Exercise Sheet 1

Category Theory & Representation Theory Seminar

Atonu Roy Chowdhury

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Problem 1. We have verified that $\mathrm{Hom}_{\mathcal{C}}(X,-):\mathcal{C}\to\mathbf{Sets}$ and $\mathrm{Hom}_{\mathcal{C}}(-,X):\mathcal{C}^{\mathrm{op}}\to\mathbf{Sets}$ are functors. Now show that

$$\mathrm{Hom}_{\mathcal{C}}(-,-):\mathcal{C}^{\mathrm{op}}\times\mathcal{C}\to\mathbf{Sets}$$

is a functor.

Problem 2. Show that the canonical isomorphism between a finite dimensional $\mathbb{K}$-vector space V and its double dual V^{**}, given by

$$\eta_V:V\to V^{**},\qquad \eta_V(\mathbf{v})(\varphi)=\varphi(\mathbf{v})$$

defines a natural isomorphism. In other words, given any linear map $f:V\to W$, show the commutativity of the following square:

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\eta_V"'] & W \arrow[d, "\eta_W"] \\ V^{**} \arrow[r, "(f^T)^T"'] & W^{**} \end{tikzcd}$$

Problem 3. Show the Hom-Tensor adjunction: first define the isomorphism $\eta_{(U,W)}$ between the sets $\mathrm{Hom}_{\mathbf{Vect}_{\mathbb{K}}}(U\otimes V,W)$ and $\mathrm{Hom}_{\mathbf{Vect}_{\mathbb{K}}}(U,\mathrm{Hom}(V,W))$. Then given an arrow $(\alpha_1^{\mathrm{op}},\alpha_2):(U,W)\to(U',W')$ in $\mathbf{Vect}_{\mathbb{K}}^{\mathrm{op}}\times\mathbf{Vect}_{\mathbb{K}}$, show the commutativity of the following diagram in the category $\mathbf{Sets}$:

$$\begin{tikzcd} \mathrm{Hom}_{\mathbf{Vect}_{\mathbb{K}}}(U\otimes V,W) \arrow[r, "\eta_{(U,W)}"] \arrow[d, "F(\alpha_1^{\mathrm{op}},\alpha_2)"'] & \mathrm{Hom}_{\mathbf{Vect}_{\mathbb{K}}}(U,\mathrm{Hom}(V,W)) \arrow[d, "G(\alpha_1^{\mathrm{op}},\alpha_2)"] \\ \mathrm{Hom}_{\mathbf{Vect}_{\mathbb{K}}}(U'\otimes V,W') \arrow[r, "\eta_{(U',W')}"'] & \mathrm{Hom}_{\mathbf{Vect}_{\mathbb{K}}}(U',\mathrm{Hom}(V,W')) \end{tikzcd}$$

where $F=\mathrm{Hom}_{\mathbf{Vect}_{\mathbb{K}}}(-\otimes V,-)$ and $G=\mathrm{Hom}_{\mathbf{Vect}_{\mathbb{K}}}(-,\mathrm{Hom}(V,-))$.

Problem 4. We all know the first isomorhism theorem for groups: if $f:G\to H$ is a group homomorphism, then $\mathrm{im}\,f\cong G/\mathrm{Ker}\,f$. Can you see this as a natural isomorhism between two functors? If so, what are the functors?

Problem 5. Recall the statement of Yoneda lemma that there is a natural isomorphism

$$\begin{tikzcd}
\mathcal{C}^{\mathrm{op}} \times \mathrm{Fun}\left(\mathcal{C}^{\mathrm{op}}, \mathbf{Sets}\right) \arrow[r, bend left=50, "{(X,F)\mapsto \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathbf{Sets})}(\mathrm{Hom}_{\mathcal{C}}(-,X),F)}"] \arrow[r, bend right=50, "{(X,F)\mapsto F(X)}"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathbf{Sets}
\end{tikzcd}$$

Here, the functors in question are defined by their actions on the objects only. Define the functors' action on arrows as well, and verify that they are functors indeed.

Problem 6. We showed that the **Yoneda embedding**

$$\begin{aligned}
\mathscr{Y} : \mathcal{C} &\to \mathrm{Fun}\left(\mathcal{C}^{\mathrm{op}}, \mathbf{Sets}\right) \\
X &\mapsto \mathrm{Hom}\left(-, X\right) \\
(f : X \to Y) &\mapsto \left(f_* : \mathrm{Hom}_{\mathcal{C}}\left(-, X\right) \to \mathrm{Hom}_{\mathcal{C}}\left(-, Y\right)\right)
\end{aligned}$$

is fully faithful. The arrows in the codomain category $\mathrm{Fun}\left(\mathcal{C}^{\mathrm{op}}, \mathbf{Sets}\right)$ are natural transformations. In order for $\mathscr{Y}$ to be a functor, f_* has to be a natural transformation. Verify that f_* is indeed a natural transformation. Also, verify that $\mathscr{Y}$ is indeed a functor.

Problem 7. While proving that $\mathscr{Y}$ is fully faithful, we have taken $F = \mathrm{Hom}_{\mathcal{C}}\left(-, Y\right)$ in Yoneda lemma. Then we argued that by Yoneda lemma, there is a bijection

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathbf{Sets})}\left(\mathrm{Hom}_{\mathcal{C}}(-, X), \mathrm{Hom}_{\mathcal{C}}(-, Y)\right) \cong \mathrm{Hom}_{\mathcal{C}}\left(X, Y\right).$$

But this argument was incomplete, because just showing a bijection between these two sets is not enough. In order to show that $\mathscr{Y}$ is a fully faithful functor, we need to show that

$$\mathscr{Y}_{X,Y} : \mathrm{Hom}_{\mathcal{C}}\left(X, Y\right) \to \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathbf{Sets})}\left(\mathrm{Hom}_{\mathcal{C}}(-, X), \mathrm{Hom}_{\mathcal{C}}(-, Y)\right),$$

defined by $f \mapsto f_*$, is a bijection. Show it.

Problem 8. Show that Ind is the left adjoint of Res, i.e. show the natural isomorphism

$$\begin{tikzcd}
\mathrm{Rep}(H)^{\mathrm{op}} \times \mathrm{Rep}(G) \arrow[r, bend left=50, "{\mathrm{Hom}_{\mathrm{Rep}(G)}(\mathrm{Ind}(-),-)}"] \arrow[r, bend right=50, "{\mathrm{Hom}_{\mathrm{Rep}(H)}(-,\mathrm{Res}(-))}"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathbf{Sets}
\end{tikzcd}$$

First, clearly define the functors in question by their actions on both objects and arrows. Then show the commutativity of the naturality square.

Problem 9. Show that Coind is the right adjoint of Res, i.e. show the natural isomorphism

$$\begin{tikzcd}
\mathrm{Rep}(G)^{\mathrm{op}} \times \mathrm{Rep}(H) \arrow[r, bend left=50, "{\mathrm{Hom}_{\mathrm{Rep}(H)}(\mathrm{Res}(-),-)}"] \arrow[r, bend right=50, "{\mathrm{Hom}_{\mathrm{Rep}(G)}(-,\mathrm{Coind}(-))}"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathbf{Sets}
\end{tikzcd}$$

First, clearly define the functors in question by their actions on both objects and arrows. Then show the commutativity of the naturality square.