Algebriac Topology III (MAT484)

Lecture Notes

Preface

This series of lecture notes has been prepared for aiding students who took the BRAC University course **Algebraic Topology III (MAT484)** in Spring 2023 semester. These notes were typeset under the supervision of mathematician **Dr. Syed Hasibul Hassan Chowdhury**. The main goal of this typeset is to have an organized digital version of the notes, which is easier to share and handle. Lecture notes of the previous Algebraic Topology courses can be found in the following links.

- **Algebraic Topology I (MAT431)**: https://atonurc.github.io/assets/MAT431_AT1.pdf
- **Algebraic Topology II (MAT432)**: https://atonurc.github.io/assets/MAT432_AT2.pdf

If you see any mistakes or typos, please send me an email at atonuroychowdhury@gmail.com

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References:

- *Elements of Algebraic Topology*, by James R. Munkres
- *Foundations of Algebraic Topology*, by Samuel Eilenberg & Norman E. Steenrod
- *Axiomatic Approach to Homology Theory*, by Samuel Eilenberg & Norman E. Steenrod. Link to the paper: https://www.pnas.org/content/pnas/31/4/117.full.pdf

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1 Singular Homology Theory

§1.1 Singular Homology Groups

Let $\mathbb{R}^\infty$ denote the generalized Euclidean space $\mathbb{E}^J$, with J being the set of positive integers. An element of the vector space $\mathbb{R}^\infty$ is an infinite sequence of real numbers (functions from $\mathbb{N}$ to $\mathbb{R}$) with finitely many nonzero entries. Let Δ_p denote the p-simplex in $\mathbb{R}^\infty$ having vertices

$$
\begin{aligned}
\varepsilon_0 &= (1,0,0,\ldots,0,\ldots)\,, \\
\varepsilon_1 &= (0,1,0,\ldots,0,\ldots)\,, \\
&\cdots \\
\varepsilon_p &= (0,0,0,\ldots,\underbrace{1}_{(p+1)\text{-th entry}},\ldots)\,.
\end{aligned}
$$

We call Δ_p the **standard p-simplex**. In this notation, Δ_{p-1} is a face of Δ_p.

> **Definition 1.1** (Singular p-simplex)**.** Let X be a topological space. We define a **singular p-simplex** of X to be a continuous map $T : \Delta_p \to X$. The free abelian group generated by singular p-simplices of X is denoted by $S_p(X)$, and is called the **singular chain group** of X in dimension p. We shall denote an element of $S_p(X)$ by a $\mathbb{Z}$-linear combination of singular p-simplices of X.

Singular means that T could be a "bad" map, i.e. it may not be an imbedding. All we want that T is just continuous. Now, recall that

$$
\Delta_p = \left\{ (x_0, x_1, \ldots, x_p, 0, \ldots) \in \mathbb{R}^\infty | 0 \le x_i \le 1 \text{ and } \sum_{i=0}^{p} x_i = 1 \right\}. \tag{1.1}
$$

Given $a_0, a_1, \ldots, a_p \in \mathbb{R}^\infty$, there is a unique affine map $l_{(a_0,\ldots,a_p)} : \Delta_p \to \mathbb{R}^\infty$ that maps ε_i to a_i. It is defined by

$$
\begin{aligned}
l_{(a_0,\ldots,a_p)}(x_0, x_1, \ldots, x_p, 0, \ldots) &= \sum_{i=0}^{p} x_i a_i = \sum_{i=0}^{p} x_i a_i + a_0 - \sum_{i=0}^{p} x_i a_0 \\
&= a_0 + \sum_{i=0}^{p} x_i (a_i - a_0)\,.
\end{aligned} \tag{1.2}
$$

We call this map the **linear singular simplex** determined by $a_0, a_1, \ldots, a_p \in \mathbb{R}^\infty$. Now, what is $l_{(\varepsilon_0,\ldots,\varepsilon_p)}$? Observe that

$$
l_{(\varepsilon_0,\ldots,\varepsilon_p)}\varepsilon_i = l_{(\varepsilon_0,\ldots,\varepsilon_p)}(0,\ldots,0,\underbrace{1}_{(i+1)\text{-th entry}},0,\ldots) = \varepsilon_i. \tag{1.3}
$$

Therefore, $l_{(\varepsilon_0,\ldots,\varepsilon_p)}$ maps ε_i to itself, for every $i = 0, 1, \ldots, p$. Also,

$$
l_{(\varepsilon_0,\ldots,\varepsilon_p)}(x_0, x_1, \ldots, x_p, 0, \ldots) = \sum_{i=0}^{p} x_i \varepsilon_i = (x_0, x_1, \ldots, x_p, 0, \ldots)\,. \tag{1.4}
$$

Therefore, $l_{(\varepsilon_0,\ldots,\varepsilon_p)}$ is just the inclusion map of Δ_p into $\mathbb{R}^\infty$. Now, suppose $(x_0, x_1, \ldots, x_{p-1}, 0, \ldots) \in \Delta_{p-1}$, so that $\sum_{i=0}^{p-1} x_i = 1$. Then

$$
\begin{aligned}
l_{(\varepsilon_0,\ldots,\widehat{\varepsilon_i},\ldots,\varepsilon_p)}(x_0, x_1, \ldots, x_{p-1}, 0, \ldots) &= x_0\varepsilon_0 + \cdots + x_{i-1}\varepsilon_{i-1} + 0 \cdot \varepsilon_i + x_{i+1}\varepsilon_{i+1} + \cdots + x_{p-1}\varepsilon_p \\
&= (x_0, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_{p-1}, 0, \ldots)\,,
\end{aligned} \tag{1.5}
$$

which is a point on the face of Δ_p opposite to the vertex ε_i. In fact, $l_{(\varepsilon_0,\dots,\widehat{\varepsilon_i},\dots,\varepsilon_p)}$ is a linear homomorphism of Δ_{p-1} into the face of Δ_p that is opposite to the vertex ε_i. In other words,

$$l_{(\varepsilon_0,\dots,\widehat{\varepsilon_i},\dots,\varepsilon_p)} : \Delta_{p-1} \to \Delta_p$$

maps Δ_{p-1} to the face of Δ_p opposite to the vertex ε_i. Therefore, given a singular p-simplex $T : \Delta_p \to X$, one can form the composite

$$T \circ l_{(\varepsilon_0,\dots,\widehat{\varepsilon_i},\dots,\varepsilon_p)} : \Delta_{p-1} \to X,$$

which is a singular $(p-1)$-simplex. We think of it as the i-th face of the singular p-simplex T.

Definition 1.2 (Boundary homomorphism)**.** We define $\partial : S_p(X) \to S_{p-1}(X)$ as follows. If $T : \Delta_p \to X$ is a singular p-simplex, we define ∂T to be

$$\partial T = \sum_{i=0}^{p} (-1)^i \, T \circ l_{(\varepsilon_0,\dots,\widehat{\varepsilon_i},\dots,\varepsilon_p)}. \tag{1.6}$$

In other words, ∂T is a formal sum of singular simplices of dimension $p-1$, which are the faces of T.

Remark 1.1 (IMPORTANT!)**.** Note that only the singular p-simplices are maps, not the singular p-chains. The p-chains are just formal sum of continuous maps from Δ_p to X. If T_1 and T_2 are two singular p-simplices, i.e. continuous maps $\Delta_p \to X$, then $T_1 + T_2$ is **NOT** a map. The sum present here is nothing but a formal notation. So one cannot act $T_1 + T_2$ on a point of Δ_p. For the same reason, ∂T_1 is not a map. It is merely a formal linear combination of the continuous maps $T \circ l_{(\varepsilon_0,\dots,\widehat{\varepsilon_i},\dots,\varepsilon_p)}$.

If $f : X \to Y$ is a continuous map, we define a group homomorphism $f_\# : S_p(X) \to S_p(Y)$ by defining it on singular p-simplices by the equation

$$f_\#(T) = f \circ T \tag{1.7}$$

for a singular p-simplex T.

$$\begin{tikzcd} \Delta_p \arrow[r, "T"] \arrow[rr, bend right, "f\circ T"'] & X \arrow[r, "f"] & Y \end{tikzcd}$$

Theorem 1.1

The homomorphism $f_\#$ commutes with ∂. Furthermore, $\partial^2 = 0$.

Proof. Given a singular p-simplex T,

$$\partial f_\#(T) = \partial(f \circ T) = \sum_{i=0}^{p} (-1)^i (f \circ T) \circ l_{(\varepsilon_0,\dots,\widehat{\varepsilon_i},\dots,\varepsilon_p)}. \tag{1.8}$$

$$f_\#(\partial T) = f_\#\left(\sum_{i=0}^{p} (-1)^i \, T \circ l_{(\varepsilon_0,\dots,\widehat{\varepsilon_i},\dots,\varepsilon_p)}\right) = \sum_{i=0}^{p} (-1)^i f \circ T \circ l_{(\varepsilon_0,\dots,\widehat{\varepsilon_i},\dots,\varepsilon_p)}. \tag{1.9}$$

Therefore, $\partial f_\#(T) = f_\#(\partial T)$. Now, to prove $\partial^2 = 0$, we first compute ∂ for linear singular simplices $l_{(a_0,\dots,a_p)}$.

$$\partial l_{(a_0,\dots,a_p)} = \sum_{i=0}^{p} (-1)^i \, l_{(a_0,\dots,a_p)} \circ l_{(\varepsilon_0,\dots,\widehat{\varepsilon_i},\dots,\varepsilon_p)}. \tag{1.10}$$

Observe that

$$\begin{aligned} l_{(a_0,\dots,a_p)} \circ l_{(\varepsilon_0,\dots,\widehat{\varepsilon}_i,\dots,\varepsilon_p)} (x_0,\dots,x_{p-1},0,\dots) &= l_{(a_0,\dots,a_p)} (x_0,\dots,x_{i-1},0,x_i x_{p-1},0,) \\ &= x_0 a_0 + \dots + x_{i-1}a_{i-1} + 0 \cdot a_i + x_i a_{i+1} + \cdots x_{p-1}a_p \\ &= l_{(a_0,\dots,\widehat{a}_i,\dots a_p)} (x_0,\dots,x_{p-1},0,\dots). \end{aligned} \tag{1.11}$$

Hence,

$$l_{(a_0,\dots,a_p)} \circ l_{(\varepsilon_0,\dots,\widehat{\varepsilon}_i,\dots,\varepsilon_p)} = l_{(a_0,\dots,\widehat{a}_i,\dots a_p)}. \tag{1.12}$$

Therefore, from 1.10, it follows that

$$\partial l_{(a_0,\dots,a_p)} = \sum_{i=0}^{p} (-1)^i \, l_{(a_0,\dots,\widehat{a}_i,\dots a_p)}. \tag{1.13}$$

Let's now evaluate $\partial\partial l_{(a_0,\dots,a_p)}$.

$$\begin{aligned} \partial\partial l_{(a_0,\dots,a_p)} &= \sum_{i=0}^{p} (-1)^i \, \partial l_{(a_0,\dots,\widehat{a}_i,\dots a_p)} \\ &= \sum_{i=0}^{p} (-1)^i \sum_{j<i} (-1)^j \, l_{(a_0,\dots,\widehat{a}_j,\dots \widehat{a}_i,\dots a_p)} + \sum_{i=0}^{p} (-1)^i \sum_{j>i} (-1)^{j-1} \, l_{(a_0,\dots,\widehat{a}_i,\dots \widehat{a}_j,\dots a_p)} \\ &= \sum_{i=0}^{p} \sum_{j<i} (-1)^{i+j} \, l_{(a_0,\dots,\widehat{a}_j,\dots \widehat{a}_i,\dots a_p)} - \sum_{i=0}^{p} \sum_{j>i} (-1)^{i+j} \, l_{(a_0,\dots,\widehat{a}_i,\dots \widehat{a}_j,\dots a_p)}. \end{aligned} \tag{1.14}$$

Now fix $0 \le j_0 < i_0 \le p$. In the first summand of 1.14, the contribution of $i = i_0, j = j_0$ is

$$(-1)^{i_0+j_0} \, l_{\left(a_0,\dots,\widehat{a_{j_0}},\dots \widehat{a_{i_0}},\dots a_p\right)}. \tag{1.15}$$

On the other hand, in the second summand of 1.14, the contribution of $i = j_0, j = i_0$ is also

$$(-1)^{i_0+j_0} \, l_{\left(a_0,\dots,\widehat{a_{j_0}},\dots \widehat{a_{i_0}},\dots a_p\right)}. \tag{1.16}$$

These two contributions cancel each other. This way, one finds that the RHS of 1.14 vanishes. Hence,

$$\partial\partial l_{(a_0,\dots,a_p)} = 0. \tag{1.17}$$

In particular,

$$\partial\partial l_{(\varepsilon_0,\dots,\varepsilon_p)} = 0. \tag{1.18}$$

Now, $l_{(\varepsilon_0,\dots,\varepsilon_p)} : \Delta_p \to \Delta_p$ is continuous, so $l_{(\varepsilon_0,\dots,\varepsilon_p)} \in S_p(\Delta_p)$. Furthermore, it is the identity map as we have seen in 1.4. Since $T : \Delta_p \to X$ is continuous, we can form $T_\# : S_p(\Delta_p) \to S_p(X)$.

$$T_\# \left(l_{(\varepsilon_0,\dots,\varepsilon_p)} \right) = T \circ l_{(\varepsilon_0,\dots,\varepsilon_p)} = T \circ \mathrm{id}_{\Delta_p} = T. \tag{1.19}$$

Therefore, using the fact that $T_\#$ commutes with ∂, we obtain

$$\partial\partial T = \partial\partial T_\# \left(l_{(\varepsilon_0,\dots,\varepsilon_p)} \right) = T_\# \left(\partial\partial l_{(\varepsilon_0,\dots,\varepsilon_p)} \right) = 0. \tag{1.20}$$

Hence, $\partial^2 T = 0$. ■

Definition 1.3 (Singular homology groups)**.** Th family of groups $S_p(X)$ and homomorphisms $\partial_p : S_p(X) \to S_{p-1}(X)$ is called **singular chain complex** of X, and is denoted by $\mathcal{S}(X)$.

$$\cdots \longrightarrow S_{p+1}(X) \xrightarrow{\partial_{p+1}} S_p(X) \xrightarrow{\partial_p} S_{p-1}(X) \longrightarrow \cdots$$

The homology groups of this chain complex are called the **singular homology groups** of X, and are denoted by $H_p(X)$.

Definition 1.4 (Augmentation map)**.** The chain complex $\mathcal{S}(X)$ is augmented by the homomorphism $\epsilon : S_0(X) \to \mathbb{Z}$ defined by setting $\epsilon(T) = 1$ for each singular 0-simplex $T : \Delta_0 \to X$. (A generic singular 0-chain is a $\mathbb{Z}$-linear combination of singular 0-simplices.)

It's immediate that if T is a singular 1-simplex, then $\epsilon(\partial T) = 0$. Indeed,

$$\epsilon(\partial T) = \epsilon\left(T \circ l_{(\widehat{\varepsilon_0}, \varepsilon_1)}\right) - \epsilon\left(T \circ l_{(\varepsilon_0, \widehat{\varepsilon_1})}\right) = 0. \tag{1.21}$$

Definition 1.5 (Reduced homology groups)**.** The homology groups of $\{\mathcal{S}(X), \epsilon\}$ are called the **reduced singular homology groups** of X, and are denoted by $\widetilde{H}_p(X)$.

Now, given continuous map $f : X \to Y$ and $T : \Delta_0 \to X$ a singular 0-simplex on X, then $f_\#(T) = f \circ T : \Delta_0 \to Y$.

$$\begin{tikzcd} \Delta_0 \arrow[r, "T"] \arrow[rr, bend right, "f\circ T"'] & X \arrow[r, "f"] & Y \end{tikzcd}$$

Now, consider the augmented singular chain complexes $\left\{\mathcal{S}(X), \epsilon^X\right\}$ and $\left\{\mathcal{S}(Y), \epsilon^Y\right\}$. Noting continuous $T : \Delta_0 \to X$ and $f_\#(T) : \Delta_0 \to Y$, one obtains $\epsilon^X(T) = 1$ and $\epsilon^Y(f_\#(T)) = 1$. In other words, the following diagram commutes

$$\begin{tikzcd} S_0(X) \arrow[r, "\epsilon^X"] \arrow[d, "(f_\#)_0"'] & \mathbb{Z} \arrow[d, "\mathrm{id}"] \\ S_0(Y) \arrow[r, "\epsilon^Y"] & \mathbb{Z} \end{tikzcd}$$

Therefore, $f_\# : S_p(X) \to S_p(Y)$ is an **augmentation preserving chain map** between $\left\{\mathcal{S}(X), \epsilon^X\right\}$ and $\left\{\mathcal{S}(Y), \epsilon^Y\right\}$. Thus, $f_\#$ induces a homomorphism f_* in both ordinary and reduced singular homology.

In Theorem 1.1, we saw that the chain map $f_\#$ commutes with the boundary operator ∂. In other words, $(f_\#)_p : S_p(X) \to S_p(Y)$ takes cycles to cycles and boundaries to boundaries. Suppose $c_p \in Z_p(X) = \operatorname{Ker} \partial_p^X$, so that $\partial_p^X c_p = 0$. Now,

$$\partial_p^Y\left((f_\#)_p c_p\right) = (f_\#)_{p-1}\left(\partial_p^X c_p\right) = 0. \tag{1.22}$$

Hence, $(f_\#)_p c_p \in Z_p(Y)$. On the other hand, let $b_p \in B_p(X) = \operatorname{Im} \partial_{p+1}^X$. Then $b_p = \partial_{p+1}^X d_{p+1}$ for some $d_{p+1} \in S_{p+1}(X)$. Then

$$(f_\#)_p b_p = (f_\#)_p\left(\partial_{p+1}^X d_{p+1}\right) = \partial_{p+1}^Y\left((f_\#)_{p+1} d_{p+1}\right). \tag{1.23}$$

In other words, $(f_\#)_p b_p \in B_p(Y)$. This reflects the fact that $(f_\#)_p : S_p(X) \to S_p(Y)$ induces a homomorphism between the singular homology groups $(f_*)_p : H_p(X) \to H_p(Y)$. $(f_*)_p$ is given by

$$(f_*)_p(c_p + B_p(X)) = (f_\#)_p c_p + B_p(Y). \tag{1.24}$$

If the reduced homology groups of X vanishes in all dimensions, we say that X is **acyclic** (in singular homology).

Theorem 1.2

If $i : X \to X$ is the identity, then so is $(i_*)_p : H_p(X) \to H_p(X)$. If $f : X \to Y$ and $g : Y \to Z$ are continuous, then $((g \circ f)_*)_p = (g_*)_p \circ (f_*)_p$.

Proof. It is sufficient to show that the equations hold at the chain level. We know from the definition of $(f_\#)_p : S_p(X) \to S_p(Y)$ that it maps $T \in S_p(X)$ to $f \circ T \in S_p(Y)$. Since $i : X \to X$ is the identity map,

$$(i_\#)_p(T) = i \circ T = T. \tag{1.25}$$

So $(i_\#)_p : S_p(X) \to S_p(X)$ is the identity homomorphism. As a result,

$$(i_*)_p(c_p + B_p(X)) = (i_\#)_p c_p + B_p(X) = c_p + B_p(X). \tag{1.26}$$

Therefore, $(i_*)_p = \mathrm{id}_{H_p(X)}$.

Given continuous $f : X \to Y$ and $g : Y \to Z$, $\left((g \circ f)_\#\right)_p : S_p(X) \to S_p(Z)$ is defined by

$$\left((g \circ f)_\#\right)_p T = (g \circ f) \circ T = g \circ (f \circ T) = (g_\#)_p \left((f_\#)_p T\right). \tag{1.27}$$

Therefore, $\left((g \circ f)_\#\right)_p = (g_\#)_p \circ (f_\#)_p$. Now, at the homology level, for $c_p + B_p(X) \in H_p(X) = Z_p(X)/B_p(X)$

$$((g \circ f)_*)_p(c_p + B_p(X)) = \left((g \circ f)_\#\right)_p c_p + B_p(Z) = (g_\#)_p \left((f_\#)_p c_p\right) + B_p(Z). \tag{1.28}$$

Also,

$$(g_*)_p \circ (f_*)_p(c_p + B_p(X)) = (g_*)_p \left((f_\#)_p c_p + B_p(Y)\right) = (g_\#)_p \left((f_\#)_p c_p\right) + B_p(Z). \tag{1.29}$$

From 1.28 and 1.29, we can deduce that $((g \circ f)_*)_p = (g_*)_p \circ (f_*)_p$. ■

Corollary 1.3

If $h : X \to Y$ is a homeomorphims, then $(h_*)_p : H_p(X) \to H_p(Y)$ is an isomorphism.

Proof. Both $h : X \to Y$ and $h^{-1} : Y \to X$ are continuous, and $h \circ h^{-1} = \mathrm{id}_Y$. Therefore,

$$(h_*)_p \circ \left((h^{-1})_*\right)_p = \left((h \circ h^{-1})_*\right)_p = ((\mathrm{id}_Y)_*)_p = \mathrm{id}_{H_p(Y)}. \tag{1.30}$$

Similarly, starting with $h^{-1} \circ h = \mathrm{id}_X$, we will get $\left((h^{-1})_*\right)_p \circ (h_*)_p = \mathrm{id}_{H_p(X)}$. Therefore, $\left((h^{-1})_*\right)_p$ is the inverse of $(h_*)_p$. In other words, $(h_*)_p$ is an invertible homomorphism, i.e. an isomorphism. ■

Theorem 1.4

Let X be a topological space. Then $H_0(X)$ is free abelian. If $\{X_\alpha\}$ is the collection of path components of X, and if T_α is a singular 0-simplex with image in X_α for each α, then the homology classes of the chains T_α form a basis for $H_0(X)$. The group $\widetilde{H}_0(X)$ is also free abelian; it vanishes if X is path connected. Otherwise, let α_0 be a fixed index, then the homology classes of the chains $T_\alpha - T_{\alpha_0}$ for $\alpha \neq \alpha_0$ form a basis for $\widetilde{H}_0(X)$.

Proof. Let $x_\alpha = T_\alpha(\Delta_0) \in X_\alpha$, with $T_\alpha : \Delta_0 \to X$ being a singular 0-simplex. Here, Δ_0 consists of the point $\varepsilon_0 = (1, 0, 0, \ldots) \in \mathbb{R}^\infty$. Also, let $T : \Delta_0 \to X$ be any singular 0-simplex such that $T(\Delta_0) \in X_\alpha$. Since X_α is path connected, there is a path connecting $T(\Delta_0)$ and $T_\alpha(\Delta_0)$. In other words, there is a singular 1-simplex $f : \Delta_1 \to X$ such that

$$f(1, 0, 0 \ldots) = T(\Delta_0) \text{ and } f(0, 1, 0 \ldots) = T_\alpha(\Delta_0). \tag{1.31}$$

Then we have

$$\partial_1 f = f \circ l_{(\widehat{\varepsilon_0}, \varepsilon_1)} - f \circ l_{(\varepsilon_0, \widehat{\varepsilon_1})}. \tag{1.32}$$

Now,

$$f \circ l_{(\varepsilon_0, \widehat{\varepsilon_1})}(1, 0, 0, \ldots) = f(1, 0, 0, \ldots) = T(\Delta_0) = T(1, 0, 0, \ldots), \tag{1.33}$$
$$f \circ l_{(\widehat{\varepsilon_0}, \varepsilon_1)}(1, 0, 0, \ldots) = f(0, 1, 0, \ldots) = T_\alpha(\Delta_0) = T_\alpha(1, 0, 0, \ldots). \tag{1.34}$$

Therefore, $\partial_1 f = T_\alpha - T$.

An arbitrary singular 0-chain is a $\mathbb{Z}$-linear combination of singular 0-simplices. Let's take $c \in S_0(X)$. Then $c = \sum_\beta m_\beta T'_\beta$, with $m_\beta \in \mathbb{Z}$ and T'_β being singular 0-simplices. Each $T'_\beta(\Delta_0)$ belongs to some X_α, and hence homologous to T_α. Therefore, c is homologous to some $\mathbb{Z}$-linear combination $\sum_\alpha n_\alpha T_\alpha$ of the T_α's. We will now show that no such nontrivial 0-chain $\sum_\alpha n_\alpha T_\alpha$ bounds.

Assume the contrary that $\sum_\alpha n_\alpha T_\alpha = \partial_1 d$ for some $d \in S_1(X)$. Now, the singular 1-chain d is a formal linear combination of singular 1-simplices with path connected image, i.e. the image lies in one of the path components X_α. Thus we can write $d = \sum_\alpha d_\alpha$, where d_α consists of the terms whose images are in X_α. Therefore,

$$\sum_\alpha n_\alpha T_\alpha = \partial_1 d = \sum_\alpha \partial_1 d_\alpha. \tag{1.35}$$

Hence, we get

$$n_\alpha T_\alpha = \partial_1 d_\alpha \tag{1.36}$$

for each α. Applying ϵ to both sides of 1.36, we get

$$\epsilon(n_\alpha T_\alpha) = \epsilon(\partial_1 d_\alpha) \implies n_\alpha = 0. \tag{1.37}$$

Therefore, no non-trivial 0-chain $\sum_\alpha n_\alpha T_\alpha$ bounds. Since every 0-chain is automatically a 0-cycle, an element of $H_0(X)$ is homologous to a 0-chain of the form $\sum_\alpha n_\alpha T_\alpha$. Hence, the homology classes of the singular 0-simplices $\{T_\alpha\}$ form a basis for the free abelian group $H_0(X)$.

$$S_1(X) \xrightarrow{\partial_1} S_0(X) \xrightarrow{\epsilon} \mathbb{Z}$$

$\widetilde{H}_0(X)$ is defined as $\widetilde{H}_0(X) = \operatorname{Ker}\epsilon / \operatorname{Im}\partial_1$. Given a singular 0-chain $T \in S_0(X)$, we've seen that T is homologous to a 0-chain of the form $T' = \sum_\alpha n_\alpha T_\alpha$; and T' bounds iff $T' = 0$, i.e. $n_\alpha = 0$ for every α. If further $T \in \operatorname{Ker}\epsilon$, then $\epsilon(T) = 0$. Since T and T' are homologous, $T = T' + \partial_1 d$ for some $d \in S_1(X)$. Therefore,

$$0 = \epsilon(T) = \epsilon(T') + \epsilon(\partial_1 d) = \epsilon\left(\sum_\alpha n_\alpha T_\alpha\right) = \sum_\alpha n_\alpha. \tag{1.38}$$

If X is path connected, there is only one component, and hence there is only one n_α involved. Thus $n_\alpha = 0$ from 1.38. This gives us $T' = 0$, leading to the fact that every $T \in \operatorname{Ker}\epsilon$ is homologous to 0, i.e. $T = 0 + \partial_1 d$ for some $d \in S_1(X)$. So $\operatorname{Ker}\epsilon = \operatorname{Im}\partial_1$. Therefore, $\widetilde{H}_0(X) = 0$, when X is path connected.

Now, suppose X has more than one path components. Fix α_0. Then from 1.38, we get

$$0 = \sum_\alpha n_\alpha = n_{\alpha_0} + \sum_{\alpha \neq \alpha_0} n_\alpha \implies n_{\alpha_0} = -\sum_{\alpha \neq \alpha_0} n_\alpha. \tag{1.39}$$

Then T' is

$$T' = \sum_{\alpha} n_\alpha T_\alpha = \sum_{\alpha\neq\alpha_0} n_\alpha T_\alpha + n_{\alpha_0} T_{\alpha_0} = \sum_{\alpha\neq\alpha_0} n_\alpha T_\alpha - \sum_{\alpha\neq\alpha_0} n_\alpha T_{\alpha_0} = \sum_{\alpha\neq\alpha_0} n_\alpha \left(T_\alpha - T_{\alpha_0}\right). \tag{1.40}$$

1.40 suggests that T' is a linear combination of the singular 0-chains $\{T_\alpha - T_{\alpha_0}\}_{\alpha\neq\alpha_0}$. And T' bounds iff it is trivial, as shown earlier. Therefore, the homology classes of 0-chains $\{T_\alpha - T_{\alpha_0}\}_{\alpha\neq\alpha_0}$ form a basis for $\widetilde{H}_0(X)$. ■

Theorem 1.4 illustrates the following result:

$$H_p(X) = \begin{cases} \widetilde{H}_p(X) & \text{if } p > 0 \\ \widetilde{H}_0(X) \oplus \mathbb{Z} & \text{if } p = 0 \end{cases}. \tag{1.41}$$

§1.2 Bracket Operation

> **Definition 1.6** (Star convex set). A set $X \subseteq \mathbb{E}^J$ is said to be **star convex** relative to the point $w \in X$, if for each $x \in X$, the line segment from x to w lies in X.

> **Definition 1.7** (Bracket operation). Suppose $X \in \mathbb{E}^J$ is star convex relative to w. We define bracket operation on singular chains of X. Let us first define it for singular p-simplices. Let $T : \Delta_p \to X$ be a singular p-simplex of X. Define a singular $(p+1)$-simplex
>
> $$[T, w] : \Delta_{p+1} \to X$$
>
> by letting $[T, w]$ carry the line segment from x to ε_{p+1}, for $x \in\in \Delta_p$ (the collection of all such line segments as x varies in Δ_p constitutes Δ_{p+1}), linearly onto the line segment $T(x)$ to w in X. In other words,
>
> $$[T, w](t\varepsilon_{p+1} + (1-t)x) = tw + (1-t)T(x), \tag{1.42}$$
>
> for $t \in [0, 1]$. Now, extend the definition of bracket operation to arbitrary p-chains as follows: if $c = \sum n_i T_i$ is a singular p-chain of X with each T_i being a singular p-simplex, then we define
>
> $$[c, w] = \sum n_i [T_i, w]. \tag{1.43}$$
>
> In other words, $[\cdot\,, w] : S_p(X) \to S_{p+1}(X)$, $c \mapsto [c, w]$ is a homomorphism.

From Figure 1.1, it's immediate that the restriction of $[T, w]$ to the face Δ_p of Δ_{p+1} is just the map T. Now, consider the case when T is the linear singular simplex $l_{(a_0,\ldots,a_p)}$ for $a_0, \ldots, a_p \in \mathbb{R}^\infty$. We want to calculate what $\left[l_{(a_0,\ldots,a_p)}, w\right]$ is.

Recall that $l_{(a_0,\ldots,a_p)} : \Delta_p \to \mathbb{R}^\infty$ is defined as

$$l_{(a_0,\ldots,a_p)}(x_0, \ldots, x_p) = \sum_{i=0}^{p} x_i a_i. \tag{1.44}$$

Consider a point $(x_0, \ldots, x_p, x_{p+1}, 0, \ldots) \in \Delta_{p+1}$. We want to see where $\left[l_{(a_0,\ldots,a_p)}, w\right]$ takes this point to. Since $(x_0, \ldots, x_p, x_{p+1}, 0, \ldots) \in \Delta_{p+1}$, each x_i is nonnegative with $\sum_{i=0}^{p+1} x_i = 1$. Now,

$$\sum_{i=0}^{p} \frac{x_i}{1 - x_{p+1}} = 1, \tag{1.45}$$

so $\left(\frac{x_0}{1-x_{p+1}}, \frac{x_1}{1-x_{p+1}}, \ldots, \frac{x_p}{1-x_{p+1}}, 0, \ldots\right) \in \Delta_p$. Therefore,

$$(x_0, \ldots, x_p, x_{p+1}, 0, \ldots) = (1 - x_{p+1})\left(\frac{x_0}{1-x_{p+1}}, \frac{x_1}{1-x_{p+1}}, \ldots, \frac{x_p}{1-x_{p+1}}, 0, \ldots\right) + x_{p+1}\varepsilon_{p+1}. \tag{1.46}$$

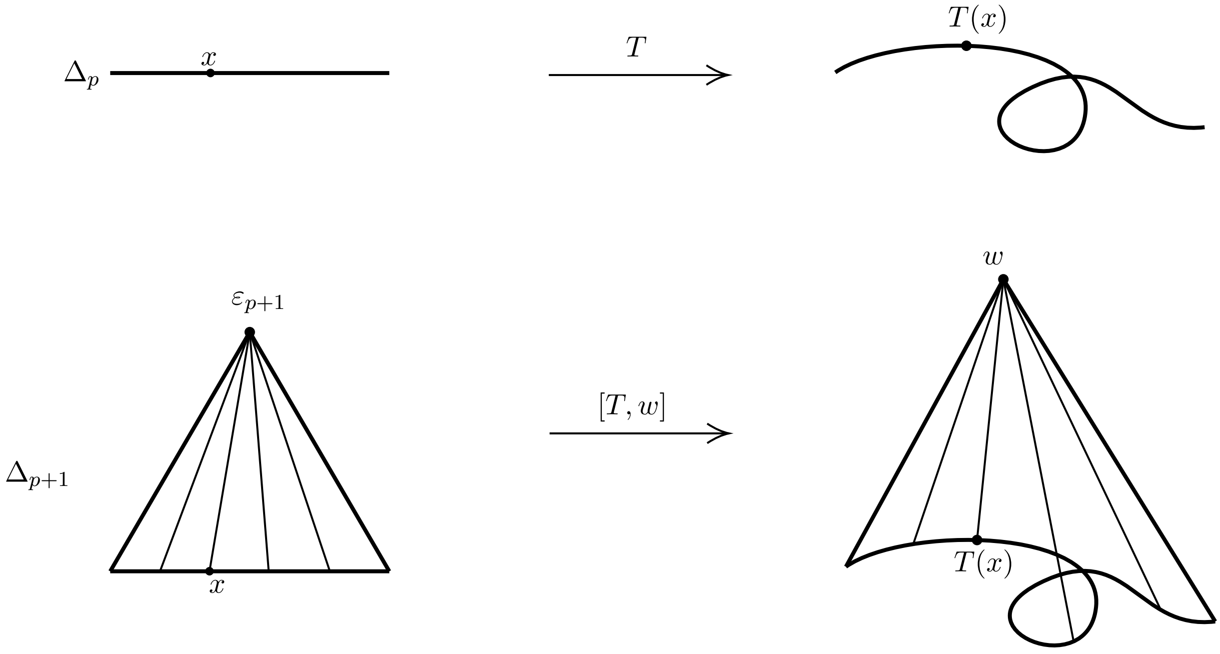

Figure 1.1

By the definition of bracket operation,

$$\begin{aligned}
& \left[l_{(a_0,\ldots,a_p)},w\right](x_0,\ldots,x_p,x_{p+1},0,\ldots) \\
=\ & (1-x_{p+1})\, l_{(a_0,\ldots,a_p)}\left(\frac{x_0}{1-x_{p+1}},\frac{x_1}{1-x_{p+1}},\ldots,\frac{x_p}{1-x_{p+1}},0,\ldots\right)+x_{p+1}w \\
=\ & (1-x_{p+1})\sum_{i=0}^{p}\frac{x_i}{1-x_{p+1}}a_i+x_{p+1}w \\
=\ & \sum_{i=0}^{p}x_ia_i+x_{p+1}w. \qquad (1.47)
\end{aligned}$$

Furthermore,

$$l_{(a_0,\ldots,a_p,w)}(x_0,\ldots,x_p,x_{p+1},0,\ldots)=x_0a_0+\cdots+x_pa_p+x_{p+1}w=\sum_{i=0}^{p}x_ia_i+x_{p+1}w. \qquad (1.48)$$

Equating 1.47 and 1.48, we get

$$\left[l_{(a_0,\ldots,a_p)},w\right]=l_{(a_0,\ldots,a_p,w)}. \qquad (1.49)$$

Now we will showo that $[T,w]:\Delta_{p+1}\to X$ is continuous. We have seen earlier that given $x\in\Delta_p$, a point in Δ_{p+1} is expressed as $t\varepsilon_{p+1}+(1-t)\,x$, with $0\le t\le 1$. Hence, we are concerned with the following quotient map $\pi:\Delta_p\times[0,1]\to\Delta_{p+1}$ defined by

$$\pi(x,t)=t\varepsilon_{p+1}+(1-t)\,x. \qquad (1.50)$$

If $x=(x_0,\ldots,x_p,0,\ldots)\in\Delta_p$, then 1.50 takes the familiar form

$$\pi((x_0,\ldots,x_p,0,\ldots),t)=((1-t)\,x_0,\ldots,(1-t)\,x_p,t,0,\ldots). \qquad (1.51)$$

Observe that $\pi\big|_{\Delta_p\times[0,1)}:\Delta_p\times[0,1)\to\Delta_{p+1}$ is 1-1, and $\pi(\Delta_p\times\{1\})=\{\varepsilon_{p+1}\}$, showing that π collapses $\Delta_p\times\{1\}$ to the $(p+1)$-th vertex ε_{p+1} of Δ_{p+1}. Now, the continuous map $f:\Delta_p\times[0,1]\to X$ defined by

$$f(x,t)=tw+(1-t)\,T(x) \qquad (1.52)$$

is constant on $\Delta_p \times \{1\}$. In fact, $f(\Delta_p \times \{1\}) = \{w\}$. Since π is $1-1$ for other points, f is seen to be constant for $\pi^{-1}(y)$ with $y \in \Delta_{p+1} \setminus \{\varepsilon_{p+1}\}$. In other words, $f : \Delta_p \times [0,1] \to X$ is constant for each $\pi^{-1}(y)$ with $y \in \Delta_{p+1}$. Therefore, f induces a unique continuous map $\widetilde{f} : \Delta_{p+1} \to X$ such that the following diagram commutes

$$\begin{tikzcd}
\Delta_p \times [0,1] \arrow[d, "\pi"'] \arrow[dr, "f"] & \\
\Delta_{p+1} \arrow[r, "\widetilde{f}"] & X
\end{tikzcd}$$

This unique map $\widetilde{f}$ is precisely $[T,w]$, since

$$([T,w] \circ \pi)(x,t) = [T,w](t\varepsilon_{p+1} + (1-t)x) = tw + (1-t)T(x) = f(x,t). \tag{1.53}$$

Therefore, $\widetilde{f} = [T,w]$, and hence it is continuous. So $[T,w]$ is indeed a singular $(p+1)$-simplex.

Lemma 1.5

Let X be a star convex set with respect to w; let c be a singular p-chain of X. Then

$$\partial[c,w] = \begin{cases} [\partial c, w] + (-1)^{p+1}c & \text{if } p > 0 \\ \epsilon(c)T_w - c & \text{if } p = 0 \end{cases}, \tag{1.54}$$

where T_w is the singular 0-simplex mappting Δ_0 to w.

Proof. If T is a singular 0-simplex, $[T,w]$ is a singular 1-simplex. Then

$$\partial[T,w] = [T,w] \circ l_{(\widehat{\varepsilon}_0,\varepsilon_1)} - [T,w] \circ l_{(\varepsilon_0,\widehat{\varepsilon}_1)}. \tag{1.55}$$

Now, recall $[T,w] : \Delta_1 \to X$ maps the line joining ε_1 to ε_0 to the line joining w to $T(\varepsilon_0)$. So

$$[T,w](1-t,t,0,\ldots) = tw + (1-t)T(\varepsilon_0). \tag{1.56}$$

Now,

$$\left([T,w] \circ l_{(\widehat{\varepsilon}_0,\varepsilon_1)}\right)(1,0,\ldots) = [T,w](0,1,0,\ldots) = w = T_w(1,0,\ldots). \tag{1.57}$$

Therefore, $\left([T,w] \circ l_{(\widehat{\varepsilon}_0,\varepsilon_1)}\right) = T_w$.

$$\left([T,w] \circ l_{(\varepsilon_0,\widehat{\varepsilon}_1)}\right)(1,0,\ldots) = [T,w](1,0,\ldots) = T(\varepsilon_0) = T(1,0,\ldots), \tag{1.58}$$

so $[T,w] \circ l_{(\varepsilon_0,\widehat{\varepsilon}_1)} = T$. By 1.55, we get

$$\partial[T,w] = T_w - T. \tag{1.59}$$

Now, let $c = \sum_i n_i T_i$ be a singular 0-chain with T_i being singular 0-simplices. Then

$$\partial\left[\sum_i n_i T_i, w\right] = \sum_i n_i \partial[T_i,w] = \sum_i n_i(T_w - T_i) = \left(\sum_i n_i\right)T_w - \sum_i n_i T_i. \tag{1.60}$$

Now, applying the augmentation map to c, we get

$$\epsilon(c) = \epsilon\left(\sum_i n_i T_i\right) = \sum_i n_i \epsilon(T_i) = \sum_i n_i. \tag{1.61}$$

Therefore, 1.60 gives us

$$\partial[c,w] = \epsilon(c)T_w - c. \tag{1.62}$$

Now we shall consider the case when T is a singular p-simplex, and we shall prove that $\partial\,[T,w] = [\partial T, w] + (-1)^{p+1}\,T$.

$$\begin{aligned}\partial\,[T,w] &= \sum_{i=0}^{p+1} (-1)^i\,[T,w] \circ l_{(\varepsilon_0,\ldots\widehat{\varepsilon}_i,\ldots,\varepsilon_{p+1})}\\ &= \sum_{i=0}^{p} (-1)^i\,[T,w] \circ l_{(\varepsilon_0,\ldots\widehat{\varepsilon}_i,\ldots,\varepsilon_{p+1})} + (-1)^{p+1}\,[T,w] \circ l_{(\varepsilon_0,\ldots,\varepsilon_p,\widehat{\varepsilon}_{p+1})}.\end{aligned} \tag{1.63}$$

$l_{(\varepsilon_0,\ldots,\varepsilon_p,\widehat{\varepsilon}_{p+1})}$ is the inclusion map of Δ_p into Δ_{p+1}. So $[T,w]\circ l_{(\varepsilon_0,\ldots,\varepsilon_p,\widehat{\varepsilon}_{p+1})}$ is nothing but the restriction of $[T,w]$ to Δ_p, which is the same as T. Now we want to show that

$$[T,w] \circ l_{(\varepsilon_0,\ldots\widehat{\varepsilon}_i,\ldots,\varepsilon_{p+1})} = \left[T \circ l_{(\varepsilon_0,\ldots\widehat{\varepsilon}_i,\ldots,\varepsilon_p)}, w\right]. \tag{1.64}$$

Both sides of 1.64 are maps from Δ_p to X. Let $(x_0,\ldots,x_p,0,\ldots) \in \Delta_p$. Then

$$\left([T,w] \circ l_{(\varepsilon_0,\ldots\widehat{\varepsilon}_i,\ldots,\varepsilon_{p+1})}\right)(x_0,\ldots,x_p,0,\ldots) = [T,w]\,(x_0,\ldots,x_{i-1},0,x_i,\ldots,x_{p-1},x_p,0,\ldots). \tag{1.65}$$

Now, $(x_0,\ldots,x_{i-1},0,x_i,\ldots,x_{p-1},x_p,0,\ldots)$ is a point in Δ_{p+1}. We can write it as

$$(x_0,\ldots,x_{i-1},0,x_i,\ldots,x_{p-1},x_p,0,\ldots) = (1-x_p)\left(\frac{x_0}{1-x_p},\ldots,\frac{x_{i-1}}{1-x_p},0,\frac{x_i}{1-x_p},\ldots,\frac{x_{p-1}}{1-x_p},0,\ldots\right) + x_p\varepsilon_{p+1}. \tag{1.66}$$

Now, $\left(\frac{x_0}{1-x_p},\ldots,\frac{x_{i-1}}{1-x_p},0,\frac{x_i}{1-x_p},\ldots,\frac{x_{p-1}}{1-x_p},0,\ldots\right)$ is a point in Δ_p since its nonzero components are all non-negative and they add to 1. Therefore,

$$\begin{aligned}&[T,w]\,(x_0,\ldots,x_{i-1},0,x_i,\ldots,x_{p-1},x_p,0,\ldots)\\ =\ &(1-x_p)\,T\left(\frac{x_0}{1-x_p},\ldots,\frac{x_{i-1}}{1-x_p},0,\frac{x_i}{1-x_p},\ldots,\frac{x_{p-1}}{1-x_p},0,\ldots\right) + x_p w.\end{aligned} \tag{1.67}$$

On the other hand, we can write $(x_0,\ldots,x_p,0,\ldots)$ as

$$(x_0,\ldots,x_p,0,\ldots) = (1-x_p)\left(\frac{x_0}{1-x_p},\ldots,\frac{x_{p-1}}{1-x_p},0,\ldots\right) + x_p\varepsilon_p, \tag{1.68}$$

where $\left(\frac{x_0}{1-x_p},\ldots,\frac{x_{p-1}}{1-x_p},0,\ldots\right) \in \Delta_{p-1}$. So

$$\begin{aligned}&\left[T \circ l_{(\varepsilon_0,\ldots\widehat{\varepsilon}_i,\ldots,\varepsilon_p)}, w\right](x_0,\ldots,x_p,0,\ldots)\\ =\ &x_p w + (1-x_p)\left(T \circ l_{(\varepsilon_0,\ldots\widehat{\varepsilon}_i,\ldots,\varepsilon_p)}\right)\left(\frac{x_0}{1-x_p},\ldots,\frac{x_{p-1}}{1-x_p},0,\ldots\right)\\ =\ &x_p w + (1-x_p)\,T\left(\frac{x_0}{1-x_p},\ldots,\frac{x_{i-1}}{1-x_p},0,\frac{x_i}{1-x_p},\ldots,\frac{x_{p-1}}{1-x_p},0,\ldots\right).\end{aligned} \tag{1.69}$$

Combining 1.65, 1.67 and 1.69, we get that 1.64 indeed holds, i.e.

$$[T,w] \circ l_{(\varepsilon_0,\ldots\widehat{\varepsilon}_i,\ldots,\varepsilon_{p+1})} = \left[T \circ l_{(\varepsilon_0,\ldots\widehat{\varepsilon}_i,\ldots,\varepsilon_p)}, w\right].$$

Now, from 1.63, we then get

$$\begin{aligned}\partial\,[T,w] &= \sum_{i=0}^{p} (-1)^i\left[T \circ l_{(\varepsilon_0,\ldots\widehat{\varepsilon}_i,\ldots,\varepsilon_p)}, w\right] + (-1)^{p+1}\,T\\ &= \left[\sum_{i=0}^{p} (-1)^i\,T \circ l_{(\varepsilon_0,\ldots\widehat{\varepsilon}_i,\ldots,\varepsilon_p)}, w\right] + (-1)^{p+1}\,T\\ &= [\partial T, w] + (-1)^{p+1}\,T.\end{aligned} \tag{1.70}$$

Now, if $c = \sum_i n_i T_i$ is a singular p-chain with T_i being singular 0-simplices, then

$$\partial\,[c,w] = \sum_i n_i \partial\,[T_i,w] = \sum_i n_i\,[\partial T_i, w] + (-1)^{p+1}\sum_i n_i T_i = [\partial c, w] + (-1)^{p+1}\,c. \tag{1.71}$$

■

Theorem 1.6

Let $X \subseteq \mathbb{E}^J$ be star convex with respect to w. Then X is acyclic in singular homology.

Proof. To show that $\widetilde{H}_0(X) = 0$, let $c \in \operatorname{Ker} \epsilon$.

$$S_1(X) \xrightarrow{\partial_1} S_0(X) \xrightarrow{\epsilon} \mathbb{Z}$$

So $\epsilon(c) = 0$. Now, by Lemma 1.5,

$$\partial_1 [c, w] = \epsilon(c) T_w - c = -c. \tag{1.72}$$

Hence, $c \in \operatorname{Im} \partial_1$ leading to $\operatorname{Ker} \epsilon \subseteq \operatorname{Im} \partial_1$. We already know Hence, $\operatorname{Im} \partial_1 \subseteq \operatorname{Ker} \epsilon$. Therefore, $\widetilde{H}_0(X) = 0$.

Now we shall show that $H_p(X) = 0$ for $p > 0$. Let $z \in \operatorname{Ker} \partial_p$. Then $\partial_p z = 0$. By Lemma 1.5 again,

$$\partial_{p+1} [z, w] = [\partial_p z, w] + (-1)^{p+1} z = (-1)^{p+1} z. \tag{1.73}$$

Hence, $z \in \operatorname{Im} \partial_{p+1}$. Therefore, $H_p(X) = 0$. In other words, $\widetilde{H}_p(X) = 0$ for all p, i.e. X is acyclic. ■

Corollary 1.7

Any simplex is acyclic in singular homology.

2 Axioms of Singular Homology

§2.1 Relative Homology Groups

If X is a space and A is a subspace of X, there is a natural inclusion $S_p(A) \hookrightarrow S_p(X)$. The group of **relative singular chains** is defined by

$$S_p(X, A) = S_p(X) / S_p(A) . \tag{2.1}$$

The boundary operator $\partial_p^X : S_p(X) \to S_{p-1}(X)$ restricts to the boundary operator on $S_p(A)$, i.e. $\partial_p^X\big|_{S_p(A)} : S_p(A) \to S_{p-1}(A)$. It, therefore, induces a boundary operator at the relative singular chain level:

$$\begin{aligned} \partial_p^{(X,A)} &: S_p(X, A) \to S_{p-1}(X, A), \\ T + S_p(A) &\mapsto \partial_p^X T + S_{p-1}(A), \end{aligned} \tag{2.2}$$

with $T = \sum_\alpha n_\alpha T_\alpha$ being a singular p-chain, where $n_\alpha \in \mathbb{Z}$ and T_α singular p-simplices. If any of the T_α's are such that $T_\alpha(\Delta_p) \subseteq A$, then $T_\alpha \in S_p(A)$. So, we can assume $T_\alpha(\Delta_p) \setminus A \neq \varnothing$. Such T_α's generate the group $S_p(X, A)$, and so $S_p(X, A)$ is a free abelian group.

The family of groups $S_p(X, A)$ and homomorphisms $\partial_p^{(X,A)}$ is called **the singular chain complex** of the pair (X, A), and is denoted by $\mathcal{S}(X, A)$. The homology groups of the chain complex $\mathcal{S}(X, A)$ of the pair (X, A) are called the **singular homology groups** of the pair (X, A), and are denoted by $H_p(X, A)$.

The chain complex $\mathcal{S}(X, A)$ is free, i.e. $S_p(X, A)$ is free for each p. The group $S_p(X, A)$ has as basis all the cosets of the form $T + S_p(A)$, where T is a singular p-simplex with $T(\Delta_p) \setminus A \neq \varnothing$.

If $f : (X, A) \to (Y, B)$ is a continuous map (recall that by the continuity of f between pairs (X, A) and (Y, B), we actually mean that $f : X \to Y$ is continuous, with $f(A) \subseteq B$), then homomorphisms $(f_\#)_p : S_p(X) \to S_p(Y)$ carries singular p-chains of A into singular p-chains of B. So it induces a homomorphism (also denoted by $(f_\#)_p$) at the level of relative singular p-chains:

$$\begin{aligned} (f_\#)_p &: S_p(X, A) \to S_p(Y, B), \\ T + S_p(A) &\mapsto (f_\#)_p T + S_p(B) = f \circ T + S_p(B) . \end{aligned} \tag{2.3}$$

where T is a singular p-simplex with $T(\Delta_p) \setminus A \neq \varnothing$. This map can be seen to commute with the boundary oeprator at the relative singular chain level. To be precise,

$$(f_\#)_{p-1} \circ \partial_p^{(X,A)} = \partial_p^{(Y,B)} \circ (f_\#)_p . \tag{2.4}$$

In other words, the following diagram commutes.

$$\begin{tikzcd} S_p(X, A) \arrow[r, "\partial_p^{(X,A)}"] \arrow[d, "(f_\#)_p"'] & S_{p-1}(X, A) \arrow[d, "(f_\#)_{p-1}"] \\ S_p(Y, B) \arrow[r, "\partial_p^{(Y,B)}"'] & S_{p-1}(Y, B) \end{tikzcd}$$

Therefore, $f_\#$ induces a homomorphism

$$\begin{aligned} (f_*)_p &: H_p(X, A) \to H_p(Y, B), \\ c + \operatorname{Im} \partial_{p+1}^{(X,A)} &\mapsto (f_\#)_p c + \operatorname{Im} \partial_{p+1}^{(Y,B)} . \end{aligned} \tag{2.5}$$

Theorem 2.1

If $i : (X, A) \to (X, A)$ is the identity, then so is $(i_*)_p : H_p (X, A) \to H_p (X, A)$. If $h : (X, A) \to (Y, B)$ and $k : (Y, B) \to (Z, C)$ are continuous, then $((k \circ h)_*)_p = (k_*)_p \circ (h_*)_p$.

Proof. Since $(i_\#)_p : S_p (X) \to S_p (X)$ is the identity map (as proven while proving Theorem 1.2), so is $(i_\#)_p : S_p (X, A) \to S_p (X, A)$. Then from 2.5, we get that $(i_*)_p : H_p (X, A) \to H_p (X, A)$ is the identity, i.e. $(i_*)_p = \mathrm{id}_{H_p(X,A)}$.

Now, let us prove $\left((k \circ h)_\#\right)_p = (k_\#)_p \circ (h_\#)_p$. The equality at the homology level will then follow from 2.5.

$$(h_\#)_p : S_p (X, A) \to S_p (Y, B)\ ,\ (k_\#)_p : S_p (Y, B) \to S_p (Z, C)\,.$$

We choose a singular p-simplex T such that $T (\Delta_p) \setminus A \neq \varnothing$. Then the cosets of the form $T + S_p (A)$ form a basis of $S_p (X, A)$.

$$\begin{tikzcd} \Delta_p \arrow[r, "T"] & X \arrow[r, "h"] \arrow[rr, bend right, "k\circ h"'] & Y \arrow[r, "k"] & Z \end{tikzcd}$$

Using 2.3, we get

$$(h_\#)_p (T + S_p (A)) = h \circ T + S_p (B)\,, \tag{2.6}$$

$$(k_\#)_p \left((h_\#)_p (T + S_p (A))\right) = (k_\#)_p (h \circ T + S_p (B)) = k \circ h \circ T + S_p (C)\,, \tag{2.7}$$

$$\left((k \circ h)_\#\right)_p (T + S_p (A)) = k \circ h \circ T + S_p (C)\,. \tag{2.8}$$

Therefore, we can conclude that $\left((k \circ h)_\#\right)_p = (k_\#)_p \circ (h_\#)_p$. ■

Theorem 2.2

There is a homomorphism $(\partial_*)_p : H_p (X, A) \to H_{p-1} (A)$, defined for $A \subset X$ and all p, such that the sequence

$$\cdots \longrightarrow H_p(A) \xrightarrow{(i_*)_p} H_p(X) \xrightarrow{(\pi_*)_p} H_p(X, A) \xrightarrow{(\partial_*)_p} H_{p-1}(A) \qquad \cdots$$

is exact, where i and π are the inclusions

$$(A, \varnothing) \xhookrightarrow{i} (X, \varnothing) \xhookrightarrow{\pi} (X, A).$$

The same holds if reduced homology is used for X and A, provided $A \neq \varnothing$.

A continuous map $f : (X, A) \to (Y, B)$ induces a homomorphism of the corresponding exact sequences in singular homology, either ordinary or reduced.

Proof. Let us recall the Zig-Zag lemma (Lemma 4.4.1 in the lecture note of AT2). Given a short exact sequence of chain complexes $\mathcal{C} = \left\{C_p, \partial^C_p\right\}$, $\mathcal{D} = \left\{D_p, \partial^D_p\right\}$ and $\mathcal{E} = \left\{E_p, \partial^E_p\right\}$, i.e.

$$0 \longrightarrow \mathcal{C} \xrightarrow{\phi} \mathcal{D} \xrightarrow{\psi} \mathcal{E} \longrightarrow 0$$

with ϕ and ψ being chain maps, i.e. family of homomorphisms $\{\phi_p\}$ and $\{\psi_p\}$ such that

$$0 \longrightarrow C_p \xrightarrow{\phi_p} D_p \xrightarrow{\psi_p} E_p \longrightarrow 0$$

is exact for each p, then there is a long exact homology sequence

$$\begin{tikzcd}
 & \cdots & \cdots \arrow[dll] \\
H_p\left(\mathcal{C}\right) \arrow[r, "\left(\phi_p\right)_*"'] & H_p\left(\mathcal{D}\right) \arrow[r, "\left(\psi_p\right)_*"] & H_p\left(\mathcal{E}\right) \arrow[dll, "\left(\partial_*\right)_p"] \\
H_{p-1}\left(\mathcal{C}\right) \arrow[r, "\left(\phi_{p-1}\right)_*"'] & H_{p-1}\left(\mathcal{D}\right) \arrow[r] & \cdots
\end{tikzcd}$$

We shall use Zig-Zag lemma with $C_p = S_p\left(A\right)$, $D_p = S_p\left(X\right)$ and $E_p = S_p\left(X, A\right)$, with chain maps given as follows:

$$0 \longrightarrow S_p\left(A\right) \xrightarrow{\left(i_\#\right)_p} S_p\left(X\right) \xrightarrow{\left(\pi_\#\right)_p} S_p\left(X, A\right) \longrightarrow 0.$$

Then the above sequence is exact, since $S_p\left(X, A\right) = S_p\left(X\right)/S_p\left(A\right)$. Now, Zig-Zag lemma guarantees the existence of the homomorphism $\left(\partial_*\right)_p : H_p\left(X, A\right) \to H_{p-1}\left(A\right)$ and the following long-exact sequence

$$\cdots \longrightarrow H_p(A) \xrightarrow{\left(i_*\right)_p} H_p(X) \xrightarrow{\left(\pi_*\right)_p} H_p(X, A) \xrightarrow{\left(\partial_*\right)_p} H_{p-1}(A) \qquad \cdots.$$

Now, given a continuous map $f : (X, A) \to (Y, B)$, we shall verify that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & S_p(A) \arrow[r, "\left(i_\#\right)_p"] \arrow[d, "\left(\left(f\big|_A\right)_\#\right)_p"'] & S_p(X) \arrow[r, "\left(\pi_\#\right)_p"] \arrow[d, "\left(\left(f\big|_X\right)_\#\right)_p"'] & S_p(X, A) \arrow[r] \arrow[d, "\left(f_\#\right)_p"] & 0 \\
0 \arrow[r] & S_p(B) \arrow[r, "\left(i'_\#\right)_p"'] & S_p(Y) \arrow[r, "\left(\pi'_\#\right)_p"'] & S_p(Y, B) \arrow[r] & 0
\end{tikzcd}$$

Here, by $f\big|_X$, we mean the map $f : X \to Y$. First, let's show the commutativity of the left hand square. Let's take a singular p-simplex T of A, i.e. $T : \Delta_p \to A$ is continuous. Then

$$\left(i_\#\right)_p T = i \circ T = T\,,\ \left(f_\#\right)_p \left(\left(i_\#\right)_p T\right) = f \circ T\,. \tag{2.9}$$

$$\left(\left(f\big|_A\right)_\#\right)_p T = f\big|_A \circ T = f \circ T\,,\ \left(i'_\#\right)_p \left(\left(\left(f\big|_A\right)_\#\right)_p T\right) = i' \circ f \circ T = f \circ T. \tag{2.10}$$

$f\big|_A \circ T = f \circ T$ because the image of T lies entirely in A. Therefore, the left hand square commutes. Now we shall show that the right hand square commutes as well. Let's take a singular p-simplex T of X, i.e. $T : \Delta_p \to X$ is continuous.

$$\left(\pi_\#\right)_p T = T + S_p\left(A\right)\,,\ \left(f_\#\right)_p \left(\left(\pi_\#\right)_p T\right) = \left(f_\#\right)_p T + S_p\left(B\right) = \left(\pi'_\#\right)_p \left(\left(f_\#\right)_p T\right). \tag{2.11}$$

Therefore, the right hand square commutes. So the diagram is commutative. Now, applying Theorem 5.1.1 from the lecture note of AT2, we obtain that the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r] & H_p(A) \arrow[r, "\left(i_*\right)_p"] \arrow[d, "\left(\left(f\big|_A\right)_*\right)_p"'] & H_p(X) \arrow[r, "\left(\pi_*\right)_p"] \arrow[d, "\left(f_*\right)_p"'] & H_p(X, A) \arrow[r, "\left(\partial_*\right)_p"] \arrow[d, "\left(f_*\right)_p"] & H_{p-1}(A) \arrow[r] \arrow[d, "\left(\left(f\big|_A\right)_*\right)_{p-1}"] & \cdots \\
\cdots \arrow[r] & H_p(B) \arrow[r, "\left(i'_*\right)_p"'] & H_p(Y) \arrow[r, "\left(\pi'_*\right)_p"'] & H_p(Y, B) \arrow[r, "\left(\partial'_*\right)_p"'] & H_{p-1}(B) \arrow[r] & \cdots
\end{tikzcd}$$

This establishes the induced homomorphisms between the respective long exact sequences of the singular homology. Following the same procedure, one can show that the same result holds in reduced homology. ■

Theorem 2.3

If P is a one-point space, then $H_p(P) = 0$ for $p \neq 0$, and $H_0(P) \cong \mathbb{Z}$.

Proof. We provide a direct proof here. We first compute the chain complex $\mathcal{S}(P)$. Observe that there is exactly one singular p-simplex in each non-negative dimention $p \geq 0$: $T_p : \Delta_p \to P$, because P is a singleton. Therefore, the group of p-chains $S_p(P) \cong \mathbb{Z}$, which is infinite cyclic. Each of the "faces" of $T_p : \Delta_p \to P$ is given

$$T_p \circ l_{(\varepsilon_0,\ldots,\widehat{\varepsilon}_i,\ldots,\varepsilon_p)} : \Delta_p \to P$$

and is precisely T_{p-1}. All $(p+1)$ faces of T_p are just T_{p-1}. Therefore, if p is even, then the singular p-simplex $(p+1)$ faces, which is an odd number. Hence, in the formula

$$\partial_p T_p = \sum_{i=0}^{p} (-1)^i \, T_p \circ l_{(\varepsilon_0,\ldots,\widehat{\varepsilon}_i,\ldots,\varepsilon_p)}, \tag{2.12}$$

only one term will survive, the others will cancel in pairs. Hence, we find that $\partial_p T_p = T_{p-1}$, when p is even.

On the other hand, when p is odd, T_p will have an even number of faces, and all the terms in 2.12 will cancel in pairs. Therefore, $\partial_p T_p = 0$, when p is odd. The chain complex $\mathcal{S}(P)$ is, thus, of the following form:

$$\cdots \longrightarrow S_{2k}(P) \longrightarrow S_{2k-1}(P) \longrightarrow \cdots \longrightarrow S_1(P) \longrightarrow S_0(P) \longrightarrow 0$$

$$\cdots \longrightarrow \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{\bar{0}} \cdots \longrightarrow \mathbb{Z} \xrightarrow{\bar{0}} \mathbb{Z} \longrightarrow 0$$

Here, $\bar{0}$ maps everything to 0. In dimension $(2k-1)$, every $(2k-1)$-chain is a cycle, and every $(2k-1)$-chain can be seen to be a boundary of a $2k$-chain. Hence, there is no nontrivial $(2k-1)$-cycle that is not a $(2k-1)$-boundary. Therefore, $H_{2k-1}(P) = 0$.

In dimension $2k$, for $k > 0$, there is no nontrivial chain that is a cycle. Hence, $H_{2k} = 0$. In dimension 0, every chain is a cycle, and no nontrivial 0-chain is a bounday. Therefore, $H_0(P) \cong \mathbb{Z}$. ■

§2.2 Compact Support Axiom

In this section, we shall verify that singular homology theory satisfies the compact support axiom[1].

Definition 2.1 (Minimal carrier). If $T : \Delta_p \to X$ is a singular p-simplex of X, then the **minimal carrier** of T is defined to be the image set $T(\Delta_p)$. If $c = \sum n_i T_i$ is a singular p-chain, with T_i being singular p-simplices and each n_i nonzero, then the minimal carrier of c is defined to be the union of the minimal carriers of the singular p-simplices T_i.

A singular p-simplex T is a continuous map from Δ_p to X. Since Δ_p is compact, so is $T(\Delta_p)$ since continuous map takes compact sets to compact sets. Now, a finite union of compact sets is also compact. Therefore, the minimal carrier of a singular p-chain is compact.

Theorem 2.4

Given $\alpha \in H_p(X, A)$, there is a compact pair $(X_0, A_0) \subseteq (X, A)$, with $\iota : (X_0, A_0) \hookrightarrow (X, A)$ such that $(\iota_*)_p(\beta) = \alpha$ for some $\beta \in H_p(X_0, A_0)$, where $(\iota_*)_p : H_p(X_0, A_0) \to H_p(X, A)$ is the homomorphism induced by the inclusion ι.

[1]The axiom of compact support is one of the Eilenberg-Steenrod axioms. See chapter 6 of `https://atonurc.github.io/assets/MAT432_AT2.pdf`

Proof. Given $\alpha \in H_p(X, A) = Z_p(X, A)/B_p(X, A)$, α is of the form $C + B_p(X, A)$, with $C \in Z_p(X, A) \subset S_p(X, A) = S_p(X)/S_p(A)$. Therefore,

$$\alpha = (c_p + S_p(A)) + B_p(X, A), \tag{2.13}$$

where $c_p \in S_p(X)$ such that $\partial_p c_p$ is carried by A. The minimal carrier of $\partial_p c_p$ is a compact set contained in A. Let us denote this compact set by A_0. On the other hand, c_p is minimally carried by a compact set X_0 contained in X. Now, we define

$$D = c_p + S_p(A_0) \in S_p(X_0, A_0). \tag{2.14}$$

Since $\partial_p c_p$ is carried by A_0, $D \in Z_p(X_0, A_0)$. Now, we claim that

$$\beta = D + B_p(X_0, A_0) = (c_p + S_p(A_0)) + B_p(X_0, A_0) \in H_p(X_0, A_0) \tag{2.15}$$

is the required element of $H_p(X_0, A_0)$ whose image under $(\iota_*)_p$ is α. Now,

$$(\iota_*)_p(\beta) = (\iota_*)_p((c_p + S_p(A_0)) + B_p(X_0, A_0)) = \Big((\iota_\#)_p c_p + S_p(A)\Big) + B_p(X, A). \tag{2.16}$$

If $c_p = \sum n_i T_i$, with T_i being singular p-simplices, then

$$(\iota_\#)_p c_p = \sum n_i (\iota_\#)_p(T_i) = \sum n_i(\iota \circ T_i) = \sum n_i T_i = c_p. \tag{2.17}$$

Therefore,

$$(\iota_*)_p(\beta) = (c_p + S_p(A)) + B_p(X, A) = \alpha. \tag{2.18}$$

■

Theorem 2.5

Let $i : (X_0, A_0) \hookrightarrow (X, A)$ be inclusion, where (X_0, A_0) is a compact pair. If $\alpha \in H_p(X_0, A_0)$ with $(i_*)_p(\alpha) = 0$, then there are a compact pais (X_1, A_1) and inclusions

$$(X_0, A_0) \xhookrightarrow{j} (X_1, A_1) \xhookrightarrow{k} (X, A)$$

such that $(j_*)_p(\alpha) = 0$.

Proof. Let $\alpha = (c_p + S_p(A_0)) + B_p(X_0, A_0) \in H_p(X_0, A_0)$, where $c_p \in S_p(X_0)$ and $\partial_p c_p$ is carried by A_0. Now, $(i_*)_p : H_p(X_0, A_0) \to H_p(X, A)$, so $(i_*)_p(\alpha) = 0 + B_p(X, A)$.

$$0 + B_p(X, A) = (i_*)_p(\alpha) = \Big((i_\#)_p c_p + S_p(A)\Big) + B_p(X, A). \tag{2.19}$$

Using a similar method as in 2.17, one can show that $(i_\#)_p c_p = c_p$. So 2.19 reads

$$0 + B_p(X, A) = (c_p + S_p(A)) + B_p(X, A). \tag{2.20}$$

Therefore, $c_p + S_p(A) \in B_p(X, A)$. In other words, there exists a $(p+1)$-chain d_{p+1} such that $c_p - \partial_{p+1} d_{p+1}$ is carried by A. Now, d_{p+1} is carried by

$$X_1 = X_0 \cup (\text{minimal carrier of } d_{p+1}),$$

and $c_p - \partial_{p+1} d_{p+1}$ is carried by

$$A_1 = A_0 \cup (\text{minimal carrier of } c_p - \partial_{p+1} d_{p+1}).$$

Consider the inclusion maps

$$\begin{tikzcd} (X_0, A_0) \arrow[r, hook, "j"] \arrow[rr, bend right, "i=k\circ j"'] & (X_1, A_1) \arrow[r, hook, "k"] & (X, A). \end{tikzcd}$$

Then $(j_*)_p(\alpha)$ is

$$(j_*)_p(\alpha) = (j_*)_p((c_p + S_p(A_0)) + B_p(X_0, A_0)) = \Big((j_\#)_p c_p + S_p(A_1)\Big) + B_p(X_1, A_1). \tag{2.21}$$

Again, similarly as in 2.17, one can show that $(j_\#)_p c_p = c_p$.

$$(j_*)_p(\alpha) = (c_p + S_p(A_1)) + B_p(X_1, A_1). \tag{2.22}$$

$c_p - \partial_{p+1} d_{p+1}$ is carried by A_1, so $c_p - \partial_{p+1} d_{p+1} \in S_p(A_1)$. Therefore,

$$\begin{aligned} c_p + S_p(A_1) &= c_p - (c_p - \partial_{p+1} d_{p+1}) + S_p(A_1) = \partial_{p+1} d_{p+1} + S_p(A_1) \\ &= \partial_{p+1}(d_{p+1} + S_{p+1}(A_1)) \in B_p(X_1, A_1). \end{aligned} \tag{2.23}$$

Combining 2.22 and 2.23, we get

$$(j_*)_p(\alpha) = \partial_{p+1}(d_{p+1} + S_{p+1}(A_1)) + B_p(X_1, A_1) = 0 + B_p(X_1, A_1). \tag{2.24}$$

■

§2.3 Chain Homotopy

Definition 2.2. Given chain complexes $\mathcal{C} = \{C_p, \partial_p\}$ and $\mathcal{C}' = \{C'_p, \partial'_p\}$ and chain maps $\phi, \psi : \mathcal{C} \to \mathcal{C}'$, a **chain homotopy** of ϕ to ψ is a family of homomorphisms $D_p : C_p \to C'_{p+1}$ such that the following holds

$$\partial'_{p+1} D_p + D_{p-1} \partial_p = \psi_p - \phi_p. \tag{2.25}$$

The following diagram might be useful for to understand the above formula in 2.25. Note that this is **NOT** a commutative diagram.

$$\begin{tikzcd} & C'_{p+1} \arrow[d, "\partial'_{p+1}"] \\ C_p \arrow[ur, "D_p"] \arrow[r, shift left, "\phi_p"] \arrow[r, shift right, "\psi_p"'] \arrow[d, "\partial_p"'] & C'_p \\ C_{p-1} \arrow[ur, "D_{p-1}"'] & \end{tikzcd}$$

Now, consider the inclusions $i, j : X \to X \times I$ (I is the unit interval $[0, 1]$) given by

$$i(x) = (x, 0) \text{ and } j(x) = (x, 1). \tag{2.26}$$

The corresponding chain maps are denoted by $(i_\#)_p, (j_\#)_p : S_p(X) \to S_p(X \times I)$. Construct a chain homotopy D^X between the chain map $i_\#$ and $j_\#$ as follows:

$$\begin{aligned} D^X &: \mathcal{S}(X) \to \mathcal{S}(X \times I), \\ D^X_p &: S_p(X) \to S_p(X \times I). \end{aligned} \tag{2.27}$$

For D^X to be a chain homotopy, the following equation must hold:

$$\partial^{X \times I}_{p+1} \circ D^X_p + D^X_{p-1} \circ \partial^X_p = (j_\#)_p - (i_\#)_p. \tag{2.28}$$

$$
\begin{tikzcd}
 & S_{p+1}(X \times I) \arrow[d, "\partial_{p+1}^{X \times I}"] \\
S_p(X) \arrow[ur, "D_p^X"] \arrow[r, shift left, "(i_\#)_p"] \arrow[r, shift right, "(j_\#)_p"'] \arrow[d, "\partial_p^X"'] & S_p(X \times I) \\
S_{p-1}(X) \arrow[ur, "D_{p-1}^X"'] &
\end{tikzcd}
$$

One can now construct the following diagram to find that $F_\# \circ D^X$ is a chain homotopy between the chain maps $f_\#, g_\# : \mathcal{S}(X) \to \mathcal{S}(Y)$, where X and Y are topological spaces and F is a homotopy between the maps $f, g : X \to Y$, i.e. $F : X \times I \to Y$ is a continuous map such that

$$F(x, 0) = f(x) \text{ and } F(x, 1) = g(x).$$

Using 2.26, we then have

$$F \circ i = f \text{ and } F \circ j = g. \tag{2.29}$$

$F_\# : \mathcal{S}(X \times I) \to \mathcal{S}(Y)$. In order to show that $F_\# \circ D^X$ is a chain homotopy between $f_\#$ and $g_\#$, one needs to prove that

$$\partial_{p+1}^Y \circ (F_\#)_{p+1} \circ D_p^X + (F_\#)_p \circ D_{p-1}^X \circ \partial_p^X = (g_\#)_p - (f_\#)_p. \tag{2.30}$$

$$
\begin{tikzcd}
 & & S_{p+1}(Y) \arrow[dd, "\partial_{p+1}^Y"] \\
 & S_{p+1}(X \times I) \arrow[ur, "(F_\#)_{p+1}"] & \\
S_p(X) \arrow[ur, "D_p^X"] \arrow[rr, shift left, "(f_\#)_p"] \arrow[rr, shift right, "(g_\#)_p"'] \arrow[dd, "\partial_{p+1}^X"'] & & S_p(Y) \\
 & S_p(X \times I) \arrow[ur, "(F_\#)_p"'] & \\
S_{p-1}(X) \arrow[ur, "D_{p-1}^X"'] & &
\end{tikzcd}
$$

Let us quickly see how 2.30 comes from 2.28. Since chain maps commute with the boundary operator, we have the following commutative diagram:

$$
\begin{tikzcd}
S_{p+1}(X \times I) \arrow[r, "(F_\#)_{p+1}"] \arrow[d, "\partial_{p+1}^{X \times I}"'] & S_{p+1}(Y) \arrow[d, "\partial_{p+1}^Y"] \\
S_p(X \times I) \arrow[r, "(F_\#)_p"'] & S_p(Y)
\end{tikzcd}
$$

i.e. $\partial_{p+1}^Y \circ (F_\#)_{p+1} = (F_\#)_p \circ \partial_{p+1}^{X \times I}$. Therefore, one obtains

$$
\begin{aligned}
\partial_{p+1}^Y \circ (F_\#)_{p+1} \circ D_p^X &= (F_\#)_p \circ \partial_{p+1}^{X \times I} \circ D_p^X \\
&= (F_\#)_p \circ \left[(j_\#)_p - (i_\#)_p - D_{p-1}^X \circ \partial_p^X \right] \\
&= \Big((F \circ j)_\# \Big)_p - \Big((F \circ i)_\# \Big)_p - (F_\#)_p \circ D_{p-1}^X \circ \partial_p^X \\
&= (g_\#)_p - (f_\#)_p (F_\#)_p \circ D_{p-1}^X \circ \partial_p^X,
\end{aligned} \tag{2.31}
$$

which can be rearranged to obtain 2.30. The existence of the chain map $D^X : \mathcal{S}(X) \to \mathcal{S}(X \times I)$ is governed by the following lemma.

Lemma 2.6

There exists, for each space X, and each non-negative integer p, a homomorphism $D_p^X : S_p(X) \to S_{p+1}(X \times I)$ having the following properties:

(a) If $T : \Delta_p \to X$ is a singular p-simplex then

$$\partial_{p+1}^{X\times I} D_p^X T + D_{p-1}^X \partial_p^X T = (j_\#)_p T - (i_\#)_p T. \tag{2.32}$$

Here, the map $i : X \to X \times I$ carries x to $(x,0)$ and the map $j : X \to X \times I$ carries x to $(x,1)$.

(b) D_p^X is natural; i.e. given $f : X \to Y$ continuous, the following diagram commutes:

$$\begin{tikzcd} S_p(X) \arrow[r, "D_p^X"] \arrow[d, "(f_\#)_p"'] & S_{p+1}(X \times I) \arrow[d, "((f\times \mathrm{id}_I)_\#)_{p+1}"] \\ S_p(Y) \arrow[r, "D_p^Y"] & S_{p+1}(Y \times I) \end{tikzcd}$$

Note that continuous $f : X \to Y$ induces a continnuous map $f \times \mathrm{id}_I : X \times I \to Y \times I$ given by $(x,t) \mapsto (f(x), t)$. Hence there is a group homomorphism

$$\left((f \times \mathrm{id}_I)_\#\right)_p : S_p(X \times I) \to S_p(Y \times I)$$

for each non-negative integer p.

Proof of the lemma is omitted.

Theorem 2.7

If $f, g : (X,A) \to (Y,B)$ are homotopic, then $(f_*)_p = (g_*)_p$ for all p, with $(f_*)_p, (g_*)_p : H_p(X,A) \to H_p(Y,B)$ group homomorphisms. The same holds in the reduced homology if $A = B = \varnothing$.

Proof. Let $F : (X \times I, A \times I) \to (Y \times I, B \times I)$ be the homotopy between $f, g : (X,A) \to (Y,B)$. Let $i, j : (X,A) \to (X \times I, A \times I)$ be given by $i(x) = (x,0)$ and $j(x) = (x,1)$, for $x \in X$. Let $D_p^X : S_p(X) \to S_p(X \times I)$ be the group homomorphism associated with the chain homotopy $D^X : \mathcal{S}(X) \to \mathcal{S}(X \times I)$ constructed in Lemma 2.6. Naturality of D^X with respect to the inclusion map $\iota : A \hookrightarrow X$ dictates that the following diagram commutes:

$$\begin{tikzcd} S_p(A) \arrow[r, "D_p^A"] \arrow[d, "(\iota_\#)_p"'] & S_{p+1}(A \times I) \arrow[d, "((\iota\times \mathrm{id}_I)_\#)_{p+1}"] \\ S_p(X) \arrow[r, "D_p^X"] & S_{p+1}(X \times I) \end{tikzcd}$$

Consider $T \in S_{p+1}(A \times I)$ such that T is a $(p+1)$-singular simplex of $A \times I$, i.e. $T : \Delta_{p+1} \to A \times I$ is continuous. For a given $x \in \Delta_{p+1}$, let $T(x) = (a,t) \in A \times I$. Now,

$$\left((\iota \times \mathrm{id}_I)_\#\right)_{p+1} T(x) = (\iota \times \mathrm{id}_I) \circ T(x) = (\iota \times \mathrm{id}_I)(a,t) = (a,t) = T(x). \tag{2.33}$$

Hence, $\left((\iota \times \mathrm{id}_I)_\#\right)_{p+1} T = T$. So, we have

$$\left((\iota \times \mathrm{id}_I)_\#\right)_{p+1} \circ D_p^A = D_p^A. \tag{2.34}$$

Now, commutativity of the above diagram yields

$$\left((\iota \times \mathrm{id}_I)_\#\right)_{p+1} \circ D_p^A = D_p^X \circ (\iota_\#)_p = D_p^X\big|_{S_p(A)}. \tag{2.35}$$

Therefore, combining 2.34 and 2.35, we get

$$D_p^X\big|_{S_p(A)} = D_p^A. \tag{2.36}$$

In other words, $D_p^X : S_p(X) \to S_{p+1}(X \times I)$ carries $S_p(A)$ into $S_p(X \times I)$, and thus induces a chain homotopy on the relative level. The constituent group homomorphisms are given by

$$D_p^{(X,A)} : S_p(X, A) \to S_{p+1}(X \times I, A \times I). \tag{2.37}$$

Now, 2.32 indeed holds for $D_p^{(X,A)}$ as it is induced by D_p^X. Then we have

$$(F_\#)_{p+1} \circ D_p^{(X,A)} : S_p(X, A) \to S_{p+1}(Y, B),$$

where the homomorphism $(F_\#)_{p+1}$ associated with the chain map $F_\# : \mathcal{S}(X \times I, A \times I) \to \mathcal{S}(Y, B)$ is

$$(F_\#)_{p+1} : S_{p+1}(X \times I, A \times I) \to S_{p+1}(Y, B).$$

Then

$$\begin{aligned}
\partial_{p+1}^Y \circ (F_\#)_{p+1} \circ D_p^{(X,A)} &= (F_\#)_p \circ \partial_{p+1}^{X\times I} \circ D_p^{(X,A)} \\
&= (F_\#)_p \circ \left[(j_\#)_p - (i_\#)_p - D_{p-1}^{(X,A)} \circ \partial_p^X\right] \\
&= \left((F \circ j)_\#\right)_p - \left((F \circ i)_\#\right)_p - (F_\#)_p \circ D_{p-1}^{(X,A)} \circ \partial_p^X \\
&= (g_\#)_p - (f_\#)_p - (F_\#)_p \circ D_{p-1}^{(X,A)} \circ \partial_p^X.
\end{aligned} \tag{2.38}$$

This proves that $F_\# \circ D^{(X,A)} : \mathcal{S}(X, A) \to \mathcal{S}(Y, B)$ is a chain homotopy between $f_\#, g_\# : \mathcal{S}(X, A) \to \mathcal{S}(Y, B)$. It now remains to prove that $(f_*)_p = (g_*)_p$ for all p.

Let $\alpha \in Z_p(X, A)$. It suffices to show that $(f_\#)_p(\alpha)$ and $(g_\#)_p(\alpha)$ differ by a boundary term. Given $\alpha \in Z_p(X, A)$, $\alpha = c_p + S_p(A)$ for some $c_p \in S_p(X)$ such that $\partial_p c_p$ is carried by A. By 2.38,

$$\begin{aligned}
(g_\#)_p(\alpha) - (f_\#)_p(\alpha) &= \partial_{p+1}^Y \circ (F_\#)_{p+1} \circ D_p^{(X,A)}(\alpha) + (F_\#)_p \circ D_{p-1}^{(X,A)} \circ \partial_p^X(\alpha) \\
&= \partial_{p+1}^Y \circ (F_\#)_{p+1} \circ D_p^{(X,A)}(\alpha),
\end{aligned} \tag{2.39}$$

proving that $(f_\#)_p(\alpha)$ and $(g_\#)_p(\alpha)$ differ by a boundary term. Therefore, $(f_*)_p(\alpha + B_p(X, A)) = (f_*)_p(\alpha + B_p(X, A))$.

The result in reduced homology is left as an exercise. ■

§2.4 Homotopy Equivalence

Definition 2.3 (Retraction). Let $A \subset X$. A **retraction** of X onto A is a continuous map $r : X \to A$ such that $r(a) = a$ for every $a \in A$, i.e. $r\big|_A = \mathrm{id}_A$. If there is a retraction of X onto A, we say that A is a retract of X,

Definition 2.4 (Deformation retraction). A **deformation retraction** of X onto A is a continuous map $F : X \times I \to X$ such that

$$F(x, 0) = x, \quad F(x, 1) \in A, \quad \text{and } F(a, t) = a \tag{2.40}$$

for all $x \in X, a \in A, t \in I$.

If F is a deformation retraction of X onto A, then one can define

$$r(x) = F(x,1). \tag{2.41}$$

Then 2.40 tells us that r is a map from X to A, and $r(a) = a$ for all $a \in A$. Hence, r is indeed a retraction of X onto A. Now, 2.40 also tells us that

$$F(x,0) = x = \mathrm{id}_X(x) \text{ and } F(x,1) = j \circ r(x), \tag{2.42}$$

where $j : A \hookrightarrow X$ is the inclusion. Therefore, F is a homotopy between the identity map $\mathrm{id}_X : X \to X$ and $j \circ r : X \to X$.

Definition 2.5. Let $f : (X,A) \to (Y,B)$ be continuous. If there is a continuous map $g : (Y,B) \to (X,A)$ such that $g \circ f$ is homotopic to the identity map $\mathrm{id}_{(X,A)} : (X,A) \to (X,A)$ and $f \circ g$ is homotopic to the identity map $\mathrm{id}_{(Y,B)} : (Y,B) \to (Y,B)$, then we call f a **homotopy equivalence**, and we call g a **homotopy inverse** for f.

Theorem 2.8

Let $f : (X,A) \to (Y,B)$ be continuous.

(a) If f is a homotopy equivalence, then f_* is an isomorphism in relative homology.

(b) More generally, if $f : X \to Y$ and $f\big|_A : A \to B$ are homotopy equivalences, then f_* is an isomorphism in relative homology.

Proof. Let $f : (X,A) \to (Y,B)$ be a homotopy equivalence, and $g : (Y,B) \to (X,A)$ its homotopy inverse. Then $f \circ g \simeq \mathrm{id}_{(Y,B)}$ and $g \circ f \simeq \mathrm{id}_{(X,A)}$. Then by Theorem 2.7,

$$((f \circ g)_*)_p = \left(\left(\mathrm{id}_{(Y,B)}\right)_*\right)_p \text{ and } ((g \circ f)_*)_p = \left(\left(\mathrm{id}_{(X,A)}\right)_*\right)_p.$$

In other words,

$$(f_*)_p \circ (g_*)_p = \mathrm{id}_{H_p(Y,B)} \text{ and } (g_*)_p \circ (f_*)_p = \mathrm{id}_{H_p(X,A)}. \tag{2.43}$$

Therefore, $(f_*)_p : H_p(X,A) \to H_p(Y,B)$ is an isomorphism.

Now we shall prove (b). Consider the long exact sequence of the pairs (X,A) and (Y,B), separately with $(f_*)_p$ being the respective connecting homomorphisms.

$$\begin{tikzcd}
\cdots \arrow[r] & H_p(A) \arrow[r, "(i_*)_p"] \arrow[d, "\left(\left(f|_A\right)_*\right)_p"'] & H_p(X) \arrow[r, "(\pi_*)_p"] \arrow[d, "(f_*)_p"'] & H_p(X,A) \arrow[r, "(\partial_*)_p"] \arrow[d, "(f_*)_p"] & H_{p-1}(A) \arrow[r, "(i_*)_{p-1}"] \arrow[d, "\left(\left(f|_A\right)_*\right)_{p-1}"] & H_{p-1}(X) \arrow[r] \arrow[d, "(f_*)_{p-1}"] & \cdots \\
\cdots \arrow[r] & H_p(B) \arrow[r, "(i'_*)_p"'] & H_p(Y) \arrow[r, "(\pi'_*)_p"'] & H_p(Y,B) \arrow[r, "(\partial'_*)_p"'] & H_{p-1}(B) \arrow[r, "(i'_*)_{p-1}"'] & H_{p-1}(X) \arrow[r] & \cdots
\end{tikzcd}$$

By hypothesis, $f : (X,\varnothing) \to (Y,\varnothing)$ is a homotopy equivalence, and hence $(f_*)_p : H_p(X) \to H_p(Y)$ is an isomorphism. Similarly, by hypothesis, $f\big|_A : (A,\varnothing) \to (B,\varnothing)$ is a homotopy equivalence, and hence $\left(\left(f|_A\right)_*\right)_p : H_p(A) \to H_p(B)$ is an isomorphism. Now, applying Steenrod five lemma to the diagram above, one obtains that

$$(f_*)_p : H_p(X,A) \to H_p(Y,B)$$

is an isomorphism. ■

Remark 2.1. If $f : (X,A) \to (Y,B)$ is a homotopy equivalence, then $f : X \to Y$ and $f\big|_A : A \to B$ are automaatically homotopy equivalences. However, the converse is not true. One counterexample is presented below.

Example 2.1

Consider the inclusion map $j : \left(B^n, S^{n-1}\right) \hookrightarrow \left(\mathbb{R}^n, \mathbb{R}^n \setminus \{\mathbf{0}\}\right)$. $j : B^n \hookrightarrow \mathbb{R}^n$ has a homotopy inverse, so that B^n and $\mathbb{R}^n$ are homotopy equivalent. The homotopy inverse is given by $f : \mathbb{R}^n \to B^n$,

$$f\left(\mathbf{x}\right) = \begin{cases} \mathbf{x} & \text{if } \left\|\mathbf{x}\right\| \leq 1 \\ \frac{\mathbf{x}}{\left\|\mathbf{x}\right\|} & \text{if } \left\|\mathbf{x}\right\| > 1 \end{cases}. \tag{2.44}$$

Then $f\left(j\left(\mathbf{x}\right)\right) = \mathbf{x}$, so $f \circ j = \mathrm{id}_{B^n}$. $j\left(f\left(\mathbf{x}\right)\right) = f(\mathbf{x}) \in B^n$. So $F : \mathbb{R}^n \times I \to \mathbb{R}^n$ given by

$$F\left(\mathbf{x}, t\right) = \left(1-t\right)\mathbf{x} + tj \circ f(\mathbf{x}) \tag{2.45}$$

is a homotopy between $\mathrm{id}_{\mathbb{R}^n}$ and $j \circ f$. Therefore, f is the homotopy inverse of j.

In a similar manner, one can show that $j\big|_{S^{n-1}} : S^{n-1} \hookrightarrow \mathbb{R}^n \setminus \{\mathbf{0}\}$ also has a homotopy inverse. The homotopy inverse is $h : \mathbb{R}^n \setminus \{\mathbf{0}\} \to S^{n-1}$ given by

$$h\left(\mathbf{x}\right) = \frac{\mathbf{x}}{\left\|\mathbf{x}\right\|}. \tag{2.46}$$

Then $h \circ j\big|_{S^{n-1}} = \mathrm{id}_{S^{n-1}}$. Furthermore, $G : \left(\mathbb{R}^n \setminus \{\mathbf{0}\}\right) \times I \to \mathbb{R}^n \setminus \{\mathbf{0}\}$ given by

$$G\left(\mathbf{x}, t\right) = \left(1-t\right)\mathbf{x} + tj\big|_{S^{n-1}} \circ h\left(\mathbf{x}\right) = \left(\left(1-t\right) + \frac{t}{\left\|\mathbf{x}\right\|}\right)\mathbf{x} \tag{2.47}$$

is a homotopy between $\mathrm{id}_{\mathbb{R}^n \setminus \{\mathbf{0}\}}$ and $j\big|_{S^{n-1}} \circ h$. Therefore, h is the homotopy inverse of j.

However, $j : \left(B^n, S^{n-1}\right) \hookrightarrow \left(\mathbb{R}^n, \mathbb{R}^n \setminus \{\mathbf{0}\}\right)$ has no homotopy inverse although both $j : B^n \hookrightarrow \mathbb{R}^n$ and $j\big|_{S^{n-1}} : S^{n-1} \hookrightarrow \mathbb{R}^n \setminus \{\mathbf{0}\}$ have homotopy inverses. To show this, assume the contrary that $g : \left(\mathbb{R}^n, \mathbb{R}^n \setminus \{\mathbf{0}\}\right) \to \left(B^n, S^{n-1}\right)$ is a homotopy inverse of j. Then g is continuous, and it maps $\mathbb{R}^n \setminus \{\mathbf{0}\}$ into S^{n-1}. But $\mathbf{0}$ is a limit point of $\mathbb{R}^n \setminus \{\mathbf{0}\}$, and S^{n-1} is closed. Therefore, $g\left(\mathbf{0}\right) \in S^{n-1}$. In other words, g maps all of $\mathbb{R}^n$ into S^{n-1}. Hence, the composite

$$g \circ j : \left(B^n, S^{n-1}\right) \to \left(B^n, S^{n-1}\right) \tag{2.48}$$

maps all of B^n to S^{n-1}. If $T : \Delta_p \to B^n$ is a singular p-simplex, then for $T + S_p\left(S^{n-1}\right) \in S_p\left(B^n, S^{n-1}\right)$,

$$\left(\left(g \circ j\right)_{\#}\right)_p \left(T + S_p\left(S^{n-1}\right)\right) = g \circ j \circ T + S_p\left(S^{n-1}\right). \tag{2.49}$$

But the image of $g \circ j \circ T$ lies entirely on S^{n-1}. So $\left(\left(g \circ j\right)_{\#}\right)_p$ is the trivial chain map. Therefore, $\left(\left(g \circ j\right)_*\right)_p : H_p\left(B^n, S^{n-1}\right) \to H_p\left(B^n, S^{n-1}\right)$ is the trivial map. However, since $g \circ j$ is homotopic with $\mathrm{id}_{\left(B^n, S^{n-1}\right)}$, $\left(\left(g \circ j\right)_*\right)_p$ is the identity homomorphism on $H_p\left(B^n, S^{n-1}\right)$. This can only be true if $H_p\left(B^n, S^{n-1}\right) = 0$. We shall soon see this is not true.

§2.5 Subdivision