Differential Geometry II (MAT401)

Lecture Notes

Preface

This series of lecture notes has been prepared for aiding students who took the BRAC University course **Differential Geometry II (MAT401)** in Fall 2023 semester. These notes were typeset under the supervision of mathematician **Dr. Syed Hasibul Hassan Chowdhury**. The main goal of this typeset is to have an organized digital version of the notes, which is easier to share and handle. If you see any mistakes or typos, please send me an email at atonuroychowdhury@gmail.com

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References:

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- *An Introduction to Differentiable Manifolds and Riemannian Geometry*, by **William Boothby**
- *Introduction to Smooth Manifolds*, by **John M. Lee**
- *Lectures on Differential geometry*, by **S.S Chern, W.H. Chen and K.S. Lam**
- *Geometry of Differential Forms*, by **Shigeyuki Morita**
- *From Calculus to Cohomology: De Rham Cohomology and Characteristic Classes*, by **Ib Madsen and Jxrgen Tornehave**.

Contents

1 Review of Multilinear Algebra

§1.1 Dual Space

Let V and W be real vector spaces. We denote by $\mathrm{Hom}\,(V, W)$ the vector space of all linear maps $f : V \to W$. In particular, if we choose $W = \mathbb{R}$, we get the **dual space** V^*.

$$V^* = \mathrm{Hom}\,(V, \mathbb{R})\,.$$

The elements of V^* are called covectors on V. In the rest of the lecture, we will assume V to be a finite dimensional vector space. Let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be a basis for V. Then every $\mathbf{v} \in V$ is a unique linear combination

$$\mathbf{v} = \sum_{i=1}^{n} v^i \mathbf{e}_i, \tag{1.1}$$

with $v^i \in \mathbb{R}$. v^i's are called the coordinates of $\mathbf{v}$ relative to the basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$. Let $\widehat{\alpha}^i$ be the linear function on V that picks up the i-th coordinate of the vector, i.e.

$$\widehat{\alpha}^i\,(\mathbf{v}) = \widehat{\alpha}^i \left(\sum_{i=1}^{n} v^i \mathbf{e}_i \right) = v^i. \tag{1.2}$$

When $\mathbf{v}$ is one of the basis vectors,

$$\widehat{\alpha}^i\,(\mathbf{e}_j) = \delta^i{}_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{1.3}$$

Proposition 1.1

The functions $\widehat{\alpha}^1, \ldots, \widehat{\alpha}^n$ form a basis for V^*.

Proof. Suppose $f \in V^*$. Then for any $\mathbf{v} = \sum_{i=1}^{n} v^i \mathbf{e}_i \in V$,

$$f\,(\mathbf{v}) = f\left(\sum_{i=1}^{n} v^i \mathbf{e}_i \right) = \sum_{i=1}^{n} v^i f\,(\mathbf{e}_i) = \sum_{i=1}^{n} f\,(\mathbf{e}_i)\,\widehat{\alpha}^i\,(\mathbf{v})\,.$$

Since this holds for any $\mathbf{v} \in V$,

$$f = \sum_{i=1}^{n} f\,(\mathbf{e}_i)\,\widehat{\alpha}^i. \tag{1.4}$$

Therefore, $\widehat{\alpha}^1, \ldots, \widehat{\alpha}^n$ span V^*. As for linear independence, suppose

$$\sum_{i=1}^{n} c_i \widehat{\alpha}^i = \mathbf{0}, \tag{1.5}$$

where $\mathbf{0}$ is the function that takes all of V to $0 \in \mathbb{R}$. If we evaluate (1.5) at $\mathbf{e}_j$, we get

$$0 = \sum_{i=1}^{n} c_i \widehat{\alpha}^i\,(\mathbf{e}_j) = \sum_{i=1}^{n} c_i \delta^i{}_j = c_j. \tag{1.6}$$

So $c_j = 0$, and this holds for each $j = 1, 2, \ldots, n$. Therefore, $\{\widehat{\alpha}^1, \ldots, \widehat{\alpha}^n\}$ is a linearly independent set that spans V^*, i.e. a basis. ■

Corollary 1.2

The dual space V^* of a finite dimensional vector space has the same dimension as V.

The basis $\{\widehat{\alpha}^1, \ldots, \widehat{\alpha}^n\}$ for V^* is said to be dual to the basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ for V.

§1.2 Permutations

Fix a positive integer k. A permutation of the set $A = \{1, 2, \ldots, k\}$ is a bijection $\sigma : A \to A$. The product of two permutations τ and σ is the composition $\tau \circ \sigma : A \to A$. The **cyclic permutation** $(a_1\ a_2\ \cdots\ a_r)$ is the permutation σ such that

$$\sigma(a_1) = a_2,\ \sigma(a_2) = a_3,\ \cdots,\ \sigma(a_{r-1}) = a_r,\ \text{and } \sigma(a_r) = 1,$$

leaving all other elements unchanged, i.e. $\sigma(j) = j$ if j is not one of the a_i's. A cyclic permutation $(a_1\ a_2\ \cdots\ a_r)$ is also called a **cycle** of length r or an r-cycle. A **transposition** is a permutation of the form $(a\ b)$ that interchanges a and b, leaving all other elements of A fixed.

A permutation $\sigma : A \to A$ can be described by

$$\begin{bmatrix} 1 & 2 & \cdots & k \\ \sigma(1) & \sigma(2) & \cdots & \sigma(k) \end{bmatrix}.$$

We also write it as

$$[1\ 2\ \cdots\ k] \overset{\sigma}{\mapsto} [\sigma(1)\ \sigma(2)\ \cdots\ \sigma(k)].$$

Example 1.1. Suppose $\sigma : \{1, 2, 3, 4, 5\} \to \{1, 2, 3, 4, 5\}$ is the permutation given by

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 5 & 1 & 3 \end{bmatrix}.$$

In other words, $\sigma(1) = 2$, $\sigma(2) = 4$, $\sigma(3) = 5$, $\sigma(4) = 1$, and $\sigma(5) = 3$.

$$[1\ 2\ 3\ 4\ 5] \overset{\sigma}{\mapsto} [2\ 4\ 5\ 1\ 3].$$

Observe that the cyclic permutation $\sigma' = (1\ 2\ 4)$ acts as $\sigma'(1) = 2$, $\sigma'(2) = 4$ and $\sigma'(4) = 1$, keeping 3 and 5 unchanged, i.e. $\sigma'(3) = 3$ and $\sigma'(5) = 5$.

$$[1\ 2\ 3\ 4\ 5] \xmapsto{(1\ 2\ 4)} [2\ 4\ 3\ 1\ 5].$$

Now the transposition $\sigma'' = (3\ 5)$ acts as $\sigma''(3) = 5$ and $\sigma''(5) = 3$, keeping $1, 2, 4$ unchanged. Therefore,

$$\begin{tikzcd}
{[1\ 2\ 3\ 4\ 5]} \arrow[r, mapsto, "(1\ 2\ 4)"] \arrow[rr, bend right, "(3\ 5)(1\ 2\ 4)"'] & {[2\ 4\ 3\ 1\ 5]} \arrow[r, mapsto, "(3\ 5)"] & {[2\ 4\ 5\ 1\ 3]}
\end{tikzcd}$$

so that $\sigma = (3\ 5)(1\ 2\ 4)$.

Let S_k be the group of permutations of the set $\{1, 2, \ldots, k\}$. The order of this group is $k!$. A permutation is *even* or *odd* depending on whether it is the product of an even or an odd number of transpositions. The sign of a permutation σ is 1 if the permutation is even, and -1 otherwise. It is denoted by $\operatorname{sgn}\sigma$. For example, in Example 1.1, $\sigma = (3\ 5)(1\ 2\ 4)$. Note that we can write $(1\ 2\ 4)$ as a product of two transpositions:

$$\begin{tikzcd}
{[1\ 2\ 3\ 4\ 5]} \arrow[r, mapsto, "(1\ 2)"] \arrow[rr, bend right, "(1\ 4)(1\ 2)=(1\ 2\ 4)"'] & {[2\ 1\ 3\ 4\ 5]} \arrow[r, mapsto, "(1\ 4)"] & {[2\ 4\ 3\ 1\ 5]}
\end{tikzcd}$$

In other words, $\sigma = (3\ 5)(1\ 4)(1\ 2)$. Hence, $\operatorname{sgn}\sigma = -1$. One can easily check that

$$\operatorname{sgn}(\sigma\tau) = (\operatorname{sgn}\sigma)(\operatorname{sgn}\tau). \tag{1.7}$$

So $\operatorname{sgn} : S_k \to \{1, -1\}$ is a group homomorphism.

Example 1.2. Observe that the 5-cycle (1 2 3 4 5) can be written as

$$(1\ 2\ 3\ 4\ 5) = (1\ 5)\,(1\ 4)\,(1\ 3)\,(1\ 2)\,.$$

Indeed,

$$\begin{tikzcd}
{[1\ 2\ 3\ 4\ 5]} \arrow[r, mapsto, "(1\ 2)"] \arrow[rrrr, bend right, "(1\ 2\ 3\ 4\ 5)=(1\ 5)(1\ 4)(1\ 3)(1\ 2)"'] & {[2\ 1\ 3\ 4\ 5]} \arrow[r, mapsto, "(1\ 3)"] & {[2\ 3\ 1\ 4\ 5]} \arrow[r, mapsto, "(1\ 4)"] & {[2\ 3\ 4\ 1\ 5]} \arrow[r, mapsto, "(1\ 5)"] & {[2\ 3\ 4\ 5\ 1]}
\end{tikzcd}$$

Therefore, $\operatorname{sgn}(1\ 2\ 3\ 4\ 5) = 1$.

An **inversion** in a permutation σ is an ordered pair $(\sigma(i), \sigma(j))$ such that $i < j$ but $\sigma(i) > \sigma(j)$. In Example 1.1, $\sigma(1) = 2$, $\sigma(2) = 4$, $\sigma(3) = 5$, $\sigma(4) = 1$, and $\sigma(5) = 3$. So, the inversions in this permutation are

$$(\sigma(1), \sigma(4))\,,\ (\sigma(2), \sigma(4))\,,\ (\sigma(2), \sigma(5))\,,\ (\sigma(3), \sigma(4))\,,\ (\sigma(3), \sigma(5))\,.$$

Hence, there are 5 inversions associated with the permutation σ. There is an efficient way of determining the sign of a permutation.

Proposition 1.3

A permutation is even if and only if it has an even number of inversions.

Proof. Let $\sigma \in S_k$ with n inversions. We shall prove that we can multiply σ by n transpositions and get the identity permutation. This will prove that $\operatorname{sgn}\sigma = (-1)^n$.

Suppose $\sigma(j_1) = 1$. Then for each $i < j_1$, $(\sigma(i), \sigma(j_1))$ is an inversion, and there are $j_1 - 1$ many of them. These are all the inversions with 1 in the second slot of the ordered pair of inversion. If we now multiply σ by the j_1-cycle

$$(\sigma(1)\ \ 1)\,(\sigma(2)\ \ 1)\cdots(\sigma(j_1-1)\ \ 1)$$

to the left of σ, the resulting permutation σ_1 would be

$$\begin{bmatrix} 1 & 2 & 3 & \cdots & j_1 & j_1+1 & \cdots & k \\ 1 & \sigma(1) & \sigma(2) & \cdots & \sigma(j_1-1) & \sigma(j_1+1) & \cdots & \sigma(k)\,. \end{bmatrix}$$

This permutation has no inversion with 1 in the second slot of the ordered pair of inversion. Suppose now that $\sigma(j_2) = 2$. Now observe that if $(\sigma_1(i), 2)$ is an inversion in σ_1, then either $(\sigma(i), 2)$ (if $i \geq j_1 + 1$) or $\sigma(i-1), 2$ (if $i \leq j_1 - 1$) is an inversion in σ. Therefore, the number of inversions in σ_1 ending in 2 is precisely the same as the number of inversions in σ ending in 2. So following a similar procedure as above, we can multiply σ_1 by i_2-many transpositions to the left (i_2 is the number of transpositions ending in 2) and get

$$\begin{bmatrix} 1 & 2 & 3 & \cdots & k \\ 1 & 2 & \sigma(1) & \cdots & \sigma(k)\,. \end{bmatrix}$$

We can continue these steps for each $j = 1, 2, \ldots, k$, and the number of transpositions required to move j to its natural position is the same as the number of inversions ending in j. In the end we achieve the identity permutation. Therefore, $\operatorname{sgn}\sigma = (-1)^n$, where n is the number of inversions. ∎

§1.3 Multilinear Functions

Definition 1.1. Let V^k be the cartesian product of k-copies of a real vector space V.

$$V^k = \underbrace{V \times V \times \cdots \times V}_{k\text{-copies}}$$

A function $f : V^k \to \mathbb{R}$ is called k-linear if it is linear in each of its k arguments:

$$f(\ldots, a\mathbf{v} + b\mathbf{w}, \ldots) = a\, f(\ldots, \mathbf{v}, \ldots) + b\, f(\ldots, \mathbf{w}, \ldots), \tag{1.8}$$

for $a, b \in \mathbb{R}$ and $\mathbf{v}, \mathbf{w} \in V$.

Instead of 2-linear and 3-linear, it's customary to call "bilinear" and "trilinear", respectively. A k-linear function on V is called a **k-tensor** on V. We will denote the vector space of all k-tensors on V by $L_k(V)$. The vector addition and scalar multiplication of the real vector space $L_k(V)$ is the straightforward pointwise operation.

Example 1.3. The dot product $f(\mathbf{v}, \mathbf{w}) = \mathbf{v} \cdot \mathbf{w}$ on $\mathbb{R}^n$ is bilinear: if $\mathbf{v} = \sum_{i=1}^n v^i \mathbf{e}_i$ and $\mathbf{w} = \sum_{i=1}^n w^i \mathbf{e}_i$, then

$$f(\mathbf{v}, \mathbf{w}) = \mathbf{v} \cdot \mathbf{w} = \sum_{i=1}^n v^i w^i.$$

Example 1.4. The determinant

$$f(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n) = \det \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}$$

viewed as a function of the n column vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is n-linear.

Definition 1.2 (Symmetric and alternating function)**.** A k-linear function $f : V^k \to \mathbb{R}$ is **symmetric** if

$$f\left(\mathbf{v}_{\sigma(1)}, \mathbf{v}_{\sigma(2)}, \ldots, \mathbf{v}_{\sigma(k)}\right) = f(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k), \tag{1.9}$$

for all permutations $\sigma \in S_k$. It is **alternating** if

$$f\left(\mathbf{v}_{\sigma(1)}, \mathbf{v}_{\sigma(2)}, \ldots, \mathbf{v}_{\sigma(k)}\right) = (\operatorname{sgn} \sigma)\, f(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k), \tag{1.10}$$

for all permutations $\sigma \in S_k$.

The dor product function on $\mathbb{R}^n$ in Theorem 1.3 is symmetric, and the determinant function on $\mathbb{R}^n$ in Theorem 1.4 is alternating.

We are especially interested in the vector space $A_k(V)$ of all alternating k-linear functions on a vector space V, for $k > 0$. The elements of $A_k(V)$ are called alternating k-tensors (also known as k-covectors). We define $A_0(V)$ to be $\mathbb{R}$. The elements of $A_0(V)$ are simply constants, which we call 0-covectors. The elements of $A_1(V)$ are simply covectors, i.e. the elements of V^*.

Permutation action on k-linear functions

If $f \in L_k(V)$ and $\sigma \in S_k$, define $\sigma f \in L_k(V)$ as follows:

$$(\sigma f)(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k) = f\left(\mathbf{v}_{\sigma(1)}, \mathbf{v}_{\sigma(2)}, \ldots, \mathbf{v}_{\sigma(k)}\right). \tag{1.11}$$

Thus, f is symmetric if and only if $f = \sigma f$ for all $\sigma \in S_k$; and f is alternating if and only if $\sigma f = (\operatorname{sgn} \sigma) f$ for all $\sigma \in S_k$. When $k = 1$, S_k only has the identity permutation. In that case, a 1-linear function or simply linear function on V is both symmetric and alternating. In particular,

$$A_1(V) = L_1(V) = V^*.$$

Lemma 1.4
If $\sigma, \tau \in S_k$ and $f \in L_k(V)$, then $\tau(\sigma f) = (\tau\sigma) f$.

Proof. For any $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in V$,

$$\begin{aligned}
(\tau(\sigma f))(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k) &= (\sigma f)\left(\mathbf{v}_{\tau(1)}, \mathbf{v}_{\tau(2)}, \ldots, \mathbf{v}_{\tau(k)}\right) \\
&= (\sigma f)(\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k) && [\mathbf{w}_i = \mathbf{v}_{\tau(i)}] \\
&= f\left(\mathbf{w}_{\sigma(1)}, \mathbf{w}_{\sigma(2)}, \ldots, \mathbf{w}_{\sigma(k)}\right) \\
&= f\left(\mathbf{v}_{\tau(\sigma(1))}, \mathbf{v}_{\tau(\sigma(2))}, \ldots, \mathbf{v}_{\tau(\sigma(k))}\right) \\
&= ((\tau\sigma) f)(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k).
\end{aligned}$$

Therefore, $\tau(\sigma f) = (\tau\sigma) f$. ■

Definition 1.3. If G is a group and X is a set, a map

$$\begin{aligned} G \times X &\to X \\ (g, x) &\mapsto g \cdot x \end{aligned}$$

is called a **left action** of G on X if

(i) $e \cdot x = x$, where e is the identity element in G and x is any element in X; and

(ii) $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$, for all $g_1, g_2 \in G$ and $x \in X$.

Similarly, a **right action** of G on X is a map

$$\begin{aligned} X \times G &\to X \\ (x, g) &\mapsto x \cdot g \end{aligned}$$

such that

(i) $x \cdot e = x$, for all $x \in X$; and

(ii) $(x \cdot g_1) \cdot g_2 = x \cdot (g_1 g_2)$, for all $g_1, g_2 \in G$ and $x \in X$.

Symmetrizing and alternating operators

Given $f \in L_k(V)$, there is a way to make it a symmetric k-linear function $\mathcal{S} f$ from it:

$$(\mathcal{S} f)(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k) = \sum_{\sigma \in S_k} f\left(\mathbf{v}_{\sigma(1)}, \mathbf{v}_{\sigma(2)}, \ldots, \mathbf{v}_{\sigma(k)}\right). \tag{1.12}$$

In other words,

$$\mathcal{S} f = \sum_{\sigma \in S_k} \sigma f. \tag{1.13}$$

Similarly, there is a way to make an alternating k-linear function from f:

$$\mathcal{A} f = \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma)\, \sigma f. \tag{1.14}$$

Proposition 1.5 (i) The k-linear function $\mathcal{S} f$ is symmetric.

(ii) The k-linear function $\mathcal{A} f$ is alternating.

Proof. (i) Let $\tau \in S_k$. Then

$$\tau\left(\mathcal{S}f\right) = \tau\left(\sum_{\sigma\in S_k} \sigma f\right). \tag{1.15}$$

The group action of S_k on $L_k\left(V\right)$ is distributive over the vector space addition. Therefore,

$$\tau\left(\mathcal{S}f\right) = \sum_{\sigma\in S_k} \tau(\sigma f) = \sum_{\sigma\in S_k} (\tau\sigma)f. \tag{1.16}$$

As σ varies over S_k, $\tau\sigma$ also varies over S_k. Therefore, $\sum_{\sigma\in S_k}(\tau\sigma)f = \mathcal{S}f$. In other words,

$$\tau\left(\mathcal{S}f\right) = \mathcal{S}f, \tag{1.17}$$

i.e. $\mathcal{S}f$ is symmetric.

(ii) Let $\tau \in S_k$. Then

$$\tau\left(\mathcal{A}f\right) = \tau\left(\sum_{\sigma\in S_k} \left(\operatorname{sgn}\sigma\right)\sigma f\right) = \sum_{\sigma\in S_k} \left(\operatorname{sgn}\sigma\right)\tau(\sigma f) = \sum_{\sigma\in S_k} \left(\operatorname{sgn}\sigma\right)(\tau\sigma)f. \tag{1.18}$$

Since $\left(\operatorname{sgn}\tau\right)^2 = 1$,

$$\begin{aligned}\tau\left(\mathcal{A}f\right) &= \sum_{\sigma\in S_k} \left(\operatorname{sgn}\tau\right)^2\left(\operatorname{sgn}\sigma\right)(\tau\sigma)f \\ &= \left(\operatorname{sgn}\tau\right)\sum_{\sigma\in S_k} \left(\operatorname{sgn}\tau\right)\left(\operatorname{sgn}\sigma\right)(\tau\sigma)f \\ &= \left(\operatorname{sgn}\tau\right)\sum_{\sigma\in S_k} \left(\operatorname{sgn}(\tau\sigma)\right)(\tau\sigma)f. \end{aligned} \tag{1.19}$$

As σ varies over S_k, $\tau\sigma$ also varies over S_k. Therefore, $\sum_{\sigma\in S_k}\left(\operatorname{sgn}(\tau\sigma)\right)(\tau\sigma)f = \mathcal{A}f$. In other words,

$$\tau\left(\mathcal{A}f\right) = \mathcal{A}f, \tag{1.20}$$

i.e. $\mathcal{A}f$ is alternating.

■

Lemma 1.6

If $f \in A_k\left(V\right)$, then $\mathcal{A}f = \left(k!\right)f$.

Proof. Since f is alternating,

$$\mathcal{A}f = \sum_{\sigma\in S_k} \left(\operatorname{sgn}\sigma\right)\sigma f = \sum_{\sigma\in S_k} \left(\operatorname{sgn}\sigma\right)^2 f = \sum_{\sigma\in S_k} f = \left(k!\right)f, \tag{1.21}$$

because the order of S_k is $k!$. ■

§1.4 Tensor Product and Wedge Product

Definition 1.4 (Tensor Product)**.** Let f be a k-linear function and g an l-linear function on a vector space V. Their tensor product $f\otimes g$ is the $(k+l)$-linear function defined by

$$\left(f\otimes g\right)\left(\mathbf{v}_1,\ldots,\mathbf{v}_k,\mathbf{v}_{k+1},\ldots,\mathbf{v}_{k+l}\right) = f\left(\mathbf{v}_1,\ldots,\mathbf{v}_k\right)g\left(\mathbf{v}_{k+1},\ldots,\mathbf{v}_{k+l}\right). \tag{1.22}$$

$(k+l)$-linearity of $f\otimes g$ follows from k-linearity of f and l-linearity of g.

Lemma 1.7 (Associativity of Tensor Product)
Let $f \in L_k(V)$, $g \in L_l(V)$ and $h \in L_m(V)$. Then

$$(f \otimes g) \otimes h = f \otimes (g \otimes h).$$

Proof. For $\mathbf{v}_1, \ldots, \mathbf{v}_{k+l+m}$,

$$\begin{aligned}[(f \otimes g) \otimes h](\mathbf{v}_1, \ldots, \mathbf{v}_{k+l+m}) &= (f \otimes g)(\mathbf{v}_1, \ldots, \mathbf{v}_{k+l})\, h(\mathbf{v}_{k+l+1}, \ldots, \mathbf{v}_{k+l+m}) \\ &= f(\mathbf{v}_1, \ldots, \mathbf{v}_k)\, g(\mathbf{v}_{k+1}, \ldots, \mathbf{v}_{k+l})\, h(\mathbf{v}_{k+l+1}, \ldots, \mathbf{v}_{k+l+m}). \end{aligned} \tag{1.23}$$

$$\begin{aligned}[f \otimes (g \otimes h)](\mathbf{v}_1, \ldots, \mathbf{v}_{k+l+m}) &= f(\mathbf{v}_1, \ldots, \mathbf{v}_k)\,(g \otimes h)(\mathbf{v}_{k+1}, \ldots, \mathbf{v}_{k+l+m}) \\ &= f(\mathbf{v}_1, \ldots, \mathbf{v}_k)\, g(\mathbf{v}_{k+1}, \ldots, \mathbf{v}_{k+l})\, h(\mathbf{v}_{k+l+1}, \ldots, \mathbf{v}_{k+l+m}). \end{aligned} \tag{1.24}$$

Therefore, $(f \otimes g) \otimes h = f \otimes (g \otimes h)$, i.e. tensor product is associative. ■

Example 1.5. Let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be the standard basis for $\mathbb{R}^n$, and $\{\widehat{\alpha}^1, \ldots, \widehat{\alpha}^n\}$ its dual basis. The Euclidean inner product on $\mathbb{R}^n$ is the bilinear function

$$\langle\ ,\ \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$

defined by

$$\langle\ ,\ \rangle(\mathbf{v}, \mathbf{w}) = \langle \mathbf{v}, \mathbf{w} \rangle = \sum_{i=1}^n v^i w^i,$$

for $\mathbf{v} = \sum_{i=1}^n v^i \mathbf{e}_i$ and $\mathbf{w} = \sum_{i=1}^n w^i \mathbf{e}_i$. We can express $\langle\ ,\ \rangle$ in terms of tensor product as follows:

$$\langle\ ,\ \rangle(\mathbf{v}, \mathbf{w}) = \sum_{i=1}^n v^i w^i = \sum_{i=1}^n \widehat{\alpha}^i(\mathbf{v})\, \widehat{\alpha}^i(\mathbf{w}) = \sum_{i=1}^n \left(\widehat{\alpha}^i \otimes \widehat{\alpha}^i\right)(\mathbf{v}, \mathbf{w}).$$

Since $\mathbf{v}, \mathbf{w}$ are arbitrary,

$$\langle\ ,\ \rangle = \sum_{i=1}^n \left(\widehat{\alpha}^i \otimes \widehat{\alpha}^i\right). \tag{1.25}$$

If $f \in A_k(V)$ and $g \in A_l(V)$, then it's not true that $f \otimes g \in A_{k+l}(V)$, in general. We need to construct a product that is also alternating.

Definition 1.5 (Wedge Product). For $f \in A_k(V)$ and $g \in A_l(V)$, the wedge product of f and g is defined as follows:

$$f \wedge g = \frac{1}{k!l!} \mathcal{A}(f \otimes g). \tag{1.26}$$

Explicitly,

$$\begin{aligned}(f \wedge g)(\mathbf{v}_1, \ldots, \mathbf{v}_{k+l}) &= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma)\, \sigma(f \otimes g)(\mathbf{v}_1, \ldots, \mathbf{v}_{k+l}) \\ &= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma)(f \otimes g)\left(\mathbf{v}_{\sigma(1)}, \ldots, \mathbf{v}_{\sigma(k+l)}\right) \\ &= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma)\, f\left(\mathbf{v}_{\sigma(1)}, \ldots, \mathbf{v}_{\sigma(k)}\right) g\left(\mathbf{v}_{\sigma(k+1)}, \ldots, \mathbf{v}_{\sigma(k+l)}\right). \end{aligned} \tag{1.27}$$

When $k=0$, the element $f \in A_0(V)$ is simply a constant $c \in \mathbb{R}$ as discussed earlier. In this case, the wedge product $c \wedge g$ is just scalar multiplication as is evident from (1.27).

$$\begin{aligned}
(c \wedge g)(\mathbf{v}_1, \ldots, \mathbf{v}_l) &= \frac{1}{l!} \sum_{\sigma \in S_l} (\operatorname{sgn} \sigma)\, c\, g\left(\mathbf{v}_{\sigma(1)}, \ldots, \mathbf{v}_{\sigma(l)}\right) \\
&= \frac{1}{l!} \sum_{\sigma \in S_l} (\operatorname{sgn} \sigma)\, c\, (\operatorname{sgn} \sigma)\, g(\mathbf{v}_1, \ldots, \mathbf{v}_l) \\
&= \frac{1}{l!} \sum_{\sigma \in S_l} c\, g(\mathbf{v}_1, \ldots, \mathbf{v}_l) \\
&= \frac{1}{l!} l! c\, g(\mathbf{v}_1, \ldots, \mathbf{v}_l) \\
&= c\, g(\mathbf{v}_1, \ldots, \mathbf{v}_l).
\end{aligned}$$

Thus $c \wedge g = cg$, for $c \in \mathbb{R}$ and $g \in A_l(V)$.

Example 1.6. For $f \in A_2(V)$ and $g \in A_1(V)$,

$$\begin{aligned}
\mathcal{A}(f \otimes g)(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = {} & f(\mathbf{v}_1, \mathbf{v}_2)\, g(\mathbf{v}_3) - f(\mathbf{v}_1, \mathbf{v}_3)\, g(\mathbf{v}_2) - f(\mathbf{v}_2, \mathbf{v}_1)\, g(\mathbf{v}_3) \\
& - f(\mathbf{v}_3, \mathbf{v}_2)\, g(\mathbf{v}_1) + f(\mathbf{v}_2, \mathbf{v}_3)\, g(\mathbf{v}_1) + f(\mathbf{v}_3, \mathbf{v}_1)\, g(\mathbf{v}_2).
\end{aligned}$$

Among these 6 terms, there are 3 pairs of equal terms due to the alternating nature of f.

$$\begin{aligned}
f(\mathbf{v}_1, \mathbf{v}_2)\, g(\mathbf{v}_3) &= -f(\mathbf{v}_2, \mathbf{v}_1)\, g(\mathbf{v}_3), \\
f(\mathbf{v}_3, \mathbf{v}_1)\, g(\mathbf{v}_2) &= -f(\mathbf{v}_1, \mathbf{v}_3)\, g(\mathbf{v}_2), \\
f(\mathbf{v}_2, \mathbf{v}_3)\, g(\mathbf{v}_1) &= -f(\mathbf{v}_3, \mathbf{v}_2)\, g(\mathbf{v}_1).
\end{aligned}$$

Therefore,

$$\mathcal{A}(f \otimes g)(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = 2f(\mathbf{v}_1, \mathbf{v}_2)\, g(\mathbf{v}_3) + 2f(\mathbf{v}_3, \mathbf{v}_1)\, g(\mathbf{v}_2) + 2f(\mathbf{v}_2, \mathbf{v}_3)\, g(\mathbf{v}_1). \tag{1.28}$$

Hence,

$$\begin{aligned}
(f \wedge g)(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) &= \frac{1}{2!1!} \mathcal{A}(f \otimes g)(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \\
&= f(\mathbf{v}_1, \mathbf{v}_2)\, g(\mathbf{v}_3) + f(\mathbf{v}_3, \mathbf{v}_1)\, g(\mathbf{v}_2) + f(\mathbf{v}_2, \mathbf{v}_3)\, g(\mathbf{v}_1). \qquad (1.29)
\end{aligned}$$

Example 1.7 (Wedge product of 2 covectors)**.** If $f, g \in A_1(V)$, and $\mathbf{v}_1, \mathbf{v}_2 \in V$, then

$$(f \wedge g)(\mathbf{v}_1, \mathbf{v}_2) = \frac{1}{1!1!} \mathcal{A}(f \otimes g)(\mathbf{v}_1, \mathbf{v}_2).$$

S_2 has 2 elements: the identity element e and $(1\ 2)$. Therefore,

$$(f \wedge g)(\mathbf{v}_1, \mathbf{v}_2) = \mathcal{A}(f \otimes g)(\mathbf{v}_1, \mathbf{v}_2) = f(\mathbf{v}_1)\, g(\mathbf{v}_2) - f(\mathbf{v}_2)\, g(\mathbf{v}_1).$$

Proposition 1.8 (Anticommutativity of wedge product)

The wedge product is anticommutative: if $f \in A_k(V)$ and $g \in A_l(V)$, then

$$f \wedge g = (-1)^{kl}\, g \wedge f.$$

Proof. Define $\tau \in S_{k+l}$ to be the following permutation:

$$\begin{bmatrix} 1 & 2 & \cdots & l & l+1 & l+2 & \cdots & l+k \\ k+1 & k+2 & \cdots & k+l & 1 & 2 & \cdots & k \end{bmatrix}.$$

In other words,

$$\tau(i) = \begin{cases} k+i & \text{if } 1 \le i \le l, \\ i-l & \text{if } l+1 \le i \le l+k. \end{cases}$$

Then for any $\sigma \in S_{k+l}$,

$$\sigma(j) = \begin{cases} \sigma(\tau(l+j)) & \text{if } 1 \le j \le k, \\ \sigma(\tau(j-k)) & \text{if } k+1 \le j \le k+l. \end{cases} \tag{1.30}$$

Now, for any $\mathbf{v}_1, \ldots, \mathbf{v}_{k+l} \in V$,

$$\begin{aligned} \mathcal{A}(f \otimes g)(\mathbf{v}_1, \ldots, \mathbf{v}_{k+l}) &= \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma) f\left(\mathbf{v}_{\sigma(1)}, \ldots, \mathbf{v}_{\sigma(k)}\right) g\left(\mathbf{v}_{\sigma(k+1)}, \ldots, \mathbf{v}_{\sigma(k+l)}\right) \\ &= \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma) f\left(\mathbf{v}_{\sigma(\tau(l+1))}, \ldots, \mathbf{v}_{\sigma(\tau(l+k))}\right) g\left(\mathbf{v}_{\sigma(\tau(1))}, \ldots, \mathbf{v}_{\sigma(\tau(l))}\right) \\ &= (\operatorname{sgn} \tau) \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma)(\operatorname{sgn} \tau)\, g\left(\mathbf{v}_{\sigma(\tau(1))}, \ldots, \mathbf{v}_{\sigma(\tau(l))}\right) f\left(\mathbf{v}_{\sigma(\tau(l+1))}, \ldots, \mathbf{v}_{\sigma(\tau(l+k))}\right) \\ &= (\operatorname{sgn} \tau) \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma\tau)\, g\left(\mathbf{v}_{\sigma(\tau(1))}, \ldots, \mathbf{v}_{\sigma(\tau(l))}\right) f\left(\mathbf{v}_{\sigma(\tau(l+1))}, \ldots, \mathbf{v}_{\sigma(\tau(l+k))}\right). \end{aligned}$$

Again, as σ varies over S_{k+l}, $\sigma\tau$ also varies over S_{k+l}. Therefore,

$$\mathcal{A}(f \otimes g)(\mathbf{v}_1, \ldots, \mathbf{v}_{k+l}) = (\operatorname{sgn} \tau)\, \mathcal{A}(g \otimes f)(\mathbf{v}_1, \ldots, \mathbf{v}_{k+l}). \tag{1.31}$$

Now, let us evaluate the sign of the permutation τ. Let $(\tau(i), \tau(j))$ be an inversion of τ. Then it's not possible that $1 \le i < j \le l$, or $l+1 \le i < j \le l+k$; because if we have $1 \le i < j \le l$ or $l+1 \le i < j \le l+k$, then $\tau(i) < \tau(j)$. Therefore, i must be in between 1 and l (inclusive), and j must be in between $l+1$ and $l+k$ (inclusive). So there are l options for i, and k options for j. Therefore, τ has kl many inversions. So $\operatorname{sgn} \tau = (-1)^{kl}$. Using (1.31),

$$\mathcal{A}(f \otimes g) = (-1)^{kl} \mathcal{A}(g \otimes f). \tag{1.32}$$

Dividing by $k!l!$, we obtain

$$f \wedge g = (-1)^{kl} g \wedge f. \tag{1.33}$$

■

Corollary 1.9

If f is a k-covector on V, i.e. $f \in A_k(V)$, and k is odd, then $f \wedge f = 0$.

Proof. By anticommutativity of wedge product,

$$f \wedge f = (-1)^{k^2} f \wedge f = -f \wedge f.$$

Therefore, $f \wedge f = 0$. ■

If f is a k-covector and g is an l-covector, i.e. $f \in A_k(V)$ and $g \in A_l(V)$, then we have defined their wedge product to be the $(k+l)$-covector

$$f \wedge g = \frac{1}{k!l!} \mathcal{A}(f \otimes g). \tag{1.34}$$

We have the following lemmas associated with the alternating operator $\mathcal{A}$.

Lemma 1.10

Suppose $f \in L_k(V)$ and $g \in L_l(V)$. Then

(i) $\mathcal{A}(\mathcal{A}(f) \otimes g) = k!\mathcal{A}(f \otimes g)$.

(ii) $\mathcal{A}(f \otimes \mathcal{A}(g)) = l!\mathcal{A}(f \otimes g)$.

Proof. (i) By definition,

$$\begin{aligned}\mathcal{A}(\mathcal{A}(f) \otimes g) &= \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma)\, \sigma\, (\mathcal{A}(f) \otimes g) \\ &= \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma)\, \sigma \left[\sum_{\tau \in S_k} (\operatorname{sgn} \tau)\, (\tau f) \otimes g\right]. \end{aligned} \tag{1.35}$$

We can view $\tau \in S_k$ as a permutation in the following way: define $\tau' \in S_{k+l}$ as follows

$$\tau'(i) = \begin{cases} \tau(i) & \text{if } i \leq k, \\ i & \text{if } i > k. \end{cases} \tag{1.36}$$

Then for $\mathbf{v}_1, \ldots, \mathbf{v}_{k+l}$, we have

$$\begin{aligned}[(\tau f) \otimes g]\,(\mathbf{v}_1, \ldots, \mathbf{v}_{k+l}) &= (\tau f)(\mathbf{v}_1, \ldots, \mathbf{v}_k)\, g(\mathbf{v}_{k+1}, \ldots, \mathbf{v}_{k+l}) \\ &= f\left(\mathbf{v}_{\tau(1)}, \ldots, \mathbf{v}_{\tau(k)}\right) g(\mathbf{v}_{k+1}, \ldots, \mathbf{v}_{k+l}) \\ &= f\left(\mathbf{v}_{\tau'(1)}, \ldots, \mathbf{v}_{\tau'(k)}\right) g\left(\mathbf{v}_{\tau'(k+1)}, \ldots, \mathbf{v}_{\tau'(k+l)}\right) \\ &= [\tau'(f \otimes g)]\,(\mathbf{v}_1, \ldots, \mathbf{v}_{k+l}). \end{aligned}$$

Therefore, $(\tau f) \otimes g = \tau'(f \otimes g)$. Furthermore, $\operatorname{sgn} \tau = \operatorname{sgn} \tau'$ since the inversions $(\tau'(i), \tau'(j))$ occur only when $1 \leq i < j \leq k$, so that the τ and τ' has the same number of inversions.

Let us abuse notation a bit and denote by S_k the subgroup of permutations in S_{k+l} by keeping the last l arguments fixed. This subgroup of S_{k+l} is indeed isomorphic to S_k, so we will denote both these groups by S_k. Therefore, from (1.35),

$$\begin{aligned}\mathcal{A}(\mathcal{A}(f) \otimes g) &= \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma)\, \sigma \left[\sum_{\tau' \in S_k \subseteq S_{k+l}} (\operatorname{sgn} \tau')\, \tau'(f \otimes g)\right] \\ &= \sum_{\sigma \in S_{k+l}} \sum_{\tau' \in S_k \subseteq S_{k+l}} (\operatorname{sgn} \sigma)(\operatorname{sgn} \tau')\, \sigma\tau'(f \otimes g) \\ &= \sum_{\tau' \in S_k \subseteq S_{k+l}} \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma \operatorname{sgn} \tau')\, ((\sigma\tau')(f \otimes g)). \end{aligned}$$

For a fixed τ', as σ varies over S_{k+l}, $\sigma\tau'$ also varies over S_{k+l}. Therefore,

$$\mathcal{A}(\mathcal{A}(f) \otimes g) = \sum_{\tau' \in S_k \subseteq S_{k+l}} \mathcal{A}(f \otimes g) = k!\mathcal{A}(f \otimes g). \tag{1.37}$$

(ii) By (1.32),

$$\begin{aligned}\mathcal{A}(f \otimes \mathcal{A}(g)) &= \mathcal{A}\left((-1)^{kl} \mathcal{A}(g) \otimes f\right) \\ &= (-1)^{kl} \mathcal{A}(\mathcal{A}(g) \otimes f) \\ &= (-1)^{kl}\, l!\mathcal{A}(g \otimes f) \\ &= l!\mathcal{A}\left((-1)^{kl} g \otimes f\right) \\ &= l!\mathcal{A}(f \otimes g). \end{aligned} \tag{1.38}$$

■

Proposition 1.11 (Associativity of wedge product)
Let V be a real vector space and f, g, h be alternating multilinear functions on V of degree k, l, m, respectively. Then
$$(f \wedge g) \wedge h = f \wedge (g \wedge h) .$$

Proof. Using the definition of wedge product,
$$\begin{aligned}
(f \wedge g) \wedge h &= \frac{1}{(k+l)!m!} \mathcal{A}\left[(f \wedge g) \otimes h\right] \\
&= \frac{1}{(k+l)!m!} \mathcal{A}\left[\frac{1}{k!l!} \mathcal{A}(f \otimes g) \otimes h\right] \\
&= \frac{1}{(k+l)!k!l!m!} \mathcal{A}\left[\mathcal{A}(f \otimes g) \otimes h\right] \\
&= \frac{(k+l)!}{(k+l)!k!l!m!} \mathcal{A}\left[(f \otimes g) \otimes h\right] \\
&= \frac{1}{k!l!m!} \mathcal{A}\left[(f \otimes g) \otimes h\right] .
\end{aligned}$$

On the other hand,
$$\begin{aligned}
f \wedge (g \wedge h) &= \frac{1}{k!\,(l+m)!} \mathcal{A}\left[f \otimes (g \wedge h)\right] \\
&= \frac{1}{k!\,(l+m)!} \mathcal{A}\left[f \otimes \left(\frac{1}{l!m!} \mathcal{A}(g \otimes h)\right)\right] \\
&= \frac{1}{k!\,(l+m)!l!m!} \mathcal{A}\left[f \otimes \mathcal{A}(g \otimes h)\right] \\
&= \frac{(l+m)!}{k!\,(l+m)!l!m!} \mathcal{A}\left[f \otimes (g \otimes h)\right] \\
&= \frac{1}{k!l!m!} \mathcal{A}\left[f \otimes (g \otimes h)\right] .
\end{aligned}$$

Since tensor product is associative (by Lemma 1.7), we conclude that
$$(f \wedge g) \wedge h = f \wedge (g \wedge h) . \tag{1.39}$$

■

By associativity, we can omit the parenthesis and write univocally $f \wedge g \wedge h$ instead of $(f \wedge g) \wedge h$ or $f \wedge (g \wedge h)$.

Corollary 1.12
Under the hypothesis of Proposition 1.11,
$$f \wedge g \wedge h = \frac{1}{k!l!m!} \mathcal{A}\left[f \otimes g \otimes h\right] . \tag{1.40}$$

This easily generalizes to an arbitrary number of factors: if $f_i \in A_{d_i}(V)$ for $i = 1, 2, \ldots, r$, i.e. f_i is an alternating d_i-linear function on V, then
$$f_1 \wedge \cdots \wedge f_r = \frac{1}{d_1! \cdots d_r!} \mathcal{A}(f_1 \otimes \cdots \otimes f_r) . \tag{1.41}$$