Differential Geometry (MAT313)

Lecture Notes from Fall 2020

Preface

This series of lecture notes has been prepared for aiding students who took the BRAC University course **Differential Geometry (MAT313)** in Fall 2020 semester. These notes were typeset under the supervision of mathematician **Dr. Syed Hasibul Hassan Chowdhury**. The recorded video lectures can be found here. The main goal of this typeset is to have an organized digital version of the notes, which is easier to share and handle. If you see any mistakes or typos, please send me an email at atonuroychowdhury@gmail.com

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References:

- *An Introduction to Manifolds*, by Loring W. Tu
- *An Introduction to Differentiable Manifolds and Riemannian Geometry*, by William Boothby
- *Introduction to Smooth Manifolds*, by John M. Lee
- *Lectures on Differential geometry*, by S.S Chern, W.H. Chen and K.S. Lam

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1 Topology Review

§1.1 Euclidean Space $\mathbb{R}^n$

Before embarking on the concept of general topological space, let us look at the Euclidean space $\mathbb{R}^n$. $\mathbb{R}^n$ is equipped with the notion of distance between 2 points p and q.

Definition 1.1.1 (Distance). Let the coordinates of p and q be $(p^1, p^2, ...p^n)$ and $(q^1, q^2, ...q^n)$, respectively. The distance between p and q is given by

$$d(p,q) = \left[\sum_{i=1}^{n}(p^i - q^i)^2\right]^{\frac{1}{2}}$$

Definition 1.1.2 (Open ball). An open ball $B(p,r)$ in $\mathbb{R}^n$ with center $p \in \mathbb{R}^n$ and radius $r > 0$ is defined as the set

$$B(p,r) = \{x \in \mathbb{R}^n \ : \ d(x,p) < r\}$$

A set equipped with the notion of distance between its elements is called a metric space[1]. Thus the Euclidean space $\mathbb{R}^n$ is a metric space. And we can talk about open balls in $\mathbb{R}^n$ using this metric. We can define open sets in $\mathbb{R}^n$ using open balls $B(p,r)$ defined above.

Definition 1.1.3 (Open Set in $\mathbb{R}^n$). A set U in $\mathbb{R}^n$ is said to be open if for every p in U, there is an open ball $B(p,r)$ such that $B(p,r) \subseteq U$.

Proposition 1.1.1

The union of an arbitrary collection of $\{U_\alpha\}$ of open sets is open. The intersection of finite collection of open sets is open.

Proof. Trivial. ■

Example 1.1.1

The intervals $\left(-\frac{1}{n}, \frac{1}{n}\right)$, $n = 1, 2, 3, ...$ are all open in $\mathbb{R}$ but their intersection

$$\bigcap_{n\in\mathbb{N}} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$$

is not open.

The metric d in $\mathbb{R}^n$ allows us to define open sets in $\mathbb{R}^n$. In other words, given a subset of $\mathbb{R}^n$, we can tell if it is open or not. This situation is a special case called **metric topology in $\mathbb{R}^n$**.

§1.2 Topology

[1]There are some properties that a metric (distance) function should have. We won't go into much details

Definition 1.2.1 (Topology)**.** A topology on a set S is a collection $\mathcal{T}$ of subsets of S containing both the empty set $\varnothing$ and the S such that $\mathcal{T}$ is closed under arbitrary union and finite intersection. In other words,

- If $U_\alpha \in \mathcal{T}$ for all α in an index set A, then $\bigcup\limits_{\alpha \in A} U_\alpha \in \mathcal{T}$
- If $U_i \in \mathcal{T}$ for $i \in \{1, 2, ..., n\}$, then $\bigcap\limits_{i=1}^{n} U_i \in \mathcal{T}$

The elements of $\mathcal{T}$ are called open sets.

Definition 1.2.2 (Topological Space)**.** The pair $(S, \mathcal{T})$ consisting of a set S together with a topology $\mathcal{T}$ on S is called a **topological space**.

Abuse of Notation. We shall often say "S is a topological space" in short. But there is always a topology $\mathcal{T}$ on S, which we recall when necessary.

Definition 1.2.3 (Neighborhood)**.** A **neighbourhood** of a point $p \in S$ is called an open set U containing p.

Definition 1.2.4 (Closed Set)**.** The complement of an open set is called a **closed set**.

Proposition 1.2.1

The union of a finite collection of closed sets is closed. The intersection of an arbitrary collection of closed sets is closed.

Proof. Let $\{F_i\}_{i=1}^n$ be a finite collection of closed sets. Then, $\{S \setminus F_i\}_{i=1}^n$ is a finite collection of open sets. The intersection of a finite collection of open sets is open, therefore $\bigcap\limits_{i=1}^{n} (S \setminus F_i)$ is open. By De Morgan's law,

$$\bigcap_{i=1}^{n} (S \setminus F_i) = S \setminus \left(\bigcup_{i=1}^{n} F_i \right) \quad \text{is open} \implies \bigcup_{i=1}^{n} F_i \quad \text{is closed}$$

Therefore, the union of a finite collection of closed sets is closed.
Now, let $\{F_\alpha\}_{\alpha \in A}$ be an arbitrary collection of closed sets with A being an index set. Then $\{S \setminus F_\alpha\}_{\alpha \in A}$ is an arbitrary collection of open sets. We know that the union of an arbitrary collection of open sets is open, therefore $\bigcup_{\alpha \in A} (S \setminus F_\alpha)$ is open. By De Morgan's law,

$$\bigcup_{\alpha \in A} (S \setminus F_\alpha) = S \setminus \left(\bigcap_{\alpha \in A} F_\alpha \right) \quad \text{is open} \implies \bigcap_{\alpha \in A} F_\alpha \quad \text{is closed}$$

Therefore, the intersection of an arbitrary collection of closed sets is closed. ■

Definition 1.2.5 (Subspace Topology)**.** Let $(S, \mathcal{T})$ be a topological space and A a subset of S. Define $\mathcal{T}_A$ to be the collection of subsets

$$\mathcal{T}_A = \{U \cap A \mid U \in \mathcal{T}\}$$

$\mathcal{T}_A$ is called the **subspace topology** of A in S.

It is not hard to see that $\mathcal{T}_A$ satisfies the conditions of a Topology. Firstly, $\mathcal{T}_A$ contains both $\varnothing$ and A. For these, taking $U = \varnothing$ and $U = S$, respecitvely, suffices. By the distributive property of union and intersection

$$\bigcup_{\alpha}(U_\alpha \cap A) = \left(\bigcup_{\alpha} U_\alpha\right) \cap A \text{ and } \bigcap_{i=1}^{n}(U_i \cap A) = \left(\bigcap_{i=1}^{n} U_i\right) \cap A$$

which shows that $\mathcal{T}_A$ is closed under arbitrary union and finite intersection. So $\mathcal{T}$ is a Topology indeed.

Example 1.2.1

Consider the subset $A = [0,1]$ of $\mathbb{R}$. In the subspace topology, the half-open interval $\left[0, \frac{1}{2}\right)$ is an open subset of A, because $\left[0, \frac{1}{2}\right) = \left(-\frac{1}{2}, \frac{1}{2}\right) \cap [0,1]$

Lemma 1.2.2

Let Y be a subspace of X (that is Y has the subspace topology inherited from X). If U is open in Y and Y is open in X, then U is open in X.

Proof. Since U is open in Y, $U = Y \cap V$ for some V open in X. Both Y and V are open in X, hence $Y \cap V = U$ is also open in X. ∎

§1.3 Bases and Countability

Definition 1.3.1 (Basis and Basic Open Sets)**.** A subcollection $\mathcal{B}$ of a topology $\mathcal{T}$ is a **basis** for $\mathcal{T}$ if given an open set U and a point p in U, there is an open set $\mathrm{B} \in \mathcal{B}$ such that $p \in \mathrm{B} \subseteq U$. An element of $\mathcal{B}$ is called a **basic open set**.

Example 1.3.1

The collection of all open balls $B(p, r)$ in $\mathbb{R}^n$ with $p \in \mathbb{R}^n$ and $r > 0$ is a basis for the standard topology (metric topology) on $\mathbb{R}^n$.

Proposition 1.3.1

A collection $\mathcal{B}$ of open sets of S is a basis if and only if every open set in S is a union of sets in $\mathcal{B}$.

Proof. ($\Rightarrow$) We are given a collection of $\mathcal{B}$ of open sets of S that is a basis. U is any open set in S. Also, let $p \in U$. Therefore, there is a basic open set $B_p \in \mathcal{B}$ such that $p \in B_p \subseteq U$. Hence, one can show that $U = \bigcup\limits_{p \in U} B_p$.

($\Leftarrow$) Suppose, every open set in S is a union of open sets in $\mathcal{B}$. Now, given an open set U and a point $p \in U$, since $U = \bigcup\limits_{B_\alpha \in \mathcal{B}} B_\alpha$, there is a $B_\alpha \in \mathcal{B}$, such that $p \in B_\alpha \subseteq U$. Hence $\mathcal{B}$ is a basis. ∎

Proposition 1.3.2

A collection $\mathcal{B}$ of subsets of a set S is a basis for some topology $\mathcal{T}$ on S if and only if

(i) S is the union of all the sets in $\mathcal{B}$, and

(ii) given any two sets B_1 and $B_2 \in \mathcal{B}$ and a point $p \in B_1 \cap B_2$, there is a set $B \in \mathcal{B}$ such that $p \in B \subset B_1 \cap B_2$.

Proof. (⇒) (i) follows from Proposition 1.3.1.

(ii) If $\mathcal{B}$ is a basis, then B_1 and B_2 are open sets and hence so is $B_1 \cap B_2$. By the definition of a basis, there is a $B \in \mathcal{B}$ such that $p \in B \subseteq B_1 \cap B_2$.

(⇐) Define $\mathcal{T}$ to be the collection consisting of all sets that are unions of sets in $\mathcal{B}$. Then the empty set $\varnothing$ and the set S are in $\mathcal{T}$ and $\mathcal{T}$ is clearly closed under arbitrary union. To show that $\mathcal{T}$ is closed under finite intersection, let $U = \bigcup_\mu B_\mu$ and $V = \bigcup_\nu B_\nu$ be in $\mathcal{T}$, where $B_\mu, B_\nu \in \mathcal{B}$. Then

$$U \cap V = \left(\bigcup_\mu B_\mu\right) \cap \left(\bigcup_\nu B_\nu\right) = \bigcup_{\mu,\nu} (B_\mu \cap B_\nu) \ .$$

Thus, any p in $U \cap V$ is in $B_\mu \cap B_\nu$ for some μ, ν. By (ii) there is a set B_p in $\mathcal{B}$ such that $p \in B_p \subseteq B_\mu \cap B_V$. Therefore,

$$U \cap V = \bigcup_{p \in U \cap V} B_p \in \mathcal{T} \,.$$

Therefore, $\mathcal{B}$ generates a topology on S. ■

We say that a point in $\mathbb{R}^n$ is rational if all of its coordinates are rational numbers. Let $\mathbb{Q}$ be the set of rational numbers and $\mathbb{Q}^+$ the set of positive rational numbers.

Lemma 1.3.3

Every open set in $\mathbb{R}^n$ contains a rational point.

Proof. An open set U in $\mathbb{R}^n$ contains an open ball $B(p, r)$ which, in turn, contains an open cube $\prod_{i=1}^n I_i$ where I_i is the open interval $\left(p^i - \frac{r}{\sqrt{n}}, p^i + \frac{r}{\sqrt{n}}\right)$. Here is a visual example for $n = 2$.

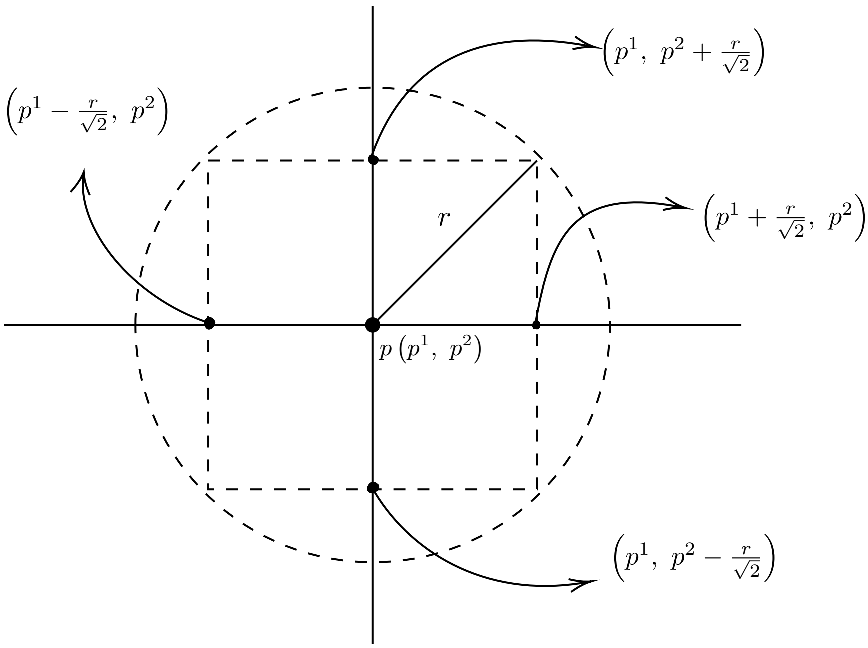


Figure 1.1: $B(p,r)$ contains $\left(p^1 - \frac{r}{\sqrt{n}}, p^1 + \frac{r}{\sqrt{n}}\right) \times \left(p^2 - \frac{r}{\sqrt{n}}, p^2 + \frac{r}{\sqrt{n}}\right)$

Now back to general n. For each i, let q^i be a rational number in I_i. Then $(q^1, q^2, ..., q^n)$ is a rational point in $\prod_{i=1}^n I_i \subseteq B(p, r)$. Therefore, every open set contains a rational point. ■

Proposition 1.3.4

The collection $\mathcal{B}_{\mathbb{Q}}$ of all open balls in $\mathbb{R}^n$ with rational centers and rational radii is a basis for $\mathbb{R}^n$.

Proof. Given an open set U in $\mathbb{R}^n$ and $p \in U$, there is an open ball $B(p, r')$ with positive real radius r' such that $p \in B(p, r') \subseteq U$. Take a rational number $r \in (0, r')$. Then we have

$$p \in B(p, r) \subseteq B(p, r') \subseteq U$$

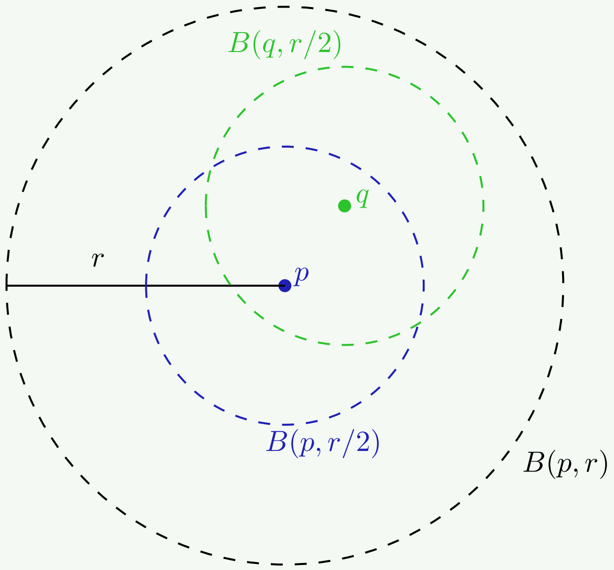
By Lemma 1.3.3 , there is a rational point in the smaller ball $B\left(p, \frac{r}{2}\right)$.

Claim — $p \in B\left(q, \frac{r}{2}\right) \subseteq B(p, r)$

Proof. Since $d(p, q) < \frac{r}{2}$, we have $p \in B\left(q, \frac{r}{2}\right)$. Next, if $x \in B\left(q, \frac{r}{2}\right)$, then by triangle inequality

$$d(x, p) \leq d(x, q) + d(q, p) < \frac{r}{2} + \frac{r}{2} = r$$

Therefore, $x \in B(p, r)$.



So, $p \in B\left(q, \frac{r}{2}\right)$ and $B\left(q, \frac{r}{2}\right) \subseteq B(p, r)$. □

As a result, $p \in B\left(q, \frac{r}{2}\right) \subseteq B(p, r) \subseteq B\left(p, r'\right) \subseteq U$. Hence we proved,

$$p \in B\left(q, \frac{r}{2}\right) \subseteq U$$

In other words, the collection $\mathcal{B}_{\mathbb{Q}}$ of open balls with rational centers and rational radii is a basis for $\mathbb{R}^n$. ■

Both the sets $\mathbb{Q}$ and $\mathbb{Q}^+$ are countable. Since the centers of the open balls in $\mathcal{B}_{\mathbb{Q}}$ are indexed by $\mathbb{Q}^n$, a countable set, and the radii are indexed by $\mathbb{Q}^+$, also a countable set, the collection $\mathcal{B}_{\mathbb{Q}}$ is countable.

Definition 1.3.2 (Second Countable). A topological space is said to be second countable if it has a countable basis.

Proposition 1.3.4 shows that $\mathbb{R}^n$ with its standard topology is second countable.

Proposition 1.3.5

Let $\mathcal{B} = \{B_\alpha\}$ be a basis for S, and A a subspace of S. Then $\{B_\alpha \cap A\}$ is a basis for A.

Proof. Let U' be any open set in A and $p \in U'$. By the definition of subspace topology, $U' = U \cap A$ for some open set U in S. Since $p \in U \cap A \subset U$, there is a basic open set B_α such that $p \in B_\alpha \subset U$. Then

$$p \in B_\alpha \cap A \subset U \cap A = U',$$

which proves that the collection $\{B_\alpha \cap A \mid B_\alpha \in \mathcal{B}\}$ is a basis for A. ■

Corollary 1.3.6

Subspace of a second countable space is also second countable.

Definition 1.3.3 (Neighborhood Basis)**.** Let S be a topological space and p be a point in S. A **basis of neighbourhoods** or a **neighbourhood basis** at p is a collection $\mathcal{B} = \{B_\alpha\}$ of neighbourhoods of p such that for any neighbourhood U of p there is a $B_\alpha \in \mathcal{B}$ such that $p \in B_\alpha \subseteq U$.

Definition 1.3.4 (First Countable)**.** A topological space S is first countable if it has a countable basis of neighbourhoods at every point $p \in S$.

Example 1.3.2

For $p \in \mathbb{R}^n$, let $B\left(p, \frac{1}{n}\right)$ be the open ball of center p and radius $\frac{1}{n}$ in $\mathbb{R}^n$. Then $\left\{B\left(p, \frac{1}{n}\right)\right\}_{n=1}^{\infty}$ is a neighbourhood basis at p. Thus $\mathbb{R}^n$ is first countable.

An important note: An uncountable discrete topological space is first countable but not second countable. A second countable topological space is always first countable.

§1.4 Hausdorff Space

Definition 1.4.1 (Hausdorff Space)**.** A topological space S is Hausdorff if given any 2 distinct points x, y in S there exist disjoint open sets U, V such that $x \in U$ and $y \in V$.

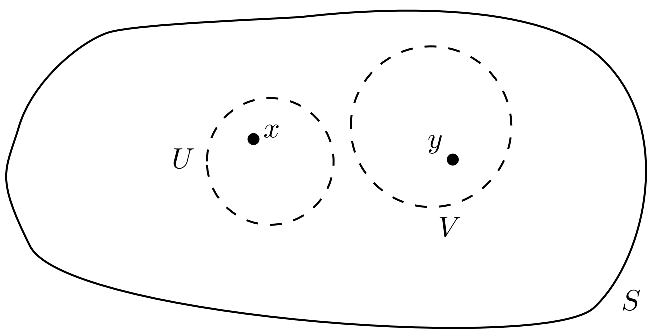


Figure 1.2: Here S is a Hausdorff space, U and V are disjoint open sets containing x and y respectively.

Proposition 1.4.1

Every singleton set (a one-point set) in a Hausdorff space S is closed.

Proof. Let $x \in S$. We want to prove that $\{x\}$ is closed, i.e. $S \setminus \{x\}$ is open.

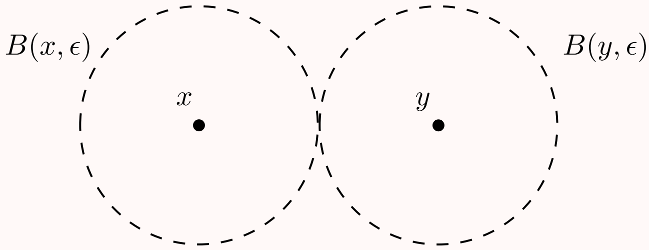
Let $y \in S \setminus \{x\}$. Since S is Hausdorff, we can find disjoint open sets U_y and V_y such that $x \in U_y$ and $y \in V_y$. No such V_y contains x. Therefore

$$S \setminus \{x\} = \bigcup_{y \in S \setminus \{x\}} V_y$$

So $S \setminus \{x\}$ is union of open sets, hence open. So $\{x\}$ is closed. ■

Example 1.4.1

The Euclidean space $\mathbb{R}^n$ (equipped with standard/ metric topology) is Hausdorff, for given distinct points x, y in $\mathbb{R}^n$, if $\epsilon = \frac{1}{2}d(x, y)$, then the open balls $B(x, \epsilon)$ and $B(y, \epsilon)$ will be disjoint.



In a similar manner, one can show that every metric space is Hausdorff.

Lemma 1.4.2

Let A be a subspace of X. If X is a Hausdorff space, then so is A.

Proof. Take $x, y \in A \subseteq X$ with $x \neq y$. As X is Hausdorff, we can find disjoint open sets U and V in X, such that $U \ni x$ and $V \ni y$. $x \in A$ and $x \in U$, so $x \in A \cap U$. Similarly, $y \in A \cap V$.

Now, both $A \cap U$ and $A \cap V$ are open in A, with respect to the subspace topology. Furthermore, $(A \cap U) \cap (A \cap V) = A \cap (U \cap V) = \varnothing$. Therefore, for $x, y \in A$ we've found disjoint open sets $A \cap U$ and $A \cap V$, containing x and y respectively. So A is Hausdorff. ■

§1.5 Continuity and Homeomorphism

Definition 1.5.1 (Continuous Maps)**.** Let $f : X \to Y$ be a map of topological spaces. f is said to be continuous if for each open subset V of Y, the set $f^{-1}(V)$ is an open subset of X.

Proposition 1.5.1

$f : X \to Y$ is continuous if and only if for every closed subset B of Y, the set $f^{-1}(B)$ will be closed in X.

Proof. ($\Rightarrow$) Suppose f is continuous. B is closed, so $Y \setminus B$ is open in Y. Therefore, by the continuity of f, $f^{-1}(Y \setminus B) = X \setminus f^{-1}(B)$ is open in X, so $f^{-1}(B)$ is closed.

($\Leftarrow$) Suppose $f^{-1}(B)$ is closed in X for any closed $B \subseteq Y$. Take any open set U in Y. Choose $B = Y \setminus U$. Then by the assumption $f^{-1}(Y \setminus U) = X \setminus f^{-1}(U)$ is closed in X. This gives us $f^{-1}(U)$ is open. So f is continuous. ■

Definition 1.5.2 (Homeomorphism)**.** Let X and Y be topological spaces; let $f : X \to Y$ be a bijection. If both f and the inverse function $f^{-1} : Y \to X$ are continuous, then f is called a **homeomorphism**.

Example 1.5.1

The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 3x + 1$ is a homeomorphism. We define $g : \mathbb{R} \to \mathbb{R}$ by $g(y) = \frac{1}{3}(y - 1)$. Then we have

$$f(g(y)) = y \text{ and } g(f(x)) = x \qquad \forall\, x, y \in \mathbb{R}$$

This proves $g = f^{-1}$. It is easy to see that both f and g are continuous functions. Therefore f is a homeomorphism.

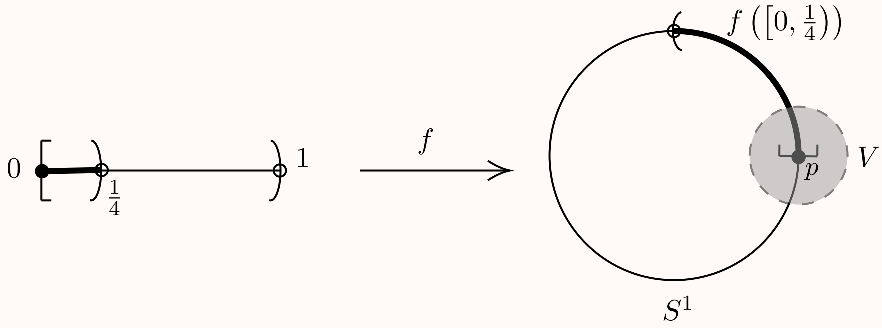
However, a bijective function can be continuous without being a homeomorphism.

Example 1.5.2

Let S^1 denote the unit circle in $\mathbb{R}^2$; that is $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$, considered as a **subspace**[a] of the space $\mathbb{R}^2$. Let $f : [0, 1) \to S^1$ be the

$$f(t) = (\cos 2\pi t, \sin 2\pi t)$$

It is left as an exercise for the reader to show that f is a continuous bijective function. But the function f^{-1} is not continuous.



$U = \left[0, \frac{1}{4}\right)$ is an open set in $[0, 1)$ according to the subspace topology. We want to show that $f(U)$ is not open in S^1. That would prove the discontinuity of f^{-1}.
Let p be the point $f(0)$. And $p \in f(U)$. We need to find an open set of S^1 in subspace topology containing $p = f(0)$ and contained in $f(U)$ to show that $f(U)$ is open in S^1, i.e we have to find an open set in V of $\mathbb{R}^2$ such that $f(0) = p \in V \cap S^1 \subseteq f(U)$. But it is impossible as is evident from the figure above. No matter what V we choose, some part of $V \cap S^1$ would lie outside $f(U)$.

[a] Subset of $\mathbb{R}^2$ equipped with subspace topology.

Lemma 1.5.2 (Pasting Lemma)

Let $X = A \cup B$, where A and B are closed in X. Let $f : A \to Y$ and $g : B \to Y$ be continuous. If $f(x) = g(x)$ for every $x \in A \cap B$, then f and g combine to give a continuous function $h : X \to Y$ defined by

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B \end{cases}$$

Proof. Let C be a closed subset of Y. Now,

$$h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$$

Since f is continuous, $f^{-1}(C)$ is closed in A, hence closed in X. Similarly, $g^{-1}(C)$ is closed in X. So $h^{-1}(C)$ is the union of two closed sets in X, hence it is closed in X. Therefore, h is continuous. ■

Lemma 1.5.3

Let X, Y be topological spaces and $f : X \to Y$. Then f is continuous if and only if for every $x \in X$ and each neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subseteq V$.

Proof. ($\Rightarrow$) Suppose f is continuous. Let $x \in X$ and $V \ni f(x)$ is open in Y. We take $U = f^{-1}(V)$. Since f is open and U is preimage of open set, so U is open. Also,

$$f(x) \in V \implies x \in f^{-1}(V) = U \text{ and } f(U) = f\left(f^{-1}(V)\right) \subseteq V$$

($\Leftarrow$) Let $V \subseteq Y$ be open. We need to show that $f^{-1}(V)$ is open. Take $x \in f^{-1}(V)$. Then $f(x) \in V$, so V is a neighborhood of $f(x)$. By assumption, there exists open $U \ni x$ such that

$$f(U) \subseteq V \implies U \subseteq f^{-1}(V)$$

So for every $x \in f^{-1}(V)$, there exists a neighborhood of x that is contained in $f^{-1}(V)$. So $f^{-1}(V)$ is open, and hence f is continuous. ∎

§1.6 Product Topology

The Cartesian product of two sets A and B is the set $A \times B$ of all ordered pairs (a, b) with $a \in A$ and $b \in B$. Given two topological spaces X and Y, consider the collection $\mathcal{B}$ of subsets of $X \times Y$ of the form $U \times V$, with U open in X and V open in Y. We will call elements of $\mathcal{B}$ basic open sets in $X \times Y$. If $U_1 \times V_1$ and $U_2 \times V_2$ are in $\mathcal{B}$, then

$$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \times V_2)\ ,$$

which is also in $\mathcal{B}$. From this, it follows easily that $\mathcal{B}$ satisfies the conditions of Proposition 1.3.2 for a basis and generates a topology on $X \times Y$, called the **product topology**. Unless noted otherwise, this will always be the topology we assign to the product of two topological spaces.

Proposition 1.6.1

Let $\{U_i\}$ and $\{V_j\}$ be bases for the topological spaces X and Y, respectively. Then $\{U_i \times V_j\}$ is a basis for $X \times Y$.

Proof. Given an open set W in $X \times Y$ and point $(x, y) \in W$, we can find a basic open set $U \times V$ in $X \times Y$ such that $(x, y) \in U \times V \subset W$. Since U is open in X and $\{U_i\}$ is a basis for X, $x \in U_i \subset U$ for some U_i. Similarly, $y \in V_j \subset V$ for some V_j. Therefore,

$$(x, y) \in U_i \times V_j \subset U \times V \subset W.$$

By the definition of a basis, $\{U_i \times V_j\}$ is a basis for $X \times Y$. ∎

Corollary 1.6.2

The product of two second-countable spaces is second countable.

Proposition 1.6.3

The product of two Hausdorff spaces X and Y is Hausdorff.

Proof. Given two distinct points $(x_1, y_1), (x_2, y_2)$ in $X \times Y$, without loss of generality we may assume that $x_1 \neq x_2$. Since X is Hausdorff, there exist disjoint open sets U_1, U_2 in X such that $x_1 \in U_1$ and $x_2 \in U_2$. Then $U_1 \times Y$ and $U_2 \times Y$ are disjoint neighborhoods of (x_1, y_1) and (x_2, y_2), so X ×Y is Hausdorff. ∎

The product topology can be generalized to the product of an arbitrary collection $\{X_\alpha\}_{\alpha\in\mathrm{A}}$ of topological spaces. Whatever the definition of the product topology, the projection maps

$$\pi_{\alpha_i} : \prod_\alpha X_\alpha \to X_{\alpha_i} \ , \pi_{\alpha_i}\left(\prod x_\alpha\right) = x_{\alpha_i}$$

should all be continuous. Thus, for each open set U_{α_i} in X_{α_i}, the inverse image $\pi_{\alpha_i}^{-1}(U_{\alpha_i})$ should be open in $\prod_\alpha X_\alpha$. By the properties of open sets, a finite intersection $\bigcap_{i=1}^r \pi_{\alpha_i}^{-1}(U_{\alpha_i})$ should also be open. Such a finite intersection is a set of the form $\prod_{\alpha\in A} U_\alpha$, where U_α is open in X_α and $U_\alpha = X_\alpha$ for all but finitely many $\alpha \in A$. We define the product topology on the Cartesian product $\prod_{\alpha\in\mathrm{A}} X_\alpha$ to be the topology with basis consisting of sets of this form.

Theorem 1.6.4

Let $f : A \to \prod_{\alpha\in J} X_\alpha$ be given by the equation

$$f(a) = (f_\alpha(a))_{\alpha\in J} \ ,$$

where $f_\alpha : A \to X_\alpha$ for each α. Let $\prod X_\alpha$ have the product topology. Then the function f is continuous if and only if each function f_α is continuous.

Proof. ($\Rightarrow$) Suppose f is continuous. Then $f_\alpha = \pi_\alpha \circ f$ is the composition of two continuous maps, hence continuous.

($\Leftarrow$) Now suppose f_α is continuous for every α. Let $U \subseteq \prod_{\alpha\in J} X_\alpha$ be a basic open set. Then U is of the form $\prod U_\alpha$ where U_α is open in X_α for every α, and $U_\alpha \neq X_\alpha$ for only fintely many α's. Then we have

$$\begin{aligned} f^{-1}(U) = \bigcap_\alpha f_\alpha^{-1}(U_\alpha) &= \left(\bigcap_{U_\alpha\neq X_\alpha} f_\alpha^{-1}(U_\alpha)\right) \cap \left(\bigcap_{U_\alpha = X_\alpha} f_\alpha^{-1}(U_\alpha)\right) \\ &= \left(\bigcap_{U_\alpha\neq X_\alpha} f_\alpha^{-1}(U_\alpha)\right) \cap \left(\bigcap_{U_\alpha = X_\alpha} A\right) \\ &= \bigcap_{U_\alpha\neq X_\alpha} f_\alpha^{-1}(U_\alpha) \end{aligned}$$

Since each f_α is continuous, $f_\alpha^{-1}(U_\alpha)$ is open in A. Therefore, as a finite intersection of open sets, $f^{-1}(U)$ is open, proving the continuity of f. ■

§1.7 Quotient Topology

Quotient topology is defined using an equivalence relation. An equivalence relation is a binary relation on a set that has some properties.

Definition 1.7.1 (Equivalence Relation and Equivalence Class). An equivalence relation $\sim$ on a set S is a binary relation which is reflexive, symmetric and transitive. That is

(i) $a \sim a$ for every $a \in S$

(ii) $a \sim b \implies b \sim a$

(iii) $a \sim b \ , \ b \sim c \implies a \sim c$

The equivalence class $[x]$, if $x \in S$, is the set of all elements in S equivalent to x.

An equivalence relation on S partitions S into disjoint equivalence classes. We denote the set of all equivalence classes with $S/\!\sim$ and call this the quotient of S by the equivalence relation $\sim$. There is a natural projection map $\pi : S \to S/\!\sim$ which projects $x \in S$ to its own equivalence class $[x] \in S/\!\sim$.

Abuse of Notation. Ideally $[x]$ denotes a point in $S/\sim$. But we will use the same notation $[x]$ to identify a set in S whose elements are all equivalent to each other under the given equivalence relation.

> **Definition 1.7.2** (Quotient Topology). Let S be a topological space. We define a topology called **quotient topology** on $S/\sim$ by declaring a set U in $S/\sim$ to be open if and only if $\pi^{-1}(U)$ is open in S.

It's not hard to see that quotient topology is a well defined topology. Note that $\pi^{-1}(\varnothing) = \varnothing$ and $\pi^{-1}(S/\sim) = S$ and hence $\varnothing$ and $S/\sim$ are both open sets in quotient topology. Now let $\{U_\alpha\}_{\alpha\in A}$ be an arbitrary collection of open sets in $S/\sim$. Then $\left\{\pi^{-1}(U_\alpha)\right\}_{\alpha\in A}$ is an an arbitrary collection of open sets in S. So,

$$\bigcup_{\alpha\in A}\pi^{-1}(U_\alpha) = \pi^{-1}\left(\bigcup_{\alpha\in A}U_\alpha\right) \text{ is open in } S \implies \bigcup_{\alpha\in A}U_\alpha \text{ open in } S/\sim$$

So arbitrary union of open sets is open in $S/\sim$. Now for a finite collection of open sets $\{U_i\}_{i=1}^n$ in $S/\sim$, $\left\{\pi^{-1}(U_i)\right\}_{i=1}^n$ is a finite collection of open sets in S. So,

$$\bigcap_{i=1}^{n}\pi^{-1}(U_i) = \pi^{-1}\left(\bigcap_{i=1}^{n}U_\alpha\right) \text{ is open in } S \implies \bigcap_{i=1}^{n}U_\alpha \text{ open in } S/\sim$$

So finite intersection of open sets is open in $S/\sim$. Therefore, we've verified that the open sets defined on $S/\sim$ indeed form a topology.

Continuity on Quotient Topology

Let $\sim$ be a equivalence relation on the topological space S and give $S/\sim$ the quotient topology. Suppose that the function $f : S \to Y$ is continuous from S to another topological space Y. Further assume that f is constant on each equivalence class. Then f induces a map

$$\bar{f} : S/\sim \to Y \ ; \ \bar{f}([p]) = f(p) \quad \forall\, p\in S$$

Note that this latter function $\bar{f}$ wouldn't be well-defined had we not assumed f to be constant on each equivalence class in $S/\sim$.

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "\pi"'] & Y \\ S/\sim \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

$$f = \bar{f}\circ\pi$$
$$f(p) = \bar{f}(\pi(p)) = f([p])$$

> **Proposition 1.7.1**
> The induced map $\bar{f} : S/\sim \to Y$ is continuous if and only if the map $f : S \to Y$ is continuous.

Proof. ($\Rightarrow$). Suppose $f : S \to Y$ is continuous. Let V be open in Y. Then $f^{-1}(V) = \pi^{-1}\left(\bar{f}^{-1}(V)\right)$ is open in S. Therefore, by the definition of quotient topology, then $\bar{f}^{-1}(V)$ is open in $S/\sim$. Hence, we've shown that for a given open set V in Y, $\bar{f}^{-1}(V)$ is open in $S/\sim$. So, $\bar{f} : S/\sim \to Y$ is continuous.

($\Leftarrow$). If $\bar{f} : S/\sim \to Y$ is continuous, then $f = \bar{f}\circ\pi$ is the composition of two continuous maps, hence continuous. ■

Identification of a subset to a point

If A is a subspace of a topological space S, we can define a relation $\sim$ on S by declaring

$$x \sim x \ , \ \forall \ x \in S \qquad \text{and} \qquad x \sim y \ , \ \forall \ x, y \in A$$

It is immediate that $\sim$ is an equivalence relation. We say that the quotient space $S/\sim$ is obtained from S by identifying A to a point.

§1.8 Compactness

Definition 1.8.1 (Open Cover)**.** Let S be a topological space. A collection $\{U_\alpha\}_{\alpha \in I}$ of open subsets of S is said to be an open cover of S if

$$S \subseteq \bigcup_{\alpha \in I} U_\alpha$$

Since the open sets are in the topology of S and consequently $U_\alpha \subseteq S$ for every $\alpha \in I$, one has $\bigcup_{\alpha \in I} U_\alpha \subseteq S$. Therefore, the open cover condition in this case reduces to $S = \bigcup_{\alpha \in I} U_\alpha$.

With the subspace topology, a subset A of a topological space S is a topological space by its own right. The subspace A can be covered by **open sets in** A or **by open sets in** S.

- An **open cover of** A **in** S is a collection $\{U_\alpha\}_\alpha$ of open sets in S that covers A. In other words, $A \subseteq \bigcup_\alpha U_\alpha$ (Note that in this case $A = \bigcup_\alpha U_\alpha$ might not hold in general).
- An **open cover of** A **in** A is a collection $\{U_\alpha\}_\alpha$ of open sets in A in subsapce topology that covers A. In other words, $A \subseteq \bigcup_\alpha U_\alpha$ (Here, in fact, $A = \bigcup_\alpha U_\alpha$ as each $U_\alpha \subseteq A$).

Definition 1.8.2 (Compact Set)**.** Let S be a topological space and $A \subseteq S$. A is **compact** if and only if every open cover of A in A has finite subcover.

Proposition 1.8.1

A subspace A of a topological space S is **compact** if and only if every **open cover of** A **in** S has a finite subcover.

Proof. ($\Rightarrow$) Assume A is compact and let $\{U_\alpha\}$ be an open cover of A in S. This means that $A \subseteq \bigcup\limits_\alpha U_\alpha$. Hence,

$$A \subseteq \left(\bigcup_\alpha U_\alpha\right) \bigcap A = \bigcup_\alpha \left(U_\alpha \bigcap A\right)$$

Now, $\{U_\alpha \cap A\}_\alpha$ is an open cover of A in A. Since A is compact, every open cover of A in A has a finite subcover. Let the finite sub-cover be $\{U_{\alpha_i} \cap A\}_{i=1}^n$. Thus,

$$A \subseteq \bigcup_{i=1}^n U_{\alpha_i}$$

which means that $\{U_{\alpha_i}\}_{i=1}^n$ is a finite sub-cover of the open cover $\{U_\alpha\}_\alpha$ of A in S.

($\Leftarrow$) Suppose every open cover of A in S has a finite subcover, and let $\{V_\alpha\}_\alpha$ be an open cover of A in A. Then each V_α is an open set of A in subspace topology. According to the definition of subspace topology, there is an open set U_α in S such that $V_\alpha = U_\alpha \cap A$. Now,

$$A \subseteq \bigcup_\alpha V_\alpha = \bigcup_\alpha (U_\alpha \cap A) = \left(\bigcup_\alpha U_\alpha\right) \cap A \subseteq \bigcup_\alpha U_\alpha$$

Therefore, $\{U_\alpha\}_\alpha$ is an open cover of A in S. By hypothesis, there are finitely many sets $\{U_{\alpha_i}\}_{i=1}^n$ such that $A \subseteq \bigcup\limits_{i=1}^{n} U_{\alpha_i}$. Hence,

$$A \subseteq \left(\bigcup_{i=1}^{n} U_{\alpha_i}\right) \cap A = \bigcup_{i=1}^{n} (U_{\alpha_i} \cap A) = \bigcup_{i=1}^{n} V_{\alpha_i}$$

So $\{V_{\alpha_i}\}_{i=1}^n$ is a finite subcover of $\{V_\alpha\}$ that covers A in A. Therefore, A is compact. ■

Proposition 1.8.2

Every compact subset of K of a Hausdorff space S is closed.

Proof. We shall prove that $S \setminus K$ is open. Let's take $x \in S \setminus K$. We claim that there is a neighborhood U_x of x that is disjoint from K.

Since S is Hausdorff, for each $y \in K$, we can choose disjoint open sets U_y and V_y such that $U_y \ni x$ and $V_y \ni y$. The collection $\{V_y \ : \ y \in K\}$ is an open cover of K in S. Since K is compact, there exists a finite subcover $\{V_{y_i}\}_{i=1}^n$. That is $K \subseteq \bigcup_{i=1}^n V_{y_i}$. Since $U_{y_i} \cap V_{y_i}$ for every i, we have

$$\left(\bigcap_{i=1}^{n} U_{y_i}\right) \cap \left(\bigcup_{i=1}^{n} V_{y_i}\right) = \varnothing \implies U_x \cap K = \varnothing \text{ where } U_x = \bigcap_{i=1}^{n} U_{y_i}$$

U_x is the finite intersection of open sets, hence open. Also, every U_{y_i} contains x, hence their intersection U_x also contains x. So U_x is the desired open set that is disjoint from K, in other words $x \in U_x \subseteq S \setminus K$. As a result,

$$S \setminus K \subseteq \bigcup_{x \in S \setminus K} U_x \subseteq S \setminus K \implies S \setminus K = \bigcup_{x \in S \setminus K}$$

$S \setminus K$ is the union of open sets, hence open. Therefore K is closed. ■

Proposition 1.8.3

The image of a compact set under a continuous map is compact.

Proof. Let $f : X \to Y$ be a continuous and K a compact subset of X. Suppose $\{U_\alpha\}$ is an open cover of $f(K)$ by open subsets of Y. Since, f is continuous, the inverse images of $f^{-1}(U_\alpha)$ are all open in X. Moreover,

$$K \subseteq f^{-1}(f(K)) \subseteq f^{-1}\left(\bigcup_{\alpha} U_\alpha\right) = \bigcup_{\alpha} f^{-1}(U_\alpha)$$

So $\left\{f^{-1}(U)_\alpha\right\}$ is an open cover of K in X. By Proposition 1.8.1, there is a finite sub-collection $\left\{f^{-1}(U_{\alpha_i})\right\}_{i=1}^n$ such that

$$K \subseteq \bigcup_{i=1}^{n} f^{-1}(U_{\alpha_i}) = f^{-1}\left(\bigcup_{i=1}^{n} U_{\alpha_i}\right) \implies f(K) \subseteq f\left(f^{-1}\left(\bigcup_{i=1}^{n} U_{\alpha_i}\right)\right) \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$$

Thus $f(K)$ is compact. ■

Lemma 1.8.4

A closed subset F of a compact topological space S is compact.

Proof. Let $\{U_\alpha\}_\alpha$ be an open cover of F in S. The collection $\{U_\alpha, S \setminus F\}$ is an open cover of S itself. By compactness of S, there is a finite sub-cover $\{U_{\alpha_i}, S \setminus F\}_{i=1}^n$ of S, that is,

$$F \subseteq S \subseteq \left(\bigcup_{i=1}^n U_{\alpha_i} \right) \cup (S \setminus F) \implies F \subseteq \bigcup_{i=1}^n U_{\alpha_i}$$

Therefore, $\{U_{\alpha_i}\}_{i=1}^n$ is a finite subcover of the open cover $\{U_\alpha\}$ of F in S. Hence, F is also compact. ■

Proposition 1.8.5

A continuous map $f : X \to Y$ form a compact space X to a Hausdorff space Y is a closed map (a map that takes closed sets to closed sets).

Proof. Let $F \subseteq X$ be closed. Then F is compact by Lemma 1.8.4. Since $f : X \to Y$ is a continuous map, by Proposition 1.8.3, $f(F)$ is compact in Y. Since Y is Hausdorff, by Proposition 1.8.2, $f(F)$ is closed in Y. Hence, f is a closed map. ■

Corollary 1.8.6

A continuous bijection $f : X \to Y$ from a compact space X to a Hausdorff space is a homeomorphism.

Proof. We want to show thaat $f^{-1} : Y \to X$ is continuous. And in order to that it suffices to show that for every closed set F in X, $\left(f^{-1}\right)^{-1}(F) = f(F)$ is closed in Y. In other words, it suffices to show that f is a closed map. The corollary then follows from Proposition 1.8.5. ■

Definition 1.8.3 (Bounded Set). A subset A of $\mathbb{R}^n$ is said to be bounded if it is contained in some open ball $B(p, r)$. otherwise, it is unbounded.

Theorem 1.8.7 (Heine-Borel Theorem)

A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

Definition 1.8.4 (Diameter of Set). Let $A \subseteq X$ be a bounded subset of a metric space (X, d). The diameter of A is defined by

$$\operatorname{diam}(A) := \sup \{d(a_1, a_2) \ : \ a_1, a_2 \in A\}$$

Lemma 1.8.8 (Lebesgue Number Lemma)

Let (X, d) be a compact metric space. Given an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in J}$ of X, there exists a number $\delta > 0$ — called the Lebesgue number associated with the cover — such that for a given $A \subseteq X$ with $\operatorname{diam}(A) < \delta$, one must have $A \subseteq U_\alpha$ for some $\alpha \in J$.

Proof. Take $x \in X$. As $\mathcal{U}$ covers X, we can find $U_\alpha \in \mathcal{U}$ such that $x \in U_\alpha$. Since U_α is open and $x \in U_\alpha$, there exists $r_x > 0$ such that

$$B(x, r_x) \subseteq U_\alpha$$

We do this for every $x \in X$. So we get an open cover of X

$$X = \bigcup_{x \in X} B\left(x, \frac{r_x}{2}\right)$$

Since X is compact, there exists a finite subcover of this open cover. So

$$X = \bigcup_{i=1}^{n} B\left(x_i, \frac{r_{x_i}}{2}\right)$$

We define $\delta > 0$ in the following way:

$$\delta = \min\left\{\frac{r_{x_i}}{2} \ : \ i = 1, 2, \ldots, n\right\}$$

We claim that this δ is our desired Lebesgue number of the open cover $\mathcal{U}$. Let $A \subseteq X$ with $\operatorname{diam}(A) < \delta$. Fix $a \in A$. Then there exists $j \in \{1, 2, \ldots, n\}$ such that

$$a \in B\left(x_j, \frac{r_{x_j}}{2}\right) \implies \boxed{d\left(x_j, a\right) < \frac{r_{x_j}}{2}}$$

By the construction of r_{x_j}, there exists $U_\beta \in \mathcal{U}$ such that $B\left(x_j, r_{x_j}\right) \subseteq U_\beta$. We claim that $A \subseteq U_\beta$. Take any $b \in A$.

$$d\left(a, b\right) \leq \operatorname{diam}(A) < \delta \leq \frac{r_{x_j}}{2} \implies \boxed{d\left(a, b\right) < \frac{r_{x_j}}{2}}$$

$$d\left(x_j, b\right) \leq d\left(x_j, a\right) + d\left(a, b\right) < \frac{r_{x_j}}{2} + \frac{r_{x_j}}{2} = r_{x_j} \implies b \in B\left(x_j, r_{x_j}\right)$$

For every $b \in A$, we have $b \in B\left(x_j, r_{x_j}\right)$. Therefore, $A \subseteq B\left(x_j, r_{x_j}\right) \subseteq U_\beta$. ■

§1.9 Quotient Topology Continued

Let I be the closed interval $[0, 1]$ in the standard topology of $\mathbb{R}^n$ and $I/\sim$ be the quotient space obtained from I by identifying the 2 points $\{0, 1\}$ to a point. Denote by S^1 the unit circle in the complex plane. Define f by $f(x) = e^{2\pi i x}$.

Now the function $f : I \to S^1$ defined above assumes the same value at 0 and 1 and based on the discussion prior to Proposition 1.7.1, f induces the map $\bar{f} : I/\sim \to S^1$.

Proposition 1.9.1

The function $\bar{f} : I/\sim \to S^1$ is a homeomorphism.

Proof. The function $f : I \to S^1$ defined by $f(x) = e^{2\pi i x}$ is continuous (check!). Therefore, by Proposition 1.7.1, $\bar{f} : I \to S^1$ is also continuous.

Note that $I = [0, 1]$ in $\mathbb{R}$ is closed and bounded and hence by Heine-Borel Theorem, I is compact. Since the projection $\pi : I \to I/\sim$ is continuous, by Proposition 1.8.3, the image of I under π, *i.e.*, $I/\sim$ is compact.

It should also be obvious that $\bar{f} : I/\sim \to S^1$ is a bijection. Since S^1 is a of the Hausdorff space $\mathbb{R}^2$, by Lemma 1.4.2, S^1 is also Hausdorff. Hence, $\bar{f}$ is a continuous bijection from the compact space $I/\sim$ to the Hausdorff topological space S^1. Therefore, by Corollary 1.8.6, $\bar{f} : I/\sim \to S^1$ is a homeomorphism. ■

Necessary Condition for a Hausdorff quotient

Even if S is a Hausdorff space, the quotient space $S/\sim$ may fail to be Hausdorff.

Proposition 1.9.2

If the quotient space $S/\sim$ is Hausdorff, then the equivalence class $[p]$ of any point p in S is closed in S.

Proof. By Proposition 1.4.1, every singleton set is closed in a Hausdorff topological space. Now, consider the canonical projection map $\pi : S \to S/{\sim}$. For a point $p \in S$, $\{\pi(p)\}$ is a singleton set in $S/{\sim}$.

Since, by hypothesis $S/\sim$ is Hausdorff, $\{\pi(p)\}$ must be closed in $S/\sim$ with respect to quotient topology. By continuity of π, $\pi^{-1}(\{\pi(p)\})$ is closed in S. But $\pi^{-1}(\{\pi(p)\}) = [p]$. Hence, $[p]$ is a closed set in S. ■

Remark. In order to prove that a quotient space $S/{\sim}$ is not Hausdorff it is sufficient to prove that the equivalence class $[p]$ of some point $p \in S$ is not closed in S. We have the following example to elucidate this remark.

Example 1.9.1

Define an equivalence relation $\sim$ on $\mathbb{R}$ by identifying the open interval $(0, \infty)$ to a point. The resulting quotient space $\mathbb{R}/{\sim}$ is not Hausdorff since the equivalence class $(0, \infty)$ is not a closed subset of $\mathbb{R}$.

§1.10 Open Equivalence Relations

Definition 1.10.1. An equivalence relation $\sim$ on a topological space S is said to be open if the underlying projection map $\pi : S \to S/{\sim}$ is open (maps open sets to open sets).

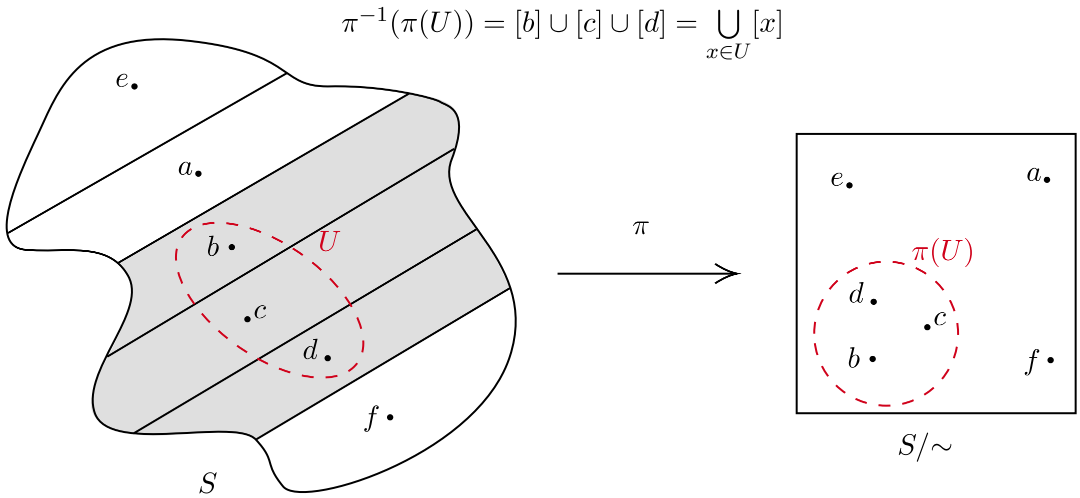


Figure 1.3: Indeed $\pi^{-1}(\pi(U)) = \bigcup_{x \in U} [x]$

In other words, the equivalence relation $\sim$ on S is open if and only if for every open set $U \in S$, the set $\pi(U) \in S/{\sim}$ is open. Or equivalently, by definition of quotient topology,

$$\pi^{-1}(\pi(U)) = \bigcup_{x \in U} [x] \quad \text{is open in } S$$

$\bigcup_{x \in U} [x]$ denotes all points equivalent to some point of U (shaded region in Figure 1.3).

Example 1.10.1

The projection map onto a quotient space is, in general, not open. For example, let $\sim$ be the equivalence relation on the real line $\mathbb{R}$ that identifies the two points 1 and -1, and $\pi : \mathbb{R} \to \mathbb{R}/{\sim}$ the projection map.

The map π is open if and only if for every open set V in $\mathbb{R}$, its image $\pi(V)$ is open in $\mathbb{R}/{\sim}$, or equivalently $\pi^{-1}(\pi(V))$ is open in $\mathbb{R}$. Let V be the open interval $(-2, 0)$ in $\mathbb{R}$. Then,

$$\pi^{-1}(\pi(V)) = (-2, 0) \cup \{1\} \,, \qquad [\text{Since } \pi(1) \in \pi(V)]$$

which is not open in $\mathbb{R}$ and hence π is not an open map. In other words, the equivalence relation $\sim$ is not open.

Definition 1.10.2 (Graph of Equivalence Relation)**.** Given an equivalence relation $\sim$ on S, let R be the subset of $S \times S$ that defines the relation $R = \{(x, y) \in S \times S \mid x \sim y\}$. We call R the **graph** of the equivalence relation $\sim$.

We have a necessary and sufficient condition for a quotient space to be Hausdorff if the underlying equivalence relation is an open equivalence relation. We state this condition by means of a theorem.

Theorem 1.10.1

Suppose $\sim$ is an open equivalence relation on a topological space S. Then the quotient space $S/\sim$ is Hausdorff if and only if the graph R of $\sim$ is closed in $S \times S$.

Proof. **($\Leftarrow$)** Suppose R is closed in $S \times S$. Then $R^c = (S \times S) \setminus R$ is open. Therefore, for every $(x, y) \in R^c$, there exists basic open set $U \times V$ containing (x, y) such that $U \times V \subseteq R^c$. This is equivalent to saying, no element of U is equivalent to any element of V, and vice versa.

Since $\sim$ is an open equivalence relation, $\pi(U)$ and $\pi(V)$ are open sets containing $[x]$ and $[y]$, respectively. Since no element of U is equivalent to any element of V, $\pi(U)$ and $\pi(V)$ are disjoint. Therefore, for $[x] \neq [y]$, we have found their disjoint neighborhoods. Hence, $S/\sim$ is Hausdorff.

($\Rightarrow$) Now suppose $S/\sim$ is Hausdorff. Take $[x] \neq [y]$ from $S/\sim$. Then there exist disjoint neighborhoods $A \ni [x]$ and $B \ni [y]$. A and B are open, so $U = \pi^{-1}(A)$ and $V = \pi^{-1}(B)$ are open in S.

$$\pi(U) = \pi\left(\pi^{-1}(A)\right) = A \quad \text{and } \pi(V) = \pi\left(\pi^{-1}(B)\right) = B \,.$$

So $\pi(U)$ and $\pi(V)$ are disjoint. In other words, no element of U is equivalent to any element of V. Therefore, $U \times V \subseteq R^c$. $[x] \in A$, and $U = \pi^{-1}(A)$, so $x \in U$. Similarly, $y \in V$. Therefore,

$$(x, y) \in U \times V \subseteq R^c \,.$$

So R^c is open, and hence R is closed. ■

If the equivalence relation $\sim$ is equality, *i.e.*, $x \sim y$ iff $x = y$, then the quotient space $S/\sim$ is S itself and the graph R of $\sim$ is simply the diagonal $\Delta = \{(x, x) \in S \times S\}$.

Corollary 1.10.2

A topological space is Hausdorff if and only if the diagonal Δ is closed in $S \times S$.

Theorem 1.10.3

Let $\sim$ be an open equivalence relation on a topological space S with projection $\pi : S \to S/\sim$. If $\mathcal{B} = \{B_\alpha\}$ is a basis for S, then its image $\{\pi(B_\alpha)\}$ under π is a basis for $S/\sim$.

Proof. Since π is open, $\{\pi(B_\alpha)\}$ is a collection of open sets in $S/\sim$. Let W be an open set in $S/\sim$ and $[x] \in W$ with $x \in S$. So $\pi(x) \in W$, *i.e.*, $x \in \pi^{-1}(W)$. Since $\pi^{-1}(W)$ is open in S, there is a basic open set $B \in \mathcal{B}$ such that, $x \in B \subseteq \pi^{-1}(W)$. Hence

$$[x] = \pi(x) \in \pi(B) \subseteq \pi(\pi^{-1}(W)) \subseteq W$$

Now, we have seen that given W open in $S/\sim$ and $[x] \in W$, there exists an open set $\pi(B)$ in the collection $\{\pi(B_\alpha)\}$ such that $[x] \in \pi(B) \subseteq W$. This proves that $\{\pi(B_\alpha)\}$ is a basis for $S/\sim$. ■

Corollary 1.10.4

If $\sim$ is an open equivalence relation on a second-countable topological space, then the quotient space $S/\sim$ is second countable.

2 Multivariable Calculus Review

§2.1 Differentiabiliy

Consider $f : \mathbb{R}^2 \to \mathbb{R}$ defined as follows:

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

For the piecewise defined function stated above, note that along the x-axis $y = 0$. So $f(x,0) = 0$ for every $x \in \mathbb{R}$. In other words, f is constant and identically 0 on the x-axis. Therefore,

$$\frac{\partial f}{\partial x}(x,y)\bigg|_{y=0} = 0\,.$$

Similarly, along the y-axis $x = 0$. So $f(0,y) = 0$ for every $y \in \mathbb{R}$. In other words, f is constant and identically 0 on the y-axis. Therefore,

$$\frac{\partial f}{\partial y}(x,y)\bigg|_{x=0} = 0\,.$$

Therefore, both the partial derivatives exist at $(0,0)$, and are equal to 0. We will now show that f is not even continuous at $(0,0)$. Consider the line $y = x$, and we shall evaluate the limit of $f(x,y)$ as $(x,y) \to (0,0)$ along this line.

$$\lim_{x\to 0} f(x,x) = \lim_{x\to 0} \frac{x \cdot x}{x^2 + x^2} = \frac{1}{2} \neq 0\,.$$

So we get,

$$\begin{aligned} \lim_{(x,y)\to(0,0)} f(x,y) &= 0\ ,\ \text{along } x\text{-axis;} \\ \lim_{(x,y)\to(0,0)} f(x,y) &= 0\ ,\ \text{along } y\text{-axis;} \\ \lim_{(x,y)\to(0,0)} f(x,y) &= \frac{1}{2}\ ,\ \text{along the line } y = x. \end{aligned}$$

Therefore, f is not even continuous at $(0,0)$, let alone being differentiable. Therefore, mere existence of partial derivatives of order doesn't guarantee differentiability at a given point.

We will, first, consider functions whose domain is $U \subseteq \mathbb{R}^n$ and codomain is $\mathbb{R}$. If $f : U \to \mathbb{R}^n$ is such a function, then $f(\vec{x}) = f\left(x^1, x^2, \ldots, x^n\right)$ denotes its value at $\vec{x} \equiv \left(x^1, x^2, \ldots, x^n\right) \in U$. We also assume that the underlying domain of f is an open set $U \subseteq \mathbb{R}^n$. At each $\vec{a} \in U$, the partial derivative $\frac{\partial f}{\partial x^j}\Big|_{\vec{a}}$ of f with respect to x^j is the following limit, if it exists

$$\frac{\partial f}{\partial x^j}\bigg|_{\vec{x}=\vec{a}} = \lim_{h\to 0} \frac{f\left(a^1, \ldots, a^j + h, \ldots, a^n\right) - f\left(a^1, \ldots, a^j, \ldots, a^n\right)}{h}\,.$$

If $\frac{\partial f}{\partial x^j}$ is defined, that is, the limit above exists at each point of U for $1 \leq j \leq n$, this defines n functions on U. Should these functions be continuous on U for $1 \leq j \leq n$, f is said to be continuously differentiable on U, denoted by $f \in C^1(U)$.

We shall say that f is differentiable at $\vec{a} \in U$ if there is a homogenous linear expression $\sum\limits_{i=1}^{n} b_i \left(x^i - a^i\right)$ such that the inhomogenous expression $f(\vec{a}) + \sum\limits_{i=1}^{n} b_i \left(x^i - a^i\right)$ approximates $f(\vec{x})$ near $\vec{a}$ in the following sense:

$$\lim_{\vec{x} \to \vec{a}} \frac{f(\vec{x}) - f(\vec{a}) - \sum\limits_{i=1}^{n} b_i \left(x^i - a^i\right)}{\|\vec{x} - \vec{a}\|} = 0 \,.$$

In other words, if there exist constants $b_1, b_2, \ldots, b_n$ and a real valued function $r(\vec{x}, \vec{a})$ defined on a neghborhood V of $\vec{a} \in U$ such that the following two conditions hold:

$$f(\vec{x}) = f(\vec{a}) + \sum_{i=1}^{n} b_i \left(x^i - a^i\right) + \|\vec{x} - \vec{a}\| \, r(\vec{x}, \vec{a}) \quad \text{and} \quad \lim_{\vec{x} \to \vec{a}} r(\vec{x}, \vec{a}) = 0 \,.$$

b_i's are uniquely determined, and they are the partial derivatives at $\vec{a}$:

$$b_i = \left. \frac{\partial f}{\partial x^i} \right|_{\vec{x} = \vec{a}} \,.$$

In fact,

$$f(\vec{x}) = f(\vec{a}) + \sum_{i=1}^{n} \left. \frac{\partial f}{\partial x^i} \right|_{\vec{x} = \vec{a}} \left(x^i - a^i\right) + \|\vec{x} - \vec{a}\| \, r(\vec{x}, \vec{a}) \,.$$

Actually, existence of partial derivatives and their continuity guarantees differentiability at a given point $\vec{a} \in U \subseteq \mathbb{R}^n$.

§2.2 Chain Rule

By a differentiable curve in $\mathbb{R}^n$, we mean $f : (a, b) \to \mathbb{R}^n$, with $f(t) = \left(x^1(t), x^2(t), \ldots, x^n(t)\right)$, where the n coordinate functions $x^i(t)$ are all differentiable on (a, b). Recall that, for a function of one variable, differentiability is equivalent to existence of derivative.

Here, $\left(x^i(t)\right)$ are real valued functions of one variable. And you must be familiar with the notion of C^r-differentiability of real valued functions of one variable. For example, $h(t) = t^{\frac{1}{3}}$ is not C^1, because its derivative does not exist at $t = 0$. Similarly, $k(t) = t^{\frac{4}{3}}$ is C^1, but not C^2.

Now, let's suppose $f : (a, b) \to \mathbb{R}^n$ is a C^r differentiable curve in the sense that all the n coordinate functions $x^i(t)$ are C^r differentiable. Take t_0 with $a < t_0 < b$, and $f : (a, b) \to U \subseteq \mathbb{R}^n$. Let g be a C^r-differentiable function from U to $\mathbb{R}$. In particular, $g : U \to \mathbb{R}$ is differentiable at $f(t_0) \in U$. Then $g \circ f : (a, b) \to \mathbb{R}$ is differentiable at t_0, and the derivative is given by:

$$\left. \frac{\mathrm{d}}{\mathrm{d}t} (g \circ f)(t) \right|_{t = t_0} = \sum_{i=1}^{n} \left. \frac{\partial g(f(t))}{\partial x^i} \right|_{f(t_0)} \cdot \left. \frac{\mathrm{d}x^i(t)}{\mathrm{d}t} \right|_{t = t_0} \,.$$

This result is known as the chain rule for real-valued functions.

Now, we can generalize this idea to functions on subsets U of $\mathbb{R}^n$, whose range is not in $\mathbb{R}$, but in $\mathbb{R}^n$. In other words, we consider $F : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^m$.

$$\vec{x} \equiv \left(x^1, x^2, \ldots, x^n\right) \in U \;;\; F(\vec{x}) = \left(F^1(\vec{x}), F^2(\vec{x}), \ldots, F^m(\vec{x})\right) \,.$$

Now take a point $\vec{p} \in U$ with coordinate $\left(p^1, p^2, \ldots, p^n\right)$. Then $F(\vec{p})$ is a point in V with coordinate $\left(F^1(\vec{p}), F^2(\vec{p}), \ldots, F^m(\vec{p})\right)$. Now let $G : V \subseteq R^m \to R^l$. Write a point $\vec{y} \equiv \left(y^1, y^2, \ldots, y^m\right) \in V \subseteq \mathbb{R}^m$. Then

$$G(\vec{y}) = \Big(G^1(\vec{x}), G^2(\vec{x}), \ldots, G^l(\vec{x})\Big) \,.$$

In other words, $G^i : V \to \mathbb{R}$. Then we have $G^i \circ F : U \subseteq \mathbb{R}^n \to \mathbb{R}$. In this case, the chain rule is

$$\frac{\partial \left(G^i \circ F\right)}{\partial x^j}(\vec{p}) = \sum_{k=1}^{m} \frac{\partial G^i}{\partial y^k}(F(\vec{p})) \cdot \frac{\partial F^k}{\partial x^j}(\vec{p}) \,.$$

§2.3 Differential of a Map

Let $F : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^m$. Let $T_p\mathbb{R}^n$ denote the tangent space on $\mathbb{R}^n$ to the point $p \in \mathbb{R}^n$. (For convenience, we'll drop arrows in $\vec{p}$) The differential of F at p is a map $DF_p : T_p\mathbb{R}^n \to T_{F(p)}\mathbb{R}^m$. $T_p\mathbb{R}^n$ is clearly isomorphic to $\mathbb{R}^n$ as vector space. Hence, $DF_p : \mathbb{R}^n \to \mathbb{R}^m$. Let's try to see that DF_p is related to the Jacobian matrix of $F : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^m$.

$$\left\{ \left.\frac{\partial}{\partial x^1}\right|_p, \left.\frac{\partial}{\partial x^2}\right|_p, \ldots, \left.\frac{\partial}{\partial x^n}\right|_p \right\}$$

is a basis of $T_p\mathbb{R}^n$, which can be treated as $\mathbb{R}^n$ with origin at p. Similarly,

$$\left\{ \left.\frac{\partial}{\partial y^1}\right|_{F(p)}, \left.\frac{\partial}{\partial y^2}\right|_{F(p)}, \ldots, \left.\frac{\partial}{\partial y^m}\right|_{F(p)} \right\}$$

is a basis of $T_{F(p)}\mathbb{R}^m$, which can be treated as $\mathbb{R}^m$ with origin at $F(p)$.

Geometric tangent vectors like $\left.\frac{\partial}{\partial x^i}\right|_p$ or $\left.\frac{\partial}{\partial y^j}\right|_{F(p)}$ act on smooth functions of $\mathbb{R}^n$ or $\mathbb{R}^m$, respectively, and spit out real numbers.

$$\left.\frac{\partial}{\partial x^i}\right|_p f = \frac{\partial f}{\partial x^i}(p) \in \mathbb{R}\,.$$

Since DF_p is a linear map between two vector spaces, in order to express DF_p as a matrix, we need to find where the basis vectors are getting mapped. So we want to find $DF_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right)$. This is a vector in $T_{F(p)}\mathbb{R}^m$, and hence can be written as a linear combination of $\left.\frac{\partial}{\partial y^j}\right|_{F(p)}$'s. Now we wish to find the coefficients in the linear combination.

$DF_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right)$ acts on $f \in C^\infty(\mathbb{R}^m)$ and yields a real number.

$$DF_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) f := \left.\frac{\partial}{\partial x^i}\right|_p (f \circ F)\,.$$

This makes perfect sense as $f \circ F : U \subseteq \mathbb{R}^n \to \mathbb{R}$. By chain rule,

$$\left.\frac{\partial}{\partial x^i}\right|_p (f \circ F) = \frac{\partial (f \circ F)}{\partial x^i}(p) = \sum_{j=1}^{m} \left.\frac{\partial f}{\partial y^j}\right|_{F(p)} \left.\frac{\partial F^j}{\partial x^i}\right|_p\,.$$

$$\therefore DF_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) f = \sum_{j=1}^{m} \left.\frac{\partial F^j}{\partial x^i}\right|_p \left.\frac{\partial}{\partial y^j}\right|_{F(p)} f \implies \boxed{DF_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = \sum_{j=1}^{m} \frac{\partial F^j}{\partial x^i}(p) \cdot \left.\frac{\partial}{\partial y^j}\right|_{F(p)}}$$

Therefore, DF_p can be represented by the following $m \times n$ matrix:

$$\begin{bmatrix} \frac{\partial F^1}{\partial x^1}(p) & \frac{\partial F^1}{\partial x^2}(p) & \cdots & \frac{\partial F^1}{\partial x^n}(p) \\ \frac{\partial F^2}{\partial x^1}(p) & \frac{\partial F^2}{\partial x^2}(p) & \cdots & \frac{\partial F^2}{\partial x^n}(p) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F^m}{\partial x^1}(p) & \frac{\partial F^m}{\partial x^2}(p) & \cdots & \frac{\partial F^m}{\partial x^n}(p) \end{bmatrix}$$

F is differentiable at $p \in U \subseteq \mathbb{R}^n$ if all the entries in the $m \times n$ matrix DF exist and are continuous at p. If F is differentiable at every $p \in U$, we say that F is of class C^1. DF is called the total derivative in the language of multivariable calculus.

Similarly, if all the second order partial derivatives exist and are continuous at p, then we say F is twice differentiable at p. If F is twice differentiable at every $p \in U$, we say F is of class C^2. In a similar manner, we define maps of class C^r. If a map F is of class C^r for every $r \in \mathbb{N}$, we say F is **smooth** or **infinitely differentiable**, or F belongs in the class C^∞.

§2.4 Inverse Function Theorem

Definition 2.4.1. Let U and V be open subsets of $\mathbb{R}^n$. A map $F: U \to V$ is said to be a $\boldsymbol{C^r}$**-diffeomorphism** if F is a homeomorphism, and both F and F^{-1} are of class C^r. When $r = \infty$, we just say F is a **diffeomorphism**.

Theorem 2.4.1 (Inverse Function Theorem)

Let W be an open subset of $\mathbb{R}^n$ and $F: W \to \mathbb{R}^n$ a C^∞ mapping. If $p \in W$ and DF_p is nonsingular, then there exists a neighborhood U of p in W such that $V = F(U)$ is open and $F: U \to V$ is a diffomorphism. If $x \in U$, then

$$DF^{-1}_{F(x)} = (DF_x)^{-1} .$$

We are not going to prove it here. We will see an example now.

Example 2.4.1. Let's consider the conversion of polar to rectangular coordinate. $F: \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$F\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} .$$

Then the differential DF is

$$DF = \begin{bmatrix} \dfrac{\partial F^1}{\partial r} & \dfrac{\partial F^1}{\partial \theta} \\ \dfrac{\partial F^2}{\partial r} & \dfrac{\partial F^2}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

Hence, $\det DF = r$. So $DF_{(r,\theta)}$ is differentiable for $r \neq 0$. Choose $r = \sqrt{2}$ and $\theta = \frac{\pi}{4}$. Then

$$F\begin{pmatrix} \sqrt{2} \\ \frac{\pi}{4} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} .$$

$$DF_{(\sqrt{2},\frac{\pi}{4})} = \begin{bmatrix} \dfrac{1}{\sqrt{2}} & -1 \\ \dfrac{1}{\sqrt{2}} & 1 \end{bmatrix} .$$

By the Inverse Function Theorem, there is a local inverse

$$DF^{-1}_{(1,1)} = \left(DF_{(\sqrt{2},\frac{\pi}{4})}\right)^{-1} .$$

Now, F^{-1} is given by

$$F^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sqrt{x^2+y^2} \\ \tan^{-1}\left(\frac{y}{x}\right) \end{pmatrix} .$$

Therefore,

$$DF^{-1} = \begin{bmatrix} \dfrac{2x}{2\sqrt{x^2+y^2}} & \dfrac{2y}{2\sqrt{x^2+y^2}} \\ \dfrac{-y}{x^2+y^2} & \dfrac{x}{x^2+y^2} \end{bmatrix} = \begin{bmatrix} \dfrac{x}{\sqrt{x^2+y^2}} & \dfrac{y}{\sqrt{x^2+y^2}} \\ \dfrac{-y}{x^2+y^2} & \dfrac{x}{x^2+y^2} \end{bmatrix} .$$

As a result,

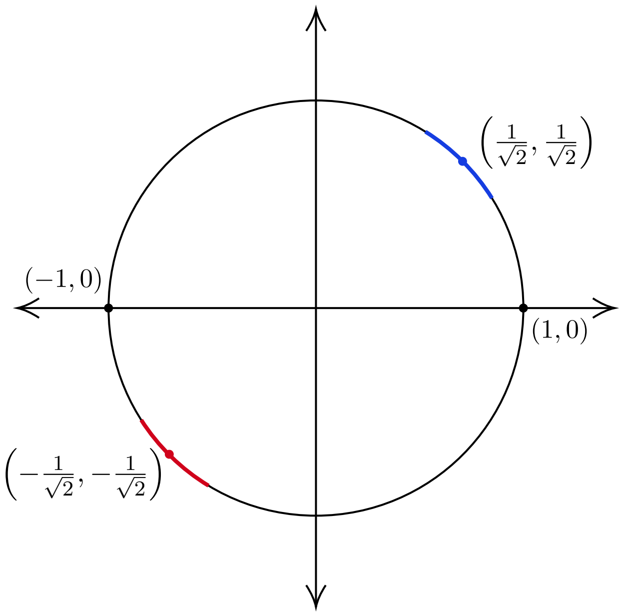
$$DF^{-1}_{(1,1)} = \begin{bmatrix} \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \\ \dfrac{-1}{2} & \dfrac{1}{2} \end{bmatrix} .$$

One can indeed check that

$$\begin{bmatrix} \dfrac{1}{\sqrt{2}} & -1 \\ \dfrac{1}{\sqrt{2}} & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \\ \dfrac{-1}{2} & \dfrac{1}{2} \end{bmatrix} .$$

§2.5 Implicit Function Theorem

Let us consider the equation of a unit circle in $\mathbb{R}^2$; $x^2 + y^2 = 1$.



The graph of the unit circle above does not represent a function. Because, for a given value of x, there are 2 values for y that satisfy the equation. Choose a point, say $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, on the unit circle. Then one can consider an arc (colored blue in the figure above) containing $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ that indeed represents a function given by $y = \sqrt{1 - x^2}$. Had we started with the point $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$, we could find an arc (colored red in the figure above) containing $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ that represents a function given by $y = -\sqrt{1 - x^2}$. The only problematic points are $(1, 0)$ and $(-1, 0)$. No matter how small an arc we choose about these points, it is not going to be represented by a function. Because, for those arcs, for a given x, there will be multiple values for y.

Now let us address the following 2-dimensional problem: Given an equation $F(x, y) = 0$, which is not globally a functional relationship (in the unit circle example, $F(x, y) = x^2 + y^2 - 1$), does there exist a point (x_0, y_0) satisfying $F(x_0, y_0) = 0$ so that there exists a neighborhood of (x_0, y_0) where y can be written as $y = f(x)$ for some real valued function f of one variable? In other words, $F(x, f(x)) = 0$ should hold for all values of x in that neighborhood. In the unit circle example, this f was given by $f(x) = \sqrt{1 - x^2}$ or $f(x) = -\sqrt{1 - x^2}$, depending on the choice of the point (x_0, y_0) in the upper or lower semicircle, respectively. The Implicit Function Theorem guarantees the **local existence** of such a function provided the initial point (x_0, y_0) was chosen *appropriately*. In the unit circle example, $(1, 0)$ and $(-1, 0)$ were two *inappropriate* points. As required by the Implicit Function Theorem, one must have

$$\frac{\partial F}{\partial y}(x_0, y_0) \neq 0 \,.$$

But in this case, for $F(x, y) = x^2 + y^2 - 1$,

$$\frac{\partial F}{\partial y} = 2y \implies \frac{\partial F}{\partial y}(1, 0) = 0 = \frac{\partial F}{\partial y}(1, 0) \,.$$

Therefore, in the light of Implicit Function Theorem, $(1, 0)$ and $(-1, 0)$ are not *appropriate* points on the unit circle around which we can construct a locally functional relationship. Now we state the most general form of Implicit Function Theorem.

Theorem 2.5.1 (Implicit Function Theorem)

Let U be an open set in $\mathbb{R}^n \times \mathbb{R}^m$ and $F : U \to \mathbb{R}^m$ a C^∞ map. Write $(x, y) = \left(x^1, \ldots, x^n, y^1, \ldots, y^m\right)$ for a point in U. Suppose the matrix

$$\left[\frac{\partial F^i}{\partial y^j}(x_0, y_0)\right]_{1 \le i, j \le m}$$

is non-singular for a point $(x_0, y_0) \in U$ satisfying $F(x_0, y_0) = 0$. Then there exists a neighborhood $X \times Y$ of (x_0, y_0) in U and a unique C^∞ map $f : X \to Y$ such that in $X \times Y \subseteq U \subseteq \mathbb{R}^n \times \mathbb{R}^m$,

$$F(x, y) = 0 \iff y = f(x) \ .$$

3 Manifolds

§3.1 Topological Manifolds

Definition 3.1.1 (Locally Euclidean Space). A topological space M is **locally Euclidean** of dimension n if every point in M has a neighborhood U such that there is a homeomorphism φ from U onto an open subset of $\mathbb{R}^n$. We call the pair $(U, \varphi : U \to \mathbb{R}^n)$ a **chart**, U a **coordinate neighborhood** and φ a **coordinate system** on U. We also say that a chart (U, φ) is centered at $p \in U$ if $\varphi(p) = \vec{0}$.

Definition 3.1.2 (Topological Manifold). A **topological manifold** of dimension n is a Hausdorff, second countable, locally Euclidean space of dimension n.

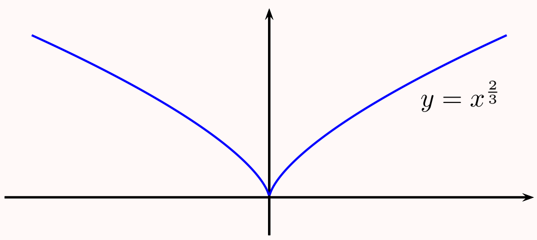
Example 3.1.1

The Euclidean space $\mathbb{R}^n$ is covered by a single chart $(\mathbb{R}^n, \mathbb{1}_{\mathbb{R}^n})$, where $\mathbb{1}_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n$ is the identity map. Every open subset U of $\mathbb{R}^n$ is also a topological manifold with the chart $(U, \mathbb{1}_U)$.

Recall that the hausdorff condition and second countability are "hereditary properties". That is, they are inherited by subspaces: a subspace of a Hausdorff space is also Hausdorff, and a subspace of a second countable space is also second countable. Hence, any subspace of $\mathbb{R}^n$ is Hausdorff and second countable.

Example 3.1.2 (The Cusp)

The graph of $y = x^{\frac{2}{3}}$ in $\mathbb{R}^2$ is a topological manifold.

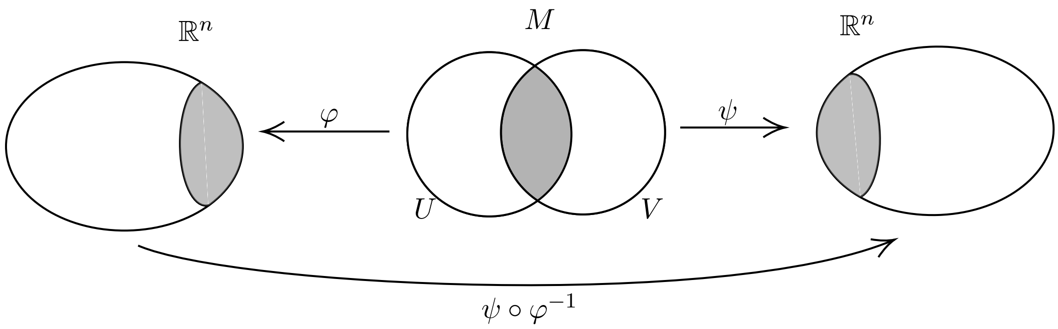


As a subspace of $\mathbb{R}^2$, it is Hausdorff and second countable. It is locally Euclidean, because it is homeomorphic to $\mathbb{R}$ via the map $\left(x, x^{2/3}\right) \mapsto x$. This map is continuous since it is just the projection onto first coordinate. The inverse map $x \mapsto \left(x, x^{2/3}\right)$ is continuous, as both $x \mapsto x$ and $x \mapsto x^{2/3}$ are continuous.

Definition 3.1.3 (Compatible Charts). Two charts $(U, \varphi : U \to \mathbb{R}^n)$ and $(V, \psi : V \to \mathbb{R}^n)$ of a topological manifold are ***C*$^\infty$-compatible** if the two maps

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V) \quad \text{and} \quad \varphi \circ \psi^{-1} : \psi(U \cap V) \to \varphi(U \cap V)$$

are both C^∞. These two maps are called **transition functions** between the charts. If $U \cap V$ is empty, then the two charts are automatically compatible.



Definition 3.1.4 (Atlas). A ***C^∞*-atlas** or simply an **atlas** on a locally Euclidean space M is a collection $\mathscr{U} = \{(U_\alpha, \varphi_\alpha)\}$ of pairwise C^∞-compatible charts that cover M. In other words,

$$M = \bigcup_\alpha U_\alpha \,.$$

Example 3.1.3

The unit circle S^1 in the complex plane can be described as the set of points $\left\{e^{it} \in \mathbb{C} \mid 0 \leq t < 2\pi\right\}$. Let U_1 and U_2 be the following two open subsets of S^1:

$$U_1 = \left\{e^{it} \in \mathbb{C} \mid -\pi < t < \pi\right\} \text{ and } U_2 \left\{e^{it} \in \mathbb{C} \mid 0 < t < 2\pi\right\} .$$

Define $\varphi_i : U_i \to \mathbb{R}$ by

$$\begin{aligned} \varphi_1\left(e^{it}\right) &= t\,, \quad -\pi < t < \pi\,; \\ \varphi_2\left(e^{it}\right) &= t\,, \quad 0 < t < 2\pi\,. \end{aligned}$$

(U_1, φ_1) and (U_2, φ_2) are charts on S^1. Their intersection $U_1 \cap U_2$ consists of two disjoint subsets of S^1 denoted by A and B.

$$A = \left\{e^{it} \in \mathbb{C} \mid -\pi < t < 0\right\} \text{ and } B \left\{e^{it} \in \mathbb{C} \mid 0 < t < \pi\right\} .$$

$U_1 \cap U_2 = A \sqcup B$. Now,

$$\begin{aligned} \varphi_1\left(U_1 \cap U_2\right) &= \varphi_1\left(A \sqcup B\right) = \varphi_1\left(A\right) \sqcup \varphi_1\left(B\right) = (-\pi, 0) \sqcup (0, \pi) \\ \varphi_2\left(U_1 \cap U_2\right) &= \varphi_2\left(A \sqcup B\right) = \varphi_2\left(A\right) \sqcup \varphi_2\left(B\right) = (\pi, 2\pi) \sqcup (0, \pi) \end{aligned}$$

Now, the transition function $\varphi_2 \circ \varphi_1^{-1} : \varphi_1\left(U_1 \cap U_2\right) \to \varphi_2\left(U_1 \cap U_2\right)$ is given by:

$$\left(\varphi_2 \circ \varphi_1^{-1}\right)(t) = \begin{cases} t + 2\pi & \text{for } t \in (-\pi, 0) \\ t & \text{for } t \in (0, \pi) \end{cases}$$

Similarly, the transition function $\varphi_1 \circ \varphi_2^{-1} : \varphi_2\left(U_1 \cap U_2\right) \to \varphi_1\left(U_1 \cap U_2\right)$ is given by:

$$\left(\varphi_1 \circ \varphi_2^{-1}\right)(t) = \begin{cases} t - 2\pi & \text{for } t \in (\pi, 2\pi) \\ t & \text{for } t \in (0, \pi) \end{cases}$$

These two transition functions are C^∞. Therefore, (U_1, φ_1) and (U_2, φ_2) are C^∞-compatible charts on S^1 and form an atlas.

Remark. Although the C^∞-compatibility of charts is clearly reflexive and symmetric, it is not transitive. The reason is as follows. Suppose (U_1, φ_1) is C^∞-compatible with (U_2, φ_2), and (U_2, φ_2) is C^∞-compatible with (U_3, φ_3). Note that the three coordinate functions are simultaneously defined only on the triple intersection $U_1 \cap U_2 \cap U_3$. Thus, the composite

$$\varphi_3 \circ \varphi_1^{-1} = \left(\varphi_3 \circ \varphi_2^{-1}\right) \circ \left(\varphi_2 \circ \varphi_1^{-1}\right)$$

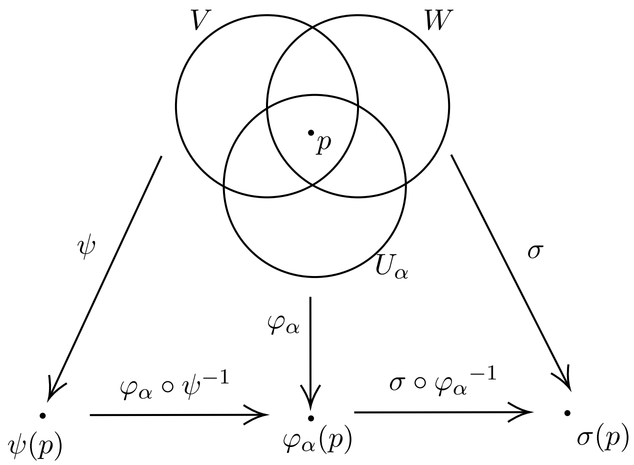
is C^∞, but only on $\varphi_1(U_1 \cap U_2 \cap U_3)$, not necessarily on $\varphi_1(U_1 \cap U_3)$. A priori we know nothing about $\varphi_3 \circ \varphi_1^{-1}$ on $\phi_1((U_1 \cap U_3) \setminus (U_1 \cap U_2 \cap U_3))$ and so we cannot conclude that (U_1, ϕ_1) and (U_3, ϕ_3) are C^∞-compatible.

We say that a chart (V, ψ) is compatible with an atlas $\{(U_\alpha, \varphi_\alpha)\}$ if it is compatible with all the charts $(U_\alpha, \varphi_\alpha)$ of the atlas.

Lemma 3.1.1

Let $\{(U_\alpha, \varphi_\alpha)\}$ be an atlas on a locally Euclidean space M. If two charts (V, ψ) and (W, σ) are both compatible with the atlas $\{(U_\alpha, \varphi_\alpha)\}$, then they are compatible with each other.

Proof. Let $p \in V \cap W$. First, we need to show that $\sigma \circ \psi^{-1}$ is C^∞ at $\psi(p)$.



Since $\{(U_\alpha, \varphi_\alpha)\}$ is an atlas for M, $p \in U_\alpha$ for some α. Hence, $p \in V \cap W \cap U_\alpha$. By the remark above,

$$\sigma \circ \psi^{-1} = \left(\sigma \circ \varphi_\alpha^{-1}\right) \circ \left(\varphi_\alpha \circ \psi^{-1}\right)$$

is C^∞ on $\psi(V \cap W \cap U_\alpha)$, and hence at $\psi(p)$. Since p was an arbitrary point of $V \cap W$, this proves that $\sigma \circ \psi^{-1}$ is C^∞ on $\psi(V \cap W)$. Similarly, $\psi \circ \sigma^{-1}$ is C^∞ on $\sigma(V \cap W)$. ■

Remark. In the equality $\sigma \circ \psi^{-1} = \left(\sigma \circ \varphi_\alpha^{-1}\right) \circ \left(\varphi_\alpha \circ \psi^{-1}\right)$, the maps on the two sides of the equality sign have different domains. What the equality means is that the two maps are equal on their common domain.

§3.2 Smooth Manifold

Definition 3.2.1 (Maximal Atlas)**.** An atlas $\mathscr{M}$ on a locally Euclidean space is said to be **maximal** if it is not contained in a larger atlas. In other words, if $\mathscr{U}$ is any other atlas containing $\mathscr{M}$, then $\mathscr{U} = \mathscr{M}$.

Definition 3.2.2 (Smooth Manifold)**.** A **smooth** or $\boldsymbol{C^\infty}$ **manifold** is a topological manifold M together with a maximal atlas. The maximal atlas is also called a *differentiable structure* on M.

In practice, to check that a topological manifold M is a smooth manifold, it is not necessary to exhibit a maximal atlas. The existence of *any* atlas on M will do, because of the following proposition.

Proposition 3.2.1

Any atlas $\mathscr{U} = \{(U_\alpha, \varphi_\alpha)\}$ on a locally Euclidean space is contained in a unique maximal atlas.

Proof. Adjoin to the atlas $\mathscr{U}$ all the charts (V_i, ψ_i) that are compatible with $\mathscr{U}$. By Lemma 3.1.1, the charts (V_i, ψ_i) are compatible with one another. So the enlarged collection of charts is an atlas. Can we enlarge this new atlas any further? Any chart compatible with the new atlas (that we wish to adjoin to the new atlas) must be compatible with the original atlas $\mathscr{U}$ and so by construction belongs to the new atlas. This proves that the new atlas is maximal.

Now we need to prove the uniqueness. Let $\mathscr{M}$ be the maximal atlas that we constructed in the preceeding paragraph. If $\mathscr{M}'$ is another maximal atlas containing $\mathscr{U}$, then all the charts in $\mathscr{M}'$ are compatible with $\mathscr{U}$ and so by construction must belong to $\mathscr{M}$. This proves that $\mathscr{M}' \subseteq \mathscr{M}$. $\mathscr{M}'$ is a maximal atlas contained in another atlas, so $\mathscr{M}$ and $\mathscr{M}$ must be the same. Therefore, the maximal atlas containing $\mathscr{U}$ is unique. ■

In summary, to show that a topological space M is a smooth manifold, it suffices to check that

(i) M is Hausdorff and second countable,

(ii) M has a C^∞ atlas (not necessarily maximal).

Some Notations

From now on, a "manifold" will mean a "smooth manifold". Also we shall use the terms "smooth" and "C^∞" interchangeably. Let $\vec{v} \in \mathbb{R}^n$ be a vector, or an n-tuple. The function $r^i : \mathbb{R}^n \to \mathbb{R}$ is defined as $r^i(\vec{v}) = v^i$. Let (U, φ) be a chart of the n-dimensional manifold M and let $p \in U$. Since $\varphi : U \to \mathbb{R}^n$, we write

$$\varphi(p) = \left(x^1(p), x^2(p), \ldots, x^n(p)\right),$$

with each component x^i of φ being a real valued function $x^i : U \to \mathbb{R}$ such that $x^i = r^i \circ \varphi$. The functions $x^1, x^2, \ldots, x^n$ are called *coordinates* or *local coordinates* on U. We sometimes write $\varphi = \left(x^1, x^2, \ldots, x^n\right)$ and the chart $(U, \varphi) = \left(U, x^1, x^2, \ldots, x^n\right)$.

Example 3.2.1 (Euclidean Space)

The Euclidean space $\mathbb{R}^n$ is a smooth manifold with a single chart $\left(\mathbb{R}^n, r^1, r^2, \ldots, r^n\right)$, where $r^1, r^2, \ldots, r^n$ are the standard coordinates on $\mathbb{R}^n$.

Example 3.2.2 (Open Subset of a Manifold)

Any open subset V of a manifold M is also a manifold. If $\{(U_\alpha, \varphi_\alpha)\}$ is an atlas for M, then

$$\mathscr{U}_V = \left\{\left(U_\alpha \cap V, \varphi_\alpha\big|_{U_\alpha \cap V}\right)\right\}$$

is an atlas for V. Notice that V, equipped with the subspace topology inherited from M, is indeed Hausdorff and second countable. It is a topological manifold because $U_\alpha \cap V$ is open in M, and φ_α is an open map; hence, as a restriction of a homeomorphism, $\varphi_\alpha\big|_{U_\alpha \cap V}$ is a homeomorphism mapping $U_\alpha \cap V$ to an open subset of $\mathbb{R}^n$. Now we are left to show that any two charts in the collection $\mathscr{U}_V$ are compatible.

$$\varphi_\alpha\big|_{U_\alpha \cap V} \circ \varphi_\beta\big|_{U_\beta \cap V}^{-1} = \left(\varphi_\alpha \circ \varphi_\beta^{-1}\right)\big|_{\varphi_\beta(U_\alpha \cap U_\beta \cap V)}.$$

As a restriction of a C^∞ map, this is also a C^∞ map. Hence $\mathscr{U}_V$ is truly an atlas for V.

Example 3.2.3 (General Linear Groups)

For any two positive integers m and n, let $\mathbb{R}^{m\times n}$ be the vector space of all $m \times n$ matrices. Since $\mathbb{R}^{m\times n}$ is isomorphic to $\mathbb{R}^{mn}$, we give it the topology of $\mathbb{R}^{mn}$. The definition of general linear group $\mathrm{GL}\,(n,\mathbb{R})$ is as follows:

$$\mathrm{GL}\,(n,\mathbb{R}) := \left\{A \in \mathbb{R}^{n\times n} \mid \det A \neq 0\right\} .$$

Consider the determinant function $\det : \mathbb{R}^{n\times n} \to \mathbb{R}$. It is a polynomial of the entries, hence continuous. In terms of this continuous function, the pre-image of $\mathbb{R}\setminus\{0\}$ is precisely $\mathrm{GL}\,(n,\mathbb{R})$.

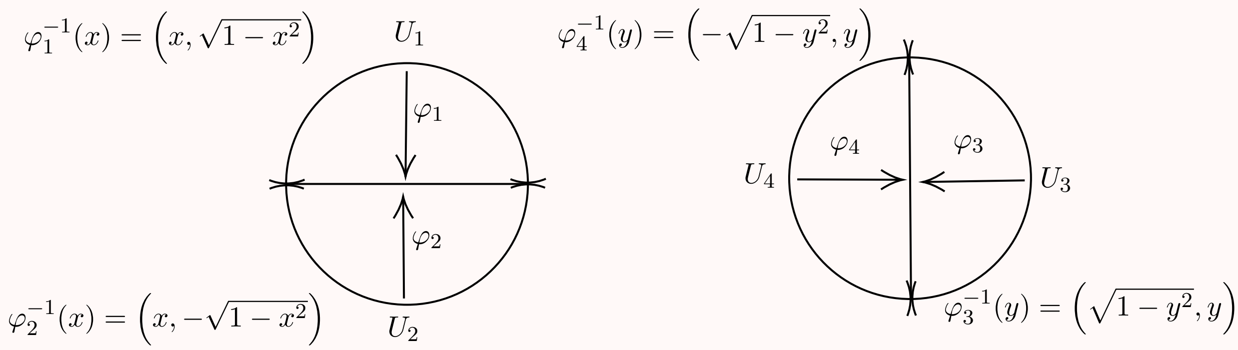
$$\mathrm{GL}\,(n,\mathbb{R}) = \overset{-1}{\det}\,(\mathbb{R}\setminus\{0\}) .$$

Since det is a continuous function from $\mathbb{R}^{n\times n} \cong \mathbb{R}^{n^2}$ to $\mathbb{R}$, and $\mathbb{R}\setminus\{0\}$ is open in $\mathbb{R}$, $\det^{-1}(\mathbb{R}\setminus\{0\})$ will be open in $\mathbb{R}^{n^2}$. Therefore, by Example 3.2.2, $\mathrm{GL}\,(n,\mathbb{R})$ is a manifold.

Example 3.2.4 (Unit circle in the (x, y)-plane)

In Example 3.1.3, we found a C^∞ atlas with 2 charts on the unit circle S^1 in the complex plane $\mathbb{C}$. We'll now view S^1 as the unit circle in $\mathbb{R}^2$ with defining equation $x^2 + y^2 = 1$. We can cover S^1 with 4 open sets: the upper and lower semicirles U_1 and U_2, the right and left semicircles U_3 and U_4. The homeomorphisms are:

$$\varphi_i : U_i \to (-1,1) \ , \ \varphi_i(x,y) = \begin{cases} x & \text{if } i = 1,2 \\ y & \text{if } i = 3,4 \end{cases}$$



Let us check that on $U_1 \cap U_3$,

$$\left(\varphi_3 \circ \varphi_1^{-1}\right)(\varphi_1(x,y)) = \left(\varphi_3 \circ \varphi_1^{-1}\right)(x) = \varphi_3\left(x, \sqrt{1-x^2}\right) = \sqrt{1-x^2} .$$

Since $(1,0) \notin U_1 \cap U_3$, we can conclude that $\varphi_3 \circ \varphi_1^{-1}$ is C^∞. Also, on $U_2 \cap U_4$,

$$\left(\varphi_2 \circ \varphi_4^{-1}\right)(\varphi_4(x,y)) = \left(\varphi_2 \circ \varphi_4^{-1}\right)(y) = \varphi_2\left(-\sqrt{1-y^2}, y\right) = -\sqrt{1-y^2} .$$

Since $(0,-1) \notin U_2 \cap U_4$, we can conclude that $\varphi_2 \circ \varphi_4^{-1}$ is C^∞. In a similar manner, one can check that $\varphi_i \circ \varphi_j^{-1}$ is C^∞ for every i, j. Therefore, $\{(U_i, \varphi_i) \mid 1 \leq i \leq 4\}$ is indeed a C^∞ atlas on S^1.

If M and N are manifolds, it's natural to think that $M \times N$ should also be a manifold. Now we shall demonstrate it. $M \times N$ with its product topology is Hausdorff and second countable (Proposition 1.6.3 and Corollary 1.6.2). To show that $M \times N$ is a manifold, it remains to exhibit an atlas on it. Recall that the product of two set maps $f : X \to X'$ and $g : Y \to Y'$ is

$$f \times g : X \times Y \to X' \times Y' \ , \ (f \times g)(x,y) = (f(x), g(y)) .$$

Proposition 3.2.2 (Atlas for Product Manifold)
If $\{(U_\alpha, \varphi_\alpha)\}$ and $\{(V_i, \psi_i)\}$ are C^∞ atlases for the manifolds M and N of dimensions m and n, respectively, then the collection

$$\{(U_\alpha \times V_i, \varphi_\alpha \times \psi_i : U_\alpha \times V_i \to \mathbb{R}^m \times \mathbb{R}^n)\}$$

of charts is a C^∞ atlas on $M \times N$. Therefore, $M \times N$ is a C^∞ manifold of dimension $m + n$.

Proof. φ_α is a homeomorphism of U_α onto $\varphi_\alpha(U_\alpha) = \overline{U_\alpha} \subseteq \mathbb{R}^m$, and ψ_i is a homeomorphism of V_i onto $\psi_i(V_i) = \overline{V_i} \subseteq \mathbb{R}^n$. Now,

$$(\varphi_\alpha \times \psi_i)(a, b) = (\varphi_\alpha(a), \psi_i(b)) = ((\varphi_\alpha \circ \pi_1)(a, b), (\psi_i \circ \pi_2)(a, b)) \,,$$

where π_1 and π_2 are projection on first and second coordinate, respectively. Both $\varphi_\alpha \circ \pi_1$ and $\psi_i \circ \pi_2$ are composition of continuous maps, hence continuous. Therefore, by Theorem 1.6.4, $\varphi_\alpha \times \psi_i$ is continuous. One can show that

$$(\varphi_\alpha \times \psi_i)^{-1} = \varphi_\alpha^{-1} \times \psi_i^{-1} \,.$$

Using an analogous argument as above, $\varphi_\alpha^{-1} \times \psi_i^{-1}$ is continuous. Therefore, $\varphi_\alpha \times \psi_i : U_\alpha \times V_i \to \overline{U_\alpha} \times \overline{V_i} \subseteq \mathbb{R}^{m+n}$ is a homeomorphism. Furthermore,

$$\bigcup_{\alpha,i} (U_\alpha \times V_i) = \bigcup_\alpha \left(U_\alpha \times \left(\bigcup_i V_i \right) \right) = \bigcup_\alpha (U_\alpha \times N) = \left(\bigcup_\alpha U_\alpha \right) \times N = M \times N \,.$$

Now, we are only left to show that any two charts are compatible with each other. It suffices to show that $(\varphi_\alpha \times \psi_i) \circ (\varphi_\beta \times \psi_j)^{-1} : \overline{U_\beta} \times \overline{V_j} \to \overline{U_\alpha} \times \overline{V_i}$ is a C^∞ map.

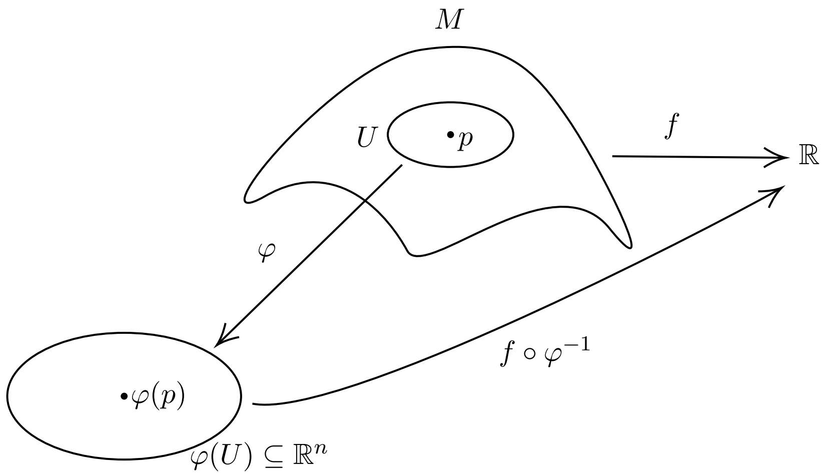
$$\begin{aligned}
(\varphi_\alpha \times \psi_i) \circ (\varphi_\beta \times \psi_j)^{-1} &= (\varphi_\alpha \times \psi_i) \circ \left(\varphi_\beta^{-1} \times \psi_j^{-1} \right) \\
\left((\varphi_\alpha \times \psi_i) \circ \left(\varphi_\beta^{-1} \times \psi_j^{-1} \right) \right)(x, y) &= (\varphi_\alpha \times \psi_i) \left(\varphi_\beta^{-1}(x), \psi_j^{-1}(y) \right) \\
&= \left(\left(\varphi_\alpha \circ \varphi_\beta^{-1} \right)(x), \left(\psi_i \circ \psi_j^{-1} \right)(y) \right) \\
\therefore (\varphi_\alpha \times \psi_i) \circ (\varphi_\beta \times \psi_j)^{-1} &= \left(\varphi_\alpha \circ \varphi_\beta^{-1} \right) \times \left(\psi_i \circ \psi_j^{-1} \right)
\end{aligned}$$

Both $\varphi_\alpha \circ \varphi_\beta^{-1}$ and $\psi_i \circ \psi_j^{-1}$ are C^∞ maps since $\{(U_\alpha, \varphi_\alpha)\}$ and $\{(V_i, \psi_i)\}$ are C^∞ atlases. Therefore, as a cartesian product of C^∞ maps, $(\varphi_\alpha \times \psi_i) \circ (\varphi_\beta \times \psi_j)^{-1}$ is also C^∞. This completes the proof. ■

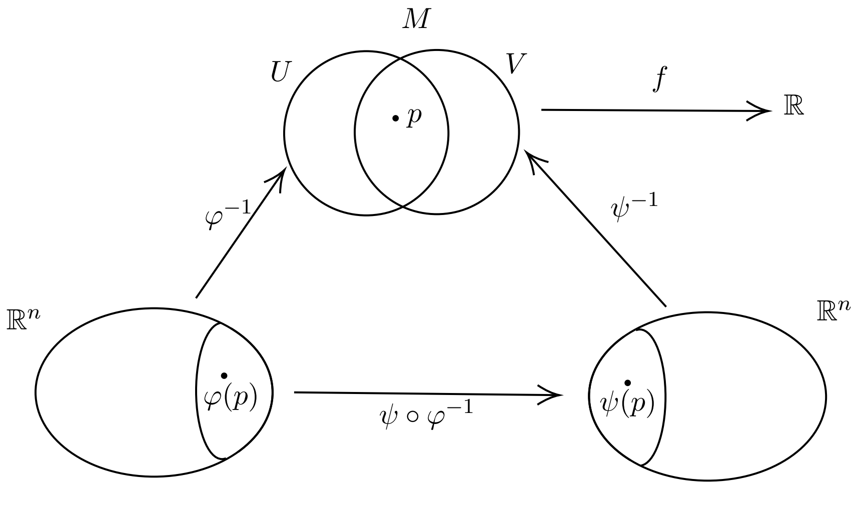
4 Smooth Maps on Manifold

§4.1 Smooth Functions on Manifold

Definition 4.1.1. Let M be a smooth manifold of dimension n. A function $f : M \to \mathbb{R}$ is said to be C^∞ or smooth at a point $p \in M$ if there is a chart (U, φ) about p in M such that $f \circ \varphi^{-1} : \varphi(U) \subseteq \mathbb{R}^n \to \mathbb{R}$ is C^∞ at $\varphi(p)$. The function f is said to be C^∞ on M if it is C^∞ at every point of M.



Remark. The definition of the smoothness of a function f at a given point on the manifold is independent of the chart (U, φ). Let us check this.



Suppose that $f \circ \varphi^{-1}$ is C^∞ at $\varphi(p)$ for a given chart (U, φ) about $p \in M$. Let (V, ψ) be any other chart about p. Then on $\psi(U \cap V)$,

$$f \circ \psi^{-1} = \left(f \circ \varphi^{-1}\right) \circ \left(\varphi \circ \psi^{-1}\right) .$$

$\varphi \circ \psi^{-1}$ is C^∞ by compatibility of charts. Therefore, as a composition of C^∞ maps, $f \circ \psi^{-1}$ is C^∞ at $\psi(p)$. This proves the independence of chart to determine the smoothness of a function at a given point.

Proposition 4.1.1

Let M be a manifold of dimension n, and $f : M \to \mathbb{R}$ a real-valued function on M. The following are equivalent:

(i) The function $f : M \to \mathbb{R}$ is C^∞.

(ii) The manifold M has an atlas such that for every chart (U, φ) in the atlas, $f \circ \varphi^{-1} : \varphi(U) \subseteq \mathbb{R}^n \to \mathbb{R}$ is C^∞.

(iii) For every chart (V, ψ) on M, the function $f \circ \psi^{-1} : \psi(V) \subseteq \mathbb{R}^n \to \mathbb{R}$ is C^∞.

Proof. **(ii)⇒(i):** Since (ii) holds, one can find for every $p \in M$, a coordinate neighborhood (U, φ) such that $f \circ \varphi^{-1}$ is C^∞ at $\varphi(p)$. Therefore, from the definition of C^∞ function on a manifold, $f : M \to \mathbb{R}$ is C^∞.

(i)⇒(iii): Let (V, ψ) be an arbitrary chart on M and $p \in V$. Since (i) holds, $f \circ \psi^{-1}$ is C^∞ at $\psi(p)$ (by the remark). Since p is an arbitrary point on V, $f \circ \psi^{-1}$ is C^∞ on $\psi(V)$.

(iii)⇒(ii): Obvious. ∎

Definition 4.1.2 (Pullback)**.** Let $F : N \to M$ be a map and h a function on M. The **pullback** of h by F, denoted by $F * h$, is the composite function $h \circ F$.

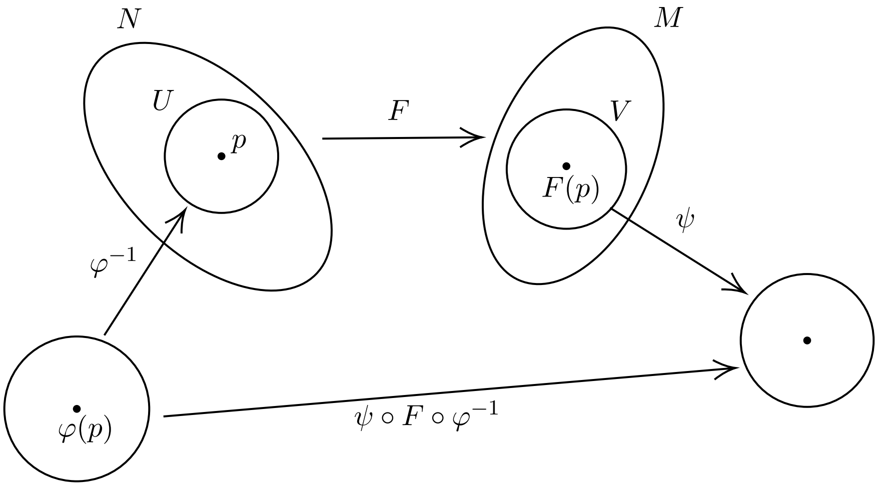
Using this terminology of pullback, a function f on M is C^∞ on a chart (U, φ) if and only if its pullback $\left(\varphi^{-1}\right) * f$ by φ^{-1} is C^∞ on the subset $\varphi(U)$ of Euclidean space.

§4.2 Smooth Maps Between Manifolds

Definition 4.2.1. Let N and M be manifolds of dimension n and m, respectively. A continuous map $F : N \to M$ is C^∞ at a point $p \in N$ if there are charts (V, ψ) about $F(p) \in M$ and (U, φ) about $p \in N$ such that the composition

$$\psi \circ F \circ \varphi^{-1} : \varphi\left(F^{-1}(V) \cap U\right) \subseteq \mathbb{R}^n \to \mathbb{R}^m$$

is C^∞ at $\varphi(p)$. The continuous map $F : N \to M$ is said to be C^∞ if it is C^∞ at every point of N.



Remark. Note that in the definition of smooth map between manifolds, one must have a continuous map to start with. We require $F : N \to M$ to be continuous so that $F^{-1}(V)$ is open and $\varphi\left(F^{-1}(V) \cap U\right)$ becomes an open subset of $\mathbb{R}^n$.

Proposition 4.2.1

Suppose $F : N \to M$ is C^∞ at $p \in N$. If (U, φ) is any chart about $p \in N$ and (V, ψ) is any chart about $F(p) \in M$, then $\psi \circ F \circ \varphi^{-1}$ is C^∞ at $\varphi(p)$.

Proof. Since F is C^∞ at $p \in N$, there are charts $(U_\alpha, \varphi_\alpha)$ about $p \in N$ and (V_β, ψ_β) about $F(p) \in M$ such that $\psi_\beta \circ F \circ \varphi_\alpha^{-1}$ is C^∞ at $\varphi_\alpha(p)$. By the C^∞ compatibility of charts in a differentiable structure, both $\varphi_\alpha \circ \varphi^{-1}$ and $\psi \circ \psi_\beta^{-1}$ are C^∞ on open subsets of Euclidean spaces. Hence, the composite

$$\psi \circ F \circ \varphi^{-1} = \left(\psi \circ \psi_\beta^{-1}\right) \circ \left(\psi_\beta \circ F \circ \varphi_\alpha^{-1}\right) \circ \left(\varphi_\alpha \circ \varphi^{-1}\right)$$

is C^∞ at $\varphi(p)$. ■

Proposition 4.2.2 (Smoothness of a map in terms of charts)

Let N and M be smooth manifolds, and $F : N \to M$ a continuous map. The following are equivalent:

(i) The map $F : N \to M$ is C^∞.

(ii) There are atlases $\mathscr{U}$ for N and $\mathscr{V}$ for M such that for every chart (U, φ) in $\mathscr{U}$ and (V, ψ) in $\mathscr{V}$, the map

$$\psi \circ F \circ \varphi^{-1} : \varphi\left(U \cap F^{-1}(V)\right) \to \mathbb{R}^m$$

is C^∞.

(iii) For every chart (U, φ) on N and (V, ψ) on M, the map

$$\psi \circ F \circ \varphi^{-1} : \varphi\left(U \cap F^{-1}(V)\right) \to \mathbb{R}^m$$

is C^∞.

Proof. **(ii)⇒(i):** Let $p \in N$. Suppose (U, φ) is a chart about p in $\mathscr{U}$ and (V, ψ) is a chart about $F(p)$ in $\mathscr{V}$. Now, (ii) implies that $\psi \circ F \circ \varphi^{-1}$ is C^∞ at $\varphi(p)$. By the definition of a C^∞ map, $F : N \to M$ is C^∞ at p. Since p was an arbitrary point of N, the map $F : N \to M$ is C^∞.

(i)⇒(iii): Suppose (U, φ) and (V, ψ) are charts on N and M, respectively, such that $U \cap F^{-1}(V) \neq \varnothing$. Let $p \in U \cap F^{-1}(V)$ so that $p \in U$ and $F(p) \in V$. Then (U, φ) is a chart about p and (V, ψ) is a chart about $F(p)$. By Proposition 4.2.1, $\psi \circ F \circ \varphi^{-1}$ is C^∞ at $\varphi(p)$. Since $\varphi(p)$ was an arbitrary point of $\varphi\left(U \cap F^{-1}(V)\right)$, the map $\psi \circ F \circ \varphi^{-1} : \varphi\left(U \cap F^{-1}(V)\right) \to \mathbb{R}^m$ is C^∞.

(iii)⇒(ii): Take $\mathscr{U}$ and $\mathscr{V}$ to be the maximal atlases of N and M, respectively. ■

Proposition 4.2.3 (Composition of C^∞ maps)

If $F : N \to M$ and $G : M \to P$ are C^∞ maps of manifolds, then the composite $G \circ F : N \to P$ is C^∞.

Proof. Let $(U, \varphi), (V, \psi)$, and (W, σ) be charts on N, M, and P, respectively. Then

$$\sigma \circ (G \circ F) \circ \varphi^{-1} = \left(\sigma \circ G \circ \psi^{-1}\right) \circ \left(\psi \circ F \circ \varphi^{-1}\right).$$

Since F and G are C^∞, by Proposition 4.2.2 (i)⇒(iii), $\sigma \circ G \circ \psi^{-1}$ and $\psi \circ F \circ \varphi^{-1}$ are C^∞ maps on their respective domains. As a composite of C^∞ maps of open subsets of Euclidean spaces, $\sigma \circ (G \circ F) \circ \varphi^{-1}$ is C^∞. In particular,

$$\sigma \circ (G \circ F) \circ \varphi^{-1} : \varphi\left(U \cap F^{-1}(V)\right) \cap \psi\left(V \cap G^{-1}(W)\right) \to \mathbb{R}^p$$

is C^∞ provided N, M and P are of dimension n, m and p, respectively. By (iii)⇒(i) of Proposition 4.2.2, $G \circ F$ is C^∞. ■

Definition 4.2.2 (Diffeomorphism). A **diffeomorphism** of manifolds is a bijective C^∞ map $F : N \to M$ whose inverse F^{-1} is also C^∞.

Proposition 4.2.4

If (U, φ) is a chart on a manifold M of dimension n, then the coordinate map $\varphi : U \to \varphi(U) \subseteq \mathbb{R}^n$ is a diffeomorphism.

Proof. By definition, φ is a homeomorphism. So it suffices to check that both φ and φ^{-1} are smooth. In order to check the smoothness of $\varphi : U \to \varphi(U)$, we shall use the atlas $\{(U, \varphi)\}$ on the manifold U, and the atlas $\left\{\left(\varphi(U), \mathbb{1}_{\varphi(U)}\right)\right\}$ on the manifold $\varphi(U)$. Observe that,

$$\mathbb{1}_{\varphi(U)} \circ \varphi \circ \varphi^{-1} : \varphi(U) \to \varphi(U)$$

is just the identity map on $\varphi(U)$, hence C^∞. Therefore, by (ii)⇒(i) of Proposition 4.2.2, φ is C^∞.

We shall use the same atlas as above to show the smoothness of $\varphi^{-1} : \varphi(U) \to U$. Now,

$$\varphi \circ \varphi^{-1} \circ \mathbb{1}_{\varphi(U)}^{-1} = \mathbb{1}_{\varphi(U)} : \varphi(U) \to \varphi(U) .$$

Identity map is C^∞, hence by (ii)⇒(i) of Proposition 4.2.2, φ^{-1} is C^∞. ■

Proposition 4.2.5 (Smoothness of a vector-valued function)

Let N be a manifold and $F : N \to \mathbb{R}^m$ a continuous map. The following are equivalent:

(i) The map $F : N \to \mathbb{R}^m$ is C^∞.

(ii) The manifold N has an atlas such that for every chart (U, φ) in the atlas, the map $F \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^m$ is C^∞.

(iii) For every chart (U, φ) on N, the map $F \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^m$ is C^∞.

Proof. **(ii)⇒(i):** In Proposition 4.2.2(ii), take the atlas $\mathscr{V}$ of $\mathbb{R}^m$ to be $\{(\mathbb{R}^m, \mathbb{1}_{\mathbb{R}^m})\}$. Now, $\varphi(U) = \varphi(U \cap N) = \varphi\left(U \cap F^{-1}(\mathbb{R}^m)\right)$. Therefore,

$$\mathbb{1}_{\mathbb{R}^m} \circ F \circ \varphi^{-1} : \varphi\left(U \cap F^{-1}(\mathbb{R}^m)\right) \to \mathbb{R}^m$$

is the same as $F \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^m$, which is C^∞. Hence by (ii)⇒(i) of Proposition 4.2.2, F is C^∞.

(i)⇒(iii): In Proposition 4.2.2(iii), let (V, ψ) be the chart $(\mathbb{R}^m, \mathbb{1}_{\mathbb{R}^m})$ on $\mathbb{R}^m$. Hence,

$$\mathbb{1}_{\mathbb{R}^m} \circ F \circ \varphi^{-1} : \varphi\left(U \cap F^{-1}(\mathbb{R}^m)\right) \to \mathbb{R}^m$$

is C^∞, which is the same as $F \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^m$.

(iii)⇒(ii): Choose the maximal atlas of N. ■

Proposition 4.2.6 (Smoothness in terms of components)

Let N be a manifold. A vector-valued function $F : N \to \mathbb{R}^m$ is C^∞ if and only if its component functions $F^1, \ldots, F^m : N \to \mathbb{R}$ are all C^∞.

Proof. $F : N \to \mathbb{R}^m$ is C^∞ if and only if for every chart (U, φ) on N, $F \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^m$ is C^∞ (Proposition 4.2.5). Now, $F \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^m$ is C^∞ if and only if $r^i \circ \left(F \circ \varphi^{-1}\right)$ is C^∞ for every $1 \leq i \leq m$.

$$r^i \circ \left(F \circ \varphi^{-1}\right) = F^i \circ \varphi^{-1} : \varphi(U) \to \mathbb{R} .$$

Therefore, F being smooth is equivalent to each $F^i \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}$ being smooth for every chart (U, φ). By Proposition 4.1.1, this is equivalent to each $F^i \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}$ being smooth. Therefore, $F : N \to \mathbb{R}^m$ is C^∞ if and only if each $F^i \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}$ is C^∞. ■

Example 4.2.1

Let M and N be manifolds and $\pi : M \times N \to M$, $\pi(p,q) = p$ be the projection onto the first factor. We want to show that π is a C^∞ map.

Let (p,q) be an arbitrary point of $M \times N$. Suppose $(U,\varphi) = \left(U, x^1, \ldots, x^m\right)$ and $(V,\psi) = \left(V, y^1, \ldots, y^n\right)$ are coordinate neighborhoods of p and q in M and N, respectively. By Proposition 3.2.2,

$$(U \times V, \varphi \times \psi) = \left(U \times V, x^1, \ldots, x^m, y^1, \ldots, y^n\right)$$

is a coordinate neighborhood of (p,q). Therefore, given $\left(a^1, \ldots, a^m, b^1, \ldots, b^n\right) \in (\varphi \times \psi)(U \times V) \subseteq \mathbb{R}^{m+n}$,

$$\begin{aligned}\left(\varphi \circ \pi \circ (\varphi \times \psi)^{-1}\right)\left(a^1, \ldots, a^m, b^1, \ldots, b^n\right) &= (\varphi \circ \pi)\left(\varphi^{-1}\left(a^1, \ldots, a^m\right), \psi^{-1}\left(b^1, \ldots, b^n\right)\right) \\ &= \varphi\left(\varphi^{-1}\left(a^1, \ldots, a^m\right)\right) \\ &= \left(a^1, \ldots, a^m\right)\end{aligned}$$

Therefore, $\varphi \circ \pi \circ (\varphi \times \psi)^{-1} : (\varphi \times \psi)(U \times V) \subseteq \mathbb{R}^{m+n} \to \mathbb{R}^m$ is just the projection onto the first m coordinates, which is a C^∞ map. Hence, $\pi : M \times N \to M$ is C^∞ at (p,q). Since (p,q) was chosen arbitrarily from $M \times N$, $\pi : M \times N \to M$ is C^∞ on $M \times N$.

Lemma 4.2.7

Let M_1, M_2 and N be manifolds of dimensions m_1, m_2 and n, respectively. Prove that a map $(f_1, f_2) : N \to M_1 \times M_2$ is C^∞ if and only if $f_i : N \to M_i$, $i = 1,2$ are both C^∞.

Proof. Let $(f_1, f_2) = f$, and $\pi_i : M_1 \to M_2 \to M_i$ be projection maps for $i = 1,2$. Both π_i are smooth, as proved in Example 4.2.1. If f is smooth, then $f_i = \pi_i \circ f : N \to M_i$ is composition of smooth maps, hence smooth.

Conversely, suppose both $f_i : N \to M_i$ are smooth. Then both f_i are continuous, hence so is f (Theorem 1.6.4). Let $p \in N$, and take coordinate neighborhoods (U,φ), (V_1,ψ_1), (V_2,ψ_2) of p, $f_1(p)$, $f_2(p)$, respectively. We can choose U sufficiently small so that $f(U) \subseteq V_1 \times V_2$. Since f_i is smooth,

$$\psi_i \circ f_i \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^{m_i}$$

is smooth at $\varphi(p)$. Now, $(V_1 \times V_2, \psi_1 \times \psi_2)$ is a coordinate neighborhood of $f(p) = (f_1(p), f_2(p))$. Given $a \in \varphi(U) \subseteq \mathbb{R}^n$,

$$\begin{aligned}\left((\psi_1 \times \psi_2) \circ f \circ \varphi^{-1}\right)(a) &= (\psi_1 \times \psi_2)\left(\left(f_1 \circ \varphi^{-1}\right)(a), \left(f_2 \circ \varphi^{-1}\right)(a)\right) \\ &= \left(\left(\psi_1 \circ f_1 \circ \varphi^{-1}\right)(a), \left(\psi_2 \circ f_2 \circ \varphi^{-1}\right)(a)\right) \\ \therefore (\psi_1 \times \psi_2) \circ f \circ \varphi^{-1} &= \left(\psi_1 \circ f_1 \circ \varphi^{-1}, \psi_2 \circ f_2 \circ \varphi^{-1}\right)\end{aligned}$$

Both $\psi_i \circ f_i \circ \varphi^{-1}$ are smooth at $\varphi(p)$. Therefore, $(\psi_1 \times \psi_2) \circ f \circ \varphi^{-1}$ is also smooth at $\varphi(p)$. In other words, f is smooth at p. Since p was chosen arbitrarily from N, f is smooth on N. ■

§4.3 Partial Derivatives

On a manifold M of dimension n, let (U,φ) be a chart and $f : M \to \mathbb{R}$ a C^∞ function. As a function into $\mathbb{R}^n$, φ has n components: $x^1, x^2, \ldots, x^n$. Let $r^1.r^2, \ldots, r^n$ be standard coordinates on $\mathbb{R}^n$. That is, if $\vec{v} \equiv \left(v^1, v^2, \ldots, v^n\right) \in \mathbb{R}^n$, then $r^i(\vec{v}) = v^i$ for $1 \le i \le n$.

Now, $x^i = r^i \circ \varphi$. For $p \in U$, one defines the partial derivative $\frac{\partial f}{\partial x^i}$ of f with respect to x^i at p to be

$$\left.\frac{\partial}{\partial x^i}\right|_p f := \frac{\partial f}{\partial x^i}(p) := \frac{\partial\left(f \circ \varphi^{-1}\right)}{\partial r^i}(\varphi(p)) = \left.\frac{\partial}{\partial r^i}\right|_{\varphi(p)}\left(f \circ \varphi^{-1}\right).$$

Since $p = \varphi^{-1}(\varphi(p))$, the equation can be rewritten as

$$\frac{\partial f}{\partial x^i}\left(\varphi^{-1}(\varphi(p))\right) = \frac{\partial\left(f\circ\varphi^{-1}\right)}{\partial r^i}(\varphi(p)) \implies \left(\frac{\partial f}{\partial x^i}\circ\varphi^{-1}\right)(\varphi(p)) = \frac{\partial\left(f\circ\varphi^{-1}\right)}{\partial r^i}(\varphi(p)) .$$

Thus, as functions on $\varphi(U)$,

$$\frac{\partial f}{\partial x^i}\circ\varphi^{-1} = \frac{\partial\left(f\circ\varphi^{-1}\right)}{\partial r^i} .$$

The partial derivative $\frac{\partial f}{\partial x^i}$ is C^∞ on U because its pullback $\frac{\partial f}{\partial x^i}\circ\varphi^{-1}$ is C^∞ on $\varphi(U)$.

Proposition 4.3.1

Suppose $\left(U, x^1, \ldots, x^n\right)$ is a chart on a manifold. Then $\dfrac{\partial x^i}{\partial x^j} = \delta^i_j$.

Proof. At a point $p \in U$, using $x^i = r^i \circ \varphi$,

$$\frac{\partial x^i}{\partial x^j}(p) = \frac{\partial\left(x^i\circ\varphi^{-1}\right)}{\partial r^j}(\varphi(p)) = \frac{\partial\left(r^i\circ\varphi\circ\varphi^{-1}\right)}{\partial r^j}(\varphi(p)) = \frac{\partial r^i}{\partial r^j}(\varphi(p)) = \delta^i_j .$$

■

Definition 4.3.1. Let $F : N \to M$ be a smooth map, and let $(U, \varphi) = \left(U, x^1, \ldots, x^n\right)$ and $(V, \psi) = \left(V, y^1, \ldots, y^m\right)$ be charts on N and M respectively such that $F(U) \subset V$. Denote by

$$F^i := y^i \circ F = r^i \circ \psi \circ F : U \to \mathbb{R}$$

the i-th component of F in the chart (V, ψ). Then the $m \times n$ matrix $\left[\frac{\partial F^i}{\partial x^j}\right]$ is called the **Jacobian matrix** of F relative to the charts (U, φ) and (V, ψ). In case N and M have the same dimension, the determinant of the Jacobian matrix is called the **Jacobian determinant** of F relative to the two charts. The Jacobian determinant is also written as

$$\det\left[\frac{\partial F^i}{\partial x^j}\right] = \frac{\partial\left(F^1, \ldots, F^n\right)}{\partial\left(x^1, \ldots, x^n\right)} .$$

Example 4.3.1 (Jacobian matrix of a transition map)

Let $(U, \varphi) = \left(U, x^1, \ldots, x^n\right)$ and $(V, \psi) = \left(V, y^1, \ldots, y^n\right)$ be overlapping charts on a manifold M. The transition map $\psi\circ\varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is a diffeomorphism of open subsets of $\mathbb{R}^n$. Then its Jacobian matrix $J\left(\psi\circ\varphi^{-1}\right)$ at $\varphi(p)$ is the matrix $\left[\frac{\partial y^i}{\partial x^j}\right]$ of partial derivatives at p.

Since $\psi\circ\varphi^{-1}$ is a map between two open subsets of Euclidean spaces, $J\left(\psi\circ\varphi^{-1}\right) = \left[\frac{\partial\left(\psi\circ\varphi^{-1}\right)^i}{\partial r^j}\right]$.

$$\begin{aligned}
\frac{\partial\left(\psi\circ\varphi^{-1}\right)^i}{\partial r^j}(\varphi(p)) &= \frac{\partial\left(r^i\circ\psi\circ\varphi^{-1}\right)}{\partial r^j}(\varphi(p)) \\
&= \frac{\partial\left(y^i\circ\varphi^{-1}\right)}{\partial r^j}(\varphi(p)) \\
&= \frac{\partial y^i}{\partial x^j}(p)
\end{aligned}$$

Definition 4.3.2. A C^∞ map $F : N \to M$ is **locally invertible** at $p \in N$ if p has a neighborhood U on which $F\big|_U : U \to F(U)$ is a diffeomorphism.

Theorem 4.3.2 (Inverse Function Theorem for Manifolds)
Let $F : N \to M$ be a C^∞ map between two manifolds of the same dimension, and $p \in N$. Suppose for some charts $(U, \varphi) = \left(U, x^1, \ldots, x^n\right)$ about $p \in N$ and $(V, \psi) = \left(V, y^1, \ldots, y^n\right)$ about $F(p) \in M$, $F(U) \subseteq V$. Set $F^i = y^i \circ F$. Then F is locally invertible at p if and only if its Jacobian determinant $\det \left[\frac{\partial F^i}{\partial x^j}(p)\right]$ is nonzero.

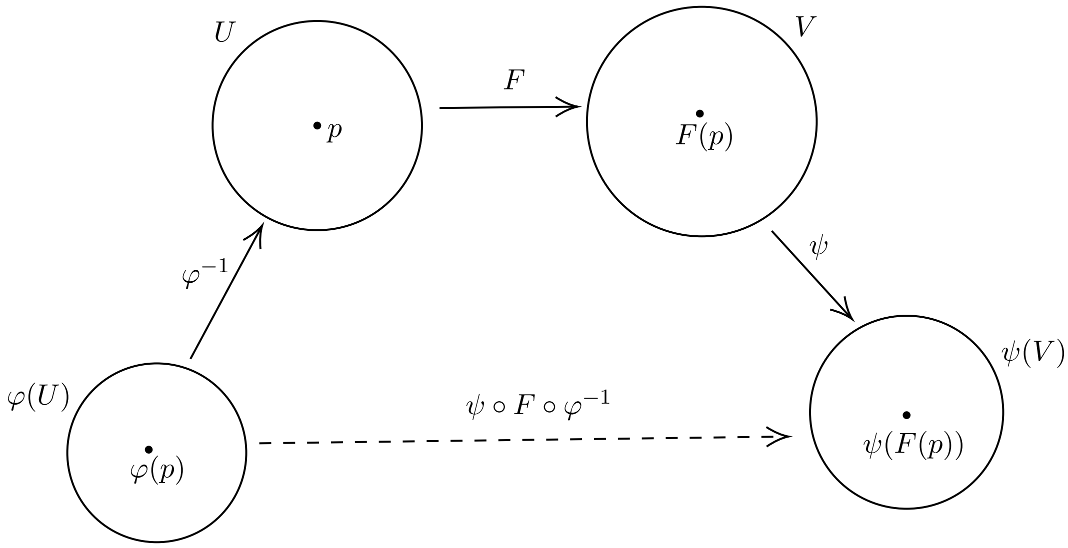
Proof. Since $F^i = y^i \circ F = r^i \circ \psi \circ F$, the Jacobian matrix of F relative to the charts (U, φ) and (V, ψ) is

$$\left[\frac{\partial F^i}{\partial x^j}(p)\right] = \left[\frac{\partial \left(r^i \circ \psi \circ F\right)}{\partial x^j}(p)\right] = \left[\frac{\partial \left(r^i \circ \psi \circ F \circ \varphi^{-1}\right)}{\partial r^j}(\varphi(p))\right] =$$

which is the Jacobian matrix of the map

$$\psi \circ F \circ \varphi^{-1} : \varphi(U) \subseteq \mathbb{R}^n \to \psi(V) \subseteq \mathbb{R}^n$$

between two open subsets of $\mathbb{R}^n$.



By Inverse Function Theorem for $\mathbb{R}^n$, $\psi \circ F \circ \varphi^{-1}$ is locally invertible at $\varphi(p)$ if and only if

$$\det \left[\frac{\partial F^i}{\partial x^j}(p)\right] = \det \left[\frac{\partial \left(\psi \circ F \circ \varphi^{-1}\right)^i}{\partial r^j}(\varphi(p))\right] \neq 0\,.$$

By Proposition 4.2.4, φ and ψ are diffeomorphisms. Therefore, local invertibility of $\psi \circ F \circ \varphi^{-1}$ at $\varphi(p)$ is equivalent to local invertibility of F at p. ∎

5 Some Interesting Manifolds

§5.1 Real Projective Space

Define an equivalence relation $\sim$ on $\mathbb{R}^{n+1} \setminus \{0\}$ by

$$x \sim y \iff y = tx \text{ for some nonzero real number } t\,,$$

where $x, y \in \mathbb{R}^{n+1} \setminus \{0\}$. The **real projective space** is the quotient space of $\mathbb{R}^{n+1} \setminus \{0\}$ by this equivalence relation and is denoted by $\mathbb{R}P^n$. We denote the equivalence class of a point $\left(a^0, a^1, \ldots, a^n\right) \in \mathbb{R}^{n+1} \setminus \{0\}$ by $\left[a^0, a^1, \ldots, a^n\right]$ and let $\pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}P^n$ be the underlying projection map. We call $\left[a^0, a^1, \ldots, a^n\right]$ homogenous coordinates on $\mathbb{R}P^n$.

Geometrically, two nonzero points of $\mathbb{R}^{n+1}$ are equivalent if and only if they lie on the same line through the origin. So $\mathbb{R}P^n$ can be though of as the set of all lines through the origin in $\mathbb{R}^{n+1}$. A line through the origin in $\mathbb{R}^{n+1}$ is just a point in $\mathbb{R}P^n$.

Each line through the origin in $\mathbb{R}^{n+1}$ meets the unit sphere S^n in a pair of antipodal points. Conversely, a pair of antipodal points on S^n determines a unique line in $\mathbb{R}^{n+1}$.

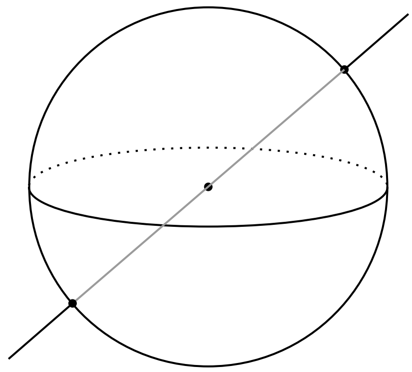


Figure 5.1: A line through 0 in $\mathbb{R}^3$ corresponds to a pair of antipodal points on S^2.

This suggests that we can define an equivalence relation $\sim$ on S^n by identifying the antipodal points:

$$x \sim y \iff x = \pm y\,, \quad x, y \in S^n\,.$$

We then have a bijection $\mathbb{R}P^n \leftrightarrow S^n/\sim$. We shall now see that this bijection is a homeomorphism.

Lemma 5.1.1
$\mathbb{R}P^n$ is homeomorphic to $S^n/\sim$.

Proof. Consider $f : \mathbb{R}^{n+1} \setminus \{0\} \to S^n$ defined by $f(x) = \frac{x}{\|x\|}$. Then f is continuous. Note that, for a nonzero real t,

$$f(tx) = \frac{tx}{\|tx\|} = \frac{t}{|t|}\frac{x}{\|x\|} = \begin{cases} f(x) & \text{if } t > 0 \\ -f(x) & \text{if } t < 0 \end{cases}$$

Now we define $\overline{f} : \mathbb{R}P^n \to S^n/\sim$ by $\overline{f}([x]) = [f(x)]$. This map is well-defined, since

$$\overline{f}([tx]) = [f(tx)] = [\pm f(x)] = [f(x)] = \overline{f}([x])\,.$$

Let $\pi_1 : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}P^n$ and $\pi_2 : S^n \to S^n/\sim$ be the respective projection maps. Now we have a commutative diagram.

$$\begin{tikzcd} \mathbb{R}^{n+1}\setminus\{0\} \arrow[r, "f"] \arrow[d, "\pi_1"'] \arrow[dr, "\pi_2\circ f"] & S^n \arrow[d, "\pi_2"] \\ \mathbb{R}P^n \arrow[r, "\overline{f}"'] & S^n/\sim \end{tikzcd}$$

Now, $\pi_2 \circ f$ is the composition of two continuous maps, hence continuous. Therefore, by Proposition 1.7.1, $\overline{f}$ is continuous.

Now, let $g : S^n \to \mathbb{R}^{n+1}\setminus\{0\}$ be the inclusion map given by $g(x) = x$. We know that g is continuous. It induces another map $\overline{g} : S^n/\sim \to \mathbb{R}P^n$ defined by $\overline{g}([x]) = [x]$. $\overline{g}$ is well-defined, because

$$\overline{g}([-x]) = [-x] = [x] = \overline{g}([x]) \ .$$

As before, we have another commutative diagram.

$$\begin{tikzcd} S^n \arrow[r, "g"] \arrow[d, "\pi_2"'] \arrow[dr, "\pi_1\circ g"] & \mathbb{R}^{n+1}\setminus\{0\} \arrow[d, "\pi_1"] \\ S^n/\sim \arrow[r, "\overline{g}"'] & \mathbb{R}P^n \end{tikzcd}$$

$\pi_1 \circ g$ is the composition of two continuous maps, hence continuous. Therefore, by Proposition 1.7.1, $\overline{g}$ is continuous. Now we are only left to show that $\overline{f}$ and $\overline{g}$ are inverses of one another. For $[x] \in \mathbb{R}P^n$,

$$\left(\overline{g}\circ\overline{f}\right)[x] = \overline{g}\left[\frac{x}{\|x\|}\right] = \left[\frac{x}{\|x\|}\right] = [x] \ ,$$

because $x \sim \frac{x}{\|x\|}$ in $\mathbb{R}^{n+1}\setminus\{0\}$, where the value of the nonzero real t is $\frac{1}{\|x\|}$. Furthermore, for $[x] \in S^n/\sim$, $x \in S^n$, so $\|x\| = 1$.

$$\left(\overline{f}\circ\overline{g}\right)[x] = \overline{f}\,[x] = \left[\frac{x}{\|x\|}\right] = [x] \ .$$

Hence, $\overline{g}$ is indeed the inverse of $\overline{f}$. Therefore, $\overline{f} : \mathbb{R}P^n \to S^n/\sim$ is a homeomorphism. ■

Proposition 5.1.2

The equivalence relation $\sim$ on $\mathbb{R}^{n+1}\setminus\{0\}$ is an open equivalence relation.

Proof. For an open set $U \subseteq \mathbb{R}^{n+1}\setminus\{0\}$, $\pi(U)$ is open in $\mathbb{R}P^n$ if and only if $\pi^{-1}(\pi(U))$ is open in $\mathbb{R}^{n+1}\setminus\{0\}$ (definition of Quotient Topology). Now, if we take an arbitrary point of U and then take its nonzero multiple, both the points will belong to the same equivalence class in $\mathbb{R}P^n$. In other words, for $x \in U$, tx and x will be mapped to the same point in $\pi(U)$. Hence,

$$\pi^{-1}(\pi(U)) = \bigcup_{t\in\mathbb{R}^\times} tU = \bigcup_{t\in\mathbb{R}^\times} \{tx \mid x \in U\} \ ,$$

where $\mathbb{R}^\times = \mathbb{R}\setminus\{0\}$. The map of multiplication by a nonzero real t is a homeomorphism from $\mathbb{R}^{n+1}\setminus\{0\}$ to itself. Hence, tU is open in $\mathbb{R}^{n+1}\setminus\{0\}$ for any nonzero t. Therefore, their union

$$\bigcup_{t\in\mathbb{R}^\times} tU = \pi^{-1}(\pi(U))$$

is also open in $\mathbb{R}^{n+1}\setminus\{0\}$. ■

Corollary 5.1.3
The real projective space $\mathbb{R}P^n$ is second countable.

Proof. Follows from the fact that $\mathbb{R}^{n+1} \setminus \{0\}$ is second countable and Corollary 1.10.4. ■

Proposition 5.1.4
$\mathbb{R}P^n$ is Hausdorff.

Proof. Let $S = \mathbb{R}^{n+1} \setminus \{0\}$. Now, consider the set

$$R = \left\{(x, y) \in S \times S \mid y = tx \text{ for some } t \in \mathbb{R}^\times\right\} = \{(x, y) \in S \times S \mid x \sim y\} \ .$$

R is the graph of $\sim$. We want to show that R is closed in $S \times S$. Consider the real valued function $f : S \times S \to \mathbb{R}$ defined by

$$f(x, y) = f\left(x^0, \dots, x^n, y^0, \dots, y^n\right) = \sum_{i \neq j} \left(x^i y^j - x^j y^i\right)^2 \ .$$

Note that f is continuous and vanishes if and only if $y = tx$ for some $t \in \mathbb{R}^\times$, since

$$\begin{aligned} f(x, y) = 0 &\iff \left(x^i y^j - x^j y^i\right)^2 \text{ for every } i \neq j \\ &\iff x^i y^j = x^j y^i \text{ for every } i \neq j \\ &\iff \frac{x^i}{y^i} = \frac{x^j}{y^j} \text{ for every } i \neq j \\ &\iff y = tx \text{ for some } t \in \mathbb{R}^\times \end{aligned}$$

Therefore, $R = f^{-1}(\{0\})$. $\{0\}$ is closed in $\mathbb{R}$ and f is continuous. Hence, R is closed in $S \times S$. Therefore, by Theorem 1.10.1, $S/{\sim} = \mathbb{R}P^n$ is Hausdorff. ■

The Standard Atlas on Real Projective Space

Let $\left[a^0, a^1, \dots, a^n\right]$ be homogenous coordinates on projective space $\mathbb{R}P^n$. Consider the set

$$U_0 = \left\{\left[a^0, a^1, \dots, a^n\right] \in \mathbb{R}P^n \mid a^0 \neq 0\right\} \ .$$

Let us denote by $\widetilde{U_0}$ the following set

$$\widetilde{U_0} = \left\{\left(a^0, a^1, \dots, a^n\right) \in \mathbb{R}^{n+1} \setminus \{0\} \mid a^0 \neq 0\right\} \ .$$

The projection map $p_0 : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}$ onto first coordinate is continuous, and $\widetilde{U_0} = p_0^{-1}(\mathbb{R}^\times)$. $\mathbb{R}^\times$ is open in $\mathbb{R}$, so $\widetilde{U_0}$ is open in $\mathbb{R}^{n+1} \setminus \{0\}$. Note that $U_0 = \pi\left(\widetilde{U_0}\right)$. Since $\sim$ is an open equivalence relation and $\widetilde{U_0}$ is open, U_0 is also open in $\mathbb{R}P^n$. In a similar manner, we can also define the following open subsets of $\mathbb{R}P^n$ for each $i = 1, \dots, n$.

$$U_i = \left\{\left[a^0, a^1, \dots, a^n\right] \in \mathbb{R}P^n \mid a^i \neq 0\right\} \ .$$

It is trivial that

$$\bigcup_{i=0}^{n} U_0 = \mathbb{R}P^n \, .$$

Now, define $\widetilde{\varphi_0} : \widetilde{U_0} \to \mathbb{R}^n$ by

$$\varphi_0\left(a^0, a^1, \dots, a^n\right) = \left(\frac{a^1}{a^0}, \frac{a^2}{a^0}, \dots, \frac{a^n}{a^0}\right) \ .$$

$\widetilde{\varphi_0}$ is continuous since $a^0 \neq 0$. This induces a map $\varphi_0 : U_0 \to \mathbb{R}^n$ by

$$\varphi_0\left(\left[a^0, a^1, \ldots, a^n\right]\right) = \varphi_0\left(a^0, a^1, \ldots, a^n\right) = \left(\frac{a^1}{a^0}, \frac{a^2}{a^0}, \ldots, \frac{a^n}{a^0}\right) .$$

This map is well-defined since

$$\varphi_0\left(\left[ta^0, ta^1, \ldots, ta^n\right]\right) = \left(\frac{ta^1}{ta^0}, \frac{ta^2}{ta^0}, \ldots, \frac{ta^n}{ta^0}\right) = \varphi_0\left(\left[a^0, a^1, \ldots, a^n\right]\right) .$$

Due to Proposition 1.7.1, continuity of $\widetilde{\varphi_0}$ implies continuity of φ_0. φ_0 has a continuous inverse $\varphi_0^{-1} : \mathbb{R}^n \to U_0$ given by

$$\varphi_0^{-1}\left(b^1, b^2, \ldots, b^n\right) = \pi\left(1, b^1, b^2, \ldots, b^n\right) = \left[1, b^1, b^2, \ldots, b^n\right] .$$

φ_0^{-1} is continuous because it is the composition of π and the continuous map $\left(b^1, \ldots, b^n\right) \mapsto \left(1, b^1, b^2, \ldots, b^n\right)$. Now we shall check that φ_0^{-1} is indeed the inverse of φ_0.

$$\left(\varphi_0 \circ \varphi_0^{-1}\right)\left(b^1, b^2, \ldots, b^n\right) = \varphi_0\left(\left[1, b^1, b^2, \ldots, b^n\right]\right) = \left(b^1, b^2, \ldots, b^n\right) .$$

$$\left(\varphi_0^{-1} \circ \varphi_0\right)\left[a^0, a^1, \ldots, a^n\right] = \varphi_0^{-1}\left(\frac{a^1}{a^0}, \frac{a^2}{a^0}, \ldots, \frac{a^n}{a^0}\right) = \left[1, \frac{a^1}{a^0}, \frac{a^2}{a^0}, \ldots, \frac{a^n}{a^0}\right] = \left[a^0, a^1, \ldots, a^n\right] .$$

Hence, φ_0^{-1} is indeed the inverse of φ. Therefore, φ_0 is a homeomorphism. Similarly, there are homeomorphisms $\varphi_i : U_i \to \mathbb{R}^n$ for each $i = 1, \ldots, n$.

$$\varphi_i\left(\left[a^0, \ldots, a^n\right]\right) = \left(\frac{a^0}{a^i}, \ldots, \widehat{\frac{a^i}{a^i}}, \ldots, \frac{a^n}{a^i}\right) ,$$

where the caret sign $\widehat{}$ over $\frac{a^i}{a^i}$ means that this entry is to be omitted. This proves that $\mathbb{R}P^n$ is locally Euclidean with $(U_i.\varphi_i)$ as charts.

Now, on the intersection $U_0 \cap U_1$, there are two charts. For $\left[a^0, a^1, \ldots, a^n\right] \in U_0 \cap U_1$, we have $a_0 \neq 0$ and $a_1 \neq 0$.

$$\begin{tikzcd}
& \left[a^0, a^1, \ldots, a^n\right] \arrow[dl, maps to, "\varphi_0"'] \arrow[dr, maps to, "\varphi_1"] & \\
\left(\frac{a^1}{a^0}, \frac{a^2}{a^0}, \ldots, \frac{a^n}{a^0}\right) & & \left(\frac{a^0}{a^1}, \frac{a^2}{a^1}, \ldots, \frac{a^n}{a^1}\right)
\end{tikzcd}$$

On $U_0 \cap U_1$, one has $\varphi_0\left(U_0 \cap U_1\right) = \left\{\left(b^1, b^2, \ldots, b^n\right) \in \mathbb{R}^n \mid b^1 \neq 0\right\}$. Given $\left(b^1, b^2, \ldots, b^n\right) \in \varphi_0\left(U_0 \cap U_1\right)$, one obtains

$$\left(\varphi_1 \circ \varphi_0^{-1}\right)\varphi_0\left(U_0 \cap U_1\right) = \varphi_1\left(\left[1, b^1, b^2, \ldots, b^n\right]\right) = \left(\frac{1}{b^1}, \frac{b^2}{b^1}, \ldots, \frac{b^n}{b^1}\right) .$$

This is a C^∞ map between open subsets of $\mathbb{R}^n$ since $b^1 \neq 0$ for $\left(b^1, b^2, \ldots, b^n\right) \in \varphi_0\left(U_0 \cap U_1\right)$. In a similar manner, one can show that $\varphi_i \circ \varphi_j^{-1}$ is C^∞ for every i, j. Therefore,

$$\{(U_i, \varphi_i) \mid 0 \leq i \leq n\}$$

is a C^∞ atlas on $\mathbb{R}P^n$, called the **standard atlas**. So we have shown that $\mathbb{R}P^n$ is second countable, Hausdorff locally Euclidean space equipped with a C^∞ atlas. Therefore, $\mathbb{R}P^n$ is a smooth manifold.

§5.2 The Grassmannian

The Grassmannianin $G(k,n)$ is the set of all k-planes through the origin in $\mathbb{R}^n$. Such a k-plane is a linear subspace of dimension k of $\mathbb{R}^n$ and has a basis consisting of k linearly independent vectors $a_1, a_2, \ldots, a_k$ in $\mathbb{R}^n$. It is therefore completely specified by an $n \times k$ matrix

$$A = \begin{bmatrix} a_1 & a_2 & \cdots & a_k \end{bmatrix} \text{ such that } \operatorname{rank} A = k\,.$$

This matrix is called a matrix representative of the k-plane. Two bases $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$ determine the same k-plane if there is a change-of-basis matrix $g = [g_{ij}] \in \mathrm{GL}(k, \mathbb{R})$ such that

$$b_j = \sum_i a_i g_{ij}\,.\text{ In matrix notation, } B = Ag\,.$$

Let $F(k,n)$ be the set of all $n \times k$ matrices of rank k, topologized as a subspace of $\mathbb{R}^{n\times k}$. We define an equivalence relation $\sim$ on $F(k,n)$ as follows:

$$A \sim B \iff \text{ there is a matrix } g \in \mathrm{GL}(k,\mathbb{R}) \text{ such that } B = Ag\,.$$

There is a bijection between $G(k,n)$ and the quotient space $F(k,n)/\!\sim$. We give the Grassmannian $G(k,n)$ the quotient topology on $F(k,n)/\sim$. Let $\pi : F(k,n) \to F(k,n)/\sim$ be the quotient map.

Lemma 5.2.1

Let A be an $m \times n$ matrix (not necessarily square), and k a positive integer. Then $\operatorname{rank} A \geq k$ if and only if A has a nonsingular $k \times k$ submatrix. Equivalently, $\operatorname{rank} A \leq k-1$ if and only if all $k \times k$ minors of A vanish. (A $k \times k$ minor of a matrix A is the determinant of a $k \times k$ submatrix of A.)

Proof. **($\Rightarrow$):** Suppose $\operatorname{rank} A \geq k$. Then one can find k linearly independent columns, which we call $a_1, \ldots, a_k$. Since the $m \times k$ matrix $B = \begin{bmatrix} a_1 & \cdots & a_k \end{bmatrix}$ has rank k, it has k linearly independent rows $b_1, \ldots, b_k$. The submatrix C of B whose rows are $b_1, \ldots, b_k$ is a $k \times k$ submatrix of A, and $\operatorname{rank} C = k$. In other words, C is a nonsingular $k \times k$ submatrix of A.

($\Leftarrow$): Suppose A has a nonsingular $k \times k$ submatrix B. Let $a_1, \ldots, a_k$ be the columns of A such that the submatrix $\begin{bmatrix} a_1 & \cdots & a_k \end{bmatrix}$ contains B. Since $\begin{bmatrix} a_1 & \cdots & a_k \end{bmatrix}$ has k linearly independent rows, it also has k linearly independent columns. Thus, $\operatorname{rank} A \geq k$. ■

Lemma 5.2.2

If $A \in F(k,n)$, then $Ag \in F(k,n)$ for every $g \in \mathrm{GL}(k,\mathbb{R})$.

Proof. Using the Linear Algebra fact that $\operatorname{rank}(XY) \leq \min\{\operatorname{rank} X, \operatorname{rank} Y\}$, we get that

$$\operatorname{rank}(Ag) \leq \min\{\operatorname{rank} A, \operatorname{rank} g\} = k\,.$$

Using the same result on $A = Ag\,g^{-1}$, we get

$$k = \operatorname{rank}(A) \leq \min\left\{\operatorname{rank}(Ag), \operatorname{rank} g^{-1}\right\} = \min\{\operatorname{rank}(Ag), k\} = \operatorname{rank}(Ag)\,.$$

Therefore, $\operatorname{rank}(Ag) = k$. ■

So we get, the multiplication-by-g-map m_g that maps $A \in F(k,n)$ to Ag is truly a map from $F(k,n)$ to itself. m_g is continuous, because the components are nothing but polynomials in the entries. The inverse map of m_g is $m_{g^{-1}}$, since

$$\left(m_g \circ m_{g^{-1}}\right)(A) = m_g\left(Ag^{-1}\right) = Ag^{-1}g = A\,,$$

$$\left(m_{g^{-1}} \circ m_g\right)(A) = m_{g^{-1}}(Ag) = Agg^{-1} = A\,.$$

Therefore, $m_{g^{-1}} = m_g^{-1}$. It is also continuous by a similar reasoning. Therefore, m_g is a homeomorphism from $F(k,n)$ to itself.

Proposition 5.2.3

The equivalence relation $\sim$ on $F(k,n)$ is an open equivalence relation.

Proof. We shall mimic the proof of Proposition 5.1.2. For an open set $U \subseteq F(k,n)$, $\pi(U)$ is open in $G(k,n)$ if and only if $\pi^{-1}(\pi(U))$ is open in $F(k,n)$ (definition of Quotient Topology).

$$\pi^{-1}(\pi(U)) = \bigcup_{A\in U} [A] = \bigcup_{A\in U} \{Ag \mid g \in \mathrm{GL}(k,\mathbb{R})\} = \bigcup_{g\in\mathrm{GL}(k,\mathbb{R})} \{Ag \mid A \in U\} = \bigcup_{g\in\mathrm{GL}(k,\mathbb{R})} m_g(U) \,.$$

The map $m_g : F(k,n) \to F(k,n)$ is a homeomorphism, as shown above. Therefore, it is an open map. So $m_g(U)$ is open in $F(k,n)$ for every $g \in \mathrm{GL}(k,\mathbb{R})$. Hence, their union

$$\bigcup_{g\in\mathrm{GL}(k,\mathbb{R})} m_g(U) = \pi^{-1}(\pi(U))$$

is also open in $F(k,n)$. ■

Corollary 5.2.4

The Grassmannian $G(k,n)$ is second countable.

Proof. $F(k,n)$ is a subspace of the second countable space $\mathbb{R}^{n\times k}$, hence it is also second countable. $\sim$ is an open equivalence relation. Therefore, by Corollary 1.10.4, $F(k,n)/\sim = G(k,n)$ is second countable. ■

Proposition 5.2.5

$G(k,n)$ is Hausdorff.

Proof. Let $S = F(k,n)$. Now, consider the set

$$R = \{(A,B) \in S\times S \mid B = Ag \text{ for some } g \in \mathrm{GL}(k,\mathbb{R})\} = \{(A,B) \in S \times S \mid A \sim B\} \,.$$

R is the graph of $\sim$. We want to show that R is closed. Take $A = \begin{bmatrix} a_1 & \cdots & a_k \end{bmatrix}$ and $B = \begin{bmatrix} b_1 & \cdots & b_k \end{bmatrix}$ from S. $A \sim B$ if and only if the columns of B can be expressed as a linear combination of the columns of A. In that case, we would have that the $n \times 2k$ matrix $M_{A,B} = \begin{bmatrix} A & B \end{bmatrix}$ has k linearly independent columns. In other words, rank $M_{A,B} = k \le k$. By Lemma 5.2.1, this is equivalent to **all** $(k+1)\times(k+1)$ minors of $M_{A,B}$ being 0.

Let $I = (i_0, i_1, \ldots, i_k)$ and $J = (j_0, j_1, \ldots, j_k)$ be $(k+1)$-tuples of integers such that

$$1 \le i_0 < i_1 < \cdots < i_k \le n \quad\text{and}\quad 1 \le j_0 < j_1 < \cdots < j_k \le 2k \,.$$

Define $f_{I,J} : S\times S \to \mathbb{R}$ such that it takes two matrices A and B and returns the $(k+1)\times(k+1)$ minor of $M_{A,B}$ corresponding to rows $i_0, i_1, \ldots, i_k$ and columns $j_0, j_1, \ldots, j_k$.

We have seen before that $A \sim B$ if and only if **all** the $(k+1)\times(k+1)$ minors of $M_{A,B}$ are 0. Therefore,

$$A \sim B \iff f_{I,J}(A,B) = 0 \text{ for every } I, J \,.$$

So, we can write R as

$$R = \bigcap_{I,J} f_{I,J}^{-1}(\{0\}) \,.$$

$f_{I,J} : S \times S \to \mathbb{R}$ is continuous since determinant is nothing but a polynomial of the entries. $\{0\}$ is closed in $\mathbb{R}$. Therefore, $f_{I,J}^{-1}(\{0\})$ is closed in $S \times S$. There are only finitely many choices for I, J. In fact, there are total

$$\binom{n}{k+1}\binom{2k}{k+1}$$

ways one can choose I, J. Intersection of finitely many closed sets is closed. Therefore, R is closed in $S \times S$. Hence, by Theorem 1.10.1, $S/\sim = G(k,n)$ is Hausdorff. ■

Next we want to find a C^∞ atlas on the Grassmannian $G(k,n)$. For simplicity, we specialize to $G(2,4)$. For any 4×2 matrix A, let A_{ij} be the 2×2 submatrix consisting of its i-th row and j-th row. Define

$$V_{ij} = \{A \in F(2,4) \mid A_{ij} \text{ is nonsingular }\} = \{A \in F(2,4) \mid \det A_{ij} \neq 0\} .$$

The map $A \mapsto \det A_{ij}$ is a continuous real-valued function. V_{ij} is the pre-image of $\mathbb{R}\setminus\{0\}$ under this continuous function. So, we can conclude that V_{ij} is an open subset of $F(2,4)$.

Lemma 5.2.6

If $A \in V_{ij}$ then $Ag \in V_{ij}$ for every $g \in \mathrm{GL}(2,\mathbb{R})$.

Proof. Let A_i be the i-th row of A.

$$A = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} \implies Ag = \begin{bmatrix} A_1 g \\ A_2 g \\ A_3 g \\ A_4 g \end{bmatrix} \implies (Ag)_{ij} = \begin{bmatrix} A_i g \\ A_j g \end{bmatrix} = A_{ij} g .$$

Therefore, $\det (Ag)_{ij} = \det A_{ij} \det g \neq 0$. So $Ag \in V_{ij}$. ■

Define $U_{ij} = V_{ij}/\!\sim\, = \pi(V_{ij})$. Since $\sim$ is an open equivalence relation, U_{ij} is an open subset of $G(2,4)$. Also, the collection $\{V_{ij}\}$ covers $F(2,4)$, so $\{U_{ij}\}$ covers $G(2,4)$. Now, for $A \in V_{12}$, $A_{12} \in \mathrm{GL}(2,\mathbb{R})$.

$$A \sim A A_{12}^{-1} = \begin{bmatrix} \mathbb{I}_2 \\ A_{34} A_{12}^{-1} \end{bmatrix}$$

So we define a map $\widetilde{\varphi_{12}} : V_{12} \to \mathbb{R}^{2\times 2}$ by $\widetilde{\varphi_{12}}(A) = A_{34}A_{12}^{-1}$. This induces a map $\varphi_{12} : U_{12} \to \mathbb{R}^{2\times 2}$, $\varphi_{12}[A] = \widetilde{\varphi_{12}}(A) = A_{34}A_{12}^{-1}$. This map is well-defined, since

$$\varphi_{12}[Ag] = (Ag)_{34}(Ag)_{12}^{-1} = A_{34}g(A_{12}g)^{-1} = A_{34}gg^{-1}A_{12}^{-1} = A_{34}A_{12}^{-1} = \varphi_{12}[A] ,$$

for any $g \in \mathrm{GL}(2,\mathbb{R})$. $\widetilde{\varphi_{12}}$ is continuous since it is just rational function on the entries. In the light of Proposition 1.7.1, continuity of $\widetilde{\varphi_{12}}$ implies the continuity of φ_{12}. φ_{12} has a continuous inverse $\varphi_{12}^{-1} : \mathbb{R}^{2\times 2} \to U_{12}$ given by

$$\varphi_{12}^{-1}(g) = \pi\left(\begin{bmatrix} \mathbb{I}_2 \\ g \end{bmatrix}\right) = \left[\begin{bmatrix} \mathbb{I}_2 \\ g \end{bmatrix}\right]$$

φ_{12}^{-1} is continuous because it is the composition of π and the continuous map that takes g to $\begin{bmatrix} \mathbb{I}_2 \\ g \end{bmatrix}$. Now we shall check that φ_{12}^{-1} is indeed the inverse of φ_{12}.

$$\left(\varphi_{12} \circ \varphi_{12}^{-1}\right)(g) = \varphi_{12}\left[\begin{bmatrix} \mathbb{I}_2 \\ g \end{bmatrix}\right] = g\mathbb{I}_2^{-1} = g .$$

$$\left(\varphi_{12}^{-1} \circ \varphi_{12}\right)[A] = \varphi_{12}^{-1}\left(A_{34}A_{12}^{-1}\right) = \left[\begin{bmatrix} \mathbb{I}_2 \\ A_{34}A_{12}^{-1} \end{bmatrix}\right] = [A] .$$

Hence, φ_{12}^{-1} is indeed the inverse of φ_{12}. Therefore, φ_{12} is a homeomorphism. Similarly, there are homeomorphisms $\varphi_{ij} : U_{ij} \to \mathbb{R}^{2\times 2}$ for every i,j. This proves $G(2,4)$ is locally Euclidean with (U_{ij}, φ_{ij}) as charts.

Now, on the intersection $U_{12} \cap U_{23}$, there are two charts. For $[A] \in U_{12} \cap U_{23}$, both A_{12} and A_{23} are invertible. Take $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \varphi_{23}(U_{12} \cap U_{23})$. Then we have

$$\left(\varphi_{12} \circ \varphi_{23}^{-1}\right)\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \varphi_{12}\left[\begin{bmatrix} a & b \\ 1 & 0 \\ 0 & 1 \\ c & d \end{bmatrix}\right]$$

Since $\left[\begin{bmatrix} a & b \\ 1 & 0 \\ 0 & 1 \\ c & d \end{bmatrix}\right] \in U_{12} \cap U_{23}$, $\begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix}$ is invertible. So b cannot be 0.

$$\left(\varphi_{12} \circ \varphi_{23}^{-1}\right) \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{b} & -\frac{a}{b} \\ \frac{d}{b} & c - \frac{da}{b} \end{bmatrix}$$

This is a C^∞ map between open subsets of $\mathbb{R}^{2\times 2} \cong \mathbb{R}^4$ since $b \neq 0$. In a similar manner, one can show that $\varphi_{ij} \circ \varphi_{pq}^{-1}$ is smooth for every i, j, p, q. Therefore,

$$\{(U_{ij}, \varphi_{ij}) \mid 1 \le i < j \le 4\}$$

is a C^∞ atlas on $G(2,4)$. So we have shown that $G(2,4)$ is second countable, Hausdorff locally Euclidean space equipped with a C^∞ atlas. Therefore, $G(2,4)$ is a smooth manifold.

Now it can be generalized in a similar manner. Let I be a strictly ascending multi-index $1 \le i_1 < i_2 < \cdots < i_k \le n$. Let A_I be the $k \times k$ submatrix of A consisting of i_1-th, i_2-th, …, i_k-th rows of A. Define

$$V_I = \{A \in F(k,n) \mid \det A_I \neq 0\} .$$

V_I is an open subset of $F(k,n)$ because it is the pre-image of $\mathbb{R} \setminus \{0\}$ under the continuos real-valued function $A \mapsto \det A_I$. As before, it can be easily seen that $A \in V_I$ implies $Ag \in V_I$ for every $g \in \mathrm{GL}(k,\mathbb{R})$.

$$(Ag)_I = A_I g \implies \det (Ag)_I = \det A_I \det g \neq 0 .$$

Let $U_I = V_I/\!\sim = \pi(V_I)$. Since $\sim$ is an open equivalence relation, $U_I = \pi(V_I)$ is open in $G(k,n)$. Also, the collection $\{V_I\}$ covers $F(k,n)$, so $\{U_I\}$ covers $G(k,n)$.

Next, we define $\widetilde{\varphi_I} : V_I \to \mathbb{R}^{(n-k)\times k}$ as follows

$$\widetilde{\varphi_I}(A) = \left(AA_I^{-1}\right)_{I'} ,$$

where $()_{I'}$ denotes the $(n-k) \times k$ submatrix obtained from the complement I' of the multi-index I. This induces a map $\varphi_I : U_I \to \mathbb{R}^{(n-k)\times k}$, $\varphi_I[A] = \widetilde{\varphi_I}(A) = \left(AA_I^{-1}\right)_{I'}$. One can easily check the well-definedness of φ_I.

$$\varphi_I[Ag] = \left(Ag(Ag)_I^{-1}\right)_{I'} = \left(Ag(A_I g)^{-1}\right)_{I'} = \left(Agg^{-1}A_I^{-1}\right)_{I'} = \left(AA_I^{-1}\right)_{I'} ,$$

for every $g \in \mathrm{GL}(k,\mathbb{R})$. $\widetilde{\varphi_I}$ is continuous. Therefore, by Proposition 1.7.1, φ_I is continuous.

Let $\varphi_I^{-1} : \mathbb{R}^{(n-k)\times k} \to U_I$ be defined as follows: for $X \in \mathbb{R}^{(n-k)\times k}$, $\varphi_I^{-1}(X) = [A]$, where $A_I = \mathbb{I}_k$ and $A_{I'} = X$. Then φ_I^{-1} is easily seen to be continuous. Also, one can easily check that φ_I^{-1} is indeed the inverse of φ_I.

$$\left(\varphi_I \circ \varphi_I^{-1}\right)(X) = \varphi_I[A] = \left(AA_I^{-1}\right)_{I'} = \left(A\mathbb{I}_k^{-1}\right)_{I'} = A_{I'} = X .$$

$$\left(\varphi_I^{-1} \circ \varphi_I\right)[A] = \varphi_I^{-1}\left(AA_I^{-1}\right)_{I'} = [B] ,$$

where $B_I = \mathbb{I}_k$ and $B_{I'} = \left(AA_I^{-1}\right)_{I'}$. Therefore, $B = AA_I^{-1}$.

$$\left(\varphi_I^{-1} \circ \varphi_I\right)[A] = \left[AA_I^{-1}\right] = \left[AA_I^{-1}A_I\right] = [A] .$$

Therefore, φ_I^{-1} is indeed the inverse of φ_I. This completes the proof that φ_I is a homeomorphism. Hence, $G(k,n)$ is a locally Euclidean space with charts (U_I, φ_I).

Now, we want to show that $\varphi_I \circ \varphi_J^{-1}$ is a smooth map between open subsets of Euclidean space. For $[A] \in U_I \cap U_J$, both A_I and A_J are invertible. Now, take some $X \in \varphi_J(U_I \cap U_J)$. Then we have

$$\left(\varphi_I \circ \varphi_J^{-1}\right)(X) = \varphi_I[A] ,$$

with $A_J = \mathbb{I}_k$ and $A_{J'} = X$. Since $[A] \in U_I \cap U_J$, $\det A_I \neq 0$.

$$\left(\varphi_I \circ \varphi_J^{-1}\right)(X) = \left(AA_I^{-1}\right)_{I'} .$$

Now, the entries of $\left(AA_I^{-1}\right)_{I'}$ can be expressed as rational functions on the entries of X, with the denominator being $\frac{1}{\det A_I} \neq 0$. Therefore, we can conclude that $\varphi_I \circ \varphi_J^{-1}$ is a smooth map between open subsets of Euclidean space. Therefore,

$$\{(U_I, \varphi_I) \mid I \text{ is strictly asecnding multi-index } 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}$$

is a C^∞ atlas on $G(k,n)$. So we have shown that $G(k,n)$ is second countable, Hausdorff locally Euclidean space equipped with a C^∞ atlas. Therefore, $G(k,n)$ is a smooth manifold.