname{sgn}\sigma = (-1)^n$.

Suppose $\sigma(j_1) = 1$. Then for each $i < j_1$, $(\sigma(i), \sigma(j_1))$ is an inversion, and there are $j_1 - 1$ many of them. These are all the inversions with 1 in the second slot of the ordered pair of inversion. If we now multiply σ by the j_1-cycle

$$(\sigma(1)\ \ 1)\,(\sigma(2)\ \ 1) \cdots (\sigma(j_1 - 1)\ \ 1)$$

to the left of σ, the resulting permutation σ_1 would be

$$\begin{bmatrix} 1 & 2 & 3 & \cdots & j_1 & j_1 + 1 & \cdots & k \\ 1 & \sigma(1) & \sigma(2) & \cdots & \sigma(j_1 - 1) & \sigma(j_1 + 1) & \cdots & \sigma(k)\,. \end{bmatrix}$$

This permutation has no inversion with 1 in the second slot of the ordered pair of inversion. Suppose now that $\sigma(j_2) = 2$. Now observe that if $(\sigma_1(i), 2)$ is an inversion in σ_1, then either $(\sigma(i), 2)$ (if $i \geq j_1 + 1$) or $\sigma(i-1), 2$ (if $i \leq j_1 - 1$) is an inversion in σ. Therefore, the number of inversions in σ_1 ending in 2 is precisely the same as the number of inversions in σ ending in 2. So following a similar procedure as above, we can multiply σ_1 by i_2-many transpositions to the left (i_2 is the number of transpositions ending in 2) and get

$$\begin{bmatrix} 1 & 2 & 3 & \cdots & k \\ 1 & 2 & \sigma(1) & \cdots & \sigma(k)\,. \end{bmatrix}$$

We can continue these steps for each $j = 1, 2, \ldots, k$, and the number of transpositions required to move j to its natural position is the same as the number of inversions ending in j. In the end we achieve the identity permutation. Therefore, $\operatorname{sgn}\sigma = (-1)^n$, where n is the number of inversions. ■

§1.3 Multilinear Functions

Definition 1.1. Let V^k be the cartesian product of k-copies of a real vector space V.

$$V^k = \underbrace{V \times V \times \cdots \times V}_{k\text{-copies}}$$

A function $f : V^k \to \mathbb{R}$ is called k-linear if it is linear in each of its k arguments:

$$f(\ldots, a\mathbf{v} + b\mathbf{w}, \ldots) = a\, f(\ldots, \mathbf{v}, \ldots) + b\, f(\ldots, \mathbf{w}, \ldots), \tag{1.8}$$

for $a, b \in \mathbb{R}$ and $\mathbf{v}, \mathbf{w} \in V$.

Instead of 2-linear and 3-linear, it's customary to call "bilinear" and "trilinear", respectively. A k-linear function on V is called a k**-tensor** on V. We will denote the vector space of all k-tensors on V by $L_k(V)$. The vector addition and scalar multiplication of the real vector space $L_k(V)$ is the straightforward pointwise operation.

Example 1.3. The dot product $f(\mathbf{v}, \mathbf{w}) = \mathbf{v} \cdot \mathbf{w}$ on $\mathbb{R}^n$ is bilinear: if $\mathbf{v} = \sum_{i=1}^n v^i \mathbf{e}_i$ and $\mathbf{w} = \sum_{i=1}^n w^i \mathbf{e}_i$, then

$$f(\mathbf{v}, \mathbf{w}) = \mathbf{v} \cdot \mathbf{w} = \sum_{i=1}^n v^i w^i.$$

Example 1.4. The determinant

$$f(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n) = \det \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}$$

viewed as a function of the n column vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is n-linear.

Definition 1.2 (Symmetric and alternating function)**.** A k-linear function $f : V^k \to \mathbb{R}$ is **symmetric** if

$$f\left(\mathbf{v}_{\sigma(1)}, \mathbf{v}_{\sigma(2)}, \ldots, \mathbf{v}_{\sigma(k)}\right) = f(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k), \tag{1.9}$$

for all permutations $\sigma \in S_k$. It is **alternating** if

$$f\left(\mathbf{v}_{\sigma(1)}, \mathbf{v}_{\sigma(2)}, \ldots, \mathbf{v}_{\sigma(k)}\right) = (\operatorname{sgn} \sigma)\, f(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k), \tag{1.10}$$

for all permutations $\sigma \in S_k$.

The dor product function on $\mathbb{R}^n$ in Theorem 1.3 is symmetric, and the determinant function on $\mathbb{R}^n$ in Theorem 1.4 is alternating.

We are especially interested in the vector space $A_k(V)$ of all alternating k-linear functions on a vector space V, for $k > 0$. The elements of $A_k(V)$ are called alternating k-tensors (also known as k-covectors). We define $A_0(V)$ to be $\mathbb{R}$. The elements of $A_0(V)$ are simply constants, which we call 0-covectors. The elements of $A_1(V)$ are simply covectors, i.e. the elements of V^*.

Permutation action on k-linear functions

If $f \in L_k(V)$ and $\sigma \in S_k$, define $\sigma f \in L_k(V)$ as follows:

$$(\sigma f)(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k) = f\left(\mathbf{v}_{\sigma(1)}, \mathbf{v}_{\sigma(2)}, \ldots, \mathbf{v}_{\sigma(k)}\right). \tag{1.11}$$

Thus, f is symmetric if and only if $f = \sigma f$ for all $\sigma \in S_k$; and f is alternating if and only if $\sigma f = (\operatorname{sgn} \sigma) f$ for all $\sigma \in S_k$. When $k = 1$, S_k only has the identity permutation. In that case, a 1-linear function or simply linear function on V is both symmetric and alternating. In particular,

$$A_1(V) = L_1(V) = V^*.$$

Lemma 1.4

If $\sigma, \tau \in S_k$ and $f \in L_k(V)$, then $\tau(\sigma f) = (\tau\sigma) f$.

Proof. For any $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in V$,

$$\begin{aligned}
(\tau(\sigma f))(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k) &= (\sigma f)\left(\mathbf{v}_{\tau(1)}, \mathbf{v}_{\tau(2)}, \ldots, \mathbf{v}_{\tau(k)}\right) \\
&= (\sigma f)(\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k) && [\mathbf{w}_i = \mathbf{v}_{\tau(i)}] \\
&= f\left(\mathbf{w}_{\sigma(1)}, \mathbf{w}_{\sigma(2)}, \ldots, \mathbf{w}_{\sigma(k)}\right) \\
&= f\left(\mathbf{v}_{\tau(\sigma(1))}, \mathbf{v}_{\tau(\sigma(2))}, \ldots, \mathbf{v}_{\tau(\sigma(k))}\right) \\
&= ((\tau\sigma) f)(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k).
\end{aligned}$$

Therefore, $\tau(\sigma f) = (\tau\sigma) f$. ∎

Definition 1.3. If G is a group and X is a set, a map

$$\begin{aligned} G \times X &\to X \\ (g, x) &\mapsto g \cdot x \end{aligned}$$

is called a **left action** of G on X if

(i) $e \cdot x = x$, where e is the identity element in G and x is any element in X; and

(ii) $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$, for all $g_1, g_2 \in G$ and $x \in X$.

Similarly, a **right action** of G on X is a map

$$\begin{aligned} X \times G &\to X \\ (x, g) &\mapsto x \cdot g \end{aligned}$$

such that

(i) $x \cdot e = x$, for all $x \in X$; and

(ii) $(x \cdot g_1) \cdot g_2 = x \cdot (g_1 g_2)$, for all $g_1, g_2 \in G$ and $x \in X$.

Symmetrizing and alternating operators

Given $f \in L_k(V)$, there is a way to make it a symmetric k-linear function $\mathcal{S} f$ from it:

$$(\mathcal{S} f)(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k) = \sum_{\sigma \in S_k} f\left(\mathbf{v}_{\sigma(1)}, \mathbf{v}_{\sigma(2)}, \ldots, \mathbf{v}_{\sigma(k)}\right). \tag{1.12}$$

In other words,

$$\mathcal{S} f = \sum_{\sigma \in S_k} \sigma f. \tag{1.13}$$

Similarly, there is a way to make an alternating k-linear function from f:

$$\mathcal{A} f = \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma)\, \sigma f. \tag{1.14}$$

Proposition 1.5 (i) The k-linear function $\mathcal{S} f$ is symmetric.

(ii) The k-linear function $\mathcal{A} f$ is alternating.

Proof. (i) Let $\tau \in S_k$. Then

$$\tau(\mathcal{S}f) = \tau\left(\sum_{\sigma \in S_k} \sigma f\right). \tag{1.15}$$

The group action of S_k on $L_k(V)$ is distributive over the vector space addition. Therefore,

$$\tau(\mathcal{S}f) = \sum_{\sigma \in S_k} \tau(\sigma f) = \sum_{\sigma \in S_k} (\tau\sigma) f. \tag{1.16}$$

As σ varies over S_k, $\tau\sigma$ also varies over S_k. Therefore, $\sum_{\sigma \in S_k}(\tau\sigma)f = \mathcal{S}f$. In other words,

$$\tau(\mathcal{S}f) = \mathcal{S}f, \tag{1.17}$$

i.e. $\mathcal{S}f$ is symmetric.

(ii) Let $\tau \in S_k$. Then

$$\tau(\mathcal{A}f) = \tau\left(\sum_{\sigma \in S_k} (\operatorname{sgn}\sigma)\,\sigma f\right) = \sum_{\sigma \in S_k} (\operatorname{sgn}\sigma)\,\tau(\sigma f) = \sum_{\sigma \in S_k} (\operatorname{sgn}\sigma)\,(\tau\sigma) f. \tag{1.18}$$

Since $(\operatorname{sgn}\tau)^2 = 1$,

$$\begin{aligned}\tau(\mathcal{A}f) &= \sum_{\sigma \in S_k} (\operatorname{sgn}\tau)^2 (\operatorname{sgn}\sigma)(\tau\sigma) f \\ &= (\operatorname{sgn}\tau) \sum_{\sigma \in S_k} (\operatorname{sgn}\tau)(\operatorname{sgn}\sigma)(\tau\sigma) f \\ &= (\operatorname{sgn}\tau) \sum_{\sigma \in S_k} (\operatorname{sgn}(\tau\sigma))(\tau\sigma) f. \end{aligned} \tag{1.19}$$

As σ varies over S_k, $\tau\sigma$ also varies over S_k. Therefore, $\sum_{\sigma \in S_k}(\operatorname{sgn}(\tau\sigma))(\tau\sigma)f = \mathcal{A}f$. In other words,

$$\tau(\mathcal{A}f) = \mathcal{A}f, \tag{1.20}$$

i.e. $\mathcal{A}f$ is alternating.

■

Lemma 1.6

If $f \in A_k(V)$, then $\mathcal{A}f = (k!)\,f$.

Proof. Since f is alternating,

$$\mathcal{A}f = \sum_{\sigma \in S_k} (\operatorname{sgn}\sigma)\,\sigma f = \sum_{\sigma \in S_k} (\operatorname{sgn}\sigma)^2 f = \sum_{\sigma \in S_k} f = (k!)\,f, \tag{1.21}$$

because the order of S_k is $k!$. ■

§1.4 Tensor Product and Wedge Product

Definition 1.4 (Tensor Product)**.** Let f be a k-linear function and g an l-linear function on a vector space V. Their tensor product $f \otimes g$ is the $(k+l)$-linear function defined by

$$(f \otimes g)(\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{v}_{k+1}, \dots, \mathbf{v}_{k+l}) = f(\mathbf{v}_1, \dots, \mathbf{v}_k)\, g(\mathbf{v}_{k+1}, \dots, \mathbf{v}_{k+l}). \tag{1.22}$$

$(k+l)$-linearity of $f \otimes g$ follows from k-linearity of f and l-linearity of g.

Lemma 1.7 (Associativity of Tensor Product)
Let $f \in L_k(V)$, $g \in L_l(V)$ and $h \in L_m(V)$. Then

$$(f \otimes g) \otimes h = f \otimes (g \otimes h).$$

Proof. For $\mathbf{v}_1, \ldots, \mathbf{v}_{k+l+m}$,

$$\begin{aligned} [(f \otimes g) \otimes h](\mathbf{v}_1, \ldots, \mathbf{v}_{k+l+m}) &= (f \otimes g)(\mathbf{v}_1, \ldots, \mathbf{v}_{k+l})\, h(\mathbf{v}_{k+l+1}, \ldots, \mathbf{v}_{k+l+m}) \\ &= f(\mathbf{v}_1, \ldots, \mathbf{v}_k)\, g(\mathbf{v}_{k+1}, \ldots, \mathbf{v}_{k+l})\, h(\mathbf{v}_{k+l+1}, \ldots, \mathbf{v}_{k+l+m}). \end{aligned} \tag{1.23}$$

$$\begin{aligned} [f \otimes (g \otimes h)](\mathbf{v}_1, \ldots, \mathbf{v}_{k+l+m}) &= f(\mathbf{v}_1, \ldots, \mathbf{v}_k)\,(g \otimes h)(\mathbf{v}_{k+1}, \ldots, \mathbf{v}_{k+l+m}) \\ &= f(\mathbf{v}_1, \ldots, \mathbf{v}_k)\, g(\mathbf{v}_{k+1}, \ldots, \mathbf{v}_{k+l})\, h(\mathbf{v}_{k+l+1}, \ldots, \mathbf{v}_{k+l+m}). \end{aligned} \tag{1.24}$$

Therefore, $(f \otimes g) \otimes h = f \otimes (g \otimes h)$, i.e. tensor product is associative. ■

Example 1.5. Let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be the standard basis for $\mathbb{R}^n$, and $\{\widehat{\alpha}^1, \ldots, \widehat{\alpha}^n\}$ its dual basis. The Euclidean inner product on $\mathbb{R}^n$ is the bilinear function

$$\langle\ ,\ \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$

defined by

$$\langle\ ,\ \rangle(\mathbf{v}, \mathbf{w}) = \langle \mathbf{v}, \mathbf{w} \rangle = \sum_{i=1}^{n} v^i w^i,$$

for $\mathbf{v} = \sum_{i=1}^n v^i \mathbf{e}_i$ and $\mathbf{w} = \sum_{i=1}^n w^i \mathbf{e}_i$. We can express $\langle\ ,\ \rangle$ in terms of tensor product as follows:

$$\langle\ ,\ \rangle(\mathbf{v}, \mathbf{w}) = \sum_{i=1}^{n} v^i w^i = \sum_{i=1}^{n} \widehat{\alpha}^i(\mathbf{v})\, \widehat{\alpha}^i(\mathbf{w}) = \sum_{i=1}^{n} \left(\widehat{\alpha}^i \otimes \widehat{\alpha}^i\right)(\mathbf{v}, \mathbf{w}).$$

Since $\mathbf{v}, \mathbf{w}$ are arbitrary,

$$\langle\ ,\ \rangle = \sum_{i=1}^{n} \left(\widehat{\alpha}^i \otimes \widehat{\alpha}^i\right). \tag{1.25}$$

If $f \in A_k(V)$ and $g \in A_l(V)$, then it's not true that $f \otimes g \in A_{k+l}(V)$, in general. We need to construct a product that is also alternating.

Definition 1.5 (Wedge Product). For $f \in A_k(V)$ and $g \in A_l(V)$, the wedge product of f and g is defined as follows:

$$f \wedge g = \frac{1}{k!l!} \mathcal{A}(f \otimes g). \tag{1.26}$$

Explicitly,

$$\begin{aligned} (f \wedge g)(\mathbf{v}_1, \ldots, \mathbf{v}_{k+l}) &= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma)\, \sigma(f \otimes g)(\mathbf{v}_1, \ldots, \mathbf{v}_{k+l}) \\ &= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma)\, (f \otimes g)\left(\mathbf{v}_{\sigma(1)}, \ldots, \mathbf{v}_{\sigma(k+l)}\right) \\ &= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma)\, f\left(\mathbf{v}_{\sigma(1)}, \ldots, \mathbf{v}_{\sigma(k)}\right) g\left(\mathbf{v}_{\sigma(k+1)}, \ldots, \mathbf{v}_{\sigma(k+l)}\right). \end{aligned} \tag{1.27}$$

When $k=0$, the element $f \in A_0(V)$ is simply a constant $c \in \mathbb{R}$ as discussed earlier. In this case, the wedge product $c \wedge g$ is just scalar multiplication as is evident from (1.27).

$$\begin{aligned}
(c \wedge g)(\mathbf{v}_1, \ldots, \mathbf{v}_l) &= \frac{1}{l!} \sum_{\sigma \in S_l} (\operatorname{sgn} \sigma)\, c\, g\left(\mathbf{v}_{\sigma(1)}, \ldots, \mathbf{v}_{\sigma(l)}\right) \\
&= \frac{1}{l!} \sum_{\sigma \in S_l} (\operatorname{sgn} \sigma)\, c\, (\operatorname{sgn} \sigma)\, g(\mathbf{v}_1, \ldots, \mathbf{v}_l) \\
&= \frac{1}{l!} \sum_{\sigma \in S_l} c\, g(\mathbf{v}_1, \ldots, \mathbf{v}_l) \\
&= \frac{1}{l!} l! c\, g(\mathbf{v}_1, \ldots, \mathbf{v}_l) \\
&= c\, g(\mathbf{v}_1, \ldots, \mathbf{v}_l)\,.
\end{aligned}$$

Thus $c \wedge g = cg$, for $c \in \mathbb{R}$ and $g \in A_l(V)$.

Example 1.6. For $f \in A_2(V)$ and $g \in A_1(V)$,

$$\begin{aligned}
\mathcal{A}(f \otimes g)(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = {} & f(\mathbf{v}_1, \mathbf{v}_2)\, g(\mathbf{v}_3) - f(\mathbf{v}_1, \mathbf{v}_3)\, g(\mathbf{v}_2) - f(\mathbf{v}_2, \mathbf{v}_1)\, g(\mathbf{v}_3) \\
& - f(\mathbf{v}_3, \mathbf{v}_2)\, g(\mathbf{v}_1) + f(\mathbf{v}_2, \mathbf{v}_3)\, g(\mathbf{v}_1) + f(\mathbf{v}_3, \mathbf{v}_1)\, g(\mathbf{v}_2)\,.
\end{aligned}$$

Among these 6 terms, there are 3 pairs of equal terms due to the alternating nature of f.

$$\begin{aligned}
f(\mathbf{v}_1, \mathbf{v}_2)\, g(\mathbf{v}_3) &= -f(\mathbf{v}_2, \mathbf{v}_1)\, g(\mathbf{v}_3)\,, \\
f(\mathbf{v}_3, \mathbf{v}_1)\, g(\mathbf{v}_2) &= -f(\mathbf{v}_1, \mathbf{v}_3)\, g(\mathbf{v}_2)\,, \\
f(\mathbf{v}_2, \mathbf{v}_3)\, g(\mathbf{v}_1) &= -f(\mathbf{v}_3, \mathbf{v}_2)\, g(\mathbf{v}_1)\,.
\end{aligned}$$

Therefore,

$$\mathcal{A}(f \otimes g)(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = 2f(\mathbf{v}_1, \mathbf{v}_2)\, g(\mathbf{v}_3) + 2f(\mathbf{v}_3, \mathbf{v}_1)\, g(\mathbf{v}_2) + 2f(\mathbf{v}_2, \mathbf{v}_3)\, g(\mathbf{v}_1)\,. \tag{1.28}$$

Hence,

$$\begin{aligned}
(f \wedge g)(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) &= \frac{1}{2!1!} \mathcal{A}(f \otimes g)(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \\
&= f(\mathbf{v}_1, \mathbf{v}_2)\, g(\mathbf{v}_3) + f(\mathbf{v}_3, \mathbf{v}_1)\, g(\mathbf{v}_2) + f(\mathbf{v}_2, \mathbf{v}_3)\, g(\mathbf{v}_1)\,. \qquad (1.29)
\end{aligned}$$

Example 1.7 (Wedge product of 2 covectors)**.** If $f, g \in A_1(V)$, and $\mathbf{v}_1, \mathbf{v}_2 \in V$, then

$$(f \wedge g)(\mathbf{v}_1, \mathbf{v}_2) = \frac{1}{1!1!} \mathcal{A}(f \otimes g)(\mathbf{v}_1, \mathbf{v}_2)\,.$$

S_2 has 2 elements: the identity element e and $(1\ 2)$. Therefore,

$$(f \wedge g)(\mathbf{v}_1, \mathbf{v}_2) = \mathcal{A}(f \otimes g)(\mathbf{v}_1, \mathbf{v}_2) = f(\mathbf{v}_1)\, g(\mathbf{v}_2) - f(\mathbf{v}_2)\, g(\mathbf{v}_1)\,.$$

Proposition 1.8 (Anticommutativity of wedge product)

The wedge product is anticommutative: if $f \in A_k(V)$ and $g \in A_l(V)$, then

$$f \wedge g = (-1)^{kl}\, g \wedge f.$$

Proof. Define $\tau \in S_{k+l}$ to be the following permutation:

$$\begin{bmatrix} 1 & 2 & \cdots & l & l+1 & l+2 & \cdots & l+k \\ k+1 & k+2 & \cdots & k+l & 1 & 2 & \cdots & k \end{bmatrix}.$$

In other words,

$$\tau(i) = \begin{cases} k+i & \text{if } 1 \le i \le l, \\ i-l & \text{if } l+1 \le i \le l+k. \end{cases}$$

Then for any $\sigma \in S_{k+l}$,

$$\sigma(j) = \begin{cases} \sigma(\tau(l+j)) & \text{if } 1 \le j \le k, \\ \sigma(\tau(j-k)) & \text{if } k+1 \le j \le k+l. \end{cases} \tag{1.30}$$

Now, for any $\mathbf{v}_1, \ldots, \mathbf{v}_{k+l} \in V$,

$$\begin{aligned} \mathcal{A}(f \otimes g)(\mathbf{v}_1, \ldots, \mathbf{v}_{k+l}) &= \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma) f\left(\mathbf{v}_{\sigma(1)}, \ldots, \mathbf{v}_{\sigma(k)}\right) g\left(\mathbf{v}_{\sigma(k+1)}, \ldots, \mathbf{v}_{\sigma(k+l)}\right) \\ &= \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma) f\left(\mathbf{v}_{\sigma(\tau(l+1))}, \ldots, \mathbf{v}_{\sigma(\tau(l+k))}\right) g\left(\mathbf{v}_{\sigma(\tau(1))}, \ldots, \mathbf{v}_{\sigma(\tau(l))}\right) \\ &= (\operatorname{sgn} \tau) \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma)(\operatorname{sgn} \tau)\, g\left(\mathbf{v}_{\sigma(\tau(1))}, \ldots, \mathbf{v}_{\sigma(\tau(l))}\right) f\left(\mathbf{v}_{\sigma(\tau(l+1))}, \ldots, \mathbf{v}_{\sigma(\tau(l+k))}\right) \\ &= (\operatorname{sgn} \tau) \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma\tau)\, g\left(\mathbf{v}_{\sigma(\tau(1))}, \ldots, \mathbf{v}_{\sigma(\tau(l))}\right) f\left(\mathbf{v}_{\sigma(\tau(l+1))}, \ldots, \mathbf{v}_{\sigma(\tau(l+k))}\right). \end{aligned}$$

Again, as σ varies over S_{k+l}, $\sigma\tau$ also varies over S_{k+l}. Therefore,

$$\mathcal{A}(f \otimes g)(\mathbf{v}_1, \ldots, \mathbf{v}_{k+l}) = (\operatorname{sgn} \tau)\, \mathcal{A}(g \otimes f)(\mathbf{v}_1, \ldots, \mathbf{v}_{k+l}). \tag{1.31}$$

Now, let us evaluate the sign of the permutation τ. Let $(\tau(i), \tau(j))$ be an inversion of τ. Then it's not possible that $1 \le i < j \le l$, or $l+1 \le i < j \le l+k$; because if we have $1 \le i < j \le l$ or $l+1 \le i < j \le l+k$, then $\tau(i) < \tau(j)$. Therefore, i must be in between 1 and l (inclusive), and j must be in between $l+1$ and $l+k$ (inclusive). So there are l options for i, and k options for j. Therefore, τ has kl many inversions. So $\operatorname{sgn} \tau = (-1)^{kl}$. Using (1.31),

$$\mathcal{A}(f \otimes g) = (-1)^{kl} \mathcal{A}(g \otimes f). \tag{1.32}$$

Dividing by $k!l!$, we obtain

$$f \wedge g = (-1)^{kl} g \wedge f. \tag{1.33}$$

■

Corollary 1.9

If f is a k-covector on V, i.e. $f \in A_k(V)$, and k is odd, then $f \wedge f = 0$.

Proof. By anticommutativity of wedge product,

$$f \wedge f = (-1)^{k^2} f \wedge f = -f \wedge f.$$

Therefore, $f \wedge f = 0$. ■

If f is a k-covector and g is an l-covector, i.e. $f \in A_k(V)$ and $g \in A_l(V)$, then we have defined their wedge product to be the $(k+l)$-covector

$$f \wedge g = \frac{1}{k!l!} \mathcal{A}(f \otimes g). \tag{1.34}$$

We have the following lemmas associated with the alternating operator $\mathcal{A}$.

Lemma 1.10

Suppose $f \in L_k(V)$ and $g \in L_l(V)$. Then

(i) $\mathcal{A}(\mathcal{A}(f) \otimes g) = k!\mathcal{A}(f \otimes g)$.

(ii) $\mathcal{A}(f \otimes \mathcal{A}(g)) = l!\mathcal{A}(f \otimes g)$.

Proof. (i) By definition,

$$\begin{aligned}\mathcal{A}(\mathcal{A}(f) \otimes g) &= \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma)\, \sigma (\mathcal{A}(f) \otimes g) \\ &= \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma)\, \sigma \left[\sum_{\tau \in S_k} (\operatorname{sgn} \tau) (\tau f) \otimes g \right]. \end{aligned} \tag{1.35}$$

We can view $\tau \in S_k$ as a permutation in the following way: define $\tau' \in S_{k+l}$ as follows

$$\tau'(i) = \begin{cases} \tau(i) & \text{if } i \leq k, \\ i & \text{if } i > k. \end{cases} \tag{1.36}$$

Then for $\mathbf{v}_1, \ldots, \mathbf{v}_{k+l}$, we have

$$\begin{aligned}[(\tau f) \otimes g](\mathbf{v}_1, \ldots, \mathbf{v}_{k+l}) &= (\tau f)(\mathbf{v}_1, \ldots, \mathbf{v}_k)\, g(\mathbf{v}_{k+1}, \ldots, \mathbf{v}_{k+l}) \\ &= f\left(\mathbf{v}_{\tau(1)}, \ldots, \mathbf{v}_{\tau(k)}\right) g(\mathbf{v}_{k+1}, \ldots, \mathbf{v}_{k+l}) \\ &= f\left(\mathbf{v}_{\tau'(1)}, \ldots, \mathbf{v}_{\tau'(k)}\right) g\left(\mathbf{v}_{\tau'(k+1)}, \ldots, \mathbf{v}_{\tau'(k+l)}\right) \\ &= [\tau'(f \otimes g)](\mathbf{v}_1, \ldots, \mathbf{v}_{k+l}).\end{aligned}$$

Therefore, $(\tau f) \otimes g = \tau'(f \otimes g)$. Furthermore, $\operatorname{sgn} \tau = \operatorname{sgn} \tau'$ since the inversions $(\tau'(i), \tau'(j))$ occur only when $1 \leq i < j \leq k$, so that the τ and τ' has the same number of inversions.

Let us abuse notation a bit and denote by S_k the subgroup of permutations in S_{k+l} by keeping the last l arguments fixed. This subgroup of S_{k+l} is indeed isomorphic to S_k, so we will denote both these groups by S_k. Therefore, from (1.35),

$$\begin{aligned}\mathcal{A}(\mathcal{A}(f) \otimes g) &= \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma)\, \sigma \left[\sum_{\tau' \in S_k \subseteq S_{k+l}} (\operatorname{sgn} \tau')\, \tau'(f \otimes g) \right] \\ &= \sum_{\sigma \in S_{k+l}} \sum_{\tau' \in S_k \subseteq S_{k+l}} (\operatorname{sgn} \sigma)(\operatorname{sgn} \tau')\, \sigma\tau'(f \otimes g) \\ &= \sum_{\tau' \in S_k \subseteq S_{k+l}} \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma \operatorname{sgn} \tau')\left((\sigma\tau')(f \otimes g)\right).\end{aligned}$$

For a fixed τ', as σ varies over S_{k+l}, $\sigma\tau'$ also varies over S_{k+l}. Therefore,

$$\mathcal{A}(\mathcal{A}(f) \otimes g) = \sum_{\tau' \in S_k \subseteq S_{k+l}} \mathcal{A}(f \otimes g) = k!\mathcal{A}(f \otimes g). \tag{1.37}$$

(ii) By (1.32),

$$\begin{aligned}\mathcal{A}(f \otimes \mathcal{A}(g)) &= \mathcal{A}\left((-1)^{kl} \mathcal{A}(g) \otimes f\right) \\ &= (-1)^{kl} \mathcal{A}(\mathcal{A}(g) \otimes f) \\ &= (-1)^{kl}\, l!\mathcal{A}(g \otimes f) \\ &= l!\mathcal{A}\left((-1)^{kl} g \otimes f\right) \\ &= l!\mathcal{A}(f \otimes g).\end{aligned} \tag{1.38}$$

■

Proposition 1.11 (Associativity of wedge product)

Let V be a real vector space and f, g, h be alternating multilinear functions on V of degree k, l, m, respectively. Then

$$(f \wedge g) \wedge h = f \wedge (g \wedge h) .$$

Proof. Using the definition of wedge product,

$$\begin{aligned}
(f \wedge g) \wedge h &= \frac{1}{(k+l)!m!} \mathcal{A} \left[(f \wedge g) \otimes h \right] \\
&= \frac{1}{(k+l)!m!} \mathcal{A} \left[\frac{1}{k!l!} \mathcal{A} \left(f \otimes g \right) \otimes h \right] \\
&= \frac{1}{(k+l)!k!l!m!} \mathcal{A} \left[\mathcal{A} \left(f \otimes g \right) \otimes h \right] \\
&= \frac{(k+l)!}{(k+l)!k!l!m!} \mathcal{A} \left[(f \otimes g) \otimes h \right] \\
&= \frac{1}{k!l!m!} \mathcal{A} \left[(f \otimes g) \otimes h \right] .
\end{aligned}$$

On the other hand,

$$\begin{aligned}
f \wedge (g \wedge h) &= \frac{1}{k!\,(l+m)!} \mathcal{A} \left[f \otimes (g \wedge h) \right] \\
&= \frac{1}{k!\,(l+m)!} \mathcal{A} \left[f \otimes \left(\frac{1}{l!m!} \mathcal{A} \left(g \otimes h \right) \right) \right] \\
&= \frac{1}{k!\,(l+m)!l!m!} \mathcal{A} \left[f \otimes \mathcal{A} \left(g \otimes h \right) \right] \\
&= \frac{(l+m)!}{k!\,(l+m)!l!m!} \mathcal{A} \left[f \otimes (g \otimes h) \right] \\
&= \frac{1}{k!l!m!} \mathcal{A} \left[f \otimes (g \otimes h) \right] .
\end{aligned}$$

Since tensor product is associative (by Lemma 1.7), we conclude that

$$(f \wedge g) \wedge h = f \wedge (g \wedge h) . \tag{1.39}$$

■

By associativity, we can omit the parenthesis and write univocally $f \wedge g \wedge h$ instead of $(f \wedge g) \wedge h$ or $f \wedge (g \wedge h)$.

Corollary 1.12

Under the hypothesis of Proposition 1.11,

$$f \wedge g \wedge h = \frac{1}{k!l!m!} \mathcal{A} \left[f \otimes g \otimes h \right] . \tag{1.40}$$

This easily generalizes to an arbitrary number of factors: if $f_i \in A_{d_i}(V)$ for $i = 1, 2, \ldots, r$, i.e. f_i is an alternating d_i-linear function on V, then

$$f_1 \wedge \cdots \wedge f_r = \frac{1}{d_1! \cdots d_r!} \mathcal{A} \left(f_1 \otimes \cdots \otimes f_r \right) . \tag{1.41}$$

Proposition 1.13

Let $\widehat{\alpha}^1, \widehat{\alpha}^2, \ldots, \widehat{\alpha}^k$ be linear functions on a real vector space V (i.e. $\widehat{\alpha}^i : V \to \mathbb{R}$) and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in V$. Then

$$\begin{aligned}\left(\widehat{\alpha}^1 \wedge \cdots \wedge \widehat{\alpha}^k\right)(\mathbf{v}_1, \ldots, \mathbf{v}_k) &= \det\left[\widehat{\alpha}^i(\mathbf{v}_j)\right] \\ &= \det\begin{bmatrix} \widehat{\alpha}^1(\mathbf{v}_1) & \widehat{\alpha}^1(\mathbf{v}_2) & \cdots & \widehat{\alpha}^1(\mathbf{v}_k) \\ \widehat{\alpha}^2(\mathbf{v}_1) & \widehat{\alpha}^2(\mathbf{v}_2) & \cdots & \widehat{\alpha}^2(\mathbf{v}_k) \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{\alpha}^k(\mathbf{v}_1) & \widehat{\alpha}^k(\mathbf{v}_2) & \cdots & \widehat{\alpha}^k(\mathbf{v}_k) \end{bmatrix}.\end{aligned}$$

Proof. By 1.41,

$$\left(\widehat{\alpha}^1 \wedge \cdots \wedge \widehat{\alpha}^k\right)(\mathbf{v}_1, \ldots, \mathbf{v}_k) = \mathcal{A}\left(\widehat{\alpha}^1 \otimes \cdots \otimes \widehat{\alpha}^k\right)(\mathbf{v}_1, \ldots, \mathbf{v}_k).$$

By the definition of the action of alternating operator,

$$\mathcal{A}\left(\widehat{\alpha}^1 \otimes \cdots \otimes \widehat{\alpha}^k\right)(\mathbf{v}_1, \ldots, \mathbf{v}_k) = \sum_{\sigma \in S_k} (\operatorname{sgn}\sigma)\, \widehat{\alpha}^1\left(\mathbf{v}_{\sigma(1)}\right) \cdots \widehat{\alpha}^k\left(\mathbf{v}_{\sigma(k)}\right). \tag{1.42}$$

By the definition of determinant of a $k \times k$ matrix $A = [a_{ij}]$,

$$\det A = \sum_{\sigma \in S_k} (\operatorname{sgn}\sigma)\, a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{k\sigma(k)}. \tag{1.43}$$

Using (1.43) in (1.42), we get

$$\mathcal{A}\left(\widehat{\alpha}^1 \otimes \cdots \otimes \widehat{\alpha}^k\right)(\mathbf{v}_1, \ldots, \mathbf{v}_k) = \det\left[\widehat{\alpha}^i(\mathbf{v}_j)\right]. \tag{1.44}$$

■

§1.5 A Basis for $A_k(V)$

Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be a basis for a real vector space V, and let $\{\widehat{\alpha}^1, \ldots, \widehat{\alpha}^n\}$ be the dual basis for V^*. Introduce the multi-index notation

$$I = (i_1, i_2, \ldots, i_k)$$

and write $\mathbf{e}_I$ for $(\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \ldots, \mathbf{e}_{i_k})$ and $\widehat{\alpha}^I$ for $\widehat{\alpha}^{i_1} \wedge \widehat{\alpha}^{i_2} \wedge \cdots \wedge \widehat{\alpha}^{i_k}$.

A k-linear function f on V is completely determined by its values on all k-tuples $(\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \ldots, \mathbf{e}_{i_k})$. If f is alternating, then f is completely determined by its values on all k-tuples $(\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \ldots, \mathbf{e}_{i_k})$ with

$$1 \le i_1 < i_2 < \cdots < i_k \le n.$$

In other words, it's sufficient to consider $\mathbf{e}_I$ with I in ascending order.

Lemma 1.14

Suppose I and J are ascending multi-indices of length k. Then

$$\widehat{\alpha}^I(\mathbf{e}_J) = \delta^I{}_J := \begin{cases} 1 & \text{if } I = J, \\ 0 & \text{if } I \neq J. \end{cases}$$

Proof. Suppose $I = (i_1, \ldots, i_k)$ and $J = (j_1, \ldots, j_k)$. Using (1.42), we get

$$\begin{aligned}\widehat{\alpha}^I(\mathbf{e}_J) &= \left(\widehat{\alpha}^{i_1} \wedge \widehat{\alpha}^{i_2} \wedge \cdots \wedge \widehat{\alpha}^{i_k}\right)(\mathbf{e}_{j_1}, \mathbf{e}_{j_2}, \ldots, \mathbf{e}_{j_k}) \\ &= \sum_{\sigma \in S_k} (\operatorname{sgn}\sigma)\, \widehat{\alpha}^{i_1}\left(\mathbf{e}_{j_{\sigma(1)}}\right) \cdots \widehat{\alpha}^{i_k}\left(\mathbf{e}_{j_{\sigma(k)}}\right) \\ &= \sum_{\sigma \in S_k} (\operatorname{sgn}\sigma)\, \delta^{i_1}{}_{j_{\sigma(1)}} \cdots \delta^{i_k}{}_{j_{\sigma(k)}}.\end{aligned} \tag{1.45}$$

The terms in the sum (1.45) contribute $\operatorname{sgn}\sigma$ if and only if

$$(i_1, i_2, \ldots, i_k) = \left(j_{\sigma(1)}, j_{\sigma(2)}, \ldots, j_{\sigma(k)}\right);$$

otherwise they contribute 0 to the sum. Both I and J are ascending multi-indices. Permuting the elements of J no longer gives an ascending multi-index (unless the permutation σ is the identity permutation). Therefore, in (1.45), all the summands corresponding to σ being a non-identity permutation contribute 0.

$$\widehat{\alpha}^I(\mathbf{e}_J) = \sum_{\sigma\in S_k} (\operatorname{sgn}\sigma)\, \delta^{i_1}{}_{j_{\sigma(1)}} \cdots \delta^{i_k}{}_{j_{\sigma(k)}} = \delta^{i_1}{}_{j_1} \cdots \delta^{i_k}{}_{j_k} = \begin{cases} 1 & \text{if } I = J, \\ 0 & \text{if } I \neq J. \end{cases} \tag{1.46}$$

■

Proposition 1.15

The alternating k-linear functions $\widehat{\alpha}^I$, $I = (i_1, \ldots, i_k)$, with $1 \le i_1 < \cdots < i_k \le n$ form a basis for the space $A_k(V)$ of alternating k-linear functions on V.

Proof. Let us first show linear independence. Suppose

$$\sum_I c_I \widehat{\alpha}^I = \mathbf{0}, \tag{1.47}$$

$c_I \in \mathbb{R}$ with I running over ascending multi-indices of length k. Applying $\mathbf{e}_J$ to both sides, we get

$$0 = \sum_I c_I \widehat{\alpha}^I(\mathbf{e}_J) = \sum_I c_I \delta^I{}_J = c_J. \tag{1.48}$$

Therefore, $\left\{\widehat{\alpha}^I \mid I \text{ is ascending multi-index of length } k\right\}$ is a linearly independent set. Now let us prove that this set spans $A_k(V)$. Let $f \in A_k(V)$. We claim that

$$f = \sum_I f(\mathbf{e}_I)\, \widehat{\alpha}^I. \tag{1.49}$$

Let $g = \sum_I f(\mathbf{e}_I)\, \widehat{\alpha}^I$. We need to prove that $f = g$. By k-linearity and alternating property, if two k-covectors agree on all $\mathbf{e}_J$ where J is an ascending multi-index, then they are equal. Now,

$$g(\mathbf{e}_J) = \sum_I f(\mathbf{e}_I)\, \widehat{\alpha}^I(\mathbf{e}_J) = \sum_I f(\mathbf{e}_I)\, \delta^I{}_J = f(\mathbf{e}_J). \tag{1.50}$$

Therefore, $f = g = \sum_I f(\mathbf{e}_I)\, \widehat{\alpha}^I$. ■

Corollary 1.16

If the vector space V has dimension n, then the vector space $A_k(V)$ of k-covectors on V has dimension $\binom{n}{k}$.

Proof. An ascending multi-index $I = (i_1, i_2, \ldots, i_k)$, $1 \le i_1 < i_2 < \cdots < i_k \le n$ is obtained by choosing a k-element subset of $\{1, 2, \ldots, n\}$. This can be done in $\binom{n}{k}$ ways. ■

Corollary 1.17

If $k > \dim V$, then $A_k(V) = 0$.

Proof. If $k > \dim V = n$, then in the expression

$$\widehat{\alpha}^{i_1} \wedge \widehat{\alpha}^{i_2} \wedge \cdots \wedge \widehat{\alpha}^{i_k}$$

with each $i \in \{1, 2, \ldots, n\}$, there must be a repeated i_j's, say $\widehat{\alpha}^r$. Then $\widehat{\alpha}^r \wedge \widehat{\alpha}^r$ arises in the expression $\widehat{\alpha}^{i_1} \wedge \widehat{\alpha}^{i_2} \wedge \cdots \wedge \widehat{\alpha}^{i_k}$. But $\widehat{\alpha}^r \wedge \widehat{\alpha}^r = 0$ by Corollary 1.9. Hence, $\widehat{\alpha}^{i_1} \wedge \widehat{\alpha}^{i_2} \wedge \cdots \wedge \widehat{\alpha}^{i_k} = 0$. Therefore, the basis set of $A_k(V)$ is empty, meaning $A_k(V) = 0$. ■

2 Differential Forms on $\mathbb{R}^n$

Given an open set $U \subseteq \mathbb{R}^n$ and $p \in U$, T_pU is the set of tangent vectors at $p \in U$ is identified with the point derivations of C_p^∞ (germs of smooth functions at p), i.e. a tangent vector $X_p \in T_pU$ is a map $X_p : C_p^\infty \to \mathbb{R}$ such that X_p is $\mathbb{R}$-linear:

$$X_p\left(\alpha f + g\right) = \alpha\left(X_p f\right) + X_p g; \tag{2.1}$$

and satisfies the Leibniz condition:

$$X_p\left(fg\right) = \left(X_p f\right) g(p) + f(p)\left(X_p g\right).$$

In contrast to the notion of point derivation, there is this notion of derivation of aa algebra. If X is a C^∞ vector field on an open subset $U \subseteq \mathbb{R}^n$, i.e. $X \in \mathfrak{X}(U)$, and f is a C^∞ function on U, i.e. $f \in C^\infty(U)$, then $Xf \in C^\infty(U)$ defined by

$$\left(Xf\right)(p) = X_p f.$$

Remember that f in (2.1) and (2) is a representative of an equivalence class, the equivalence class of germs of C^∞ functions at $p \in U$. These equivalence classes constitute $C_p^\infty(U)$. It is of course an $\mathbb{R}$-algebra. While in (2), $f \in C^\infty(U)$, the algebra of C^∞ functions on U with no reference of p whatsoever.

From the discussion above, a C^∞ vector field X gives rise to an $\mathbb{R}$-linear map $C^\infty(U) \to C^\infty(U)$ by $f \mapsto Xf$ that additionally has to satisfy the following Leibniz condition:

$$X\left(fg\right) = \left(Xf\right) g + f\left(Xg\right). \tag{2.2}$$

Note that a derivation at p is not a derivation of the algebra C_p^∞. A derivation at p is a map from $C_p^\infty \to \mathbb{R}$ that satisfies (2), while a derivation of the algebra C_p^∞ is supposed to be a map from C_p^∞ to itself obeying Leibniz condition.

§2.1 1 form

From any C^∞ function $f : U \to \mathbb{R}$, one can construct a 1-form (dual notion of C^∞ vector field) $\mathrm{d}f$, the restriction of which to a given point $p \in U$ yields a covector $\left(\mathrm{d}f\right)_p \in T_p^*U$, the dual space of T_pU, in the following way:

$$\left(\mathrm{d}f\right)_p\left(X_p\right) = X_p f. \tag{2.3}$$

Proposition 2.1
If $x^1, x^2, \ldots, x^n$ are the standard coordinates on $\mathbb{R}^n$, then at each point $p \in \mathbb{R}^n$,

$$\left\{\left(\mathrm{d}x^1\right)_p, \left(\mathrm{d}x^2\right)_p, \ldots, \left(\mathrm{d}x^n\right)_p\right\}$$

is the basis for the cotangent space $T_p^*\mathbb{R}^n$ dual to the basis $\left\{\left.\frac{\partial}{\partial x^1}\right|_p, \ldots, \left.\frac{\partial}{\partial x^n}\right|_p\right\}$ for the tangent space $T_p\mathbb{R}^n$.

Proof. $\left(\mathrm{d}x^i\right)_p : T_p^*\mathbb{R}^n \to \mathbb{R}$ is a linear map for each i. Now,

$$\left(\mathrm{d}x^i\right)_p\left(\left.\frac{\partial}{\partial x^j}\right|_p\right) = \left.\frac{\partial}{\partial x^j}\right|_p\left(x^i\right) = \delta^i{}_j. \tag{2.4}$$

Therefore, $\left\{(\mathrm{d}x^1)_p, (\mathrm{d}x^2)_p, \ldots, (\mathrm{d}x^n)_p\right\}$ is the basis of $T_p^*\mathbb{R}^n$ dual to the basis $\left\{\frac{\partial}{\partial x^1}\Big|_p, \ldots, \frac{\partial}{\partial x^n}\Big|_p\right\}$ for $T_p\mathbb{R}^n$. ■

If ω is a 1-form on an open subset $U \subseteq \mathbb{R}^n$, then by Proposition 2.1, there is a linear combination

$$\omega_p = \sum_{i=1}^{n} a_i(p) \left(\mathrm{d}x^i\right)_p, \tag{2.5}$$

for some $a_i(p) \in \mathbb{R}$. As p varies over U, the coefficients a_i become functions on U, and we may write

$$\omega = \sum_{i=1}^{n} a_i \mathrm{d}x^i. \tag{2.6}$$

The 1-form ω is said to be C^∞ on U if the coefficient functions a_i are all C^∞ functions on U.

Proposition 2.2 (The differential in terms of coordinates)
If $f : U \to \mathbb{R}$ is a C^∞ function on an open set $U \subseteq \mathbb{R}^n$, then

$$\mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} \mathrm{d}x^i.$$

Proof. By Proposition 2.1, at each point $p \in U$,

$$(\mathrm{d}f)_p = \sum_{i=1}^{n} a_i(p) \left(\mathrm{d}x^i\right)_p, \tag{2.7}$$

for some constants $a_i(p)$ depending on p. Thus

$$\mathrm{d}f = \sum_{i=1}^{n} a_i \ \mathrm{d}x^i, \tag{2.8}$$

for some functions a_i on U. To evaluate a_j, apply both sides of (2.8) to the coordinate vector field $\frac{\partial}{\partial x^j}$:

$$\mathrm{d}f\left(\frac{\partial}{\partial x^j}\right) = \sum_{i=1}^{n} a_i \ \mathrm{d}x^i \left(\frac{\partial}{\partial x^j}\right) = \sum_{i=1}^{n} a_i {\delta^i}_j = a_j. \tag{2.9}$$

On the other hand, using $(\mathrm{d}f)_p (X_p) = X_p f = (Xf)(p)$, we get $(\mathrm{d}f)(X) = Xf$. So

$$\mathrm{d}f\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial f}{\partial x^j}. \tag{2.10}$$

Therefore, $a_j = \frac{\partial f}{\partial x^j}$. Hence,

$$\mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} \ \mathrm{d}x^i. \tag{2.11}$$

■

(2.11) tells us that $\mathrm{d}f$ will be a C^∞ 1-form if $\frac{\partial f}{\partial x^i}$ is C^∞ on U. Hence, it is sufficient to have f as a C^∞ function on U in order to have $\mathrm{d}f$ as a C^∞ 1-form.

§2.2 Differential k-forms

A differential form ω of degree k (or a k-form) on an open subset $U \subseteq \mathbb{R}^n$ is a map that assigns to each point $p \in U$, an alternating k-linear function on the tangent space $T_p\mathbb{R}^n$, i.e.

$$\omega_p \in A_k\left(T_p\mathbb{R}^n\right).$$

By Proposition 1.15, a basis for $A_k\left(T_p\mathbb{R}^n\right)$ is

$$\left(\mathrm{d}x^I\right)_p = \left(\mathrm{d}x^{i_1}\right)_p \wedge \left(\mathrm{d}x^{i_2}\right)_p \wedge \cdots \wedge \left(\mathrm{d}x^{i_k}\right)_p,$$

where $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. Therefore, at each point $p \in U$, ω_p is a linear combination

$$\omega_p = \sum_I a_I(p)\left(\mathrm{d}x^I\right)_p, \tag{2.12}$$

and a k-form ω on U is a linear combination

$$\omega = \sum_I a_I\ \mathrm{d}x^I, \tag{2.13}$$

with function coefficients $a_I : U \to \mathbb{R}$. We say that a k-form ω is **smooth** on U if all the coeffieicnts a_I are C^∞ functions on U.

Denote by $\Omega^k(U)$ the vector space of C^∞ k-forms on U. A 0-form on U assigns to each point $p \in U$ an element of $A_0\left(T_p\mathbb{R}^n\right) = \mathbb{R}$. Thus a 0-form on U is simply a real-valued function on U, and $\Omega^0\left(U\right) = C^\infty(U)$.

Since one can multiply a C^∞ k-form by a C^∞ function on U from the left, the set $\Omega^k(U)$ of C^∞ k-forms on U is both a real vector space and a $C^\infty(U)$-module. With the wedge product as multiplication, the direct sum

$$\Omega^*(U) = \bigoplus_{k=0}^{n} \Omega^k(U)$$

becomes an algebra over $\mathbb{R}$ as well as a module over $C^\infty(U)$. As an algebra, it is anticommutative and associative.

Example 2.1. Let x, y, z be the coordinates on $\mathbb{R}^3$. The C^∞ 1-forms on $\mathbb{R}^3$ are

$$P\left(x,y,z\right)\ \mathrm{d}x + Q\left(x,y,z\right)\ \mathrm{d}y + R\left(x,y,z\right)\ \mathrm{d}z,$$

where P, Q, R range over all C^∞ functions on $\mathbb{R}^3$. The C^∞ 2-forms are

$$A\left(x,y,z\right)\ \mathrm{d}y \wedge \mathrm{d}z + B\left(x,y,z\right)\ \mathrm{d}x \wedge \mathrm{d}z + C\left(x,y,z\right)\ \mathrm{d}x \wedge \mathrm{d}y;$$

and the C^∞ 1-forms are

$$a\left(x,y,z\right)\ \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z.$$

Example 2.2 (A basis for 3-covectors)**.** Let x^1, x^2, x^3, x^4 be the standard coordinates on $\mathbb{R}^4$, and $p \in \mathbb{R}^4$. A basis for $A_3\left(T_p\mathbb{R}^4\right)$ is

$$\Big\{\left(\mathrm{d}x^1\right)_p \wedge \left(\mathrm{d}x^2\right)_p \wedge \left(\mathrm{d}x^3\right)_p,\ \left(\mathrm{d}x^1\right)_p \wedge \left(\mathrm{d}x^2\right)_p \wedge \left(\mathrm{d}x^4\right)_p,$$
$$\left(\mathrm{d}x^1\right)_p \wedge \left(\mathrm{d}x^3\right)_p \wedge \left(\mathrm{d}x^4\right)_p, \left(\mathrm{d}x^2\right)_p \wedge \left(\mathrm{d}x^3\right)_p \wedge \left(\mathrm{d}x^4\right)_p\Big\}.$$

So $\dim\left(A_3\left(T_p\mathbb{R}^n\right)\right) = 4$.

§2.3 Exterior Derivative

Before defining exterior derivative of a C^∞ k-form on an open subset $U \subseteq \mathbb{R}^n$, we first define it on 0-forms. The exterior derivative of a C^∞ function $f \in C^\infty(U)$ is its differential:

$$\mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} \mathrm{d}x^i \in \Omega^1(U).$$

Definition 2.1 (Exterior Derivative). If $\omega = \sum_I a_I \ \mathrm{d}x^I \in \omega^K(U)$, then its exterior derivative is defined as follows:

$$\mathrm{d}\omega = \sum_I \mathrm{d}a_I \wedge \mathrm{d}x^I = \sum_I \left(\sum_{j=1}^{n} \frac{\partial a_I}{\partial x^j} \ \mathrm{d}x^j \right) \wedge \mathrm{d}x^I \in \Omega^{k+1}(U). \tag{2.14}$$

Example 2.3. Let ω be the 1 form $f\mathrm{d}x + g\mathrm{d}y$ on $\mathbb{R}^2$, where f and g are C^∞ functions on $\mathbb{R}^2$. Let us write $f_x = \frac{\partial f}{\partial x}$ and $f_y = \frac{\partial f}{\partial y}$. Then

$$\begin{aligned} \mathrm{d}\omega &= \mathrm{d}f \wedge \mathrm{d}x + \mathrm{d}g \wedge \mathrm{d}y \\ &= (f_x\mathrm{d}x + f_y\mathrm{d}y) \wedge \mathrm{d}x + (g_x\mathrm{d}x + g_y\mathrm{d}y) \wedge \mathrm{d}y \\ &= -f_y \ \mathrm{d}x \wedge \mathrm{d}y + g_x \ \mathrm{d}x \wedge \mathrm{d}y \\ &= (g_x - f_y)\,\mathrm{d}x \wedge \mathrm{d}y. \end{aligned}$$

Definition 2.2 (Graded Algebra). An algebra A over a field $\mathbb{K}$ is said to be **graded** if it can be written as a direct sum

$$A = \bigoplus_{k=0}^{\infty} A^k$$

of vector spaces over $\mathbb{K}$ so that the multiplication map sends $A^k \times A^l$ to A^{k+l}.

The notation $A = \bigoplus_{k=0}^{\infty} A^k$ means that each element of A is uniquely a **finite sum**

$$a = a_{i_1} + a_{i_2} + \cdots + a_{i_m},$$

where $a_{i_j} \in A^{i_j}$.

Example 2.4. The polynomial algebra

$$\mathbb{R}\left[x, y\right] = \bigoplus_{k=0}^{\infty} A^k$$

with A^k being the vector space of homogenous polynomials of degree k in x and y. Observe that the 0 polynomial is trivially homogenous of any degree, and hence belongs to A^k for all $k \geq 0$. Multiplication of degree k homogenous polynomial with a degree l homogenous polynomial in x and y will result in a homogenous polynomial of degree $k + l$ in x and y.

Example 2.5. The algebra $\Omega^*(U)$ of C^∞ differential forms on U is also graded by the degree of differential forms. Each $\Omega^k(U)$ is a vector space. Multiplication of differential forms is defined by wedge product between them. The wedge product of a degree k differential form on U with a degree l differential form results in a degree $k + l$ differential form.

Definition 2.3 (Anti-derivation). Let $A = \bigoplus_{k=0}^{\infty} A^k$ be a graded algebra over a field $\mathbb{K}$. An **anti-derivation** of the graded algebra A is a $\mathbb{K}$-linear map $D : A \to A$ such that for $\omega \in A^k$ and $\tau \in A^l$, one has

$$D(\omega\tau) = (D\omega)\,\tau + (-1)^k\,\omega\,(D\tau)\,. \tag{2.15}$$

If the antiderivation D sends $\omega \in A^k$ to $D\omega \in A^{k+m}$, we say that it is an antiderivation of degree m.

Proposition 2.3 (i) The exterior derivative $\mathrm{d} : \Omega^*(U) \to \Omega^*(U)$ is an antiderivation of degree 1:

$$\mathrm{d}(\omega \wedge \tau) = (\mathrm{d}\omega) \wedge \tau + (-1)^{\deg \omega}\,\omega \wedge \mathrm{d}\tau. \tag{2.16}$$

(ii) $\mathrm{d}^2 = 0$.

(iii) If $f \in \Omega^0(U) = C^\infty(U)$ and $X \in \mathfrak{X}(U)$ (the space of C^∞ vector fields), then $(\mathrm{d}f)(X) = Xf$.

Proof. (i) Since the exterior derivative operator $\mathrm{d} : \Omega^k(U) \to \Omega^{k+1}(U)$ is linear, it suffices to check the equality (2.16) for $\omega = f\,\mathrm{d}x^I$ and $\tau = g\,\mathrm{d}x^J$ with $I = (i_1, \ldots, i_k)$ and $J = (j_1, \ldots, j_l)$ being strictly ascending multi-indices.

$$\begin{aligned}
\mathrm{d}(\omega \wedge \tau) &= \mathrm{d}\left(fg\mathrm{d}x^I \wedge \mathrm{d}x^J\right) \\
&= \sum_{i=1}^n \frac{\partial(fg)}{\partial x^i}\mathrm{d}x^i \wedge \mathrm{d}x^I \wedge \mathrm{d}x^J \\
&= \sum_{i=1}^n \frac{\partial f}{\partial x^i} \cdot g\mathrm{d}x^i \wedge \mathrm{d}x^I \wedge \mathrm{d}x^J + \sum_{i=1}^n f \cdot \frac{\partial g}{\partial x^i}\mathrm{d}x^i \wedge \mathrm{d}x^I \wedge \mathrm{d}x^J \\
&= \sum_{i=1}^n \frac{\partial f}{\partial x^i}\mathrm{d}x^i \wedge \mathrm{d}x^I \wedge g\mathrm{d}x^J + \sum_{i=1}^n \frac{\partial g}{\partial x^i}\mathrm{d}x^i \wedge f\,\mathrm{d}x^I \wedge \mathrm{d}x^J.
\end{aligned} \tag{2.17}$$

Now, in the second sum in (2.17), one has to push $\frac{\partial g}{\partial x^i}\mathrm{d}x^i$ through the k-fold wedge product $\mathrm{d}x^I$ and hence in the process picks out a sign $(-1)^k$. Therefore,

$$\sum_{i=1}^n \frac{\partial g}{\partial x^i}\mathrm{d}x^i \wedge f\,\mathrm{d}x^I \wedge \mathrm{d}x^J = (-1)^k\, f\,\mathrm{d}x^I \wedge \frac{\partial g}{\partial x^i}\mathrm{d}x^i \wedge \mathrm{d}x^J. \tag{2.18}$$

Now, observe that

$$\mathrm{d}\omega = \mathrm{d}\left(f\,\mathrm{d}x^I\right) = \sum_{i=1}^n \frac{\partial f}{\partial x^i}\mathrm{d}x^i \wedge \mathrm{d}x^I, \text{ and} \tag{2.19}$$

$$\mathrm{d}\tau = \mathrm{d}\left(g\,\mathrm{d}x^J\right) \sum_{i=1}^n \frac{\partial g}{\partial x^i}\mathrm{d}x^i \wedge \mathrm{d}x^J. \tag{2.20}$$

Therefore,

$$\mathrm{d}(\omega \wedge \tau) = \mathrm{d}\omega \wedge \tau + (-1)^k\,\omega \wedge \mathrm{d}\tau. \tag{2.21}$$

(ii) Again, by $\mathbb{R}$-linearity of d, it suffices to show that $\mathrm{d}^2\omega = 0$ for $\omega = f\mathrm{d}x^I$.

$$\begin{aligned}
\mathrm{d}^2\left(f\mathrm{d}x^I\right) &= \mathrm{d}\left(\sum_{i=1}^n \frac{\partial f}{\partial x^i}\mathrm{d}x^i \wedge \mathrm{d}x^I\right) \\
&= \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 f}{\partial x^j \partial x^i}\mathrm{d}x^j \wedge \mathrm{d}x^i \wedge \mathrm{d}x^I.
\end{aligned} \tag{2.22}$$

If $i = j$, then $\mathrm{d}x^j \wedge \mathrm{d}x^i = 0$. If $i \neq j$, then $\frac{\partial^2 f}{\partial x^j \partial x^i}$ is symmetric in i and j, but $\mathrm{d}x^j \wedge \mathrm{d}x^i$ is alternating in i and j. Therefore, the terms with $i \neq j$ pair up and cancel out.

(iii) Let $X = \sum_{i=1}^{n} a^i \frac{\partial}{\partial x^i}$. Then

$$\begin{aligned}(\mathrm{d}f)(X) &= \left(\sum_{j=1}^{n} \frac{\partial f}{\partial x^j}\mathrm{d}x^j\right)\left(\sum_{i=1}^{n} a^i \frac{\partial}{\partial x^i}\right) \\ &= \sum_{j=1}^{n}\sum_{i=1}^{n} a^i \frac{\partial f}{\partial x^j}\mathrm{d}x^j\left(\frac{\partial}{\partial x^i}\right) \\ &= \sum_{j=1}^{n}\sum_{i=1}^{n} a^i \frac{\partial f}{\partial x^j}\delta^j{}_i \\ &= \sum_{i=1}^{n} a^i \frac{\partial f}{\partial x^i} = Xf. \end{aligned} \tag{2.23}$$

■

Proposition 2.4 (Characterization of exterior derivative)

The 3 properties of Proposition 2.3 uniquely characterize exterior derivative on an open set $U \subseteq \mathbb{R}^n$. In other words, if $D : \Omega^*(U) \to \Omega^*(U)$ is an antiderivation of degree 1 such that $D^2 = 0$ and for $f \in C^\infty(U)$ and $X \in \mathfrak{X}(U)$, $(Df)(X) = Xf$, then $D = \mathrm{d}$.

Proof. Since every k-form on U is a sum of terms such as $f\mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2} \wedge \cdots \wedge \mathrm{d}x^{i_k}$, by linearity of d, it suffices to show that $D = \mathrm{d}$ on a k-form of this type. Applying property (iii) for $f = x^i$, one has

$$Dx^i(X) = X\left(x^i\right).$$

Writing $X = \sum_{j=1}^{n} a^j \frac{\partial}{\partial x^j}$, we get

$$Dx^i\left(\sum_{j=1}^{n} a^j \frac{\partial}{\partial x^j}\right) = \sum_{j=1}^{n} a^j \frac{\partial}{\partial x^j}\left(x^i\right) = a^i = \mathrm{d}x^i\left(\sum_{j=1}^{n} a^j \frac{\partial}{\partial x^j}\right).$$

Therefore,

$$Dx^i = \mathrm{d}x^i. \tag{2.24}$$

Now,

$$\begin{aligned} D\left(f\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}\right) &= D\left(fDx^{i_1} \wedge \cdots \wedge Dx^{i_k}\right) \\ &= Df \wedge \left(Dx^{i_1} \wedge \cdots \wedge Dx^{i_k}\right) + (-1)^0 fD\left(Dx^{i_1} \wedge \cdots \wedge Dx^{i_k}\right). \end{aligned} \tag{2.25}$$

Now, since $\mathrm{d}f(X) = Xf = Df(X)$ for any $X \in \mathfrak{X}(U)$, $\mathrm{d}f = Df$. Furthermore, $D\left(Dx^{i_1}\right) = 0$, and

$$\begin{aligned} D\left(Dx^{i_1} \wedge \cdots \wedge Dx^{i_k}\right) &= D^2x^{i_1} \wedge Dx^{i_2} \wedge \cdots \wedge Dx^{i_k} - Dx^{i_1} \wedge D\left(Dx^{i_2} \wedge \cdots \wedge Dx^{i_k}\right) \\ &= -Dx^{i_1} \wedge D\left(Dx^{i_2} \wedge \cdots \wedge Dx^{i_k}\right). \end{aligned} \tag{2.26}$$

Therefore, by induction on k,

$$D\left(Dx^{i_1} \wedge \cdots \wedge Dx^{i_k}\right) = 0. \tag{2.27}$$

Hence, from (2.25),

$$\begin{aligned} D\left(f\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}\right) &= Df \wedge \left(Dx^{i_1} \wedge \cdots \wedge Dx^{i_k}\right) \\ &= \mathrm{d}f \wedge \left(\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}\right) \\ &= \mathrm{d}\left(f\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}\right). \end{aligned} \tag{2.28}$$

So $D = \mathrm{d}$ on $\Omega^*(U)$. ■

Closed Forms and Exact Forms

A k-form ω on U is **closed** if $\mathrm{d}\omega = 0$; it's **exact** if there is a $(k-1)$-form τ on U such that $\omega = \mathrm{d}\tau$. Since $\mathrm{d}^2 = 0$, every exact form is closed. But in general, a closed form may fail to be exact. We will see how non-exact closed forms capture the geometry of a manifold when we do de Rham cohomology on a manifold.

Example 2.6. Define a 1-form ω on $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ by

$$\omega = \frac{1}{x^2+y^2}\left(-y\mathrm{d}x + x\mathrm{d}y\right). \tag{2.29}$$

Then ω is closed.

A collection of vector spaces $\left\{V^k\right\}_{k=0}^{\infty}$ with linear maps $d_k : V^k \to V^{k+1}$ such that $d_{k+1} \circ d_k = 0$ is called a **differential complex** or a **cochain complex**. For aany open set $U \subseteq \mathbb{R}^n$, the exterior derivative d makes the vector space $\Omega^*(U)$ of C^∞ forms on U into a cochain complex, called the **de Rham complex** on U:

$$\Omega^0(U) \xrightarrow{\mathrm{d}} \Omega^1(U) \xrightarrow{\mathrm{d}} \Omega^2(U) \xrightarrow{\mathrm{d}} \cdots$$

The closed forms are precisely the elements of the kernel of d and the exact forms are the elements of the image of d. In the language of cohomology, d is also called the coboundary operator.

§2.4 Applications to Vector Calculus

The theory of differential forms unifies many theorems in vector calculus on $\mathbb{R}^3$. A vector valued function on $\mathbb{R}^3$ is the same as a vector field. Recall the 3 operators on scalar and vector-valued functions on $\mathbb{R}^3$.

$$\{\text{scalar function}\} \xrightarrow{\text{grad}} \{\text{vector function}\} \xrightarrow{\text{curl}} \{\text{vector function}\} \xrightarrow{\text{div}} \{\text{scalar function}\}.$$

Let f be a scalar function and $\begin{bmatrix} P \\ Q \\ R \end{bmatrix}$ be a vector field on $\mathbb{R}^3$, where each of P, Q, R is a scalar function on $\mathbb{R}^3$. Then

$$\begin{aligned} \operatorname{grad} f &= \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix}, \\ \operatorname{curl} \begin{bmatrix} P \\ Q \\ R \end{bmatrix} &= \begin{bmatrix} R_y - Q_z \\ P_z - R_x \\ Q_x - P_y \end{bmatrix}, \\ \operatorname{div} \begin{bmatrix} P \\ Q \\ R \end{bmatrix} &= P_x + Q_y + R_z. \end{aligned} \tag{2.30}$$

Then one has the following results.

Proposition 2.5

$$\operatorname{curl}\left(\operatorname{grad} f\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{2.31}$$

Proposition 2.6

$$\operatorname{div}\left(\operatorname{curl}\begin{bmatrix}P\\Q\\R\end{bmatrix}\right)=0. \tag{2.32}$$

Proposition 2.7
On $\mathbb{R}^3$, a vector field $\mathbf{F}(x,y,z)$ is the gradient of some scalar function if and only if $\operatorname{curl}\mathbf{F}=\mathbf{0}$.

A 1-form on $\mathbb{R}^3$ can be written as

$$P(x,y,z)\,\mathrm{d}x+Q(x,y,z)\,\mathrm{d}y+R(x,y,z)\,\mathrm{d}z.$$

This 1-form on $\mathbb{R}^3$ can be identified with the vector field $\begin{bmatrix}P\\Q\\R\end{bmatrix}$.

Similarly, the 2-forms on $\mathbb{R}^3$ given by

$$A(x,y,z)\,\mathrm{d}y\wedge\mathrm{d}z+B(x,y,z)\,\mathrm{d}z\wedge\mathrm{d}x+C(x,y,z)\,\mathrm{d}x\wedge\mathrm{d}y$$

can be identified with the vector field $\begin{bmatrix}A\\B\\C\end{bmatrix}$ on $\mathbb{R}^3$.

In terms of these identifications, the exterior derivative of a 0-form f (scalar function) is

$$\mathrm{d}f=\frac{\partial f}{\partial x}\mathrm{d}x+\frac{\partial f}{\partial y}\mathrm{d}y+\frac{\partial f}{\partial z}\mathrm{d}z,$$

which can be identified with the vector field

$$\begin{bmatrix}\frac{\partial f}{\partial x}\\\frac{\partial f}{\partial y}\\\frac{\partial f}{\partial z}\end{bmatrix}=\operatorname{grad}f.$$

The exterior derivative of a 1-form on $\mathbb{R}^3$ is

$$\begin{aligned}&\mathrm{d}(P\mathrm{d}x+Q\mathrm{d}y+R\mathrm{d}z)\\=\;&\frac{\partial P}{\partial y}\mathrm{d}y\wedge\mathrm{d}x+\frac{\partial P}{\partial z}\mathrm{d}z\wedge\mathrm{d}x+\frac{\partial Q}{\partial x}\mathrm{d}x\wedge\mathrm{d}y+\frac{\partial Q}{\partial z}\mathrm{d}z\wedge\mathrm{d}y+\frac{\partial R}{\partial x}\mathrm{d}x\wedge\mathrm{d}z+\frac{\partial R}{\partial y}\mathrm{d}y\wedge\mathrm{d}z\\=\;&(R_y-Q_z)\,\mathrm{d}y\wedge\mathrm{d}z+(P_z-R_x)\,\mathrm{d}z\wedge\mathrm{d}x+(Q_x-P_y)\,\mathrm{d}x\wedge\mathrm{d}y,\end{aligned}$$

which corresponds to

$$\begin{bmatrix}R_y-Q_z\\P_z-R_x\\Q_x-P_y\end{bmatrix}=\operatorname{curl}\begin{bmatrix}P\\Q\\R\end{bmatrix}.$$

The exterior derivative of a 2-form is

$$\begin{aligned}&\mathrm{d}(A\mathrm{d}y\wedge\mathrm{d}z+B\mathrm{d}z\wedge\mathrm{d}x+C\mathrm{d}x\wedge\mathrm{d}y)\\=\;&A_x\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z+B_y\mathrm{d}y\wedge\mathrm{d}z\wedge\mathrm{d}x+C_z\mathrm{d}z\wedge\mathrm{d}x\wedge\mathrm{d}y\\=\;&(A_x+B_y+C_z)\,\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z,\end{aligned}$$

which corresponds to

$$A_x+B_y+C_z=\operatorname{div}\begin{bmatrix}A\\B\\C\end{bmatrix}.$$

In summary, exterior derivative d on 0-forms is identified with **gradient**; exterior derivative d on 1-forms is identified with **curl**; exterior derivative d on 2-forms is identified with **divergence**. Using de Rham complex on U:

$$\Omega^0(U) \xrightarrow{\mathrm{d}} \Omega^1(U) \xrightarrow{\mathrm{d}} \Omega^2(U) \xrightarrow{\mathrm{d}} \Omega^3(U).$$

Using vector calculus language,

$$C^\infty(U) \xrightarrow{\text{grad}} \mathfrak{X}(U) \xrightarrow{\text{curl}} \mathfrak{X}(U) \xrightarrow{\text{div}} C^\infty(U).$$

Remark 2.1. Proposition 2.5 and Proposition 2.6 express the property $\mathrm{d}^2 = 0$ of exterior derivative. A vector field $\begin{bmatrix} P \\ Q \\ R \end{bmatrix}$ is the gradient of a C^∞ function f if and only if the corresponding 1-form $P\mathrm{d}x + Q\mathrm{d}y + R\mathrm{d}z$ is $\mathrm{d}f$. Proposition 2.7 expresses the fact that a 1-form on $\mathbb{R}^3$ is exact if and only if it is closed. It's worth remarking at this stage that Proposition 2.7 need not hold true on a region other than $\mathbb{R}^3$, as the following well-known example from calculus suggests.

Example 2.7. Suppose $U = \mathbb{R}^3 \setminus \{z\text{-axis}\}$, and $\mathbf{F}(x, y, z)$ is the vector field

$$\mathbf{F} = \begin{bmatrix} \frac{-y}{x^2+y^2} \\ \frac{x}{x^2+y^2} \\ 0 \end{bmatrix}$$

on U. Then $\operatorname{curl} \mathbf{F} = \mathbf{0}$. Indeed,

$$\begin{aligned} \operatorname{curl} \mathbf{F} &= \begin{bmatrix} \frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}\left(\frac{x}{x^2+y^2}\right) \\ \frac{\partial}{\partial z}\left(\frac{-y}{x^2+y^2}\right) - \frac{\partial}{\partial x}(0) \\ \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ \frac{(x^2+y^2) - x\cdot 2x}{(x^2+y^2)^2} - \frac{-(x^2+y^2)+y\cdot 2y}{(x^2+y^2)^2} \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0}. \end{aligned}$$

But $\mathbf{F}$ is not the gradient of a C^∞ function on U. Recall the theorem from vector calculus that the line integral of the gradient of a function along a curve gives the total change in the value of the function from the start to the end of the curve. In other words, if $\mathbf{r} : [a, b] \to \mathbb{R}^3$ is a curve and $f : \mathbb{R}^3 \to \mathbb{R}$ is a scalar function, then

$$\int_a^b (\boldsymbol{\nabla} f) \cdot \mathrm{d}\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)). \tag{2.33}$$

Then if $\mathbf{F}$ is the gradient of a smooth scalar function, then the line integral

$$\oint_C \frac{-y}{x^2+y^2}\,\mathrm{d}x + \frac{x}{x^2+y^2}\,\mathrm{d}y$$

over any closed curve would become 0. Let us take the closed curve to be the unit circle: $x = \cos t$,

$y = \sin t$, $z = 0$ for $t \in [0, 2\pi]$. Then

$$\begin{aligned}
&\oint_C \frac{-y}{x^2+y^2}\,\mathrm{d}x + \frac{x}{x^2+y^2}\,\mathrm{d}y \\
&= \int_0^{2\pi} -\sin t\,\mathrm{d}\left(\cos t\right) + \int_0^{2\pi} \cos t\,\mathrm{d}\left(\sin t\right) \\
&= \int_0^{2\pi} \sin^2 t\,\mathrm{d}t + \int_0^{2\pi} \cos^2 t\,\mathrm{d}t \\
&= 2\pi.
\end{aligned}$$

Hence, although $\operatorname{curl}\mathbf{F} = \mathbf{0}$, there is no C^∞ function f on U such that $\mathbf{F} = \operatorname{grad} f$. In the language of differential forms, the 1-form

$$\omega = \frac{-y}{x^2+y^2}\mathrm{d}x + \frac{x}{x^2+y^2}\mathrm{d}y$$

is closed but not exact.

It turns out that whether Proposition 2.7 is true for a region $U \subseteq \mathbb{R}^3$ depends on the topology of U. One measure of the failure of a closed k-form to be exact is the quotient vector space

$$H^k\left(U\right) = \frac{\{\text{closed } k\text{-forms on } U\}}{\{\text{exact } k\text{-forms on } U\}},$$

called the k-th de Rham cohomology of U. The generalization of Proposition 2.7 to any differential form on $\mathbb{R}^n$ is called the **Poincare lemma**:

> For $k \geq 1$, every closed k-form on $\mathbb{R}^n$ is exact.

This statement is equivalent to the vanishing of the k-th de Rham cohomology $H^k\left(\mathbb{R}^n\right)$ for $k \geq 1$.

3 Differential Forms on Manifold

§3.1 Definition and Local Expression

Let M be a smooth manifold and $p \in M$. The **cotangent space** of M at p, denoted by T_p^*M is the dual space of the tangent space T_pM. An element in T_p^*M is called a covector at p. Thus, a covector $\omega_p \in T_p^*M$ is a linear function

$$\omega_p : T_pM \to \mathbb{R}.$$

A 1-form on M is a functiion that assigns to each $p \in M$, a covector at p.

Definition 3.1 (Differential of a function)**.** Let $f : M \to \mathbb{R}$ be a C^∞ function on a manifold M. Its **differential** is defined to be the 1-form $\mathrm{d}f$ on M such that for any $p \in M$ and $X_p \in T_pM$,

$$(\mathrm{d}f)_p (X_p) = X_p f. \tag{3.1}$$

Proposition 3.1

If $f : M \to \mathbb{R}$ is a C^∞ function, then for $p \in M$ and $X_p \in T_pM$,

$$f_{*,p} (X_p) = (\mathrm{d}f)_p (X_p) \left.\frac{\partial}{\partial x}\right|_{f(p)}.$$

Proof. Since $f_{*,p} (X_p) \in T_{f(p)}\mathbb{R}$, there is a real number c such that

$$f_{*,p} (X_p) = c \left.\frac{\partial}{\partial x}\right|_{f(p)}. \tag{3.2}$$

(Here the chart chosen on $\mathbb{R}$ is $(\mathbb{R}, \mathbb{1}_\mathbb{R})$ so that x is the coordinate of this chart, i.e. $x = \mathbb{1}_\mathbb{R}$.) To evaluate c, apply both sides of (3.2) to the function $x \in C^\infty(\mathbb{R})$. Then

$$f_{*,p} (X_p) (x) = c \left.\frac{\partial}{\partial x}\right|_{f(p)} (x) = c.$$

Therefore,

$$c = f_{*,p} (X_p) (x) = X_p (x \circ f) = X_p f = (\mathrm{d}f)_p (X_p), \tag{3.3}$$

since $x = \mathbb{1}_R$. Therefore, substituting the value of c into (3.2),

$$f_{*,p} (X_p) = (\mathrm{d}f)_p (X_p) \left.\frac{\partial}{\partial x}\right|_{f(p)}. \tag{3.4}$$

■

Let $(U, \varphi) \equiv (U, x^1, x^2, \ldots, x^n)$ be a coordinate chart on M. Here $x^i = r^i \circ \varphi$, where r^i is the i-th coordinate function of a vector in $\mathbb{R}^n$. Then the differentials $\mathrm{d}x^1, \mathrm{d}x^2, \ldots, \mathrm{d}x^n$ are 1-forms on U.

Proposition 3.2

At each point $p \in U$, the covectors $(\mathrm{d}x^1)_p, \ldots, (\mathrm{d}x^n)_p$ form a basis for the cotangent space T_p^*M, dual to the basis $\left\{ \left.\frac{\partial}{\partial x^1}\right|_p, \ldots, \left.\frac{\partial}{\partial x^n}\right|_p \right\}$ for the tangent space T_pM.

Proof. Observe that
$$\left(\mathrm{d}x^i\right)_p\left(\frac{\partial}{\partial x^j}\bigg|_p\right)=\frac{\partial}{\partial x^j}\bigg|_p\left(x^i\right)=\delta^i{}_j. \tag{3.5}$$

So $\left\{\left(\mathrm{d}x^1\right)_p,\ldots,\left(\mathrm{d}x^n\right)_p\right\}$ is the dual basis to $\left\{\frac{\partial}{\partial x^1}\Big|_p,\ldots,\frac{\partial}{\partial x^n}\Big|_p\right\}$. ∎

Thus, every 1-form ω on U can be wrotten as a linear combination
$$\omega=\sum_{i=1}^n a_i\ \mathrm{d}x^i,$$
where a_i are functions on U. In particular, if f is a C^∞ function on M, then the 1-form $\mathrm{d}f$, when restricted to U, must be a linear combination
$$\mathrm{d}f=\sum_{i=1}^n a_i\ \mathrm{d}x^i. \tag{3.6}$$

If we evaluate both sides of (3.6) on $\frac{\partial}{\partial x^j}$,
$$(\mathrm{d}f)\left(\frac{\partial}{\partial x^j}\right)=\sum_{i=1}^n a_i\ \mathrm{d}x^i\left(\frac{\partial}{\partial x^j}\right)=\sum_{i=1}^n a_i\delta^i{}_j=a_j.$$

Then
$$a_j=(\mathrm{d}f)\left(\frac{\partial}{\partial x^j}\right)=\frac{\partial f}{\partial x^j}.$$

Therefore,
$$\mathrm{d}f=\sum_{i=1}^n\frac{\partial f}{\partial x^i}\ \mathrm{d}x^i. \tag{3.7}$$

§3.2 The Cotangent Bundle

The underlying set of the **cotangent bundle** is the disjoint union of the cotangent spaces at all points of M:
$$T^*M=\bigsqcup_{p\in M}T_p^*M=\bigcup_{p\in M}\{p\}\times T_p^*M. \tag{3.8}$$

Let us give T^*M a topology in the following way: let $\left(U,x^1,\ldots,x^n\right)$ be a chart on M and $p\in U$. Then each $\omega_p\in T_p^*M$ can be written uniquely as a linear combination
$$\omega_p=\sum_{i=1}^n c_i\left(\omega_p\right)\left(\mathrm{d}x^i\right)_p,$$
with $c_i\left(\omega_p\right)\in\mathbb{R}$. This gives rise to a bijection
$$\begin{aligned}\widetilde{\varphi}:T^*U&\to\varphi(U)\times\mathbb{R}^n\\(p,\omega_p)&\mapsto\left(\varphi(p),c_1\left(\omega_p\right),c_2\left(\omega_p\right),\ldots,c_n\left(\omega_p\right)\right).\end{aligned}$$

We use this bijection $\widetilde{\varphi}$ to transfer the topology of $\varphi(U)\times\mathbb{R}^n$ to T^*U: a set $A\subseteq T^*U$ is said to be open if and only if $\widetilde{\varphi}(A)$ is open in $\varphi(U)\times\mathbb{R}^n$, where $\varphi(U)\times\mathbb{R}^n$ is given the subspace topology of $\mathbb{R}^{2n}$. Now, let $\mathscr{A}=\{(U_\alpha,\varphi_\alpha)\}_{\alpha\in I}$ be the maximal atlas of M. Now, let
$$\begin{aligned}\mathcal{B}&=\bigcup_{\alpha\in I}\{A\mid A\subseteq T^*U_\alpha\text{ is open in }A\subseteq T^*U_\alpha\}\\&=\{A\mid A\subseteq T^*U_\alpha\text{ is open in }A\subseteq T^*U_\alpha,\ \alpha\in I\}.\end{aligned}$$

It can be shown using the same technique of tangent bundle that $\mathcal{B}$ forms a basis for topology. We give T^*M the topology generated by the basis $\mathcal{B}$. We declare $A \subseteq T^*M$ to be open if and only if there exists a subfamily $\{B_\lambda\}_\lambda \subseteq \mathcal{B}$ such that

$$A = \bigcup_\lambda B_\lambda.$$

Furthermore, T^*M has the structure of a C^∞ manifold. An atlas for T^*M is

$$\{(T^*U_\alpha, \widetilde{\varphi}_\alpha)\}_{\alpha \in I}.$$

If two coordinate open sets U_α and U_β intersect, suppose $U_{\alpha\beta} = U_\alpha \cap U_\beta$. Then for $p \in U_{\alpha\beta}$, each $\omega_p \in T_p^*M$ has two basis expansions:

$$\omega_p = \sum_{i=1}^n a_i \left(\mathrm{d}x^i\right)_p = \sum_{j=1}^n b_i \left(\mathrm{d}y^i\right)_p. \tag{3.9}$$

(Here $(U_\alpha, x^1, \ldots, x^n)$ and $(U_\beta, y^1, \ldots, y^n)$ are charts.) Now applying $\left.\frac{\partial}{\partial y^k}\right|_p$ to both sides of (3.9),

$$b_k = \sum_{i=1}^n a_i \left(\mathrm{d}x^i\right)_p \left(\left.\frac{\partial}{\partial y^k}\right|_p\right) = \sum_{i=1}^n a_i \left.\frac{\partial x^i}{\partial y^k}\right|_p.$$

Therefore, $\widetilde{\varphi}_\beta \circ \widetilde{\varphi}_\alpha^{-1} : \varphi_\alpha(U_{\alpha\beta}) \times \mathbb{R}^n \to \varphi_\beta(U_{\alpha\beta}) \times \mathbb{R}^n$ is given by

$$(\varphi_\alpha(p), a_1, \ldots, a_n) \mapsto \left(\left(\varphi_\beta \circ \varphi_\alpha^{-1}\right)(\varphi_\alpha(p)), \sum_{i=1}^n a_i \left.\frac{\partial x^i}{\partial y^1}\right|_p, \ldots, \sum_{i=1}^n a_i \left.\frac{\partial x^i}{\partial y^n}\right|_p\right).$$

$\varphi_\beta \circ \varphi_\alpha^{-1}$ is smooth, and each $\frac{\partial x^i}{\partial y^j}$ is smooth. Therefore, the transition map $\widetilde{\varphi}_\beta \circ \widetilde{\varphi}_\alpha^{-1}$ is smooth, making T^*M a smooth manifold.

T^*M is, in fact, a **vector bundle** of rank n over M. It has a natural projection $\pi : T^*M \to M$ given by $(p, \omega_p) \mapsto p$. In terms of cotangent bundle, a 1-form on M is simply a section of the cotangent bundle T^*M, i.e. it is a map $\omega : M \to T^*M$ such that $\pi \circ \omega = \mathbb{1}_M$. We say that a 1-form is **smooth** if it is C^∞ as a map $\omega : M \to T^*M$ between two manifolds.

§3.3 Characterization of Smooth 1-forms

By definition, a 1-form on an open set $U \subseteq M$ is C^∞ if it is C^∞ as a section of the cotangent bundle T^*M over U.

Lemma 3.3

Let $(U, \varphi) \equiv (U, x^1, \ldots, x^n)$ be a chart on a manifold M. A 1-form $\omega = \sum a_i \, \mathrm{d}x^i$ on U is smooth if and only if the coefficient functions a_i are all smooth on U.

Proof. This is a special case of *Proposition 9.4.2* of DG1 which states that:

> Let $\pi : E \to M$ be a C^∞ vector bundle and U an open subset of M. Suppose $s_1, \ldots, s_r$ is a C^∞ frame for E over U. Then a section $s = \sum_{j=1}^r c^j s_j$ of E over U is C^∞ if and only if the coefficients c^j are C^∞ functions on U.

Here we take E to be the cotangent bundle T^*M, and $\{s_i\}_{i=1}^r$ the C^∞ frame for E over U to be the coordinate 1-forms $\{(\mathrm{d}x^i)\}_{i=1}^n$. ∎

Proposition 3.4

Let ω be a 1-form on a manifold M. Then the following are equivalent:

(i) ω is C^∞.

(ii) For every point $p \in M$, there is a chart $(U, x^1, \ldots, x^n)$ about p such that if $\omega = \sum_{i=1}^n a_i \, \mathrm{d}x^i$ on U, then the functions a_i are C^∞ on U.

(iii) For any chart $(U, x^1, \ldots, x^n)$ on M, if $\omega = \sum_{i=1}^n a_i \, \mathrm{d}x^i$ on U, then the functions a_i are C^∞ on U.

Proof. **(ii)⇒(i):** By Lemma 3.3, for every point $p \in M$, there is a chart $(U, x^1, \ldots, x^n)$ about p such that ω is smooth on U. In particular, the section $\omega : M \to T^*M$ is smooth at p, for every $p \in M$. Therefore, $\omega : M \to T^*M$ is a smooth map between manifolds.

(i)⇒(iii): If $\omega : M \to T^*M$ is a smooth map between manifolds, ω is smooth at every chart of M. Therefore, by Lemma 3.3, if $\omega = \sum_{i=1}^n a_i \, \mathrm{d}x^i$ on a chart $(U, x^1, \ldots, x^n)$, each a_i is smooth on U.

(iii)⇒(ii): Obvious. ∎

Proposition 3.5

A 1-form ω on a manifold M is C^∞ if and only if for every C^∞ vector field X, the function $\omega(X)$ is C^∞ on M.

Proof. **(⇒)** Suppose ω is a C^∞ 1-form and X is a C^∞ vector field. Let $(U, x^1, \ldots, x^n)$ be a chart on M. Then

$$\omega = \sum_{i=1}^n a_i \, \mathrm{d}x^i \quad \text{and} \quad X = \sum_{i=1}^n b^j \frac{\partial}{\partial x^j}, \tag{3.10}$$

for C^∞ functions a_i and b^j on U. Then on U, one has

$$\omega(X) = \left(\sum_{i=1}^n a_i \, \mathrm{d}x^i\right)\left(\sum_{i=1}^n b^j \frac{\partial}{\partial x^j}\right) = \sum_{i=1}^n \sum_{j=1}^n a_i b^j \delta^i{}_j = \sum_{i=1}^n a_i b^i, \tag{3.11}$$

which is a C^∞ function on U. Since U was chosen to be an arbitrary coordinate open set, $\omega(X)$ is a smooth function on all of M.

(⇐) Suppose ω is a 1-form on M such that for every C^∞ vector field X on M, the function $\omega(X)$ is smooth on M. For a given $p \in M$, choose a coordinate neighborhood $(U, \varphi) \equiv (U, x^1, \ldots, x^n)$ about p. Then one has

$$\omega = \sum_{i=1}^n a_i \, \mathrm{d}x^i$$

on U. Now fix an integer $j \in \{1, 2, \ldots, n\}$. We can extend the C^∞ vector field $\frac{\partial}{\partial x^j}$ on U to a C^∞ vector field X on the whole of M that agrees with $\frac{\partial}{\partial x^j}$ in a neighborhood V of p (not necessarily the whole of U, but possibly a smaller neighborhood) contained in U (*Proposition 11.1.4* of DG1). The extended vector field is defined in the following way: let $\sigma : M \to \mathbb{R}$ be a C^∞ bump function which is identically 1 on a neighborhood V of p and which has support contained in U. Now, define the vector field $q \mapsto X_q \in T_qM$, denoted by X, in terms of the bump function σ in the following way:

$$X_q = \begin{cases} \sigma(q) \left.\frac{\partial}{\partial x^j}\right|_q & \text{if } q \in U, \\ \mathbf{0} & \text{if } q \notin U. \end{cases} \tag{3.12}$$

The vector field X is smooth in the whole of M, as proved in *Proposition 11.1.4* of DG1. Now, by the hypothesis, $\omega(X)$ is C^∞ on M. In particular, $\omega(X)$ is smooth on V. Therefore,

$$\omega(X) = \left(\sum_{i=1}^n a_i \, \mathrm{d}x^i\right)\left(\frac{\partial}{\partial x^j}\right) = \sum_{i=1}^n a_i \delta^i{}_j = a_j$$

is smooth on V. We, therefore, see that the coefficient functions a_i's appearing in $\omega = \sum_{i=1}^n a_i \, \mathrm{d}x^i$ are smooth on $V \subseteq U$. It means that for a given point p, we can find a chart $(V, \widetilde{x}^1, \ldots, \widetilde{x}^n)$, where

$$\widetilde{x}^i = r^i \circ \varphi|_V,$$

such that $\omega = \sum_{i=1}^n a_i|_V \, \mathrm{d}\widetilde{x}^i$ on V, with each $a_i|_V$ smooth on V. Therefore, by Proposition 3.4, ω is C^∞. ■

§3.4 Pullback of 1-forms

Recall that the differential of a smooth map $F : N \to M$ at $p \in N$ is a linear map $F_{*,p} : T_pN \to T_{F(p)}M$ defined by

$$[F_{*,p}(X_p)](f) = X_p(f \circ F), \tag{3.13}$$

where $f \in C^\infty_{F(p)}(M)$. Indeed, $f \circ F \in C^\infty_p(N)$. Analogously, the **codifferential** (the dual of a differential) at $F(p) \in M$ is a linear map

$$F^{*,p} : T^*_{F(p)}M \to T^*_pN.$$

One observes that the differential $F_{*,p}$ pushes forward a tangent vector at $p \in N$ while the codifferential $F^{*,p}$ pulls back a covector from $T^*_{F(p)}M$ at $F(p) \in M$ to T^*_pN.

Remark 3.1. Note that a vector field, in general, cannot be pushed forward under a smooth map $F : N \to M$. Suppose $F : N \to M$ is a smooth map of manifolds. Also suppose $F(p) = F(q) = z \in M$ so that F is not injective. Now, the differentials

$$F_{*,p} : T_pN \to T_zM \text{ and } F_{*,q} : T_qN \to T_zM$$

are linear maps. Now, let $X \in \mathfrak{X}(N)$ be a C^∞ vector field on N so that X_p under $F_{*,p}$ is pushed forward to $F_{*,p}(X_p) \in T_zM$ and X_q is pushed forward to $F_{*,q}(X_q) \in T_zM$ under $F_{*,q}$. There is no reason for $F_{*,p}(X_p)$ and $F_{*,q}(X_q)$ to be the same tangent vector in T_zM. In other words, in general,

$$F_{*,p}(X_p) \neq F_{*,q}(X_q),$$

so that $z \mapsto F_{*,p}(X_p) := Y_z \in T_zM$ and $z \mapsto F_{*,q}(X_q) := Y'_z \in T_zM$ are distinct vector fields on M, denoted by Y and Y', respectively. Therefore, if there were push forward of vector fields $F_* : \mathfrak{X}(N) \to \mathfrak{X}(M)$ associated with the non-injective smooth map $F : N \to M$, there is an ambiguity regarding which vector field X gets mapped to.

Furthermore, if F is not surjective, there is $z \in M$ such that $z \neq F(p)$ for any $p \in N$. In that case as well, defining the push forward vector field $F_*(X)$ at the point z is impossible. However, when $F : N \to M$ is a diffeomorphism, one can define the push forward of a vector field.

Contrary to the non-existence of push forward of a vector field associated with a generic smooth map $F : N \to M$, one can always talk about pullback of a 1-form ω on M:

$$(F^*\omega)_p(X_p) = \omega_{F(p)}(F_{*,p}(X_p)). \tag{3.14}$$

Here, $\omega \in \Omega^1(M)$, $X_p \in T_pN$, $p \in N$. Note that $(F^*\omega)_p$ is simply the image of the covector $\omega_{F(p)} \in T^*_{F(p)}M$ under the codifferential $F^{*,p} : T^*_{F(p)}M \to T^*_pN$. In other words,

$$(F^*\omega)_p = F^{*,p}\left(\omega_{F(p)}\right). \tag{3.15}$$

4 Differential k-forms

§4.1 Definition and Local Expression

We denoted by $A_k(V)$ the vector space of alternating k-tensors on V. We have also seen that if $\{\widehat{\alpha}^1, \ldots, \widehat{\alpha}^n\}$ is a basis for 1-tensors on V, then a basis element of $A_k(V)$ is

$$\widehat{\alpha}^{i_1} \wedge \widehat{\alpha}^{i_2} \wedge \cdots \wedge \widehat{\alpha}^{i_k},$$

where $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. We apply this construction to the tangent space T_pM of a manifold M at a point $p \in M$. The vector space $A_k(T_pM)$, usually denoted by $\Lambda^k\left(T_p^*M\right)$, is the space of all alternating k-tensors on the tangent space T_pM.

> **Definition 4.1** (Differential k-form). A **differential k-form** on a manifold M is a function ω that assigns to each point $p \in M$, a k-covector $\omega_p \in \Lambda^k\left(T_p^*M\right)$. An n-from on a manifold of dimension n is called a **top degree form**.

Example 4.1. On $\mathbb{R}^n$, at each point p, there is a standard basis for the tangent space $T_p\mathbb{R}^n$:

$$\left\{ \left.\frac{\partial}{\partial r^1}\right|_p, \left.\frac{\partial}{\partial r^2}\right|_p, \ldots, \left.\frac{\partial}{\partial r^n}\right|_p \right\}.$$

Let $\left\{(\mathrm{d}r^1)_p, \ldots, (\mathrm{d}r^n)_p\right\}$ be the dual basis of $T_p^*\mathbb{R}^n$.

$$\left(\mathrm{d}r^i\right)_p \left(\left.\frac{\partial}{\partial r^j}\right|_p \right) = \delta^i{}_j.$$

As p varies over $\mathbb{R}^n$, we get differential forms $\mathrm{d}r^1, \ldots, \mathrm{d}r^n$ on $\mathbb{R}^n$. By Proposition 1.15, a basis element of alternating k-tensors $\Lambda^k\left(T_p^*\mathbb{R}^n\right)$ is

$$\left(\mathrm{d}r^{i_1}\right)_p \wedge \left(\mathrm{d}r^{i_2}\right)_p \wedge \cdots \wedge \left(\mathrm{d}r^{i_k}\right)_p,$$

where $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. If ω is a k-form on $\mathbb{R}^n$, then at each point $p \in \mathbb{R}^n$, ω_p is the following linear combination:

$$\omega_p = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} a_{i_1 \cdots i_k} \left(\mathrm{d}r^{i_1}\right)_p \wedge \left(\mathrm{d}r^{i_2}\right)_p \wedge \cdots \wedge \left(\mathrm{d}r^{i_k}\right)_p. \tag{4.1}$$

Omitting the point p, we write

$$\omega = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} a_{i_1 \cdots i_k} \, \mathrm{d}r^{i_1} \wedge \mathrm{d}r^{i_2} \wedge \cdots \wedge \mathrm{d}r^{i_k}. \tag{4.2}$$

In the expression above, $a_{i_1 \cdots i_k}$ are functions on $\mathbb{R}^n$. To simplify the notations, we use multi-indices to write (4.2) as

$$\omega = \sum_I a_I \, \mathrm{d}r^I, \tag{4.3}$$

where $\mathrm{d}r^I = \mathrm{d}r^{i_1} \wedge \mathrm{d}r^{i_2} \wedge \cdots \wedge \mathrm{d}r^{i_k}$, and $I = (i_1, i_2, \ldots, i_k)$ is a strictly ascending multi-index.

Suppose $(U, x^1, \dots, x^n)$ is a coordinate chart on a manifold M. We have already defined the 1-forms $\mathrm{d}x^1, \dots, \mathrm{d}x^n$ on U. Since at each point $p \in U$, $\left\{(\mathrm{d}x^1)_p, \dots, (\mathrm{d}x^n)_p\right\}$ is a basis for T_p^*M, by Proposition 1.15, a basis for $\Lambda^k\left(T_p^*\mathbb{R}^n\right)$ is

$$\left(\mathrm{d}x^{i_1}\right)_p \wedge \left(\mathrm{d}x^{i_2}\right)_p \wedge \dots \wedge \left(\mathrm{d}x^{i_k}\right)_p,$$

where $1 \le i_1 < i_2 < \dots < i_k \le n$. Thus, locally a k-form on U will be a linear combination

$$\omega = \sum_I a_I \, \mathrm{d}x^I, \tag{4.4}$$

where $\mathrm{d}x^I = \mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2} \wedge \dots \wedge \mathrm{d}x^{i_k}$, $I = (i_1, i_2, \dots, i_k)$ is a strictly ascending multi-index, and a_I are functions on U.

§4.2 The Bundle Point of View

Let V be a real vector space. Another common notation for the vector space $A_k(V)$ of alternating k-linear functionos on V is $\Lambda^k(V^*)$.

$$\begin{aligned}
\Lambda^0(V^*) &= A_0(V) = \mathbb{R}, \\
\Lambda^1(V^*) &= A_1(V) = V^*, \\
\Lambda^2(V^*) &= A_2(V),
\end{aligned}$$

and so on. Now, $\Lambda^k(T^*M)$ is defined to be the disjoint union of the vector spaces $\Lambda^k\left(T_p^*M\right)$ as p varies over M. So

$$\begin{aligned}
\Lambda^k(T^*M) &= \bigsqcup_{p\in M} \Lambda^k\left(T_p^*M\right) = \bigsqcup_{p\in M} A_k(T_pM) \\
&= \bigcup_{p\in M} \{p\} \times A_k(T_pM),
\end{aligned} \tag{4.5}$$

which is the set of all alternating k-tensors at all points of M. This set is called the k-th **exterior power** of the cotangent bundle T^*M.

If (U, φ) is a coordinate chart on M, then there is a bijection $\overline{\varphi} : \Lambda^k(T^*U) \to \varphi(U) \times \mathbb{R}^{\binom{n}{k}}$ defined as follows: a generic element of $\Lambda^k(T^*U)$ is (p, ω_p), where $\omega_p \in \Lambda^k\left(T_p^*U\right)$. Then ω_p is a unique linear combination

$$\omega_p = \sum_I a_I(p) \, \mathrm{d}x^I,$$

where I runs over the set of strictly ascending multi-indices of length k. There are $\binom{n}{k}$ many such multi-indices. If we fix a labeling of the multi-indices once and for all, then we have a $\binom{n}{k}$-tuple $(a_I)_I$. Then we define

$$\overline{\varphi}(p, \omega_p) = \left(\varphi(p), (a_I)_I\right) \in \varphi(U) \times \mathbb{R}^{\binom{n}{k}}.$$

Thus, $\Lambda^k(T^*U)$ is in a bijective correspondence with $\varphi(U) \times \mathbb{R}^{\binom{n}{k}}$. Using this bijective correspondence, one transfers the topology of $\varphi(U) \times \mathbb{R}^{\binom{n}{k}}$ to $\Lambda^k(T^*U)$. By varying the open set U in the charts contained in the maximal atlas of M, one can obtain a basis that generates the topology on the whole of $\Lambda^k(T^*M)$.

$\Lambda^k(T^*M)$ can, in fact, be shown to be a C^∞ vector bundle of rank $\binom{n}{k}$ over M, i.e. $\pi : \Lambda^k(T^*U) \to M$ is a C^∞ vector bundle of rank $\binom{n}{k}$ over M. A differential k-form is a section of this vector bundle. We define a k-form to be C^∞ if it is C^∞ as a section of the vector bundle $\Lambda^k(T^*M)$.

Notation. If $\pi : E \to M$ is a C^∞ vector bundle, then the vector space of C^∞ sections of E is denoted by $\Gamma(E)$, or $\Gamma(M, E)$. The vector space of all C^∞ k-forms, i.e. all C^∞ sections of the bundle $\Lambda^k(T^*M)$ is usually denoted by $\Omega^k(M)$. Thus,

$$\Omega^k(M) = \Gamma\left(\Lambda^k(T^*M)\right) = \Gamma\left(M, \Lambda^k(T^*M)\right).$$

Lemma 4.1

Let $(U, x^1, \ldots, x^n)$ be a chart on a manifold M. A k-form $\omega = \sum a_I dx^I$ on U is smooth if and only if the coefficient functions a_I are all smooth on U.

Proposition 4.2 (Characterization of a smooth k-form)

Let ω be a k-form on a manifold M. The following are equivalent:

(i) The k-form ω is C^∞ on M.

(ii) The manifold M has an atlas such that on every chart $(U, \phi) = (U, x^1, \ldots, x^n)$ in the atlas, the coefficients a_I of $\omega = \sum a_I \, \mathrm{d}x^I$ relative to the coordinate frame $\left\{\mathrm{d}x^I\right\}_I$ are all C^∞.

(iii) On every chart $(U, \phi) = (U, x^1, \ldots, x^n)$ on M, the coefficients a_I of $\omega = \sum a_I \, \mathrm{d}x^I$ relative to the coordinate frame $\left\{\mathrm{d}x^I\right\}_I$ are all C^∞.

(iv) For any k smooth vector fields $X_1, \ldots, X_k$ on M, the function $\omega(X_1, \ldots, X_k)$ is C^∞ on M.

Example 4.2. We defined the 0-tensors and the 0-covectors as constants, i.e. for a real vector space V, $A_0(V) = L_0(V) = \mathbb{R}$. Now, recall that

$$\Lambda^k(T^*M) = \bigcup_{p \in M} \{p\} \times \Lambda^k\left(T_p^*M\right).$$

Since $\Lambda^0\left(T_p^*M\right) = \mathbb{R}$ for every $p \in M$, one has

$$\Lambda^0(T^*M) = \bigcup_{p \in M} \{p\} \times \mathbb{R} = M \times \mathbb{R}. \tag{4.6}$$

Hence,

$$\Omega^0(M) = \Gamma\left(\Lambda^0(T^*M)\right) = \Gamma(M, M \times \mathbb{R}). \tag{4.7}$$

A C^∞ section of the 0-th exterior power of the tangent bundle T^*M is nothing but a C^∞ sectio of the globally trivial C^∞ vector bundle $M \times \mathbb{R}$ over M. Such a section maps $p \in M$ to a pair $(p, \sigma(p))$ with $\sigma(p) \in \mathbb{R}$. Therefore, such a section is nothing but a smooth assignment $p \mapsto \sigma(p)$, i.e. $\sigma \in C^\infty(M, \mathbb{R})$. So

$$\Omega^0(M) = \Gamma(M, M \times \mathbb{R}) = C^\infty(M, \mathbb{R}).$$

§4.3 Pullback of k-forms

Let $F : N \to M$ be a smooth map of manifolds. Recall that a 1-form $\omega \in \Omega^1(M)$ can be pulled back to $\Omega^1(N)$ via the pullback $F^* : \Omega^1(M) \to \Omega^1(N)$ defined by

$$(F^*\omega)_p(X_p) = \omega_{F(p)}(F_{*,p}(X_p)). \tag{4.8}$$

For 0-forms, i.e. functions, the pullback is defined by composition:

$$N \xrightarrow{F} M \xrightarrow{f} \mathbb{R}$$

Given $f \in C^\infty(M, \mathbb{R})$, its pullback is defined to be

$$F^*(f) = f \circ F \in C^\infty(N, \mathbb{R}), \tag{4.9}$$

so that indeed $F^* : \Omega^0(M) \to \Omega^0(N)$.

For a k-form ω on M, we define its pullback $F^*\omega$ as follows: if $p \in N$ and $X_p^1, X_p^2, \ldots, X_p^k \in T_pN$ are k tangent vectors, then

$$(F^*\omega)_p\left(X_p^1, X_p^2, \ldots, X_p^k\right) = \omega_{F(p)}\left(F_{*,p}\left(X_p^1\right), F_{*,p}\left(X_p^2\right), \ldots, F_{*,p}\left(X_p^k\right)\right). \tag{4.10}$$

Proposition 4.3 (Linearity of pullback)
Let $F : N \to M$ be a C^∞ map. If ω, τ are k-forms on M and α is a real number, then

(i) $F^*(\omega + \tau) = F^*\omega + F^*\tau$.

(ii) $F^*(\alpha\omega) = \alpha F^*\omega$.

§4.4 The Wedge Product

If $\omega \in \Omega^k(M)$ and $\tau \in \Omega^l(M)$, then for any $p \in M$, $\omega_p \in \Lambda^k\left(T_p^*M\right)$ and $\tau_p \in \Lambda^l\left(T_p^*M\right)$ and $\omega_p \wedge \tau_p \in \Lambda^{k+l}\left(T_p^*M\right)$. Then we define the wedge product of ω and τ to be the $(k+l)$-form $\omega \wedge \tau$ such that

$$(\omega \wedge \tau)_p = \omega_p \wedge \tau_p. \tag{4.11}$$

Proposition 4.4
If ω and τ are C^∞ forms on M, then so is $\omega \wedge \tau$.

Proof. Let $(U, x^1, \ldots, x^n)$ be a chart on M. On U,

$$\omega = \sum_I a_I \,\mathrm{d}x^I, \quad \tau = \sum_J b_J \,\mathrm{d}x^J \tag{4.12}$$

for C^∞ functions a_I, b_J on U. Their Wedge product is

$$\begin{aligned}\omega \wedge \tau &= \left(\sum_I a_I \,\mathrm{d}x^I\right) \wedge \left(\sum_J b_J \,\mathrm{d}x^J\right) \\ &= \sum_{I,J} a_I b_J \,\mathrm{d}x^I \wedge \mathrm{d}x^J \ldots \end{aligned}\tag{4.13}$$

In (4.13), $\mathrm{d}x^I \wedge \mathrm{d}x^J = 0$ if I and J have at least an index in common. If I and J are disjoint, i.e., have none of their indices to be common, then

$$\mathrm{d}x^I \wedge \mathrm{d}x^J = \pm \mathrm{d}x^K, \tag{4.14}$$

where $K = I \cup J$ but reordered as an increasing sequence. Thus,

$$\omega \wedge \tau = \sum_K \left(\sum_{I \cup J = K} \pm a_I b_J\right) \mathrm{d}x^K. \tag{4.15}$$

Since the coefficients of $\mathrm{d}x^K$ in (4.15) are C^∞, by Proposition 4.2, $\omega \wedge \tau$ is C^∞ on M. ∎

Proposition 4.5 (Pullback of wedge product)
If $F : N \to M$ is a C^∞ map of manifolds and ω are τ are differential forms on M, then

$$F^*(\omega \wedge \tau) = F^*(\omega) \wedge F^*(\tau). \tag{4.16}$$

We define the vector space $\Omega^*(M)$ of C^∞ differential forms on a manifold M of dimension n to be the direct sum

$$\Omega^*(M) = \bigoplus_{k=0}^n \Omega^k(M). \tag{4.17}$$

Each element of $\Omega^*(M)$ is uniquely a formal sum $\sum_{i=1}^r \omega_{k_i}$ with $\omega_{k_i} \in \Omega^{k_i}(M)$. With the wedge product, the vector space $\Omega^*(M)$ becomes a **graded algebra**, graded by the the degree of differential forms. Proposition 4.3 and Proposition 4.5 tells us that the pullback map $F^* : \Omega^*(M) \to \Omega^*(N)$ is a homomorphism of graded algebras.

5 Exterior Derivative

The basic objects in differential geometry are differential forms. Our goal will be to learn how we can differentiate and integrate differential forms on manifolds. Recall that an antiderivation on a graded algebra $A = \bigoplus_{k=0}^{\infty} A^k$ is an $\mathbb{R}$-linear map $D : A \to A$ such that

$$D(\omega \cdot \tau) = (D\omega) \cdot \tau + (-1)^k \omega \cdot (D\tau),$$

for $\omega \in A^k$ and $\tau \in A^l$, and $\cdot$ is the multiplication of the graded algebra. In the graded algebra A, an element of A^k is called a **homogenous element of degree** k. The antiderivation D is of degree m if

$$\deg(D\omega) = \deg \omega + m$$

for all homogenous elements $\omega \in A$.

Now, let M be a manifold and $\Omega^*(M)$ the graded algebra of C^∞ differential forms on M. Now, we'll see that on the graded algebra $\Omega^*(M)$, there is a uniquely and intrinsically defined anti-derivation called exterior derivative.

Definition 5.1 (Exterior derivative)**.** An **exterior derivative** on a manifold M is an $\mathbb{R}$-linear map

$$D : \Omega^*(M) \to \Omega^*(M)$$

such that

(i) D is an antiderivation of degree 1,

(ii) $D \circ D = 0$,

(iii) if f is a C^∞ function and X is a C^∞ vector field on M, then $(Df)(X) = Xf$.

Remark 5.1. Condition (iii) in the definition above says that on 0-forms, i.e. C^∞ functions on M, an exterior derivative agrees with the differential $\mathrm{d}f$ of a function f. We have learned earlier that in a coordinate chart $(U, x^1, \ldots, x^n)$, the 1-form $\mathrm{d}f$ can be expressed as

$$\mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} \mathrm{d}x^i.$$

Hence, in the chart $(U, x^1, \ldots, x^n)$,

$$Df = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} \mathrm{d}x^i.$$

We now prove the existence and uniqueness of the exterior differentiation on a manifold.

Lemma 5.1

Let $D\Omega^*(M) \to \Omega^*(M)$ be an exterior derivative on M. If $f^1, \ldots, f^k$ are smooth functions on U, then

$$D\left(Df^1 \wedge Df^2 \wedge \cdots \wedge Df^k\right) = 0.$$

Proof. We prove it by induction on k. The base case $k = 1$ follows trivially from $D \circ D = 0$.

Suppose $D\left(Df^1 \wedge Df^2 \wedge \cdots \wedge Df^{k-1}\right) = 0$. Then

$$\begin{aligned} D\left(Df^1 \wedge Df^2 \wedge \cdots \wedge Df^k\right) &= D\left(\left(Df^1 \wedge Df^2 \wedge \cdots \wedge Df^{k-1}\right) \wedge Df^k\right) \\ &= D\left(Df^1 \wedge Df^2 \wedge \cdots \wedge Df^{k-1}\right) \wedge Df^k + (-1)^{k-1}\left(Df^1 \wedge Df^2 \wedge \cdots \wedge Df^{k-1}\right) \wedge \\ &= 0. \end{aligned} \tag{5.1}$$

Therefore, $D\left(Df^1 \wedge Df^2 \wedge \cdots \wedge Df^k\right) = 0$ for any $k \geq 1$. ■

§5.1 Exterior Derivative on a Coordinate Chart

Suppose $(U, x^1, \ldots, x^n)$ is a coordinate chart on a manifold M. Then any k-form ω on U is uniquely a linear combination

$$\omega = \sum_I a_I \, \mathrm{d}x^I,$$

where $a_I \in C^\infty(U)$, and the sum runs over all strictly ascending multi-indices I of length k. The $\mathbb{R}$-linear map $\mathrm{d} : \Omega^*(U) \to \Omega^*(U)$ can be constructed to be an exterior derivative on U. In fact, d acts on a homogenous element $\omega \in \Omega^k(U)$ in the following way:

$$\begin{aligned} \mathrm{d}\omega = \mathrm{d}\left(\sum_I a_I \, \mathrm{d}x^I\right) &= \sum_I \mathrm{d}a_I \wedge \mathrm{d}x^I + (-1)^0 \sum_I a_I \, \mathrm{d}\mathrm{d}x^I \\ &= \sum_I \mathrm{d}a_I \wedge \mathrm{d}x^I + \sum_I a_I \, \mathrm{d}\left(\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}\right) \\ &= \sum_I \mathrm{d}a_I \wedge \mathrm{d}x^I \\ &= \sum_I \sum_j \frac{\partial a_I}{\partial x^j} \, \mathrm{d}x^j \wedge \mathrm{d}x^I. \end{aligned} \tag{5.2}$$

(5.2) suggests that $\mathrm{d}\omega \in \Omega^{k+1}(U)$, and it can be written in the chart $(U, x^1, \ldots, x^n)$ using (5.2). This proves the existence of the exterior derivative $\mathrm{d} : \Omega^*(U) \to \Omega^*(U)$, on an open set U of M. The uniqueness of $\mathrm{d} : \Omega^*(U) \to \Omega^*(U)$ can be shown exactly the same way we proved it for the Euclidean case in Proposition 2.4.

Sometimes we write $\mathrm{d}_U\omega$ instead of $\mathrm{d}\omega$ to emphasize that it is the **unique** exterior derivative on the open set $U \subseteq M$. In other words, if (U, x^i) and (U, y^j) are two charts on M, and $\omega = \sum a_I \, \mathrm{d}x^I = \sum b_J \, \mathrm{d}y^J$, then

$$\mathrm{d}_U\omega = \sum_I \sum_i \frac{\partial a_I}{\partial x^i} \, \mathrm{d}x^i \wedge \mathrm{d}x^I = \sum_J \sum_i \frac{\partial b_J}{\partial y^i} \, \mathrm{d}y^i \wedge \mathrm{d}y^I. \tag{5.3}$$

This reveals that the expression $\mathrm{d}_U\omega$ is chart independent.

§5.2 Local Operators

An endomorphism of a vector space W (a linear transformation from W to itself) is often called an operator on W. For example, if $W = C^\infty(\mathbb{R})$, the vector space of C^∞ functions on $\mathbb{R}$, then $\frac{\mathrm{d}}{\mathrm{d}x}$ is an operator on W:

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) = f'(x).$$

The derivative has the desired property that the value of f' at a point p depends only on the values of f in a small neighborhood of p. More precisely, if $f = g$ on an open set $U \subseteq \mathbb{R}$, then $f' = g'$ on U. We say that the derivative is a local operator on $C^\infty(\mathbb{R})$.

Definition 5.2 (Local operator). An operator $D : \Omega^*(M) \to \Omega^*(M)$ is said to be **local** if for all $k \geq 0$, whenever a k-form $\omega \in \Omega^k(M)$ restricts to 0 on an open set U (i.e. $\omega_p = 0$ at every $p \in U$), then $D\omega \equiv 0$ on U (i.e. $(D\omega)_p = 0$ at every $p \in U$).

An equivalent definition of local operator is that for all $k \geq 0$, whenever two k-forms $\omega, \tau \in \Omega^k(M)$ agree on an open set U, then $D\omega \equiv D\tau$ on U (i.e. $(D\omega)_p = (D\tau)_p$ at every $p \in U$).

Proposition 5.2

Any antiderivation D on $\Omega^*(M)$ is a local operator.

Proof. Suppose $\omega \in \Omega^*(M)$ and $\omega \equiv 0$ on an open subset U. Let $p \in U$. It suffices to show that $(D\omega)_p = 0$. Take a bump function f at p supported in U, i.e. $\operatorname{supp} f \subseteq U$. In particular, $f \equiv 1$ in a neighborhood V of p in U, so that $V \subset \operatorname{supp} f \subseteq U$. Then $f\omega \equiv 0$ on M. This can be seen by noting that if $q \in U$,

$$(f\omega)_q = f(q)\omega_q = 0,$$

since $\omega_q = 0$ by hypothesis. On the other hand, if $q \notin U$, then $q \notin \operatorname{supp} f$, so $f(q) = 0$, which yields

$$(f\omega)_q = f(q)\omega_q = 0.$$

Therefore, $f\omega \equiv 0$ on M. Applying D on $f\omega = f \wedge \omega$, we get

$$D(f\omega) = (Df) \wedge \omega + (-1)^0 f \wedge D\omega. \tag{5.4}$$

By the linearity of D, $D(f\omega) = 0$. Now, we evaluate the RHS of (5.4) at $p \in U$, and use the fact that $f(p) = 1$ and $\omega_p = 0$. As a result,

$$\begin{aligned} & (Df)_p \wedge \omega_p + f(p) \wedge (D\omega)_p = 0 \\ \implies & (D\omega)_p = 0. \end{aligned} \tag{5.5}$$

Since $p \in U$ is arbitrary, $D\omega \equiv 0$ on U. ■

Sometimes we are given a differential form τ that is defined only on an open subset U of a manifold M. We can use bump functions to extend τ to a global form $\tilde{\tau}$ on M that agrees with τ near some point.

Proposition 5.3

Suppose τ is a C^∞ differential k-form on an open subset U of M (such a differential form is called a local differential form). For any $p \in U$. There is a C^∞ global form $\tilde{\tau}$ on M (can be defined anywhere on M using its charts) that agrees with τ on a neighborhood of p contained in U.

Proof. Choose a smooth bump function f at p supported in U, i.e. $\operatorname{supp} f \subseteq U$. In particular, $f \equiv 1$ in a neighborhood V of p in U, so that $V \subset \operatorname{supp} f \subseteq U$. Then we define

$$\tilde{\tau}_q = \begin{cases} f(q)\,\tau_q & \text{if } q \in U, \\ \mathbf{0}_{\Lambda^k(T_q^*M)} & \text{if } q \notin U. \end{cases}$$

By the definition of $\tilde{\tau}$, it agrees with τ on V. By *Proposition 9.3.1(ii)* of DG1, $\tilde{\tau}$ is smooth on U. Now, let $q \notin U$. We want to show that $\tilde{\tau}$ is smooth at q.

Since $\operatorname{supp} f \subseteq U$, $q \notin U$ implies $q \in M \setminus U \subseteq M \setminus \operatorname{supp} f$. Since $\operatorname{supp} f$ is closed, $M \setminus \operatorname{supp} f$ is open. Hence, we can find a coordinate chart (W, φ) about q such that $W \subseteq M \setminus \operatorname{supp} f$. Then, for $r \in W$, $\tilde{\tau}_r = \mathbf{0}_{\Lambda^k(T_r^*M)}$. Also, $\left(\Lambda^k(T^*U), \overline{\varphi}\right)$ is a chart on $\Lambda^k(T^*M)$ about $\mathbf{0}_{\Lambda^k(T_r^*M)}$.

$$(\overline{\varphi} \circ \tilde{\tau})(r) = (\varphi(r), \underbrace{0, 0, \ldots, 0}_{\binom{n}{k} \text{ 0-s}}).$$

φ is smooth. Therefore, $\tilde{\tau}$ is smooth on W. In particular, $\tilde{\tau}$ is smooth at q. Since $q \notin U$ was arbitrary, $\tilde{\tau}$ is smooth at every $q \notin U$. Therefore, $\tilde{\tau}$ is smooth on all of M. ■

§5.3 Existence and Uniqueness of an Exterior Differentiation

To define an exterior derivative $\mathrm{d} : \Omega^*(M) \to \Omega^*(M)$, let $\omega \in \Omega^k(M)$ and $p \in M$. Choose a chart $(U, x^1, \ldots, x^n)$ about p. Suppose $\omega = \sum_I a_I \, \mathrm{d}x^I$ on U. Now, $\mathrm{d}\omega$ is supposed to be a $(k+1)$-form on M, i.e. $\mathrm{d}\omega \in \Omega^{k+1}(M)$. Define $\mathrm{d}\omega \in \Omega^{k+1}(M)$ such that at $p \in U$, $(\mathrm{d}\omega)_p$ is expressed as

$$(\mathrm{d}\omega)_p = \left(\sum_I \mathrm{d}a_I \wedge \mathrm{d}x^I \right)_p . \tag{5.6}$$

It needs to be proven that the definition (5.6) is independent of chart. If $(V, y^1, \ldots, y^n)$ is another chart about p, and $\omega = \sum_J b_J \, \mathrm{d}y^J$ on V, then on $U \cap V$,

$$\sum_I a_I \, \mathrm{d}_{U\cap V} x^I = \sum_J b_J \, \mathrm{d}_{U\cap V} y^J ,$$

where $\mathrm{d}_{U\cap V}$ is the unique exterior derivative $\mathrm{d}_{U\cap V} : \Omega^*(U\cap V) \to \Omega^*(U \cap V)$. Then by the locality of exterior derivative,

$$\mathrm{d}_{U\cap V} \left(\sum_I a_I \, \mathrm{d}_{U\cap V} x^I \right) = \mathrm{d}_{U\cap V} \left(\sum_J b_J \, \mathrm{d}_{U\cap V} y^J \right) . \tag{5.7}$$

Reading off the antiderivation $\mathrm{d}_{U\cap V}$ in the chart $(U \cap V, x^1, \ldots, x^n)$ using (5.6), the LHS of (5.7) can be recast into

$$\sum_I \mathrm{d}_{U\cap V} a_I \, \mathrm{d}_{U\cap V} x^I .$$

On the other hand, the antiderivation $\mathrm{d}_{U\cap V}$ in the chart $(U \cap V, y^1, \ldots, y^n)$ can be expressed using (5.6) to compute the RHS of (5.7):

$$\sum_J \mathrm{d}_{U\cap V} b_J \, \mathrm{d}_{U\cap V} y^J .$$

Therefore,

$$\sum_I \mathrm{d}_{U\cap V} a_I \, \mathrm{d}_{U\cap V} x^I = \sum_J \mathrm{d}_{U\cap V} b_J \, \mathrm{d}_{U\cap V} y^J , \tag{5.8}$$

on $U \cap V$. In particular, for $p \in U \cap V$,

$$\left(\sum_I \mathrm{d}_{U\cap V} a_I \, \mathrm{d}_{U\cap V} x^I \right)_p = \left(\sum_J \mathrm{d}_{U\cap V} b_J \, \mathrm{d}_{U\cap V} y^J \right)_p ,$$

proving that the definition (5.6) is indeed chart independent. As p varies over all of M, (5.6) defines an operator

$$\mathrm{d} : \Omega^*(M) \to \Omega^*(M).$$

It's straightforward to verify that the 3 desired conditions of exterior derivative are fullfilled by the definition (5.6).

Now we prove the uniqueness of exterior derivative. Suppose $D : \Omega^*(M) \to \Omega^*(M)$ is an exterior derivative. We will now show that D coincides with the exterior derivative defined by (5.6).

Let $\omega \in \Omega^k(M)$, and $p \in M$. Choose a chart $(U, x^1, \ldots, x^n)$ about p, and suppose $\omega = \sum_I a_I \, \mathrm{d}x^I$ on U. Extend the functions $a_I, x^1, \ldots, x^n$ to C^∞ functions $\widetilde{a}_I, \widetilde{x}^1, \ldots, \widetilde{x}^n$ that agrees with $a_I, x^1, \ldots, x^n$ in a neighborhood V of p. Define

$$\widetilde{\omega} = \sum_I \widetilde{a}_I \, \mathrm{d}\widetilde{x}^I . \tag{5.9}$$

Then $\omega \equiv \widetilde{\omega}$ on V. Since D is a local operator, one must have $D\omega \equiv D\widetilde{\omega}$ on V. Thus,

$$(D\omega)_p = (D\widetilde{\omega})_p = \left[D \left(\sum_I \widetilde{a}_I \, \mathrm{d}\widetilde{x}^I \right) \right]_p . \tag{5.10}$$

Since D is an exterior derivative operator on $\Omega^* M$, and d is the exterior derivative operator defined by (5.6), for $f \in C^\infty(M)$,

$$(Df)(X) = Xf = (\mathrm{d}f)(X),$$

for any C^∞ vector field X. In particular,

$$D\widetilde{a}_I = \mathrm{d}\widetilde{a}_I, \text{ and } D\widetilde{x}^i = \mathrm{d}\widetilde{x}^i,$$

so that $D\widetilde{x}^I = d\widetilde{x}^I$, for a strictly ascending multi-index I of length k. Hence, (5.10) reduces to

$$\begin{aligned}(D\omega)_p &= \left[D\left(\sum_I \widetilde{a}_I\, \mathrm{d}\widetilde{x}^I\right)\right]_p \\ &= \left[D\left(\sum_I \widetilde{a}_I\, D\widetilde{x}^I\right)\right]_p \\ &= \left(\sum_I D\widetilde{a}_I \wedge D\widetilde{x}^I\right)_p \\ &= \left(\sum_I \mathrm{d}\widetilde{a}_I \wedge \mathrm{d}\widetilde{x}^I\right)_p .\end{aligned}$$

Now, since $\widetilde{a}_I = a_I$ and $\widetilde{x}^i = x^i$ in a neighborhood of p, we have $\mathrm{d}\widetilde{a}_I = \mathrm{d}a_I$ and $\mathrm{d}\widetilde{x}^I = \mathrm{d}x^I$ at p. Therefore,

$$(D\omega)_p = \left(\sum_I \mathrm{d}\widetilde{a}_I \wedge \mathrm{d}\widetilde{x}^I\right)_p = \left(\sum_I \mathrm{d}a_I \wedge \mathrm{d}x^I\right)_p = (\mathrm{d}\omega)_p . \tag{5.11}$$

So $D = d$, and hence the exterior derivative is unique.

The restriction of a k-form to a submanifold

Let S be a regular submanifold of a manifold M, and ω is a k-form on M, i.e. $\omega \in \Omega^k(M)$. Then the restriction of ω to S is the k-form $\omega\big|_S$ on S defined by

$$\left(\omega\big|_S\right)_p \left(X_p^1, \ldots, X_p^k\right) = \omega_p \left(X_p^1, \ldots, X_p^k\right), \tag{5.12}$$

for $X_p^1, \ldots, X_p^k \in T_pS \subseteq T_pM$. Thus, $\left(\omega\big|_S\right)_p$ is obtained from ω_p by restricting its domain to $T_pS \times T_pS \times \cdots \times T_pS$ (k-times).

Example 5.1. If S is a smooth curve in $\mathbb{R}^2$ defined by the non-constant function $f(x, y) = 0$ (f could be $x^2 + y^2 - 1$, defining the unit circle in $\mathbb{R}^2$), then

$$\mathrm{d}f = \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y$$

is a nonzero 1-form on $\mathbb{R}^2$. But since f is identically 0 on S, $(\mathrm{d}f)\big|_S = 0$. So a nonzero form on M can be restricted to a zero form on a submanifold S.

A form that is not identically zero is called a **nonzero form**. On the other hand, a form ω that is nowhere zero,i.e. $\omega_p \neq 0$ for all $p \in M$, is called a **nowhere vanishing form**.

Example 5.2 (A nowhere vanishing 1-form on S^1)**.** Let S^1 be the unit circle defined by $x^2 + y^2 = 1$ in $\mathbb{R}^2$. The 1-form $\mathrm{d}x$ restricts from $\mathbb{R}^2$ to a 1-form on S^1. When restricted to S^1, the domain of the covector $\left((\mathrm{d}x)\big|_{S^1}\right)_p$ is T_pS^1 instead of $T_p\mathbb{R}^2$:

$$\left((\mathrm{d}x)\big|_{S^1}\right)_p : T_pS^1 \to \mathbb{R}.$$

Now, from $x^2 + y^2 = 1$, one obtains

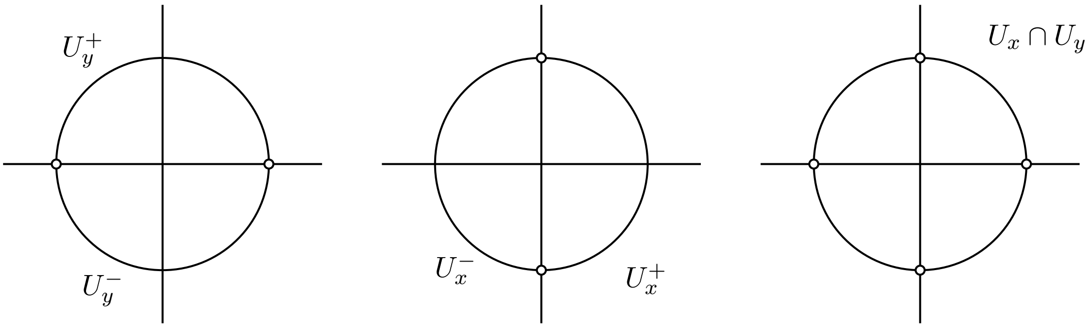
$$2x\,\mathrm{d}x + 2y\,\mathrm{d}y = 0. \tag{5.13}$$

At $p = (1, 0)$, (5.13) reduces to $(\mathrm{d}x)_p = 0$. It shows that although $\mathrm{d}x$ is a nowhere vanishing 1-form on $\mathbb{R}^2$, it vanishes at $(1, 0)$ when restricted to S^1.

To find a nowhere vanishing 1-form on S^1, we again take exterior derivative of both sides of the equation $x^2 + y^2 - 1 = 0$ to arrive at

$$2x\,\mathrm{d}x + 2y\,\mathrm{d}y = 0. \tag{5.14}$$

Let $U_x = \{(x, y) \in S^1 \mid x \neq 0\}$, and $U_y = \{(x, y) \in S^1 \mid y \neq 0\}$.



By (5.14), then one obtains on $U_x \cap U_y$,

$$\frac{\mathrm{d}y}{x} = -\frac{\mathrm{d}x}{y}. \tag{5.15}$$

Now we define a 1-form ω on S^1 by

$$\omega = \begin{cases} \frac{\mathrm{d}y}{x} & \text{on } U_x, \\ -\frac{\mathrm{d}x}{y} & \text{on } U_y. \end{cases} \tag{5.16}$$

Since $\frac{\mathrm{d}y}{x} = -\frac{\mathrm{d}x}{y}$ on $U_x \cap U_y$, ω is a well-defined 1-form on $S^1 = U_x \cup U_y$. To show that ω is C^∞ and nowhere vanishing, we need charts.

$$U_x^+ = \left\{(x, y) \in S^1 \mid x > 0\right\}, U_x^- = \left\{(x, y) \in S^1 \mid x < 0\right\},$$
$$U_y^+ = \left\{(x, y) \in S^1 \mid y > 0\right\}, U_y^- = \left\{(x, y) \in S^1 \mid y < 0\right\}.$$

On U_x^+, the local coordinates are the y-coordinates, so that $(\mathrm{d}y)_p$ is a basis for the cotangent space $T_p^* S^1$ at each $p \in U_x^+$. Now, since $\omega = \frac{\mathrm{d}y}{x}$ on U_x^+, ω is C^∞ and nowhere zero on U_x^+. Similarly, $\omega = \frac{\mathrm{d}y}{x}$ on U_x^- is also C^∞ and nowhere zero on U_x^-. One can show using similar argument that $\omega = -\frac{\mathrm{d}x}{y}$ is C^∞ and nowhere vanishing on U_y^+ and U_y^-. Hence, ω is C^∞ and nowhere zero on S^1.