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Representation Theory (MAT440)

Lecture Notes

Preface

This series of lecture notes has been prepared for aiding students who took the BRAC University course **Representation Theory (MAT440)** in Summer 2024 semester. These notes were typeset under the supervision of mathematician **Dr. Syed Hasibul Hassan Chowdhury**. The main goal of this typeset is to have an organized digital version of the notes, which is easier to share and handle. If you see any mistakes or typos, please send me an email at atonuroychowdhury@gmail.com

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References:

- *Representation Theory: A First Course*, by **Joe Harris and William Fulton**
- *Representations of Finite and Compact Groups*, by **Barry Simon**
- *Introduction to Representation Theory*, by **Pavel Etingof et al.**

Contents

1 Representation of Finite Groups

§1.1 Definitions

Definition 1.1 (Representation). A **representation** of a finite group G on a finite dimensional complex vector space V is a homomorphism $\rho : G \to \mathrm{GL}(V)$ of G to the group of invertible linear transformations on V. We often say that such a homomorphism gives V the structure of a G-module. The dimension of V is sometimes called the **degree** of the representation ρ. We also sometimes call V itself a representation of G.

Definition 1.2. A **map** φ between two representations V and W of G is a linear map $\varphi : V \to W$ such that the following diagram commutes for every $g \in G$:

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "\rho(g)"'] & W \arrow[d, "\sigma(g)"] \\ V \arrow[r, "\varphi"'] & W \end{tikzcd}$$

In other words, $\varphi \circ \rho(g) = \sigma(g) \circ \varphi$. Here, $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$ are two group homomorphisms in question. We distinguish such a linear map $\varphi : V \to W$ between two representations of G from an ordinary linear map between vector spaces by calling it a **G-linear map**.

One can then define G-module structure on $\operatorname{Ker}\varphi$ and $\operatorname{im}\varphi$ by restricting the group homomorphisms $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$, namely,

$$\rho_1 : G \to \mathrm{GL}\left(\operatorname{Ker}\varphi\right) \text{ and } \sigma_1 : G \to \mathrm{GL}\left(\operatorname{im}\varphi\right).$$

Suppose $\mathbf{v} \in \operatorname{Ker}\varphi$. Then $\rho\left(g\right)\left(\mathbf{v}\right) \in \operatorname{Ker}\varphi$, because

$$\varphi\left(\rho\left(g\right)\left(\mathbf{v}\right)\right) = \sigma\left(g\right)\left(\varphi\left(\mathbf{v}\right)\right) = \sigma\left(g\right)\left(\mathbf{0}\right) = \mathbf{0}. \tag{1.1}$$

Also, let $\mathbf{w} \in \operatorname{im}\varphi$. Then $\mathbf{w} = \varphi(\mathbf{v})$ for some $\mathbf{v} \in V$. Then $\sigma\left(g\right)\left(\mathbf{w}\right) \in \operatorname{im}\varphi$, because

$$\sigma\left(g\right)\varphi(\mathbf{v}) = \varphi\left(\rho\left(g\right)\left(\mathbf{v}\right)\right) \in \operatorname{im}\varphi. \tag{1.2}$$

One can also give the quotient vector space $W/\operatorname{im}\varphi = \operatorname{Coker}\varphi$ a G-module structure by introducing the group homomorphism $\sigma_2 : G \to \mathrm{GL}(\operatorname{Coker}\varphi)$. Given $\mathbf{w} + \operatorname{im}\varphi \in \operatorname{Coker}\varphi$ and $g \in G$, one defines

$$\sigma_2\left(g\right)\left(\mathbf{w} + \operatorname{im}\varphi\right) = \sigma(g)\left(\mathbf{w}\right) + \operatorname{im}\varphi \in \operatorname{Coker}\varphi. \tag{1.3}$$

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The space of all G-linear maps from V to W is denoted $\mathrm{Hom}_G(V, W)$. It has a vector space structure. Suppose $\varphi, \psi \in \mathrm{Hom}_G(V, W)$ and $z \in \mathbb{C}$. Then we have the following commutative squares:

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "\rho(g)"'] & W \arrow[d, "\sigma(g)"] \\ V \arrow[r, "\varphi"'] & W \end{tikzcd} \qquad \begin{tikzcd} V \arrow[r, "\psi"] \arrow[d, "\rho(g)"'] & W \arrow[d, "\sigma(g)"] \\ V \arrow[r, "\psi"'] & W \end{tikzcd}$$

Then one can show that $z\varphi + \psi$ is also a G-linear map. Indeed,

$$\begin{aligned}\sigma(g) \circ (z\varphi + \psi)(\mathbf{v}) &= z\sigma(g)(\varphi(\mathbf{v})) + \sigma(g)(\psi(\mathbf{v})) \\ &= z\varphi(\rho(g)(\mathbf{v})) + \psi(\rho(g)(\mathbf{v})) \\ &= (z\varphi + \psi)(\rho(g)\mathbf{v}).\end{aligned}$$

This proves the commutativity of the following square:

$$\begin{tikzcd} V \arrow[r, "z\varphi+\psi"] \arrow[d, "\rho(g)"'] & W \arrow[d, "\sigma(g)"] \\ V \arrow[r, "z\varphi+\psi"'] & W \end{tikzcd}$$

Therefore, $z\varphi + \psi \in \mathrm{Hom}_G(V, W)$, i.e. $\mathrm{Hom}_G(V, W)$ is a complex vector space.

Definition 1.3 (Subrepresentation)**.** Suppose one is given a representation V of G with the help of the group homomorphism $\rho : G \to \mathrm{GL}(V)$ and $W \subset V$ be a vector subspace. One calls W **invariant** under the action of G if for all $g \in G$ and all $\mathbf{w} \in W$, one has $\rho(g)\mathbf{w} \in W$.

A **subrepresentation** of a representation V of G is a vector subspace W of V that is invariant under the action of G. A representation V of G is called **irreducible** if there is no proper nonzero invariant subspace W of V, i.e., there is no invariant subsapce $W \subset V$ such that $W \neq \{\mathbf{0}\}$ and $W \neq V$.

§1.2 Linear algebra revisited

Definition 1.4 (Tensor product)**.** The **tensor product** of two complex vector spaces V and W is another complex vector space $V \otimes W$ equipped with a bilinear map $\theta : V \times W \to V \otimes W$ that is *universal*: for any bilinear map $\beta : V \times W \to U$ to a complex vector space U, there exists a unique linear map $\alpha : V \otimes W \to U$ such that the following diagram commutes:

$$\begin{tikzcd} V \times W \arrow[r, "\theta"] \arrow[d, "\beta"'] & V \otimes W \arrow[dl, dashed, "\exists!\,\alpha"] \\ U & \end{tikzcd}$$

In other words, $\beta = \alpha \circ \theta$.

If we want the ground field $\mathbb{C}$ to be mentioned, we write the tensor product by $V \otimes_{\mathbb{C}} W$. If $\{\mathbf{e}_i\}$ and $\{\mathbf{f}_j\}$ are bases of V and W, respectively, $\{\mathbf{e}_i \otimes \mathbf{f}_j\}$ form a basis for $V \otimes W$. Similarly, one can form the tensor product $V_1 \otimes \cdots \otimes V_n$ of n vector spaces, with the universal (in the above sense) multilinear map

$$\begin{aligned}\theta : V_1 \times \cdots \times V_n &\to V_1 \otimes \cdots \otimes V_n \\ (\mathbf{v}_1, \ldots, \mathbf{v}_n) &\mapsto \mathbf{v}_1 \otimes \cdots \otimes \mathbf{v}_n.\end{aligned} \tag{1.4}$$

In particular, one can construct

$$V^{\otimes n} = \underbrace{V \otimes \cdots \otimes V}_{n\text{-copies}},$$

for a fixed complex vector space V. If $\{\mathbf{e}_i \mid i = 1, 2, \ldots, m\}$ is a basis for V, then the set

$$\{\mathbf{e}_{i_1} \otimes \mathbf{e}_{i_2} \otimes \cdots \otimes \mathbf{e}_{i_n} \mid i_1, \ldots, i_n \in \{1, 2, \ldots, m\}\} \tag{1.5}$$

is a basis for $V^{\otimes n}$. It follows that $\dim V^{\otimes n} = m^n$.

Let $\mathfrak{S}_n$ be the symmetric group on the set $\{1, 2, \ldots, n\}$. It is a finite group of order $n!$ that consists of all the permutations (i.e. bijections) on the set $\{1, 2, \ldots, n\}$. An alternating multilinear map $\beta : V \times \cdots \times V \to U$ satisfies

$$\beta\left(\mathbf{v}_{\sigma(1)}, \ldots, \mathbf{v}_{\sigma(n)}\right) = \operatorname{sgn}\sigma \beta\left(\mathbf{v}_1, \ldots, \mathbf{v}_n\right), \tag{1.6}$$

for every $\sigma \in \mathfrak{S}_n$.

Definition 1.5 (Exterior power)**.** The **exterior power** of a complex vector spaces V is another complex vector space $\Lambda^n V$ equipped with an alternating multilinear map

$$\begin{aligned} \kappa : V \times \cdots \times V &\to \Lambda^n V \\ (\mathbf{v}_1, \ldots, \mathbf{v}_n) &\mapsto \mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_n, \end{aligned}$$

that is *universal*: for any alternating multilinear map $\beta : V \times \cdots \times V \to U$ to a complex vector space U, there exists a unique linear map $\alpha : \Lambda^n V \to U$ such that the following diagram commutes:

$$\begin{tikzcd} V \times \cdots \times V \arrow[r, "\kappa"] \arrow[d, "\beta"'] & \Lambda^n V \arrow[dl, dashed, "\exists!\,\alpha"] \\ U & \end{tikzcd}$$

In other words, $\beta = \alpha \circ \kappa$.

If $\{\mathbf{e}_i \mid i = 1, 2, \ldots, m\}$ is a basis for V, then the set

$$\{\mathbf{e}_{i_1} \wedge \mathbf{e}_{i_2} \wedge \cdots \wedge \mathbf{e}_{i_n} \mid 1 \leq i_1 < i_2 < \cdots < i_n \leq m\} \tag{1.7}$$

is a basis for $\Lambda^n V$. It follows that $\dim \Lambda^n V = \binom{m}{n}$.

A symmetric multilinear map $\beta : V \times \cdots \times V \to U$ satisfies

$$\beta\left(\mathbf{v}_{\sigma(1)}, \ldots, \mathbf{v}_{\sigma(n)}\right) = \beta\left(\mathbf{v}_1, \ldots, \mathbf{v}_n\right), \tag{1.8}$$

for every $\sigma \in \mathfrak{S}_n$.

Definition 1.6 (Symmetric power)**.** The **symmetric power** of a complex vector spaces V is another complex vector space $\operatorname{Sym}^n V$ equipped with an symmetric multilinear map

$$\begin{aligned} \delta : V \times \cdots \times V &\to \operatorname{Sym}^n V \\ (\mathbf{v}_1, \ldots, \mathbf{v}_n) &\mapsto \mathbf{v}_1 \odot \cdots \odot \mathbf{v}_n, \end{aligned}$$

that is *universal*: for any symmetric multilinear map $\beta : V \times \cdots \times V \to U$ to a complex vector space U, there exists a unique linear map $\alpha : \operatorname{Sym}^n V \to U$ such that the following diagram commutes:

$$\begin{tikzcd} V \times \cdots \times V \arrow[r, "\delta"] \arrow[d, "\beta"'] & \operatorname{Sym}^n V \arrow[dl, dashed, "\exists!\,\alpha"] \\ U & \end{tikzcd}$$

In other words, $\beta = \alpha \circ \delta$.

If $\{\mathbf{e}_i \mid i = 1, 2, \ldots, m\}$ is a basis for V, then the set

$$\{\mathbf{e}_{i_1} \odot \mathbf{e}_{i_2} \odot \cdots \odot \mathbf{e}_{i_n} \mid 1 \leq i_1 \leq i_2 \leq \cdots \leq i_n \leq m\} \tag{1.9}$$

is a basis for $\operatorname{Sym}^n V$. It follows that $\dim \operatorname{Sym}^n V = \binom{m+n-1}{n}$.

§1.3 New representations from old ones

If V and W are representations of G, then so are the direct sum $V \oplus W$ and the tensor product $V \otimes W$. More explicitly, suppose $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$ are the relevant group homomorphisms. Then, one defines $\rho \oplus \sigma : G \to \mathrm{GL}(V \oplus W)$ by

$$(\rho \oplus \sigma)(g)(\mathbf{v} \oplus \mathbf{w}) = \rho(g)\mathbf{v} \oplus \sigma(g)\mathbf{w}, \tag{1.10}$$

for $g \in G$. Similarly, one can define the group homomorphism $\rho \otimes \sigma : G \to \mathrm{GL}(V \otimes W)$ by

$$(\rho \otimes \sigma)(g)(\mathbf{v} \otimes \mathbf{w}) = \rho(g)\mathbf{v} \otimes \sigma(g)\mathbf{w} \tag{1.11}$$

for $g \in G$.

For a representation V of G, the nth tensor power $V^{\otimes n}$ is again a representation of G:

$$(\rho \otimes \rho \otimes \cdots \otimes \rho)(g)(\mathbf{v}_1 \otimes \mathbf{v}_2 \otimes \cdots \otimes \mathbf{v}_n) = \rho(g)\mathbf{v}_1 \otimes \rho(g)\mathbf{v}_2 \otimes \cdots \otimes \rho(g)\mathbf{v}_n, \tag{1.12}$$

for $g \in G$. The exterior power $\Lambda^n(V)$ and the symmetric power $\operatorname{Sym}^n(V)$ are subrepresentations of $V^{\otimes n}$. Given the group homomorphism $\rho : G \to \mathrm{GL}(V)$, we defined the nth tensor power representation $\rho^{\otimes n} : G \to \mathrm{GL}\left(V^{\otimes n}\right)$ by (1.12). Now, the exterior power representation $\Lambda^n \rho : G \to \mathrm{GL}(\Lambda^n V)$, being a subrepresentation of $V^{\otimes n}$, can be defined as follows:

$$(\Lambda^n \rho)(g)(\mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \cdots \wedge \mathbf{v}_n) = \rho(g)\mathbf{v}_1 \wedge \rho(g)\mathbf{v}_2 \wedge \cdots \wedge \rho(g)\mathbf{v}_n. \tag{1.13}$$

One can now write down the group homomorphism $\operatorname{Sym}^n \rho : G \to \mathrm{GL}(\operatorname{Sym}^n V)$ associated with the subrepresentation $\operatorname{Sym}^n V$ of the representation $V^{\otimes n}$ of G:

$$(\operatorname{Sym}^n \rho)(g)(\mathbf{v}_1 \odot \mathbf{v}_2 \odot \cdots \odot \mathbf{v}_n) = \rho(g)\mathbf{v}_1 \odot \rho(g)\mathbf{v}_2 \odot \cdots \odot \rho(g)\mathbf{v}_n. \tag{1.14}$$

Now, let us define $\rho^* : G \to \mathrm{GL}(V^*)$, given $\rho : G \to \mathrm{GL}(V)$. Suppose $\{\mathbf{e}_i\}_{i=1}^m$ and $\{\widehat{\alpha}^i\}_{i=1}^m$ are bases of V and V^*, respectively. Here, $V^* = \operatorname{Hom}(V, \mathbb{C})$, the dual vector space of linear functionals on V. Any linear functional $\widehat{\omega} \in V^*$ can be written as

$$\widehat{\omega} = \sum_{i=1}^m \omega_i \widehat{\alpha}^i \tag{1.15}$$

Also, any vector $\mathbf{v} \in V$ can be written as

$$\mathbf{v} = \sum_{i=1}^m v^i \mathbf{e}_i. \tag{1.16}$$

In a given basis $\{\mathbf{e}_i\}_{i=1}^m$ of V and its dual basis $\{\widehat{\alpha}^i\}_{i=1}^m$ of V^*, $\omega \in V^*$ can be coordinated as a column vector $\begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_m \end{bmatrix}$, whereas a vector $\mathbf{v} \in V$ can be coordinated as $\begin{bmatrix} v^1 \\ v^2 \\ \vdots \\ v^m \end{bmatrix}$. We will simply denote the column vector $\begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_m \end{bmatrix}$ by $\widehat{\omega}$, and the column vector $\begin{bmatrix} v^1 \\ v^2 \\ \vdots \\ v^m \end{bmatrix}$ by $\mathbf{v}$. We then write the dual pairing

$$\langle \widehat{\omega}, \mathbf{v} \rangle = \widehat{\omega}(\mathbf{v}) = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_m \end{bmatrix}^T \begin{bmatrix} v^1 \\ v^2 \\ \vdots \\ v^m \end{bmatrix} = \widehat{\omega}^T \mathbf{v}. \tag{1.17}$$

Now, we want the dual representation V^* of V to satisfy

$$\langle \rho^* g(\widehat{\omega}), \rho(g)\mathbf{v} \rangle = \langle \widehat{\omega}, \mathbf{v} \rangle \tag{1.18}$$

for $g \in G$, $\mathbf{v} \in V$ and $\widehat{\omega} \in V^*$. Now, we claim that $\rho^* : V^* \to V^*$ defined by

$$\rho^*(g)(\widehat{\omega}) = \left[\rho\left(g^{-1}\right)\right]^T \widehat{\omega} \tag{1.19}$$

satisfies (1.18). Indeed,

$$\begin{aligned} \langle \rho^* g(\widehat{\omega}), \rho(g)\mathbf{v} \rangle &= \rho^* g(\widehat{\omega})\,(\rho(g)\mathbf{v}) \\ &= \left[\rho\left(g^{-1}\right)\right]^T \widehat{\omega}\,[\rho(g)\mathbf{v}] \\ &= \widehat{\omega}\left(\rho\left(g^{-1}\right)\rho(g)\mathbf{v}\right) \\ &= \widehat{\omega}(\mathbf{v}) = \langle \widehat{\omega}, \mathbf{v} \rangle. \end{aligned}$$

Here we used the following definition of transpose: given a linear map $f : V \to W$, its transpose map $f^T : W^* \to V^*$ is defined as $f^T(\widehat{\omega})(\mathbf{v}) = \widehat{\omega}(f(\mathbf{v}))$. In light of this, we can also write (1.19) as

$$\rho^*(g)(\widehat{\omega})(\mathbf{v}) = \left[\rho\left(g^{-1}\right)\right]^T \widehat{\omega}(\mathbf{v}) = \widehat{\omega}\left(\rho\left(g^{-1}\right)\mathbf{v}\right). \tag{1.20}$$

Now, if V and W are representations of G, then so is $\mathrm{Hom}(V,W)$. In order to see this, we shall use the fact that

$$\mathrm{Hom}(V,W) \cong V^* \otimes W. \tag{1.21}$$

Note here that both V and W are finite dimensional complex vector spaces. Consider the group homomorphisms $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$. Now, the group homomorphism associated with dual representation on V^* of G is given by $\rho^* : G \to GL(V^*)$. Note that for $\widehat{\omega} \in V^*$, one has $\widehat{\omega}(\mathbf{e}_i) = \omega_i$, and for $\mathbf{v} \in V$, $\widehat{\alpha}^i(\mathbf{v}) = v^i$, where $\{\mathbf{e}_i\}_{i=1}^m$ is a basis for V and $\{\widehat{\alpha}^i\}_{i=1}^m$ is the dual basis for V^*. Note that $\widehat{\alpha}^i(\mathbf{e}_j) = \delta^i{}_j$.

Given $\varphi \in \mathrm{Hom}(V,W)$, define $\widetilde{g} : \mathrm{Hom}(V,W) \to V^* \otimes W$ by

$$\widetilde{g}(\varphi) = \sum_{i=1}^m \widehat{\alpha}^i \otimes \varphi(\mathbf{e}_i). \tag{1.22}$$

On the other hand, define $\widetilde{f} : V^* \otimes W \to \mathrm{Hom}(V,W)$ by

$$\widetilde{f}(\widehat{\kappa} \otimes \mathbf{w})(\mathbf{v}) = \widehat{\kappa}(\mathbf{v})\,\mathbf{w}, \tag{1.23}$$

where $\widehat{\kappa} \in V^*$, $\mathbf{v} \in V$, $\mathbf{w} \in W$. Then observe that $\widetilde{f}$ and $\widetilde{g}$ are inverses of each other. In fact,

$$\begin{aligned}
\widetilde{f}\left(\widetilde{g}\left(\varphi\right)\right)\left(\mathbf{v}\right) &= \widetilde{f}\left(\sum_{i=1}^{m} \widehat{\alpha}^i \otimes \varphi\left(\mathbf{e}_i\right)\right)\left(\mathbf{v}\right) \\
&= \sum_{i=1}^{m} \widetilde{f}\left(\widehat{\alpha}^i \otimes \varphi\left(\mathbf{e}_i\right)\right)\left(\mathbf{v}\right) \\
&= \sum_{i=1}^{m} \widehat{\alpha}^i\left(\mathbf{v}\right)\varphi\left(\mathbf{e}_i\right) \\
&= \sum_{i=1}^{m} v^i\varphi\left(\mathbf{e}_i\right) \\
&= \varphi\left(\sum_{i=1}^{m} v^i\mathbf{e}_i\right) \\
&= \varphi\left(\mathbf{v}\right).
\end{aligned}$$

Therefore,

$$\widetilde{f} \circ \widetilde{g} = \mathbb{1}_{\mathrm{Hom}(V,W)}\,. \tag{1.24}$$

Now, for a given $\widehat{\kappa} \otimes \mathbf{w} \in V^* \otimes W$,

$$\begin{aligned}
\widetilde{g}\left(\widetilde{f}\left(\widehat{\kappa} \otimes \mathbf{w}\right)\right) &= \sum_{i=1}^{m} \widehat{\alpha}^i \otimes \widetilde{f}\left(\widehat{\kappa} \otimes \mathbf{w}\right)\left(\mathbf{e}_i\right) \\
&= \sum_{i=1}^{m} \widehat{\alpha}^i \otimes \widehat{\kappa}\left(\mathbf{e}_i\right)\mathbf{w} \\
&= \sum_{i=1}^{m} \widehat{\kappa}\left(\mathbf{e}_i\right)\widehat{\alpha}^i \otimes \mathbf{w} \\
&= \sum_{i=1}^{m} \kappa_i\widehat{\alpha}^i \otimes \mathbf{w} \\
&= \widehat{\kappa} \otimes \mathbf{w}.
\end{aligned}$$

Therefore,

$$\widetilde{g} \circ \widetilde{f} = \mathbb{1}_{V^* \otimes W}\,. \tag{1.25}$$

(1.24) and (1.25) together imply that $\mathrm{Hom}\left(V, W\right) \cong V^* \otimes W$. We now define the representation of G on $\mathrm{Hom}(V, W)$ via the representation of G on $V^* \otimes W$. In fact, G acts on $V^* \otimes W$ via the map $\rho^* \otimes \sigma : G \to \mathrm{GL}\left(V^* \otimes W\right)$, so that $\left(\rho^* \otimes \sigma\right)g\left(\widehat{\kappa} \otimes \mathbf{w}\right) \in V^* \otimes W$. Then via $\widetilde{f} : V^* \otimes W \to \mathrm{Hom}(V, W)$, one has $\widetilde{f}\left(\left(\rho^* \otimes \sigma\right)g\left(\widehat{\kappa} \otimes \mathbf{w}\right)\right) \in \mathrm{Hom}(V, W)$. This is, by definition, the representation of G on $\mathrm{Hom}(V, W)$. In other words, $\gamma : G \to \mathrm{GL}\left(\mathrm{Hom}(V, W)\right)$ is defined by

$$\begin{aligned}
\gamma\left(g\right)\left(\widetilde{f}\left(\widehat{\kappa} \otimes \mathbf{w}\right)\right)\left(\mathbf{v}\right) &= \widetilde{f}\left(\left(\rho^* \otimes \sigma\right)g\left(\widehat{\kappa} \otimes \mathbf{w}\right)\right)\left(\mathbf{v}\right) \\
&= \widetilde{f}\left(\rho^*\left(g\right)\widehat{\kappa} \otimes \sigma(g)\mathbf{w}\right)\left(\mathbf{v}\right) \\
&= \left(\rho^*\left(g\right)\widehat{\kappa}\right)\left(\mathbf{v}\right)\sigma(g)\mathbf{w} \\
&= \widehat{\kappa}\left(\rho\left(g^{-1}\right)\mathbf{v}\right)\sigma(g)\mathbf{w} \\
&= \sigma\left(g\right)\left(\widehat{\kappa}\left(\rho\left(g^{-1}\right)\mathbf{v}\right)\mathbf{w}\right).
\end{aligned} \tag{1.26}$$

Now, let us write $\widetilde{f}\left(\widehat{\kappa} \otimes \mathbf{w}\right) = \varphi \in \mathrm{Hom}(V, W)$. So we have

$$\varphi\left(\mathbf{v}\right) = \widetilde{f}\left(\widehat{\kappa} \otimes \mathbf{w}\right)\left(\mathbf{v}\right) = \widehat{\kappa}\left(\mathbf{v}\right)\mathbf{w}. \tag{1.27}$$

As a result,

$$\varphi\left(\rho\left(g^{-1}\right)\mathbf{v}\right) = \widehat{\kappa}\left(\rho\left(g^{-1}\right)\mathbf{v}\right)\mathbf{w}. \tag{1.28}$$

(1.26) and (1.28) together imply that

$$(\gamma\,(g)\,\varphi)\,(\mathbf{v}) = \sigma\,(g)\,\left(\varphi\left(\rho\left(g^{-1}\right)\mathbf{v}\right)\right). \tag{1.29}$$

(1.29) can be expressed by means of the commutativity of the following diagram:

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "\rho(g)"'] & W \arrow[d, "\sigma(g)"] \\ V \arrow[r, "\gamma(g)\varphi"'] & W \end{tikzcd}$$

Proposition 1.1

Given representations $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$ of a finite group G, $\widetilde{f} : V^* \otimes W \to \mathrm{Hom}(V, W)$ is an isomorphism of representations.

Proof. We have already shown that $\widetilde{f} : V^* \otimes W \to \mathrm{Hom}(V, W)$ is an isomorphism of vector spaces. We now need to show that $\widetilde{f}$ is a map between the representations $\rho^* \otimes \sigma$ and γ. For that purpose, we need to show the commutativity of the following square:

$$\begin{tikzcd} V^* \otimes W \arrow[r, "\widetilde{f}"] \arrow[d, "(\rho^*\otimes\sigma)(g)"'] & \mathrm{Hom}(V,W) \arrow[d, "\gamma(g)"] \\ V^* \otimes W \arrow[r, "\widetilde{f}"'] & \mathrm{Hom}(V,W) \end{tikzcd} \tag{1.30}$$

Given any $\widehat{\kappa} \in V^*$ and $\mathbf{w} \in W$, we need to show that

$$\gamma(g) \circ \widetilde{f}\,(\widehat{\kappa} \otimes \mathbf{w}) = \widetilde{f} \circ (\rho^* \otimes \sigma)\,(g)\,(\widehat{\kappa} \otimes \mathbf{w})\,. \tag{1.31}$$

Both sides of (1.31) are in $\mathrm{Hom}(V, W)$. In order to show their equality, we need to show the equality of them evaluated at an arbitrary $\mathbf{v} \in V$. So, we are going to show that

$$\left[\gamma(g) \circ \widetilde{f}\,(\widehat{\kappa} \otimes \mathbf{w})\right](\mathbf{v}) = \left[\widetilde{f} \circ (\rho^* \otimes \sigma)\,(g)\,(\widehat{\kappa} \otimes \mathbf{w})\right](\mathbf{v})\,. \tag{1.32}$$

The RHS of (1.32) is

$$\begin{aligned} \mathrm{RHS} &= \left[\widetilde{f} \circ (\rho^* \otimes \sigma)\,(g)\,(\widehat{\kappa} \otimes \mathbf{w})\right](\mathbf{v}) \\ &= \left[\widetilde{f}\,(\rho^*(g)\widehat{\kappa} \otimes \sigma(g)\mathbf{w})\right](\mathbf{v}) \\ &= (\rho^*(g)\widehat{\kappa})\,(\mathbf{v}) \cdot \sigma(g)\,(\mathbf{w}) && [\,\cdot \text{ is the scalar multiplication in } W] \\ &= \widehat{\kappa}\left(\rho\,(g)^{-1}\,\mathbf{v}\right) \cdot \sigma(g)\,(\mathbf{w}) \end{aligned}$$

Before computing the LHS of (1.32), let us quickly recall the definition of γ. $\gamma : G \to \mathrm{GL}\,(\mathrm{Hom}(V, W))$ is defined so that the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "\rho(g)"'] & W \arrow[d, "\sigma(g)"] \\ V \arrow[r, "\gamma(g)\varphi"'] & W \end{tikzcd}$$

In other words,

$$\gamma(g)(\varphi) = \sigma(g) \circ \varphi \circ \rho(g)^{-1}. \tag{1.33}$$

Now, the LHS of (1.32) is

$$\begin{aligned}
\text{LHS} &= \left[\gamma(g) \circ \widetilde{f}(\widehat{\kappa} \otimes \mathbf{w})\right](\mathbf{v}) \\
&= \left[\sigma(g) \circ \left(\widetilde{f}(\widehat{\kappa} \otimes \mathbf{w})\right) \circ \rho(g)^{-1}\right](\mathbf{v}) \\
&= \sigma(g)\left(\widetilde{f}(\widehat{\kappa} \otimes \mathbf{w})\left(\rho(g)^{-1}\mathbf{v}\right)\right) \\
&= \sigma(g)\left(\widehat{\kappa}\left(\rho(g)^{-1}\mathbf{v}\right) \cdot \mathbf{w}\right) && [\,\cdot \text{ is the scalar multiplication in } W] \\
&= \widehat{\kappa}\left(\rho(g)^{-1}\mathbf{v}\right) \cdot \sigma(g)(\mathbf{w}).
\end{aligned}$$

Therefore, LHS = RHS, so (1.32) holds. As a result, (1.30) commutes, and hence, $\widetilde{f}$ is a G-linear map, as required. ■

§1.4 Complete reducibility

Definition 1.7 (Hermitian inner product)**.** If V is a complex vector space, then a **Hermitian inner product** is a positive definite sesquilinear map $H : V \times V \to \mathbb{C}$ that satisfies the following:

(i) $H(a\mathbf{u} + b\mathbf{v}, \mathbf{w}) = \bar{a}H(\mathbf{u}, \mathbf{w}) + \bar{b}H(\mathbf{v}, \mathbf{w})$ and $H(\mathbf{w}, a\mathbf{u} + b\mathbf{v}) = aH(\mathbf{w}, \mathbf{u}) + bH(\mathbf{w}, \mathbf{v})$ for all $a, b \in \mathbb{C}$, $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$.

(ii) $H(\mathbf{u}, \mathbf{v}) = \overline{H(\mathbf{v}, \mathbf{u})}$, for all $\mathbf{u}, \mathbf{v} \in V$.

(iii) $H(\mathbf{u}, \mathbf{u}) > 0$, for every $\mathbf{u} \in V \setminus \{\mathbf{0}\}$ (positive definite).

If $W \subset V$ is a vector subspace of a complex vector space with a Hermitian inner product, we define the following subspace:

$$W^{\perp} = \{\mathbf{v} \in V \mid H(\mathbf{v}, \mathbf{w}) = 0, \text{ for all } \mathbf{w} \in W\}. \tag{1.34}$$

If V is a finite dimensional complex vector space, then we can write $V = W \oplus W^{\perp}$, i.e. $W^{\perp}$ is the the orthogonal complement of W. We also say that $W^{\perp}$ is the complementary subspace of V.

Definition 1.8. A Hermitian inner product H on a finite dimensional representation V of a finite group G ($\rho : G \to \mathrm{GL}(V)$) is said to be **preserved under group action** if

$$H(\rho(g)\mathbf{u}, \rho(g)\mathbf{w}) = H(\mathbf{u}, \mathbf{w}) \tag{1.35}$$

for all $g \in G$ and $\mathbf{u}, \mathbf{w} \in V$. H is then called a G**-invariant** Hermitian inner product.

If H is a G-invariant Hermitian inner product on a finite dimensional representation V of a finite group G, then we have

$$\begin{aligned}
H(\rho(g)\mathbf{v}, \mathbf{w}) &= H\left(\rho(g)\mathbf{v}, \rho(g)\rho\left(g^{-1}\right)\mathbf{w}\right) \\
&= H\left(\mathbf{v}, \rho\left(g^{-1}\right)\mathbf{w}\right).
\end{aligned} \tag{1.36}$$

Lemma 1.2

If $H : V \times V \to \mathbb{C}$ is a G-invariant Hermitian inner product on a finite dimensional representation V of a finite group G and $W \subset V$ is a subrepresentation, then $W^{\perp}$ is a G-invariant complement to W.

Proof. Since we are dealing with finite dimensional complex vector spaces, $W^\perp$ is a complement to W. It, therefore, suffices to show that $W^\perp$ is G-invariant.

Suppose $g \in G$, $\mathbf{u} \in W^\perp$, and $\mathbf{w} \in W$. Let us denote the group homomorphism associated with the finite dimensional complex representation by $\rho : G \to \mathrm{GL}(V)$. Since the Hermitian inner product $H : V \times V \to \mathbb{C}$ is G-invariant, one has

$$H(\rho(g)\mathbf{u}, \mathbf{w}) = H(\mathbf{u}, \rho(g^{-1})\mathbf{w}). \tag{1.37}$$

Since W is a subrepresentation of V, one must have $\rho(g^{-1})\mathbf{w} \in W$ for any $g \in G$ and $\mathbf{w} \in W$. Hence, $H(\mathbf{u}, \rho(g^{-1})\mathbf{w}) = 0$ in (1.37) leads to

$$H(\rho(g)\mathbf{u}, \mathbf{w}) = 0 \tag{1.38}$$

This is true for all $\mathbf{w} \in W$. Therefore, from the definition of $W^\perp$, one then must have $\rho(g)\mathbf{u} \in W^\perp$ for any $g \in G$, which then implies that the subspace $W^\perp$ is G-invariant. ■

Proposition 1.3

If V is a complex representation of a finite group G, then there is a G-invariant Hermitian inner product on V.

Proof. Pick a Hermitian inner product $H_0 : V \times V \to \mathbb{C}$ on the finite dimensional complex vector space V with respect to which a given basis of V is orthonormal, i.e., choose a basis $\{\mathbf{e}_i\}_{i=1}^m$ of V and define $H_0(\mathbf{e}_i, \mathbf{e}_j) = \delta_{ij}$ and extend H_0 to all of $V \times V$ sesquilinearly. Given $\mathbf{v} = \sum_{i=1}^m v^i \mathbf{e}_i$ and $\mathbf{w} = \sum_{j=1}^m w^j \mathbf{e}_j$, we then have

$$H_0(\mathbf{v}, \mathbf{w}) = H_0\left(\sum_{i=1}^m v^i \mathbf{e}_i, \sum_{j=1}^m w^j \mathbf{e}_j\right) = \sum_{i=1}^m \overline{v^i} w^i. \tag{1.39}$$

Then define a new Hermitian inner product $H_1 : V \times V \to \mathbb{C}$ by averaging over all of G via representation $\rho : G \to \mathrm{GL}(V)$:

$$H_1(\mathbf{v}, \mathbf{w}) = \frac{1}{|G|} \sum_{g \in G} H_0(\rho(g)\mathbf{v}, \rho(g)\mathbf{w}). \tag{1.40}$$

Using the Hermitian inner product properties of H_0, one can verify that H_1 is also a Hermitian inner product on V. Additionally,

$$\begin{aligned} H_1(\rho(h)\mathbf{v}, \rho(h)\mathbf{w}) &= \frac{1}{|G|} \sum_{g \in G} H_0(\rho(g)\rho(h)\mathbf{v}, \rho(g)\rho(h)\mathbf{w}) \\ &= \frac{1}{|G|} \sum_{g \in G} H_0(\rho(gh)\mathbf{v}, \rho(gh)\mathbf{w}) \\ &= \frac{1}{|G|} \sum_{g' \in G} H_0(\rho(g')\mathbf{v}, \rho(g')\mathbf{w}) \quad (\text{where } g' = gh) \\ &= H_1(\mathbf{v}, \mathbf{w}). \end{aligned} \tag{1.41}$$

Then (1.41) implies that the Hermitian inner product $H_1 : V \times V \to \mathbb{C}$ defined by (1.40) on V is G-invariant. ■

Corollary 1.4

If W is a subrepresentation of a finite dimensional complex representation V of a finite group G, then there exists a complementary invariant subspace $W^\perp$ of V so that $V = W \oplus W^\perp$.

Proof. Given that V is a complex representation of a finite group G, there is a G-invariant Hermitian inner product on V by Proposition 1.3. Now, if W is a subrepresentation of V, then by Lemma 1.2, the complementary subspace $W^\perp$ is G-invariant, i.e., $V = W \oplus W^\perp$. ■

Corollary 1.5 (Maschke's theorem)
Any complex representation of a finite group can be expressed as a direct sum of irreducible representations.

Remark 1.1. The property of a representation being expressed as a direct sum of irreducibles is called complete reducibility (semisimplicity). Maschke's theorem tells us that any complex representation of a finite group is semisimple. The additive group $\mathbb{R}$, being an infinite group, doesn't have this property; for example, the representation

$$a \mapsto \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$$

is not semisimple.

The extent to which the decomposition of an arbitrary complex representation into a direct sum of irreducibles is unique is one of the consequences of the following.

Lemma 1.6 (Schur's lemma)
Recall that $\mathrm{Hom}_G(V, W)$ is the vector space of G-linear maps between two finite dimensional complex representations V and W of the finite group G. Suppose V and W are irreducible complex representations of G. Then

(a) Every element of $\mathrm{Hom}_G(V, W)$ is either 0 or an isomorphism.

(b) $\dim_{\mathbb{C}} \mathrm{Hom}_G(V, W) = 0$ or 1.

Proof. (a) Let $\varphi : V \to W$ be a non-zero G-linear map. We have verified in (1.1) that $\mathrm{Ker}\,\varphi \subseteq V$ is a G-invariant subspace of V. Since V is irreducible, by hypothesis, one has

$$\mathrm{Ker}\,\varphi = \{\mathbf{0}\}\,, \tag{1.42}$$

because $\mathrm{Ker}\,\varphi \neq V$, as φ is chosen to be nonzero.

We also know from (1.2) that $\mathrm{im}\,\varphi \subseteq W$ is a G-invariant subspace of W, i.e., $\mathrm{Im}\varphi$ is a subrepresentation of W. Since W is also irreducible, by hypothesis, one must have

$$\mathrm{im}\,\varphi = W, \tag{1.43}$$

because $\mathrm{im}\,\varphi \neq \{\mathbf{0}\}$ as φ is chosen to be nonzero.

Now, $\mathrm{Ker}\,\varphi = \{\mathbf{0}\}$ and $\mathrm{im}\,\varphi = W$ together imply that $\varphi : V \to W$ is a bijective linear map from V to W, i.e., φ is an isomorphism between vector spaces.

(b) Suppose $\varphi_1, \varphi_2 \in \mathrm{Hom}_G(V, W)$ with both being nonzero. Then by (a), φ_1 and φ_2 are both isomorphisms. Since $\varphi_1^{-1} : W \to V$ and $\varphi_2 : V \to W$, one can compose them to obtain $\varphi = \varphi_1^{-1} \circ \varphi_2 \in \mathrm{Hom}_G(V, V)$.

Now, $\varphi : V \to V$ is a linear operator on the finite dimensional complex vector space V. Also, since $\mathbb{C}$ is algebraically closed, $\det(\varphi - \lambda\, \mathbb{1}_V) = 0$ has a solution (here $\varphi - \lambda\, \mathbb{1}_V$ is considered a square matrix) which implies that $\mathrm{Ker}(\varphi - \lambda\, \mathbb{1}_V) \neq \{\mathbf{0}\}$, i.e., $\varphi - \lambda\, \mathbb{1}_V$ is not an isomorphism belonging to the vector space $\mathrm{Hom}_G(V, V)$. Then, by (a), one concludes that $\varphi - \lambda\, \mathbb{1}_V$ must be the 0-map in $\mathrm{Hom}_G(V, V)$, i.e.,

$$\varphi = \varphi_1^{-1} \circ \varphi_2 = \lambda\, \mathbb{1}_V\,.$$

In other words, $\varphi_2 = \lambda \varphi_1$. Since this is true for any pair of G-linear maps $\varphi_1, \varphi_2 \in \mathrm{Hom}_G(V, W)$, we have $\dim_{\mathbb{C}} \mathrm{Hom}_G(V, W) = 1$.

∎

Lemma 1.7

Suppose V_1, V_2, W are finite dimensional complex representation of the finite group G. Then one has the following vector space isomorphisms:

$$\mathrm{Hom}_G(V_1 \oplus V_2, W) \cong \mathrm{Hom}_G(V_1, W) \oplus \mathrm{Hom}_G(V_2, W),$$
$$\mathrm{Hom}_G(W, V_1 \oplus V_2) \cong \mathrm{Hom}_G(W, V_1) \oplus \mathrm{Hom}_G(W, V_2).$$

Proof. Following are the required linear maps that can easily be verified to be isomorphisms:

$$\begin{aligned} s : \mathrm{Hom}_G(V_1, W) \oplus \mathrm{Hom}_G(V_2, W) &\to \mathrm{Hom}_G(V_1 \oplus V_2, W), \\ s(\varphi_1, \varphi_2)(\mathbf{v}_1, \mathbf{v}_2) &= \varphi_1(\mathbf{v}_1) + \varphi_2(\mathbf{v}_2). \end{aligned} \tag{1.44}$$

$$\begin{aligned} u : \mathrm{Hom}_G(V_1 \oplus V_2, W) &\to \mathrm{Hom}_G(V_1, W) \oplus \mathrm{Hom}_G(V_2, W) \\ u(\varphi) &= (\varphi \circ i_1, \varphi \circ i_2), \end{aligned} \tag{1.45}$$

where $i_1 : V_1 \to V_1 \oplus V_2$ and $i_2 : V_2 \to V_1 \oplus V_2$ are the canoncial inclusions defined by

$$i_1(\mathbf{v}_1) = (\mathbf{v}_1, \mathbf{0}_{V_2}) \quad \text{and} \quad i_2(\mathbf{v}_2) = (\mathbf{0}_{V_1}, \mathbf{v}_2).$$

Now, one can check that $u \circ s = \mathbb{1}_{\mathrm{Hom}_G(V_1,W) \oplus \mathrm{Hom}_G(V_2,W)}$ and $s \circ u = \mathbb{1}_{\mathrm{Hom}_G(V_1 \oplus V_2, W)}$. Indeed,

$$\begin{aligned} (u \circ s)(\varphi_1, \varphi_2) &= u(s(\varphi_1, \varphi_2)) \\ &= (s(\varphi_1, \varphi_2) \circ i_1, s(\varphi_1, \varphi_2) \circ i_2). \end{aligned}$$

Now,

$$\begin{aligned} (s(\varphi_1, \varphi_2) \circ i_1)(\mathbf{v}_1) &= s(\varphi_1, \varphi_2)(i_1(\mathbf{v}_1)) \\ &= s(\varphi_1, \varphi_2)(\mathbf{v}_1, \mathbf{0}_{V_2}) \\ &= \varphi_1(\mathbf{v}_1) + \varphi_2(\mathbf{0}_{V_2}) \\ &= \varphi_1(\mathbf{v}_1). \end{aligned}$$

Therefore, $s(\varphi_1, \varphi_2) \circ i_1 = \varphi_1$. Similarly, $s(\varphi_1, \varphi_2) \circ i_2 = \varphi_2$. Hence,

$$(u \circ s)(\varphi_1, \varphi_2) = (s(\varphi_1, \varphi_2) \circ i_1, s(\varphi_1, \varphi_2) \circ i_2) = (\varphi_1, \varphi_2).$$

So we have

$$u \circ s = \mathbb{1}_{\mathrm{Hom}_G(V_1,W) \oplus \mathrm{Hom}_G(V_2,W)}. \tag{1.46}$$

On the other hand, given $\varphi \in \mathrm{Hom}_G(V_1 \oplus V_2, W)$,

$$\begin{aligned} [(s \circ u)(\varphi)](\mathbf{v}_1, \mathbf{v}_2) &= [s(\varphi \circ i_1, \varphi \circ i_2)](\mathbf{v}_1, \mathbf{v}_2) \\ &= (\varphi \circ i_1)(\mathbf{v}_1) + (\varphi \circ i_2)(\mathbf{v}_2) \\ &= \varphi(\mathbf{v}_1, \mathbf{0}_{V_2}) + \varphi(\mathbf{0}_{V_1}, \mathbf{v}_2) \\ &= \varphi(\mathbf{v}_1, \mathbf{v}_2). \end{aligned}$$

Therefore,

$$s \circ u = \mathbb{1}_{\mathrm{Hom}_G(V_1 \oplus V_2, W)}. \tag{1.47}$$

So $s : \mathrm{Hom}_G(V_1, W) \oplus \mathrm{Hom}_G(V_2, W) \to \mathrm{Hom}_G(V_1 \oplus V_2, W)$ is an isomorphism.

Now consider the following linear maps

$$\begin{aligned} t : \mathrm{Hom}_G(W, V_1) \oplus \mathrm{Hom}_G(W, V_2) &\to \mathrm{Hom}_G(W, V_1 \oplus V_2) \\ t(\varphi_1, \varphi_2)(\mathbf{w}) &= (\varphi_1(\mathbf{w}), \varphi_2(\mathbf{w})). \end{aligned} \tag{1.48}$$

$$\begin{aligned} v : \mathrm{Hom}_G(W, V_1 \oplus V_2) &\to \mathrm{Hom}_G(W, V_1) \oplus \mathrm{Hom}_G(W, V_2) \\ v(\varphi) &= (q_1 \circ \varphi, q_2 \circ \varphi), \end{aligned} \tag{1.49}$$

where $q_1 : V_1 \oplus V_2 \to V_1$ and $q_2 : V_1 \oplus V_2 \to V_2$ are the canonical projections, defined by

$$q_1(\mathbf{v}_1, \mathbf{v}_2) = \mathbf{v}_1 \quad \text{and} \quad q_2(\mathbf{v}_1, \mathbf{v}_2) = \mathbf{v}_2.$$

Now, one can check that $v \circ t = \mathbb{1}_{\mathrm{Hom}_G(W,V_1)\oplus\mathrm{Hom}_G(W,V_2)}$ and $t \circ v = \mathbb{1}_{\mathrm{Hom}_G(W,V_1\oplus V_2)}$. Indeed,

$$\begin{aligned} (v \circ t)(\varphi_1, \varphi_2) &= v(t(\varphi_1, \varphi_2)) \\ &= (q_1 \circ t(\varphi_1, \varphi_2), q_2 \circ t(\varphi_1, \varphi_2)). \end{aligned}$$

Now,

$$\begin{aligned} (q_1 \circ t(\varphi_1, \varphi_2))(\mathbf{w}) &= q_1[t(\varphi_1, \varphi_2)\mathbf{w}] \\ &= q_1(\varphi_1(\mathbf{w}), \varphi_2(\mathbf{w})) \\ &= \varphi_1(\mathbf{w}). \end{aligned}$$

Therefore, $q_1 \circ t(\varphi_1, \varphi_2) = \varphi_1$. Similarly, $q_2 \circ t(\varphi_1, \varphi_2) = \varphi_2$. Hence,

$$(v \circ t)(\varphi_1, \varphi_2) = (q_1 \circ t(\varphi_1, \varphi_2), q_2 \circ t(\varphi_1, \varphi_2)) = (\varphi_1, \varphi_2).$$

So we have

$$v \circ t = \mathbb{1}_{\mathrm{Hom}_G(W,V_1)\oplus\mathrm{Hom}_G(W,V_2)}. \tag{1.50}$$

On the other hand, given $\varphi \in \mathrm{Hom}_G(W, V_1 \oplus V_2)$, let $\varphi(\mathbf{w}) = (\mathbf{v}_1, \mathbf{v}_2)$. Then

$$\begin{aligned} [(t \circ v)(\varphi)](\mathbf{w}) &= t(q_1 \circ \varphi, q_2 \circ \varphi)(\mathbf{w}) \\ &= ((q_1 \circ \varphi)(\mathbf{w}), (q_2 \circ \varphi)(\mathbf{w})) \\ &= (\mathbf{v}_1, \mathbf{v}_2) = \varphi(\mathbf{w}). \end{aligned}$$

Therefore,

$$t \circ v = \mathbb{1}_{\mathrm{Hom}_G(W,V_1\oplus V_2)}. \tag{1.51}$$

So $t : \mathrm{Hom}_G(W, V_1) \oplus \mathrm{Hom}_G(W, V_2) \to \mathrm{Hom}_G(W, V_1 \oplus V_2)$ is an isomorphism. ■

Now, let G be a finite group and V be a finite dimensional complex representation of G. Since V is a direct sum of irreducible representations by Maschke's theorem, up to isomorphism we can group together the isomorphic representations and say that

$$V \cong V_1^{r_1} \oplus \cdots \oplus V_m^{r_m} \tag{1.52}$$

Here $V_i^{r_i}$ is the shorthand for r_i fold direct sum of V_i with itself.

$$V_i^{r_i} = \underbrace{V_i \oplus V_i \oplus \cdots \oplus V_i}_{r_i\text{-fold direct sum}}. \tag{1.53}$$

Here, for distinct i and j, V_i and V_j are non-isomorphic, and the integers $r_i \geq 1$.

Remark 1.2. While grouping together in (1.52), we are grouping isomorphic representations together, NOT isomorphic vector spaces. V_1 and V_2 may be isomorphic as vector spaces, but we don't group them together unless they are isomorphic representations. In other words, if $\rho : G \to \mathrm{GL}(V)$ is the said representation of G into V, we group two irreducible subrepresentations W_1 and W_2 together while writing (1.52) if there exists a vector space isomorphism $\psi : W_1 \to W_2$ such that the following diagram commutes for every $g \in G$:

$$\begin{tikzcd} W_1 \arrow[r, "\psi"] \arrow[d, "\rho(g)\big|_{W_1}"'] & W_2 \arrow[d, "\rho(g)\big|_{W_2}"] \\ W_1 \arrow[r, "\psi"'] & W_2 \end{tikzcd}$$

When we say V_i and V_j are not isomorphic for $i \neq j$ in (1.52), we mean that they are not isomorphic as representations, i.e. there is no isomorphism in $\mathrm{Hom}_G(V_i, V_j)$. In principle, they can be isomorphic as vector spaces, but that's not our concern here.

Proposition 1.8
In (1.52), $r_i = \dim_{\mathbb{C}} \mathrm{Hom}_G(V_i, V) = \dim_{\mathbb{C}} \mathrm{Hom}_G(V, V_i)$.

Proof. By Lemma 1.7,

$$\mathrm{Hom}_G(V_i, V) \cong \mathrm{Hom}_G\left(V_i, \bigoplus_{j=1}^{m} V_j^{r_j}\right) \cong \bigoplus_{j=1}^{m} \mathrm{Hom}_G\left(V_i, V_j^{r_j}\right). \tag{1.54}$$

But $\mathrm{Hom}_G\left(V_i, V_j^{r_j}\right)$ is

$$\mathrm{Hom}_G\left(V_i, V_j^{r_j}\right) = \mathrm{Hom}_G\left(V_i, \underbrace{V_j \oplus \cdots \oplus V_j}_{r_j\text{-fold direct sum}}\right) \cong \underbrace{\mathrm{Hom}_G(V_i, V_j) \oplus \cdots \oplus \mathrm{Hom}_G(V_i, V_j)}_{r_j\text{-fold direct sum}}. \tag{1.55}$$

Since V_i's are pairwise non-isomorphic for $j \neq i$, we have $\mathrm{Hom}_G(V_i, V_j) = \{\mathbf{0}\}$, so that

$$\dim_{\mathbb{C}} \mathrm{Hom}_G(V_i, V_j) = 0 \quad \text{and} \quad \dim_{\mathbb{C}} \mathrm{Hom}_G(V_i, V_j) = 1. \tag{1.56}$$

So we have

$$\begin{aligned} \dim_{\mathbb{C}} \mathrm{Hom}_G(V_i, V) &= \dim_{\mathbb{C}}\left(\bigoplus_{j=1}^{m} \mathrm{Hom}_G\left(V_i, V_j^{r_j}\right)\right) \\ &= \sum_{j=1}^{m} \dim_{\mathbb{C}} \mathrm{Hom}_G\left(V_i, V_j^{r_j}\right) \\ &= \dim_{\mathbb{C}} \mathrm{Hom}_G(V_i, V_i^{r_i}) \\ &= \dim_{\mathbb{C}}\left(\underbrace{\mathrm{Hom}_G(V_i, V_i) \oplus \cdots \oplus \mathrm{Hom}_G(V_i, V_i)}_{r_i\text{-fold direct sum}}\right) \\ &= \underbrace{1 + 1 + \cdots + 1}_{r_i\text{-fold sum}} \\ &= r_i. \end{aligned} \tag{1.57}$$

Similarly, $\dim_{\mathbb{C}} \mathrm{Hom}_G(V, V_i) = r_i$. ■

Proposition 1.9
The decomposition (1.52) is unique up to replacement of each V_i by an isomorphic representation.

Proof. Suppose

$$V \cong V_1^{r_1} \oplus \cdots \oplus V_m^{r_m} \cong W_1^{s_1} \oplus \cdots \oplus W_n^{s_n} \tag{1.58}$$

are two decompositions into non-isomorphic irreducible representations of G. By Proposition 1.8, for $i_0 \in \{1, 2, \ldots, m\}$,

$$\begin{aligned} r_{i_0} &= \dim_{\mathbb{C}} \mathrm{Hom}_G \left(V_{i_0}, V\right) \\ &= \dim_{\mathbb{C}} \mathrm{Hom}_G \left(V_{i_0}, \bigoplus_{j=1}^{n} W_j^{s_j}\right) \\ &= \dim_{\mathbb{C}} \left(\bigoplus_{j=1}^{n} \mathrm{Hom}_G \left(V_{i_0}, W_j^{s_j}\right)\right) \\ &= \sum_{j=1}^{n} s_j \dim \mathrm{Hom}_G \left(V_{i_0}, W_j\right). \end{aligned} \tag{1.59}$$

Since $r_{i_0} > 0$, there must exist some $j_0 \in \{1, 2, \ldots, n\}$ such that $\mathrm{Hom}_G (V_{i_0}, W_{j_0}) \neq \{\mathbf{0}\}$, i.e. it is nontrivial. Then by Schur's lemma, $W_{j_0} \cong V_{i_0}$. The j_0 must also be unique because W_j's are pairwise non-isomorphic. In other words, the only nonvanishing contribution in the sum (1.59) is due to the unique value $j = j_0$, for which

$$\dim_{\mathbb{C}} \mathrm{Hom}_G (V_{i_0}, W_{j_0}) = 1 \quad \text{and} \quad \dim_{\mathbb{C}} \mathrm{Hom}_G (V_{i_0}, W_j) = 0 \text{ for } j \neq j_0. \tag{1.60}$$

Hence, by (1.59) and (1.60), $r_{i_0} = s_{j_0}$. Thus we have an injection $\sigma : \{1, 2, \ldots, m\} \to \{1, 2, \ldots, n\}$ such that $V_{i_0} \cong W_{j_0} = W_{\sigma(i_0)}$ and $r_{i_0} = s_{j_0} = s_{\sigma(i_0)}$ for each i_0.

In a similar manner, interchanging V_i and W_j throughout above, we have an injection $\tau : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, m\}$ such that $W_{j_0} \cong V_{\tau(j_0)}$ and $s_{j_0} = r_{\tau(i_0)}$ for each j_0. The first injection σ implies that $m \leq n$. The latter injection τ gives $n \leq m$. Therefore, $m = n$, and σ and τ are permutations, i.e. $\sigma \in \mathfrak{S}_n$. Hence, (1.52) is unique up to replacement of each V_{i_0} by an isomorphic representation W_{j_0}. ■

Corollary 1.10

The irreducible complex representations of a finite abelian group G are all 1-dimensional.

Proof. Let V be a complex irreducible representation of a finite group G and $\rho : G \to \mathrm{GL}(V)$ be the underlying group homomorphism. Then, for each $g \in G$, the map $\rho(g) : V \to V$ is G-linear:

$$\begin{tikzcd} V \arrow[r, "\rho(g)"] \arrow[d, "\rho(h)"'] & V \arrow[d, "\rho(h)"] \\ V \arrow[r, "\rho(g)"'] & V \end{tikzcd}$$

The diagram above is commutative for all $h \in G$ for a given $g \in G$. Indeed,

$$\rho(g)\rho(h) = \rho(gh) = \rho(hg) = \rho(h)\rho(g).$$

We, therefore, have $\rho(g) \in \mathrm{Hom}_G(V, V)$. By Schur's lemma, $\dim_{\mathbb{C}} \mathrm{Hom}_G(V, V) = 1$, so $\rho(g) = \lambda_g \mathbb{1}_V$ for some $\lambda_g \in \mathbb{C}$.

Now, choose a non-zero vector $\mathbf{v} \in V$ and consider the 1-dimensional subspace

$$\langle \mathbf{v} \rangle = \mathbb{C}\mathbf{v} \subset V,$$

by taking all complex multiples of the nonzero vector $\mathbf{v}$. Observe that $\langle \mathbf{v} \rangle$ is G-invariant. Indeed,

$$\rho(g)\mathbf{v} = \lambda_g \mathbb{1}_V \mathbf{v} = \lambda_g \mathbf{v} \in \langle \mathbf{v} \rangle,$$

i.e. $\langle \mathbf{v} \rangle$ is a G-invariant subspace of V, i.e. a subrepresentation. But V is irreducible by hypothesis. Hence, $\langle \mathbf{v} \rangle = V$. In other words, V is 1-dimensional. ■

Definition 1.9 (Faithful representation)**.** A complex representation V of a finite group G is called **faithful** if the homomorphism $\rho : G \to \mathrm{GL}(V)$ is injective.

Corollary 1.11

If G has a faithful complex irreducible representation, then $Z(G)$ is cyclic.

Proof. Let $\rho : G \to \mathrm{GL}(V)$ be the injective group homomorphism associated with a faithful irreducible complex representation V of a finite group G. Now, let $z \in Z(G)$ so that $zg = gz$ for all $g \in G$. Now consider the map $\rho(z) : V \to V$. Since z commutes with all $g \in G$, the following diagram commutes:

$$\begin{tikzcd}
V \arrow[r, "\rho(z)"] \arrow[d, "\rho(g)"'] & V \arrow[d, "\rho(g)"] \\
V \arrow[r, "\rho(z)"'] & V
\end{tikzcd}$$

Hence, $\rho(z) \in \mathrm{Hom}_G(V, V)$. By Schur's lemma, $\dim_{\mathbb{C}} \mathrm{Hom}_G(V, V) = 1$, so $\rho(z) = \lambda_z \mathbb{1}_V$ for some $\lambda_z \in \mathbb{C}^\times := \mathbb{C} \setminus \{0\}$.

Now, the map $Z(G) \to \mathbb{C}^\times = \mathrm{GL}(\mathbb{C})$ given by $z \mapsto \lambda_z$ is a representation of the subgroup $Z(G)$ of G. Moreover, this representation is faithful, because

$$\begin{aligned}
\lambda_z = \lambda_{z'} &\implies \lambda_z \mathbb{1}_V = \lambda_{z'} \mathbb{1}_V \\
&\implies \rho(z) = \rho(z') \\
&\implies z = z',
\end{aligned}$$

since ρ is injective. Therefore, the map $Z(G) \to \mathbb{C}^\times = \mathrm{GL}(\mathbb{C})$ given by $z \mapsto \lambda_z$ is injective. So $Z(G)$ is isomorphic to a finite subgroup of $\mathbb{C}^\times$. Finite subgroups of the multiplicative group of a field is a cyclic group. Hence, $Z(G)$ is cyclic. ■

One also knows from elementary group theory that every finite abelian group is isomorphic to a direct product of cyclic groups. In other words, if G is a finite abelian group, then we can write G as

$$G = C_{n_1} \times \cdots \times C_{n_r}, \tag{1.61}$$

where each C_{n_i} is a cyclic group of order n_i.

Proposition 1.12

A finite abelian group G has precisely $|G|$-many irreducible complex representations.

Proof. We write G as a direct product of cyclic groups as follows:

$$G = \langle x_1 \rangle \times \cdots \times \langle x_r \rangle, \tag{1.62}$$

where $|\langle x_j \rangle| = n_j$, and x_j generates the cyclic group $\langle x_j \rangle$. Suppose $\rho : G \to \mathbb{C}^\times$ is an irreducible representation of the finite abelian group G (which is 1-dimensional by Corollary 1.10). Let

$$\rho(e_1, \ldots, e_{j-1}, x_j, e_{j+1}, \ldots, e_r) = \lambda_j \in \mathbb{C}^\times, \tag{1.63}$$

where e_k's are the identity elements of the cyclic group $C_{n_k} = \langle x_k \rangle$. Since $x_j^{n_j} = e_j$, and since $\rho : G \to \mathbb{C}^\times$ is a group homomorphism, one must have

$$1 = \rho(e_1, \ldots, e_r) = \rho\left(e_1, \ldots, e_{j-1}, x_j^{n_j}, e_{j+1}, \ldots, e_r\right) = \lambda_j^{n_j}. \tag{1.64}$$

Then $\lambda_j^{n_j} = 1$ gives us that λ_j is a n_j-th root of unity. Also, observe that

$$\rho\left(x_1^{j_1}, \ldots, x_r^{j_r}\right) = \lambda_1^{j_1} \cdots \lambda_r^{j_r}, \tag{1.65}$$

for $1 \leq j_k \leq n_k$ for each k. Thus, the r-tuple $(\lambda_1, \ldots, \lambda_r)$ completely determines the homomorphism $\rho : G \to \mathbb{C}^\times$. There are n_j many n_j-th root of unity, so there are n_j many choices for λ_j. Hence, there are total $n_1 \cdots n_r$ many choices for the r-tuple $(\lambda_1, \ldots, \lambda_r)$. Therefore, there are $n_1 \cdots n_r$ many irreducible representations $\rho : G \to \mathbb{C}^\times$. But

$$|G| = |\langle x_1 \rangle \times \cdots \times \langle x_r \rangle| = \prod_{j=1}^{r} |\langle x_j \rangle| = \prod_{j=1}^{r} n_j. \tag{1.66}$$

Hence, there are $|G|$ many irreducible complex representation of the finite abelian group G. ■

Example 1.1 (Example of finite abelian group representations)**.** (i) Consider the finite abelian group $G = C_2 \times C_2 = \langle x_1 \rangle \times \langle x_2 \rangle$, with $x_1^2 = e_1$ and $x_2^2 = e_2$.[1]

We are concerned with the 2nd roots of unity, namely 1 and -1. There are 4 possible choices for (λ_1, λ_2), they are $(1,1), (1,-1), (-1,1), (-1,-1)$. Corresponding to these 4 choices, there are 4 irreducible representations $\rho_1, \rho_2, \rho_3, \rho_4$. The way these 4 irreducible representations map is illustrates in the following table:

(λ_1, λ_2)	(e_1, e_2)	(x_1, e_2)	(e_1, x_2)	(x_1, x_2)
$\rho_1 \equiv (1,1)$	1	1	1	1
$\rho_2 \equiv (1,-1)$	1	1	-1	-1
$\rho_3 \equiv (-1,1)$	1	-1	1	-1
$\rho_4 \equiv (-1,-1)$	1	-1	-1	1

From this table, we can see that there is no irreducible faithful representation of G.

(ii) Now consider the cyclic group $G = C_4 = \langle x \rangle$. This group has 4 elements: e, x, x^2, x^3, and $x^4 = e$. There are 4 roots of unity, namely $1, -1, i, -i$. Corresponding to these 4 roots of unity, there are 4 irreducible representations $\rho_1, \rho_2, \rho_3, \rho_4$. The way these 4 irreducible representations map is illustrates in the following table:

λ	e	x	x^2	x^3
$\rho_1 \equiv 1$	1	1	1	1
$\rho_2 \equiv -1$	1	-1	1	-1
$\rho_3 \equiv i$	1	i	-1	$-i$
$\rho_4 \equiv -i$	1	$-i$	-1	i

From the table, we can see that ρ_3 and ρ_4 are faithful.

[1]This is the Klein four-group. Geometrically, it represents the group of all symmetries of a non-square rectangle.

2 Character Theory

§2.1 Characters

Definition 2.1. Let V be a finite dimensional complex representation of a finite group G and $\rho : G : \mathrm{GL}(V)$ be the corresponding group homomorphism. Then the **character** χ_ρ of the representation V is the function $\chi_\rho : G \to \mathbb{C}$ defined by

$$\chi_\rho(g) = \operatorname{Tr} \rho(g). \tag{2.1}$$

The right side of (2.1) is unambiguous. In fact, $\rho(g) \in \mathrm{GL}(V)$ is a invertible linear transformation on the finite dimensional vector space V. In different bases of V, $\rho(g)$ can be represented by different $n \times n$ complex matrices if the dimension of V is n. But $\operatorname{Tr} \rho(g)$ will be the same for all these matrices following from the invariance of trace under conjugation: denote the $n \times n$ complex matrix $[\rho(g)]_{\mathcal{B}}$ representing the invertible linear transformation $\rho(g) \in \mathrm{GL}(V)$ in the basis $\mathcal{B}$ of the finite dimensional complex vector space V. Also, let $[\rho(g)]_{\mathcal{B}'}$ be the matrix representation of $\rho(g) \in \mathrm{GL}(V)$ with respect to the basis $\mathcal{B}'$ of V. We know from basic linear algebra that there exists an invertible $n \times n$ complex matrix T such that

$$[\rho(g)]_{\mathcal{B}'} = T^{-1}[\rho(g)]_{\mathcal{B}} T. \tag{2.2}$$

The cyclicity of trace (i.e. $\operatorname{Tr}(ABC) = \operatorname{Tr}(CAB) = \operatorname{Tr}(BCA)$) then guarantees

$$\operatorname{Tr}[\rho(g)]_{\mathcal{B}'} = \operatorname{Tr}[\rho(g)]_{\mathcal{B}}. \tag{2.3}$$

The basis independent complex number given by (2.3) is precisely the right side of (2.1), namely $\operatorname{Tr} \rho(g)$.

Remark 2.1. In general, not every invertible linear map has an eigenbasis, i.e. not every linear map is diagonalizable. But the situation is much simpler when we are dealing with representations of finite groups. Since $|G|$ is finite, $g^{|G|} = e$, for every $g \in G$. Therefore,

$$\rho(g)^{|G|} = \rho(e) = \mathbb{1}_V, \tag{2.4}$$

i.e. $\rho(g)$ is of finite order. Linear maps that are of finite order are diagonalizable, because of the following theorem from linear algebra:

> A linear map is diagonalizable if and only if its minimal polynomial doesn't have repeated roots.

Since $\rho(g)$ satisfies $\rho(g)^{|G|} - \mathbb{1}_V = 0$, it is the zero of the polynomial $x^{|G|} - 1$. Therefore, the minimal polynomial of $\rho(g)$ divides $x^{|G|} - 1$. But the roots of $x^{|G|} - 1$ are the $|G|$-th roots of unity. In particular, the roots of $x^{|G|} - 1$ are all distinct. Therfore, the minimal polynomial of $\rho(g)$ can't have repeated roots. As a result, we can pick a basis of V using eigenvectors of $\rho(g)$. In this basis, the trace of $\rho(g)$ is the sum of its eigenvalues. So we can write

$$\chi_\rho(g) = \sum_{\lambda \text{ eigenvalues of } \rho(g)} \lambda. \tag{2.5}$$

Furthermore, the roots of the minimal polynomial of $\rho(g)$ are also roots of $x^{|G|} - 1$, which are the $|G|$-th roots of unity. So the eigenvalues of $\rho(g)$ have modulus 1.

Remark 2.2. Note that ther character $\chi_\rho : G \to \mathbb{C}$ of the representation $\rho : G \to \mathrm{GL}(V)$ is constant on the conjugacy classes of G. In other words,

$$\chi_\rho\left(h^{-1}gh\right) = \chi_\rho(g) \tag{2.6}$$

for every $h \in G$. Also,

$$\chi_\rho(e) = \operatorname{Tr}\rho(e) = \operatorname{Tr}\mathbb{1}_V = \dim V, \tag{2.7}$$

where $e \in G$ is the identity element.

Proposition 2.1

Let V and W be representations of G with $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$ being the respective group homomorphisms. Then

(a) $\chi_{\rho\oplus\sigma} = \chi_\rho + \chi_\sigma$;

(b) $\chi_{\rho\otimes\sigma} = \chi_\rho \cdot \chi_\sigma$;

(c) $\chi_{\rho^*}(g) = \overline{\chi_\rho(g)}$ for every $g \in G$;

(d) $\chi_{\Lambda^2\rho}(g) = \frac{1}{2}\left[(\chi_\rho(g))^2 - \chi_\rho\left(g^2\right)\right]$ for every $g \in G$.

Proof. (a) Suppose $n = \dim V$ and $m = \dim W$. Recall that $\rho \oplus \sigma : G \to \mathrm{GL}\left(V \oplus W\right)$ is defined as $\left(\rho \oplus \sigma\right) g\left(\mathbf{v}, \mathbf{w}\right) = (\rho(g)\mathbf{v}, \sigma(g)\mathbf{w})$, for $\mathbf{v} \in V$ and $\mathbf{w} \in W$. Let $\mathcal{B}_1$ be a basis for V and $\mathcal{B}_2$ be a basis for W so that $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ is a basis for $V \oplus W$.

Now, $\rho(g) \in \mathrm{GL}(V)$ can be represented by the $n \times n$ complex matrix $[\rho(g)]_{\mathcal{B}_1}$, and $\sigma(g) \in \mathrm{GL}(W)$ can be represented by the $m \times m$ complex matrix $[\sigma(g)]_{\mathcal{B}_2}$. Then $\left(\rho \oplus \sigma\right) g \in \mathrm{GL}\left(V \oplus W\right)$ can be represented by an $(m + n) \times (m + n)$ complex matrix

$$[\left(\rho \oplus \sigma\right) g]_{\mathcal{B}} = \begin{bmatrix} [\rho(g)]_{\mathcal{B}_1} & 0_{n\times m} \\ 0_{m\times n} & [\sigma(g)]_{\mathcal{B}_2} \end{bmatrix}. \tag{2.8}$$

From (2.8), it follows that

$$\chi_{\rho\oplus\sigma}(g) = \operatorname{Tr}\left[\left(\rho \oplus \sigma\right) g\right]_{\mathcal{B}} = \operatorname{Tr}\left[\rho(g)\right]_{\mathcal{B}_1} + \operatorname{Tr}\left[\sigma(g)\right]_{\mathcal{B}_2} = \chi_\rho(g) + \chi_\sigma(g). \tag{2.9}$$

(b) Recall that $\left(\rho \otimes \sigma\right) g\left(\mathbf{v} \otimes \mathbf{w}\right) = \rho(g)\mathbf{v} \otimes \sigma(g)\mathbf{w}$, for $\mathbf{v} \in V$ and $\mathbf{w} \in W$. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be an eigenbasis of V with respect to $\rho(g) \in \mathrm{GL}(V)$ and $\{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ be an eigenbasis of W with respect to $\sigma(g) \in \mathrm{GL}(W)$. Then

$$\rho(g)\mathbf{v}_i = \lambda_i\mathbf{v}_i \quad \text{and} \quad \sigma(g)\mathbf{w}_j = \mu_j\mathbf{w}_j, \tag{2.10}$$

for $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Then

$$\left(\rho \otimes \sigma\right) g\left(\mathbf{v}_i \otimes \mathbf{w}_j\right) = \rho(g)\mathbf{v}_i \otimes \sigma(g)\mathbf{w}_j = \lambda_i\mathbf{v}_i \otimes \mu_j\mathbf{w}_j = \lambda_i\mu_j\, \mathbf{v}_i \otimes \mathbf{w}_j. \tag{2.11}$$

Therefore, $\mathbf{v}_i \otimes \mathbf{w}_j$ is an eigenvector of $\left(\rho \otimes \sigma\right) g$ with the eigenvalue $\lambda_i\mu_j$. We, therefore, see that $\{\mathbf{v}_i \otimes \mathbf{w}_j \mid i = 1, \ldots, n; j = 1, \ldots, m\}$ forms an eigenbasis of $V \otimes W$. Therefore,

$$\begin{aligned}\chi_{\rho\otimes\sigma}(g) &= \sum_{\lambda \text{ eigenvalues of } (\rho\otimes\sigma)g} \lambda \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} \lambda_i\mu_j \\ &= \sum_{i=1}^{n} \lambda_i \sum_{j=1}^{m} \mu_j \\ &= \chi_\rho\left(g\right) \cdot \chi_\sigma\left(g\right). \end{aligned} \tag{2.12}$$

(c) Recall that $\rho^* : G \to \mathrm{GL}(V^*)$ is defined by $(\rho^*(g)\,\widehat{\omega})(\mathbf{v}) = \widehat{\omega}\left(\rho\left(g^{-1}\right)\mathbf{v}\right)$, for $\widehat{\omega} \in V^*$ and $\mathbf{v} \in V$. The relevant eigenvalue equations are $\rho(g)\mathbf{v}_i = \lambda_i \mathbf{v}_i$.

Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be an eigenbasis of V with respect to $\rho(g) \in \mathrm{GL}(V)$, and let $\{\widehat{\alpha}^1, \ldots, \widehat{\alpha}^n\}$ be the associated dual basis of V^*. Then

$$\begin{aligned}\left(\rho^*(g)\,\widehat{\alpha}^j\right)(\mathbf{v}_i) &= \widehat{\alpha}^j\left(\rho\left(g^{-1}\right)\mathbf{v}_i\right)\\ &= \widehat{\alpha}^j\left(\frac{1}{\lambda_i}\mathbf{v}_i\right)\\ &= \frac{1}{\lambda_j}\,\widehat{\alpha}^j(\mathbf{v}_i),\end{aligned}$$

since $\widehat{\alpha}^j(\mathbf{v}_i) = \delta^j{}_i$. In other words,

$$\rho^*(g)\,\widehat{\alpha}^j = \frac{1}{\lambda_j}\,\widehat{\alpha}^j. \tag{2.13}$$

So $\{\widehat{\alpha}^1, \ldots, \widehat{\alpha}^n\}$ is an eigenbasis of V^* with respect to $\rho^*(g) \in \mathrm{GL}(V^*)$. The eigenvalues are $\frac{1}{\lambda_j}$. By Remark 2.1, $|\lambda_j| = 1$, so $\frac{1}{\lambda_j} = \overline{\lambda_j}$. So we have

$$\chi_{\rho^*}(g) = \sum_{j=1}^n \overline{\lambda_j} = \overline{\sum_{j=1}^n \lambda_j} = \overline{\chi_\rho(g)}. \tag{2.14}$$

(d) Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be an eigenbasis of V with respect to $\rho(g) \in \mathrm{GL}(V)$. The relevant eigenvalue equations are $\rho(g)\mathbf{v}_i = \lambda_i \mathbf{v}_i$, for $i = 1, \ldots, n$. Then for $1 \leq i < j \leq n$,

$$\Lambda^2\rho(g)(\mathbf{v}_i \wedge \mathbf{v}_j) = \rho(g)\mathbf{v}_i \wedge \rho(g)\mathbf{v}_j = \lambda_i\mathbf{v}_i \wedge \lambda_i\mathbf{v}_j = \lambda_i\lambda_j\,\mathbf{v}_i \wedge \mathbf{v}_j. \tag{2.15}$$

So $\{\mathbf{v}_i \wedge \mathbf{v}_j\}_{1\leq i<j\leq n}$ forms an eigenbasis of $\Lambda^2 V$ with respect to $\Lambda^2\rho(g)$. Therefore,

$$\chi_{\Lambda^2\rho}(g) = \sum_{1\leq i<j\leq n} \lambda_i\lambda_j. \tag{2.16}$$

Now, the eigenvalues of $\rho\left(g^2\right)$ are λ_i^2.

$$(\chi_\rho(g))^2 - \chi_\rho\left(g^2\right) = \left(\sum_{i=1}^n \lambda_i\right)^2 - \sum_{i=1}^n \lambda_i^2 = 2\sum_{1\leq i<j\leq n} \lambda_i\lambda_j. \tag{2.17}$$

Therefore,

$$\chi_{\Lambda^2\rho}(g) = \frac{1}{2}\left[(\chi_\rho(g))^2 - \chi_\rho\left(g^2\right)\right]. \tag{2.18}$$

■

Remark 2.3. One can similarly compute the character of the second symmetric power of a given representation, namely

$$\chi_{\mathrm{Sym}^2\rho}(g) = \frac{1}{2}\left[(\chi_\rho(g))^2 + \chi_\rho\left(g^2\right)\right]. \tag{2.19}$$

Indeed, $V^{\otimes 2} = \Lambda^2 V \oplus \mathrm{Sym}^2 V$, and $\rho \otimes \rho = \Lambda^2\rho \oplus \mathrm{Sym}^2\rho$ so that we have

$$\chi_{\rho\otimes\rho} = \chi_{\Lambda^2\rho} + \chi_{\mathrm{Sym}^2\rho}. \tag{2.20}$$

For any $g \in G$, we then compute

$$\begin{aligned}\chi_{\mathrm{Sym}^2\rho}(g) &= \chi_{\rho\otimes\rho}(g) - \chi_{\Lambda^2\rho}(g)\\ &= \chi_\rho(g)\,\chi_\rho(g) - \chi_{\Lambda^2\rho}(g)\\ &= \chi_\rho(g)^2 - \frac{1}{2}\left[(\chi_\rho(g))^2 - \chi_\rho\left(g^2\right)\right]\\ &= \frac{1}{2}\left[(\chi_\rho(g))^2 + \chi_\rho\left(g^2\right)\right].\end{aligned} \tag{2.21}$$

§2.2 Permutation representation and regular representation

Let X be a finite set and $\sigma : G \to \mathrm{Aut}(X)$ is a group homomorphism from the finite group G to the premutation group of X. That is, given $g \in G$ and $x \in X$, $\sigma(g) : X \to X$ is a bijection, so that $\sigma(g)x \in X$. In other words, $\sigma(g)$ permutes the elements of X.

Now, construct the $|X|$-dimensional complex vector space V as follows: V is the vector space with basis $\{e_x \mid x \in X\}$. Now, define the representation $\rho : G \to \mathrm{GL}(V)$ by

$$\rho(g)\left(\sum_{x\in X} a_x e_x\right) = \sum_{x\in X} a_x e_{\sigma(g)x}, \tag{2.22}$$

with $a_x \in \mathbb{C}$. The representation of G on the complex vector space V constructed above is called the **permutation representation**.

Lemma 2.2

If V is the permutation representation associated with the action of a group G on a finite set X, where $\rho : G \to \mathrm{GL}(V)$ is the corresponding group homomorphism, then $\chi_\rho(g)$ is the number of elements of X fixed by g.

Proof. We need to show that $\chi_\rho(g)$ is the number of elements of X fixed by $\sigma(g)$. Suppose we have enumerated the elements of X:

$$X = \{x_1, x_2, \dots, x_n\}. \tag{2.23}$$

Then the n-dimensional vector space V has an ordered basis:

$$\mathcal{B} = \{e_{x_1}, e_{x_2}, \dots, e_{x_n}\}. \tag{2.24}$$

Now let's consider the $n \times n$ matrix representation of $\rho(g)$ in the basis $\mathcal{B}$. Suppose $[\rho(g)]_{\mathcal{B}} = [A_{ij}]_{i,j=1}^n$. We claim that

$$A_{ii} = \begin{cases} 1 & \text{if } \sigma(x_i) = x_i, \\ 0 & \text{otherwise.} \end{cases} \tag{2.25}$$

The i-th column of $[A_{ij}]_{i,j=1}^n$ looks like $\begin{bmatrix} A_{1i} \\ A_{2i} \\ \vdots \\ A_{ni} \end{bmatrix}$. It signifies that the coordinate of $\rho(g)(e_{x_i})$ in the aforementioned basis is $\begin{bmatrix} A_{1i} \\ A_{2i} \\ \vdots \\ A_{ni} \end{bmatrix}$. In other words,

$$\rho(g)(e_{x_i}) = \sum_{j=1}^n A_{ji} e_{x_j}. \tag{2.26}$$

But $\rho(g)(e_{x_i}) = e_{\sigma(g)(x_i)}$. So we have

$$\rho(g)(e_{x_i}) = \sum_{j=1}^n A_{ji} e_{x_j} = e_{\sigma(g)(x_i)}. \tag{2.27}$$

Since every vector in a vector space can be **uniquely** written as a linear combination of the basis vecctors, we can conclude from (2.27) that

$$A_{ji} = \begin{cases} 1 & \text{if } x_j = \sigma(x_i), \\ 0 & \text{otherwise.} \end{cases} \tag{2.28}$$

Hence,

$$A_{ii} = \begin{cases} 1 & \text{if } \sigma\left(x_i\right) = x_i, \\ 0 & \text{otherwise;} \end{cases} \tag{2.29}$$

and our claim is proved. Therefore,

$$\chi_\rho\left(g\right) = \operatorname{Tr}\rho(g) = \operatorname{Tr}\left[\rho(g)\right]_{\mathcal{B}} = \sum_{i=1}^{n} A_{ii}. \tag{2.30}$$

We have shown that $A_{ii} = 1$ if and only if $\sigma(g)$ fixes x_i, and $A_{ii} = 0$ otherwise. Therefore, $\sum_{i=1}^{n} A_{ii}$ is equal to the number of x_i's such that $\sigma(g)$ fixes x_i. So

$$\chi_\rho(g) = \sum_{i=1}^{n} A_{ii} = \left|\{x \in X \mid \sigma(g) \text{ fixes } x\}\right|. \tag{2.31}$$

■

There is another important representation called the **regular representation** of a given finite group G, which is actually a special case of permutation representation. In this case, $X = G_{\mathbf{Set}}$, the underlying set of the finite group G, and $\sigma : G \to \operatorname{Aut}\left(G_{\mathbf{Set}}\right) \cong \mathfrak{S}_n$, where $n = |G|$. Here $\operatorname{Aut}\left(G_{\mathbf{Set}}\right)$ is the group of all bijections from the set $G_{\mathbf{Set}}$ to itself. Since $|G| = n$, there is a bijection from G to $\{1, 2, \ldots, n\}$. So we can actually identify $\operatorname{Aut}\left(G_{\mathbf{Set}}\right)$ to $\mathfrak{S}_n$.

Take $V = \mathbb{C}[G]$, the group algebra corresponding to the finite group G. An element $x \in \mathbb{C}[G]$ is a complex valued function on the finite set G. $\mathbb{C}[G]$ is easily seen to be a complex vector space with basis $\{\delta_g \mid g \in G\}$, where $\delta_g : G \to \mathbb{C}$ is defined by

$$\delta_g\left(h\right) = \begin{cases} 1 & \text{if } h = g, \\ 0 & \text{if } h \neq g. \end{cases} \tag{2.32}$$

A generic element $f \in \mathbb{C}[G]$ can be represented as

$$\alpha = \sum_{g \in G} a_g \delta_g, \tag{2.33}$$

with $a_g \in \mathbb{C}$ is the value α takes at $g \in G$, i.e. $a_g = \alpha(g)$. We don't talk about the algebra structure of $\mathbb{C}[G]$ at the moment. All we need here is the vector space structure of $\mathbb{C}[G]$. With these given data, the regular representation of the finite group G is the associated permutation representation. If $\rho : G \to \operatorname{GL}\left(\mathbb{C}[G]\right)$ is the representation, then for a given $h \in G$, $\rho(h) : \mathbb{C}[G] \to \mathbb{C}[G]$ is a linear map, and $\rho\left(h\right)\left(\sum_{g \in G} a_g \delta_g\right)$ is a function from G to $\mathbb{C}$. This is defined as follows: given $k \in G$,

$$\begin{aligned} \rho\left(h\right)\left(\sum_{g \in G} a_g \delta_g\right)\left(k\right) &= \sum_{g \in G} a_g \delta_{\sigma(h)g}\left(k\right) \\ &= a_g \qquad \text{such that } \sigma(h)g = k \\ &= a_{\sigma\left(h^{-1}\right)k} \\ &= \sum_{g \in G} a_g \delta_g\left(\sigma\left(h^{-1}\right)k\right). \end{aligned} \tag{2.34}$$

If we denote $\sum_{g \in G} a_g \delta_g$ by α, then we can rewrite (2.34) as

$$\left(\rho\left(h\right)\alpha\right)\left(k\right) = \alpha\left(\sigma\left(h^{-1}\right)k\right). \tag{2.35}$$

For the **left-regular representation**, we define the homomorphism $\sigma : G \to \operatorname{Aut}(G)$ as

$$\sigma\left(g\right)\left(h\right) = gh. \tag{2.36}$$

In this case, (2.35) reads

$$(\rho(h)\alpha)(k) = \alpha\left(h^{-1}k\right). \tag{2.37}$$

In a similar manner, we can also define the **right-regular representation**, where $\sigma : G \to \mathrm{Aut}(G)$ is defined as

$$\sigma(g)(h) = hg^{-1}. \tag{2.38}$$

In this case, (2.35) reads

$$(\rho(h)\alpha)(k) = \alpha(kh). \tag{2.39}$$

§2.3 An example of $\mathfrak{S}_3$

Consider $G = \mathfrak{S}_3$. It has 6 elements, $1, (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)$. There are 3 conjugacy classes:

$$\{1\}, \quad \{(1\ 2), (1\ 3), (2\ 3)\}, \quad \{(1\ 2\ 3), (1\ 3\ 2)\}. \tag{2.40}$$

Here, $G = \mathrm{Aut}(X)$ with $X = \{1, 2, 3\}$. Consider V to be the vector space of all complex valued functions on X. It is isomorphic to $\mathbb{C}^3$, and the basis we choose for V is $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

Here $\mathbf{e}_x$ can be seen as a complex valued function on $X = \{1, 2, 3\}$, i.e. $\mathbf{e}_x : X \to \mathbb{C}$, defined by

$$\mathbf{e}_x(y) = \begin{cases} 1 & \text{if } y = x, \\ 0 & \text{if } y \neq x. \end{cases} \tag{2.41}$$

So the linear combination $\sum_{x\in X} a_x \mathbf{e}_x$ is also seen as a complex valued function on X. Now, (2.22) reads

$$\rho(g)\left(\sum_{x\in X} a_x \mathbf{e}_x\right) = \sum_{x\in X} a_x \mathbf{e}_{\sigma(g)x}, \tag{2.42}$$

so that for $y \in X$,

$$\begin{aligned} \rho(g)\left(\sum_{x\in X} a_x \mathbf{e}_x\right)(y) &= \sum_{x\in X} a_x \mathbf{e}_{\sigma(g)x}(y) \\ &= a_x \qquad \text{such that } \sigma(g)x = y \\ &= a_{\sigma(g^{-1})y} \\ &= \left(\sum_{x\in X} a_{\sigma(g^{-1})x} \mathbf{e}_x\right)(y). \end{aligned}$$

Therefore,

$$\rho(g)\left(\sum_{x\in X} a_x \mathbf{e}_x\right) = \sum_{x\in X} a_{\sigma(g^{-1})x} \mathbf{e}_x. \tag{2.43}$$

We can identify a complex valued function on $X = \{1, 2, 3\}$ by the column vector $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \mathbb{C}^3$, and the action of $g \in \mathfrak{S}_3$ on this triple is realized as

$$\rho(g)\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_{\sigma(g^{-1})1} \\ a_{\sigma(g^{-1})2} \\ a_{\sigma(g^{-1})3} \end{bmatrix}. \tag{2.44}$$

For $g = (1\ 2\ 3)$, $g^{-1} = (1\ 3\ 2)$.

$$\rho((1\ 2\ 3))\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_{\sigma((1\ 3\ 2))1} \\ a_{\sigma((1\ 3\ 2))2} \\ a_{\sigma((1\ 3\ 2))3} \end{bmatrix} = \begin{bmatrix} a_3 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}. \tag{2.45}$$

Therefore, $\chi_\rho((1\ 2\ 3)) = 0$. Similarly, for $g = (1\ 2)$, $g^{-1} = (1\ 2)$.

$$\rho((1\ 2)) \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_{\sigma((1\ 2))1} \\ a_{\sigma((1\ 2))2} \\ a_{\sigma((1\ 2))3} \end{bmatrix} = \begin{bmatrix} a_2 \\ a_1 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}. \tag{2.46}$$

So $\chi_\rho((1\ 2)) = 1$. Finally,

$$\rho(1) \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \tag{2.47}$$

So $\chi_\rho(1) = 3$.

The permutation representation $\mathbb{C}^3$ associated with the group homomorphism $\rho : \mathfrak{S}_3 \to \mathrm{GL}(\mathbb{C}^3)$ that we studied above is not irreducible. If we take the subspace

$$\left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \mathbb{C}^3 \mid a_1 = a_2 = a_3 \right\},$$

which is a 1-dimensional subspace of $\mathbb{C}^3$, it is invariant under the action of the permutation group as all the coefficients a_1, a_2, a_3 are the same. This 1-dimensional subspace of $\mathbb{C}^3$ is spanned by $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. T1he complementary subspace of this one-dimensional subspace is given by the set

$$V = \left\{ \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \in \mathbb{C}^3 \mid z_1 + z_2 + z_3 = 0 \right\}.$$

This is a 2-dimensional vector subspace of $\mathbb{C}^3$ that is also left invariant under the action of the permutation group by Corollary 1.4. One can verify that the subrepresentations mentioned above are irreducible representations of $\mathfrak{S}_3$. The 2-dimensional irreducible representation of $\mathfrak{S}_3$ is called the **standard representation** of $\mathfrak{S}_3$.

Let us denote the group homomorphism associated with the standard representation V of $\mathfrak{S}_3$ by ρ_V. Observe that $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ form a basis $\mathcal{B}_V$ for the 2-dimensional subspace V of $\mathbb{C}^3$. Since

$$\rho_V(1\ 2\ 3) \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \rho_V(1\ 2\ 3) \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \tag{2.48}$$

one has the matrix representation of $\rho_V(1\ 2\ 3)$ in the above basis as

$$[\rho_V((1\ 2\ 3))]_{\mathcal{B}_V} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}. \tag{2.49}$$

Similarly,

$$[\rho_V((1\ 2))]_{\mathcal{B}_V} = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}, \text{ and } [\rho_V(1)]_{\mathcal{B}_V} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{2.50}$$

so that

$$\chi_{\rho_V}((1\ 2\ 3)) = -1,\ \chi_{\rho_V}((1\ 2)) = 0,\ \chi_{\rho_V}(1) = 2. \tag{2.51}$$

Recall that an element of $\mathfrak{S}_3$ is even or odd if it can be written as a product of an even or odd number of transpositions. The sign of an element of $\mathfrak{S}_3$ is 1 if it is even and is -1 if it is odd. For example,

$\mathrm{sgn}((1\ 2\ 3)) = 1$ as $(1\ 2\ 3) = (1\ 2)(1\ 3)$. Now, the alternating representation $\sigma' : \mathfrak{S}_3 \to \mathrm{GL}(\mathbb{C}) \cong \mathbb{C}^\times$ is given by

$$\sigma'(g)v = \mathrm{sgn}(g)v, \tag{2.52}$$

for $g \in \mathfrak{S}_3$ and $v \in \mathbb{C}$. This is indeed a representation as

$$\sigma'(g')(\sigma'(g)v) = \sigma'(g')(\mathrm{sgn}(g)v) = \mathrm{sgn}(g')\mathrm{sgn}(g)v = \mathrm{sgn}(g'g)v = \sigma'(g'g)v.$$

Explicitly, considering $\mathrm{GL}(\mathbb{C}) \cong \mathbb{C}^\times$,

$$\sigma'(1) = 1, \quad \sigma'((1\ 2)) = \sigma'((1\ 3)) = \sigma'((2\ 3)) = -1, \quad \sigma'((1\ 2\ 3)) = \sigma'((1\ 3\ 2)) = 1.$$

And the character of the alternating representation is given by

$$\chi_{\sigma'}(1) = 1, \quad \chi_{\sigma'}((1\ 2)) = -1, \quad \chi_{\sigma'}((1\ 2\ 3)) = 1. \tag{2.53}$$

The alternating representation is a 1-dimensional (irreducible) representation of $\mathfrak{S}_3$. And there is this trivial 1-dimensional representation of $\mathfrak{S}_3$, $\sigma : \mathfrak{S}_3 \to \mathrm{GL}(\mathbb{C}) \cong \mathbb{C}^\times$ given by

$$\sigma(g) = 1, \quad \forall\, g \in \mathfrak{S}_3. \tag{2.54}$$

Then the character is given by

$$\chi_\sigma(1) = 1, \quad \chi_\sigma((1\ 2)) = 1, \quad \chi_\sigma((1\ 2\ 3)) = 1. \tag{2.55}$$

Now, take an arbitrary representation W of $\mathfrak{S}_3$ whose associated homomorphism is given by $\rho_W : \mathfrak{S}_3 \to \mathrm{GL}(W)$. Now, $\mathfrak{S}_3$ has an abelian subgroup of order 3, that is generated by a 3-cycle, say $(1\ 2\ 3)$. This finite abelian group is isomorphic to $\mathbb{Z}_3$. Let us denote this finite abelian subgroup by $\mathfrak{A}_3$. Let us denote by g_1 one of the two 3-cycles that generate $\mathfrak{A}_3$, i.e. $\mathfrak{A}_3 = \langle g_1 \rangle$. Then W is also a representation of $\mathfrak{A}_3$.

The complex vector space W has an eigenbasis with respect to $\rho(g_1) \in \mathrm{GL}(V)$. By Remark 2.1, the eiganvalues are cubic roots of unity, namely $1, \omega, \omega^2$. Then we write the respective eigenvalue equations as

$$\rho(g_1)\,\mathbf{v}_i = \omega^{\alpha_i}\mathbf{v}_i, \tag{2.56}$$

with $\{\mathbf{v}_i\}_{i=1}^n$ being the eigenbasis. Thus the representation W of $\mathfrak{A}_3$ is decomoposed into one dimensional complex vector spaces:

$$W = \bigoplus_{i=1}^n V_i,$$

where $V_i = \mathbb{C}\mathbf{v}_i$. This decomposition only refers to 3 elements: $g_1 = (1\ 2\ 3), g_1^2 = (1\ 3\ 2), g_1^3 = e$ of $\mathfrak{S}_3$. How does the decomposition (2.3) respond to when the rest of the elements of $\mathfrak{S}_3$ are considered? Choose a transposition, say $(1\ 2)$ of $\mathfrak{S}_3$ and denote it by g_2. Observe that $g_2 = (1\ 2)$ and $g_1 = (1\ 2\ 3)$ generate the whole of $\mathfrak{S}_3$. Indeed, one has $g_2g_1g_2 = g_1^2$, i.e. $g_1g_2 = g_2g_1^2$, since $g_2 = g_2^{-1}$. We are trying to find proper $\mathfrak{S}_3$-invariant subspace that can't be further decomposed. Now, for $\mathbf{v}_i \in W$ satisfying (2.56), one has

$$\begin{aligned}
\rho_W(g_1)\,(\rho_W(g_2)\mathbf{v}_i) &= \rho_W\,(g_1g_2)\,\mathbf{v}_i \\
&= \rho_W\left(g_2g_1^2\right)\mathbf{v}_i \\
&= \rho_W\,(g_2)\,\rho_W\left(g_1^2\right)\mathbf{v}_i \\
&= \rho_W(g_2)\left(\omega^{2\alpha_i}\mathbf{v}_i\right) \\
&= \omega^{2\alpha_i}\,(\rho_W(g_2)\mathbf{v}_i)\,.
\end{aligned} \tag{2.57}$$

So $\rho_W(g_2)\mathbf{v}_i$ is an eigenvector of $\rho_W(g_1)$ with eigenvalue $\omega^{2\alpha_i}$. To check $\mathfrak{S}_3$-invariance of a proper subspace of the complex vector space W, it is sufficient to verify the invariance of the subspace in question under the action of $\rho_W(g_1)$ and $\rho_W(g_2)$, as g_1 and g_2 generate $\mathfrak{S}_3$. Also, let

$$\mathbf{s} = \begin{bmatrix} \omega \\ 1 \\ \omega^2 \end{bmatrix} \quad \text{and} \quad \mathbf{t} = \begin{bmatrix} 1 \\ \omega \\ \omega^2 \end{bmatrix}$$

with $\mathbf{s}, \mathbf{t}$ being a basis for the 2-dimensional vector subspace V of $\mathbb{C}^3$ that is known as the standard representation of $\mathfrak{S}_3$. Recall that the permutation representation $\rho : \mathfrak{S}_3 \to \mathrm{GL}(\mathbb{C}^3)$ satisfies

$$\rho(g_1)\begin{bmatrix}\omega\\1\\\omega^2\end{bmatrix} = \begin{bmatrix}\omega^2\\\omega\\1\end{bmatrix} = \omega\begin{bmatrix}\omega\\1\\\omega^2\end{bmatrix};$$

$$\rho(g_1)\begin{bmatrix}1\\\omega\\\omega^2\end{bmatrix} = \begin{bmatrix}\omega^2\\1\\\omega\end{bmatrix} = \omega^2\begin{bmatrix}1\\\omega\\\omega^2\end{bmatrix};$$

$$\rho(g_2)\begin{bmatrix}\omega\\1\\\omega^2\end{bmatrix} = \begin{bmatrix}1\\\omega\\\omega^2\end{bmatrix};$$

$$\rho(g_1)\begin{bmatrix}1\\\omega\\\omega^2\end{bmatrix} = \begin{bmatrix}\omega\\1\\\omega^2\end{bmatrix}.$$

Altogether, one has the following:

$$\rho\left(g_1\right)\mathbf{s} = \omega\mathbf{s}, \quad \rho\left(g_1\right)\mathbf{t} = \omega^2\mathbf{t}, \quad \rho\left(g_2\right)\mathbf{s} = \mathbf{t}, \quad \rho\left(g_2\right)\mathbf{t} = \mathbf{s}. \tag{2.58}$$

Suppose that we start with an eigenvector $\mathbf{v}$ of $\rho_W\left(g_1\right)$. Then we have the following possibilities:

1. The eigenvalue of $\rho_W(g_1)$ corresponding to the eigenvactor $\mathbf{v}$ is ω^i, where $\omega^i \neq 1$. Then $\omega^{2i} \neq \omega^i$. In terms of the eigenvalue equations, one has

$$\rho_W\left(g_1\right)\mathbf{v} = \omega^i\mathbf{v} \text{ and } \rho_W\left(g_1\right)\rho_W\left(g_2\right)\mathbf{v} = \omega^{2i}\rho_W\left(g_2\right)\mathbf{v}. \tag{2.59}$$

Since $\mathbf{v}$ and $\rho_W\left(g_2\right)\mathbf{v}$ are eigenvectors of two different eigenvalues, they are linearly independent. In other words, $\operatorname{span}\left\{\mathbf{v}, \rho_W\left(g_2\right)\mathbf{v}\right\} =: V'$ is a 2-dimensional vector subspace of W that is invariant under the action of $\mathfrak{S}_3$ (as g_1 and g_2 generate $\mathfrak{S}_3$).

Furthermore, this 2-dimensional representation V' is isomorphic to the standard representation V of $\mathfrak{S}_3$. In order to show this isomorphism, we need to prove the commutativity of the following square for each $g \in \mathfrak{S}_3$:

$$\begin{tikzcd} V \arrow[r, "j"] \arrow[d, "\rho(g)"'] & V' \arrow[d, "\rho_W(g)"] \\ V \arrow[r, "j"'] & V', \end{tikzcd}$$

where $j : V \to V'$ is a vector space isomorphism. It suffices to verify the commutativity for g_1 and g_2 as they generate $\mathfrak{S}_3$.

Here $V = \operatorname{span}\left\{\mathbf{s}, \mathbf{t}\right\}$ and $V' = \operatorname{span}\left\{\mathbf{v}, \rho_W\left(g_2\right)\mathbf{v}\right\}$. Consider the case $i = 1$ first. Then we define $j\left(\mathbf{s}\right) = \mathbf{v}$ and $j\left(\mathbf{t}\right) = \rho_W\left(g_2\right)\mathbf{v}$. This is an isomorphism of vector spaces.

$$\begin{aligned}\left(\rho_W\left(g_2\right)\circ j\right)\left(c_1\mathbf{s} + c_2\mathbf{t}\right) &= c_1\rho_W\left(g_2\right)\left(j\left(\mathbf{s}\right)\right) + c_2\rho_W\left(g_2\right)\left(j\left(\mathbf{t}\right)\right)\\ &= c_1\rho_W\left(g_2\right)\mathbf{v} + c_2\rho_W\left(g_2\right)\rho_W\left(g_2\right)\mathbf{v}\\ &= c_1\rho_W\left(g_2\right)\mathbf{v} + c_2\mathbf{v}\\ \left(j\circ\rho\left(g_2\right)\right)\left(c_1\mathbf{s} + c_2\mathbf{t}\right) &= c_1 j\left(\rho\left(g_2\right)\mathbf{s}\right) + c_2 j\left(\rho\left(g_2\right)\mathbf{t}\right)\\ &= c_1 j\left(\mathbf{t}\right) + c_2 j\left(\mathbf{s}\right)\\ &= c_1\rho_W\left(g_2\right)\mathbf{v} + c_2\mathbf{v}.\end{aligned}$$

$$
\begin{aligned}
(\rho_W(g_1)\circ j)(c_1\mathbf{s}+c_2\mathbf{t}) &= c_1\rho_W(g_1)(j(\mathbf{s}))+c_2\rho_W(g_1)(j(\mathbf{t}))\\
&= c_1\rho_W(g_1)\mathbf{v}+c_2\rho_W(g_1)\rho_W(g_2)\mathbf{v}\\
&= c_1\omega\mathbf{v}+c_2\omega^2\rho_W(g_2)\mathbf{v}\\
(j\circ\rho(g_1))(c_1\mathbf{s}+c_2\mathbf{t}) &= c_1 j(\rho(g_1)\mathbf{s})+c_2 j(\rho(g_1)\mathbf{t})\\
&= c_1 j(\omega\mathbf{s})+c_2 j\left(\omega^2\mathbf{t}\right)\\
&= c_1\omega\mathbf{v}+c_2\omega^2\rho_W(g_2)\mathbf{v}.
\end{aligned}
$$

Therefore, the following diagrams commute

$$
\begin{tikzcd}
V \arrow[r, "j"] \arrow[d, "\rho(g_2)"'] & V' \arrow[d, "\rho_W(g_2)"] \\
V \arrow[r, "j"'] & V',
\end{tikzcd}
\qquad
\begin{tikzcd}
V \arrow[r, "j"] \arrow[d, "\rho(g_1)"'] & V' \arrow[d, "\rho_W(g_1)"] \\
V \arrow[r, "j"'] & V'.
\end{tikzcd}
$$

So $j : V \to V'$ is an isomorphism of representations.

Now we are left with the case $i = 2$. We define $j(\mathbf{s}) = \rho_W(g_2)\mathbf{v}$ and $j(\mathbf{t}) = \mathbf{v}$. This is an isomorphism of vector spaces.

$$
\begin{aligned}
(\rho_W(g_2)\circ j)(c_1\mathbf{t}+c_2\mathbf{s}) &= c_1\rho_W(g_2)(j(\mathbf{t}))+c_2\rho_W(g_2)(j(\mathbf{s}))\\
&= c_1\rho_W(g_2)\mathbf{v}+c_2\rho_W(g_2)\rho_W(g_2)\mathbf{v}\\
&= c_1\rho_W(g_2)\mathbf{v}+c_2\mathbf{v}\\
(j\circ\rho(g_2))(c_t\mathbf{s}+c_2\mathbf{s}) &= c_1 j(\rho(g_2)\mathbf{t})+c_2 j(\rho(g_2)\mathbf{s})\\
&= c_1 j(\mathbf{s})+c_2 j(\mathbf{t})\\
&= c_1\rho_W(g_2)\mathbf{v}+c_2\mathbf{v}.
\end{aligned}
$$

$$
\begin{aligned}
(\rho_W(g_1)\circ j)(c_1\mathbf{t}+c_2\mathbf{s}) &= c_1\rho_W(g_1)(j(\mathbf{t}))+c_2\rho_W(g_1)(j(\mathbf{s}))\\
&= c_1\rho_W(g_1)\mathbf{v}+c_2\rho_W(g_1)\rho_W(g_2)\mathbf{v}\\
&= c_1\omega^2\mathbf{v}+c_2\omega^4\rho_W(g_2)\mathbf{v}\\
(j\circ\rho(g_1))(c_1\mathbf{t}+c_2\mathbf{s}) &= c_1 j(\rho(g_1)\mathbf{t})+c_2 j(\rho(g_1)\mathbf{s})\\
&= c_1 j\left(\omega^2\mathbf{t}\right)+c_2 j(\omega\mathbf{s})\\
&= c_1\omega^2\mathbf{v}+c_2\omega\rho_W(g_2)\mathbf{v}.
\end{aligned}
$$

Therefore, the following diagrams commute

$$
\begin{tikzcd}
V \arrow[r, "j"] \arrow[d, "\rho(g_2)"'] & V' \arrow[d, "\rho_W(g_2)"] \\
V \arrow[r, "j"'] & V',
\end{tikzcd}
\qquad
\begin{tikzcd}
V \arrow[r, "j"] \arrow[d, "\rho(g_1)"'] & V' \arrow[d, "\rho_W(g_1)"] \\
V \arrow[r, "j"'] & V'.
\end{tikzcd}
$$

So $j : V \to V'$ is an isomorphism of representations in $i = 2$ case as well.

Therefore, the 2-dimensional representation V' is isomorphic to V for both cases. Since V is irreducible, so is V'.

2. Now suppose the eigenvalue of $\rho_W(g_1)$ corresponding to the eigenvector $\mathbf{v}$ is 1. By (2.57),

$$\rho_W(g_1)(\rho_W(g_2)\mathbf{v}) = \rho_W(g_2)\mathbf{v}. \tag{2.60}$$

In other words, $\rho_W(g_2)\mathbf{v}$ is an eigenvector of $\rho_W(g_1)$ with eigenvalue 1. But $\mathbf{v}$ is also an eigenvector of $\rho_W(g_1)$ with eigenvalue 1.

Case 2(i): If $\mathbf{v}$ and $\rho_W(g_2)\mathbf{v}$ are linearly dependent, then $\mathbf{v}$ is an eigenvector of $\rho_W(g_2)$. Since $g_2^2 = e$, the eigenvalue of $\rho_W(g_2)$ corresponding to the eigenvector $\mathbf{v}$ will be 1 or -1.

If the eigenvalue is 1, then $\rho_W(g_1)\mathbf{v} = \mathbf{v}$ and $\rho_W(g_2)\mathbf{v} = \mathbf{v}$. Since g_1 and g_2 generate $\mathfrak{S}_3$, $\rho_W(g)\mathbf{v} = \mathbf{v}$ for every $g \in \mathfrak{S}_3$. So $\mathbb{C}\mathbf{v}$ is a 1-dimensional representation of $\mathfrak{S}_3$ isomorphic to the trivial representation.

If the eigenvalue is -1, then $\rho_W(g_1)\mathbf{v} = \mathbf{v}$ and $\rho_W(g_2)\mathbf{v} = -\mathbf{v}$. Then the equation $\rho_W(g)\mathbf{v} = (\operatorname{sgn} g)\mathbf{v}$ holds for $g = g_1, g_2$. Since g_1, g_2 generate $\mathfrak{S}_3$, this holds for all $g \in \mathfrak{S}_3$. Therefore, $\mathbb{C}\mathbf{v}$ is a 1-dimensional representation of $\mathfrak{S}_3$ isomorphic to the alternating representation.

Case 2(ii): Now suppose $\mathbf{v}$ and $\rho_W(g_2)\mathbf{v}$ are linearly independent. Then $\mathbf{v} + \rho_W(g_2)\mathbf{v}$ span a 1-dimensional representation of $\mathfrak{S}_3$ iromosphic to the trivial representation of $\mathfrak{S}_3$. Indeed,

$$\rho_W(g_1)(\mathbf{v} + \rho_W(g_2)\mathbf{v}) = \mathbf{v} + \rho_W(g_2)\mathbf{v}, \quad \rho_W(g_2)(\mathbf{v} + \rho_W(g_2)\mathbf{v}) = \rho_W(g_2)\mathbf{v} + \mathbf{v}. \tag{2.61}$$

Since g_1 and g_2 generate $\mathfrak{S}_3$, $\rho_W(g)(\rho_W(g_2)\mathbf{v} + \mathbf{v}) = \rho_W(g_2)\mathbf{v} + \mathbf{v}$ for every $g \in \mathfrak{S}_3$. Therefore, $\operatorname{span}\{\rho_W(g_2)\mathbf{v} + \mathbf{v}\}$ is a 1-dimensional representation of $\mathfrak{S}_3$ isomosphic to the trivial representation of $\mathfrak{S}_3$. On the other hand,

$$\rho_W(g_1)(\mathbf{v} - \rho_W(g_2)\mathbf{v}) = \mathbf{v} - \rho_W(g_2)\mathbf{v}, \quad \rho_W(g_2)(\mathbf{v} - \rho_W(g_2)\mathbf{v}) = \rho_W(g_2)\mathbf{v} - \mathbf{v}. \tag{2.62}$$

The equation $\rho_W(g)(\mathbf{v} - \rho_W(g_2)\mathbf{v}) = (\operatorname{sgn} g)(\mathbf{v} - \rho_W(g_2)\mathbf{v})$ holds for $g = g_1, g_2$. Since g_1, g_2 generate $\mathfrak{S}_3$, this holds for all $g \in \mathfrak{S}_3$. Therefore, $\operatorname{span}\{\mathbf{v} - \rho_W(g_2)\mathbf{v}\}$ is a 1-dimensional representation of $\mathfrak{S}_3$ isomorphic to the alternating representation of $\mathfrak{S}_3$.

In conclusion, there are 3 irreducible subrepresentations of W of $\mathfrak{S}_3$: the 2-dimensional irreducible representation isomorphic to the standard representation; the 1-dimensional irreducible representation isomorphic to the trivial representation; the 1-dimensional irreducible representation isomorphic to the alternating representation. By Maschke's theorem, W can be expressed as a direct sum of the 3 irreducible representations stated above:

$$\rho_W \cong \sigma^{\otimes a} \oplus (\sigma')^{\otimes b} \oplus \rho_V^{\otimes c}. \tag{2.63}$$

Here, $\sigma^{\otimes a}$ stands for the a-fold direct sum of the trivial representation $\sigma : \mathfrak{S}_3 \to \mathrm{GL}(\mathbb{C})$ with itself; $(\sigma')^{\otimes b}$ stands for the b-fold direct sum of the alternating representation $\sigma' : \mathfrak{S}_3 \to \mathrm{GL}(\mathbb{C})$ with itself; $\rho_V^{\otimes c}$ stands for the c-fold direct sum of the standard representation $\rho_V : \mathfrak{S}_3 \to \mathrm{GL}(V)$ with itself. Now, how do we determine the multiplicities a, b, c?

Suppose $\mathbf{v} \in V$ is an eigenvector of $\rho_V(g_1) \in \mathrm{GL}(V)$ with eigenvalue ω. Then $\rho_V(g_1)\mathbf{v} = \omega\mathbf{v}$. Take $\rho_V^{\otimes c}(g_1) \in \mathrm{GL}(V^c)$. There is a $\mathbf{v}$ in each copy of V in V^c. There are c-many linearly independent eigenvectors in W of $\rho_W(g_1)$ with eigenvalue ω, namely

$$\begin{bmatrix} \mathbf{0}_{a\times 1} \\ \mathbf{0}_{b\times 1} \\ \mathbf{v} \\ \mathbf{0}_{2\times 1} \\ \cdots \\ \mathbf{0}_{2\times 1} \end{bmatrix}, \begin{bmatrix} \mathbf{0}_{a\times 1} \\ \mathbf{0}_{b\times 1} \\ \mathbf{0}_{2\times 1} \\ \mathbf{v} \\ \cdots \\ \mathbf{0}_{2\times 1} \end{bmatrix}, \ldots, \begin{bmatrix} \mathbf{0}_{a\times 1} \\ \mathbf{0}_{b\times 1} \\ \mathbf{0}_{2\times 1} \\ \cdots \\ \mathbf{0}_{2\times 1} \\ \mathbf{v} \end{bmatrix}.$$

Therefore, the number of linearly independent eigenvectors in W of $\rho_W(g_1)$ with eigenvalue ω is equal to c. Now, $\rho_W(g_2)$ has eigenvalues 1 or -1. It has $a + c$ eigenvectors of eigenvalue 1, namely

$$\begin{bmatrix} 1 \\ 0 \\ \cdots \\ 0 \\ \mathbf{0}_{b\times 1} \\ \mathbf{0}_{2c\times 1} \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \\ \cdots \\ 1 \\ \mathbf{0}_{b\times 1} \\ \mathbf{0}_{2c\times 1} \end{bmatrix}, \begin{bmatrix} \mathbf{0}_{a\times 1} \\ \mathbf{0}_{b\times 1} \\ \mathbf{v} + \rho_V(g_2)\mathbf{v} \\ \mathbf{0}_{2\times 1} \\ \cdots \\ \mathbf{0}_{2\times 1} \end{bmatrix}, \ldots, \begin{bmatrix} \mathbf{0}_{a\times 1} \\ \mathbf{0}_{b\times 1} \\ \mathbf{0}_{2\times 1} \\ \cdots \\ \mathbf{0}_{2\times 1} \\ \mathbf{v} + \rho_V(g_2)\mathbf{v} \end{bmatrix}.$$

Finally, $\rho_W(g_2)$ has $b+c$ eigenvectors of eigenvalue -1, namely

$$\begin{bmatrix} \mathbf{0}_{a\times 1} \\ 1 \\ 0 \\ \cdots \\ 0 \\ \mathbf{0}_{2c\times 1} \end{bmatrix}, \ldots, \begin{bmatrix} \mathbf{0}_{a\times 1} \\ 0 \\ 0 \\ \cdots \\ 1 \\ \mathbf{0}_{2c\times 1} \end{bmatrix}, \begin{bmatrix} \mathbf{0}_{a\times 1} \\ \mathbf{0}_{b\times 1} \\ \mathbf{v} - \rho_V(g_2)\,\mathbf{v} \\ \mathbf{0}_{2\times 1} \\ \cdots \\ \mathbf{0}_{2\times 1} \end{bmatrix}, \ldots, \begin{bmatrix} \mathbf{0}_{a\times 1} \\ \mathbf{0}_{b\times 1} \\ \mathbf{0}_{2\times 1} \\ \cdots \\ \mathbf{0}_{2\times 1} \\ \mathbf{v} - \rho_V(g_2)\,\mathbf{v} \end{bmatrix}.$$

Hence, the nonnegative integer $a+c$ in (2.63) is the multiplicity of the eigenvalue 1 of $\rho_W(g_2)$; and $b+c$ is the multiplicity of the eigenvalue -1 of $\rho_W(g_2)$.

§2.4 Projection Formulae

Recall that if V and W are two finite dimensional complex representations of a finite group G, then $\mathrm{Hom}_G(V,W)$ is the vector space of all G-linear maps (sometimes called G-module homomorphisms) from the finite dimensional complex representation V to the finite dimensional complex representation W of the finite group G. Now, given any representation $\rho : G \to \mathrm{GL}(V)$, we define

$$V^G = \{\in V \mid \rho(g)\,\mathbf{v} = \mathbf{v} \text{ for every } g \in G\}. \tag{2.64}$$

Observe that for a given $g_0 \in G$, the automorphism $\rho(g_0) : V \to V$ is not necessarily a G-module homomorphism as $\rho(g) \circ \rho(g_0)$ and $\rho(g_0) \circ \rho(g)$ are not necessarily equal for every $g \in G$. If we, instead, take the average of all the automorphisms $\rho(g) \in \mathrm{GL}(V)$, for all $g \in G$, and denote it by φ, i.e.

$$\varphi = \frac{1}{|G|}\sum_{g\in G}\rho(g), \tag{2.65}$$

then φ is a G-module homomorphism. Indeed, for any $g' \in G$,

$$\rho(g') \circ \varphi = \frac{1}{|G|}\rho(g')\sum_{g\in G}\rho(g) = \frac{1}{|G|}\sum_{g\in G}\rho(g')\,\rho(g) = \frac{1}{|G|}\sum_{g\in G}\rho(g'g) = \frac{1}{|G|}\sum_{g\in G}\rho(g).$$

$$\varphi \circ \rho(g') = \frac{1}{|G|}\left(\sum_{g\in G}\rho(g)\right)\rho(g') = \frac{1}{|G|}\sum_{g\in G}\rho(g)\rho(g') = \frac{1}{|G|}\sum_{g\in G}\rho(g') = \frac{1}{|G|}\sum_{g\in G}\rho(g).$$

Therefore,

$$\rho(g') \circ \varphi = \varphi \circ \rho(g') = \varphi \tag{2.66}$$

for every $g' \in G$, i.e. the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "\rho(g')"'] & V \arrow[d, "\rho(g')"] \\ V \arrow[r, "\varphi"'] & V \end{tikzcd}$$

Proposition 2.3
The map $\varphi : V \to V^G$ is a projection of V onto V^G.

Proof. Let us first show that $\operatorname{im}\varphi = V^G$. Suppose $\mathbf{v} = \varphi(\mathbf{w})$. Then for any $h \in G$,

$$\rho(h)\,\mathbf{v} = [\rho(h) \circ \varphi](\mathbf{w}) = \varphi(\mathbf{w}) = \mathbf{v}, \tag{2.67}$$

since we proved $\rho(h) \circ \varphi = \varphi$ in (2.66). So we have $\rho(h)\,\mathbf{v} = \mathbf{v}$ for any $h \in G$. Therefore, $\mathbf{v} \in V^G$, i.e. $\operatorname{im}\varphi \subseteq V^G$.

Conversely, suppose $\mathbf{v} \in V^G$. Then $\rho(g)\,\mathbf{v} = \mathbf{v}$ for any $g \in G$. So

$$\varphi(\mathbf{v}) = \frac{1}{|G|}\sum_{g\in G}\rho(g)\mathbf{v} = \frac{1}{|G|}\sum_{g\in G}\mathbf{v} = \mathbf{v}. \tag{2.68}$$

So $\mathbf{v} = \varphi(\mathbf{v}) \in \operatorname{im}\varphi$, i.e. $V^G \subseteq \operatorname{im}\varphi$. Hence, $\operatorname{im}\varphi = V^G$.

Now, for $\mathbf{v} \in V$,

$$(\varphi\circ\varphi)(\mathbf{v}) = \varphi(\varphi(\mathbf{v})) = \varphi(\mathbf{v}), \tag{2.69}$$

since $\varphi(\mathbf{v}) \in V^G$ and we showed earlier that $\varphi(\mathbf{w}) = \mathbf{w}$ for $\mathbf{w} \in V^G$. Therefore, $\varphi : V \to V^G$ is a surjective map satisfying $\varphi\circ\varphi = \varphi$. So it is a projection map of V onto V^G. ■

Given a finite dimensional complex representation V of the finite group G, we want to calculate the dimension of the vector space V^G. We refer back to the projection map $\varphi : V \to V^G$. One can decompose V as $V = V^G \oplus \operatorname{Ker}\varphi$. Now, one can form a basis $\mathcal{B}$ of V by taking the union of a basis of V^G and a basis of $\operatorname{Ker}\varphi$. In this chosen basis of V, φ can be expressed as the following block-diagonal matrix:

$$[\varphi]_{\mathcal{B}} = \begin{bmatrix} \mathbb{1}_{V^G} & \\ & \mathbf{0}_{k\times k} \end{bmatrix},$$

where $k = \dim\operatorname{Ker}\varphi$. From this block-diagonal form, one obtains

$$\dim V^G = \operatorname{Tr}\varphi = \frac{1}{G}\sum_{g\in G}\operatorname{Tr}(\rho(g)) = \frac{1}{G}\sum_{g\in G}\chi_\rho(g). \tag{2.70}$$

If one denotes $\dim_{\mathbb{C}} V^G = m$, then one immediately finds that the nonnegative integer m is precisely the number of times the trivial (1-dimensional) representation of G appears in the direct sum decomposition of V. In particular, if V is an irreducible representation other than the trivial representation of G, then since there is no ptoper G-invariant subspace of V, one must have $\dim V^G = 0$. In other words, if $\rho : G \to \mathrm{GL}(V)$ is an irreducible representation (other than the trivial representation), then

$$\sum_{g\in G}\chi_\rho(g) = 0. \tag{2.71}$$

Now, given two finite dimensional representations V and W with associated group homomorphisms $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$, $\operatorname{Hom}(V, W)$ is also a representation with group homomorphism $\gamma : G \to \mathrm{GL}(\operatorname{Hom}(V, W))$ defined by

$$\gamma(g)\,\psi = \sigma(g)\circ\psi\circ\rho\left(g^{-1}\right). \tag{2.72}$$

In other words, the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "\psi"] & W \arrow[d, "\sigma(g)"] \\ V \arrow[u, "\rho(g^{-1})"] \arrow[r, "\gamma(g)\psi"'] & W \end{tikzcd}$$

Now, using the definition (2.64),

$$\operatorname{Hom}(V, W)^G = \{\psi \in \operatorname{Hom}(V, W) \mid \gamma(g)\,\psi = \psi \text{ for every } g \in G\} \tag{2.73}$$

Proposition 2.4

$\operatorname{Hom}_G(V, W) = \operatorname{Hom}(V, W)^G$.

Proof.

$$\begin{aligned}
\psi \in \mathrm{Hom}\,(V,W)^G &\iff \gamma\,(g)\,\psi = \psi \text{ for every } g \in G\\
&\iff \sigma\,(g)\circ\psi\circ\rho\left(g^{-1}\right) = \psi \text{ for every } g \in G\\
&\iff \sigma\,(g)\circ\psi = \psi\circ\rho\,(g)\ \text{ for every } g \in G\\
&\iff \psi \in \mathrm{Hom}_G\,(V,W)\,,
\end{aligned}$$

because $\sigma\,(g)\circ\psi = \psi\circ\rho\,(g)$ is equivalent to the commutativity of the following diagram:

$$\begin{tikzcd}
V \arrow[r, "\psi"] \arrow[d, "\rho(g)"'] & W \arrow[d, "\sigma(g)"] \\
V \arrow[r, "\psi"'] & W
\end{tikzcd}$$

Therefore, $\mathrm{Hom}\,(V,W)^G = \mathrm{Hom}_G\,(V,W)$. ■

Remark 2.4. Note that in Proposition 2.4, on the right side, $\mathrm{Hom}\,(V,W)$ is the representation of G given by the group homomorphism $\gamma : G \to \mathrm{GL}\,(\mathrm{Hom}\,(V,W))$ defined by (2.72). On the left hand side of Proposition 2.4, $\mathrm{Hom}_G\,(V,W)$ is the vector space of all G-module homomorphisms from the finite dimensional complex representation V to the finite dimensional complex representation W.

If V is irreducible and W is reducible with the multiplicity of V in the decomposition of W being m, i.e. $W = V^m \oplus \cdots$, then by Proposition 1.8,

$$m = \dim_{\mathbb{C}} \mathrm{Hom}_G\,(V,W) = \dim_{\mathbb{C}} \mathrm{Hom}\,(V,W)^G\,. \tag{2.74}$$

Similarly, if W is irreducible and V is reducible with the multiplicity of W in the decomposition of V being n, i.e. $V = W^n \oplus \cdots$, then by Proposition 1.8,

$$n = \dim_{\mathbb{C}} \mathrm{Hom}_G\,(V,W) = \dim_{\mathbb{C}} \mathrm{Hom}\,(V,W)^G\,. \tag{2.75}$$

When both the representations V and W of the finite group G are irreducibles, then

$$\dim_{\mathbb{C}} \mathrm{Hom}\,(V,W)^G = \begin{cases} 1 & \text{if } V \cong W \text{ as representations;}\\ 0 & \text{if } V \ncong W \text{ as representations.}\end{cases} \tag{2.76}$$

In Proposition 1.1, we showed that $\mathrm{Hom}\,(V,W)$ and $V^* \otimes W$ are isomorphic as representations, i.e. $\gamma \cong \rho^* \otimes \sigma$. Now, using Proposition 2.1, we get

$$\chi_\gamma\,(g) = \overline{\chi_\rho\,(g)}\,\chi_\sigma(g). \tag{2.77}$$

Now, using (2.70),

$$\dim_{\mathbb{C}} \mathrm{Hom}\,(V,W)^G = \frac{1}{|G|}\sum_{g\in G}\chi_\gamma\,(g) = \frac{1}{|G|}\sum_{g\in G}\overline{\chi_\rho\,(g)}\,\chi_\sigma(g). \tag{2.78}$$

In the case when both V and W are irreducible representations, with the respective group homomorphisms $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$, then

$$\frac{1}{|G|}\sum_{g\in G}\overline{\chi_\rho\,(g)}\,\chi_\sigma(g) = \begin{cases} 1 & \text{if } V \cong W,\\ 0 & \text{if } V \ncong W.\end{cases} \tag{2.79}$$

(Here, the isomorphism is isomorphism of representations.)

Definition 2.2 (Class functions). A **class function** on G is a complex valued function $f : G \to \mathbb{C}$ that is constant on the conjugacy classes of G. We will denote the space of all class functions on a finite group G by $\mathbb{C}_{\text{class}}[G]$.

Character associated with a finite dimensional representation is an example of a class function. Now we define a Hermitian inner product on $\mathbb{C}_{\text{class}}[G]$ by

$$(\alpha, \beta) = \frac{1}{|G|} \sum_{g \in G} \overline{\alpha(g)}\, \beta(g). \tag{2.80}$$

Then (2.79) translates into the following theorem.

Theorem 2.5

In terms of the inner product (2.80), the characters of the irreducible representations of G are orthonormal.

$\mathbb{C}_{\text{class}}[G]$ is, in fact, a complex inner product space endowed with the hermitial inner product given by (2.80). The dimension of $\mathbb{C}_{\text{class}}[G]$ is the number of conjugacy classes of G. Theorem 2.5 tells us that the irreducible characters are linearly independent, so that the number of irreducible representations is less than or equal to the number of conjugacy classes. We will soon prove that these two are, indeed, the same.

Corollary 2.6

Any representation is determined by its character. In other words, if $\sigma_1 : G \to \mathrm{GL}(V), \sigma_2 : G \to \mathrm{GL}(W)$ are two representations of G such that $\chi_{\sigma_1} = \chi_{\sigma_2}$, then σ_1 and σ_2 are isomorphic representations.

Proof. Suppose $\rho_i : G \to \mathrm{GL}(V_i)$ are all the irreducible representations, for $i = 1, \ldots, k$. We express V and W as direct sum of irreducible representations:

$$V = \bigoplus_{i=1}^{k} V_i^{a_i} \text{ and } W = \bigoplus_{i=1}^{k} V_i^{b_i}, \tag{2.81}$$

for $a_i, b_i \in \mathbb{Z}_{\geq 0}$.

$$\chi_{\sigma_1} = \sum_{i=1}^{k} a_i \chi_{\rho_i} \text{ and } \chi_{\sigma_2} = \sum_{i=1}^{k} b_i \chi_{\rho_i}. \tag{2.82}$$

Since $\chi_{\sigma_1} = \chi_{\sigma_2}$, we have

$$\sum_{i=1}^{k} (a_i - b_i)\, \chi_{\rho_i} = 0. \tag{2.83}$$

Since $\{\chi_{\rho_i}\}_{i=1}^{k}$ is a linearly independent set in the space of all class functions, we must have $a_i - b_i = 0$ for each i. Therefore, $a_i = b_i$ for each i, and hence, σ_1 and σ_2 are isomorphic representations. ∎

Corollary 2.7

A representation $\rho : G \to \mathrm{GL}(V)$ is irreducible if and only if $(\chi_\rho, \chi_\rho) = 1$.

Proof. We have already proved one direction: if $\rho : G \to \mathrm{GL}(V)$ is irreducible, then $(\chi_\rho, \chi_\rho) = 1$, by Theorem 2.5. Conversely, suppose $(\chi_\rho, \chi_\rho) = 1$. Suppose $\rho_i : G \to \mathrm{GL}(V_i)$ are all the irreducible representations, for $i = 1, \ldots, k$. We express V and W as direct sum of irreducible representations:

$$V = \bigoplus_{i=1}^{k} V_i^{a_i}, \tag{2.84}$$

for $a_i \in \mathbb{Z}_{\geq 0}$. Then

$$\chi_\rho = \sum_{i=1}^{k} a_i \chi_{\rho_i}. \tag{2.85}$$

Now, the sesqui-linearity of inner product along with the orthonormality of irreducible character gives us

$$\begin{aligned} 1 &= (\chi_\rho, \chi_\rho) \\ &= \left(\sum_{i=1}^{k} a_i \chi_{\rho_i}, \sum_{j=1}^{k} a_j \chi_{\rho_j} \right) \\ &= \sum_{i,j=1}^{k} \overline{a_i}\, a_j \left(\chi_{\rho_i}, \chi_{\rho_j} \right) \\ &= \sum_{i,j=1}^{k} \overline{a_i}\, a_j \; \delta_{ij} \\ &= \sum_{i=1}^{k} |a_i|^2 . \end{aligned} \tag{2.86}$$

a_i are each non-negative integers, and their square-sum is 1. This is only possible when $a_{i_0} = 1$ for some i_0, and $a_i = 0$ for other $i \neq i_0$. Therefore, $\rho = \rho_{i_0}$, and hence ρ is irreducible. ∎

Corollary 2.8
Let $\rho_i : G \to \mathrm{GL}(V_i)$ be an irreducible representation, and $\rho : G \to \mathrm{GL}(V)$ be any other representation. Then the multiplicity a_i of V_i in V is given by

$$a_i = (\chi_\rho, \chi_{\rho_i}) = (\chi_{\rho_i}, \chi_\rho) . \tag{2.87}$$

Proof. Follows trivially from (2.74), (2.75), (2.78). ∎

Corollary 2.9
Any irreducible representation V_i appears in the regular representation with multiplicity $\dim V_i$.

Proof. Let $R = \mathbb{C}[G]$ be the vector space on which the regular representation acts on, and $\rho : G \to \mathrm{GL}\left(\mathbb{C}\left[G\right]\right)$ be the associated group homomorphism. As we know that regular representation is a special case of permutation representation, with the set X being $G_{\mathbf{Set}}$.

$$\rho\left(h\right) \left(\sum_{g \in G} a_g \delta_g \right) = \sum_{g \in G} a_g \delta_{\sigma(h) g}, \tag{2.88}$$

where $\sigma(h) : G_{\mathbf{Set}} \to G_{\mathbf{Set}}$ is a bijection (i.e. permutation), which we define as $\sigma(h)g = hg$. Therefore, the character $\chi_\rho(h)$ of the regular representation indicates the number of elements of $G_{\mathbf{Set}}$ fixed by $\sigma\left(h\right)$ (Lemma 2.2).

$$\sigma(h)g = g \iff hg = g \iff h = e. \tag{2.89}$$

If $h = e$, then all the elements of $G_{\mathbf{Set}}$ are fixed by $\sigma(e)$. Otherwise, none of the elements are fixed. So

$$\chi_\rho(h) = \begin{cases} |G| & \text{if } h = e, \\ 0 & \text{otherwise.} \end{cases} \tag{2.90}$$

Let $\rho_i : G \to \mathrm{GL}\left(V_i\right)$ be an irreducible representation. Then the number of times V_i appears in the regular representation R is given by

$$\left(\chi_\rho, \chi_{\rho_i}\right) = \frac{1}{|G|}\sum_{g\in G}\overline{\chi_\rho(g)}\,\chi_{\rho_i}(g) = \frac{1}{|G|}\overline{\chi_\rho(e)}\,\chi_{\rho_i}(e) = \frac{1}{|G|}\,|G|\,\dim V_i = \dim V_i. \tag{2.91}$$

Therefore, V_i appears in R with multiplicity $\dim V_i$. ■

Proposition 2.10

Let $\alpha : G \to \mathbb{C}$ be any function on G, and V be a complex representation of G with the group homomorphism $\rho : G \to \mathrm{GL}(V)$. Let

$$\phi_{\alpha,V} = \sum_{g\in G}\alpha(g)\rho(g) : V \to V$$

be a linear map. Then $\phi_{\alpha,V} \in \mathrm{Hom}_G\left(V, V\right)$ for all V if and only if α is a class function.

Proof. Suppose α is a class function. To prove that $\phi_{\alpha,V} \in \mathrm{Hom}_G\left(V, V\right)$, we need to show that for all $h \in G$, $\phi_{\alpha,V} \circ \rho(h) = \rho(h) \circ \phi_{\alpha,V}$.

$$\phi_{\alpha,V} \circ \rho(h) = \sum_{g\in G}\alpha(g)\rho(g)\rho\left(h\right). \tag{2.92}$$

Write $h^{-1}gh = g'$ so that $g = hg'h^{-1}$. Since h is fixed, as g varies in G, g' also varies in G. Hence,

$$\phi_{\alpha,V} \circ \rho(h) = \sum_{g'\in G}\alpha\left(hg'h^{-1}\right)\rho\left(hg'h^{-1}\right)\rho\left(h\right). \tag{2.93}$$

Since α is a class function, $\alpha(hg'h^{-1}) = \alpha(g')$. So

$$\phi_{\alpha,V} \circ \rho(h) = \sum_{g'\in G}\alpha\left(g'\right)\rho\left(hg'\right) = \rho\left(h\right)\sum_{g'\in G}\alpha\left(g'\right)\rho\left(g'\right) = \rho\left(h\right)\circ\phi_{\alpha,V}. \tag{2.94}$$

So $\phi_{\alpha,V} \in \mathrm{Hom}_G\left(V, V\right)$.

Conversely, assume α is not a class function. Then we shall prove that $\phi_{\alpha,V}$ is not a G-linear map, for $V = \mathbb{C}\left[G\right]$, the regular representation. Since α is not a class function, there exists $h, k \in G$ such that $\alpha\left(h^{-1}k\right) \neq \alpha\left(kh^{-1}\right)$.

Assume for the sake of contradiction that $\phi_{\alpha,V}$ is a G-linear map. Then, $\phi_{\alpha,V} \circ \rho(h) = \rho(h) \circ \phi_{\alpha,V}$. In other words,

$$\left[\sum_{g\in G}\alpha(g)\rho(g)\right]\circ\rho h = \rho\left(h\right)\circ\left[\sum_{g\in G}\alpha(g)\rho(g)\right]. \tag{2.95}$$

We can rewrite it as follows:

$$\sum_{g\in G}\alpha(g)\rho(gh) = \sum_{g\in G}\alpha(g)\rho(hg). \tag{2.96}$$

With the change of variable $gh \to g'$ on LHS and $hg \to g'$ on RHS, we have

$$\sum_{g'\in G}\alpha\left(g'h^{-1}\right)\rho\left(g'\right) = \sum_{g\in G}\alpha\left(h^{-1}g'\right)\rho\left(g'\right). \tag{2.97}$$

Since these two are equal, they'll yield the same value when acted on $\delta_e \in \mathbb{C}[G]$. Hence,

$$\begin{aligned}\sum_{g'\in G}\alpha\left(g'h^{-1}\right)\rho\left(g'\right)\left(\delta_e\right) &= \sum_{g\in G}\alpha\left(h^{-1}g'\right)\rho\left(g'\right)\left(\delta_e\right)\\ \implies \sum_{g'\in G}\alpha\left(g'h^{-1}\right)\delta_{g'} &= \sum_{g'\in G}\alpha\left(h^{-1}g'\right)\delta_{g'}.\end{aligned} \tag{2.98}$$

Since $\left\{\delta_{g'}\right\}_{g'\in G}$ is a basis for $\mathbb{C}[G]$, (2.98) gives us that $\alpha\left(g'h^{-1}\right) = \alpha\left(h^{-1}g'\right)$ for every $g' \in G$. But we know that there exists $k \in G$ with $\alpha\left(h^{-1}k\right) \neq \alpha\left(kh^{-1}\right)$. Thus we arrive at a contradiction! Therefore, $\phi_{\alpha,V}$ is not a G-linear map, for $V = \mathbb{C}\left[G\right]$, if α is not a class function. ■

Lemma 2.11

A complex representation $\rho : G \to \mathrm{GL}(V)$ is irreducible if and only if its dual representation $\rho^* : G \to \mathrm{GL}(V^*)$ is irreducible.

Proof.

$$\begin{aligned}
\rho \text{ is irreducible} &\iff (\chi_\rho, \chi_\rho) = 1 \\
&\iff \frac{1}{|G|} \sum_{g \in G} \overline{\chi_\rho(g)}\, \chi_\rho(g) = 1 \\
&\iff \frac{1}{|G|} \sum_{g \in G} \chi_{\rho^*}(g)\, \overline{\chi_{\rho^*}(g)} = 1 \\
&\iff \frac{1}{|G|} \sum_{g \in G} (\chi_{\rho^*}, \chi_{\rho^*}) = 1 \\
&\iff \rho^* \text{ is irreducible.}
\end{aligned}$$

■

Definition 2.3 (Irreducible Characters)**.** The characters of the irreducible representations are called **irreducible characters**.

Theorem 2.12

The set of irreducible characters forms an orthonormal basis of $\mathbb{C}_{\text{class}}[G]$.

Proof. Let $\alpha \in \mathbb{C}_{\text{class}}[G]$ and $(\alpha, \chi_\rho) = 0$ for every irreducible representation $\rho : G \to \mathrm{GL}(V)$. We need to show that $\alpha = 0$. That would prove that $\{\chi_\rho\}_\rho$ spans $\mathbb{C}_{\text{class}}[G]$.

Consider $\phi_{\alpha,V} = \sum_{g \in G} \alpha(g)\rho(g) : V \to V$. By Proposition 2.10, $\phi_{\alpha,V} \in \mathrm{Hom}_G(V, V)$. Since V is an irreducible representation, by Schur's lemma, $\dim \mathrm{Hom}_G(V, V) = 1$. Since $\mathbb{1}_V \in \mathrm{Hom}_G(V, V)$, one must have $\phi_{\alpha,V} = \lambda \mathbb{1}_V$ for some $\lambda \in \mathbb{C}$. Let $n = \dim V$. Taking trace on both sides of $\phi_{\alpha,V} = \lambda \mathbb{1}_V$, we have

$$\begin{aligned}
\mathrm{Tr}\,\phi_{\alpha,V} = \lambda\,\mathrm{Tr}\,\mathbb{1}_V &\implies \mathrm{Tr}\left[\sum_{g \in G} \alpha(g)\rho(g)\right] = \lambda n \\
&\implies \sum_{g \in G} \alpha(g)\,\mathrm{Tr}\,\rho(g) = \lambda n \\
&\implies \sum_{g \in G} \alpha(g)\chi_\rho(g) = \lambda n \\
&\implies \sum_{g \in G} \overline{\alpha(g)}\,\overline{\chi_\rho(g)} = \overline{\lambda n} = \lambda n \\
&\implies \frac{1}{|G|} \sum_{g \in G} \overline{\alpha(g)}\chi_{\rho^*}(g) = \frac{n}{|G|}\lambda \\
&\implies (\alpha, \chi_{\rho^*}) = \frac{n}{|G|}\lambda
\end{aligned}$$

Since ρ is irreducible, so is ρ^*. By hypothesis, $(\alpha, \chi_\rho) = 0$ for every irreducible representation ρ. Therefore, $(\alpha, \chi_{\rho^*}) = 0$. $\frac{n}{|G|} \neq 0$, so $\lambda = 0$. This gives us $\phi_{\alpha,V} = 0$ for every irreducible representation $\rho : G \to \mathrm{GL}(V)$, i.e.

$$\sum_{g \in G} \alpha(g)\rho(g) = 0. \tag{2.99}$$

One can, therefore, conclude that for any representation $W = \bigoplus_{i=1}^{k} V_i^{r_i}$ of G, associated with the group homomorphism $\sigma : G \to \mathrm{GL}\,(W) = \mathrm{GL}\left(\bigoplus_{i=1}^{k} V_i^{r_i}\right)$,

$$\phi_{\alpha,W} = \sum_{g\in G} \alpha\,(g)\,\sigma\,(g) = 0, \tag{2.100}$$

i.e. the endomorophism $\phi_{\alpha,W}$ is the zero map. In particular, (2.100) holds for the left-regular representation $\mathbb{C}[G]$ of G. The group homomorphism associated with the left-regular representation is $\sigma : G \to \mathrm{GL}\,(\mathbb{C}[G])$. Here, $\{\delta_g \mid g \in G\}$ is a basis for $\mathbb{C}[G]$. Since $\phi_{\alpha,\mathbb{C}[G]} = 0$, it will give out 0 if acted upon δ_e. Hence,

$$0 = \left[\sum_{g\in G} \alpha\,(g)\,\sigma\,(g)\right](\delta_e) = \sum_{g\in G} \alpha\,(g)\,\delta_g. \tag{2.101}$$

$\{\delta_g \mid g \in G\}$ is a basis for $\mathbb{C}[G]$. Therefore, $\sum_{g\in G} \alpha\,(g)\,\delta_g = 0$ implies $\alpha\,(g) = 0$ for every $g \in G$, i.e. $\alpha : G \to \mathbb{C}$ has to be the 0-function. ■

Note that $\mathbb{C}_{\text{class}}\,[G]$ has a basis of complex valued functions which are 1 on a given conjugacy class and 0 otherwise (characteristic functions on conjugacy classes of the group). The number of such characteristic functions is precisely the total number of conjugacy classes of the group. Hence, the dimension of the complex vector space $\mathbb{C}_{\text{class}}[G]$ is the number of conjugacy classes of the group G. By Theorem 2.12, on the other hand, the number of irreducible characters and hence the number of irreducible representation of G is also equal to the dimension of $\mathbb{C}_{\text{class}}[G]$.

Corollary 2.13

The number of irreducible representations of G is equal to the number of conjugacy class of G.