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Category Theory (MAT434)

Lecture Notes

Preface

This series of lecture notes has been prepared for aiding students who took the BRAC University course **Category Theory (MAT434)** in Summer 2023 semester. These notes were typeset under the supervision of mathematician **Dr. Syed Hasibul Hassan Chowdhury**. The main goal of this typeset is to have an organized digital version of the notes, which is easier to share and handle. If you see any mistakes or typos, please send me an email at `atonuroychowdhury@gmail.com`

Atonu Roy Chowdhury

References:

- *Category Theory*, by **Steve Awodey**.
- *Category Theory for Scientists*, by **David Spivak**.
- *Categories for the Working Mathematician*, by **Saunders Mac Lane**.
- *Basic Category Theory*, by **Tom Leinster**.

Contents

1 Categories

§1.1 Definition of a Category

Category theory arises from the idea of a system of "functions" among some objects.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "g\circ f"'] & B \arrow[d, "g"] \\
& C
\end{tikzcd}$$

A category consists of objects $A, B, C, \ldots$ and arrows $f : A \to B, g : B \to C, \ldots$ that are closed under composition and satisfy certain conditions typical of composition of functions. Before formally defining what a category is, let us begin our discussion with the setting where the objects are sets and arrows are functions between sets.

Let f be a function from a set A to another set B. This is mathematically expressed as $f : A \to B$. Explicitly, it refers to the fact that f is defined for all of A, and all the values of f are contained in B. In other words, $\operatorname{range}(f) \subseteq B$.

Now suppose we have another function $g : B \to C$. Then there is a unique function $g \circ f : A \to C$, given by

$$(g \circ f)(a) = g(f(a)), \qquad \text{for } a \in A. \tag{1.1}$$

This unique function is called the composite of g and f, or g after f.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "g\circ f"'] & B \arrow[d, "g"] \\
& C
\end{tikzcd}$$

Now, this operation $\circ$ of composition of functions is associative. In other words, the two arrows from A to D in the following diagram are the same:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[rrr, bend left=50, "(h\circ g)\circ f"] \arrow[rr, bend right=20, "g\circ f"'] \arrow[rrr, bend right=50, "h\circ(g\circ f)"'] & B \arrow[r, "g"] \arrow[rr, bend left=30, "h\circ g"] & C \arrow[r, "h"] & D
\end{tikzcd}$$

Given $f : A \to B$, $g : B \to C$ and $h : C \to D$, one has unique compositions $h \circ g : B \to D$ and $g \circ f : A \to C$. These two composed functions can be further composed with f (from the left) and with h (from the right), respectively, to yield a unique function

$$(h \circ g) \circ f = h \circ (g \circ f), \tag{1.2}$$

from A to D as demanded by the associativity law. Using the definition of composition of functions, one verifies that this is indeed the case:

$$\begin{aligned}
((h \circ g) \circ f)(a) &= (h \circ g)(f(a)) = h(g(f(a))), \\
(h \circ (g \circ f))(a) &= h((g \circ f)(a)) = h(g(f(a))).
\end{aligned}$$

Therefore, $(h \circ g) \circ f = h \circ (g \circ f)$.

Finally, note that for every set A, there is an identity function $\mathbb{1}_A : A \to A$ given by

$$\mathbb{1}_A(a) = a. \tag{1.3}$$

These identity functions act as units for composition, i.e. given $f : A \to B$, we have

$$\begin{aligned}(f \circ \mathbb{1}_A)(a) &= f(\mathbb{1}_A(a)) = f(a), \\ (\mathbb{1}_B \circ f)(a) &= \mathbb{1}_B(f(a)) = f(a),\end{aligned}$$

for each $a \in A$. Therefore,

$$f \circ \mathbb{1}_A = \mathbb{1}_B \circ f = f. \tag{1.4}$$

The equality above is equivalent to the following commutative diagram:

$$\begin{tikzcd}
A \arrow[r, "\mathbb{1}_A"] \arrow[dr, "f\circ\mathbb{1}_A"'] & A \arrow[d, "f"'] \arrow[dr, "\mathbb{1}_B\circ f"] & \\
& B \arrow[r, "\mathbb{1}_B"'] & B
\end{tikzcd}$$

We have the following abstract version of sets and functions between sets called a **category**.

Definition 1.1 (Category). A **category** $\mathcal{C}$ consists of the following data:

- **Objects:** $A, B, C, \ldots$. The collection of objects of $\mathcal{C}$ is denoted by $\mathrm{Ob}(\mathcal{C})$.
- **Arrows:** $f, g, h, \ldots$. Given two objects A and B, the set of arrows from A to B is denoted by $\mathrm{Hom}_{\mathcal{C}}(A, B)$.
- For each arrow f, there are given objects $\mathrm{dom}(f), \mathrm{cod}(f)$ called the **domain** and **codomain** of f. We write $f : A \to B$ to indicate that $A = \mathrm{dom}(f)$ and $B = \mathrm{cod}(f)$.
- Given arrows $f : A \to B$ and $g : B \to C$, i.e. with $\mathrm{cod}(f) = \mathrm{dom}(g)$, there is a unique arrow $g \circ f : A \to C$, i.e. $g \circ f \in \mathrm{Hom}_{\mathcal{C}}(A, C)$ called the **composite** of f and g. This fact can be rephrased as the following: given $A, B, C \in \mathrm{Ob}(\mathcal{C})$, there is a function

  $$\circ : \mathrm{Hom}_{\mathcal{C}}(B, C) \times \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(A, C), \tag{1.5}$$

  with $(g, f) \mapsto g \circ f$. The well-definedness of $\circ$ is synonymous to claiming that $g \circ f \in \mathrm{Hom}_{\mathcal{C}}(A, C)$ is unique for given $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$ and $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$.
- For each $A \in \mathrm{Ob}(\mathcal{C})$, there exists an unique arrow $\mathbb{1}_A \in \mathrm{Hom}_{\mathcal{C}}(A, A)$.

The above data are required to satisfy the following laws:

- **Associativity:** For any $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$, $h \in \mathrm{Hom}_{\mathcal{C}}(C, D)$ with $A, B, C, D \in \mathrm{Ob}(\mathcal{C})$,

  $$(h \circ g) \circ f = h \circ (g \circ f), \tag{1.6}$$

- **Unit:** For any $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ with $A, B \in \mathrm{Ob}(\mathcal{C})$,

  $$f \circ \mathbb{1}_A = \mathbb{1}_B \circ f = f. \tag{1.7}$$

Remark 1.1. Suppose we have the following commutative diagram:

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[dr, "h\circ f=h\circ g"'] & B \arrow[d, "h"] \\
& C
\end{tikzcd}$$

Commutativity of this diagram doesn't violate the uniqueness of the composition $\circ$. It just means that the map $\circ$ in (1.5) is a many-to-one function.

§1.2 Examples of Categories

- Sets and functions between sets. This category is called **Sets**.
- Groups and group homomorphisms
- Vector spaces and linear mappings between them
- Graphs and graph isomorphisms
- The set of real numbers $\mathbb{R}$ as an object, and continuous functions $f : \mathbb{R} \to \mathbb{R}$ as arrows
- Open subsets $U \subseteq \mathbb{R}$ and continuous functions $f : U \to V \subseteq \mathbb{R}$ defined on them
- Differentiable manifolds and smooth (C^∞) mappings
- Posets and monotone functions.

Let us discuss the last category at length.

Definition 1.2. A **partially ordered set** or **poset** is a set A equipped with a binary relation (a subset of $A \times A$) $a \leq_A b$ (in other words, $(a, b) \in R \subset A \times A$) such that the following conditions hold for all $a, b, c \in A$:

(i) **Reflexivity:** $a \leq_A a$.

(ii) **Transitivity:** if $a \leq_A b$ and $b \leq_A c$, then $a \leq_A c$.

(iii) **Antisymmetry:** if $a \leq_A b$ and $b \leq_A a$, then $a = b$.

Remark 1.2. The antisymmetry condition tells us that if both $a \leq_A b$ and $b \leq_A a$ hold, then a and b cannot be distinct. Contrapositively, for distinct a and b in A, not both $a \leq_A b$ and $b \leq_A a$ hold true. Also, note that if $(A, \leq_A)$ is a partially ordered set, there can be elements $a, b \in A$ such that neither (a, b) nor (b, a) is in R. If it happens that given any $a, b \in A$, either (a, b) or (b, a) is in R, i.e. either $a \leq_A b$ or $b \leq_A a$, then we call A a **totally ordered set**.

Example 1.1. $(\mathbb{R}, \leq)$, the set of real numbers with the usual ordering $\leq$ is a totally ordered set.

Now we define an arrow from a poset $(A, \leq_A)$ to another poset $(B, \leq_B)$ to be a function $m : A \to B$ that is **monotone**, in the sense that for all $a, a' \in A$,

$$\text{whenever } a \leq_A a', \text{ one has } m(a) \leq_B m(a').$$

We need to verify that under this definition of arrows, we have a category. First of all, we must have $\mathbb{1}_A : A \to A$, defined by $\mathbb{1}_A(a) = a$ for each $a \in A$, to be monotone. Indeed, if $a \leq a'$ in A, then we automatically have $\mathbb{1}_A(a) \leq \mathbb{1}_A(a')$. Therefore, $\mathbb{1}_A$ is monotone.

Given monotone functions $f : A \to B$ bewteen posets $(A, \leq_A)$ and $(B, \leq_B)$, and $g : B \to C$ bewteen posets $(B, \leq_B)$ and $(C, \leq_C)$, we need to verify that the composition $g \circ f : A \to C$ is also monotone. Indeed, given $a \leq_A a'$, since f is monotone, we have

$$f(a) \leq_B f(a') . \tag{1.8}$$

Since g is monotone, this gives us

$$g(f(a)) \leq_C g(f(a')) . \tag{1.9}$$

In other words, $(g \circ f)(a) \leq_C (g \circ f)(a')$ given $a \leq_A a'$. Therefore, $g \circ f : A \to C$ is monotone.

The category thus formed is called the category of posets and monotone functions, and is denoted by **Pos**.

Finite Categories

A finite category consists of finitely many objects and finitely many arrows between them.

- The category **1** looks as follows:

  $$\begin{tikzcd} \ast \arrow[loop right] \end{tikzcd}$$

  It has one object $\ast$ and its identity arrow.

- The category **2** looks as follows:

  $$\begin{tikzcd} \star \arrow[loop left] \arrow[r] & \ast \arrow[loop right] \end{tikzcd}$$

  It has two objects $\star$ and $\ast$, their identity arrows, and exactly one arrow $\star \to \ast$.

- The category **3** looks as follows:

  $$\begin{tikzcd} \star \arrow[loop left] \arrow[r] \arrow[dr] & \ast \arrow[loop right] \arrow[d] \\ & \bullet \arrow[loop below] \end{tikzcd}$$

  It has three objects $\star, \ast, \bullet$, their respective identity arrows, and the other arrows are $\star \to \ast$, $\ast \to \bullet$, and $\star \to \bullet$ (which is the composition of the previous two arrows).

- The category **0** looks as follows:

  It has no objects or arrows.

§1.3 Functor

Definition 1.3 (Functor)**.** A **functor** $F : \mathcal{C} \to \mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$ is an assignment of $\mathrm{Ob}(\mathcal{C})$ to $\mathrm{Ob}(\mathcal{D})$ and a mapping of arrows in $\mathcal{C}$ to arrows in $\mathcal{D}$, i.e. for any $A, B \in \mathrm{Ob}(\mathcal{C})$, a mapping

$$\mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F(A), F(B)),$$

where $F(A), F(B) \in \mathrm{Ob}(\mathcal{D})$ are the assigned objects of $\mathcal{D}$ under F. In other words, for given $A, B \in \mathrm{Ob}(\mathcal{C})$ and an arrow $f : A \to B$, one has $F(A), F(B) \in \mathrm{Ob}(\mathcal{D})$ and an arrow $F(f) : F(A) \to F(B)$ such that the following hold:

(a) $F(\mathbb{1}_A) = \mathbb{1}_{F(A)}$.

(b) $F(g \circ f) = F(g) \circ F(f)$.

> In other words, F preserves domains and codomains, identity arrows and composition.

$$\begin{tikzcd}
\mathcal{C} \arrow[dd, "F"'] & & A \arrow[r, "f"] \arrow[dr, "g\circ f"'] & B \arrow[d, "g"] \\
& & & C \\
\mathcal{D} & & F(A) \arrow[r, "F(f)"] \arrow[dr, "F(g\circ f)=F(g)\circ F(f)"'] & F(B) \arrow[d, "F(g)"] \\
& & & F(C)
\end{tikzcd}$$

Now, one can see that functors compose in the expected way and that every category $\mathcal{C}$ has a distinguished functor called the identity functor $\mathbb{1}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$. Thus we have a category, namely **Cat**, the category of all categories and functors between them.

Preorder

A **preorder** is a set P equipped with a binary relation $\leq$ that is both reflexive and transitive: $a \leq a$; and if $a \leq b$ and $b \leq c$, then $a \leq c$ for any $a, b, c \in P$. Any preorder $(P, \leq)$ can be regarded as a category by taking the objects of the category to be the elements of P and taking a unique arrow

$$a \to b \text{ if and only if } a \leq b \text{ in } (P, \leq). \tag{1.10}$$

Remark 1.3. Reflexivity and transitivity property ensures that the preorder $(P, \leq)$ is indeed a category. Note that the above condition implies that there is at most one arrow from an object of $(P, \leq)$ to another. In the other direction, any category with at most one arrow from an object to another determines a preorder simply by defininig a binary relation $\leq$ on the objects by (1.10).

Remark 1.4 (On the similarities between a poset and a preorder). A poset $(P, \leq)$ is evidently a preorder with the additional condition of antisymmetry. Hence, a poset is also a category. Examples of poset include the power set $\mathscr{P}(X)$ of a given set X under the usual subset relation: $U \subseteq V$ between the subsets U, V of X.

There can be preorders that are not posets. For instance, $(\mathbb{Z}, |)$ is a preorder on the set of integers, where "$|$" is the usual divides binary relation: given $a, b \in \mathbb{Z}$, we have $a \mid b$ (read a divides b) if and only if $b = ca$ for some $c \in \mathbb{Z}$. It is clearly reflexive and transitive. Note that $a \mid b$ and $b \mid a$ imply $a = \pm b$ which is not the same as $a = b$. Hence, "$|$" is not antisymmetric. Therefore, $(\mathbb{Z}, |)$ is a preorder that is not a poset.

§1.4 Monoid

> **Definition 1.4** (Monoid). A **monoid** is a set M equipped with a binary operation $\cdot : M \times M \to M$ and a distinguished "unit" element $u \in M$ such that for each $x, y, z \in M$,
>
> $$x \cdot (y \cdot z) = (x \cdot y) \cdot z \text{ and } u \cdot x = x \cdot u = x. \tag{1.11}$$

Equivalently, a monoid is a category with just one object. The arrows of the category are the elements of the monoid. In particular, the identity arrow on the object is the unit element u. Composition of arrows is the binary operation $x \cdot y$ of the monoid.

For example, $\mathbb{N}$ (we are adopting the convention that $0 \in \mathbb{N}$), $\mathbb{Q}$, $\mathbb{R}$ with addition and 0 as the unit element. Also, $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$ with multiplication and 1 as unit are monoids. For any set X, the set of functions from X to itself, written as

$$\mathrm{Hom}_{\mathbf{Sets}}\left(X, X\right),$$

is a monoid under the operation of composition. Here **Sets** is the category of sets and functions between sets. More generally, for any object $C \in \mathrm{Ob}\left(\mathcal{C}\right)$ in a category $\mathcal{C}$, the set of arrows from C to itself, written as

$$\mathrm{Hom}_{\mathcal{C}}\left(C, C\right),$$

is a monoid under the composition of arrows in $\mathcal{C}$.

Since monoids are structured sets (sets equipped with a binary operation fulfilling associativity, unitality etc.), there is a category **Mon** whose objects are monoids and arrows are functions that preserve the monoid structure, namely monoid homomorphisms. In detail, a **monoid homomorphism** from a monoid $(M, \cdot_M)$ to a monoid $(N, \cdot_N)$ is a function $f : M \to N$ such that for all $m, n \in M$,

$$h\left(m \cdot_M n\right) = h\left(m\right) \cdot_N h\left(n\right) \text{ and } h\left(u_M\right) = u_N. \tag{1.12}$$

Here u_M and u_N are unit elements of M and N, respectively.

§1.4.i Isomorphisms

Definition 1.5. In any category $\mathcal{C}$, an arrow $f : A \to B$ is called an **isomorphism** if there is an arrow $g : B \to A$ such that

$$g \circ f = \mathbb{1}_A \text{ and } f \circ g = \mathbb{1}_B\,. \tag{1.13}$$

Suppose there is another arrow $\widetilde{g} : B \to A$ with

$$\widetilde{g} \circ f = \mathbb{1}_A \text{ and } f \circ \widetilde{g} = \mathbb{1}_B\,. \tag{1.14}$$

Then we have

$$g = g \circ \mathbb{1}_B = g \circ (f \circ \widetilde{g}) = (g \circ f) \circ \widetilde{g} = \mathbb{1}_A \circ \widetilde{g} = \widetilde{g}. \tag{1.15}$$

Hence, if an arrow $g : B \to A$ exists satisfying (1.13), then it is unique. Such unique arrow $g : B \to A$ is called the inverse of $f : A \to B$, and we write $g = f^{-1}$. When such an arrow $f : A \to B$ exists, we say that A is isomorphic to B, written $A \cong B$.

Definition 1.6 (Group). A **group** G is a monoid with an inverse g^{-1} for every element $g \in G$. Thus G is a category with one object in which every arrow is an isomorphism.

The natural numbers $\mathbb{N}$ do not form a group either under addition or multiplication. But the integers $\mathbb{Z}$ form a group under addition. So do the positive rationals $\mathbb{Q}^+$ under multiplication. For any set X, we have the group $\mathrm{Aut}\left(X\right)$ of all the automorphisms of X, i.e. isomorphisms $f : X \to X$. A **group of permutations** is a subgroup $G \subseteq \mathrm{Aut}\left(X\right)$ for some X. Thus the set G must satisfy the following:

1. The identity function $\mathbb{1}_X$ on X is in G.
2. If $g, g' \in G$, then $g \circ g' \in G$.
3. If $g \in G$, $g^{-1} \in G$.

We now have the following theorem due to Arthur Cayley.

Theorem 1.1 (Cayley's theorem)

Every group G is isomorphic to a group of permutations.

Sketch of proof. First, define the Cayley representation $\overline{G}$ of G to be the following group of permutations on a set: the set is G itself, and for each $g \in G$, one has the permutation $\overline{g} : G \to G$ defined as

$$\overline{g}\,(h) = g \cdot h \text{ for each } h \in G. \tag{1.16}$$

Indeed, $\overline{g}$ has an inverse $\overline{g}^{-1} = \overline{g^{-1}}$:

$$\overline{g}^{-1}\,(h) = g^{-1}h. \tag{1.17}$$

One, thus, verifies that $\overline{g} : G \to G$ is indeed an isomorphism, and hence a permutation on G.

Now define homomorphisms $i : G \to \overline{G}$ by $i\,(g) = \overline{g}$, and $j : \overline{G} \to G$ by $j\,(\overline{g}) = \overline{g}\,(u) = g$, with u being the identity element of the group G.

Observe that $i \circ j = \mathbb{1}_{\overline{G}}$ and $j \circ i = \mathbb{1}_G$. Indeed, for $g \in G$ and $\overline{g} \in \overline{G}$,

$$\begin{aligned}(j \circ i)\,(g) &= j\,(i\,(g)) = j\,(\overline{g}) = g,\\ (i \circ j)\,(\overline{g}) &= i\,(j\,(\overline{g})) = i\,(g) = \overline{g},\end{aligned}$$

establishing that $i : G \to \overline{G}$ is an isomorphism. ■

Remark 1.5. There are two different types of isomorphisms involved in this proof. For each $g \in G$, one defines an isomorphism $\overline{g} : G \to G$. This is an isomorphism in the category **Sets**. Later, we defined an isomorphism $i : G \to \overline{G}$, which is an isomorphism in the categorty **Groups** of groups and group homomorphisms.

Remark 1.6. The group $\overline{G}$ is the group of permutations (automorphisms) on the group G which is a subgroup of the automorphism group on G itself. This subgroup has the same unit element of that of the automorphism group on G, i.e. $\mathbb{1}_G$, the identity function on the group G. Note that this is also the unit of the group $\overline{G}$ which is not the same as $\mathbb{1}_{\overline{G}}$. This identity function $\mathbb{1}_{\overline{G}}$ on $\overline{G}$ was used to establish the required isomorphism in Cayleys theorem.

Cayley's theorem can be generalized to prove that any category not "too big" (which has the collection of objects to be a set) is isomorphic to a category in which the objects are sets and the arrows are functions between those sets. In other words, any not "too big" category is isomorphic to a subcategory of **Sets**.

Theorem 1.2 (Generalized Cayley's Theorem)

Every category $\mathcal{C}$ with a set of arrows is isomorphic to one in which the objects are sets and the arrows are functions.

Sketch of proof. Define the Cayley representation $\overline{\mathcal{C}}$ of $\mathcal{C}$ to be the following concrete category (i.e. a category where objects are sets and arrows are functions between them):

- **Objects:** objects are sets of the form

$$\overline{C} = \{f : X \to C\} = \{f \in \mathrm{Hom}_{\mathcal{C}}\,(X, C) \mid X \in \mathrm{Ob}\,(\mathcal{C})\}\,. \tag{1.18}$$

  In other words, the object $\overline{C}$ is the set of functions whose codomains are C.

- **Arrows:** arrows are functions

$$\overline{g} : \overline{C} \to \overline{D} \tag{1.19}$$

  for $g : C \to D$ in $\mathcal{C}$, defined for any $f : X \to C$ in $\overline{C}$ by $\overline{g}\,(f) = g \circ f$.

$$\begin{tikzcd} X \arrow[d, "f"'] \arrow[dr, "\overline{g}(f)=g\circ f"] & \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

Then we define a functor $\mathcal{F} : \mathcal{C} \to \overline{\mathcal{C}}$ that takes the object $C \in \mathrm{Ob}\left(\mathcal{C}\right)$ to the object $\overline{C} \in \mathrm{Ob}\left(\overline{\mathcal{C}}\right)$, and takes the arrow $g : C \to D$ in $\mathcal{C}$ to the arrow $\overline{g} : \overline{C} \to \overline{D}$ in $\overline{\mathcal{C}}$. In other words,

$$\mathcal{F}\left(C\right) = \overline{C} \text{ and } \mathcal{F}\left(g\right) = \overline{g}. \tag{1.20}$$

This is a functor since $\mathcal{F}\left(\mathbb{1}_C\right) = \mathbb{1}_{\overline{C}}$ and $\mathcal{F}\left(g \circ h\right) = \mathcal{F}\left(g\right) \circ \mathcal{F}\left(h\right)$ for composable arrows g, h in $\mathcal{C}$.

We then define another functor $\mathcal{G} : \overline{\mathcal{C}} \to \mathcal{C}$ that takes the object $\overline{C}$ to the codomains of the functions that are in $\overline{C}$, i.e. $\mathcal{G}\left(\overline{C}\right) = C$, and takes the arrow $\overline{g} : \overline{C} \to \overline{D}$ to the arrow $\overline{g}\left(\mathbb{1}_{\mathcal{G}\left(\overline{C}\right)}\right) = \overline{g}\left(\mathbb{1}_C\right) = g \circ \mathbb{1}_C = g : C \to D$ in $\mathcal{C}$. In other words,

$$\mathcal{G}\left(\overline{C}\right) = C \text{ and } \mathcal{G}\left(\overline{g}\right) = g. \tag{1.21}$$

This is a functor since $\mathcal{G}\left(\mathbb{1}_{\overline{C}}\right) = \mathbb{1}_C$ and $\mathcal{G}\left(\overline{g} \circ \overline{h}\right) = \mathcal{G}\left(\overline{g}\right) \circ \mathcal{G}\left(\overline{h}\right)$ for composable arrows $\overline{g}, \overline{h}$ in $\overline{\mathcal{C}}$. Then one can verify that

$$\mathcal{G} \circ \mathcal{F} = \mathbb{1}_{\mathcal{C}} \text{ and } \mathcal{F} \circ \mathcal{G} = \mathbb{1}_{\overline{\mathcal{C}}}. \tag{1.22}$$

Therefore, $\mathcal{C}$ and $\overline{\mathcal{C}}$ are isomorphic. ■

§1.5 Construction on Categories

1. The product of 2 categories $\mathcal{C}$ and $\mathcal{D}$, written as $\mathcal{C} \times \mathcal{D}$ has objects of the form $\left(C, D\right)$ for $C \in \mathrm{Ob}\left(\mathcal{C}\right)$ and $D \in \mathrm{Ob}\left(\mathcal{D}\right)$, and arrows of the form

$$\left(f, g\right) : \left(C, D\right) \to \left(C', D'\right),$$

with $C, C' \in \mathrm{Ob}\left(\mathcal{C}\right)$, $D, D' \in \mathrm{Ob}\left(\mathcal{D}\right)$, $f \in \mathrm{Hom}_{\mathcal{C}}\left(C, C'\right)$ and $g \in \mathrm{Hom}_{\mathcal{D}}\left(D, D'\right)$.

Composition and units are defined componentwise, i.e.

$$\left(f', g'\right) \circ \left(f, g\right) = \left(f' \circ f, g' \circ g\right) \text{ and } \mathbb{1}_{\left(C,D\right)} = \left(\mathbb{1}_C, \mathbb{1}_D\right), \tag{1.23}$$

with $C, C', C'' \in \mathrm{Ob}\left(\mathcal{C}\right)$ and $D, D', D'' \in \mathrm{Ob}\left(\mathcal{D}\right)$ and

$$\begin{tikzcd} C \arrow[loop left, "\mathbb{1}_C"] \arrow[r, "f"] \arrow[rr, bend right, "f' \circ f"'] & C' \arrow[r, "f'"] & C'' \end{tikzcd} \qquad \begin{tikzcd} D \arrow[loop above, "\mathbb{1}_D"] \arrow[r, "g"] \arrow[rr, bend right, "g' \circ g"'] & D' \arrow[r, "g'"] & D'' \end{tikzcd}$$

Then in $\mathcal{C} \times \mathcal{D}$, we have

$$\begin{tikzcd} \left(C, D\right) \arrow[loop left, "\mathbb{1}_{\left(C,D\right)} = \left(\mathbb{1}_C, \mathbb{1}_D\right)"] \arrow[r, "\left(f,g\right)"] \arrow[rr, bend right, "\left(f',g'\right) \circ \left(f,g\right) = \left(f' \circ f, g' \circ g\right)"'] & \left(C', D'\right) \arrow[r, "\left(f',g'\right)"] & \left(C'', D''\right) \end{tikzcd}$$

Then there are two **projection functors**:

$$\begin{tikzcd} \mathcal{C} & \mathcal{C} \times \mathcal{D} \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & \mathcal{D} \end{tikzcd}$$

Given $\left(C, D\right) \in \mathrm{Ob}\left(\mathcal{C} \times \mathcal{D}\right)$ and $\left(f, g\right) : \left(C, D\right) \to \left(C', D, \right)$,

$$\pi_1\left(C, D\right) = C, \;\; \pi_1\left(f, g\right) = f. \tag{1.24}$$

Similarly,

$$\pi_2\left(C, D\right) = D, \;\; \pi_2\left(f, g\right) = g. \tag{1.25}$$

2. The opposite category $\mathcal{C}^{\mathrm{op}}$ has objects that are in a one-to-one correspondence with the objects of $\mathcal{C}$. Let $C^* \in \mathrm{Ob}\left(\mathcal{C}^{\mathrm{op}}\right)$ be the object in $\mathcal{C}^{\mathrm{op}}$ that corresponds to $C \in \mathrm{Ob}\left(\mathcal{C}\right)$. Then an arrow $f : C \to D$ in $\mathcal{C}$ corresponds to an arrow $f^* : D^* \to C^*$. With this notation, one can define composition and units in $\mathcal{C}^{\mathrm{op}}$ in terms of the corresponding operations in $\mathcal{C}$, namely

$$\mathbb{1}_{C^*} = \left(\mathbb{1}_C\right)^* . \tag{1.26}$$

For $f : C \to D$, $g : D \to E$ in $\mathcal{C}$, we have $f^* : D^* \to C^*$ and $g^* : E^* \to D^*$ in $\mathcal{C}^{\mathrm{op}}$. Then their composition is defined as

$$f^* \circ g^* = \left(g \circ f\right)^* . \tag{1.27}$$

$$\begin{tikzcd}
C \arrow[loop left, "\mathbb{1}_C"] \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & D \arrow[r, "g"] & E & {} \arrow[r, leftrightarrow, "\text{Duality}"] & {} & C^* \arrow[loop above, "\mathbb{1}_{C^*}=(\mathbb{1}_C)^*"] & D^* \arrow[l, "f^*"'] & E^* \arrow[l, "g^*"'] \arrow[ll, bend left, "f^*\circ g^*=(g\circ f)^*"]
\end{tikzcd}$$

3. The slice category $\mathcal{C}/C$ of a category $\mathcal{C}$ over an object $C \in \mathrm{Ob}\left(\mathcal{C}\right)$ has
   - **Objects:** all arrows f in $\mathcal{C}$ such that $\mathrm{cod}\left(f\right) = C$. In other words, all arrows $f \in \mathrm{Hom}_{\mathcal{C}}\left(X, C\right)$ with some $X \in \mathrm{Ob}\left(\mathcal{C}\right)$.
   - **Arrows:** an arrow a from $f : X \to C$ to $f' : X' \to C$ is precisely an arrow in $\mathrm{Hom}_{\mathcal{C}}\left(X, X'\right)$ such that $f' \circ a = f$. In othe words, the following diagram commutes:

$$\begin{tikzcd}
X \arrow[rr, "a"] \arrow[dr, "f"'] & & X' \arrow[dl, "f'"] \\
& C &
\end{tikzcd}$$

Now suppose $f, g, h \in \mathrm{Ob}\,\mathcal{C}/C$ and $a \in \mathrm{Hom}_{\mathcal{C}/C}(f, g), b \in \mathrm{Hom}_{\mathcal{C}/C}(g, h)$. Then there are objects $X, X', X'' \in \mathrm{Ob}\left(\mathcal{C}\right)$ such that the two triangles in the following diagram commute:

$$\begin{tikzcd}
X \arrow[r, "a"'] \arrow[rr, bend left, "b\circ a"] \arrow[dr, "f"'] & X' \arrow[r, "b"'] \arrow[d, "g"'] & X'' \arrow[dl, "h"] \\
& C &
\end{tikzcd}$$

In other words, $g \circ a = f$ and $h \circ b = g$, so that one obtains

$$f = g \circ a = \left(h \circ b\right) \circ a = h \circ \left(b \circ a\right) . \tag{1.28}$$

Therefore, we have the following commutative diagram:

$$\begin{tikzcd}
X \arrow[rr, "b\circ a"] \arrow[dr, "f"'] & & X'' \arrow[dl, "h"] \\
& C &
\end{tikzcd}$$

Hence, $b \circ a \in \mathrm{Hom}_{\mathcal{C}/C}\left(f, h\right)$, using the definition of arrows in $\mathcal{C}/C$. For a given $f \in \mathrm{Ob}\left(\mathcal{C}/C\right)$, $\mathbb{1}_f$ is precisely the identity arrow on $\mathrm{dom}\left(f\right)$ in the category $\mathcal{C}$, which is evident from the following commutative diagram:

$$\begin{tikzcd}
\mathrm{dom}\left(f\right) \arrow[rr, "\mathbb{1}_{\mathrm{dom}(f)}"] \arrow[dr, "f"'] & & \mathrm{dom}\left(f\right) \arrow[dl, "f"] \\
& C &
\end{tikzcd}$$

If $g : C \to D$ is any arrow in $\mathcal{C}$, then there is a functor called the **composition functor**:

$$g_* : \mathcal{C}/C \to \mathcal{C}/D,$$

defined on $\mathrm{Ob}\left(\mathcal{C}/C\right)$ as

$$g_*\left(f\right) = g \circ f. \tag{1.29}$$

$$\begin{tikzcd} X \arrow[d, "f"'] \arrow[dr, "g\circ f"] & \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

Commutativity of the above diagram dictates that $g\circ f\in \mathrm{Ob}\,(\mathcal{C}/D)$. Now suppose $f, f' \in \mathrm{Ob}\,(\mathcal{C}/C)$, and consider $a \in \mathrm{Hom}_{\mathcal{C}/C}\,(f, f')$ so that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "f"'] \arrow[dr, "g\circ f"] & X' \arrow[dl, "f'"'] \arrow[d, "g\circ f'"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

$$g\circ f = g\circ (f'\circ a) = (g\circ f')\circ a, \tag{1.30}$$

so the diagram indeed commutes. So we have the following commutative diagram:

$$\begin{tikzcd} X \arrow[rr, "a"] \arrow[dr, "g\circ f"'] & & X' \arrow[dl, "g\circ f'"] \\ & D & \end{tikzcd}$$

The commutativity of this diagram dictates that $g_*\,(a) = a$. In fact, the whole construction above is a functor $\mathcal{C}/(-) : \mathcal{C} \to \textbf{Cat}$.

$$\begin{tikzcd} \boxed{\begin{array}{c}\mathcal{C}\\ \\ C \xrightarrow{g} D\end{array}} \arrow[r, "\mathcal{C}/(-)"] & \boxed{\begin{array}{c}\textbf{Cat}\\ \\ \mathcal{C}/C \xrightarrow{g_*} \mathcal{C}/D\end{array}} \end{tikzcd}$$

4. The coslice category $C/\mathcal{C}$ of a category $\mathcal{C}$ under an object $C \in \mathrm{Ob}\,(\mathcal{C})$ has as objects all arrows f of $\mathcal{C}$ such that $\mathrm{dom}\,(f) = C$. An arrow in $\mathrm{Hom}_{C/\mathcal{C}}\,(f, f')$ is an arrow $h \in \mathrm{Hom}_{\mathcal{C}}\,(X, X')$ (where $X = \mathrm{cod}\,(X)$ and $X' = \mathrm{cod}\,(f')$) such that the diagram below commutes:

$$\begin{tikzcd} & C \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ X \arrow[rr, "h"'] & & X' \end{tikzcd}$$

In other words,

$$h\circ f = f'. \tag{1.31}$$

Question. How can the coslice category be defined in terms of the slice category and the opposite construction?

Example 1.2. The category $\textbf{Sets}_*$ of pointed sets consists of sets A with a distinguished element $a \in A$, and arrows $f : (A, a) \to (B, b)$ are functions $f : A \to B$ that preserves the distinguished elements $f\,(a) = b$. Now, $\textbf{Sets}_*$ is isomorphic to the coslice categroy $1/\textbf{Sets}$ of sets under any singleton $1 = \{\star\}$.

$$\textbf{Sets}_* \cong 1/\textbf{Sets}. \tag{1.32}$$

Indeed, functions $\overline{a} : 1 \to A$ are uniquely determined by $\overline{a}\,(\star) = a \in A$, and are objects in $1/\textbf{Sets}$. Now we define a functor $\mathcal{F} : \textbf{Sets}_* \to 1/\textbf{Sets}$ by

$$\mathcal{F}\,(A, a) = \overline{a} \text{ and } \mathcal{F}\,(f) = f. \tag{1.33}$$

Then we define $\mathcal{G} : 1/\textbf{Sets} \to \textbf{Sets}_*$ by

$$\mathcal{G}\,(\overline{a}) = (A, \overline{a}\,(*)) \text{ and } \mathcal{G}\,(f) = f. \tag{1.34}$$

One can easily verify that $\mathcal{G}$ and $\mathcal{F}$ are functors, and

$$\mathcal{G}\circ\mathcal{F} = \mathbb{1}_{\textbf{Sets}_*} \text{ and } \mathcal{F}\circ\mathcal{G} = \mathbb{1}_{1/\textbf{Sets}}\,. \tag{1.35}$$

Therefore, $1/\textbf{Sets}$ and $\textbf{Sets}_*$ are isomorphic categories.

2 Free Categories

§2.1 Free Monoid

Start with an "alphabet" A of "letters" $\mathtt{a}, \mathtt{b}, \mathtt{c}, \ldots$, i.e. a set

$$A = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \ldots\}\,. \tag{2.1}$$

A word over A is a finite sequence of letters:

$$\mathtt{thisword}, \mathtt{categoriesarefun}, \mathtt{asdfghjkl}, \ldots$$

We write "-" for empty word. The **Kleene closure** of A is defined to be the set

$$A^* = \{\text{words over } A\}\,. \tag{2.2}$$

Define a binary operation $*$ on A^* by $w * w' = ww'$ for words $w, w' \in A^*$. Thus, the binary operation $*$ on A^* is just concatenation. The operation can easily be seen to be associative, and the empty word "-" is a unit. Therefore, A^* is a monoid– called the **free monoid** on the set A.

The number of letters in a word is called its **length**. The elements $\mathtt{a} \in A$ can be regarded as words of length 1. One has a function $i : A \to A^*$ defined by $i(\mathtt{a}) = \mathtt{a}$, and called the " insertion of generators". The elements of A generate the free monoid, in the sense that every $w \in A^*$ can be written as a $*$ products of elements of A, i.e.,

$$w = \mathtt{a}_1 * \mathtt{a}_2 * \cdots * \mathtt{a}_\mathtt{n},$$

for some $\mathtt{a}_1, \ldots, \mathtt{a}_\mathtt{n} \in A$.

A monoid M is **freely generated** by a subset A of M, if the following conditions hold:

(a) Every element $m \in M$ can be written as a product of elements of A:

$$m = a_1 \cdot_M a_2 \cdot_M \cdots \cdot_M a_n\,, \quad \text{where } a_i \in A.$$

(b) No "nontrivial" relations hold in M. In other words, if

$$a_1 \cdot_M \cdots \cdot_M a_n = a'_1 \cdot_M \cdots \cdot_M a'_k,$$

for $a_i, a'_j \in A$, then this is required by the axioms of monoids.

The second condition of the definition of a free monoid is made more precise in the following way:

First, every monoid N has an underlying set $|N|$, and every monoid homomorphism $f : N \to M$ has an underlying function $|f| : |N| \to |M|$. This way, one has a functor **Mon** $\to$ **Sets**. This functor is called the **forgetful functor**.

The free monoid $M(A)$ on a set A is by definition "the" monoid with the following universal mapping property or UMP:

> **Universal mapping property (UMP) of $M(A)$:**
> There is a function $i : A \to |M(A)|$; and given any monoid N and any function $f : A \to |N|$, there is a **unique** monoid homomorphism $\overline{f} : M(A) \to N$ such that $\left|\overline{f}\right| \circ i = f$, as indicated in the following diagram:

in **Mon** :
$$\begin{tikzcd} M(A) \arrow[r, dashed, "\overline{f}"] & N \end{tikzcd}$$

in **Sets** :
$$\begin{tikzcd} \lvert M(A)\rvert \arrow[r, "\lvert\overline{f}\rvert"] & \lvert N\rvert \\ A \arrow[u, "i"] \arrow[ur, "f"'] & \end{tikzcd}$$

i is called the insertion of generators.

Proposition 2.1

A^* has the UMP of the free monoid on A.

Proof. Given any monoid N and any function $f : A \to |N|$, define $\overline{f} : A^* \to N$ by $\overline{f}\,(\text{-}) = u_N$, and

$$\overline{f}\left(\mathtt{a}_1\mathtt{a}_2\cdots\mathtt{a}_n\right) = \overline{f}\left(\mathtt{a}_1 * \mathtt{a}_2 * \cdots * \mathtt{a}_n\right) := f\left(\mathtt{a}_1\right) \cdot_N \cdots \cdot_N f\left(\mathtt{a}_n\right). \tag{2.3}$$

$\overline{f} : A^* \to N$ is clearly a monoid homomorphism, with $\overline{f}\,(\mathtt{a}) = f\,(\mathtt{a})$, so that

$$\left(\left|\overline{f}\right| \circ i\right)(a) = f\,(a)\,. \tag{2.4}$$

Therefore, the following diagram commutes:

$$\begin{tikzcd} \lvert A^*\rvert \arrow[r, "\lvert\overline{f}\rvert"] & \lvert N\rvert \\ A \arrow[u, "i"] \arrow[ur, "f"'] & \end{tikzcd}$$

This proves the existence of $\overline{f} : A^* \to N$ with the required universal mapping property. Let us now prove the uniqueness. Suppose there is another monoid homomorphism $g : A^* \to N$ so that $g\,(\mathtt{a}) = f\,(\mathtt{a})$, which in turn will give us the commutative diagram exhibiting UMP. Therefore, for all $\mathtt{a}_1, \ldots, \mathtt{a}_n \in A$,

$$\begin{aligned} g\left(\mathtt{a}_1\mathtt{a}_2\cdots\mathtt{a}_n\right) &= g\left(\mathtt{a}_1 * \mathtt{a}_2 * \cdots * \mathtt{a}_n\right) \\ &= g\left(\mathtt{a}_1\right) \cdot_N \cdots \cdot_N g\left(\mathtt{a}_n\right) \\ &= f\left(\mathtt{a}_1\right) \cdot_N \cdots \cdot_N f\left(\mathtt{a}_n\right) \\ &= \overline{f}\left(\mathtt{a}_1\mathtt{a}_2\cdots\mathtt{a}_n\right). \end{aligned}$$

Therefore, $g = \overline{f}$, proving the uniqueness of $\overline{f} : A^* \to N$. ■

Remark 2.1. Existence of a monoid homomorphism $\overline{f} : M\,(A) \to N$ implies that if there is an additional equality (sometimes referred to as "noise") besides the ones imposed by associativity law and unitality law in $M\,(A)$, then the additional equality is transported to the monoid N. But N is **any** monoid to which the free monoid $M\,(A)$ is supposed to be mapped to via the monoid homomorphism $\overline{f} : M\,(A) \to N$. Hence, the existence of monoid homomorphism $\overline{f} : M\,(A) \to N$ is equivalent to the second condition of absense of any "noise" in $M\,(A)$.

Proposition 2.2

Given monoids M and N with functions $i : A \to |M|$ and $j : A \to |N|$, each with the UMP of the

> free monoid on A, there is a unique monoid isomorphism $h : M \xrightarrow{\cong} N$ such that
> $$|h| \circ i = j \text{ and } \left|h^{-1}\right| \circ j = i.$$

Proof. From $j : A \to |N|$ and the UMP of M, one has $\bar{j} : M \to N$ with $|\bar{j}| \circ i = j$.

$$\begin{tikzcd}
\text{in } \mathbf{Mon}: & M \arrow[r, dashed, "\bar{j}"] & N \\
\text{in } \mathbf{Sets}: & {|M|} \arrow[r, "|\bar{j}|"] & {|N|} \\
& A \arrow[u, "i"] \arrow[ur, "j"'] &
\end{tikzcd}$$

From $i : A \to |M|$ and the UMP of N, one has $\bar{i} : N \to M$ with $|\bar{i}| \circ j = i$.

$$\begin{tikzcd}
\text{in } \mathbf{Mon}: & N \arrow[r, dashed, "\bar{i}"] & M \\
\text{in } \mathbf{Sets}: & {|N|} \arrow[r, "|\bar{i}|"] & {|M|} \\
& A \arrow[u, "j"] \arrow[ur, "i"'] &
\end{tikzcd}$$

Combining these two, we get the following commutative diagram:

$$\begin{tikzcd}
\text{in } \mathbf{Mon}: & M \arrow[r, dashed, "\bar{j}"'] \arrow[rr, bend left, "\bar{i} \circ \bar{j}"] & N \arrow[r, dashed, "\bar{i}"'] & M \\
\text{in } \mathbf{Sets}: & {|M|} \arrow[r, "|\bar{j}|"] & {|N|} \arrow[r, "|\bar{i}|"] & {|M|} \\
& & A \arrow[ul, "i"] \arrow[u, "j"] \arrow[ur, "i"'] &
\end{tikzcd}$$

From $i : A \to |M|$ and the UMP of M, we have the existence of a unique homomorphism $f : M \to M$ such that $|f| \circ i = i$. From the above commutative diagram, we get that $f = \bar{i} \circ \bar{j}$ satisfies $|f| \circ i = i$. Furthermore, $f = \mathbb{1}_M : M \to M$ also satisfies $|f| \circ i = i$. Therefore,

$$\bar{i} \circ \bar{j} = \mathbb{1}_M \,. \tag{2.5}$$

Similarly, exchanging M and N, we get

$$\bar{j} \circ \bar{i} = \mathbb{1}_N \,. \tag{2.6}$$

Now, $\bar{j} : M \to N$ is the required monoid isomorphism h, i.e. $\bar{j} = h$ and $\bar{i} = h^{-1}$, so that we have $|h| \circ i = j$ and $\left|h^{-1}\right| \circ j = i$. ■

In light of Proposition 2.1 and Proposition 2.2, we can say that if $M(A)$ has the UMP of a free monoid on A, then $M(A)$ is isomorphic to A^*.

§2.2 Free Category

Just as a monoid has an underlying set, a category has an underlying graph. A directed graph consists of vertices and edges, each of which has a “source” and a “target” vertex. Figure 2.1 is an example of a graph.

$$\begin{tikzcd}
A \arrow[r, "z"] \arrow[dr, "u"] & B \\
C \arrow[u, "x"] & D \arrow[u, "y"']
\end{tikzcd}$$

Figure 2.1: A graph

Definition 2.1. A **(directed) graph** consists of two sets: a set E of edges, and a set V of vertices, and two functions $s : E \to V$ (called source) adn $t : E \to V$ (called target). We denote a directed graph G by a quadruple (V, E, s, t). A **path** in a graph G is a finite sequence of edges $e_1, \dots, e_n$ such that $t(e_i) = s(e_{i+1})$ for each $i = 1, \dots, n-1$.

Suppose we have a path $e_1, \dots, e_n$ in G.

$$v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} v_2 \xrightarrow{e_3} \cdots \xrightarrow{e_n} v_n$$

Put $\operatorname{dom}(e_n \cdots e_1) = s(e_1)$ and $\operatorname{cod}(e_n \cdots e_1) = t(e_n)$, and define composition by concatenation:

$$e_n \cdots e_1 \circ e'_m \cdots e'_1 = e_n \cdots e_1 e'_m \cdots e'_1, \tag{2.7}$$

where $\operatorname{dom}(e_n \cdots e_1) = \operatorname{cod}(e'_m \cdots e'_1)$.

For each vertex v, we have an "empty path" denoted by $\mathbb{1}_v$ which is to be the identity arrow at v. With all of these terminologies at our disposal, we see that every graph G generates a category $\mathcal{C}(G)$ called the **free category** on G. It is defined by taking vertices of G as objects and paths in G as arrows. For example, take the graph given in Figure 2.1 with 4 vertices A, B, C, D. The corresponding free category on G is given by:

$$G : \quad \begin{tikzcd}
A \arrow[r, "z"] \arrow[dr, "u"] & B \\
C \arrow[u, "x"] & D \arrow[u, "y"']
\end{tikzcd}$$

$$\mathcal{C}(G) \quad \begin{tikzcd}
C \arrow[r, "x"] \arrow[rr, bend left=50, "z \circ x"] \arrow[rr, bend right=30, "y \circ u \circ x"'] \arrow[rrr, bend right=50, "u \circ x"'] & A \arrow[r, "z"] \arrow[r, bend left, "y \circ u"] \arrow[rr, bend right, "u"'] & B & D \arrow[l, "y"']
\end{tikzcd}$$

Definition 2.2 (Graph Homomorphism)**.** Let $G \equiv (V, E, s, t)$ and $G' \equiv (V', E', s', t')$ be two graphs. A **graph homomorphism** f from G to G', denoted by $f : G \to G'$ consists of two functions $f_0 : V \to V'$ and $f_1 : E \to E'$ such that the following diagrams commute:

$$\begin{tikzcd}
E \arrow[r, "f_1"] \arrow[d, "s"'] & E' \arrow[d, "s'"] \\
V \arrow[r, "f_0"'] & V'
\end{tikzcd}
\qquad
\begin{tikzcd}
E \arrow[r, "f_1"] \arrow[d, "t"'] & E' \arrow[d, "t'"] \\
V \arrow[r, "f_0"'] & V'
\end{tikzcd}$$

Remark 2.2. Note that if G has only one vertex, then $\mathcal{C}(G)$ is just the free monoid on the set of edges of G. If, on the other hand, G has only vertices with no edges, then $\mathcal{C}(G)$ is the discrete category on the set of vertices of G.

Let us now see that $\mathcal{C}(G)$ has a UMP (universal mapping property). Define a "forgetful functor" $\mathcal{U} : \mathbf{Cat} \to \mathbf{Graphs}$ in the following way: the underlying graph of a (small) category has the collection of arrows as the set of edges E and the collection of objects as the set of vertices V, with $s = \mathrm{dom}$ and $t = \mathrm{cod}$.

Also, observe that we can describe a category $\mathcal{C}$ with a diagram as below:

$$\begin{tikzcd}
C_2 \arrow[r, "\circ"] & C_1 \arrow[r, shift left=2, "\mathrm{cod}"] \arrow[r, shift right=2, "\mathrm{dom}"'] & C_0, \arrow[l, "i" description]
\end{tikzcd}$$

where C_0 is the collection of objects of $\mathcal{C}_1$, C_1 is the collection of arrows, i is the identity arrow operation, and C_2 is the collection

$$C_2 = \{(f, g) \in C_1 \times C_1 \mid \operatorname{cod} f = \operatorname{dom} g\}.$$

Then a functor $F : \mathcal{C} \to \mathcal{D}$ from $\mathcal{C}$ to another category $\mathcal{D}$ (with D_2, D_1, D_0 as given above) is a pair of assignments $F_0 : C_0 \to D_0$ and $F_1 : C_1 \to D_1$ such that each similarly labeled square in the following diagram commutes:

$$\begin{tikzcd}
C_2 \arrow[r, "\circ"] \arrow[d, "F_2"'] & C_1 \arrow[r, shift left=2, "\mathrm{cod}"] \arrow[r, shift right=2, "\mathrm{dom}"'] \arrow[d, "F_1"'] & C_0 \arrow[l, "i" description] \arrow[d, "F_0"] \\
D_2 \arrow[r, "\circ"] & D_1 \arrow[r, shift left=2, "\mathrm{cod}"] \arrow[r, shift right=2, "\mathrm{dom}"'] & D_0, \arrow[l, "i" description]
\end{tikzcd}$$

where $F_2(f, g) = (F_1(f), F_1(g))$. Commutativity of the first square tells us that

$$F_1(g \circ f) = F_1(g) \circ F_1(f). \tag{2.8}$$

Commutativity of the second square is reminiscent of graph homomorphism if one removes the identity arrow operation. Note that the underlying graph of a category $\mathcal{C}$ can be depicted as

$$\begin{tikzcd}
C_1 \arrow[r, shift left, "\mathrm{cod}"] \arrow[r, shift right, "\mathrm{dom}"'] & C_0.
\end{tikzcd}$$

Therefore, at the level of objects, the forgetful functor

$$\mathcal{U} : \mathbf{Cat} \to \mathbf{Graphs} \tag{2.9}$$

sends the category (object of **Cat**)

$$\begin{tikzcd}
C_2 \arrow[r, "\circ"] & C_1 \arrow[r, shift left=2, "\mathrm{cod}"] \arrow[r, shift right=2, "\mathrm{dom}"'] & C_0 \arrow[l, "i" description]
\end{tikzcd}$$

to the underlying graph (objecr of **Graphs**)

$$\begin{tikzcd}
C_1 \arrow[r, shift left, "\mathrm{cod}"] \arrow[r, shift right, "\mathrm{dom}"'] & C_0.
\end{tikzcd}$$

And functors (arrows of **Cat**) under $\mathcal{U}$ are sent to graph homomorphisms (arrows of **Graphs**), i.e.

$$\begin{tikzcd}
C_2 \arrow[r, "\circ"] \arrow[d, "F_2"'] & C_1 \arrow[r, shift left=2, "\mathrm{cod}"] \arrow[r, shift right=2, "\mathrm{dom}"'] \arrow[d, "F_1"'] & C_0 \arrow[l, "i" description] \arrow[d, "F_0"] \\
D_2 \arrow[r, "\circ"] & D_1 \arrow[r, shift left=2, "\mathrm{cod}"] \arrow[r, shift right=2, "\mathrm{dom}"'] & D_0, \arrow[l, "i" description]
\end{tikzcd}$$

is sent to

$$\begin{tikzcd}
C_1 \arrow[r, shift left, "\mathrm{cod}"] \arrow[r, shift right, "\mathrm{dom}"'] \arrow[d, "F_1"'] & C_0 \arrow[d, "F_0"] \\
D_1 \arrow[r, shift left, "\mathrm{cod}"] \arrow[r, shift right, "\mathrm{dom}"'] & D_0
\end{tikzcd}$$

under $\mathcal{U}$. Given a category $\mathcal{C}$, its underlying graph is denoted by $|\mathcal{C}| = \mathcal{U}(\mathcal{C})$, where $\mathcal{U}$ is the forgetful functor $\mathcal{U} : \mathbf{Cat} \to \mathbf{Graphs}$. The **free category** $\mathcal{C}(G)$ on a graph G has the following universal mapping property (UMP).

> **Universal mapping property (UMP) of $\mathcal{C}(G)$:**
> There is a graph homomorphism $i : G \to |\mathcal{C}(G)|$; and given any category $\mathcal{D}$ and any graph homomorphism $h : G \to |\mathcal{D}|$, there is a **unique** functor $\overline{h} : \mathcal{C}(G) \to \mathcal{D}$ such that $\left|\overline{h}\right| \circ i = h$, as indicated in the following diagram:
>
> in **Cat** : $$\begin{tikzcd} \mathcal{C}(G) \arrow[r, dashed, "\overline{h}"] & \mathcal{D} \end{tikzcd}$$
>
> in **Graphs** : $$\begin{tikzcd} \left|\mathcal{C}(G)\right| \arrow[r, "\left|\overline{h}\right|"] & \left|\mathcal{D}\right| \\ G \arrow[u, "i"] \arrow[ur, "h"'] & \end{tikzcd}$$

The free category on a graph with just one vertex is just a free monoid on the set oof edges. The free category on a graph with two vertices and one edge between them is the finite category **2**. On the other hand, the free category on a graph of the form

$$\begin{tikzcd} A \arrow[r, shift left, "e"] & B \arrow[l, shift left, "f"] \end{tikzcd}$$

has (in addition to the identity arrows) the infinitely many arrows:

$$e, f, ef, fe, efe, fef, efef, fefe, \ldots. \tag{2.10}$$

3 Abstract Characterizations

§3.1 Epis and Monos

In **Sets**, a function $f : A \to B$ is called

- **injective** if $f(a) = f(a')$ implies $a = a'$ for all $a, a' \in A$,
- **surjective** if for each $b \in B$, there exists $a \in A$ such that $f(a) = b$.

We have the following abstract characterizations of these properties:

Definition 3.1 (Monomorphism and Epimorphism)**.** In any category $\mathcal{C}$, an arrow $f : A \to B$ is called a **monomorphism** if given any $g, h : C \to A$, $f \circ g = f \circ h$ implies $g = h$.

$$\begin{tikzcd} C \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & A \arrow[r, "f"] & B \end{tikzcd}$$

An arrow $f : A \to B$ is called an **epimorphism** if given any $i, j : B \to D$, $i \circ f = j \circ f$ implies $i = j$.

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, shift left, "i"] \arrow[r, shift right, "j"'] & D \end{tikzcd}$$

We often write $f : A \rightarrowtail B$ if f is a monomorphism and $f : A \twoheadrightarrow B$ if f is an epimorphism.

Proposition 3.1

A function $f : A \to B$ between sets is a monomorphism if and only if it is injective.

Proof. ($\Rightarrow$) Suppose $f : A \rightarrowtail B$. We need to show that f is injective. Suppose $f(a) = f(a')$ for some $a, a' \in A$. Consider functions $g, h : \{*\} \to A$ as $g(*) = a$ and $h(*) = a'$. Then

$$(f \circ g)(*) = f(g(*)) = f(a) = f(a') = f(h(*)) = (f \circ h)(*) . \tag{3.1}$$

So $f \circ g = f \circ h$. Since f is a monomorphism, we must have $g = h$. Therefore, $a = a'$, proving the injectivity of f.

($\Leftarrow$) Conversely, let $f : A \to B$ be injective. Let $g, h : C \to A$ such that $f \circ g = f \circ h$. Then for any $c \in C$,

$$f(g(c)) = (f \circ g)(c) = (f \circ h)(c) = f(h(c)) . \tag{3.2}$$

So $f(g(c)) = f(h(c))$. Since f is injective, this implies $g(c) = h(c)$. This is true for any $c \in C$. Therefore, $g = h$. In other words, $f \circ g = f \circ h$ implies $g = h$. Hence, f is a monomorphism. ∎

Proposition 3.2

A function $f : A \to B$ between sets is an epimorphism if and only if it is surjective.

Proof. ($\Rightarrow$) Suppose $f : A \twoheadrightarrow B$. We need to show that f is surjective. We construct two functions $g_1, g_2 : B \to \{0, 1\}$ as follows:

$$g_1(b) = 0 \text{ and } g_2(b) = \begin{cases} 0 & \text{if } b \in \operatorname{im} f \\ 1 & \text{otherwise.} \end{cases} \tag{3.3}$$

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] & \{0,1\} \end{tikzcd}$$

Then for each $a \in A$, $g_1(f(a)) = g_2(f(a)) = 0$. So $g_1 \circ f = g_2 \circ f$. Since f is an epimorphism, this gives us $g_1 = g_2$. In other words,

$$g_2(b) = 0 \text{ for each } b \in B. \tag{3.4}$$

Therefore, all $b \in B$ are in $\operatorname{im} f$, proving the surjectivity of f.

(⇐) Conversely, let $f : A \to B$ be surjective. Let $g_1, g_2 : B \to C$ such that $g_1 \circ f = g_2 \circ f$.

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] & \{0,1\} \end{tikzcd}$$

Since f is surjective, for any $b \in B$, there exists $a \in A$ such that $f(a) = b$. Then we have

$$g_1(b) = g_1(f(a)) = (g_1 \circ f)(a) = (g_2 \circ f)(a) = g_2(f(a)) = g_2(b). \tag{3.5}$$

So $g_1(b) = g_2(b)$, and this is true for any $b \in B$. Therefore, $g_1 = g_2$ and hence f is an epimorphism. ■

Monomorphisms are often obbreviated as monos or monic, epimorphisms are often obbreviated as epis or epic.

Proposition 3.3

A monoid homomorphism $h : M \to N$ is monic if and only if the underlying function $|h| : |M| \to |N|$ is monic (i.e. injective by Proposition 3.1).

Proof. **(⇒)** Suppose h is monic. We need to show that $|h|$ is injective. Let $|h|(m_1) = |h|(m_2)$ for $m_1, m_2 \in |M|$. Take two functions $x, y : 1 \to |M|$, where $1 = \{*\}$ is any singleton set, defined by $x(*) = m_1$ and $y(*) = m_2$. Let $M(1)$ be the free monoid generated by 1. By the UMP of a free monoid, there are unique monoid homomorphisms $\overline{x} : M(1) \to M$ and $\overline{y} : M(1) \to M$ such that the following diagrams commute:

$$\begin{tikzcd} {|M(1)|} \arrow[rr, "|\overline{x}|"] & & {|M|} \\ & 1 \arrow[ul, "i"] \arrow[ur, "x"'] & \end{tikzcd} \qquad \begin{tikzcd} {|M(1)|} \arrow[rr, "|\overline{y}|"] & & {|M|} \\ & 1 \arrow[ul, "i"] \arrow[ur, "y"'] & \end{tikzcd}$$

In other words, $|\overline{x}| \circ i = x$ and $|\overline{y}| \circ i = y$. Furthermore, by the UMP of $M(1)$ for the functions $|h| \circ x, |h| \circ y : 1 \to N$, there are unique monoid homomorphisms $f : M(1) \to N$ and $g : M(1) \to N$ such that the following diagrams commute:

$$\begin{tikzcd} {|M(1)|} \arrow[rr, "|f|"] & & {|N|} \\ & 1 \arrow[ul, "i"] \arrow[ur, "|h| \circ x"'] & \end{tikzcd} \qquad \begin{tikzcd} {|M(1)|} \arrow[rr, "|g|"] & & {|N|} \\ & 1 \arrow[ul, "i"] \arrow[ur, "|h| \circ y"'] & \end{tikzcd}$$

But since the following diagrams commute,

$$\begin{tikzcd} {|M(1)|} \arrow[rr, "|\overline{x}|"'] \arrow[rrr, bend left, "|h| \circ |\overline{x}| = |h \circ \overline{x}|"] & & {|M|} \arrow[r, "|h|"'] & {|N|} \\ & 1 \arrow[ul, "i"] \arrow[ur, "x"'] \arrow[urr, bend right, "|h| \circ x"'] & & \end{tikzcd} \qquad \begin{tikzcd} {|M(1)|} \arrow[rr, "|\overline{y}|"'] \arrow[rrr, bend left, "|h| \circ |\overline{y}| = |h \circ \overline{y}|"] & & {|M|} \arrow[r, "|h|"'] & {|N|} \\ & 1 \arrow[ul, "i"] \arrow[ur, "y"'] \arrow[urr, bend right, "|h| \circ y"'] & & \end{tikzcd}$$

the uniqueness of f and g guarantees that

$$f = h \circ \overline{x} \text{ and } g = h \circ \overline{y}. \tag{3.6}$$

However, $|h| \circ x$ and $|h| \circ y$ are equal, since

$$(|h| \circ x)(*) = |h|(m_1) = |h|(m_2) = (|h| \circ y)(*) . \tag{3.7}$$

Therefore, $f = g$, and hence,

$$h \circ \overline{x} = h \circ \overline{y}. \tag{3.8}$$

Since h is monic, $\overline{x} = \overline{y}$. Then we have

$$x = |\overline{x}| \circ i = |\overline{y}| \circ i = y. \tag{3.9}$$

Hence, $m_1 = x(*) = y(*) = m_2$, and thus $|h|$ is injective.

($\Leftarrow$) Conversely, suppose $|h| : |M| \to |N|$ is monic. Let $f, g : X \to M$ be monoid homomorphisms such that $h \circ f = h \circ g$.

$$\begin{tikzcd} X \arrow[r, shift left, "g"] \arrow[r, shift right, "f"'] & M \arrow[r, "h"] & N. \end{tikzcd}$$

In **Sets**, we then have $|h| \circ |f| = |h| \circ |g|$.

$$\begin{tikzcd} {|X|} \arrow[r, shift left, "|g|"] \arrow[r, shift right, "|f|"'] & {|M|} \arrow[r, "|h|"] & {|N|} . \end{tikzcd}$$

Since $|h|$ is monic, we must have $|f| = |g|$. Therefore, $f = g$, and hence $h : M \to N$ is monic. ■

Example 3.1. In the category **Mon** of monoids and monoid homomorphisms, there is a monic homomorphism $\overline{\mathbb{N}} \rightarrowtail \overline{\mathbb{Z}}$, where $\overline{\mathbb{N}} = (\mathbb{N}, +, 0)$ is the additive monoid of natural numbers, and $\overline{\mathbb{Z}} = (\mathbb{Z}, +, 0)$ is the additive monoid of integers. This map given by the inclusion $\mathbb{N} \subset \mathbb{Z}$ of sets is monic in **Mon**, by Proposition 3.3, since it is an injective homomorphism. We shall now show that this is also epic in **Mon**.

$$\begin{tikzcd} \overline{\mathbb{N}} \arrow[r, "i"] & \overline{\mathbb{Z}} \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \overline{M} \end{tikzcd}$$

Given any monoid $\overline{M} = (M, *, u)$, with the underlying set M, let $f, g : \overline{\mathbb{Z}} \to \overline{M}$ be monoid homomorphisms such that $f \circ i = g \circ i$. To prove that i is epic, we need to show that $f = g$.

Since $i : \overline{\mathbb{N}} \to \overline{\mathbb{Z}}$ is the inclusion, $f \circ i = g \circ i$ gives us that $f|_{\overline{\mathbb{N}}} = g|_{\overline{\mathbb{N}}}$, i.e. $f(n) = g(n)$ for each $n \in \overline{\mathbb{N}}$. We need to show that $f(-n) = g(-n)$.

$$\begin{aligned} f(-n) &= f(-n) * u = f(-n) * g(0) = f(-n) * g(n + (-n)) \\ &= f(-n) * [g(n) * g(-n)] = [f(-n) * g(n)] * g(-n) \\ &= [f(-n) * f(n)] * g(-n) = f(-n + n) * g(-n) \\ &= u * g(-n) = g(-n) . \end{aligned} \tag{3.10}$$

Therefore, $f = g$, and hence i is epic.

Proposition 3.4

Every isomorphism is both monic and epic.

Proof. Let $m : B \to C$ be an isomorphism. Then there exists $e : C \to B$ such that $m \circ e = \mathbb{1}_C$ and $e \circ m = \mathbb{1}_B$. So the two triangles in the following diagram commute:

$$\begin{tikzcd}
A \arrow[r, shift left, "x"] \arrow[r, shift right, "y"'] & B \arrow[r, "m"] \arrow[dr, "e\circ m=\mathbb{1}_B"'] & C \arrow[d, "e"] \arrow[dr, "m\circ e=\mathbb{1}_C"] & & \\
 & & B \arrow[r, "m"'] & C \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & C
\end{tikzcd}$$

Given any arrows $x, y : A \to B$ such that $m \circ x = m \circ y$, we have

$$\begin{aligned}
m \circ x = m \circ y &\implies e \circ (m \circ x) = e \circ (m \circ y) \\
&\implies (e \circ m) \circ x = (e \circ m) \circ y \\
&\implies \mathbb{1}_B \circ x = \mathbb{1}_B \circ y \\
&\implies x = y. \qquad (3.11)
\end{aligned}$$

Therefore, m is monic. Now, given arrows $f, g : C \to D$ with $f \circ m = g \circ m$, we have

$$\begin{aligned}
f \circ m = g \circ m &\implies (f \circ m) \circ e = (g \circ m) \circ e \\
&\implies f \circ (m \circ e) = g \circ (m \circ e) \\
&\implies f \circ \mathbb{1}_C = g \circ \mathbb{1}_C \\
&\implies f = g. \qquad (3.12)
\end{aligned}$$

Therefore, m is epic. ■

Remark 3.1. In **Sets**, the converse of Proposition 3.4 also holds: every monic and epic is an isomorphism. But this is not true, in general. The counterexample is provided in the context of the category **Mon** in Example 3.1. There we saw that the inclusion homomorphism $i : (\mathbb{N}, +, 0) \to (\mathbb{Z}, +, 0)$ is both monic and epic. But this arrow in **Mon** is not an iso, i.e. its inverse does not exist. In particular, there does not exist an arrow $j : \overline{\mathbb{Z}} \to \mathbb{N}$ such that $i \circ j = \mathbb{1}_{\mathbb{Z}}$ and $j \circ i = \mathbb{1}_{\overline{\mathbb{N}}}$.

Proposition 3.5

With regard to a commutative triangle,

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "h"'] & B \arrow[d, "g"] \\
 & C
\end{tikzcd}$$

in any category $\mathcal{C}$,

(a) if f and g are isos (resp. monos, resp. epis), so is h;

(b) if h is monic, so is f;

(c) if h is epic, so is g;

Proof. (a) Suppose f and g are isos. Then there are arrows $f^{-1} : B \to A$ and $g^{-1} : C \to B$ such that $f \circ f^{-1} = \mathbb{1}_A$, $f^{-1} \circ f = \mathbb{1}_B$, $g \circ g^{-1} = \mathbb{1}_C$, $g^{-1} \circ g = \mathbb{1}_B$. Since the triangle above is commutative, $h = g \circ f$. We define $k : C \to A$ as $k = f^{-1} \circ g^{-1}$. Then

$$h \circ k = (g \circ f) \circ \left(f^{-1} \circ g^{-1}\right) = g \circ \mathbb{1}_A \circ g^{-1} = \mathbb{1}_C\,. \qquad (3.13)$$

$$k \circ h = \left(f^{-1} \circ g^{-1}\right) \circ (g \circ f) = f^{-1} \circ \mathbb{1}_B \circ f = \mathbb{1}_A\,. \qquad (3.14)$$

Therefore, h is an isomorphism, and $k = h^{-1}$.

Suppose f and g are monos. We want to show that $h : A \to C$ is monic. Suppose there are arrows $x, y : X \rightrightarrows A$ such that $h \circ x = h \circ y$.

$$\begin{tikzcd} & & B \arrow[dr, "g"] & \\ X \arrow[r, shift left, "x"] \arrow[r, shift right, "y"'] & A \arrow[ur, "f"] \arrow[rr, "h=g\circ f"'] & & C \end{tikzcd}$$

$$h \circ x = h \circ y \implies (g \circ f) \circ x = (g \circ f) \circ y \implies g \circ (f \circ x) = g \circ (f \circ y)\,. \tag{3.15}$$

Since g is monic, we must have $f \circ x = f \circ y$. Again, since f is monic, we must have $x = y$. So $h \circ x = h \circ y$ implies $x = y$. Therefore, h is monic.

Suppose f and g are epis. We want to show that $h : A \to C$ is epic. Suppose there are arrows $x, y : C \rightrightarrows Y$ such that $x \circ h = y \circ h$.

$$\begin{tikzcd} & B \arrow[dr, "g"] & & \\ A \arrow[ur, "f"] \arrow[rr, "h=g\circ f"'] & & C \arrow[r, shift left, "x"] \arrow[r, shift right, "y"'] & Y \end{tikzcd}$$

$$x \circ h = y \circ h \implies x \circ (g \circ f) = y \circ x \circ (g \circ f) \implies (x \circ g) \circ f = (y \circ g) \circ f. \tag{3.16}$$

Since f is epic, we must have $x \circ g = y \circ g$. Again, since g is epic, we must have $x = y$. So $x \circ h = y \circ h$ implies $x = y$. Therefore, h is epic.

(b) We need to show that $f : A \to B$ is monic. Suppose there are arrows $x, y : X \rightrightarrows A$ such that $f \circ x = f \circ y$.

$$\begin{tikzcd} & & B \arrow[dr, "g"] & \\ X \arrow[r, shift left, "x"] \arrow[r, shift right, "y"'] & A \arrow[ur, "f"] \arrow[rr, "h=g\circ f"'] & & C \end{tikzcd}$$

$$f \circ x = f \circ y \implies g \circ f \circ x = g \circ f \circ y \implies h \circ x = h \circ y. \tag{3.17}$$

Since h is monic, $h \circ x = h \circ y$ implies $x = y$. Therefore, since $f \circ x = f \circ y$ forces $x = y$, f is monic.

(c) We need to show that $g : B \to C$ is epic. Suppose there are arrows $x, y : C \rightrightarrows Y$ such that $x \circ g = y \circ g$.

$$\begin{tikzcd} & B \arrow[dr, "g"] & & \\ A \arrow[ur, "f"] \arrow[rr, "h=g\circ f"'] & & C \arrow[r, shift left, "x"] \arrow[r, shift right, "y"'] & Y \end{tikzcd}$$

$$x \circ g = y \circ g \implies x \circ g \circ f = y \circ g \circ f \implies x \circ h = y \circ h. \tag{3.18}$$

Since h is epic, $x \circ h = y \circ h$ implies $x = y$. Therefore, since $x \circ g = y \circ g$ forces $x = y$, g is epic.

■

Remark 3.2. In the commutative triangle of Proposition 3.5, we have $h = g \circ f$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "h"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

If h is monic, then so is f as proven in Proposition 3.5. However, h being monic does not necessarily imply g is monic. Similarly, h being epic also does not imply f is epic. The following is such an example.

Example 3.2. Let us consider the category **Sets**. Take $A = C = \{0\}$ and $B = \{0, 1\}$. Define $f : A \to B$ by $f(0) = 0$, and $g : B \to C$ by $g(x) = 0$ for $x \in \{0, 1\}$.

Then $g \circ f : A \to C$ is

$$(g \circ f)(0) = g(0) = 0. \tag{3.19}$$

So $h = g \circ f$ is an isomorphism. In particular, h is monic. However, $g : B \to C$ is not monic (i.e. injective since we are in **Sets**) because $g(0) = g(1)$. Similarly, h is epic as well, but f is not.

§3.2 Sections and Retractions

We have seen in Proposition 3.4 that any iso is both monic and epic. More generally, if an arrow $f : A \to B$ has a left inverse $g : B \to A$, i.e. $g \circ f = \mathbb{1}_A$, then f must be monic and g epic.

$$\begin{tikzcd} C \arrow[r, shift left, "x"] \arrow[r, shift right, "y"'] & A \arrow[r, "f"] \arrow[dr, "g\circ f=\mathbb{1}_A"'] & B \arrow[d, "g"] \\ & & A \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & D \end{tikzcd}$$

Given arrows $x, y : C \to A$ with $f \circ x = f \circ y$,

$$\begin{aligned} f \circ x = f \circ y &\implies g \circ (f \circ x) = g \circ (f \circ y) \\ &\implies (g \circ f) \circ x = (g \circ f) \circ y \\ &\implies \mathbb{1}_A \circ x = \mathbb{1}_A \circ y \\ &\implies x = y. \end{aligned} \tag{3.20}$$

Therefore, $f : A \to B$ is monic. Now, given $p, q : A \to D$ with $p \circ g = q \circ g$,

$$\begin{aligned} p \circ g = q \circ g &\implies (p \circ g) \circ f = (q \circ g) \circ f \\ &\implies p \circ (g \circ f) = q \circ (g \circ f) \\ &\implies p \circ \mathbb{1}_A = q \circ \mathbb{1}_A \\ &\implies p = q. \end{aligned} \tag{3.21}$$

Therefore, $g : B \to A$ is epic.

> **Definition 3.2.** A **split mono** is an arrow with a left inverse. A **split epi** is an arrow with a right inverse. Given arrows $e : X \to A$ and $s : A \to X$ with $e \circ s = \mathbb{1}_A$, the arrow s is called a **section** or **splitting** of e, and e is called a **retraction** of s. The object A is called a retract of X. Therefore, section is monic and retraction is epic.

Remark 3.3. Since functors preserve identities, they also preserve split epis and split monos. Functors do not preserve epi, in general. Counterexample is the forgetful functor $\mathcal{U} : \mathbf{Mon} \to \mathbf{Sets}$, which does not preserve the epi $i : \overline{\mathbb{N}} \to \overline{\mathbb{Z}}$, i.e. $\mathcal{U}(i) : \mathbb{N} \to \mathbb{Z}$ is not an epi in **Sets** as it's not surjective.

In **Sets**, every mono splits except those of the form $\varnothing \rightarrowtail A$. The condition that every epi splits is the categorical version of the axiom of choice. Let us see this fact in detail. Consider an epi $e : E \twoheadrightarrow X$. We then have the following family of nonempty sets

$$E_x = e^{-1}(\{x\}) \quad \text{for } x \in X.$$

Each E_x is nonempty since e is surjective. Now, for each $x \in X$, one chooses an element $s(x)$ from $E_x \subset E$, and thus define the function $s : X \to E$. By construction, $e(s(x)) = x$ for each $x \in X$, i.e. $e \circ s = \mathbb{1}_X$. One, thus finds that the choice function $s : X \to E$ associated with the family $(E_x)_{x \in X}$ is exactly a splitting of e.

Let us do the reverse construction now. Given a family of nonempty subsets $(E_x)_{x \in X}$, take

$$E = \{(x, y) \mid x \in X, y \in E_x\} = \bigcup_{x \in X} (\{x\} \times E_x), \tag{3.22}$$

and define the epi $e : E \twoheadrightarrow X$ by $(x, y) \mapsto x$. A splitting $s : X \to E$ hence determines a choice function for the family $(E_x)_{x \in X}$,

Remark 3.4. A notion related to the existence of "choice function" is that of being projective. An object P is said to be **projective** if for any $e : E \twoheadrightarrow X$ and arrow $f : P \to X$, there is some (not necessarily unique) arrow $\overline{f} : P \to E$ such that $e \circ \overline{f} = f$, as indicated in the following commutative diagram:

$$\begin{tikzcd}
 & E \arrow[d, two heads, "e"] \\
P \arrow[ur, "\overline{f}"] \arrow[r, "f"'] & X
\end{tikzcd}$$

One says that f **lifts across** e.

Let P be projective and let $e : E \twoheadrightarrow P$ be an epi. Since P is projective, given $\mathbb{1}_P : P \to P$, there exists $s : P \to E$ such that the following diagram commutes:

$$\begin{tikzcd}
 & E \arrow[d, two heads, "e"] \\
P \arrow[ur, dashed, "s"] \arrow[r, "\mathbb{1}_P"'] & P
\end{tikzcd}$$

yielding $s \circ e = \mathbb{1}_P$. In other words, the epi $e : E \twoheadrightarrow P$ splits.

§3.3 Initial and Terminal Objects

Definition 3.3. In any category $\mathcal{C}$, an object

- 0 is **initial** if for any object C, there is a unique arrow $0 \to C$,
- 1 is **terminal** if for any object C, there is a unique arrow $C \to 1$.

Proposition 3.6

Initial and terminal objects are unique up to isomorphism.

Proof. Suppose both 0 and $0'$ are initial objects in some category $\mathcal{C}$. Since 0 is initial, there is a unique arrow $u : 0 \to 0'$. Again, since $0'$ is initial, there is a unique arrow $v : 0' \to 0$. Then we can form the composition $v \circ u : 0 \to 0$. But since 0 is an initial object, there is a unique arrow $0 \to 0$, which has to be the identity arrow $\mathbb{1}_0$. Therefore, $v \circ u = \mathbb{1}_0$.

$$\begin{tikzcd}
0 \arrow[r, "u"] \arrow[dr, "v\circ u=\mathbb{1}_0"'] & 0' \arrow[d, "v"] \arrow[dr, "u\circ v=\mathbb{1}_{0'}"] & \\
& 0 \arrow[r, "u"] & 0'
\end{tikzcd}$$

Similarly, $u \circ v : 0' \to 0'$ is an arrow from $0'$ to itself. But since $0'$ is an initial object, there is a unique arrow $0' \to 0'$, which has to be the identity arrow $\mathbb{1}_{0'}$. Therefore, $u \circ v = \mathbb{1}_{0'}$. So 0 and $0'$ are isomorphic via a unique isomorphism $u : 0 \to 0'$.

Let 1 and $1'$ be terminal objects of $\mathcal{C}$. Since $1'$ is terminal, there is a unique arrow $u : 1 \to 1'$. Again, since 1 is terminal, there is a unique arrow $v : 1' \to 1$. Then we can form the composition $v \circ u : 1 \to 1$. But since 1 is a terminal object, there is a unique arrow $1 \to 1$, which has to be the identity arrow $\mathbb{1}_1$. Therefore, $v \circ u = \mathbb{1}_1$.

$$\begin{tikzcd}
1 \arrow[r, "u"] \arrow[dr, "v\circ u=\mathbb{1}_1"'] & 1' \arrow[d, "v"] \arrow[dr, "u\circ v=\mathbb{1}_{1'}"] & \\
& 1 \arrow[r, "u"] & 1'
\end{tikzcd}$$

Similarly, $u \circ v : 1' \to 1'$ is an arrow from $1'$ to itself. But since $1'$ is a terminal object, there is a unique arrow $1' \to 1'$, which has to be the identity arrow $\mathbb{1}_{1'}$. Therefore, $u \circ v = \mathbb{1}_{1'}$. So 1 and $1'$ are isomorphic via a unique isomorphism $u : 1 \to 1'$. ■

Example 3.3. 1. In **Sets**, the empty set $\varnothing$ is initial, and sny singleton set is terminal. Indeed, for any set B, there is a unique function from $\varnothing$ to B, called the empty function. When $B = \varnothing$, the empty function from $\varnothing$ to $\varnothing$ is the required identity arrow on the object $\varnothing \in \mathrm{Ob}\left(\mathbf{Sets}\right)$.

There is also a unique function from any set B to a singleton set $\{*\}$, $f : B \to \{*\}$, given by $f\left(b\right) = *$ for every $b \in B$. It is unique in the sense that there can't be any other function from the same set B to the singleton set $\{*\}$. In other words, $\mathrm{Hom}_{\mathbf{Sets}}\left(B, \{*\}\right)$ for a given set $B \in \mathrm{Ob}\left(\mathbf{Sets}\right)$ is a singleton set.

2. In $\mathbf{Vect}_{\mathbb{K}}$, the category of vector spaces over the field $\mathbb{K}$ and linear transformations, the 0-dimensional vector space $\{\mathbf{0}\}$ consisting of the zero-vector (additive identity) only is both the initial and terminal objects.

Given a $\mathbb{K}$-vector space V, there is only one linear transformation from $\{\mathbf{0}\}$ to V, namely the one that takes $\mathbf{0}$ to $\mathbf{0}_V$, the additive identity of the $\mathbb{K}$-vector space V. Also, given a $\mathbb{K}$-vector space V, there is only one linear transformation from V to $\{\mathbf{0}\}$ that takes all of V to $\mathbf{0}$. In other words, for a given $V \in \mathrm{Ob}\left(\mathbf{Vect}_{\mathbb{K}}\right)$, both $\mathrm{Hom}_{\mathbf{Vect}_{\mathbb{K}}}\left(\{\mathbf{0}\}, V\right)$ and $\mathrm{Hom}_{\mathbf{Vect}_{\mathbb{K}}}\left(V, \{\mathbf{0}\}\right)$ are singleton sets.

§3.3.i The Category of Boolean Algebras

Definition 3.4 (Boolean algebra)**.** A **Boolean algebra** is a poset $(B, \leqslant)$ together with distinguished elements $0, 1$, binary operations $a \vee b$ (read a join b) and $a \wedge b$ (read a meet b) and a unary operation $\neg b$ (read b complement) such that the following conditions are satisfied for all $a, b, c \in B$:

1. $0 \leqslant a$ and $a \leqslant 1$.
2. $a \leqslant c$ and $b \leqslant c$ if and only if $a \vee b \leqslant c$.
3. $c \leqslant a$ and $c \leqslant b$ if and only if $c \leqslant a \wedge b$.
4. $a \leqslant \neg b$ if and only if $a \wedge b = 0$.
5. $\neg(\neg a) = a$.

A typical example of a Boolean algebra is the power set $\mathscr{P}(X)$ of a set X, ordered by inclusion $A \subseteq B$, with the distinguished elements $0 = \varnothing$ and $1 = X$. The binary operations join and meet are given by union and intersection of subsets, respectively, and the unary operation complementation is $\neg A = X \setminus A$.

The union and intersection of sets satisfy $A \cap B \subseteq A \subseteq A \cup B$ and $A \cap B \subseteq B \subseteq A \cup B$. The same is satisfied in any Boolean algebra.

Proposition 3.7

In a Boolean algebra B, $a \wedge b \leqslant a \leqslant a \vee b$ and $a \wedge b \leqslant b \leqslant a \vee b$ for any $a, b \in B$.

Proof. Since $(B, \leqslant)$ is a poset, by the reflexivity property, $a \vee b \leqslant a \vee b$. Then using property 2 from the definition of Boolean algebra, we get

$$a \leqslant a \vee b \text{ and } b \leqslant a \vee b. \tag{3.23}$$

Again, $a \wedge b \leqslant a \wedge b$ by reflexivity. Then using property 3 from the definition of Boolean algebra, we get

$$a \wedge b \leqslant a \text{ and } a \wedge b \leqslant b. \tag{3.24}$$

Therefore, combining (3.23) and (3.24), we get $a \wedge b \leqslant a \leqslant a \vee b$ and $a \wedge b \leqslant b \leqslant a \vee b$. ■

A special case of Boolean algebras is the 2-element Boolean algebra $\mathbf{2} = \{0, 1\}$, considered as $\mathscr{P}(\{*\})$, the power set pf the singleton set $\{*\}$ which consists of only the empty set $\varnothing$ and $\{*\}$.

The category of Boolean algebras is denoted by **BA**, where objects are Boolean algebras and arrows are Boolean homomorphisms, that is functions $h : B \to B'$ that, in addition to being monotone, satisfy

- $h(0_B) = 0_{B'}$, $h(1_B) = 1_{B'}$;
- $h(a \vee_B b) = h(a) \vee_{B'} h(b)$;
- $h(a \wedge_B b) = h(a) \wedge_{B'} h(b)$;
- $h(\neg_B a) = \neg_{B'} h(a)$,

for any $a, b \in B$. In this category, $\mathbf{2}$ is the initial object. There is this 1-element Boolean algebra $\mathbf{1} = \{0\}$, which is regarded as the power set of $\varnothing$, i.e. $\mathbf{1} \equiv \mathscr{P}(\varnothing) = \{\varnothing\}$. In this situation, the distinguished elements 0 and 1 conicide. $\mathbf{1}$ acts as the terminal object of **BA**. In other words, given a Boolean algebra B, there is exactly one boolean homomorphism from $h : \mathbf{2} \to B$, given by

$$h(0) = 0_B \text{ and } h(1) = 1_B.$$

Furthermore, there is exactly one boolean homomorphism $f : B \to \mathbf{1}$ given by

$$f(b) = 0$$

for any $b \in B$.

Definition 3.5 (Filter)**.** A **filter** in a Boolean algebra B is a nonempty subset $F \subseteq B$ that is closed upward and under meets, i.e.

- $a \in F$ and $a \leqslant b$ implies $b \in F$,
- $a, b \in F$ implies $a \wedge b \in F$.

Definition 3.6 (Ultrafilter)**.** A filter F is called **maximal** if $F \subset F'$ implies $F' = B$, i.e. the only strictly larger filter is the "improper" filter B itself. A maximal filter is called an **ultrafilter**.

One can easily verify that a filter F is an ultrafilter if and only if for every element $b \in B$, either $b \in F$ or $\neg b \in F$ and not both.

Theorem 3.8

In a Boolean algebra B, a filter F is an ultrafilter if and only if for every element $b \in B$, either $b \in F$ or $\neg b \in F$ and not both.

Proof. ($\Rightarrow$) Suppose $F \subset B$ is an ultrafilter. If both b and $\neg b$ are in F, then so is their meet $b \wedge \neg b$. But we have

$$b \leqslant b = \neg(\neg b)\,. \tag{3.25}$$

Then using property 4 from the definition of Boolean algebra, we get

$$b \wedge \neg b = 0. \tag{3.26}$$

So $0 \in F$. But $0 \leqslant a$ for each $a \in B$. This gives us that $a \in F$ for any $a \in B$, i.e. $F = B$, which contradicts with F being an ultrafilter. Therefore, both b and $\neg b$ cannot be in F. Assume for the sake of contradiction that neither b nor $\neg b$ is in F, i.e. $b \notin F$ and $\neg b \notin F$. Consider the following set:

$$G = \{v \in B \mid u \wedge b \leqslant v \text{ for some } u \in F\}\,. \tag{3.27}$$

We claim that G is a filter.

- For $v \in G$ and $v \leqslant w$ gives us that $u \wedge b \leqslant v$ for some $u \in F$. By transitivity,

  $$u \wedge b \leqslant v \leqslant w. \tag{3.28}$$

  So $u \wedge b \leqslant w$, so $w \in G$.

- Let $v_1, v_2 \in G$. Then $u_1 \wedge b \leqslant v_1$ and $u_2 \wedge b \leqslant v_2$ for some $u_1, u_2 \in F$. We claim that $(u_1 \wedge u_2) \wedge b \leqslant v_1 \wedge v_2$.

  $$(u_1 \wedge u_2) \wedge b \leqslant u_1 \wedge u_2 \leqslant u_1 \text{ and } (u_1 \wedge u_2) \wedge b \leqslant b. \tag{3.29}$$

  Therefore, using property 3 from the definition of Boolean algebra, we get

  $$(u_1 \wedge u_2) \wedge b \leqslant u_1 \wedge b \leqslant v_1. \tag{3.30}$$

  Similarly,

  $$(u_1 \wedge u_2) \wedge b \leqslant u_2 \wedge b \leqslant v_2. \tag{3.31}$$

  Again, using property 3 from the definition of Boolean algebra, we get

  $$(u_1 \wedge u_2) \wedge b \leqslant v_1 \wedge v_2. \tag{3.32}$$

  Since F is a filter and $u_1, u_2 \in F$, $u_1 \wedge u_2 \in F$, so $v_1 \wedge v_2 \in G$.

Therefore, G is a filter. It contains F, since for any $u \in F$, we have

$$u \wedge b \leqslant u. \tag{3.33}$$

Furthermore, G is not the whole B, since $0 \notin G$. Indeed, if $0 \in G$, then $u \wedge b \leqslant 0$ for some $u \in F$. Also, $0 \leqslant u \wedge b$ from the definition of Boolean algebra. Therefore, $u \wedge b = 0$. Then property 4 from the definition of Boolean algebra ensures that $u \leqslant \neg b$. But since $u \in F$ and F is a filter, this would mean that $\neg b \in F$, which contradicts our initial assumption. Therefore, $0 \notin G$, and hence G is a filter which is not equal to the whole of B. Furthermore, $F \subseteq G$, and F is a maximal filter. So $F = G$.

Now take some $u \in F$. $u \wedge b \leqslant u \wedge b$, so $u \wedge b \in G = F$. But then

$$u \wedge b \leqslant b \tag{3.34}$$

gives us that $b \in F$, since F is closed upward. Thus we arrive at a contradiction assuming neither b not $\neg b$ is in F. So exactly one of b or $\neg b$ must be in F.

(⇐) Suppose F is a filter such that for every element $b \in B$, either $b \in F$ or $\neg b \in F$ and not both. Let F' be another filter strictly containing F, i.e. $F \subset F'$. Take an element $a \in F' \setminus F$. $a \notin F$, so $\neg a \in F$. Then

$$\neg a \in F \subset F' \implies \neg a \in F'.$$

Therefore, both a and $\neg a$ are in F'. Since F' is a filter, $a \wedge \neg a \in F'$. By (3.26), $a \wedge \neg a = 0 \in F'$. But $0 \leqslant c$ for each $c \in B$. This gives us that $c \in F'$ for any $c \in B$, i.e. $F' = B$. Therefore, any strictly larger filter than F must be the whole B. So F is an ultrafilter. ■

§3.4 Generalized Elements

In **Sets**, we have the notion of an element of a set. However, in a general category, we don't have the notion of an element of an object, because the objects need not be sets. That's why we need to generalize the concept of an element to any category. One idea can be that for a given object X, we can consider all the arrows from the terminal object to X. This idea works in **Sets**, since any set X is determined uniquely by the set of all the functions from $1 = \{*\}$ to X. In other words,

$$X \cong \operatorname{Hom}_{\mathbf{Sets}}(1, X). \tag{3.35}$$

Any $x \in X$ uniquely corresponds to a function $\overline{x} : 1 \to X$ given by $\overline{x}(*) = x$. Conversely, any function $f : 1 \to X$ uniquely corresponds to an element $f(*)$ of X. Thus, we get the one-to-one correspondence (i.e. an isomorphism in **Sets**) $X \cong \operatorname{Hom}_{\mathbf{Sets}}(1, X)$.

The arrows $1 \to X$ are called **global elements** of X. However, this idea of taking all the arrows from the terminal object does not always work in other categories. For instance, in $\mathbf{Vect}_{\mathbb{K}}$, the terminal object is $\{\mathbf{0}\}$, which is also the initial object. So given a $\mathbb{K}$-vector space V, there is only one linear transformation from $\{\mathbf{0}\}$ to V, $x : \{\mathbf{0}\} \to V$, given by

$$x(\mathbf{0}) = \mathbf{0}_V.$$

Given two $\mathbb{K}$-vector spaces V and W, and two linear transformations $f, g : V \to W$, we always have $f \circ x = g \circ x$, since

$$(f \circ x)(\mathbf{0}) = f(x(\mathbf{0})) = f(\mathbf{0}_V) = \mathbf{0}_W,$$
$$(g \circ x)(\mathbf{0}) = g(x(\mathbf{0})) = g(\mathbf{0}_V) = \mathbf{0}_W.$$

No matter how different f and g are, composing them with x yields the same arrow. This is not what we want. Ideally, we want generalized elements to have the following property:

$$f = g \text{ if and only if } f \circ x = g \circ x \text{ for every generalized element } x \text{ of } \operatorname{dom} f.$$

The motivation for requiring this condition comes from sets and functions. We call two functions $f, g : X \to Y$ the same if and only if $f(x) = g(x)$ for every $x \in X$.

For this, we draw motivations from the proof of Generalized Cayley's Theorem. In the sketch of the proof, we showed that a "not too big" category is isomorphic to a concrete category (i.e. a category where objects are sets and arrows are functions between them). We've seen that under the isomorphism, the object C of the "not too big" category $\mathcal{C}$ corresponds to the set of arrows whose codomains are C. Keeping this in mind, we define the generalized elements of C to be the arrows whose codomainds are C.

> **Definition 3.7** (Generalized Element)**.** Let $\mathcal{C}$ be a category, and $A \in \mathrm{Ob}(\mathcal{C})$. The arrows whose codomains are A are called **generalized elements** of A. In other words, any arbitrary arrow $f : X \to A$ (with arbitrary domain X) is a generalized element of A.

Then these generalized elements satisfy the required property that $f = g$ if and only if $f \circ x = g \circ x$ for every generalized element x of $\operatorname{dom} f$ $(= \operatorname{dom} g)$. Let $f, g : A \to B$ be two arrows. If $f = g$, then obviously $f \circ x = g \circ x$. Conversely, suppose $f \circ x = g \circ x$ for every generalized element x of A. Then if we choose $x = \mathbb{1}_A$ (which is an arrow with codomain A, so it's a generalized element of A), we get

$$f \circ \mathbb{1}_A = g \circ \mathbb{1}_A \implies f = g. \tag{3.36}$$

Furthermore, we defined monomorphisms (generalizations of injective functions) as follows: $f : A \to B$ is a monomorphism if for any $x, y : C \to A$, $f \circ x = f \circ y$ implies $x = y$.

$$\begin{tikzcd} C \arrow[r, shift left, "y"] \arrow[r, shift right, "x"'] & A \arrow[r, "f"] & B \end{tikzcd}$$

In other words, $f : A \to B$ is a monomorphism if it is "injective on generalized elements of A".

§3.5 Products

Let us begin by considering products of sets. Given sets A and B, the cartesian product of A and B is the set of ordered pairs

$$A \times B = \{(a, b) \mid a \in A, b \in B\}.$$

Observe that there are 2 coordinate projections

$$\begin{tikzcd} A & A \times B \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & B \end{tikzcd}$$

with $\pi_1(a, b) = a$ and $\pi_2(a, b) = b$. Indeed, given any element $c \in A \times B$, one has $c = (\pi_1 c, \pi_2 c)$. This situation is captured by the following commutative diagram:

$$\begin{tikzcd} & 1 \arrow[dl, "a"'] \arrow[d, dashed, "{(a,b)}"] \arrow[dr, "b"] & \\ A & A \times B \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & B \end{tikzcd}$$

$$(a, b)(*) = (a(*), b(*)).$$

$$(\pi_1 \circ (a, b))(*) = \pi_1(a(*), b(*)) = a(*).$$

$$(\pi_2 \circ (a, b))(*) = \pi_2(a(*), b(*)) = b(*).$$

Here, $a \in \mathrm{Hom}_{\mathbf{Sets}}(1, A), b \in \mathrm{Hom}_{\mathbf{Sets}}(1, B)$ are global elements or constants of A and B, respectively. Replacing global elements by generalized elements, one has the following definition.

> **Definition 3.8** (Product)**.** In any category $\mathcal{C}$, a product diagram for the objects A and B consists of an object P and arrows
>
> $$\begin{tikzcd} A & P \arrow[l, "p_1"'] \arrow[r, "p_2"] & B \end{tikzcd}$$
>
> satisfying the following UMP: given any diagram of the form

$$A \xleftarrow{x_1} X \xrightarrow{x_2} B$$

there exists a unique morphism $u : X \to P$ such that the following diagram commutes:

$$\begin{tikzcd}
 & X \arrow[dl, "x_1"'] \arrow[d, dashed, "\exists!"', "u"] \arrow[dr, "x_2"] & \\
A & P \arrow[l, "p_1"'] \arrow[r, "p_2"] & B
\end{tikzcd}$$

i.e. $x_1 = p_1 \circ u$ and $x_2 = p_2 \circ u$.

Proposition 3.9

Products are unique up to isomorphism.

Proof. Suppose one has two product diagrams as below:

$$A \xleftarrow{p_1} P \xrightarrow{p_2} B$$
$$A \xleftarrow{q_1} Q \xrightarrow{q_2} B$$

where P, Q are both taken to be products of A and B. Since Q is a product, there is a unique $i : P \to Q$ such that $q_1 \circ i = p_1$ and $q_2 \circ i = p_2$. Again, since P is a product, there is a unique $j : Q \to P$ such that $p_1 \circ j = q_1$ and $p_2 \circ j = q_2$. In other words, the following diagram commutes:

$$\begin{tikzcd}
 & P \arrow[dl, "p_1"'] \arrow[d, dashed, "i"] \arrow[dr, "p_2"] & \\
A & Q \arrow[l, "q_1"'] \arrow[r, "q_2"] \arrow[d, dashed, "j"] & B \\
 & P \arrow[ul, "p_1"] \arrow[ur, "p_2"'] &
\end{tikzcd}$$

Then we have

$$p_1 = q_1 \circ i = p_1 \circ (j \circ i) \text{ and } p_2 = q_2 \circ i = p_2 \circ (j \circ i)\,. \tag{3.37}$$

Since P is a product, there is a unique $f : P \to P$ such that the following diagram commutes:

$$\begin{tikzcd}
 & P \arrow[dl, "p_1"'] \arrow[d, dashed, "f"] \arrow[dr, "p_2"] & \\
A & P \arrow[l, "p_1"] \arrow[r, "p_2"'] & B
\end{tikzcd}$$

$f = \mathbb{1}_P$ makes this diagram commutative, since $p_1 = p_1 \circ \mathbb{1}_P$ and $p_2 = p_2 \circ \mathbb{1}_P$. Therefore, the uniqueness of f guarantees that $f = \mathbb{1}_P$. Furthermore, (3.37) tells us that $f = j \circ i$ also makes the above diagram commutative. Therefore,

$$f = \mathbb{1}_P = j \circ i. \tag{3.38}$$

Similarly, using the following commutative diagram,

$$\begin{tikzcd}
& Q \arrow[dl, "q_1"'] \arrow[d, dashed, "j"] \arrow[dr, "q_2"] & \\
A & P \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, dashed, "i"] & B \\
& Q \arrow[ul, "q_1"] \arrow[ur, "q_2"'] &
\end{tikzcd}$$

we have

$$q_1 = p_1 \circ j = q_1 \circ (i \circ j) \text{ and } q_2 = p_2 \circ i = q_2 \circ (i \circ j)\,. \tag{3.39}$$

Since Q is a product, there is a unique $g : Q \to Q$ such that the following diagram commutes:

$$\begin{tikzcd}
& Q \arrow[dl, "q_1"'] \arrow[d, dashed, "g"] \arrow[dr, "q_2"] & \\
A & Q \arrow[l, "q_1"] \arrow[r, "q_2"'] & B
\end{tikzcd}$$

$g = \mathbb{1}_Q$ makes this diagram commutative, since $q_1 = q_1 \circ \mathbb{1}_Q$ and $q_2 = q_2 \circ \mathbb{1}_Q$. Therefore, the uniqueness of g guarantees that $g = \mathbb{1}_Q$. Furthermore, (3.39) tells us that $g = i \circ j$ also makes the above diagram commutative. Therefore,

$$g = \mathbb{1}_Q = i \circ j. \tag{3.40}$$

Therefore, we have $i \circ j = \mathbb{1}_Q$ and $j \circ i = \mathbb{1}_P$. Hence, i is an isomorphism and $P \cong Q$. ■

If A and B have a product, we write

$$A \xleftarrow{p_1} A \times B \xrightarrow{p_2} B$$

for one such product. Then given X, x_1, x_2 as below

$$A \xleftarrow{x_1} X \xrightarrow{x_2} B$$

there is a unique arrow $u : X \to A \times B$ such that $x_i = p_i \circ u$ for $i = 1, 2$. We denote $u = \langle x_1, x_2 \rangle$. Hence, $p_1 \circ \langle x_1, x_2 \rangle = x_1$ and $p_2 \circ \langle x_1, x_2 \rangle = x_2$. Now, however, a pair of objects may have different products in a category $\mathcal{C}$. For example, given a product $A \times B, p_1, p_2$ and an isomorphism $h : A \times B \to Q$, one finds that $Q, p_1 \circ h^{-1}, p_2 \circ h^{-1}$ is also a product of A and B.

$$\begin{tikzcd}
& X \arrow[ddl, bend right, "x_1"'] \arrow[d, "u"'] \arrow[dd, bend left, "\widetilde{u}"] \arrow[ddr, bend left, "x_2"] & \\
& Q \arrow[dl, "p_1 \circ h^{-1}"'] \arrow[d, "h^{-1}"'] \arrow[dr, "p_2 \circ h^{-1}"] & \\
A & A \times B \arrow[l, "p_1"] \arrow[r, "p_2"'] & B
\end{tikzcd}$$

Now, given a diagram $A \xleftarrow{x_1} X \xrightarrow{x_2} B$, there exists unique $\widetilde{u} : X \to A \times B$ such that $p_1 \circ \widetilde{u} = x_1$ and $p_2 \circ \widetilde{u} = x_2$. Define $u : X \to Q$ as $u = h \circ \widetilde{u}$. Then $h^{-1} \circ u = \widetilde{u}$. So

$$x_1 = p_1 \circ \widetilde{u} = \left(p_1 \circ h^{-1}\right) \circ u \text{ and } x_2 = p_2 \circ \widetilde{u} = \left(p_2 \circ h^{-1}\right) \circ u,$$

and the diagraam above commutes. Furthermore, u is unique. Indeed, if there is another $v : X \to Q$ such that $x_1 = (p_1 \circ h^{-1}) \circ v$ and $x_2 = (p_2 \circ h^{-1}) \circ v$, then by the uniqueness of $\widetilde{u}$,

$$\widetilde{u} = h^{-1} \circ v \implies v = h \circ \widetilde{u} = u.$$

So u is unique. This proves that $Q, p_1 \circ h^{-1}, p_2 \circ h^{-1}$ fulfills the universal property of a product diagram.

Example 3.4. 1. Products of "structured sets" like monoids or groups can often be constructed as products of the underlying sets with componentwise operations: if G and H are groups, then $G \times H$ can be constructed by taking the underlying set of $G \times H$ to be the set $\{\langle g, h\rangle \mid g \in G, h \in H\}$ and define the binary operation in $G \times H$ by

$$\langle g, h\rangle *_{G\times H} \langle g', h'\rangle = \langle g *_G g', h *_H h'\rangle ,$$

where the group G is given by the triple $(G, *_G, u_G)$ and the group H is given by the triple $(H, *_H, u_H)$. The unit of $G \times H$ is given by

$$u_{G\times H} = \langle u_G, u_H\rangle .$$

The inverse of $\langle g, h\rangle$ in $G \times H$ is

$$\langle g, h\rangle^{-1} = \left\langle g^{-1}, h^{-1}\right\rangle .$$

The projection homomorphism $G \times H \to G$ (or H) are the evident ones

$$\langle g, h\rangle \mapsto g \text{ (or } h) \; .$$

2. Let $(P, \leqslant)$ be a poset and consider a product $p \times q$ of elements $p, q \in P$. We must have projections $p \times q \to p$ and $p \times q \to q$. In a poset, these arrows indicate

$$p \times q \leqslant p \text{ and } p \times q \leqslant q$$

. Furthermore, we need $p \times q$ to fulfill the UMP of product: if for a given element $x \in P$, $x \to p$ and $x \to q$ (i.e. $x \leqslant p$ and $x \leqslant q$ in the poset), then there exists a unique arrow $x \to p \times q$ (i.e. $x \leqslant p \times q$). The product $p \times q$ is easily recognized to be the greatest lower bound $p \wedge q$.

Remark 3.5. Note that the projection arrows $p_1 : A \times B \to A$ and $p_2 : A \times B \to B$ need not be epic. In **Sets**, let A be a non-empty set. Consider the product of $\varnothing$ and A. Then the projection

$$p_2 : \varnothing \times A \to A$$

is not surjective, since the domain is empty set but the codomain is nonempty. Therefore, the projection p_2 is not an epimorphism in **Sets**.

§3.5.i Product Topological Space

Let us show that the product of two topological spaces X and Y is a product $X \times Y$ in **Top**, the category of topological spaces and continuous maps. Let $\mathcal{O}(X)$ be the collection of open sets in X. Suppose we have two topological space X and Y and the product space $X \times Y$ along with the projections

$$\begin{tikzcd} X & X \times Y \arrow[l, "p_1"'] \arrow[r, "p_2"] & Y \end{tikzcd}$$

Recall that $\mathcal{O}(X \times Y)$ is generated by basic open sets of the form $U \times V$, where $U \in \mathcal{O}(X)$ and $V \in \mathcal{O}(Y)$, so that every $W \in \mathcal{O}(X \times Y)$ is a union of such basic open sets. We have the following observations:

- Clearly $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ are continuous: for $U \in \mathcal{O}(X)$ and $V \in \mathcal{O}(Y)$,

  $$\begin{aligned} p_1^{-1}(U) &= U \times Y \in \mathcal{O}(X \times Y), \\ p_2^{-1}(V) &= X \times V \in \mathcal{O}(X \times Y). \end{aligned}$$

  So $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ are continuous.

- Given any cotinuous $f_1 : Z \to X$ and $f_2 : Z \to Y$, let $f : Z \to X \times Y$ be defined as

  $$f(z) = (f_1(z), f_2(z)).$$

  We need to verify that f is continuous. Given $W \in \mathcal{O}(X \times Y)$, W can be expressed as

  $$W = \bigcup_\alpha (U_\alpha \times V_\alpha),$$

  for $U_\alpha \in \mathcal{O}(X)$ and $V_\alpha \in \mathcal{O}(Y)$. Then

  $$f^{-1}(W) = \bigcup_\alpha f^{-1}(U_\alpha \times V_\alpha).$$

  Therefore, it suffices to show that $f^{-1}(U \times V) \in \mathcal{O}(Z)$ for $U \in \mathcal{O}(X)$ and $V \in \mathcal{O}(Y)$.

  $$\begin{aligned} f^{-1}(U \times V) &= f^{-1}((U \times Y) \cap (X \times V)) \\ &= f^{-1}(U \times Y) \cap f^{-1}(X \times V) \\ &= f^{-1}\left(p_1^{-1}(U)\right) \cap f^{-1}\left(p_2^{-1}(V)\right) \\ &= (p_1 \circ f)^{-1}(U) \cap (p_2 \circ f)^{-1}(V) \\ &= f_1^{-1}(U) \cap f_2^{-1}(V). \end{aligned}$$

  Both $f_1^{-1}(U)$ and $f_2^{-1}(V)$ are open in Z since f_1, f_2 are continuous. Hence, f is continuous.

- Therefore, given $f_1 : Z \to X$ and $f_2 : Z \to Y$ continuous, there exists a continuous $f : Z \to X \times Y$ such that $p_1 \circ f = f_1$ and $p_2 \circ f = f_2$. Now, suppose there exists another continuous $g : Z \to X \times Y$ such that $p_1 \circ g = f_1$ and $p_2 \circ g = f_2$. Let $g(z) = (g_1(z), g_2(z))$. Then

  $$f_1(z) = (p_1 \circ g)(z) = p_1(g_1(z), g_2(z)) = g_1(z), \tag{3.41}$$

  for every $z \in Z$. Similarly, $f_2(z) = g_2(z)$. Therefore, $g = f$, proving the uniqueness of f.

So $X \times Y$ is a product of X and Y in **Top**.

§3.5.ii Category with Products

Let $\mathcal{C}$ be a category that has a product diagram for every pair of objects. Such a category is said to have a binary product. Suppose we have objects and arrows as indicated below:

$$\begin{tikzcd} A \arrow[d, "f"'] & A \times A' \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[dl, "f \circ p_1"'] \arrow[d, dashed, "f \times f'"] \arrow[dr, "f' \circ p_2"] & A' \arrow[d, "f'"] \\ B & B \times B' \arrow[l, "q_1"] \arrow[r, "q_2"'] & B' \end{tikzcd} \tag{3.42}$$

Then there exists a unique arrow from the product $A \times A'$ to $B \times B'$ as follows:

$$f \times f' = \langle f \circ p_1, f' \circ p_2 \rangle. \tag{3.43}$$

In other words, $f \times f'$ is the unique arrow from $A \times A'$ to $B \times B'$ which makes the above diagram (3.42) commutative. One can, therefore, construct a functor $\times : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$[1] as follows:

$$\times\,(A, A') = A \times A' \text{ and } \times\,(f, f') = f \times f'. \tag{3.44}$$

Let us now verify that $\times$ is indeed a functor. For an object (A, A') in $\mathcal{C} \times \mathcal{C}$, let $\mathbb{1}_{(A,A')} = (\mathbb{1}_A, \mathbb{1}_{A'})$ be its identity arrow. We need to show that $\times\left(\mathbb{1}_{(A,A')}\right) = \mathbb{1}_A \times \mathbb{1}_{A'} = \mathbb{1}_{A\times A'}$.

$$\begin{tikzcd}
A \arrow[d, "\mathbb{1}_A"'] & A \times A' \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, dashed, "\mathbb{1}_A \times \mathbb{1}_{A'}"] \arrow[dl, "\mathbb{1}_A \circ p_1"'] \arrow[dr, "\mathbb{1}_{A'} \circ p_2"] & A' \arrow[d, "\mathbb{1}_{A'}"] \\
A & A \times A' \arrow[l, "p_1"] \arrow[r, "p_2"'] & A'
\end{tikzcd}$$

$\mathbb{1}_A \times \mathbb{1}_{A'}$ is the **unique** arrow $A \times A' \to A \times A'$ that makes this diagram commute. If we choose $\mathbb{1}_{A\times A'}$ in place of $\mathbb{1}_A \times \mathbb{1}_{A'}$, then also the diagram commutes:

$$\begin{tikzcd}
A \arrow[d, "\mathbb{1}_A"'] & A \times A' \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, dashed, "\mathbb{1}_{A \times A'}"] \arrow[dl, "\mathbb{1}_A \circ p_1"'] \arrow[dr, "\mathbb{1}_{A'} \circ p_2"] & A' \arrow[d, "\mathbb{1}_{A'}"] \\
A & A \times A' \arrow[l, "p_1"] \arrow[r, "p_2"'] & A'
\end{tikzcd}$$

since

$$p_1 \circ \mathbb{1}_{A\times A'} = p_1 = \mathbb{1}_A \circ p_1 \text{ and } p_2 \circ \mathbb{1}_{A\times A'} = p_2 = \mathbb{1}_{A'} \circ p_2.$$

Therefore, by the uniqueness, $\mathbb{1}_A \times \mathbb{1}_{A'} = \mathbb{1}_{A\times A'}$. Hence,

$$\times\left(\mathbb{1}_{(A,A')}\right) = \times\,(\mathbb{1}_A, \mathbb{1}_{A'}) = \mathbb{1}_A \times \mathbb{1}_{A'} = \mathbb{1}_{A\times A'}\,. \tag{3.45}$$

Let $(f, f') : (A, A') \to (B, B')$ and $(g, g') : (B, B') \to (C, C')$ be arrows in $\mathcal{C} \times \mathcal{C}$. Then $(g, g') \circ (f, f') = (g \circ f, g' \circ f')$. Consider the following diagram:

$$\begin{tikzcd}
A \arrow[d, "f"'] & A \times A' \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, dashed, "f \times f'"] \arrow[dl, "f \circ p_1"'] \arrow[dr, "f' \circ p_2"] & A' \arrow[d, "f'"] \\
B \arrow[d, "g"'] & B \times B' \arrow[l, "q_1"'] \arrow[r, "q_2"] \arrow[d, dashed, "g \times g'"] \arrow[dl, "g \circ q_1"'] \arrow[dr, "g' \circ q_2"] & B' \arrow[d, "g'"] \\
C & C \times C' \arrow[l, "r_1"] \arrow[r, "r_2"'] & C'
\end{tikzcd}$$

In this diagram, $f \times f' : A \times A' \to B \times B'$ and $g \times g'$ are the **unique** maps such that the diagram above commutes. In other words,

$$q_1 \circ (f \times f') = f \circ p_1,\ q_2 \circ f \times f' = f' \circ p_2, \text{ and} \tag{3.46}$$

$$r_1 \circ (g \times g') = g \circ q_1,\ r_2 \circ g \times g' = g' \circ q_2. \tag{3.47}$$

$g \circ f : A \to C$ and $g' \circ f' : A' \to C'$. Then $(g \circ f) \times (g' \circ f')$ is the **unique** map from $A \times A'$ to $C \times C'$ such that the following diagram commutes:

---

[1] $\mathcal{C} \times \mathcal{C}$ is the product category as discussed in Section 1.5. An object in $\mathcal{C} \times \mathcal{C}$ is of the form (A, A') with $A, A' \in \mathrm{Ob}\,(\mathcal{C})$. Given arrows $f : A \to B$ and $f' : A' \to B'$ in $\mathcal{C}$, one has $(f, f') : (A, A') \to (B, B')$ as an arrow in $\mathcal{C} \times \mathcal{C}$ between objects (A, A') and (B, B').

$$\begin{tikzcd}
A \arrow[d, "g\circ f"'] & A\times A' \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[dl, "g\circ f\circ p_1"'] \arrow[d, dashed, "(g\circ f)\times(g'\circ f')"] \arrow[dr, "g'\circ f'\circ p_2"] & A' \arrow[d, "g'\circ f'"] \\
C & C\times C' \arrow[l, "r_1"] \arrow[r, "r_2"'] & C'
\end{tikzcd}$$

In other words,

$$r_1 \circ (g\circ f)\times(g'\circ f') = g\circ f\circ p_1 \text{ and } r_2\circ(g\circ f)\times(g'\circ f') = g'\circ f'\circ p_2. \tag{3.48}$$

If we take $(g\times g')\circ(f\times f')$ in place of $(g\circ f)\times(g'\circ f')$, then also the diagram commutes, since

$$r_1\circ(g\times g')\circ(f\times f') = g\circ q_1\circ(f\times f') = g\circ f\circ p_1, \tag{3.49}$$

$$r_2\circ(g\times g')\circ(f\times f') = g'\circ q_2\circ(f\times f') = g'\circ f'\circ p_2. \tag{3.50}$$

$$\begin{tikzcd}
A \arrow[d, "g\circ f"'] & A\times A' \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[dl, "g\circ f\circ p_1"'] \arrow[d, dashed, "(g\times g')\circ(f\times f')"] \arrow[dr, "g'\circ f'\circ p_2"] & A' \arrow[d, "g'\circ f'"] \\
C & C\times C' \arrow[l, "r_1"] \arrow[r, "r_2"'] & C'
\end{tikzcd}$$

Therefore, by the uniqueness, we have $(g\times g')\circ(f\times f') = (g\circ f)\times(g'\circ f')$. Then

$$\times\left((g,g')\circ(f,f')\right) = \times\left(g\circ f, g'\circ f'\right) = (g\circ f)\times(g'\circ f') = (g\times g')\circ(f\times f') = \times\left(g,g'\right)\circ\times\left(f,f'\right). \tag{3.51}$$

Therefore, $\times : \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ is indeed a functor.

Proposition 3.10

Given arrows $f : X\to A$ and $g : X\to B$, one forms $\langle f,g\rangle : X\to A\times B$. Then for any arrow $x : Y\to X$,

$$\langle f,g\rangle\circ x = \langle f\circ x, g\circ x\rangle .$$

Proof. We have the following commutative diagram:

$$\begin{tikzcd}
 & Y \arrow[d] \arrow[ddl, bend right, "f\circ x"'] \arrow[ddr, bend left, "g\circ x"] \arrow[dd, dashed, bend left, "\langle f\circ x, g\circ x\rangle"] & \\
 & X \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[d, dashed, "\langle f,g\rangle"'] & \\
A & A\times B \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & B
\end{tikzcd}$$

$\langle f\circ x, g\circ x\rangle : Y\to A\times B$ is the **unique** arrow such that

$$\pi_1\circ\langle f\circ x, g\circ x\rangle = f\circ x \text{ and } \pi_2\circ\langle f\circ x, g\circ x\rangle = g\circ x.$$

Observe that $\langle f,g\rangle\circ x$ also has this property:

$$\pi_1\circ(\langle f,g\rangle\circ x) = (\pi_1\circ\langle f,g\rangle)\circ x = f\circ x, \tag{3.52}$$

$$\pi_2\circ(\langle f,g\rangle\circ x) = (\pi_2\circ\langle f,g\rangle)\circ x = g\circ x. \tag{3.53}$$

Therefore, by the uniqueness of $\langle f\circ x, g\circ x\rangle$,

$$\langle f,g\rangle\circ x = \langle f\circ x, g\circ x\rangle . \tag{3.54}$$

■

§3.6 Hom-sets

We are dealing with locally small categories, where given any pair of objects A and B, the collection $\mathrm{Hom}_{\mathcal{C}}(A,B)$ of all the arrows from A to B forms a set. We call such a set of arrows **Hom-set**. Now, fix $A \in \mathrm{Ob}(\mathcal{C})$ once and for all. Then we consider a functor $\mathrm{Hom}_{\mathcal{C}}(A,-) : \mathcal{C} \to \mathbf{Sets}$ as follows:

$$\begin{tikzcd}
\mathcal{C} & & \mathbf{Sets} \\
C \arrow[dd, "g"] & & \mathrm{Hom}_{\mathcal{C}}(A,C) \arrow[dd, "\mathrm{Hom}_{\mathcal{C}}(A,g)"] \\
{} \arrow[rr, "\mathrm{Hom}_{\mathcal{C}}(A,-)"] & & {} \\
D & & \mathrm{Hom}_{\mathcal{C}}(A,D)
\end{tikzcd}$$

with the function $\mathrm{Hom}_{\mathcal{C}}(A,g) : \mathrm{Hom}_{\mathcal{C}}(A,C) \to \mathrm{Hom}_{\mathcal{C}}(A,D)$ defined by

$$\mathrm{Hom}_{\mathcal{C}}(A,g)\,(f) = g \circ f,$$

for each $f \in \mathrm{Hom}_{\mathcal{C}}(A,C)$. $\mathrm{Hom}_{\mathcal{C}}(A,-)$ is called the **representable functor** of A. Let us now show that $\mathrm{Hom}_{\mathcal{C}}(A, \mathbb{1}_X) = \mathbb{1}_{\mathrm{Hom}_{\mathcal{C}}(A,X)}$.

$$\begin{tikzcd}
\mathcal{C} & & \mathbf{Sets} \\
X \arrow[dd, "\mathbb{1}_X"] & & \mathrm{Hom}_{\mathcal{C}}(A,X) \arrow[dd, "\mathrm{Hom}_{\mathcal{C}}(A,\mathbb{1}_X)"] \\
{} \arrow[rr, "\mathrm{Hom}_{\mathcal{C}}(A,-)"] & & {} \\
X & & \mathrm{Hom}_{\mathcal{C}}(A,X)
\end{tikzcd}$$

Indeed, let $x \in \mathrm{Hom}_{\mathcal{C}}(A,X)$. Then

$$\mathrm{Hom}_{\mathcal{C}}(A, \mathbb{1}_X)(x) = \mathbb{1}_X \circ x = x = \mathbb{1}_{\mathrm{Hom}_{\mathcal{C}}(A,X)}(x), \tag{3.55}$$

for any $x \in \mathrm{Hom}_{\mathcal{C}}(A,X)$. Therefore, $\mathrm{Hom}_{\mathcal{C}}(A, \mathbb{1}_X) = \mathbb{1}_{\mathrm{Hom}_{\mathcal{C}}(A,X)}$.

Now, take two composable arrows $g : Y \to Z$ and $f : X \to Y$ in $\mathcal{C}$. We want to show that

$$\mathrm{Hom}_{\mathcal{C}}(A, g \circ f) = \mathrm{Hom}_{\mathcal{C}}(A,g) \circ \mathrm{Hom}_{\mathcal{C}}(A,f) : \mathrm{Hom}_{\mathcal{C}}(A,X) \to \mathrm{Hom}_{\mathcal{C}}(A,Z).$$

Let's take any $x \in \mathrm{Hom}_{\mathcal{C}}(A,X)$. Then,

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C}}(A, g \circ f)(x) &= g \circ f \circ x = g \circ (\mathrm{Hom}_{\mathcal{C}}(A,f)(x)) \\
&= \mathrm{Hom}_{\mathcal{C}}(A,g)(\mathrm{Hom}_{\mathcal{C}}(A,f)(x)) \\
&= [\mathrm{Hom}_{\mathcal{C}}(A,g) \circ \mathrm{Hom}_{\mathcal{C}}(A,f)](x),
\end{aligned} \tag{3.56}$$

which is true for any $x \in \mathrm{Hom}_{\mathcal{C}}(A,X)$. Therefore,

$$\mathrm{Hom}_{\mathcal{C}}(A, g \circ f) = \mathrm{Hom}_{\mathcal{C}}(A,g) \circ \mathrm{Hom}_{\mathcal{C}}(A,f). \tag{3.57}$$

Hence, $\mathrm{Hom}_{\mathcal{C}}(A,-) : \mathcal{C} \to \mathbf{Sets}$ is indeed a functor.

We will now present an alternative characterization of product using Hom-sets. For any object P, a pair of arrows $p_1 : P \to A$ and $p_2 : P \to B$ determine an element (p_1, p_2) of the set

$$\mathrm{Hom}_{\mathcal{C}}(P,A) \times \mathrm{Hom}_{\mathcal{C}}(P,B).$$

Given any arrow $x : X \to P$, composing with p_1 and p_2 gives a pair of arrows $x_1 = p_1 \circ x : X \to A$ and $x_2 = p_2 \circ x : X \to B$ as indicated in the following diagram:

$$\begin{tikzcd}
& X \arrow[ld, "x_1=p_1\circ x"'] \arrow[d, "x"] \arrow[rd, "x_2=p_2\circ x"] & \\
A & P \arrow[l, "p_1"] \arrow[r, "p_2"'] & B
\end{tikzcd}$$

In this way, one has a function

$$\vartheta_X \equiv \left(\mathrm{Hom}_{\mathcal{C}}\left(X, p_1\right), \mathrm{Hom}_{\mathcal{C}}\left(X, p_2\right)\right) : \mathrm{Hom}_{\mathcal{C}}\left(X, P\right) \to \mathrm{Hom}_{\mathcal{C}}\left(X, A\right) \times \mathrm{Hom}_{\mathcal{C}}\left(X, B\right).$$

Given any $x : X \to P$, ϑ maps it to

$$\left(\mathrm{Hom}_{\mathcal{C}}\left(X, p_1\right)(x), \mathrm{Hom}_{\mathcal{C}}\left(X, p_2\right)(x)\right) = \left(p_1 \circ x, p_2 \circ x\right).$$

In other words,

$$\vartheta\left(x\right) = \left(p_1 \circ x, p_2 \circ x\right) = \left(x_1, x_2\right). \tag{3.58}$$

Proposition 3.11

A diagram of the form $A \xleftarrow{p_1} P \xrightarrow{p_2} B$ is a product diagram if and only if for every object X, the canonical function ϑ_X defined by (3.58) is an isomorphism,

$$\vartheta_X : \mathrm{Hom}_{\mathcal{C}}\left(X, P\right) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{C}}\left(X, A\right) \times \mathrm{Hom}_{\mathcal{C}}\left(X, B\right).$$

Proof. **(⇒)** Suppose P is a product. Then given any object X of $\mathcal{C}$ and a pair of arrows $x_1 : X \to A$ and $x_2 : X \to B$, there is a unique arrow $x : X \to P$ such that the following diagram commutes:

$$\begin{tikzcd}
& X \arrow[ld, "x_1"'] \arrow[d, dashed, "x"] \arrow[rd, "x_2"] & \\
A & P \arrow[l, "p_1"] \arrow[r, "p_2"'] & B
\end{tikzcd}$$

In other words, $p_1 \circ x = x_1$ and $p_2 \circ x = x_2$. Therefore, for any object X, and $x_1 \in \mathrm{Hom}_{\mathcal{C}}\left(X, A\right)$, $x_2 \in \mathrm{Hom}_{\mathcal{C}}\left(X, B\right)$, there is a unique $x \in \mathrm{Hom}_{\mathcal{C}}\left(X, P\right)$ which makes the above diagram commutative. We, therefore, have a function

$$\wp_X : \mathrm{Hom}_{\mathcal{C}}\left(X, A\right) \times \mathrm{Hom}_{\mathcal{C}}\left(X, B\right) \to \mathrm{Hom}_{\mathcal{C}}\left(X, P\right)$$

defined by

$$\wp_X\left(x_1, x_2\right) = x, \tag{3.59}$$

where x is the **unique** arrow from X to P such that $p_1 \circ x = x_1$ and $p_2 \circ x = x_2$. In particular,

$$\wp_X\left(p_1 \circ x, p_2 \circ x\right) = x. \tag{3.60}$$

Then for any $\left(x_1, x_2\right) \in \mathrm{Hom}_{\mathcal{C}}\left(X, A\right) \times \mathrm{Hom}_{\mathcal{C}}\left(X, B\right)$,

$$\left(\vartheta_X \circ \wp_X\right)\left(x_1, x_2\right) = \vartheta_X\left(x\right) = \left(p_1 \circ x, p_2 \circ x\right) = \left(x_1, x_2\right),$$

and for any $x \in \mathrm{Hom}_{\mathcal{C}}\left(X, P\right)$,

$$\left(\wp_X \circ \vartheta_X\right)\left(x\right) = \wp_X\left(p_1 \circ x, p_2 \circ x\right) = x.$$

Therefore,

$$\vartheta_X \circ \wp_X = \mathbb{1}_{\mathrm{Hom}_{\mathcal{C}}(X,A) \times \mathrm{Hom}_{\mathcal{C}}(X,B)} \quad \text{and} \quad \wp_X \circ \vartheta_X = \mathbb{1}_{\mathrm{Hom}_{\mathcal{C}}(X,P)}. \tag{3.61}$$

Therefore, ϑ_X is an isomorphism for any object X.

(⇐) Conversely, suppose, for every object X, the canonical function

$$\vartheta_X : \mathrm{Hom}_{\mathcal{C}}(X, P) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{C}}(X, A) \times \mathrm{Hom}_{\mathcal{C}}(X, B)$$

is an isomorphism. Given any $x_1 \in \mathrm{Hom}_{\mathcal{C}}(X, A)$ and $x_2 \in \mathrm{Hom}_{\mathcal{C}}(X, B)$, since ϑ_X is an isomorphism (i.e. a bijection since we are in **Sets**), there exists a **unique** $x \in \mathrm{Hom}_{\mathcal{C}}(X, P)$ such that $\vartheta_X(x) = (x_1, x_2)$. Using (3.58), we get

$$(x_1, x_2) = \vartheta_X(x) = (p_1 \circ x, p_2 \circ x). \tag{3.62}$$

Therefore, given any $A \xleftarrow{x_1} X \xrightarrow{x_2} B$, there is a unique $x : X \to P$ such that $p_1 \circ x = x_1$ and $p_2 \circ x = x_2$. In other words, the following diagram commutes:

$$\begin{tikzcd}
& X \arrow[dl, "x_1"'] \arrow[d, dashed, "x"] \arrow[dr, "x_2"] & \\
A & P \arrow[l, "p_1"] \arrow[r, "p_2"'] & B
\end{tikzcd}$$

Therefore, P is a product of A and B. ■

Definition 3.9. Let $\mathcal{C}$ and $\mathcal{D}$ be categories with binary products. A functor $F : \mathcal{C} \to \mathcal{D}$ is said to **preserve binary products** if it takes every product diagram

$$A \xleftarrow{p_1} A \times B \xrightarrow{p_2} B \qquad \text{in } \mathcal{C}$$

to a product diagram

$$F(A) \xleftarrow{F(p_1)} F(A \times B) \xrightarrow{F(p_2)} F(B) \qquad \text{in } \mathcal{D}.$$

The latter is a product diagram in $\mathcal{D}$ if and only if $F(A \times B) \cong F(A) \times F(B)$. In other words, if and only if the canonical arrow

$$\langle F(p_1), F(p_2) \rangle : F(A \times B) \xrightarrow{\cong} F(A) \times F(B)$$

is an isomorphism. Note that $\langle F(p_1), F(p_2) \rangle$ is the **unique** arrow from $F(A \times B)$ to $F(A) \times F(B)$ such that the diagram below commutes:

$$\begin{tikzcd}
& F(A \times B) \arrow[dl, "F(p_1)"'] \arrow[d, dashed, "\langle F(p_1){,}F(p_2)\rangle"] \arrow[dr, "F(p_2)"] & \\
F(A) & F(A) \times F(B) \arrow[l, "\widetilde{p}_1"] \arrow[r, "\widetilde{p}_2"'] & F(B)
\end{tikzcd}$$

Therefore, if $\mathcal{C}$ and $\mathcal{D}$ are categories with binary products, then a functor $F : \mathcal{C} \to \mathcal{D}$ preserves binary products if and only if $F(A \times B) \cong F(A) \times F(B)$ in $\mathcal{D}$ for any $A, B \in \mathrm{Ob}(\mathcal{C})$. For example, the forgetful functor $\mathcal{U}$: **Mon** → **Sets** preserves binary products.

Corollary 3.12

For any object X in a category $\mathcal{C}$ with products, the covariant representable functor

$$\mathrm{Hom}_{\mathcal{C}}(X, -) : \mathcal{C} \to \mathbf{Sets}$$

preserves products.

Proof. For $A, B \in \operatorname{Ob}(\mathcal{C})$, Proposition 3.11 says that there is a canonical isomorphism

$$\operatorname{Hom}_{\mathcal{C}}(X, A \times B) \cong \operatorname{Hom}_{\mathcal{C}}(X, A) \times \operatorname{Hom}_{\mathcal{C}}(X, B),$$

as $A \times B$ is a product of A and B. In other words,

$$\operatorname{Hom}_{\mathcal{C}}(X, -)(A \times B) \cong \operatorname{Hom}_{\mathcal{C}}(X, -)(A) \times \operatorname{Hom}_{\mathcal{C}}(X, -)(B). \tag{3.63}$$

Hence, $\operatorname{Hom}_{\mathcal{C}}(X, -) : \mathcal{C} \to$ **Sets** is indeed a binary product preserving functor. ■

§3.7 Associativity of Product

Theorem 3.13

Product is associative up to isomorphism, i.e. if A, B, and C are objects in a category with products, then $(A \times B) \times C \cong A \times (B \times C)$.

Proof. Let the following be product diagrams:

$$\begin{tikzcd} A & A \times B \arrow[l, "p_1"'] \arrow[r, "p_2"] & B \\ B & B \times C \arrow[l, "q_1"'] \arrow[r, "q_2"] & C \\ A \times B & (A \times B) \times C \arrow[l, "r_1"'] \arrow[r, "r_2"] & C \\ A & A \times (B \times C) \arrow[l, "s_1"'] \arrow[r, "s_2"] & B \times C \end{tikzcd}$$

Consider the following composition:

$$(A \times B) \times C \xrightarrow{r_1} A \times B \xrightarrow{p_2} B.$$

Given $p_2 \circ r_1 : (A \times B) \times C \to B$ and $r_2 : (A \times B) \times C \to C$, there exists a **unique** $f : (A \times B) \times C \to B \times C$ such that the following diagram commutes:

$$\begin{tikzcd} & (A \times B) \times C \arrow[dl, "p_2 \circ r_1"'] \arrow[d, dashed, "f"] \arrow[dr, "r_2"] & \\ B & B \times C \arrow[l, "q_1"] \arrow[r, "q_2"'] & C \end{tikzcd}$$

In other words,

$$q_1 \circ f = p_2 \circ r_1, \tag{3.64}$$
$$q_2 \circ f = r_2. \tag{3.65}$$

Then consider the following composition:

$$(A \times B) \times C \xrightarrow{r_1} A \times B \xrightarrow{p_1} A.$$

Given $p_1 \circ r_1 : (A \times B) \times C \to A$ and $f : (A \times B) \times C \to (B \times C)$, there exists a **unique** $g : (A \times B) \times C \to A \times (B \times C)$ such that the following diagram commutes:

$$\begin{tikzcd} & (A \times B) \times C \arrow[dl, "p_1 \circ r_1"'] \arrow[d, dashed, "g"] \arrow[dr, "f"] & \\ A & A \times (B \times C) \arrow[l, "s_1"] \arrow[r, "s_2"'] & B \times C \end{tikzcd}$$

In other words,

$$s_1 \circ g = p_1 \circ r_1, \tag{3.66}$$

$$s_2 \circ g = f. \tag{3.67}$$

Now, we consider the following composition:

$$A \times (B \times C) \xrightarrow{s_2} B \times C \xrightarrow{q_1} B.$$

Given $s_1 : A \times (B \times C) \to A$ and $q_1 \circ s_2 : A \times (B \times C) \to B$, there exists a **unique** $h : A \times (B \times C) \to A \times B$ such that the following diagram commutes:

$$\begin{tikzcd}
& A \times (B \times C) \arrow[dl, "s_1"'] \arrow[d, dashed, "h"] \arrow[dr, "q_1 \circ s_2"] & \\
A & A \times B \arrow[l, "p_1"] \arrow[r, "p_2"'] & B
\end{tikzcd}$$

In other words,

$$p_1 \circ h = s_1, \tag{3.68}$$

$$p_2 \circ h = q_1 \circ s_2. \tag{3.69}$$

Then we consider the following composition:

$$A \times (B \times C) \xrightarrow{s_2} B \times C \xrightarrow{q_2} C.$$

Given $h : A \times (B \times C) \to A \times B$ and $q_2 \circ s_2 : A \times (B \times C) \to c$, there exists a **unique** $i : A \times (B \times C) \to (A \times B) \times C$ such that the following diagram commutes:

$$\begin{tikzcd}
& A \times (B \times C) \arrow[dl, "h"'] \arrow[d, dashed, "i"] \arrow[dr, "q_2 \circ s_2"] & \\
A \times B & (A \times B) \times C \arrow[l, "r_1"] \arrow[r, "r_2"'] & C
\end{tikzcd}$$

In other words,

$$r_1 \circ i = h, \tag{3.70}$$

$$r_2 \circ i = q_2 \circ s_2. \tag{3.71}$$

We claim that $g : (A \times B) \times C \to A \times (B \times C)$ is our desired isomorphism, with inverse $i : A \times (B \times C) \to (A \times B) \times C$. For that purpose, we need to show that $g \circ i = \mathbb{1}_{A \times (B \times C)}$ and $i \circ g = \mathbb{1}_{(A \times B) \times C}$.

Now, consider the following diagram:

$$\begin{tikzcd}
& A \times (B \times C) \arrow[loop above, "g \circ i"] \arrow[dl, "q_1 \circ s_2"'] \arrow[d, shift right, "s_2"'] \arrow[d, shift left, "s_2 \circ g \circ i"] \arrow[dr, "q_1 \circ s_2"] & \\
B & B \times C \arrow[l, "q_1"] \arrow[r, "q_2"'] & C
\end{tikzcd}$$

Using (3.67), (3.64), (3.70) and (3.69), we get

$$q_1 \circ s_2 \circ g \circ i = q_1 \circ f \circ i = p_2 \circ r_1 \circ i = p_2 \circ h = q_1 \circ s_2. \tag{3.72}$$

Furthermore, using (3.67), (3.65) and (3.71),

$$q_2 \circ s_2 \circ g \circ i = q_2 \circ f \circ i = r_2 \circ i = q_2 \circ s_2. \tag{3.73}$$

So there are two arrows $s_2, s_2 \circ g \circ i : A \times (B \times C) \to B \times C$ and both of them makes the above diagram commute. Therefore, by the universal property of the product $B \times C$,

$$s_2 = s_2 \circ g \circ i. \tag{3.74}$$

Now, let's consider the following diagram:

$$\begin{tikzcd}
& A \times (B \times C) \arrow[dl, "s_1"'] \arrow[d, shift right, "\mathbb{1}_{A\times(B\times C)}"'] \arrow[d, shift left, "g\circ i"] \arrow[dr, "s_2"] & \\
A & A \times (B \times C) \arrow[l, "s_1"] \arrow[r, "s_2"'] & B \times C
\end{tikzcd}$$

Here, (3.66), (3.70) and (3.68) gives us

$$s_1 \circ g \circ i = p_1 \circ r_1 \circ i = p_1 \circ h = s_1 = s_1 \circ \mathbb{1}_{A\times(B\times C)}\,. \tag{3.75}$$

Also, from (3.74),

$$s_2 \circ g \circ i = s_2 = s_2 \circ \mathbb{1}_{A\times(B\times C)}\,. \tag{3.76}$$

So there are two arrows $\mathbb{1}_{A\times(B\times C)}, g \circ i : A \times (B \times C) \to A \times (B \times C)$ and both of them makes the above diagram commute. Therefore, by the universal property of the product $A \times (B \times C)$,

$$g \circ i = \mathbb{1}_{A\times(B\times C)}\,. \tag{3.77}$$

Similarly, consider the following diagram:

$$\begin{tikzcd}
& (A \times B) \times C \arrow[loop above, "i\circ g"] \arrow[dl, "p_1\circ r_1"'] \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_1\circ i\circ g"] \arrow[dr, "p_2\circ r_1"] & \\
A & A \times B \arrow[l, "p_1"] \arrow[r, "p_2"'] & B
\end{tikzcd}$$

Using (3.70), (3.68) and (3.66),

$$p_1 \circ r_1 \circ i \circ g = p_1 \circ h \circ g = s_1 \circ g = p_1 \circ r_1. \tag{3.78}$$

Furthermore, by (3.70), (3.69), (3.67) and (3.64),

$$p_2 \circ r_1 \circ i \circ g = p_2 \circ h \circ g = q_1 \circ s_2 \circ g = q_1 \circ f = p_2 \circ r_1. \tag{3.79}$$

So there are two arrows $r_1, r_1 \circ i \circ g : (A \times B) \times C \to A \times B$ and both of them makes the above diagram commute. Therefore, by the universal property of the product $A \times B$,

$$r_1 = r_1 \circ i \circ g. \tag{3.80}$$

Now, let's consider the following diagram:

$$\begin{tikzcd}
& (A\times B)\times C \arrow[dl, "r_1"'] \arrow[dr, "r_2"] \arrow[d, shift right, "\mathbb{1}_{(A\times B)\times C}"'] \arrow[d, shift left, "i\circ g"] & \\
A\times B & (A\times B)\times C \arrow[l, "r_1"] \arrow[r, "r_2"'] & C
\end{tikzcd}$$

From (3.80),

$$r_1 \circ i \circ g = r_1 = r_1 \circ \mathbb{1}_{(A\times B)\times C} . \tag{3.81}$$

Also, (3.71), (3.67) and (3.65) gives us

$$r_2 \circ i \circ g = q_2 \circ s_2 \circ g = q_2 \circ f = r_2 = r_2 \circ \mathbb{1}_{(A\times B)\times C} . \tag{3.82}$$

So there are two arrows $\mathbb{1}_{(A\times B)\times C}, i \circ g : (A\times B)\times C \to (A\times B)\times C$ and both of them makes the above diagram commute. Therefore, by the universal property of the product $(A\times B)\times C$,

$$i \circ g = \mathbb{1}_{(A\times B)\times C} . \tag{3.83}$$

Therefore, from (3.77) and (3.83), we can conclude that $g : (A\times B)\times C \to A\times(B\times C)$ is an isomorphism, with inverse $i : A\times(B\times C) \to (A\times B)\times C$. ■

Furthermore, this isomorphism is “natural”. To make this notion precise, we need to define what a natural transformation is.

Definition 3.10 (Natural Transformation)**.** Let F and G be two functors between the categories $\mathcal{C}$ and $\mathcal{D}$. Then a **natural transformation** $\eta : F \Rightarrow G$ is a family of arrows in $\mathcal{D}$ that satisfies the following requirements:

1. The natural transformation η must associate to each object $X \in \mathrm{Ob}(\mathcal{C})$ an arrow $\eta_X : F(X) \to G(X)$ in $\mathcal{D}$. This arrow η_X is called the component of η at X.
2. Components must be such that for every arrow $f : X \to Y$ in $\mathcal{C}$, one must have $\eta_Y \circ F(f) = G(f) \circ \eta_X$. In other words, the diagram below commutes:

$$\begin{tikzcd}
F(X) \arrow[r, "F(f)"] \arrow[d, "\eta_X"'] & F(Y) \arrow[d, "\eta_Y"] \\
G(X) \arrow[r, "G(f)"'] & G(Y)
\end{tikzcd}$$

If $F, G : \mathcal{C} \to \mathcal{D}$ are functors, and $\eta : F \Rightarrow G$ is a functor, it is denoted as follows

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=50, "F", ""{name=U, below}] \arrow[r, bend right=50, "G"', ""{name=D}] & \mathcal{D} \arrow[Rightarrow, from=U, to=D, "\eta"]
\end{tikzcd}$$

If for every object $X \in \mathrm{Ob}(\mathcal{C})$, the arrow $\eta_X : F(X) \to G(X)$ is an isomorphism in $\mathcal{D}$, then $\eta : F \Rightarrow G$ is said to be a **natural isomorphism**. Two functors F and G between two categories are called naturally isomorphic if there exists a natural isomorphism between them.

When we say the isomorphism $g : (A\times B)\times C \xrightarrow{\cong} A\times(B\times C)$ is natural, we mean that there is a natural isomorphism whose components are precisely these isomorphisms $g : (A\times B)\times C \xrightarrow{\cong} A\times(B\times C)$. To prove it, let us first define the product of three categories.

Definition 3.11. Let $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$ be categories. Then we define the product category $\mathcal{C} \times \mathcal{D} \times \mathcal{E}$ as follows:

- The objects of $\mathcal{C} \times \mathcal{D} \times \mathcal{E}$ are ordered triples (C, D, E), where $C \in \mathrm{Ob}\,(\mathcal{C})$, $D \in \mathrm{Ob}\,(\mathcal{E})$, $E \in \mathrm{Ob}\,(\mathcal{E})$.
- Given two objects $(C, D, E), (C', D', E') \in \mathrm{Ob}\,(\mathcal{C} \times \mathcal{D} \times \mathcal{E})$, an arrow from (C, D, E) to (C', D', E') is an ordered triple (f, g, h), where $f \in \mathrm{Hom}_{\mathcal{C}}\,(C, C')$, $g \in \mathrm{Hom}_{\mathcal{D}}\,(D, D')$, $h \in \mathrm{Hom}_{\mathcal{E}}\,(E, E')$.
- Given an object $(C, D, E) \in \mathrm{Ob}\,(\mathcal{C} \times \mathcal{D} \times \mathcal{E})$, its identity arrow is $(\mathbb{1}_C, \mathbb{1}_D, \mathbb{1}_E)$, i.e.

$$\mathbb{1}_{(C,D,E)} = (\mathbb{1}_C, \mathbb{1}_D, \mathbb{1}_E)\,.$$

- Given two composable arrows $(f, g, h) : (C, D, E) \to (C', D', E')$ and $(f', g', h') : (C', D', E') \to (C'', D'', E'')$, their composition is defined as

$$(f', g', h') \circ (f, g, h) = (f' \circ f, g' \circ g, h' \circ h)\,.$$

One can easily verify that $\mathcal{C} \times \mathcal{D} \times \mathcal{E}$ is indeed a category. Now, suppose $\mathcal{C}$ is a category with products. Then we define two functors from $\mathcal{C} \times \mathcal{C} \times \mathcal{C}$ to $\mathcal{C}$.

$$\begin{aligned} F : \mathcal{C} \times \mathcal{C} \times \mathcal{C} &\to \mathcal{C}, \\ (A, B, C) &\mapsto (A \times B) \times C, \\ (f, g, h) &\mapsto (f \times g) \times h. \end{aligned} \tag{3.84}$$

$$\begin{aligned} G : \mathcal{C} \times \mathcal{C} \times \mathcal{C} &\to \mathcal{C}, \\ (A, B, C) &\mapsto A \times (B \times C)\,, \\ (f, g, h) &\mapsto f \times (g \times h)\,. \end{aligned} \tag{3.85}$$

One can easily check that F and G are functors. Now we define the natural transformation α as follows:

$$\begin{tikzcd}
\mathcal{C} \times \mathcal{C} \times \mathcal{C} \arrow[r, bend left=50, "F"{name=U}] \arrow[r, bend right=50, "G"'{name=D}] & \mathcal{C} \arrow[Rightarrow, from=U, to=D, "\alpha"]
\end{tikzcd}$$

Given $(A, B, C) \in \mathrm{Ob}\,(\mathcal{C} \times \mathcal{C} \times \mathcal{C})$, $F\,(A, B, C) = (A \times B) \times C$, and $G\,(A, B, C) = A \times (B \times C)$. So the component of α at (A, B, C) is an arrow from $(A \times B) \times C$ to $A \times (B \times C)$ in $\mathcal{C}$, which we define to be the isomorphism $g : (A \times B) \times C \to A \times (B \times C)$ defined in the course of proof of Theorem 3.13.

$$\alpha_{(A,B,C)} = g : (A \times B) \times C \xrightarrow{\cong} A \times (B \times C)\,. \tag{3.86}$$

Theorem 3.14

α defined as above is a natural isomorphism.

Proof. The components of α are isomorphisms. So n order to prove that α is a natural isomorphism, we need to show the commutativity of the following square given an arrow $(a, b, c) : (A, B, C) \to (A', B', C')$ in $\mathcal{C} \times \mathcal{C} \times \mathcal{C}$.

$$\begin{tikzcd}
(A \times B) \times C \arrow[r, "\alpha_{(A,B,C)}"] \arrow[d, "(a \times b) \times c"'] & A \times (B \times C) \arrow[d, "a \times (b \times c)"] \\
(A' \times B') \times C' \arrow[r, "\alpha_{(A',B',C')}"'] & A' \times (B' \times C')
\end{tikzcd}$$

In other words, we need to prove that

$$[a \times (b \times c)] \circ \alpha_{(A,B,C)} = \alpha_{(A',B',C')} \circ [(a \times b) \times c] \,.$$

Suppose the following are product diagrams:

$$\begin{tikzcd}
A' & A' \times B' \arrow[l, "p'_1"'] \arrow[r, "p'_2"] & B' \\
B' & B' \times C' \arrow[l, "q'_1"'] \arrow[r, "q'_2"] & C' \\
A' \times B' & (A' \times B') \times C' \arrow[l, "r'_1"'] \arrow[r, "r'_2"] & C' \\
A' & A' \times (B' \times C') \arrow[l, "s'_1"'] \arrow[r, "s'_2"] & B' \times C'
\end{tikzcd}$$

Consider the following compositions:

$$(A \times B) \times C \xrightarrow{r_1} A \times B \xrightarrow{p_2} B \xrightarrow{b} B',$$

$$(A \times B) \times C \xrightarrow{r_2} C \xrightarrow{c} C'.$$

Then there is a **unique** arrow $u : (A \times B) \times C \to B' \times C'$ such that the following diagram commutes:

$$\begin{tikzcd}
& (A \times B) \times C \arrow[dl, "b \circ p_2 \circ r_1"'] \arrow[d, dashed, "u"] \arrow[dr, "c \circ r_2"] & \\
B' & B' \times C' \arrow[l, "q'_1"] \arrow[r, "q'_2"'] & C'
\end{tikzcd}$$

In other words,

$$q'_1 \circ u = b \circ p_2 \circ r_1, \tag{3.87}$$
$$q'_1 \circ u = c \circ r_2 \tag{3.88}$$

In other words, $u = \langle b \circ p_2 \circ r_1, c \circ r_2 \rangle : (A \times B) \times C \to B' \times C'$. Then consider the following composition:

$$(A \times B) \times C \xrightarrow{r_1} A \times B \xrightarrow{p_1} A \xrightarrow{a} A'.$$

Given $a \circ p_1 \circ r_1 : (A \times B) \times C \to A'$, and $\langle b \circ p_2 \circ r_1, c \circ r_2 \rangle : (A \times B) \times C \to B' \times C'$, there exists a **unique** arrow $v : (A \times B) \times C \to A' \times (B' \times C')$ such that the following diagram commutes:

$$\begin{tikzcd}
& (A \times B) \times C \arrow[dl, "a \circ p_1 \circ r_1"'] \arrow[d, dashed, "v"] \arrow[dr, "\langle b \circ p_2 \circ r_1 {,} c \circ r_2 \rangle"] & \\
A' & A' \times (B' \times C') \arrow[l, "s'_1"] \arrow[r, "s'_2"'] & B' \times C'
\end{tikzcd} \tag{3.89}$$

In other words,

$$s'_1 \circ v = a \circ p_1 \circ r_1, \tag{3.90}$$
$$s'_2 \circ v = \langle b \circ p_2 \circ r_1, c \circ r_2 \rangle \,. \tag{3.91}$$

Now we claim that choosing $v = [a \times (b \times c)] \circ \alpha_{(A,B,C)}$ makes the above diagram commutative. The uniqueness of v would then imply that v is precisely $[a \times (b \times c)] \circ \alpha_{(A,B,C)}$.

$a \times (b \times c)$ is the **unique** arrow $A \times (B \times C) \to A' \times (B' \times C')$ such that the following diagram commutes:

$$\begin{tikzcd}
A \arrow[d, "a"'] & A \times (B \times C) \arrow[l, "s_1"'] \arrow[r, "s_2"] \arrow[d, dashed, "a\times(b\times c)"] & B \times C \arrow[d, "b\times c"] \\
A' & A' \times (B' \times C') \arrow[l, "s_1'"] \arrow[r, "s_2'"'] & B' \times C'
\end{tikzcd}$$

So

$$s_1' \circ [a \times (b \times c)] \circ \alpha_{(A,B,C)} = a \circ s_1 \circ \alpha_{(A,B,C)}. \tag{3.92}$$

$\alpha_{(A,B,C)}$ is nothing but the isomorphism g in the proof of Theorem 3.13. So using (3.66), we get $s_1 \circ \alpha_{(A,B,C)} = p_1 \circ r_1$. Therefore,

$$s_1' \circ [a \times (b \times c)] \circ \alpha_{(A,B,C)} = a \circ p_1 \circ r_1. \tag{3.93}$$

Furthermore,

$$s_2' \circ [a \times (b \times c)] \circ \alpha_{(A,B,C)} = (b \times c) \circ s_2 \circ \alpha_{(A,B,C)}. \tag{3.94}$$

Using (3.67), we get we get

$$s_2 \circ \alpha_{(A,B,C)} = f = \langle p_2 \circ r_1, r_2 \rangle . \tag{3.95}$$

Now, consider the following commutative diagram:

$$\begin{tikzcd}
& (A \times B) \times C \arrow[dl, "p_2\circ r_1"'] \arrow[d, dashed, "f"] \arrow[dr, "r_2"] & \\
B \arrow[d, "b"'] & B \times C \arrow[l, "q_1"'] \arrow[r, "q_2"] \arrow[d, dashed, "b\times c"] & C \arrow[d, "c"] \\
B' & B' \times C' \arrow[l, "q_1'"] \arrow[r, "q_2'"'] & C'
\end{tikzcd}$$

f is the **unique** arrow that makes the upper triangles commute, and $b \times c$ is the **unique** arrow that makes the lower squares commute. Therefore,

$$q_1' \circ (b \times c) \circ f = b \circ q_1 \circ f = b \circ p_2 \circ r_1, \tag{3.96}$$
$$q_2' \circ (b \times c) \circ f = c \circ q_2 \circ f = c \circ r_2. \tag{3.97}$$

Therefore, by the uniqueness of $\langle b \circ p_2 \circ r_1, c \circ r_2 \rangle : (A \times B) \times C \to B' \times C'$, we can conclude that

$$(b \times c) \circ f = \langle b \circ p_2 \circ r_1, c \circ r_2 \rangle . \tag{3.98}$$

Combining (3.94), (3.95) and (3.98), we get

$$s_2' \circ [a \times (b \times c)] \circ \alpha_{(A,B,C)} = \langle b \circ p_2 \circ r_1, c \circ r_2 \rangle . \tag{3.99}$$

Using (3.93) and (3.99), and by the uniqueness of v,

$$v = [a \times (b \times c)] \circ \alpha_{(A,B,C)}. \tag{3.100}$$

Now we claim that $v = \alpha_{(A',B',C')} \circ [(a \times b) \times c]$ also makes (3.89) commutative. The uniqueness of v would then imply that v is precisely $\alpha_{(A',B',C')} \circ [(a \times b) \times c]$.

Similarly as (3.66) and (3.67),

$$s_1' \circ \alpha_{(A',B',C')} = p_1' \circ r_1' \tag{3.101}$$

$$s_2' \circ \alpha_{(A',B',C')} = f' = \langle p_2' \circ r_1', r_2' \rangle \,. \tag{3.102}$$

Using these equations, we get

$$s_1' \circ \alpha_{(A',B',C')} \circ [(a \times b) \times c] = p_1' \circ r_1' \circ [(a \times b) \times c] \,. \tag{3.103}$$

Now, $(a \times b) \times c$ is the **unique** arrow $(A \times B) \to C \to (A' \times B') \to C'$ such the the following diagram commutes:

$$\begin{tikzcd}
A \times B \arrow[d, "a\times b"'] & (A \times B) \times C \arrow[l, "r_1"'] \arrow[r, "r_2"] \arrow[d, dashed, "(a\times b)\times c"] & C \arrow[d, "c"] \\
A' \times B' & (A' \times B') \times C' \arrow[l, "r_1'"] \arrow[r, "r_2'"'] & C'
\end{tikzcd}$$

Therefore,

$$r_1' \circ [(a \times b) \times c] = (a \times b) \circ r_1 . \tag{3.104}$$

Furthermore, $a \times b$ is the **unique** arrow such that the following diagram commutes:

$$\begin{tikzcd}
A \arrow[d, "a"'] & A \times B \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, dashed, "a\times b"] & B \arrow[d, "b"] \\
A' & A' \times B' \arrow[l, "p_1'"] \arrow[r, "p_2'"'] & B'
\end{tikzcd}$$

So

$$p_1' \circ (a \times b) \circ r_1 = a \circ p_1 \circ r_1 . \tag{3.105}$$

Combining (3.103), (3.104), (3.105), we get

$$s_1' \circ \alpha_{(A',B',C')} \circ [(a \times b) \times c] = a \circ p_1 \circ r_1 .. \tag{3.106}$$

Furthermore,

$$s_2' \circ \alpha_{(A',B',C')} \circ [(a \times b) \times c] = f' \circ [(a \times b) \times c] = \langle p_2' \circ r_1', r_2' \rangle \circ [(a \times b) \times c] \,. \tag{3.107}$$

Consider the following commutative diagram:

$$\begin{tikzcd}
A \times B \arrow[d, "a\times b"'] & (A \times B) \times C \arrow[l, "r_1"'] \arrow[r, "r_2"] \arrow[d, dashed, "(a\times b)\times c"] & C \arrow[d, "c"] \\
A' \times B' \arrow[d, "p_2'"'] & (A' \times B') \times C' \arrow[l, "r_1'"'] \arrow[r, "r_2'"'] \arrow[dl, "p_2' \circ r_1'"'] \arrow[d, dashed, "f'"'] \arrow[dr, "r_2'"'] & C' \arrow[d, equal] \\
B' & B' \times C' \arrow[l, "q_1'"] \arrow[r, "q_2'"'] & C'
\end{tikzcd}$$

$(a \times b) \times c$ is the **unique** arrow that makes the upper squares commute. $f' = \langle p_2' \circ r_1', r_2' \rangle$ is the **unique** arrow that makes the lower triangles commute. Therefore,

$$q_1' \circ f' \circ [(a \times b) \times c] = p_2' \circ r_1' \circ [(a \times b) \times c] = p_2' \circ (a \times b) \circ r_1 = b \circ p_2 \circ r_1 , \tag{3.108}$$

$$q_2' \circ f' \circ [(a \times b) \times c] = r_2' \circ [(a \times b) \times c] = c \circ r_2 . \tag{3.109}$$

Therefore, by the uniqueness of $u = \langle b \circ p_2 \circ r_1, c \circ r_2 \rangle : (A \times B) \times C \to B' \times C'$,

$$f' \circ [(a \times b) \times c] = \langle b \circ p_2 \circ r_1, c \circ r_2 \rangle \,. \tag{3.110}$$

Combining (3.107) and (3.110), we can conclude that

$$s_2' \circ \alpha_{(A',B',C')} \circ [(a \times b) \times c] = \langle b \circ p_2 \circ r_1, c \circ r_2 \rangle \,. \tag{3.111}$$

Now, by (3.106) and (3.111), and invoking the uniqueness of $v : (A \times B) \times C \to A' \times (B' \times C')$, we conclude

$$v = \alpha_{(A',B',C')} \circ [(a \times b) \times c] \,. \tag{3.112}$$

Using (3.100) and (3.112), we get

$$[a \times (b \times c)] \circ \alpha_{(A,B,C)} = \alpha_{(A',B',C')} \circ [(a \times b) \times c] \,. \tag{3.113}$$

In other words, the following diagram commutes:

$$\begin{tikzcd}
(A \times B) \times C \arrow[r, "\alpha_{(A,B,C)}"] \arrow[d, "(a\times b)\times c"'] & A \times (B \times C) \arrow[d, "a\times(b\times c)"] \\
(A' \times B') \times C' \arrow[r, "\alpha_{(A',B',C')}"'] & A' \times (B' \times C')
\end{tikzcd}$$

Therefore, α is a natural transformation. Furthermore, the components of α are all isomorphisms. Hence, α is a natural isomorphism. ■

§3.8 Product and Terminal Object

Proposition 3.15

Let $\mathcal{C}$ be a category with products, and 1 is a terminal object of $\mathcal{C}$. Then for any $X \in \mathrm{Ob}(\mathcal{C})$, $X \times 1 \cong X$.

Proof. Let the following be a product diagram:

$$\begin{tikzcd}
X & X \times 1 \arrow[l, "p_1"'] \arrow[r, "p_2"] & 1
\end{tikzcd}$$

Since 1 is a terminal object, there is a **unique** arrow $u : X \to 1$. Given $\mathbb{1}_X : X \to X$ and $u : X \to 1$, there is a **unique** arrow $\langle \mathbb{1}_X, u \rangle : X \to X \times 1$ such that the following diagram commutes:

$$\begin{tikzcd}
 & X \arrow[dl, "\mathbb{1}_X"'] \arrow[d, dashed, "\langle \mathbb{1}_X, u \rangle"] \arrow[dr, "u"] & \\
X & X \times 1 \arrow[l, "p_1"] \arrow[r, "p_2"'] & 1
\end{tikzcd}$$

In other words,

$$p_1 \circ \langle \mathbb{1}_X, u \rangle = \mathbb{1}_X \text{ and } p_2 \circ \langle \mathbb{1}_X, u \rangle = u. \tag{3.114}$$

We claim that $p_1 : X \times 1 \to X$ is an isomorphism, with inverse $\langle \mathbb{1}_X, u \rangle$. Clearly, we have $p_1 \circ \langle \mathbb{1}_X, u \rangle = \mathbb{1}_X$. We need to show that $\langle \mathbb{1}_X, u \rangle \circ p_1 = \mathbb{1}_{X \times 1}$.

$p_2 \circ \langle \mathbb{1}_X, u \rangle \circ p_1$ is an arrow from $X \times 1$ to 1. Since 1 is a terminal object, there is exactly one arrow from $X \times 1$ to 1. So $p_2 \circ \langle \mathbb{1}_X, u \rangle \circ p_1 = p_2$. Therefore,

$$p_1 \circ \langle \mathbb{1}_X, u \rangle \circ p_1 = p_1 \text{ and } p_2 \circ \langle \mathbb{1}_X, u \rangle \circ p_1 = p_2. \tag{3.115}$$

So the following diagrams commute:

$$\begin{tikzcd}
& X \times 1 \arrow[dl, "p_1"'] \arrow[d, "\langle \mathbb{1}_X , u \rangle \circ p_1"] \arrow[dr, "p_2"] & \\
X & X \times 1 \arrow[l, "p_1"] \arrow[r, "p_2"'] & 1
\end{tikzcd}
\qquad
\begin{tikzcd}
& X \times 1 \arrow[dl, "p_1"'] \arrow[d, "\mathbb{1}_{X \times 1}"] \arrow[dr, "p_2"] & \\
X & X \times 1 \arrow[l, "p_1"] \arrow[r, "p_2"'] & 1
\end{tikzcd}$$

Therefore, by the universal property of products, we must have $\langle \mathbb{1}_X , u \rangle \circ p_1 = \mathbb{1}_{X \times 1}$. Therefore, $p_1 : X \times 1 \to X$ is an isomorphism, with inverse $\langle \mathbb{1}_X , u \rangle : X \to X \times 1$, i.e. $X \times 1 \cong X$. ■

However, the converse of Proposition 3.15 is not, in general, true. If A is an object in a category $\mathcal{C}$ with products such that $X \times A \cong A$ for every object $X \in \mathrm{Ob}\,(\mathcal{C})$, then A need not be a terminal object. The following is a counterexample.

Example 3.5. Let $\mathbb{Z}^{\mathbb{N}}$ denote the set of all integer sequences, which is an object of the category **Groups**. Let us consider the subcategory $\mathcal{C}$ of **Groups** consisting of only one object $\mathbb{Z}^{\mathbb{N}}$, and all the arrows from $\mathbb{Z}^{\mathbb{N}}$ to itself in **Groups** (often refered to as the *full subcategory* of **Groups**, with the object $\mathbb{Z}^{\mathbb{N}}$). Then in this category $\mathcal{C}$,

$$\mathbb{Z}^{\mathbb{N}} \times \mathbb{Z}^{\mathbb{N}} = \mathbb{Z}^{\mathbb{N}}.$$

The product diagram is as follows:

$$\begin{tikzcd}
\mathbb{Z}^{\mathbb{N}} & \mathbb{Z}^{\mathbb{N}} \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & \mathbb{Z}^{\mathbb{N}}
\end{tikzcd}$$

The homomorphisms π_1 and π_2 are defined as follows:

$$\pi_1 \left(a_0, a_1, a_2, a_3, a_4, \ldots \right) = \left(a_1, a_3, a_5, \cdots \right),$$

$$\pi_2 \left(a_0, a_1, a_2, a_3, a_4, \cdots \right) = \left(a_0, a_2, a_4, a_6, \cdots \right).$$

This is easily seen to satisfy the UMP of product. Now, in the category $\mathcal{C}$, there is only one object. So $X \times A \cong A$ for every object $X \in \mathrm{Ob}\,(\mathcal{C})$ is clearly satisfied. But $\mathbb{Z}^{\mathbb{N}}$ is **not** a terminal object in this category, since there are more than one arrows from $\mathbb{Z}^{\mathbb{N}}$ to itself in $\mathcal{C}$.

The isomorphism we showed in Proposition 3.15 is not just any isomorphism. This isomorphism is actually "natural", meaning there is a natural isomorphism whose components are the isomorphisms we saw in Proposition 3.15.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=50, "\times(-{,}1)", ""{name=U, below}] \arrow[r, bend right=50, "\mathbb{1}_{\mathcal{C}}"', ""{name=D}] & \mathcal{C}
\arrow[Rightarrow, from=U, to=D, "\eta"]
\end{tikzcd}$$

The components of η are given by

$$\eta_X = p_1 : X \times 1 \xrightarrow{\cong} X, \tag{3.116}$$

where $p_1 : X \times 1 \to X$ is the projection on X arrow. To show that η is a natural isomorphism, we just need to show the commutativity of the following square:

$$\begin{tikzcd}
X \times 1 \arrow[r, "f \times \mathbb{1}_1"] \arrow[d, "\eta_X = p_1"'] & X' \times 1 \arrow[d, "\eta_{X'} = p_1'"] \\
X \arrow[r, "f"'] & X'
\end{tikzcd}$$

But this square obviously commutes, since $f \times \mathbb{1}_1$ is the **unique** arrow $X \times 1 \to X' \times 1$ such that the following diagram commutes:

$$\begin{tikzcd}
X \arrow[d, "f"'] & X \times 1 \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, dashed, "f \times \mathbb{1}_1"] & 1 \arrow[d, "\mathbb{1}_1"] \\
X' & X' \times 1 \arrow[l, "p_1'"] \arrow[r, "p_2'"'] & 1
\end{tikzcd}$$

The naturality square is precisely the left hand square of this diagram. So naturality holds. Therefore, the isomorphism $X \times 1 \cong X$ is **natural**.

Now, the converse of Proposition 3.15 holds if we further assume that the isomorphism is natural.

Theorem 3.16

Let $\mathcal{C}$ be a category with products, and $A \in \text{Ob}(\mathcal{C})$ such that $\eta : \times(-, A) \Rightarrow \mathbb{1}_{\mathcal{C}}$ is a natural isomorphism. Then A is a terminal object of $\mathcal{C}$.

Proof. $\eta_X : X \times A \to X$ is an isomorphism for every $X \in \text{Ob}(\mathcal{C})$. Suppose the following is a product diagram:

$$\begin{tikzcd}
X & X \times A \arrow[l, "p_1"'] \arrow[r, "p_2"] & A
\end{tikzcd}$$

Let us define a transformation $\alpha : \mathbb{1}_{\mathcal{C}} \Rightarrow \mathbb{1}_{\mathcal{C}}$, which we are going to show to be a natural transformation.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=50, "\mathbb{1}_{\mathcal{C}}"{name=U}] \arrow[r, bend right=50, "\mathbb{1}_{\mathcal{C}}"'{name=D}] & \mathcal{C} \arrow[Rightarrow, from=U, to=D, "\alpha"]
\end{tikzcd}$$

The components of α are given by

$$\alpha_X = p_1 \circ \eta_X^{-1} : X \to X, \tag{3.117}$$

where p_1 is the projection $X \times A \to X$. In order to show that α is a natural transformation, we need to show the commutativity of the following square given any arrow $g : X \to X'$:

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "\alpha_X = p_1 \circ \eta_X^{-1}"'] & X' \arrow[d, "\alpha_{X'} = p_1' \circ \eta_{X'}^{-1}"] \\
X \arrow[r, "g"'] & X'
\end{tikzcd} \tag{3.118}$$

$g \times \mathbb{1}$ is the **unique** arrow $X \times A \to X' \times A$ such that the diagram below commutes:

$$\begin{tikzcd}
X \arrow[d, "g"'] & X \times A \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, dashed, "g \times \mathbb{1}_A"] & A \arrow[d, "\mathbb{1}_A"] \\
X' & X' \times A \arrow[l, "p_1'"] \arrow[r, "p_2'"'] & A
\end{tikzcd}$$

In particular,

$$g \circ p_1 = p_1' \circ (g \times \mathbb{1}_A) \,. \tag{3.119}$$

The naturality of η guarantees the commutativity of the following square:

$$\begin{tikzcd}
X \times A \arrow[r, "g \times \mathbb{1}_A"] \arrow[d, "\eta_X"'] & X' \times A \arrow[d, "\eta_{X'}"] \\
X \arrow[r, "g"'] & X'
\end{tikzcd}$$

In other words,

$$\eta_{X'} \circ (g \times \mathbb{1}_A) = g \circ \eta_X . \tag{3.120}$$

Composing by $\eta_{X'}^{-1}$ on the left, and η_X^{-1} on the right, we get

$$(g \times \mathbb{1}_A) \circ \eta_X^{-1} = \eta_{X'}^{-1} \circ g. \tag{3.121}$$

Therefore, using (3.119) and (3.121),

$$\alpha_{X'} \circ g = p_1' \circ \eta_{X'}^{-1} \circ g = p_1' \circ (g \times \mathbb{1}_A) \circ \eta_X^{-1} = g \circ p_1 \circ \eta_X^{-1} = g \circ \alpha_X . \tag{3.122}$$

So (3.118) commutes, and hence $\alpha : \mathbb{1}_{\mathcal{C}} \to \mathbb{1}_{\mathcal{C}}$ is a natural transformation. Therefore, for any arrow $u : X \to X$, by the naturality of α, the following diagram commutes:

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "\alpha_X = p_1 \circ \eta_X^{-1}"'] & X \arrow[d, "\alpha_X = p_1 \circ \eta_X^{-1}"] \\
X \arrow[r, "u"'] & X
\end{tikzcd}$$

In other words, for any $u : X \to X$,

$$p_1 \circ \eta_X^{-1} \circ u = u \circ p_1 \circ \eta_X^{-1}. \tag{3.123}$$

In order to prove that A is a terminal object, we need to show that there exists a **unique** arrow $X \to A$. Clearly, there is an arrow $X \to A$; for instance $p_2 \circ \eta_X^{-1} : X \to A$. We now need to show that the arrow $X \to A$ is unique. Let f be **any** arrow $X \to A$.

$$\begin{tikzcd}
& X \arrow[dl, "\mathbb{1}_X"'] \arrow[d, dashed, "\langle \mathbb{1}_X , f \rangle"] \arrow[dr, "f"] & \\
X & X \times A \arrow[l, "p_1"] \arrow[r, "p_2"'] & A
\end{tikzcd}$$

Choose the arrow $u : X \to X$ to be $u = \eta_X \circ \langle \mathbb{1}_X , f \rangle : X \to X$. Using (3.123), we have

$$\eta_X \circ \langle \mathbb{1}_X , f \rangle \circ p_1 \circ \eta_X^{-1} = p_1 \circ \eta_X^{-1} \circ \eta_X \circ \langle \mathbb{1}_X , f \rangle = p_1 \circ \langle \mathbb{1}_X , f \rangle = \mathbb{1}_X . \tag{3.124}$$

So, $\eta_X \circ \langle \mathbb{1}_X , f \rangle \circ p_1 \circ \eta_X^{-1} = \mathbb{1}_X$. Now, composing by η_X on the right, and η_X^{-1} on the left, we get

$$\langle \mathbb{1}_X , f \rangle \circ p_1 = \eta_X^{-1} \circ \eta_X = \mathbb{1}_{X \times A} . \tag{3.125}$$

Therefore, $p_1 : X \times A \to X$ is an isomorphism, with the inverse $\langle \mathbb{1}_X , f \rangle$. As a result,

$$f = p_2 \circ \langle \mathbb{1}_X , f \rangle = p_2 \circ p_1^{-1}. \tag{3.126}$$

Hence, given **any** arrow $f : X \to A$, f must be equal to $p_2 \circ p_1^{-1}$. Therefore, there is a unique arrow $X \to A$, and hence A is a terminal object. ■

4 Duality

§4.1 Duality Principle

The **Elementary Theory of an Abstract Category (ETAC)** consists of certain statements which involves letters $A, B, C, \ldots$ for objects and $f, g, h, \ldots$ for arrows. For example, the statement $f : A \to B$ can be phrased as "A is the domain of f, and B is the codomain of f." A sentence is a statement with all variables quantified (quantifiers are "for all A", "for all f", "there exists an A", "there exists an f" etc.), and none being free. For example,

For all f, there exists A and B such that $f : A \to B$

is a sentence. Axioms of ETAC are examples of sentences that are true in every category.

To each category $\mathcal{C}$, we also associate the **opposite category** $\mathcal{C}^{\text{op}}$. The objects of $\mathcal{C}^{\text{op}}$ are the same as those of $\mathcal{C}$, i.e. given $A \in \text{Ob}(\mathcal{C})$, one also has $A \in \text{Ob}(\mathcal{C}^{\text{op}})$.

$$\text{Ob}(\mathcal{C}) = \text{Ob}(\mathcal{C}^{\text{op}}).$$

The arrows of $\mathcal{C}^{\text{op}}$ are arrows f^{op} in the reverse direction, and hence are in 1-1 correspondence

$$f \mapsto f^{\text{op}}$$

with the arrows f of $\mathcal{C}$. In other words, if $f : A \to B$ is an arrow in $\mathcal{C}$, then the corresponding arrow in $\mathcal{C}^{\text{op}}$ is given by $f^{\text{op}} : B \to A$ so that

$$\text{dom}(f^{\text{op}}) = \text{cod}(f) \quad \text{and} \quad \text{cod}(f^{\text{op}}) = \text{dom}(f).$$

Now, given arrows

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

in $\mathcal{C}$, one has

$$\begin{tikzcd} A & B \arrow[l, "f^{\text{op}}"'] & C \arrow[l, "g^{\text{op}}"'] \arrow[ll, bend left, "f^{\text{op}} \circ g^{\text{op}} = (g \circ f)^{\text{op}}"] \end{tikzcd}$$

in $\mathcal{C}^{\text{op}}$ so that $f^{\text{op}} \circ g^{\text{op}} = (g \circ f)^{\text{op}}$ is defined in $\mathcal{C}^{\text{op}}$ exactly when the composite $g \circ f$ is defined in $\mathcal{C}$, i.e. (g, f) is a composable pair (g after f) in $\mathcal{C}$ if and only if $(f^{\text{op}}, g^{\text{op}})$ is a composable pair (f^{op} after g^{op}) in $\mathcal{C}^{\text{op}}$, and $f^{\text{op}} \circ g^{\text{op}}$ is precisely the opposite arrow of the composite arrow $g \circ f$ in $\mathcal{C}$.

Now, the dual of any statement Σ of ETAC is formed by making the following replacements throughout in Σ: "domain" by "codomain", "codomain" by "domain", "h is the composite of f with g" by "h is the composite of g with f". As a result, arrows and composites are reversed in the dual statement Σ^*. While forming the dual sentence, logic / quantifiers (and, or, for all, there exists etc.) remain unchanged. Some examples of statements Σ and their dual statements Σ^* are listed below:

Statement Σ	Dual Statement Σ^*
$f : A \to B$	$f : B \to A$
$A = \text{dom}(f)$	$A = \text{cod}(f)$
$i = \mathbb{1}_A$	$i = \mathbb{1}_A$
$h = g \circ f$	$h = f \circ g$
T is a terminal object	T is an initial object

A sentence Σ is true in $\mathcal{C}^{\text{op}}$ if and only if its dual statement Σ^* is true in $\mathcal{C}$ (arrows in $\mathcal{C}^{\text{op}}$ are read without the op prefix).

Remark 4.1. Note that the dual of the dual is the original statement ($\Sigma^{**} = \Sigma$). There could be broadly two types of statements or sentences. A statement can be consequences of the axioms of ETAC, for example, "a terminal object of a category, if it exists, is unique up to isomorphism." The other type of statements do not follow from the axioms of ETAC. For example, "$a = \operatorname{dom} f$", or $h = g \circ f$. For the latter type of statements Σ in $\mathcal{C}$, the dual statements Σ^* refer to opposite category $\mathcal{C}^{\text{op}}$. For the former type of statements, i.e. for the statements that are consequences of the axioms of ETAC, Σ and its dual Σ^* refer to the category $\mathcal{C}$. This fact is captured by the **duality principle**.

Duality Principle:

If a statement Σ of ETAC is a consequence of the axioms, so is the dual statement Σ^*.

For example, take Σ to be "a terminal object, if it exists, is unique up to isomorphism." We have the dual Σ^* that reads "an initial object, if it exists, is unique up to isomorphism." This dual statement Σ^* applies to the same category $\mathcal{C}$.

§4.2 Coproducts

Definition 4.1. A diagram $A \xrightarrow{q_1} Q \xleftarrow{q_2} B$ is called a coproduct of A and B if for any object Z and $A \xrightarrow{z_1} Z \xleftarrow{z_2} B$, there is a **unique** arrow $u : Q \to Z$ such that the following diagram commutes:

$$\begin{tikzcd}
 & Z & \\
A \arrow[ur, "z_1"] \arrow[r, "q_1"'] & Q \arrow[u, dashed, "\exists!"', "u"] & B \arrow[ul, "z_2"'] \arrow[l, "q_2"]
\end{tikzcd}$$

i.e. $u \circ q_1 = z_1$ and $u \circ q_2 = z_2$.

We usually write $A \xrightarrow{i_1} A + B \xleftarrow{i_2} B$ for the coproduct, and $[f, g]$ for the uniquely determined arrow $[f, g] : A + B \to Z$ as in the following diagram:

$$\begin{tikzcd}
 & Z & \\
A \arrow[ur, "f"] \arrow[r, "i_1"'] & A + B \arrow[u, "{[f,g]}"] & B \arrow[ul, "g"'] \arrow[l, "i_2"]
\end{tikzcd}$$

Remark 4.2. Similar to Remark 3.5, the "co-projections" $i_1 : A \to A + B$ and $i_2 : B \to A + B$ need not be monomorphisms. A coproduct of two objects is a product of the same two objects in the opposite category, as can be seen by reversing the arrows above and using the commutative diagram of a product. Therefore, given a nonempty set A, the product $\varnothing \times A$ in **Sets** is the coproduct $\varnothing + A$ in $\mathbf{Sets}^{\text{op}}$. In other words,

$$\varnothing +_{\mathbf{Sets}^{\text{op}}} A = \varnothing \times_{\mathbf{Sets}} A. \tag{4.1}$$

Then the coprojection

$$i_2 : A \to \varnothing +_{\mathbf{Sets}^{\text{op}}} A$$

is not a monomorphism in $\mathbf{Sets}^{\text{op}}$, since the opposite arrow $p_2 : \varnothing \times_{\mathbf{Sets}} A \to A$ is not an epimorphism.

The 'co-projections" $i_1 : A \to A + B$ and $i_2 : B \to A + B$ are often called coproduct injections.

Example 4.1. In **Sets**, the coproduct $A + B$ of two sets is their disjoint union which can be constreucted as

$$A + B = \{(a, 1) \mid a \in A\} \cup \{(b, 2) \mid b \in B\} = (A \times \{1\}) \cup (B \times \{2\}), \tag{4.2}$$

with evident coproduct injections $i_1(a) = (a, 1)$ and $i_2(b) = (b, 2)$. Given any functions $f : A \to Z$ and $g : B \to Z$,

$$\begin{tikzcd}
& Z & \\
A \arrow[ur, "f"] \arrow[r, "i_1"'] & A + B \arrow[u, dashed, "{[f,g]}"] & B \arrow[ul, "g"'] \arrow[l, "i_2"]
\end{tikzcd}$$

we define $[f, g] : A + B \to Z$ as follows:

$$[f, g](x, \delta) = \begin{cases} f(x) & \text{if } \delta = 1, \\ g(x) & \text{if } \delta = 2. \end{cases} \tag{4.3}$$

Uniqueness of $[f, g]$ defined in (4.3) follows by noting that if for $h : A + B \to Z$ the above diagram commutes, then one must have $h \circ i_1 = f$ and $h \circ i_2 = g$. This leads to

$$f(a) = (h \circ i_1)(a) = h(i_1(a)) = h(a, 1), \text{ and} \tag{4.4}$$
$$g(b) = (h \circ i_2)(b) = h(i_2(b)) = h(b, 2). \tag{4.5}$$

Therefore, $h(x, \delta) = [f, g](x, \delta)$, for any $(x, \delta) \in A + B$. So $[f, g] = h$ proving the uniqueness of $[f, g]$.

In **Sets**, if A is a finite set with cardinality n, then

$$A \cong \underbrace{1 + 1 + \cdots + 1}_{n\text{-times}}, \tag{4.6}$$

where 1 stands for a singleton set. Indeed, if $A = \{a_1, a_2, \ldots, a_n\}$, then

$$\{a_1\} + \{a_2\} + \cdots \{a_n\} = \{(a_1, 1), (a_2, 2), \ldots, (a_n, n)\} \cong \{a_1, a_2, \ldots, a_n\}. \tag{4.7}$$

Proposition 4.1

Coproducts are unique up to isomorphism.

We don't need to prove it again. We can just use the fact that products are unique up to isomorphism (Proposition 3.9), and then invoke duality principle.

Theorem 4.2

Coproducts are associative up to isomorphism, i.e. if A, B, and C are objects in a category with coproducts, then $(A + B) + C \cong A + (B + C)$.

This follows from Theorem 3.13 and duality principle.

Example 4.2. If $M(A)$ and $M(B)$ are free monoids on sets A and B, then in the category **Mon**, we can construct their coproduct as

$$M(A) + M(B) \cong M(A + B).$$

In other words, the coproduct of the free monoids on A and B is the free monoid on the coproduct $A+B$ of A and B in **Sets**.

We shall prove that $M\left(A+B\right)$ satisfies the UMP of coproduct of $M(A)$ and $M(B)$. Let $\mathcal{U}:\mathbf{Mon}\to\mathbf{Sets}$ be the forgetful functor, and let

$$\begin{aligned}\eta_A &: A\to\mathcal{U}\left(M\left(A\right)\right),\\ \eta_B &: B\to\mathcal{U}\left(M\left(B\right)\right),\\ \eta_{A+B} &: A+B\to\mathcal{U}\left(M\left(A+B\right)\right)\end{aligned}$$

be the respective insertions of generators. Let

$$A\xrightarrow{i_1}A+B\xleftarrow{i_2}B$$

be the coproduct diagram for A and B. Consider the following diagram in **Sets**:

$$\begin{tikzcd}
\mathcal{U}(M(A)) \arrow[r, dashed, "\mathcal{U}(j_1)"] & \mathcal{U}(M(A+B)) & \mathcal{U}(M(B)) \arrow[l, dashed, "\mathcal{U}(j_2)"'] \\
A \arrow[u, "\eta_A"] \arrow[r, "i_1"'] & A+B \arrow[u, "\eta_{A+B}"] & B \arrow[u, "\eta_B"'] \arrow[l, "i_2"]
\end{tikzcd}$$

Given the monoid $M(A+B)$ and the function $\eta_{A+B}\circ i_1 : A\to\mathcal{U}\left(M\left(A+B\right)\right)$, by the UMP of free monoid on the set A, there is a **unique** monoid homomorphism $j_1 : M\left(A\right)\to M\left(A+B\right)$ such that

$$\eta_{A+B}\circ i_1=\mathcal{U}\left(j_1\right)\circ\eta_A. \tag{4.8}$$

Similarly, using the UMP of free monoid on the set B and the function $\eta_{A+B}\circ i_2 : B\to\mathcal{U}\left(M\left(A+B\right)\right)$, there is a **unique** monoid homomorphism $j_2 : M\left(B\right)\to M\left(A+B\right)$ such that

$$\eta_{A+B}\circ i_2=\mathcal{U}\left(j_2\right)\circ\eta_B. \tag{4.9}$$

In other words, the diagram above commutes in **Sets**. We now claim that

$$M(A)\xrightarrow{j_1}M(A+B)\xleftarrow{j_2}M(B)$$

is a coproduct diagram. Let N be any monoid and let $f : M\left(A\right)\to N$ and $g : M\left(B\right)\to N$ be monoid homomorphisms.

$$\begin{tikzcd}
 & \mathcal{U}(N) & \\
\mathcal{U}(M(A)) \arrow[ur, "\mathcal{U}(f)"] & & \mathcal{U}(M(B)) \arrow[ul, "\mathcal{U}(g)"'] \\
A \arrow[u, "\eta_A"] \arrow[r, "i_1"'] & A+B \arrow[uu, dashed, "h"] & B \arrow[u, "\eta_B"'] \arrow[l, "i_2"]
\end{tikzcd}$$

Since $A+B$ is the coproduct of A and B in **Sets**, there exists a **unique** $h : A+B\to\mathcal{U}\left(N\right)$ such that

$$h\circ i_1=\mathcal{U}(f)\circ\eta_A \text{ and } h\circ i_2=\mathcal{U}(g)\circ\eta_B. \tag{4.10}$$

$$\begin{tikzcd}
& \mathcal{U}(N) & \\
\mathcal{U}(M(A)) \arrow[ur, "\mathcal{U}(f)"] \arrow[r, "\mathcal{U}(j_1)"] & \mathcal{U}(M(A+B)) \arrow[u, dashed] & \mathcal{U}(M(B)) \arrow[ul, "\mathcal{U}(g)"'] \arrow[l, "\mathcal{U}(j_2)"'] \\
A \arrow[u, "\eta_A"] \arrow[r, "i_1"'] & A+B \arrow[u, "\eta_{A+B}"'] \arrow[uu, bend left=60, "h"] & B \arrow[u, "\eta_B"'] \arrow[l, "i_2"]
\end{tikzcd}$$

Now, given the monoid N and the function $h : A+B \to \mathcal{U}(N)$, by the UMP of free monoid on the set $A+B$, there exists a **unique** monoid homomorphism $\overline{h} : M(A+B) \to N$ such that

$$h = \mathcal{U}\left(\overline{h}\right) \circ \eta_{A+B}. \tag{4.11}$$

In other words, the following diagram commutes

$$\begin{tikzcd}
\mathcal{U}(M(A+B)) \arrow[r, "\mathcal{U}(\overline{h})"] & \mathcal{U}(N) \\
A+B \arrow[u, "\eta_{A+B}"] \arrow[ur, "h"'] &
\end{tikzcd}$$

So we get the following diagram:

$$\begin{tikzcd}
& \mathcal{U}(N) & \\
\mathcal{U}(M(A)) \arrow[ur, "\mathcal{U}(f)"] \arrow[r, "\mathcal{U}(j_1)"] & \mathcal{U}(M(A+B)) \arrow[u, "\mathcal{U}(\overline{h})"'] & \mathcal{U}(M(B)) \arrow[ul, "\mathcal{U}(g)"'] \arrow[l, "\mathcal{U}(j_2)"'] \\
A \arrow[u, "\eta_A"] \arrow[r, "i_1"'] & A+B \arrow[u, "\eta_{A+B}"'] \arrow[uu, bend left=60, "h"] & B \arrow[u, "\eta_B"'] \arrow[l, "i_2"]
\end{tikzcd}$$

In this diagram, using (4.11) and (4.10), we have

$$\mathcal{U}\left(\overline{h}\right) \circ \eta_{A+B} \circ i_1 = h \circ i_1 = \mathcal{U}(f) \circ \eta_A. \tag{4.12}$$

Again, using (4.8),

$$\mathcal{U}\left(\overline{h}\right) \circ \eta_{A+B} \circ i_1 = \mathcal{U}\left(\overline{h}\right) \circ \mathcal{U}\left(j_1\right) \circ \eta_A = \mathcal{U}\left(\overline{h} \circ j_1\right) \circ \eta_A. \tag{4.13}$$

However, for the function $\mathcal{U}\left(\overline{h}\right) \circ \eta_{A+B} \circ i_1 : A \to \mathcal{U}\left(N\right)$, using the UMP of free monoid on the set A, there exists a **unique** monoid homomorphism $u : M\left(A\right) \to N$ such that

$$\mathcal{U}\left(\overline{h}\right) \circ \eta_{A+B} \circ i_1 = \mathcal{U}\left(u\right) \circ \eta_A. \tag{4.14}$$

In other words, there is a **unique** monoid homomorphism $u : M(A) \to N$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{U}(M(A)) \arrow[r, "\mathcal{U}(u)"] & \mathcal{U}(N) \\
A \arrow[u, "\eta_A"] \arrow[ur, "\mathcal{U}(\overline{h})\circ\eta_{A+B}\circ i_1"'] &
\end{tikzcd}$$

But from (4.12) and (4.13), we get that taking $u = f$ and $u = \overline{h} \circ j_1$ makes the above diagram commute. Therefore, we must have

$$f = \overline{h} \circ j_1. \tag{4.15}$$

Similarly,

$$g = \overline{h} \circ j_2. \tag{4.16}$$

Therefore, given the following diagram commutes:

$$\begin{tikzcd}
 & N & \\
M(A) \arrow[ur, "f"] \arrow[r, "j_1"'] & M(A+B) \arrow[u, "\overline{h}"] & M(B) \arrow[ul, "g"'] \arrow[l, "j_2"]
\end{tikzcd}$$

Furthermore, we need to show that $\overline{h}$ is unique. Suppose there exists another monoid homomorphism $v : M\left(A+B\right) \to N$ such that $f = v \circ j_1$ and $g = v \circ j_2$. Then using (4.8), we get

$$\mathcal{U}\left(f\right) \circ \eta_A = \mathcal{U}\left(v\right) \circ \mathcal{U}\left(j_1\right) \circ \eta_A = \mathcal{U}\left(v\right) \circ \eta_{A+B} \circ i_1. \tag{4.17}$$

Similarly, using (4.9),

$$\mathcal{U}\left(g\right) \circ \eta_B = \mathcal{U}\left(v\right) \circ \mathcal{U}\left(j_2\right) \circ \eta_B = \mathcal{U}\left(v\right) \circ \eta_{A+B} \circ i_2. \tag{4.18}$$

However, $h : A+B \to \mathcal{U}\left(N\right)$ is the **unique** map such that

$$h \circ i_1 = \mathcal{U}(f) \circ \eta_A \text{ and } h \circ i_2 = \mathcal{U}(g) \circ \eta_B.$$

The uniqueness of h along with (4.17) and (4.18) implies that

$$h = \mathcal{U}\left(v\right) \circ \eta_{A+B}. \tag{4.19}$$

On the other hand, $\overline{h}$ is the **unique** monoid homomorphism $M(A+B) \to N$ such that $h = \mathcal{U}\left(\overline{h}\right) \circ \eta_{A+B}$. Therefore, from (4.19), we must have $\overline{h} = v$. Hence, $\overline{h}$ is the **unique** monoid homomorphism $M(A+B) \to N$ such that $f = \overline{h} \circ j_1$ and $g = \overline{h} \circ j_2$. So $M\left(A+B\right)$ satisfies the UMP of coproduct of $M\left(A\right)$ and $M\left(B\right)$. Therefore,

$$M\left(A\right) + M\left(B\right) \cong M\left(A+B\right). \tag{4.20}$$

Example 4.3. Two monoids A, B have a coproduct of the form

$$A + B = M\left(|A| + |B|\right) / \sim .$$

Here, $|A| + |B|$ is the disjoint union of the underlying sets $|A|$ and $|B|$ of the monoids A and B, respectively. One then forms the free monoid $M\left(|A| + |B|\right)$ on the set $|A|+|B|$, which is the collection of words over $|A|+|B|$. Now given two words $v, w \in M\left(|A| + |B|\right)$, one declares them to be equivalent as follows:

$$\begin{aligned} \cdots x u_A y \cdots &\sim \cdots xy \cdots , \\ \cdots x u_B y \cdots &\sim \cdots xy \cdots , \\ \cdots a \cdot_A a' \cdots &\sim \cdots aa' \cdots , \\ \cdots b \cdot_B b' \cdots &\sim \cdots bb' \cdots . \end{aligned} \tag{4.21}$$

One, thus, forms quotient of the free monoid $M\left(|A| + |B|\right)$ subjected to the equivalence relation $\sim$ provided in (4.21). Two equivalence classes $[x \ldots y], [x' \ldots y'] \in M\left(|A| + |B|\right) / \sim$ have the straightforward multiplication:

$$[x \ldots y] \cdot [x' \ldots y'] = [x \ldots y x' \ldots y'] . \tag{4.22}$$

The unit in $M\left(|A| + |B|\right) / \sim$ is the equivalence class [-] consisting of the empty word. Clearly,

$$[\text{-}] = [u_A] = [u_B] .$$

The collection of all the equivalence classes thus formed is defined as the coproduct of A and B:

$$A + B = M\left(|A| + |B|\right) / \sim . \tag{4.23}$$

The coproduct injections $i_A : A \to A + B$ and $i_B : B \to A + B$ are simply

$$i_A\left(a\right) = [a] \ , \ i_B\left(b\right) = [b] , \tag{4.24}$$

which can easily be verified to be monoid homomorphism. The UMP of $A + B$ should ensure the existence of a **unique** monoid homomorphism $[f, g] : A + B \to N$ for a given monoid N and monoid homomorphisms $f : A \to N$ and $g : B \to N$ such that the following diagram commutes:

$$\begin{tikzcd} & N & \\ A \arrow[ur, "f"] \arrow[r, "i_A"'] & A + B \arrow[u, dashed, "{[f,g]}"] & B \arrow[ul, "g"'] \arrow[l, "i_B"] \end{tikzcd}$$

Let us now construct $[f, g]$. From the given monoid homomorphisms $f : A \to N$ and $g : B \to N$, one obtains the functions $|f| : |A| \to |N|$ and $|g| : |B| \to |N|$. Now, using the UMP of the coproduct $|A| + |B|$, one knows that there exists a **unique** function $h = [|f|, |g|] : |A| + |B| \to |N|$ such that the following diagram commutes:

$$\begin{tikzcd} & {|N|} & \\ {|A|} \arrow[ur, "{|f|}"] \arrow[r, "i_1"'] & {|A| + |B|} \arrow[u, dashed, "h"] & {|B|} \arrow[ul, "{|g|}"'] \arrow[l, "i_2"] \end{tikzcd}$$

where $i_1 : |A| \to |A| + |B|$ and $i_2 : |B| \to |A| + |B|$ are coproduct injections of the coproduct $|A| + |B|$ in **Sets**. Now, consider the free monoid $M\left(|A| + |B|\right)$ on the set $|A| + |B|$. Invoking the UMP of free monoid, given the monoid N and the function $h : |A| + |B| \to |N|$, there exists a **unique** monoid homomorphism $\overline{h} : M\left(|A| + |B|\right) \to N$ such that the following diagram commutes in **Sets**:

$$\begin{tikzcd}
{|M(|A|+|B|)|} \arrow[r, "\overline{h}"] & {|N|} \\
{|A|+|B|} \arrow[u, "i"] \arrow[ur, "h"'] &
\end{tikzcd}$$

where $i : |A| + |B| \to |M(|A| + |B|)|$ is the insertion of generators. We now need to verify that $\overline{h}$ "respects the equivalence relation $\sim$", i.e. $v \sim w$ implies $\overline{h}(v) = \overline{h}(w)$.

Given $a \in A$,

$$h(a) = h(i_1(a)) = f(a). \tag{4.25}$$

Since $i(a) = a$, so

$$\overline{h}(a) = \overline{h}(i(a)) = h(a) = f(a). \tag{4.26}$$

Similarly, for $b \in B$,

$$\overline{h}(b) = g(b). \tag{4.27}$$

So $\overline{h}(u_A) = \overline{h}(u_B) = u_N$. As a result,

$$\overline{h}(\cdots x u_A y \cdots) = \cdots \overline{h}(x)\,\overline{h}(u_A)\,\overline{h}(y) \cdots = \cdots \overline{h}(x)\,\overline{h}(y) \cdots = \overline{h}(\cdots xy \cdots).$$

Similarly,

$$\overline{h}(\cdots x u_B y \cdots) = \cdots \overline{h}(x)\,\overline{h}(u_B)\,\overline{h}(y) \cdots = \cdots \overline{h}(x)\,\overline{h}(y) \cdots = \overline{h}(\cdots xy \cdots).$$

Since f is a homomorphism,

$$\begin{aligned}
\overline{h}(\cdots a \cdot_A a' \cdots) &= \cdots \overline{h}(a \cdot_A a') \cdots = \cdots f(a \cdot_A a') \cdots \\
&= \cdots f(a)\, f(a') \cdots = \cdots \overline{h}(a)\,\overline{h}(a') \cdots \\
&= \overline{h}(\cdots a a' \cdots).
\end{aligned}$$

Similarly, using the fact that g is a homomorphism,

$$\begin{aligned}
\overline{h}(\cdots b \cdot_B b' \cdots) &= \cdots \overline{h}(b \cdot_B b') \cdots = \cdots g(b \cdot_B b') \cdots \\
&= \cdots g(b)\, g(b') \cdots = \cdots \overline{h}(b)\,\overline{h}(b') \cdots \\
&= \overline{h}(\cdots b b' \cdots).
\end{aligned}$$

Therefore, $v \sim w$ implies $\overline{h}(v) = \overline{h}(w)$. So we have a monoid homomorphism $\widetilde{h} : M(|A| + |B|) / \sim \to N$ defined by

$$\widetilde{h}[w] = \overline{h}(w). \tag{4.28}$$

This is well-defined since $\overline{h}$ is constant on the equivalence classes. Therefore, the following diagram commutes:

$$\begin{tikzcd}
{M(|A|+|B|)} \arrow[r, two heads, "\pi"] \arrow[dr, "\overline{h}"'] & {M(|A|+|B|)/sim} \arrow[d, "\widetilde{h}"] \\
& N
\end{tikzcd}$$

Now, this $\widetilde{h} : M(|A| + |B|) / \sim \to N$ is our desired $[f, g]$. Indeed,

$$\left(\widetilde{h} \circ i_A\right)(a) = \widetilde{h}([a]) = \overline{h}(a) = f(a). \tag{4.29}$$

$$\left(\widetilde{h} \circ i_B\right)(b) = \widetilde{h}([b]) = \overline{h}(b) = g(b). \tag{4.30}$$

Therefore, $\widetilde{h} \circ i_A = f$ and $\widetilde{h} \circ i_B = g$. In other words, the following diagram commutes:

$$\begin{tikzcd}
 & N & \\
A \arrow[ur, "f"] \arrow[r, "i_A"'] & M\left(|A|+|B|\right)/\sim \arrow[u, "\widetilde{h}"] & B \arrow[ul, "g"'] \arrow[l, "i_B"]
\end{tikzcd}$$

Furthermore, $\widetilde{h}$ is unique. If there exists another monoid homomorphism $u : M\left(|A|+|B|\right)/\sim \to N$ such that $u \circ i_A = f$ and $u \circ i_B = g$, then we must have

$$\widetilde{h}\left([a]\right) = f(a) = \left(u \circ i_A\right)(a) = u\left([a]\right), \tag{4.31}$$

and similarly,

$$\widetilde{h}\left([b]\right) = g(b) = \left(u \circ i_B\right)(b) = u\left([b]\right), \tag{4.32}$$

for any $a \in A$ and $b \in B$. So u agrees with $\widetilde{h}$ on the equivalence classes of elements of A and B. Since any element of $M\left(|A|+|B|\right)/\sim$ can be written as a finite product of equivalence classes of elements of A and B, we can conclude that u and $\widetilde{h}$ agrees everywhere in $M\left(|A|+|B|\right)/\sim$. Therefore, $\widetilde{h}$ is indeed the **unique** arrow in **Mon** such that $\widetilde{h} \circ i_A = f$ and $\widetilde{h} \circ i_B = g$. Therefore, $M\left(|A|+|B|\right)/\sim = A+B$, and $\widetilde{h} = [f, g]$.

Abuse of Notation. In Example 4.3, we did not distinguish between $a \in |A|$ and $(a, 1) \in |A|+|B|$; and between $b \in B$ and $(b, 2) \in |A|+|B|$. This can be justified by assuming without loss of generality that $|A|$ and $|B|$ are disjoint. Indeed, if they are not disjoint, we can just replace them by $|A| \times \{1\}$ and $|B| \times \{2\}$.

Example 4.4. Coproduct in groups is called the free product. Suppose A and B are groups. The free product $A * B$ consists of words of the form

$$a_1 b_1 a_2 b_2 \cdots,$$

where $a_i \in A$ and $b_j \in B$. For example,

$$\left(a_1 b_1 a_2 b_2\right) = \left(a_1 b_1\right) * \left(a_2 b_2\right) \text{ and } \left(a_2 b_2 a_1 b_1\right) = \left(a_2 b_2\right) * \left(a_1 b_1\right)$$

are both words in $A * B$, but they are unequal highlighting the non-abelian nature of the free product of $A * B$, even when A and B are both abelian groups. When A and B are both abelian groups, one defines their direct sum as the quotient

$$A \oplus B = A * B/\sim,$$

where the equivalence relation $\sim$ is defined as

$$\left(a_1 b_1 a_2 b_2 \cdots a_n b_n \cdots\right) \sim \left(a_1 a_2 \cdots a_n \cdots b_1 b_2 \cdots b_n \cdots\right). \tag{4.33}$$

On the RHS, all the a_i's are flushed to the left. Now, since A and B are both abelian, so is $A \oplus B$:

$$\left(a_1 b_1\right) * \left(a_2 b_2\right) = a_1 b_1 a_2 b_2 = a_1 a_2 b_1 b_2 = a_2 a_1 b_2 b_1 = a_2 b_2 a_1 b_1 = \left(a_2 b_2\right) * \left(a_1 b_1\right).$$

Thus, using (4.33), one can identify a word in $A \oplus B$ by a pair

$$\left(a_1 + a_2 + \cdots + a_n + \cdots, b_1 + b_2 + \cdots + b_n + \cdots\right) =\equiv (a, b)$$

with $a \in A$ and $b \in B$. But

$$|A \times B| = \{(a, b) \mid a \in A, b \in B\}.$$

Given two abelian groups A and B, their coproduct is defined as their direct sum, which has their product set as the underlying set as we have just seen. Hence, $|A+B| = |A \times B|$, for A, B abelian groups. Then the coproduct injections $i_A : A \to A+B$ and $i_B : B \to A+B$ are defined as

$$i_A(a) = \left(a, 0_B\right) \text{ and } i_B(b) = \left(0_A, b\right). \tag{4.34}$$

Given any group homomorphisms $A \xrightarrow{f} X \xleftarrow{g} B$ for any abelian group X, we construct $[f,g] : A+B \to X$ as follows:

$$[f,g]\,(a,b) = f\,(a) +_X g\,(b)\,. \tag{4.35}$$

Then,

$$\begin{aligned} ([f,g] \circ i_A)\,(a) &= [f,g]\,(a,0_B) = f\,(a) + g\,(0_B) = f\,(a)\,, \\ ([f,g] \circ i_B)\,(b) &= [f,g]\,(0_A,b) = f\,(0_A) + g\,(b) = g\,(b)\,. \end{aligned}$$

Therefore, $[f,g] \circ i_A = f$ and $[f,g] \circ i_B = g$. In other words, the following diagram commutes:

$$\begin{tikzcd}
 & X & \\
A \arrow[ur, "f"] \arrow[r, "i_A"'] & A+B \arrow[u, "{[f,g]}"'] & B \arrow[ul, "g"'] \arrow[l, "i_B"]
\end{tikzcd}$$

Furthermore, $[f,g]$ is the **unique** homomorphism $A+B \to X$ such that the diagram above commutes. If there exists another homomorphism $h : A+B \to X$ such that $h \circ i_A = f$ and $h \circ i_B = g$, then for any $a \in A$ and $b \in B$,

$$h\,(a,0_B) = h\,(i_A\,(a)) = f\,(a) \;\text{ and }\; h\,(0_A,b) = h\,(i_B\,(b)) = g\,(b)\,.$$

Therefore,

$$h\,(a,b) = h\,(a,0_B) + h\,(0_A,b) = f\,(a) + g\,(b) = [f,g]\,(a,b)\,. \tag{4.36}$$

So $[f,g] = h$, and hence $[f,g]$ is unique. Therefore, $A+B$ is indeed the coproduct of A and B.

We have just seen that $|A+B| = |A \times B|$ for A, B abelian groups. In fact, a stronger result holds in the category **Ab** of abelian groups.

Proposition 4.3

In the category **Ab** of abelian groups, there is a canonical isomorphism between the binary product and coproduct, i.e. $A+B \cong A \times B$.

Proof. The goal is to define an arrow $\vartheta : A+B \to A \times B$. In order to do so, one needs first an arrow $A \to A \times B$ (which is determined by an arrow $A \to A$ and an arrow $A \to B$) and another arrow $B \to A \times B$ (which is determined by an arrow $B \to A$ and an arrow $B \to B$). Therefore, we need 4 arrows altogether, which we choose as follows:

$$\mathbb{1}_A : A \to A\,,\; \mathbf{0}_B : A \to B\,,\; \mathbf{0}_A : B \to A\,,\; \mathbb{1}_B : B \to B,$$

where $\mathbf{0}_B : A \to B$ maps all of A to the identity $0_B \in B$, and $\mathbf{0}_A : B \to A$ maps all of B to the identity $0_A \in A$.

$$\begin{tikzcd}
 & A \arrow[dl, "\mathbb{1}_A"'] \arrow[dr, "\mathbf{0}_B"] \arrow[d, dashed, "{\langle \mathbb{1}_A, \mathbf{0}_B \rangle}"] & \\
A & A \times B \arrow[l, "p_1"] \arrow[r, "p_2"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
 & B \arrow[dl, "\mathbf{0}_A"'] \arrow[dr, "\mathbb{1}_B"] \arrow[d, dashed, "{\langle \mathbf{0}_A, \mathbb{1}_B \rangle}"] & \\
A & A \times B \arrow[l, "p_1"] \arrow[r, "p_2"'] & B
\end{tikzcd}$$

$$\begin{tikzcd}
 & A \times B & \\
A \arrow[ur, "{\langle \mathbf{0}_A, \mathbb{1}_B \rangle}"] \arrow[r, "i_A"'] & A+B \arrow[u, dashed, "{[\langle \mathbb{1}_A, \mathbf{0}_B \rangle, \langle \mathbf{0}_A, \mathbb{1}_B \rangle]}"] & B \arrow[ul, "{\langle \mathbb{1}_A, \mathbf{0}_B \rangle}"'] \arrow[l, "i_B"]
\end{tikzcd}$$

Using the commutative diagrams above, we define

$$\vartheta = \left[\langle \mathbb{1}_A, \mathbf{0}_B \rangle, \langle \mathbf{0}_A, \mathbb{1}_B \rangle\right] : A + B \to A \times B. \tag{4.37}$$

Given $(a, b) \in A + B$, one obtains

$$\begin{aligned} \vartheta(a, b) &= \left[\langle \mathbb{1}_A, \mathbf{0}_B \rangle, \langle \mathbf{0}_A, \mathbb{1}_B \rangle\right](a, b) \\ &= \langle \mathbb{1}_A, \mathbf{0}_B \rangle (a) + \langle \mathbf{0}_A, \mathbb{1}_B \rangle (b) \\ &= (\mathbb{1}_A(a), \mathbf{0}_B(a)) + (\mathbf{0}_A(b), \mathbb{1}_B(b)) \\ &= (a, 0_B) + (0_A, b) \\ &= (a, b). \end{aligned} \tag{4.38}$$

Therefore, ϑ is a bijective group homomorphism, and hence an isomorphism. ■

§4.3 Equalizers

Definition 4.2 (Equalizer)**.** In any category $\mathcal{C}$, given parallel arrows $A \underset{g}{\overset{f}{\rightrightarrows}} B$, an **equalizer** of f and g consists of an object E and an arrow $e : E \to A$ *universal* such that $f \circ e = g \circ e$, i.e. the following diagram commutes:

$$\begin{tikzcd} E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

Here $e : E \to A$ is universal in the sense that given $z : Z \to A$ with $f \circ z = g \circ z$, there exists a unique $u : Z \to E$ satisfying $e \circ u = z$. This is encapsulated in the following commutative diagram:

$$\begin{tikzcd} E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\ Z \arrow[u, dashed, "\exists! \, u"] \arrow[ur, "z"'] & & \end{tikzcd}$$

Example 4.5. Suppose one has continuous functions $f, g : \mathbb{R}^2 \to \mathbb{R}$ in **Top** defined by

$$f(x, y) = x^2 + y^2 \text{ and } g(x, y) = 1. \tag{4.39}$$

The equalizer of f and g is the subspace

$$S^1 = \left\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\right\},$$

alog with the inclusion

$$i : S^1 \hookrightarrow \mathbb{R}^2.$$

Given any generalized element $z : Z \to \mathbb{R}^2$ of $\mathbb{R}^2$, we can write

$$z(\alpha) = (z_1(\alpha), z_2(\alpha)),$$

for $\alpha \in Z$. So $z = \langle z_1, z_2 \rangle$. If $f \circ z = g \circ z$, then for any $\alpha \in Z$,

$$1 = g(z(\alpha)) = f(z(\alpha)) = z_1(\alpha)^2 + z_2(\alpha)^2. \tag{4.40}$$

So $(z_1(\alpha), z_2(\alpha)) = z(\alpha) \in S^1$ for every $\alpha \in Z$. Therefore, there is a factorization $z = i \circ \overline{z}$ through some $\overline{z} : Z \to S^1$. $\overline{z}$ is nothing but restricting the codomain of $z : Z \to \mathbb{R}$ to S^1. Then the following diagram commutes:

$$\begin{tikzcd}
S^1 \arrow[r, hook, "e"] & \mathbb{R}^2 \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \mathbb{R} \\
Z \arrow[u, dashed, "\overline{z}"] \arrow[ur, "z"'] & &
\end{tikzcd}$$

Since the inclusion i is monic, such $\overline{z} : Z \to S^1$ is unique. Hence, the unit sphere S^1 along with the inclusion $i : S^1 \hookrightarrow \mathbb{R}^2$ is indeed the equalizer of f and g in **Top**.

Example 4.6. Similarly, in **Sets**, given any functions $f, g : A \to B$, their equalizer is the inclusion of the following subset

$$E = \{x \in A \mid f(x) = g(x)\} \tag{4.41}$$

into A. Then

$$\begin{tikzcd}
E \arrow[r, hook, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B
\end{tikzcd}$$

is a commutative diagram. Furthermore, given any set Z and a function $z : Z \to A$ such that $f \circ z = g \circ z$, we have

$$f(z(\alpha)) = g(z(\alpha)),$$

for any $\alpha \in Z$. So $z(\alpha) \in E$. So we can define a function $\overline{z} : Z \to E$ by just restricting the codomain of z, then the following diagram commutes:

$$\begin{tikzcd}
E \arrow[r, hook, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\
Z \arrow[u, dashed, "\overline{z}"] \arrow[ur, "z"'] & &
\end{tikzcd}$$

Furthermore, $\overline{z}$ is unique since the inclusion $E \hookrightarrow A$ is injective. Therefore, $e : E \hookrightarrow A$ is the equalizer of f and g.

Furthermore, every subset $U \subseteq A$ is of this "equational form", i.e. every subset is an equalizer for some pair of functions. Let us undertake this canonical construction. First, let us put

$$\mathbf{2} = \{0, 1\},$$

thinking of it as the set of "truth values" (or the 2-element Boolean algebra). Then consider the characteristic function $\chi_U : A \to \mathbf{2}$ defined by

$$\chi_U(x) = \begin{cases} 1 & \text{if } x \in U, \\ 0 & \text{if } x \notin U. \end{cases} \tag{4.42}$$

As a result,

$$U = \{x \in A \mid \chi_U(x) = 1\}.$$

We can also define a constant functin $\top : A \to \mathbf{2}$ that takes all of A to $1 \in \mathbf{2}$, i.e.

$$\top(x) = 1, \tag{4.43}$$

for any $x \in A$. Then the following is a commutative diagram:

$$\begin{tikzcd}
U \arrow[r, hook, "i"] & A \arrow[r, shift left, "\top"] \arrow[r, shift right, "\chi_U"'] & \mathbf{2}
\end{tikzcd}$$

In other words, $\top \circ i = \chi_U \circ i$.

On the other hand, let $\varphi : A \to \mathbf{2}$ be given. Form the variety (i.e. "equational subset")

$$V_\varphi = \{x \in A \mid \varphi(x) = 1\} = \varphi^{-1}(\{1\}) . \tag{4.44}$$

Now one can easily verify that the inclusion $i : V_\varphi \hookrightarrow A$ is an equalizer of the parallel arrows $\begin{tikzcd} A \arrow[r, shift left, "\top"] \arrow[r, shift right, "\varphi"'] & \mathbf{2} \end{tikzcd}$, i.e. one can verify that the following diagram is commutative:

$$\begin{tikzcd} V_\varphi \arrow[r, hook, "i"] & A \arrow[r, shift left, "\top"] \arrow[r, shift right, "\varphi"'] & \mathbf{2} \end{tikzcd}$$

Indeed, for any $x \in V_\varphi$,

$$(\varphi \circ i)(x) = \varphi(x) = 1 \text{ and } (\top \circ i)(x) = 1.$$

Let us now show that indeed $\mathcal{P}(A) \cong \mathrm{Hom}_{\mathbf{Sets}}(A, \mathbf{2})$. Let $\vartheta : \mathcal{P}(A) \to \mathrm{Hom}_{\mathbf{Sets}}(A, \mathbf{2})$ be given by

$$\vartheta(U) = \chi_U, \tag{4.45}$$

for any $U \subseteq A$; and $\lambda : \mathrm{Hom}_{\mathbf{Sets}}(A, \mathbf{2}) \to \mathcal{P}(A)$ be given by

$$\lambda(\varphi) = V_\varphi = \varphi^{-1}(\{1\}), \tag{4.46}$$

for any function $\varphi : A \to \mathbf{2}$. Then from (4.45) and (4.46), we get

$$(\vartheta \circ \lambda)(\varphi) = \chi_{\lambda(\phi)} = \chi_{V_\varphi}.$$

$$\chi_{V_\varphi}(x) = \begin{cases} 1 & \text{if } x \in V_\varphi, \\ 0 & \text{if } x \notin V_\varphi \end{cases} = \begin{cases} 1 & \text{if } \varphi(x) = 1, \\ 0 & \text{if } \varphi(x) = 0 \end{cases} = \varphi(x) .$$

Therefore, $\chi_{V_\varphi} = \varphi$, and hence

$$\vartheta \circ \lambda = \mathbb{1}_{\mathrm{Hom}_{\mathbf{Sets}}(A,\mathbf{2})} . \tag{4.47}$$

On the other hand,

$$(\lambda \circ \vartheta)(U) = V_{\vartheta(U)} = V_{\chi_U}.$$

$$V_{\chi_U} = \{x \in A \mid \chi_U(x) = 1\} = \{x \in A \mid x \in U\} = U.$$

Therefore, $V_{\chi_U} = U$, and hence

$$\lambda \circ \vartheta = \mathbb{1}_{\mathcal{P}(A)} . \tag{4.48}$$

From (4.47) and (4.48), one can conclude that

$$\vartheta : \mathcal{P}(A) \to \mathrm{Hom}_{\mathbf{Sets}}(A, \mathbf{2})$$

is an isomorphism, i.e. $\mathcal{P}(A) \cong \mathrm{Hom}_{\mathbf{Sets}}(A, \mathbf{2})$.

Example 4.7. Let G and H be two groups with $h : G \to H$ being a group homomorphism. The kernel of h is the subgroup of G defined by

$$\mathrm{Ker}\, h = \{g \in G \mid h(z) = e_H\},$$

where e_H is the identity element of the group H. Then $\iota : \mathrm{Ker}\, h \hookrightarrow G$ is an equalizer of h and u, where $u : G \to H$ is the trivial group homomorphism, i.e. the homomorphism that maps all of G to the identity element e_H of H.

$$\begin{tikzcd} \mathrm{Ker}\, h = \{g \in G \mid h(g) = e_H\} \arrow[r, "\iota"] & G \arrow[r, shift left, "h"] \arrow[r, shift right, "u"'] & H \end{tikzcd}$$

Indeed, $h \circ \iota = u \circ \iota$. Suppose there is another group A and a homomorphism $a : A \to G$ such that $h \circ a = u \circ a$.

$$\begin{tikzcd}
\operatorname{Ker} h \arrow[r, "\iota"] & G \arrow[r, shift left, "h"] \arrow[r, shift right, "u"'] & H \\
A \arrow[u, dashed, "u"] \arrow[ur, "a"'] & &
\end{tikzcd}$$

Then for any $x \in A$, $h(a(x)) = u(a(x)) = e_H$, so $a(x) \in \operatorname{Ker} h$. So we can define a homomorphism $u : A \to \operatorname{Ker} h$, which is formed by restricting the codomain of a to $\operatorname{Ker} h$. For this u, we have

$$\iota \circ u = a.$$

Furthermore, u is unique since the inclusion $\iota : \operatorname{Ker} h \to G$ is monic. Therefore, $\iota : \operatorname{Ker} h \to G$ is an equalizer of $h, u : G \to H$.

Proposition 4.4

In any category, if $e : E \to A$ is an equalizer of some pair of arrows, then e is monic.

Proof. Suppose $e : E \to A$ is an equalizer of the parallel arrows $f, g : A \to B$. Then $f \circ e = g \circ e$. Suppose $x, y : Z \to E$ such that $e \circ x = e \circ y$.

$$\begin{tikzcd}
E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\
Z \arrow[u, shift left, "x"] \arrow[u, shift right, "y"'] \arrow[ur, "z"'] & &
\end{tikzcd}$$

We want to show that $x = y$. Put $z = e \circ x = e \circ y$. Then

$$f \circ z = f \circ e \circ x = g \circ e \circ x = g \circ z. \tag{4.49}$$

Then from the definition of equalizer, there is a **unique** arrow $u : Z \to E$ such that $e \circ u = z$. But $z = e \circ x = e \circ y$. Therefore, from the uniqueness of u, we have

$$u = x = y. \tag{4.50}$$

Therefore, $e : E \to A$ is monic. ■

Proposition 4.5

Equalizers are unique up to isomorphism. In other words, if $e_1 : E_1 \to A$ and $e_2 : E_2 \to A$ are equalizers of parallel arrows $f, g : A \to B$, then there is an isomorphism $i : E_1 \to E_2$ such that $e_1 = e_2 \circ i$ and $e_2 = e_1 \circ i^{-1}$.

Proof. Since $e_1 : E_1 \to A$ is an equalizer, then $f \circ e_1 = g \circ e_1$. $e_2 : E_2 \to A$ is also an equalizer, so $f \circ e_2 = g \circ e_2$. But since E_1 is an equalizer, by the UMP of equalizer, there exists a **unique** arrow $j : E_2 \to E_1$ such that the following diagram commutes:

$$\begin{tikzcd}
E_1 \arrow[r, "e_1"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\
E_2 \arrow[u, dashed, "j"] \arrow[ur, "e_2"'] & &
\end{tikzcd}$$

In other words,

$$e_1 \circ j = e_2. \tag{4.51}$$

Similarly, since E_2 is an equalizer, by the UMP of equalizer, there exists a **unique** arrow $i : E_1 \to E_2$ such that the following diagram commutes:

$$\begin{tikzcd}
E_2 \arrow[r, "e_2"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\
E_1 \arrow[u, dashed, "i"] \arrow[ur, "e_1"'] & &
\end{tikzcd}$$

In other words,

$$e_2 \circ i = e_1. \tag{4.52}$$

We claim that this $i : E_1 \to E_2$ is our desired isomorphism with inverse $j : E_2 \to E_1$. From (4.51) and (4.52), we get $e_1 = e_2 \circ i = e_1 \circ j \circ i$. So we have the following commutative diagram:

$$\begin{tikzcd}
E_1 \arrow[r, "e_1"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\
E_1 \arrow[u, shift left, "j\circ i"] \arrow[u, shift right, "\mathbb{1}_{E_2}"'] \arrow[ur, "e_1"'] & &
\end{tikzcd}$$

Since $e_1 : E_1 \to A$ is an equalizer, and $f \circ e_1 = g \circ e_1$, there exists a **unique** arrow $u : E_1 \to E_1$ such that $e_1 \circ u = e_1$. But the above diagram shows that there are two such arrows $E_1 \to E_1$, $j \circ i$ and $\mathbb{1}_{E_1}$. Therefore, by the uniqueness, we must have

$$j \circ i = \mathbb{1}_{E_1}\,. \tag{4.53}$$

Similarly, again using (4.51) and (4.52), we get $e_2 = e_1 \circ j = e_2 \circ i \circ j$. So we have the following commutative diagram:

$$\begin{tikzcd}
E_2 \arrow[r, "e_2"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\
E_2 \arrow[u, shift left, "i\circ j"] \arrow[u, shift right, "\mathbb{1}_{E_2}"'] \arrow[ur, "e_2"'] & &
\end{tikzcd}$$

Since $e_2 : E_2 \to A$ is an equalizer, and $f \circ e_2 = g \circ e_2$, there exists a **unique** arrow $v : E_2 \to E_2$ such that $e_2 \circ v = e_2$. But the above diagram shows that there are two such arrows $E_2 \to E_2$, $i \circ j$ and $\mathbb{1}_{E_2}$. Therefore, by the uniqueness, we must have

$$i \circ j = \mathbb{1}_{E_2}\,. \tag{4.54}$$

Combining (4.53) and (4.54), we conclude that $i : E_1 \to E_2$ is an isomorphism and $j = i^{-1}$. Furthermore, $e_1 = e_2 \circ i$ and $e_2 = e_1 \circ j = e_1 \circ i^{-1}$. ■

§4.4 Coequalizer

Definition 4.3 (Coequalizer)**.** For any parallel arrows $f, g : A \to B$ in a category $\mathcal{C}$, a **coequalizer** consists of an object Q and an arrow $q : B \to Q$, *universal* with the property $q \circ f = q \circ g$ as depicted in the following diagram:

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "q"] \arrow[dr, "z"'] & Q \arrow[d, dashed, "\exists!"', "u"] \\
& & Z
\end{tikzcd}$$

> That is, given any object Z and arrow $z : B \to Z$ with $z \circ f = z \circ g$, then there exists a **unique** $u : Q \to Z$ such that $u \circ q = z$.

Consider the statement given in Proposition 4.4: “if $e : E \to A$ is an equalizer of some pair of arrows, then e is monic.” This statement follows from the axioms of ETAC, and hence its dual is also a valid statement in the same category:

Proposition 4.6

In any category, if $q : B \to Q$ is a coequalizer of some pair of arrows, then q is epic.

The statement of Proposition 4.5 —equalizers are unique up to isomorphism— also follows from the axioms of ETAC. So its dual also holds in the same category.

Proposition 4.7

Coqualizers are unique up to isomorphism. In other words, if $q_1 : B \to Q_1$ and $q_2 : B \to Q_2$ are equalizers of parallel arrows $f, g : A \to B$, then there is an isomorphism $i : Q_1 \to Q_2$ such that $q_2 = i \circ q_1$ and $q_1 = i^{-1} \circ q_2$.

Example 4.8. Let $R \subseteq X \times X$ be an equivalence relation on a set X, and consider the diagram

$$\begin{tikzcd} R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & X \end{tikzcd}$$

where the r_i's are the two projections of the inclusion $i : R \hookrightarrow X \times X$, i.e. $r_1 = \pi_1 \circ i$ and $r_2 = \pi_2 \circ i$, with $\pi_1, \pi_2 : X \times X \to X$ are defined as

$$\pi_1 (x_1, x_2) = x_1 \text{ and } \pi_2 (x_1, x_2) = x_2.$$

$$\begin{tikzcd} & R \arrow[dl, "r_1"'] \arrow[d, hook, "i"'] \arrow[dr, "r_2"] & \\ X & X \times X \arrow[l, "\pi_2"] \arrow[r, "\pi_1"'] & X \end{tikzcd}$$

Let X/R be the set of all the equivalence classes of X under the equivalence relation R. The quotient projection $\pi : X \to X/R$ is defined by $x \mapsto [x]$. Then for $(x_1, x_2) \in R$, we have

$$(\pi \circ r_1)(x_1, x_2) = \pi (x_1) = [x_1] \;\text{ and }\; (\pi \circ r_2)(x_1, x_2) = \pi (x_2) = [x_2] \,.$$

Since $(x_1, x_2) \in R$, they belong to the same equivalence class, so $[x_1] = [x_2]$. Therefore,

$$\pi \circ r_1 = \pi \circ r_2. \tag{4.55}$$

Now, suppose in the following diagram we have $f : X \to Z$ such that $f \circ r_1 = f \circ r_2$.

$$\begin{tikzcd} R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & X \arrow[r, "\pi"] \arrow[dr, "f"'] & X/R \arrow[d, dashed, "\overline{f}"] \\ & & Z \end{tikzcd}$$

$f \circ r_1 = f \circ r_2$ means that for any $(x_1, x_2) \in R$,

$$\begin{aligned} & (f \circ r_1)(x_1, x_2) = (f \circ r_2)(x_1, x_2) \\ \Longrightarrow\; & f (r_1 (x_1, x_2)) = f (r_2 (x_1, x_2)) \\ \Longrightarrow\; & f (x_1) = f (x_2) \,. \end{aligned} \tag{4.56}$$

In other words, f "respects the equivalence relation R" in the sense that f is constant on the equivalence classes. So we can define a function $\overline{f} : X/R \to Z$ by

$$\overline{f}\,[x] = f\,(x)\,. \tag{4.57}$$

This is well-defined since f is constant on the equivalence classes. Therefore, we have $\overline{f}\,(\pi\,(x)) = f\,(x)$, i.e.

$$\overline{f} \circ \pi = f. \tag{4.58}$$

Furthermore, $\overline{f}$ is unique since π is surjective.

$$\begin{tikzcd}
R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & X \arrow[r, "\pi"] \arrow[dr, "f"'] & X/R \arrow[d, dashed, "\exists!"', "\overline{f}"] \\
 & & Z
\end{tikzcd}$$

To summarize, given two parallel arrows $r_1, r_2 : R \to X$, we have an arrow $\pi : X \to X/R$ satisfying $\pi \circ r_1 = \pi \circ r_2$. Furthermore, given any arrow $f : X \to Z$ such that $f \circ r_1 = f \circ r_2$, then there exists a **unique** arrow $\overline{f} : X/R \to Z$ such that $\overline{f} \circ \pi = f$. Hence, the object X/R along with the arrow $\pi : X \to X/R$ is the required coequalizer of $\begin{tikzcd} R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & X \end{tikzcd}$.

Example 4.9. Let $\alpha, \beta : A \to B$ be two parallel arrows in **Sets**. Let R be the minimal equivalence relation on B containing all the elements of the form $(\alpha\,(a)\,, \beta\,(a))$ for all $a \in A$. In other words, R is the intersection of all the equivalence relations on B containing all the elements of the form $(\alpha\,(a)\,, \beta\,(a))$.

$$R = \bigcap \{E \mid E \subseteq B \times B \text{ is an equivalence relation, and } (\alpha\,(a)\,, \beta\,(a)) \in E \text{ for all } a \in A\}\,.$$

Then we form the set of equivalence classes of B under this equivalence relation R. The set of equivalence classes is B/R. Then we claim that the coequalizer of $\alpha, \beta : A \to B$ is

$$\gamma : B \to B/R,$$

the canonical quotient map that takes each $b \in B$ to its equivalence class $[b] \in B/R$.

$$\begin{tikzcd}
A \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & B \arrow[r, "\gamma"] \arrow[dr, "\varphi"'] & B/R \arrow[d, dashed, "\psi"] \\
 & & C
\end{tikzcd}$$

Indeed, $\gamma \circ \alpha = \gamma \circ \beta$, since for each $a \in A$, we have $(\alpha\,(a)\,, \beta\,(a)) \in R$. Therefore, $[\alpha\,(a)] = [\beta\,(a)]$. In other words,

$$(\gamma \circ \alpha)\,(a) = (\gamma \circ \beta)\,(a)\,,$$

for each $a \in A$. Therefore,

$$\gamma \circ \alpha = \gamma \circ \beta. \tag{4.59}$$

Now, suppose there is another arrow $\varphi : B \to C$ satisfying $\varphi \circ \alpha = \varphi \circ \beta$. We define another equivalence relation R' on B as follows:

$$(b_1, b_2) \in R' \iff \varphi\,(b_1) = \varphi\,(b_2)\,. \tag{4.60}$$

Then for each $a \in A$, we have

$$(\varphi \circ \alpha)(a) = (\varphi \circ \beta)\,(a) \implies \varphi\,(\alpha\,(a)) = \varphi\,(\beta\,(a))\,.$$

Therefore, $(\alpha(a), \beta(a)) \in R'$ for each $a \in A$. Then by the minimality of R, we must have $R \subseteq R'$. In other words, if $(b_1, b_2) \in R$, or equivalently $[b_1] = [b_2]$ in B/R, then $\varphi(b_1) = \varphi(b_2)$. So we can define a well-defined function $\psi : B/R \to C$ by

$$\psi([b]) = \varphi(b). \tag{4.61}$$

Then we have

$$\varphi(b) = \psi(\gamma(b)) = (\psi \circ \gamma)(b),$$

for each $b \in B$. Therefore,

$$\varphi = \psi \circ \gamma.$$

Furthermore, this ψ is unique. Indeed, if there exists another $\psi' : B/R \to C$ satisfying $\varphi = \psi' \circ \gamma$, then we must have

$$\varphi(b) = (\psi' \circ \gamma)(b) = \psi'(\gamma(b)) = \psi'([b]),$$

for each $b \in B$. Therefore, $\psi' = \psi$, so $\psi : B/R \to C$ is unique. Hence, $\gamma : B \to B/R$ is a coequalizer of $\alpha, \beta : A \to B$ in **Sets**.

Example 4.10. Let $f, g : X \to Y$ be given arrows in **Top**. Then $f, g : X \to Y$ are continuous maps between the topological spaces X and Y, whose underlying sets are $|X|$ and $|Y|$, respectively. Then $|f|, |g| : |X| \to |Y|$ are parallel arrows in **Sets**. As above, we can form the coequalizer of $|f|$ and $|g|$ in **Sets**. Let $|q| : |Y| \to |Q|$ be the coequalizer of $|f|$ and $|g|$ in **Sets**. Then we topologize the set $|Q|$ as follows:

$$V \subseteq |Q| \text{ is open} \iff |q|^{-1}(V) \subseteq |Y| \text{ is open}. \tag{4.62}$$

Let us verify that this indeed forms a topology:

- $\varnothing$ is open, since $|q|^{-1}(\varnothing) = \varnothing$ is open in Y.
- $|Q|$ is open, since $|q|^{-1}(|Q|) = |Y|$ is open in Y.
- Let $\{U_i\}_{i \in I}$ be a collection of open sets in $|Q|$. Then

  $$|q|^{-1}\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} |q|^{-1}(U_i). \tag{4.63}$$

  Since U_i is open, $|q|^{-1}(U_i)$ is open in Y. Therefore, as an union of open sets, $|q|^{-1}\left(\bigcup_{i \in I} U_i\right)$ is also open in Y. Hence, $\bigcup_{i \in I} U_i$ is open in $|Q|$.
- Let U_1 and U_2 be open in $|Q|$. Then

  $$|q|^{-1}(U_1 \cap U_2) = |q|^{-1}(U_1) \cap |q|^{-1}(U_2). \tag{4.64}$$

  Since U_1, U_2 are open, $|q|^{-1}(U_1), |q|^{-1}(U_2)$ are also open in Y. Therefore, as an intersection of finitely many open sets, $|q|^{-1}(U_1 \cap U_2)$ is also open in Y. Hence, $U_1 \cap U_2$ is open in $|Q|$.

Let Q be the topological space whose underlying set is $|Q|$, and the topology is defined as in (4.62). Then the function $q : Y \to Q$ is continuous, since the preimage of every open set in Q is also open in Y. We claim that $q : Y \to Q$ is a coequalizer of $f, g : X \to Y$.

$$\begin{tikzcd}
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "q"] \arrow[dr, "z"'] & Q \arrow[d, dashed, "u"] \\
& & Z
\end{tikzcd}$$

Since $|q|$ is a coequalizer of $|f|, |g| : |X| \to |Y|$ in **Sets**, then

$$|q| \circ |f| = |q| \circ |g| \implies |q \circ f| = |q \circ g| \implies q \circ f = q \circ g. \tag{4.65}$$

Now suppose there is another continuous map $z : Y \to Z$ such that $z \circ f = z \circ g$. Then $|z| \circ |f| = |z| \circ |g|$ in **Sets**. Since $|q|$ is a coequalizer of $|f|, |g| : |X| \to |Y|$, there is a **unique** function $|u| : |Q| \to |Z|$ such that $|u| \circ |q| = |z|$. Now we need to show that $|u|$ defines a continuous map $u : Q \to Z$. Let $U \subseteq |Z|$ be open. We need to show that $|u|^{-1}(U)$ is open in Q.

$$|q|^{-1}\left(|u|^{-1}(U)\right) = |z|^{-1}(U). \tag{4.66}$$

Then $|z|^{-1}(U)$ is open in Y since z is continuous. Therefore, $|q|^{-1}\left(|u|^{-1}(U)\right)$ is open in Y. Then by (4.62), $|u|^{-1}(U)$ is open in Q. Hence, u is open.

Therefore, there exists a **unique** continuous map $u : Q \to Z$ such that $u \circ q = z$. So $q : Y \to Q$ is indeed a coequalizer of $f, g : X \to Y$ in **Top**.

§4.5 Product and Coproduct of an Arbitrary Family of Objects

Similar to the construction of product of two objects, we can form the product of finitely many objects. Given objects $A_1, A_2, A_3, \ldots, A_n$, an object P with arrows $\pi_i : P \to A_i$ for $i = 1, 2, \ldots, n$ is called a product of $A_1, A_2, A_3, \ldots, A_n$ if the following UMP is satisfied: given any object X and arrows $x_i : X \to A_i$ for $i = 1, 2, \ldots, n$, there exists a **unique** arrow $u : X \to P$ such that $x_i = \pi_i \circ u$ for all $i = 1, 2, \ldots, n$.

$$\begin{tikzcd}
 & & P \arrow[dll, "\pi_1"'] \arrow[dl, "\pi_2"] \arrow[d, "\pi_3"] \arrow[drr, "\pi_n"] & & \\
A_1 & A_2 & A_3 & \cdots & A_n \\
 & & X \arrow[ull, "x_1"] \arrow[ul, "x_2"'] \arrow[u, "x_3"'] \arrow[urr, "x_n"'] \arrow[uu, dashed, bend right, "\exists!\, u"'] & &
\end{tikzcd}$$

In other words, there exists a unique $u : X \to P$ such that each of the following triangles commute:

$$\begin{tikzcd}
A_i & P \arrow[l, "\pi_i"'] \\
 & X \arrow[ul, "x_i"] \arrow[u, dashed, "u"']
\end{tikzcd}$$

If such an object P exists, it is unique up to isomorphism. So we write the product of $A_1, \ldots, A_n$ as

$$A_1 \times \cdots \times A_n.$$

If the category admits binary products, then it also admits **any** finite product. One can easily show that

$$(\ldots((A_1 \times A_2) \times A_3) \times \ldots) \times A_n$$

satisfies the UMP of the product $A_1 \times \cdots \times A_n$.

In the similar spirit, we can actually define the product of an arbitrary family of objects. Let $(A_\alpha)_{\alpha \in I}$ be a collection of objects indexed by the index set I. An object P with arrows $\pi_\alpha : P \to A_\alpha$ for $\alpha \in I$ is called a product of $(A_\alpha)_{\alpha \in I}$ if the following UMP is satisfied: given any object X and arrows $x_\alpha : X \to A_\alpha$ for $\alpha \in I$, there exists a **unique** arrow $u : X \to P$ such that $x_\alpha = \pi_\alpha \circ u$ for all $\alpha \in I$. In other words, there exists a unique $u : X \to P$ such that each of the following triangles commute, for each $\alpha \in I$:

$$\begin{tikzcd}
A_\alpha & P \arrow[l, "\pi_\alpha"'] \\
& X \arrow[ul, "x_\alpha"] \arrow[u, dashed, "u"']
\end{tikzcd}$$

If such an object P exists, it is unique up to isomorphism. So we write the product of $(A_\alpha)_{\alpha\in I}$ as

$$\prod_{\alpha\in I} A_\alpha.$$

Now, this index set I can be **any** set, even the empty set. If $I = \varnothing$, we call the product an empty product. The empty product is actually the terminal object of the category. The terminal object 1 satisfies the UMP of the product of and I-indexed family of objects, when $I = \varnothing$. The UMP of product is:

> Given any object X and arrows $x_\alpha : X \to A_\alpha$, there exists a **unique** arrow $u : X \to P$ such that $x_\alpha = \pi_\alpha \circ u$ for all $\alpha \in I$.

When I is the empty set, there aren't any objects, and hence there aren't any projection arrows π_α. So the UMP becomes:

> Given any object X and arrows $x_\alpha : X \to A_\alpha$ (there are no A_α's, so this is satisfied vacuously), there exists a **unique** arrow $u : X \to P$ such that $x_\alpha = \pi_\alpha \circ u$ (there are no x_α's, so this commutativity is also satisfied vacuously) for all $\alpha \in I$.

Therefore, essentially the UMP is

> Given any object X, there exists a **unique** arrow $u : X \to P$.

This is precisely the definition of the terminal object. Therefore, the product of an empty collection of objects is the terminal object of the category. It is also called the **nullary** product.

We can reverse the arrows in the above definitions, and form the coproduct of a finite, or arbitrary, or even empty collection of objects. Given objects $A_1, A_2, A_3, \ldots, A_n$, an object P with arrows $i_j : A_j \to Q$ for $j = 1, 2, \ldots, n$ is called a coproduct of $A_1, A_2, A_3, \ldots, A_n$ if the following UMP is satisfied: given any object X and arrows $x_j : A_j \to X$ for $j = 1, 2, \ldots, n$, there exists a **unique** arrow $u : Q \to X$ such that $u \circ i_j = x_j$ for all $j = 1, 2, \ldots, n$.

$$\begin{tikzcd}
& & Q \arrow[dd, dashed, bend left, "u"] & & \\
A_1 \arrow[urr, "i_1"] \arrow[drr, "x_1"'] & A_2 \arrow[ur, "i_2"'] \arrow[dr, "x_2"] & A_3 \arrow[u, "i_3"] \arrow[d, "x_3"] & \cdots & A_n \arrow[ull, "i_n"'] \arrow[dll, "x_n"] \\
& & X & &
\end{tikzcd}$$

In other words, there exists a unique $u : X \to P$ such that each of the following triangles commute:

$$\begin{tikzcd}
A_j \arrow[r, "i_j"] \arrow[dr, "x_j"'] & P \arrow[d, dashed, "u"] \\
& X
\end{tikzcd}$$

If such an object P exists, it is unique up to isomorphism. So we write the coproduct of $A_1, \dots, A_n$ as

$$A_1 + A_2 + \cdots + A_n.$$

If the category admits binary coproducts, then it also admits **any** finite product. One can easily show that

$$(\dots((A_1 + A_2) + A_3) + \cdots + \dots) + A_n$$

satisfies the UMP of the coproduct $A_1 + \cdots + A_n$.

In the similar spirit, we can actually define the coproduct of an arbitrary family of objects. Let $(A_\alpha)_{\alpha \in I}$ be a collection of objects indexed by the index set I. An object Q with arrows $i_\alpha : A_\alpha \to Q$ for $\alpha \in I$ is called a coproduct of $(A_\alpha)_{\alpha \in I}$ if the following UMP is satisfied: given any object X and arrows $x_\alpha : A_\alpha \to X$ for $\alpha \in I$, there exists a **unique** arrow $u : Q \to X$ such that $x_\alpha = u \circ \pi_\alpha$ for all $\alpha \in I$. In other words, there exists a unique $u : P \to X$ such that each of the following triangles commute, for each $\alpha \in I$:

$$\begin{tikzcd}
A_\alpha \arrow[r, "i_\alpha"] \arrow[dr, "x_\alpha"'] & P \arrow[d, dashed, "u"] \\
 & X
\end{tikzcd}$$

If such an object P exists, it is unique up to isomorphism. So we write the product of $(A_\alpha)_{\alpha \in I}$ as

$$\coprod_{\alpha \in I} A_\alpha.$$

Now, this index set I can be **any** set, even the empty set. If $I = \varnothing$, we call the coproduct an empty coproduct. By duality principle, the empty coproduct is the initial object.

5 Groups and Categories

§5.1 Groups in a Category

A group arises as ab abstraction of the automorphisms of an object. Recall the definition of a concrete category where the objects are some “structured” sets, and the arrows are “structure preserving” maps. In a specific and concrete case, a group G may thus consist of certain arrows for a given object X in a category $\mathcal{C}$:

$$G \subseteq \mathrm{Hom}_{\mathcal{C}}\left(X, X\right).$$

One requires all these arrows to be isomorphisms. But the abstract group concept can also be described directly as an object in a category, equipped with a certain structure.

Definition 5.1. Let $\mathcal{C}$ be a category admitting binary products and a final object 1. A **group** in $\mathcal{C}$ consists of objects and arrows as given below

$$\begin{tikzcd}
G \times G \arrow[r, "m"] & G & G \arrow[l, "i"'] \\
& 1 \arrow[u, "u"] &
\end{tikzcd}$$

satisfying the following conditions:

1. m is associative, i.e. the following diagram commutes:

$$\begin{tikzcd}
(G \times G) \times G \arrow[rr, "\mathcal{I}", "\cong"'] \arrow[d, "m \times \mathbb{1}_G"'] & & G \times (G \times G) \arrow[d, "\mathbb{1}_G \times m"] \\
G \times G \arrow[dr, "m"'] & & G \times G \arrow[dl, "m"] \\
& G &
\end{tikzcd} \tag{5.1}$$

   where $\mathcal{I} : (G \times G) \times G \to G \times (G \times G)$ is the canonical associativity isomorphism for product (the same isomorphism we proved in Theorem 3.13). In other words,

$$m \circ (m \times \mathbb{1}_G) = m \circ (\mathbb{1}_G \circ m) \circ \mathcal{I}. \tag{5.2}$$

2. $u : 1 \to G$ is a unit for m, i.e. both triangles below commute:

$$\begin{tikzcd}
G \arrow[r, "\langle u!{,}\mathbb{1}_G \rangle"] \arrow[d, "\langle \mathbb{1}_G{,}u! \rangle"'] \arrow[dr, "\mathbb{1}_G"] & G \times G \arrow[d, "m"] \\
G \times G \arrow[r, "m"'] & G
\end{tikzcd} \tag{5.3}$$

   Here $u!$ is the “constant arrow” $u! : G \to G$ defined by the following composition:

$$G \xrightarrow{!} 1 \xrightarrow{u} G.$$

So

$$m \circ \langle u!, \mathbb{1}_G \rangle = \mathbb{1}_G = m \circ \langle \mathbb{1}_G, u! \rangle \, . \tag{5.4}$$

3. $i : G \to G$ is an inverse with respect to m, i.e. the both squares of the following diagram commute:

$$\begin{tikzcd}
G \times G \arrow[d, "\mathbb{1}_G \times i"'] & G \arrow[l, "\langle \mathbb{1}_G, \mathbb{1}_G \rangle"'] \arrow[r, "\langle \mathbb{1}_G, \mathbb{1}_G \rangle"] \arrow[d, "u!"] & G \times G \arrow[d, "i \times \mathbb{1}_G"] \\
G \times G \arrow[r, "m"'] & G & G \times G \arrow[l, "m"]
\end{tikzcd} \tag{5.5}$$

In other words,

$$m \circ (\mathbb{1}_G \times i) \circ \langle \mathbb{1}_G, \mathbb{1}_G \rangle = u! = m \circ (i \times \mathbb{1}_G) \circ \langle \mathbb{1}_G, \mathbb{1}_G \rangle \, . \tag{5.6}$$

These commutative diagrams don't seem to manifest the usual group axioms (associativity, unit element, inverse), but they actually do! Let us see them in action.

1. **Associativity:** When G is a (structured) set and x, y, z are arbitrary elements of the underlying set, then the commutativity of (5.1) yields:

$$\begin{aligned}
& m \circ (m \times \mathbb{1}_G) \left((x, y), z \right) = \left(m \circ (\mathbb{1}_G \circ m) \circ \mathcal{I} \right) \left((x, y), z \right) \\
\implies \; & m (x * y, z) = \left(m \circ (\mathbb{1}_G \circ m) \right) (x, (y, z)) \\
\implies \; & m (x * y, z) = m (x, y * z) \\
\implies \; & (x * y) * z = x * (y * z) \, ,
\end{aligned}$$

which is precisely the associativity equation of the binary operation $*$ (here we have written $m(a, b) = a * b$).

2. **Unit element:** When G is a (structured) set, the terminal object 1 is a singleton set. So $u! : G \xrightarrow{!} 1 \xrightarrow{u} G$ is a constant function. Let $e = u!(x)$. Then the commutativity of (5.3) yields:

$$\begin{aligned}
& \left(m \circ \langle u!, \mathbb{1}_G \rangle \right)(x) = \mathbb{1}_G (x) = \left(m \circ \langle \mathbb{1}_G, u! \rangle \right)(x) \\
\implies \; & m \left(u!(x), x \right) = x = m \left(x, u!(x) \right) \\
\implies \; & e * x = x = x * e.
\end{aligned}$$

This suggests that when G is a (structured) set, the element $e = u!(x)$ is the unit of the binary operation $*$.

3. **Inverse:** When G is a (structured) set, the commutativity of (5.5) yields:

$$\begin{aligned}
& \left(m \circ (\mathbb{1}_G \times i) \circ \langle \mathbb{1}_G, \mathbb{1}_G \rangle \right)(x) = u!(x) = \left(m \circ (i \times \mathbb{1}_G) \circ \langle \mathbb{1}_G, \mathbb{1}_G \rangle \right)(x) \\
\implies \; & \left(m \circ (\mathbb{1}_G \times i) \right)(x, x) = e = \left(m \circ (i \times \mathbb{1}_G) \right)(x, x) \\
\implies \; & m \left(x, i(x) \right) = e = m \left(i(x), x \right) \\
\implies \; & x * i(x) = e = i(x) * x.
\end{aligned}$$

Since $e = u!(x)$ is the identity of the binary operation $*$, this equation suggests that $i(x)$ is the inverse element of x.

Definition 5.2 (Group Homomorphism)**.** A homomorphism $h : G \to H$ of groups in $\mathcal{C}$ consists of an arrow $h : G \to H$ in $\mathcal{C}$ such that

1. h preserves m, i.e. the following diagram commutes:

$$\begin{tikzcd} G \times G \arrow[r, "h\times h"] \arrow[d, "m_G"'] & H \times H \arrow[d, "m_H"] \\ G \arrow[r, "h"'] & H \end{tikzcd} \tag{5.7}$$

   In other words,

$$h \circ m_G = m_H \circ (h \times h) \,. \tag{5.8}$$

2. h preserves u, i.e. the following diagram commutes:

$$\begin{tikzcd} G \arrow[r, "h"] & H \\ 1 \arrow[u, "u_G"] \arrow[ur, "u_H"'] & \end{tikzcd} \tag{5.9}$$

   In other words,

$$h \circ u_G = u_H. \tag{5.10}$$

3. h preserves i, i.e. the following diagram commutes:

$$\begin{tikzcd} G \arrow[r, "h"] \arrow[d, "i_G"'] & H \arrow[d, "i_H"] \\ G \arrow[r, "h"'] & H \end{tikzcd} \tag{5.11}$$

   In other words,

$$h \circ i_G = i_H \circ h. \tag{5.12}$$

This definition of homomorphism complies with the usual definition of homomorphism we see while studying group theory. Suppose G and H are now (structured) sets, and take any $x, y \in G$. Commutativity of (5.7) implies

$$\begin{aligned} & (h \circ m_G)(x, y) = (m_H \circ (h \times h))(x, y) \\ \implies & h(x *_G y) = m_H(h(x), h(y)) \\ \implies & h(x *_G y) = h(x) *_H h(y) \,. \end{aligned}$$

When G and H are (structured) sets, the terminal object 1 is a singleton set $\{*\}$. Commutativity of (5.9) gives us

$$\begin{aligned} & (h \circ u_G)(*) = u_H(*) \\ \implies & h(e_G) = e_H, \end{aligned}$$

where e_G and e_H are identities of $*_G$ and $*_H$, respectively. Furthermore, commutativity of (5.11) yields

$$\begin{aligned} & (h \circ i_G)(x) = (i_H \circ h)(x) \\ \implies & h(i_G(x)) = i_H(h(x)) \\ \implies & h\left(x^{-1}\right) = h(x)^{-1} \,. \end{aligned}$$

With the evident identities u, inverses i and multiplications m we thus have a category of groups in $\mathcal{C}$, denoted by

$$\operatorname{Group}(\mathcal{C}) .$$

With this construction,

- A **group** in the usual sense is a group in the category **Sets**.
- A **topological group** is a group in the category **Top**.
- A **Lie group** is a group in the category **Man** of smooth manifolds and smooth maps.
- A **partially ordered group** is a group in the category **Pos**.

Example 5.1. In this example, we focus on Group (**Groups**). So it refers to the diagram

$$\begin{tikzcd}
G \times G \arrow[r, "m"] & G & G \arrow[l, "i"'] \\
 & 1 \arrow[u, "u"] &
\end{tikzcd}$$

in the category of groups subjected to the properties mentioned earlier. The terminal object $1 \in \operatorname{Ob}(\mathbf{Groups})$ is the trivial group consisting of the identity element only. Here, G itself is a group. Let us denote the group operation on G by $\circ_G$. $\circ_G$ induces $\circ_{G\times G}$ on $G \times G$ in the following way:

$$(g_1, g_2) \circ_{G\times G} (h_1, h_2) = (g_1 \circ_G h_1, g_2 \circ_G h_2) . \tag{5.13}$$

Denote $m : G \times G \to G$ by $m(g_1, g_2) = g_1 \star g_2$. Since m is an arrow in **Groups**, it is a group homomorphism. So we must have

$$\begin{aligned}
& m((g_1, g_2) \circ_{G\times G} (h_1, h_2)) = m(g_1, g_2) \circ_G m(h_1, h_2) \\
\Longrightarrow\ & m(g_1 \circ_G h_1, g_2 \circ_G h_2) = (g_1 \star g_2) \circ_G (h_1 \star h_2) \\
\Longrightarrow\ & (g_1 \circ_G h_1) \star (g_2 \circ_G h_2) = (g_1 \star g_2) \circ_G (h_1 \star h_2) .
\end{aligned} \tag{5.14}$$

Write 1° for the unit of G with respect to $\circ_G$, and $1^\star$ for the unit of G wirh respect to $\star$, i.e.

$$g \circ_G 1^\circ = g = 1^\circ \circ_G g \text{ and } g \star 1^\star = g = 1^\star \star g. \tag{5.15}$$

Proposition 5.1 (Eckmann-Hilton argument)

Given any set G equipped with two binary operations $\circ, \star : G \times G \to G$ with units 1° and $1^\star$, respectively, and satisfying (5.14), the following hold:

1. $1^\circ = 1^\star$,
2. $\circ = \star$,
3. The operation $\circ = \star$ is commutative.

Proof. First one has

$$\begin{aligned}
1^\circ &= 1^\circ \circ 1^\circ && [1^\circ \text{ is the unit with respect to } \circ] \\
&= (1^\circ \star 1^\star) \circ (1^\star \star 1^\circ) && [1^\star \text{ is the unit with respect to } \star] \\
&= (1^\circ \circ 1^\star) \star (1^\star \circ 1^\circ) && [\text{Applying (5.14)}] \\
&= 1^\star \star 1^\star && [1^\circ \text{ is the unit with respect to } \circ] \\
&= 1^\star .
\end{aligned}$$

Let us write $1^\circ = 1 = 1^\star$, and observe that for any $x, y \in G$,

$$\begin{aligned} x \circ y &= (x \star 1) \circ (1 \star y) && [1 = 1^\star \text{ is the unit with respect to } \star] \\ &= (x \circ 1) \star (1 \circ y) && [\text{Applying (5.14)}] \\ &= x \star y. && [1 = 1^\circ \text{ is the unit with respect to } \circ] \end{aligned}$$

Thus for $x, y \in G$, let us write $x \circ y = x \cdot y = x \star y$. Finally, we have

$$\begin{aligned} x \cdot y &= (1 \cdot x) \cdot (y \cdot 1) && [1 \text{ is the unit with respect to } \circ = \star = \cdot] \\ &= (1 \cdot y) \cdot (x \cdot 1) && [\text{Applying (5.14)}] \\ &= y \cdot x. && [1 \text{ is the unit with respect to } \cdot] \end{aligned}$$

Therefore, $\cdot$ is commutative. ■

Corollary 5.2

The groups in the category **Groups** are exactly the abelian groups.

Proof. In Proposition 5.1, we have shown that a group in the category **Groups** of groups is abelian. In other words, Group (**Groups**) is a subcategory of **Ab**. One now has to prove the converse, i.e. any abelian group G admits homomorphic group multiplication $m : G \times G \to G$ and homomorphic inverse $i : G \to G$. In other words, for any $g, h \in G \times G$ and $x, y \in G$,

$$m\,(g \circ_{G \times G} h) = m\,(g) \circ_G m\,(h)\,, \text{ and } i\,(x \circ_G y) = i\,(x) \circ_G i\,(y)\,.$$

If G is an abelian group and $g = (g_1, g_2)\,, h = (h_1, h_2) \in G \times G$, then

$$\begin{aligned} m\,(g \circ_{G \times G} h) &= m\,((g_1, g_2) \circ_{G \times G} (h_1, h_2)) \\ &= m\,(g_1 \circ_G h_1, g_2 \circ h_2) \\ &= (g_1 \circ_G h_1) \circ_G (g_2 \circ_G h_2) \\ &= (g_1 \circ_G g_2) \circ_G (h_1 \circ_G h_2) \\ &= m\,(g_1, g_2) \circ_G m\,(h_1, h_2) \\ &= m\,(g) \circ_G m\,(h)\,. \end{aligned} \tag{5.16}$$

Furthermore, for any $x, y \in G$,

$$i\,(x \circ_G y) = (x \circ_G y)^{-1} = y^{-1} \circ_G x^{-1} = x^{-1} \circ_G y^{-1} = i\,(x) \circ_G i\,(y)\,. \tag{5.17}$$

Therefore, m and i are group homomorphisms, i.e. arrows in **Groups**. So any abelian group G is a group in the category **Groups**. Hence, Group (**Groups**) = **Ab**. ■

§5.2 Monoidal Categories

A functor from the product category $\mathcal{C} \times \mathcal{D}$ to a category $\mathcal{E}$ is called a bifunctor. We denote such a functor by T, so that

$$T : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$$

is a functor. Given objects $A \in \mathrm{Ob}\,(\mathcal{C})$ and $B \in \mathrm{Ob}\,(\mathcal{D})$, some object $C \in \mathrm{Ob}\,(\mathcal{E})$ is assigned, i.e.

$$T\,(A, B) = C.$$

To each pair of arrows $\alpha : A \to A'$ in $\mathcal{C}$, and $\beta : B \to B'$ in $\mathcal{D}$, an arrow

$$T\,(\alpha, \beta) : T\,(A, B) \to T\,(A', B')$$

in $\mathcal{E}$ is assigned, such that the following conditions are satisfied:

1. Given objects $A \in \mathrm{Ob}\,(\mathcal{C})$ and $B \in \mathrm{Ob}\,(\mathcal{D})$
$$T\left(\mathbb{1}_A, \mathbb{1}_B\right) = \mathbb{1}_{T(A,B)}\,. \tag{5.18}$$

2. If $A \xrightarrow{\alpha} A' \xrightarrow{\alpha'} A''$ in $\mathcal{C}$ and $B \xrightarrow{\beta} B' \xrightarrow{\beta'} B''$ in $\mathcal{D}$, then in the product category $\mathcal{C} \times \mathcal{D}$, we have
$$\begin{tikzcd}
(A,B) \arrow[r, "(\alpha,\beta)"] \arrow[rr, bend right, "(\alpha',\beta')\circ(\alpha,\beta)=(\alpha'\circ\alpha,\beta'\circ\beta)"'] & (A',B') \arrow[r, "(\alpha',\beta')"] & (A'',B'')
\end{tikzcd}$$
Since $T : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ is considered a functor, it has to respect the composition of arrows, i.e.
$$T\left(\alpha' \circ \alpha, \beta' \circ \beta\right) = T\left(\alpha', \beta'\right) \circ T\left(\alpha, \beta\right). \tag{5.19}$$

Example 5.2. $\mathrm{Hom}\,(-,-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to$ **Sets** is an example of a bifuntor. Suppose A and B are objects of $\mathcal{C}$, and hence objects of $\mathcal{C}^{\mathrm{op}}$. Then this functor takes (A, B) to $\mathrm{Hom}_{\mathcal{C}}\,(A, B)$, i.e.
$$\mathrm{Hom}\,(A,B) = \mathrm{Hom}_{\mathcal{C}}\,(A,B)\,. \tag{5.20}$$

Let $\alpha^{\mathrm{op}} : A \to A'$ be an arrow in $\mathcal{C}^{\mathrm{op}}$ (the corresponding arrow in $\mathcal{C}$ reads $\alpha : A' \to A$). Suppose $\beta : B \to B'$ is another arrow in $\mathcal{C}$. Using these data, let us now construct an arrow $\mathrm{Hom}\,(\alpha^{\mathrm{op}}, \beta) : \mathrm{Hom}_{\mathcal{C}}\,(A,B) \to \mathrm{Hom}\,(A', B')$ in the category **Sets** as follows:
$$\mathrm{Hom}\,(\alpha^{\mathrm{op}}, \beta)\,(\varphi) = \beta \circ \varphi \circ \alpha, \tag{5.21}$$
for $\varphi \in \mathrm{Hom}\,(A, B)$. Let us now verify that the definition in (5.21) satisfies both (5.18) and (5.19).
$$\mathrm{Hom}\,(\mathbb{1}_A^{\mathrm{op}}, \mathbb{1}_B)\,(\varphi) = \mathbb{1}_B \circ \varphi \circ \mathbb{1}_A = \varphi,$$
for any $\varphi \in \mathrm{Hom}_{\mathcal{C}}\,(A, B)$. Therefore,
$$\mathrm{Hom}\,(\mathbb{1}_A^{\mathrm{op}}, \mathbb{1}_B) = \mathbb{1}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}\,. \tag{5.22}$$

So (5.18) is satisfied. Now, let $A \xrightarrow{\alpha_1^{\mathrm{op}}} A' \xrightarrow{\alpha_2^{\mathrm{op}}} A''$ be arrows in $\mathcal{C}^{\mathrm{op}}$ and $B \xrightarrow{\beta_1} B' \xrightarrow{\beta_2} B''$ be arrows in $\mathcal{C}$.

$\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$
$$\begin{tikzcd}
(A,B) \arrow[r, "(\alpha_1^{\mathrm{op}},\beta_1)"] \arrow[rr, bend right, "(\alpha_2^{\mathrm{op}},\beta_2)\circ(\alpha_1^{\mathrm{op}},\beta_1)=(\alpha_2^{\mathrm{op}}\circ\alpha_1^{\mathrm{op}},\beta_2\circ\beta_1)"'] & (A',B') \arrow[r, "(\alpha_2^{\mathrm{op}},\beta_2)"] & (A'',B'')
\end{tikzcd}$$
$$\begin{tikzcd}
{} \arrow[d, "\mathrm{Hom}(-,-)"] \\ {}
\end{tikzcd}$$
Sets
$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[r, "\mathrm{Hom}(\alpha_1^{\mathrm{op}},\beta_1)"] \arrow[rr, bend right, "\mathrm{Hom}(\alpha_2^{\mathrm{op}}\circ\alpha_1^{\mathrm{op}},\beta_2\circ\beta_1)"'] & \mathrm{Hom}_{\mathcal{C}}(A',B') \arrow[r, "\mathrm{Hom}(\alpha_2^{\mathrm{op}},\beta_2)"] & \mathrm{Hom}_{\mathcal{C}}(A'',B'')
\end{tikzcd}$$

Now we need to verify equation (5.19) for the functor $\mathrm{Hom}\,(-,-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to$ **Sets** which entails the commutativity of the diagram in **Sets** given above.
$$\begin{aligned}
\mathrm{Hom}(\alpha_2^{\mathrm{op}} \circ \alpha_1^{\mathrm{op}}, \beta_2 \circ \beta_1)\,(\varphi) &= \mathrm{Hom}\left((\alpha_1 \circ \alpha_2)^{\mathrm{op}}, \beta_2 \circ \beta_1\right)(\varphi) \\
&= \beta_2 \circ \beta_1 \circ \varphi \circ \alpha_1 \circ \alpha_2.
\end{aligned}$$

On the other hand,

$$\begin{aligned}(\mathrm{Hom}(\alpha_2^{\mathrm{op}},\beta_2)\circ\mathrm{Hom}(\alpha_1^{\mathrm{op}},\beta_1))\,(\varphi) &= \mathrm{Hom}(\alpha_2^{\mathrm{op}},\beta_2)\,(\beta_1\circ\varphi\circ\alpha_1)\\ &= \beta_2\circ\beta_1\circ\varphi\circ\alpha_1\circ\alpha_2.\end{aligned}$$

Therefore,

$$\mathrm{Hom}(\alpha_2^{\mathrm{op}}\circ\alpha_1^{\mathrm{op}},\beta_2\circ\beta_1) = \mathrm{Hom}(\alpha_2^{\mathrm{op}},\beta_2)\circ\mathrm{Hom}(\alpha_1^{\mathrm{op}},\beta_1). \tag{5.23}$$

Therefore, $\mathrm{Hom}\,(-,-):\mathcal{C}^{\mathrm{op}}\times\mathcal{C}\to\mathbf{Sets}$ is indeed a bifunctor.

Definition 5.3 (Monoidal Category). A **monoidal category** $\mathcal{B}=\langle\mathcal{B},\square,e,\alpha,\lambda,\rho\rangle$ is a category $\mathcal{B}$ equipped with a bifunctor $\square:\mathcal{B}\times\mathcal{B}\to\mathcal{B}$, an object $e\in\mathrm{Ob}\,(\mathcal{B})$ and three natural isomorphisms α,λ,ρ with the following properties:

- Let $\mathcal{I}:\mathcal{B}\times\mathcal{B}\times\mathcal{B}\to(\mathcal{B}\times\mathcal{B})\times\mathcal{B}$ and $\mathcal{I}':\mathcal{B}\times\mathcal{B}\times\mathcal{B}\to\mathcal{B}\times(\mathcal{B}\times\mathcal{B})$ be the canonical isomorphisms. Then $\alpha:\square\circ(\square\times\mathbb{1}_{\mathcal{B}})\circ\mathcal{I}\Rightarrow\square\circ(\mathbb{1}_{\mathcal{B}}\times\square)\circ\mathcal{I}'$ is a natural isomorphism.

$$\begin{tikzcd}[column sep=huge] \mathcal{B}\times\mathcal{B}\times\mathcal{B} \arrow[r, bend left=50, "\square\circ(\square\times\mathbb{1}_{\mathcal{B}})\circ\mathcal{I}", ""{name=U, below}] \arrow[r, bend right=50, "\square\circ(\mathbb{1}_{\mathcal{B}}\times\square)\circ\mathcal{I}'"', ""{name=D, above}] & \mathcal{B} \arrow[Rightarrow, from=U, to=D, "\alpha"] \end{tikzcd}$$

  That is, given arrows $a\xrightarrow{f}a'$, $b\xrightarrow{g}b'$, $c\xrightarrow{h}c'$ in $\mathcal{B}$, the following diagram commutes:

$$\begin{tikzcd} (a\square b)\square c \arrow[r, "(f\square g)\square h"] \arrow[d, "\alpha_{a,b,c}"'] & (a'\square b')\square c' \arrow[d, "\alpha_{a',b',c'}"] \\ a\square(b\square c) \arrow[r, "f\square(g\square h)"] & a'\square(b'\square c') \end{tikzcd}$$

  Furthermore, the components $\alpha_{a,b,c}:(a\square b)\square c\to a\square(b\square c)$ are all isomorphisms.

- The bifunctor $\square:\mathcal{B}\times\mathcal{B}\to\mathcal{B}$ induces a functor in the second component by keeping the first component e, i.e. $\square\,(e,-):\mathcal{B}\to\mathcal{B}$ is a functor. $\lambda:\square\,(e,-)\Rightarrow\mathbb{1}_{\mathcal{B}}$ is a natural isomorphism.

$$\begin{tikzcd} \mathcal{B} \arrow[r, bend left=50, "\square(e{,}-)", ""{name=U, below}] \arrow[r, bend right=50, "\mathbb{1}_{\mathcal{B}}"', ""{name=D, above}] & \mathcal{B} \arrow[Rightarrow, from=U, to=D, "\lambda"] \end{tikzcd}$$

  That is, given an arrow $a\xrightarrow{f}b$ in $\mathcal{B}$, the following diagram commutes:

$$\begin{tikzcd} e\square a \arrow[r, "\mathbb{1}_e\,\square f"] \arrow[d, "\lambda_a"'] & e\square b \arrow[d, "\lambda_b"] \\ a \arrow[r, "f"'] & b \end{tikzcd}$$

  Furthermore, the components $\lambda_a:e\square a\to a$ are all isomorphisms.

- In the same manner, $\rho:\square\,(-,e)\Rightarrow\mathbb{1}_{\mathcal{B}}$ is a natural isomorphism.

$$\begin{tikzcd} \mathcal{B} \arrow[r, bend left=50, "\square(-{,}e)", ""{name=U, below}] \arrow[r, bend right=50, "\mathbb{1}_{\mathcal{B}}"', ""{name=D, above}] & \mathcal{B} \arrow[Rightarrow, from=U, to=D, "\rho"] \end{tikzcd}$$

That is, given an arrow $a \xrightarrow{f} b$ in $\mathcal{B}$, the following diagram commutes:

$$\begin{tikzcd}
a\Box e \arrow[r, "f\Box \mathbb{1}_e"] \arrow[d, "\rho_a"'] & b\Box e \arrow[d, "\rho_b"] \\
a \arrow[r, "f"'] & b
\end{tikzcd}$$

Furthermore, the components $\rho : a\Box e \to a$ are all isomorphisms.

This data is required to satisfy the following conditions:

- For all $a, b, c, d \in \operatorname{Ob}(\mathcal{B})$, the following diagram commutes:

$$\begin{tikzcd}
 & (a\Box b)\Box(c\Box d) \arrow[dr, "\alpha_{a,b,c\Box d}"] & \\
((a\Box b)\Box c)\Box d \arrow[ur, "\alpha_{a\Box b,c,d}"] \arrow[d, "\alpha_{a,b,c}\Box \mathbb{1}_d"'] & & a\Box(b\Box(c\Box d)) \\
(a\Box(b\Box c))\Box d \arrow[rr, "\alpha_{a,b\Box c,d}"'] & & a\Box((b\Box c)\Box d) \arrow[u, "\mathbb{1}_a \Box \alpha_{b,c,d}"']
\end{tikzcd}$$

- For all $a, c \in \operatorname{Ob}(\mathcal{B})$, the following diagram commutes:

$$\begin{tikzcd}
(a\Box e)\Box c \arrow[r, "\alpha_{a,e,c}"] \arrow[dr, "\rho_a \Box \mathbb{1}_c"'] & a\Box(e\Box c) \arrow[d, "\mathbb{1}_a \Box \lambda_c"] \\
 & a\Box c
\end{tikzcd}$$

- $\lambda_e = \rho_e$.

$\mathcal{B} = \langle \mathcal{B}, \Box, e\rangle$ is called a strict monoidal category when all 3 natural isomorphisms α, λ, ρ are identity natural transformation. For example, any category with finite products can be regarded as a monoidal category with the product as the monoidal product and the terminal object as the unit.

§5.3 The Category of Groups

Let us now focus on the category of groups, i.e. $\operatorname{Group}(\mathbf{Sets})$. Let G and H be two groups with $h : G \to H$ being a group homomorphism. The kernel of h is defined by the equalizer

$$\begin{tikzcd}
\operatorname{Ker} h = \{g \in G \mid h(g) = e_H\} \arrow[r, "\iota"] & G \arrow[r, shift left, "h"] \arrow[r, shift right, "u_H!"'] & H
\end{tikzcd}$$

where, again, we write $u_H! : G \to H$ as a composition:

$$u_H! : G \xrightarrow{!} 1 \xrightarrow{u_H} H,$$

i.e. $u_H! = u_H \circ !$, and e_H is the identity element of H. Also, $\iota : \operatorname{Ker} h \to G$ is the inclusion. We have seen in Example 4.7 that this specification makes $\iota : \operatorname{Ker} h \to G$ an equalizer of h and $u_H!$.

Observe that $\operatorname{Ker} h$ is a normal subgroup of G. Now, if $N \subseteq G$ is a normal subgroup of G, then the inclusion $i : N \hookrightarrow G$ is a monomorphism. We can now construct the coequalizer

$$\begin{tikzcd}
N \arrow[r, shift left, "i"] \arrow[r, shift right, "u_G!"'] & G \arrow[r, "\pi"] & G/N
\end{tikzcd}$$

π is defined by $\pi(g) = [g] = gN$. For $n \in N$,

$$\pi(n) = nN = N = [e_G],$$

where e_G is the identity element of G. So we have $\pi(i(n)) = \pi(u_G!(n))$ for each $n \in N$. Therefore,

$$\pi \circ i = \pi \circ u_G!.$$

It remains to show that π is universal. If $f : G \to H$ satisfies $f \circ i = f \circ u_G!$, in the diagram below,

$$\begin{tikzcd}
N \arrow[r, shift left, "i"] \arrow[r, shift right, "u_G!"'] & G \arrow[r, "\pi"] \arrow[rd, "f"'] & G/N \arrow[d, dashed, "\overline{f}"] \\
& & H
\end{tikzcd}$$

then $f(n) = e_H$ for each $n \in N$. Then there is a **unique** homomorphism $\overline{f} : G/N \to H$ such that $\overline{f} \circ \pi = f$, i.e. the triangle in the diagram above commutes. $\overline{f}$ is defined as follows:

$$\overline{f}([g]) = \overline{f}(gN) = f(g). \tag{5.24}$$

To check the well-definedness of $\overline{f}$, consider $g, h \in G$ such that $[g] = [h]$. Then $gN = hN$, so $h^{-1}g \in N$. Therefore,

$$f(g) = f\left(h \cdot_G h^{-1}g\right) = f(h)\, f\left(h^{-1}g\right) = f(h) \cdot_H e_H = f(h).$$

So $\overline{f}([g]) = \overline{f}([h])$, and hence $\overline{f}$ is well-defined. The uniqueness of $\overline{f}$ follows from the fact that π is an epimorphism. Indeed, if there is another arrow $\overline{f}' : G/N \to H$ such that $\overline{f}' \circ \pi = f$, then we have the following commutative diagram.

$$\begin{tikzcd}
G \arrow[r, "\pi"] \arrow[rd, "f"'] & G/N \arrow[d, shift right, "\overline{f}"'] \arrow[d, shift left, "\overline{f}'"] \\
& H
\end{tikzcd}$$

Then $\overline{f}' \circ \pi = \overline{f} \circ \pi$. Since π is an epi, $\overline{f} = \overline{f}'$, so $\overline{f}$ is unique.

Theorem 5.3

Every group homomorphism $h : G \to H$ has a kernel $\operatorname{Ker} h = h^{-1}(u_H)$ (e_H is the unit element of H), which is a normal subgroup of G, with the property that for any normal subgroup $N \subseteq G$,

$$N \subseteq \operatorname{Ker} h \iff \exists!\, \overline{h} : G/N \to H$$

such that $\overline{h} \circ \pi = h$, as indicated in the following diagram:

$$\begin{tikzcd}
G \arrow[r, "h"] \arrow[d, "\pi"'] & H \\
G/N \arrow[ru, dashed, "\overline{h}"'] &
\end{tikzcd}$$

Proof. (⇒) Suppose $N \subseteq \operatorname{Ker} h$. Consider the following diagram:

$$\begin{tikzcd}
N \arrow[r, shift left, "i"] \arrow[r, shift right, "u_G!"'] & G \arrow[r, "\pi"] \arrow[rd, "h"'] & G/N \arrow[d, dashed, "\overline{h}"] \\
& & H
\end{tikzcd}$$

Since $N \subseteq \operatorname{Ker} h$, one has for each $n \in N$,

$$h\left(i\left(n\right)\right) = h\left(n\right) = e_H = h\left(e_G\right) = h\left(u_G!\left(n\right)\right).$$

Therefore,

$$h \circ i = h \circ u_G!. \tag{5.25}$$

We have seen earlier that $\pi : G \to G/N$ is a coequalizer of the parallel arrows $i, u_G! : N \to G$. Hence, there is a **unique** homomorphism $\overline{h} : G/N \to H$ such that $\overline{h} \circ \pi = h$.

(⇐) Conversely, suppose there is a unique homomorphism $\overline{h} : G/N \to H$ such that $\overline{h} \circ \pi = h$. Then for any $n \in N$,

$$h\left(n\right) = \overline{h}\left(\pi\left(n\right)\right) = \overline{h}\left(\left[n\right]\right) = \overline{h}\left(\left[e_G\right]\right) = e_H. \tag{5.26}$$

Therefore, $n \in \operatorname{Ker} h$, proving that $N \subseteq \operatorname{Ker} h$. ■

Corollary 5.4

Every group homomorphism $h : G \to H$ factors as a quotient followed by an injective homomorphism:

$$\begin{tikzcd}
G \arrow[r, "h"] \arrow[d, two heads, "\pi"'] & H \\
G/\operatorname{Ker} h \arrow[ur, dashed, tail, "\overline{h}"'] &
\end{tikzcd}$$

In other words, $h = \overline{h} \circ \pi$, where $\overline{h}$ is injective. Thus $\overline{h} : G/\operatorname{Ker} h \xrightarrow{\cong} \operatorname{im}\left(h\right) \subseteq H$ is an isomorphism to the subgroup $\operatorname{im}\left(h\right)$.

Proof. Let us take $N = \operatorname{Ker} h$ in Theorem 5.3. Then there exists a **unique** homomorphism $\overline{h} : G/\operatorname{Ker} h \to H$ such that $\overline{h} \circ \pi = h$. So, the following diagram commutes:

$$\begin{tikzcd}
G \arrow[r, "h"] \arrow[d, two heads, "\pi"'] & H \\
G/\operatorname{Ker} h \arrow[ur, "\overline{h}"'] &
\end{tikzcd}$$

Now we need to show that $\overline{h}$ is injective. Choose $\left[x\right], \left[y\right] \in G/\operatorname{Ker} h$ such that $\overline{h}\left[x\right] = \overline{h}\left[y\right]$. Then we have

$$\begin{aligned}
\overline{h}\left(\left[x\right]\right) = \overline{h}\left(\left[y\right]\right) &\implies \overline{h}\left(\pi\left(x\right)\right) = \overline{h}\left(\pi\left(y\right)\right) \\
&\implies h\left(x\right) = h\left(y\right) \\
&\implies h\left(x\right) \cdot \left(h\left(y\right)\right)^{-1} = e_H \\
&\implies h\left(x\right) \cdot h\left(y^{-1}\right) = e_H \\
&\implies h\left(xy^{-1}\right) = e_H \\
&\implies xy^{-1} \in \operatorname{Ker} h \\
&\implies \left[x\right] = \left[y\right].
\end{aligned}$$

Therefore, $\overline{h} : G/\operatorname{Ker} h \to H$ is injective. ■

§5.4 Groups as Categories

A group is a category with one object and every arrow from the object to itself is an isomorphism. If G and H are groups, regarded as categories, then we can consider arbitrary functors between them

$$f : G \to H.$$

It then follows immediately that a functor between groups is exactly the same thing as a group homomorphism.

Consider a functor $R : G \to \mathbf{Vect}_{\mathbb{K}}$ from a group G to the category of finite dimensional $\mathbb{K}$-vector spaces. Since G is a one object category, we denote this object by $*$. Hence, $R(*) =: V$ is a finite dimensional $\mathbb{K}$-vector space. In the category G, an arrow $g : * \to *$ is an isomorphism, so that $R(g) \in \mathrm{GL}(V)$ is an invertible linear operator on V. This is precisely a **linear representation** of G.

We now want to generalize the notions of kernel of a homomorphism and quotient group by a normal subgroup, from groups to arbitrary categories.

Definition 5.4. A congruence on a category $\mathcal{C}$ is an equivalence relation $f \sim g$ on arrows such that

1. $f \sim g$ implies $\mathrm{dom}(f) = \mathrm{dom}(g)$ and $\mathrm{cod}(f) = \mathrm{cod}(g)$.

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \end{tikzcd}$$

2. $f \sim g$ implies $b \circ f \circ a \sim b \circ g \circ a$ for any arrows $a : A \to X$ and $b : Y \to B$, where $X = \mathrm{dom}(f) = \mathrm{dom}(g)$ and $Y = \mathrm{cod}(f) = \mathrm{cod}(g)$.

$$\begin{tikzcd} A \arrow[r, "a"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "b"] & B \end{tikzcd}$$

Remark 5.1. We can also rephrase the definition of congruence as follows: a congruence is a collection of equivalence relations $R_{X,Y}$ for $X, Y \in \mathrm{Ob}(\mathcal{C})$ such that

1. $R_{X,Y} \subseteq \mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}}(X,Y)$ is an equivalence relation (i.e. it's reflexive, symmetric, transitive)
2. Given $(f, g) \in R_{X,Y}$, one has

$$(b \circ f \circ a, b \circ g \circ a) \in R_{\mathrm{dom}(a),\mathrm{cod}(b)} \tag{5.27}$$

for any arrow a, b with $\mathrm{cod}(a) = X$ and $\mathrm{dom}(b) = Y$.

Such equivalence relations clearly exist, as we can just take $R_{X,Y} = \mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}}(X,Y)$. But such equivalence relations are not unique. So **any** equivalence relations $R_{X,Y}$ on $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ satisfying (5.27) will be called a congruence.

Definition 5.5 (Congruence Category)**.** Let $\sim$ be a congruence on the category $\mathcal{C}$, and define the **congruence category** $\mathcal{C}^{\sim}$ by

$$\begin{aligned} \mathrm{Ob}(\mathcal{C}^{\sim}) &= \mathrm{Ob}(\mathcal{C}), \\ \mathrm{Hom}_{\mathcal{C}^{\sim}}(X,Y) &= \{(f,g) \mid f, g \in \mathrm{Hom}_{\mathcal{C}}(X,Y) \text{ and } f \sim g\}, \\ \widetilde{\mathbb{1}}_X &= (\mathbb{1}_X, \mathbb{1}_X), \\ (f', g') \circ (f, g) &= (f' \circ f, g' \circ g). \end{aligned}$$

Note that, in this definition, $\mathrm{Hom}_{\mathcal{C}^{\sim}}(X,Y)$ is nothing but the $R_{X,Y}$ mentioned in Remark 5.1. We need to check the well-definedness of the definition of composition.

Let $(f,g) \in \mathrm{Hom}_{\mathcal{C}^{\sim}}(X,Y)$ and $(f',g') \in \mathrm{Hom}_{\mathcal{C}^{\sim}}(Y,Z)$. We need to show that $(f' \circ f, g' \circ g) \in \mathrm{Hom}_{\mathcal{C}^{\sim}}(X,Z)$, i.e. $f' \circ f \sim g' \circ g$.

$$\begin{tikzcd} X \arrow[r, "\mathbb{1}_X"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, shift left, "f'"] \arrow[r, shift right, "g'"'] & Z \arrow[r, "\mathbb{1}_Z"] & Z \end{tikzcd}$$

$(f,g) \in \mathrm{Hom}_{\mathcal{C}^{\sim}}(X,Y)$ and $(f',g') \in \mathrm{Hom}_{\mathcal{C}^{\sim}}(Y,Z)$ implies $f \sim g$ and $f' \sim g'$. Therefore,

$$\begin{aligned} f' \circ f &= \mathbb{1}_Z \circ f' \circ (f \circ \mathbb{1}_X) \\ &\sim \mathbb{1}_Z \circ g' \circ (f \circ \mathbb{1}_X) && [\text{since } f' \sim g'] \\ &= g' \circ f \circ \mathbb{1}_X \\ &\sim g' \circ g \circ \mathbb{1}_X && [\text{since } f \sim g] \\ &= g' \circ g. && (5.28) \end{aligned}$$

Therefore, $f' \circ f \sim g' \circ g$, so $(f' \circ f, g' \circ g) \in \mathrm{Hom}_{\mathcal{C}^{\sim}}(X,Z)$ and the composition is well-defined.

$$\begin{tikzcd} X \arrow[r, "(f,g)"] \arrow[rr, bend right, "(f',g')\circ(f,g)=(f'\circ f,g'\circ g)"'] & Y \arrow[r, "(f',g')"] & Z \end{tikzcd}$$

There are two obvious projection functors:

$$\begin{tikzcd} \mathcal{C}^{\sim} \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & \mathcal{C} \end{tikzcd}$$

They leave the objects unchanged, i.e. $p_i(X) = X$ for $X \in \mathrm{Ob}(\mathcal{C}^{\sim}) = \mathrm{Ob}(\mathcal{C})$, $i = 1, 2$, and

$$p_1(f,g) = f \text{ and } p_2(f,g) = g. \tag{5.29}$$

Clearly, for $i = 1, 2$,

$$p_i\left(\widetilde{\mathbb{1}}_X\right) = p_i(\mathbb{1}_X, \mathbb{1}_X) = \mathbb{1}_X. \tag{5.30}$$

Furthermore,

$$\begin{aligned} p_1((f',g') \circ (f,g)) &= p_1(f' \circ f, g' \circ g) \\ &= f' \circ f \\ &= p_1(f',g') \circ p_1(f,g). && (5.31) \\ p_2((f',g') \circ (f,g)) &= p_2(f' \circ f, g' \circ g) \\ &= g' \circ g \\ &= p_2(f',g') \circ p_2(f,g). && (5.32) \end{aligned}$$

Therefore, p_1 and p_2 are indeed functors. We now build the quotient category $\mathcal{C}/\sim$ as follows:

$$\begin{gathered} \mathrm{Ob}(\mathcal{C}/\sim) = \mathrm{Ob}(\mathcal{C}) \\ \mathrm{Hom}_{\mathcal{C}/\sim}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(X,Y)/\sim. \end{gathered}$$

The arrows of $\mathcal{C}/\sim$ have the form $[f]$, where f is an arrow in $\mathcal{C}$, and we put

$$[\mathbb{1}_X] = \mathbb{1}_{[X]}, \tag{5.33}$$

for $X \in \mathrm{Ob}(\mathcal{C})$, where the equivalence class $[X]$ consists of X only. Furthermore,

$$[g] \circ [f] = [g \circ f], \tag{5.34}$$

for $\mathrm{cod}(f) = \mathrm{dom}(g)$ in $\mathcal{C}$. We need to verify the well-definedness of this definition. Suppose $[f] = [f']$ and $[g] = [g']$. Then $f \sim f'$ and $g \sim g'$. Therefore, using (5.28), we get $g \circ f \sim g' \circ f'$. So $[g \circ f] = [g' \circ f']$ and the composition in $\mathcal{C}/\sim$ is well-defined.

Consider the quotient functor $\pi : \mathcal{C} \to \mathcal{C}/\sim$ by

$$\pi(X) = [X] \text{ and } \pi(f) = [f], \tag{5.35}$$

for objects X and arrows f. Then the following is a coequalizer diagram in **Cat**:

$$\begin{tikzcd}
\mathcal{C}^{\sim} \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & \mathcal{C} \arrow[r, "\pi"] & \mathcal{C}/\sim
\end{tikzcd}$$

The arrows p_1, p_2, π do not change the objects. So $\pi \circ p_1 = \pi \circ p_2$ clearly holds on objects. Let (f, g) be an arrow in $\mathcal{C}^{\sim}$. Then $f \sim g$, so that $[f] = [g]$. Now,

$$(\pi \circ p_1)(f, g) = \pi(f) = [f],$$
$$(\pi \circ p_2)(f, g) = \pi(g) = [g].$$

Therefore, $\pi \circ p_1 = \pi \circ p_2$ holds at the level of arrows as well. Hence,

$$\pi \circ p_1 = \pi \circ p_2. \tag{5.36}$$

Now, suppose there is another functor $F : \mathcal{C} \to \mathcal{D}$ such that $F \circ p_1 = F \circ p_2$.

$$\begin{tikzcd}
\mathcal{C}^{\sim} \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & \mathcal{C} \arrow[r, "\pi"] \arrow[dr, "F"'] & \mathcal{C}/\sim \arrow[d, dashed, "\overline{F}"] \\
 & & \mathcal{D}
\end{tikzcd} \tag{5.37}$$

Since $F \circ p_1 = F \circ p_2$, given any arrow (f, g) in $\mathcal{C}^{\sim}$, i.e. $f \sim g$, we have

$$(F \circ p_1)(f, g) = (F \circ p_2)(f, g) \implies F(f) = F(g). \tag{5.38}$$

Now we define the functor $\overline{F} : \mathcal{C}/\sim \to \mathcal{D}$ as follows: given any object $[X] \in \operatorname{Ob}(\mathcal{C}/\sim)$, we define

$$\overline{F}([X]) = F(X) \in \operatorname{Ob}(\mathcal{D}); \tag{5.39}$$

and given an arrow $[f] \in \operatorname{Hom}_{\mathcal{C}/\sim}(X, Y)$,

$$\overline{F}([f]) = F(f). \tag{5.40}$$

This definition is well-defined, since for $[f] = [g]$, i.e. $f \sim g$, we have $F(f) = F(g)$.

Now, given any $X \in \operatorname{Ob}(\mathcal{C})$, we have

$$\left(\overline{F} \circ \pi\right)(X) = \overline{F}([X]) = F(X).$$

Furthermore, given any arrow $f \in \operatorname{Hom}_{\mathcal{C}}(X, Y)$,

$$\left(\overline{F} \circ \pi\right)(f) = \overline{F}([f]) = F(f).$$

Therefore, we have

$$\overline{F} \circ \pi = F. \tag{5.41}$$

Therefore, there exists a functor $\overline{F} : \mathcal{C}/\sim \to \mathcal{D}$ such that the triangle of (5.37) commutes. Now we need to show the uniqueness of $\overline{F}$. Suppose there is another functor $G : \mathcal{C}/\sim \to \mathcal{D}$ such that $G \circ \pi = F$.

Given any object $[X] \in \operatorname{Ob}(\mathcal{C}/\sim)$,

$$G([X]) = G(\pi(X)) = F(X) = \overline{F}([X]). \tag{5.42}$$

Furthermore, given an arrow $[f] \in \operatorname{Hom}_{\mathcal{C}/\sim}(X, Y)$,

$$G([f]) = G(\pi(f)) = F(f) = \overline{F}([f]). \tag{5.43}$$

Therefore, $G = \overline{F}$, and hence $\overline{F}$ is unique. In summary, is there is a functor $F : \mathcal{C} \to \mathcal{D}$ such that $F \circ p_1 = F \circ p_2$, then there is a **unique** functor $\overline{F} : \mathcal{C}/\sim \to \mathcal{D}$ such that $\overline{F} \circ \pi = F$, i.e. the diagram (5.37) commutes. So $\pi : \mathcal{C} \to \mathcal{C}/\sim$ is indeed a coequalizer of p_1 and p_2 in **Cat**.

Let us now see how we can use the notion of coequalizer to prove analogous homomorphism theorems for categories. Suppose one has categories $\mathcal{C}$ and $\mathcal{D}$ and a functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$. Then $\mathcal{F}$ defines a congruence $\sim_{\mathcal{F}}$ on $\mathcal{C}$ by setting

$$f \sim_{\mathcal{F}} g \iff \operatorname{dom}(f) = \operatorname{dom}(g),\ \operatorname{cod}(f) = \operatorname{cod}(g),\ \text{and } \mathcal{F}(f) = \mathcal{F}(g)\,. \tag{5.44}$$

Suppose $f \sim_{\mathcal{F}} g$ with $\operatorname{dom}(f) = \operatorname{dom}(g) = X$ and $\operatorname{cod}(f) = \operatorname{cod}(g) = Y$. Then for any arrow $a : A \to X$ and $b : Y \to B$,

$$\mathcal{F}(b \circ f \circ a) = \mathcal{F}(b) \circ \mathcal{F}(f) \circ \mathcal{F}(a) = \mathcal{F}(b) \circ \mathcal{F}(g) \circ \mathcal{F}(a) = \mathcal{F}(b \circ g \circ a)\,.$$

Therefore, $b \circ f \circ a \sim_{\mathcal{F}} b \circ g \circ a$. So $\sim_{\mathcal{F}}$ is indeed a congruence. Then we can form the congruence category $\mathcal{C}^{\sim_{\mathcal{F}}}$, which we call the **kernel** category of the functor $\mathcal{F}$.

$$\begin{tikzcd}
\operatorname{Ker}\mathcal{F} = \mathcal{C}^{\sim_{\mathcal{F}}} \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & \mathcal{C}
\end{tikzcd}$$

The functors $p_1, p_2 : \mathcal{C}^{\sim_{\mathcal{F}}} \to \mathcal{C}$ are defined as (5.29). One then obtains the quotient category as the coequalizer

$$\begin{tikzcd}
\operatorname{Ker}\mathcal{F} = \mathcal{C}^{\sim_{\mathcal{F}}} \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & \mathcal{C} \arrow[r, "\pi"] & \mathcal{C}/\sim_{\mathcal{F}}
\end{tikzcd}$$

where π is defined as (5.35). The quotient category $\mathcal{C}/\sim_{\mathcal{F}}$ then has the following UMP.

Theorem 5.5

Every functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ has a kernel category $\operatorname{Ker}\mathcal{F} = \mathcal{C}^{\sim_{\mathcal{F}}}$, determined by a congruence $\sim_{\mathcal{F}}$ on $\mathcal{C}$ such that given any congruence $\sim$ on $\mathcal{C}$, one has

$$f \sim g \implies f \sim_{\mathcal{F}} g$$

if and only if there is a unique factorization $\widetilde{\mathcal{F}} : \mathcal{C}/\sim \to \mathcal{D}$ as indicated in the following commutative diagram:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\mathcal{F}"] \arrow[d, "\pi"'] & \mathcal{D} \\
\mathcal{C}/\sim \arrow[ru, dashed, "\widetilde{\mathcal{F}}"'] &
\end{tikzcd}$$

In other words, $\mathcal{F} = \widetilde{\mathcal{F}} \circ \pi$.

Proof. (⇒) Suppose that $f \sim g \implies f \sim_{\mathcal{F}} g$. Therefore, $f \sim g$ implies $\mathcal{F}(f) = \mathcal{F}(g)$. Consider the following coequalizer diagram:

$$\begin{tikzcd}
\mathcal{C}^{\sim} \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & \mathcal{C} \arrow[r, "\pi"] \arrow[rd, "\mathcal{F}"'] & \mathcal{C}/\sim \arrow[d, dashed, "\widetilde{\mathcal{F}}"] \\
& & \mathcal{D}
\end{tikzcd}$$

Given an arrow (f, g) in $\mathcal{C}^{\sim}$, we have $f \sim g$. So

$$(\mathcal{F} \circ p_1)(f, g) = \mathcal{F}(f) = \mathcal{F}(g) = (\mathcal{F} \circ p_2)(f, g)\,. \tag{5.45}$$

Therefore, $\mathcal{F} \circ p_1 = \mathcal{F} \circ p_2$. Since $\pi : \mathcal{C} \to \mathcal{C}/\sim$ is a coequalizer of p_1 and p_2 in **Cat**, and $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ is another functor such that $\mathcal{F} \circ p_1 = \mathcal{F} \circ p_2$, there exists a **unique** arrow $\widetilde{\mathcal{F}} : \mathcal{C}/\sim \to \mathcal{D}$ such that the diagram above commutes, i.e. $\mathcal{F} = \widetilde{\mathcal{F}} \circ \pi$.

(⇐) Conversely, suppose there is a unique functor $\widetilde{\mathcal{F}} : \mathcal{C}/\sim \to \mathcal{D}$ such that $\mathcal{F} = \widetilde{\mathcal{F}} \circ \pi$. Let $f \sim g$. Then $[f] = [g]$ in $\mathcal{C}/\sim$. Then we have

$$\begin{aligned} [f] = [g] &\implies \pi(f) = \pi(g) \\ &\implies \widetilde{\mathcal{F}}(\pi(f)) = \widetilde{\mathcal{F}}(\pi(g)) \\ &\implies \mathcal{F}(f) = \mathcal{F}(g) . \end{aligned} \tag{5.46}$$

Therefore, $f \sim_{\mathcal{F}} g$. ■

Corollary 5.6

Every functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ factors as $\mathcal{F} = \widetilde{\mathcal{F}} \circ \pi$, where π is bijective on objects and surjective on Hom-sets (i.e. π is a *full* functor), and $\widetilde{\mathcal{F}}$ is injective on Hom-sets (i.e. $\widetilde{\mathcal{F}}$ is a *faithful* functor):

$$\widetilde{\mathcal{F}}_{A,B} : \mathrm{Hom}_{\mathcal{C}/\sim_{\mathcal{F}}}(A, B) \rightarrowtail \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(A), \mathcal{F}(B)) , \ \forall A, B \in \mathrm{Ob}(\mathcal{C}) .$$

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\mathcal{F}"] \arrow[d, "\pi"'] & \mathcal{D} \\ \mathcal{C}/\sim_{\mathcal{F}} \arrow[ur, dashed, "\widetilde{\mathcal{F}}"'] & \end{tikzcd}$$

Proof. Let us take $\sim = \sim_{\mathcal{F}}$ in Theorem 5.5. Then there exists a **unique** functor $\widetilde{\mathcal{F}} : \mathcal{C}/\sim \to \mathcal{D}$ such that$\mathcal{F} = \widetilde{\mathcal{F}} \circ \pi$. So, the following diagram commutes:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\mathcal{F}"] \arrow[d, "\pi"'] & \mathcal{D} \\ \mathcal{C}/\sim_{\mathcal{F}} \arrow[ur, "\widetilde{\mathcal{F}}"'] & \end{tikzcd}$$

Now, clearly π is bijective on objects since the objects of $\mathcal{C}/\sim_{\mathcal{F}}$ are equivalence classes $[X]$ containing only $X \in \mathrm{Ob}(\mathcal{C})$. Furthermore, π is surjective on Hom-sets, since given any $[f] \in \mathrm{Hom}_{\mathcal{C}/\sim_{\mathcal{F}}}(X, Y)$, $[f] = \pi(f)$. We are only left to show that $\widetilde{\mathcal{F}}$ is injective on Hom-sets.

Take $[f], [g] \in \mathrm{Hom}_{\mathcal{C}/\sim_{\mathcal{F}}}(A, B)$ such that $\widetilde{\mathcal{F}}([f]) = \widetilde{\mathcal{F}}([g])$. Then we have

$$\widetilde{\mathcal{F}}([f]) = \widetilde{\mathcal{F}}(\pi(f)) = \mathcal{F}(f) .$$

A similar computation reveals that $\widetilde{\mathcal{F}}([g]) = \mathcal{F}(g)$. Therefore,

$$\begin{aligned} \widetilde{\mathcal{F}}([f]) = \widetilde{\mathcal{F}}([g]) &\implies \mathcal{F}(f) = \mathcal{F}(g) \\ &\implies f \sim_{\mathcal{F}} g \\ &\implies [f] = [g] . \end{aligned} \tag{5.47}$$

Therefore, $\widetilde{\mathcal{F}}$ is injective on Hom-sets, i.e. faithful. ■

6 Subobjects and Pullbacks

§6.1 Subobjects

Every subset $U \subseteq X$ of a set X occurs as equalizer, and the equalizers are always monomorphisms.

$$\begin{tikzcd} U \arrow[r, hook, "i"] & X \arrow[r, shift left, "\top"] \arrow[r, shift right, "\chi_U"'] & \mathbf{2}, \end{tikzcd}$$

where $\mathbf{2} = \{0, 1\}$, $\top : A \to \mathbf{2}$ that takes all of A to $1 \in \mathbf{2}$, and

$$\chi_U(x) = \begin{cases} 1 & \text{if } x \in U, \\ 0 & \text{if } x \notin U. \end{cases}$$

Since every subset $U \subseteq X$ occurs as equalizer and equalizers are monic, it is natural to treat monos as generalized subsets. For instance, monos in **Groups** can be regarded as *subgroups*, and monos in **Top** can be regarded as *subspaces*. Given a monomorphism $m : M \rightarrowtail X$ in a category $\mathcal{G}$ of structured sets of some sort (call it "gadget"), the image subset

$$\{m(y) \mid y \in M\} = m(M) \subseteq X$$

is a "sub-gadget" of X to which M is isomorphic to via m.

$$m : M \xrightarrow{\cong} m(M) \subseteq X.$$

More generally, we can think of the mono $m : M \rightarrowtail X$ itself as determining a "part" of X, even in categories where the arrows are not necessarily functions.

> **Definition 6.1** (Subobject)**.** A **subobject** of an object X in a category $\mathcal{C}$ is a monomorphism
>
> $$m : M \rightarrowtail X.$$

Given subobjects m and m' of X, a morphism $f : m \to m'$ is an arrow in $\mathcal{C}/X$. If $m : M \rightarrowtail X$ and $m' : M' \rightarrowtail X$ are subobjects of X, then an arrow $f : m \to m'$ is an arrow $f : M \to M'$ such that $m' \circ f = m$, i.e. the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[dr, "m"'] & M' \arrow[d, "m'"] \\ & X \end{tikzcd}$$

Thus we have the category $\mathrm{Sub}_{\mathcal{C}}(X)$ of subobjects of X in $\mathcal{C}$. The objects of $\mathrm{Sub}_{\mathcal{C}}(X)$ are monomorphisms of $\mathcal{C}$ with codomain X. The objects of $\mathcal{C}/X$ are, on the other hand, all arrows of $\mathcal{C}$ with codomain X.

In the construction of $\mathrm{Sub}_{\mathcal{C}}(X)$, since m' is monic, given the following commutative diagram,

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[dr, "m"'] & M' \arrow[d, "m'"] \\ & X \end{tikzcd}$$

there is at most one $f : M \to M'$. In other words, there is at most one element in the hom-set $\mathrm{Hom}_{\mathrm{Sub}_{\mathcal{C}}(X)}(m, m')$. Indeed, if there are $f, f' : m \to m'$ in $\mathrm{Sub}_{\mathcal{C}}(X)$, $f, f' : M \to M'$ are arrows in $\mathcal{C}$ such that $m' \circ f = m' \circ f' = m$.

$$\begin{tikzcd}
M \arrow[r, shift left, "f"] \arrow[r, shift right, "f'"'] \arrow[dr, "m"'] & M' \arrow[d, "m'"] \\
 & X
\end{tikzcd}$$

Since $m' \circ f = m' \circ f'$, and m' is monic, we have $f = f'$. Therefore, there is at most one element in the hom-set $\mathrm{Hom}_{\mathrm{Sub}_{\mathcal{C}}(X)}(m, m')$, so that $\mathrm{Sub}_{\mathcal{C}}(X)$ is a **preorder category**. We define the relation of inclusion of subobjects by

$$m \subseteq m' \iff \text{there exists some } f : m \to m'. \tag{6.1}$$

Finally, we say that m and m' are equivalent, written $m \equiv m'$ if and only if they are isomorphic as subobjects, i.e. $m \subseteq m'$ and $m' \subseteq m$. This holds if and only if f and f' making both triangles below commute:

$$\begin{tikzcd}
M \arrow[r, shift left, "f"] \arrow[dr, "m"'] & M' \arrow[l, shift left, "f'"] \arrow[d, "m'"] \\
 & X
\end{tikzcd}$$

In other words,

$$m' \circ f = m \text{ and } m \circ f' = m'. \tag{6.2}$$

Using (6.2), we get

$$m \circ \mathbb{1}_M = m = m' \circ f = m \circ f' \circ f. \tag{6.3}$$

So $m \circ \mathbb{1}_M = m \circ (f' \circ f)$.

$$\begin{tikzcd}
M \arrow[r, shift left, "f' \circ f"] \arrow[r, shift right, "\mathbb{1}_M"'] & M \arrow[r, "m"] & X
\end{tikzcd}$$

Since m is a monomorphism,

$$f' \circ f = \mathbb{1}_M\,. \tag{6.4}$$

Similarly,

$$f \circ f' = \mathbb{1}_{M'}\,. \tag{6.5}$$

Hence, $M \cong M'$ via f. Thus, we can see that equivalent subobjects ($m \equiv m'$) have isomorphic domains ($M \cong M'$).

Abuse of Notation. We sometimes abuse notation and language by calling M the subobject when the monomorphism $m : M \rightarrowtail X$ is clear.

It is often convenient to pass from the preorder $\mathrm{Sub}_{\mathcal{C}}(X)$ to the poset $\mathrm{Sub}_{\mathcal{C}}(X) / \equiv$ by factoring out the equivalence relation $\equiv$. Notice that two distinct monos m and m' in $\mathrm{Sub}_{\mathcal{C}}(X)$ can be equivalent, i.e. $m \equiv m'$ so that both $m \subseteq m'$ and $m' \subseteq m$ hold with m and m' being distinct. That's why $\mathrm{Sub}_{\mathcal{C}}(X)$ is just a preorder category and not a poset category. As soon as we construct $\mathrm{Sub}_{\mathcal{C}}(X) / \equiv$, distinct m and m' in $\mathrm{Sub}_{\mathcal{C}}(X)$ with $m \equiv m'$ coincide, i.e. in $\mathrm{Sub}_{\mathcal{C}}(X)$, $[m] = [m']$.

Now, in $\mathrm{Sub}_{\mathcal{C}}(X) / \equiv$, for distinct equivalence classes $[m] \neq [m']$ one will have either $[m] \subseteq [m']$ or $[m'] \subseteq [m]$ (or neither) and noth both. Hence. $\subseteq$ is antisymmetric in $\mathrm{Sub}_{\mathcal{C}}(X) / \equiv$ so that $\mathrm{Sub}_{\mathcal{C}}(X) / \equiv$ is a **poset category** with respect to the inclusion $\subseteq$.

Proposition 6.1

In **Sets**, under this notion of subobject, one then has the isomorphism

$$\mathrm{Sub}_{\mathbf{Sets}}(X)/{\equiv} \;\cong\; \mathcal{P}(X).$$

Proof. Let X be a set. If $X = \varnothing$, then there is only one mono $m : A \to \varnothing$, which is the empty function $m : \varnothing \to \varnothing$. So $\mathrm{Sub}_{\mathbf{Sets}}(X)$ has cardinality 1, and hence is isomorphic to $\mathcal{P}(\varnothing) = \{\varnothing\}$. Now suppose $X \neq \varnothing$.

We define a function $\varphi : \mathrm{Sub}_{\mathbf{Sets}}(X)/{\equiv} \to \mathcal{P}(X)$ as follows: given a subobject $m : M \rightarrowtail X$, we define

$$\varphi([m]) = \operatorname{im} m = m(M) \subseteq X. \tag{6.6}$$

Let us check that φ is well-defined. Suppose $m : M \rightarrowtail X$ and $m' : M' \rightarrowtail X$ are subobjects such that $[m] = [m']$. Then $m \equiv m'$. In other words, there is an isomorphism $f : M \to M'$ such that $m = m' \circ f$, i.e. the following diagram commutes

$$\begin{tikzcd}
M \arrow[r, "f", "\cong"'] \arrow[dr, "m"'] & M' \arrow[d, "m'"] \\
 & X
\end{tikzcd}$$

Since f is an isomorphism, $f(M) = M'$. So we have

$$\operatorname{im} m = m(M) = m'(f(M)) = m'(M') = \operatorname{im} m'. \tag{6.7}$$

Therefore,

$$\varphi([m]) = \operatorname{im} m = \operatorname{im} m' = \varphi([m']), \tag{6.8}$$

so that φ is well-defined. Now we define another function $\psi : \mathcal{P}(X) \to \mathrm{Sub}_{\mathbf{Sets}}(X)/{\equiv}$ as follows: given $U \subseteq X$, let $i_U : U \hookrightarrow X$ be the inclusion (which is injective, and hence a mono). Then we define

$$\psi(U) = [i_U]. \tag{6.9}$$

For any $U \in \mathcal{P}(X)$,

$$(\varphi \circ \psi)(U) = \varphi([i_U]) = \operatorname{im} i_U = U.$$

Therefore,

$$\varphi \circ \psi = \mathbb{1}_{\mathcal{P}(X)}. \tag{6.10}$$

Now take any $[m] \in \mathrm{Sub}_{\mathbf{Sets}}(X)/{\equiv}$. Then $m : M \rightarrowtail X$ is injective. We can define another function $\overline{m} : M \to \operatorname{im} m$ by $\overline{m}(a) = m(a)$ for each $a \in M$. $\overline{m}$ is exactly the same as m, with the codomain restricted to image. Then $\overline{m}$ is both injective (because m is injective) and surjective. So $\overline{m}$ is an isomorphism. Now, consider the following diagram:

$$\begin{tikzcd}
M \arrow[r, "\overline{m}", "\cong"'] \arrow[dr, "m"'] & \operatorname{im} M \arrow[d, hook, "i_{\operatorname{im} M}"] \\
 & X
\end{tikzcd}$$

This diagram commutes. Therefore, $m \equiv i_{\operatorname{im} M}$, i.e. $[m] = [i_{\operatorname{im} m}]$. Now,

$$(\psi \circ \varphi)[m] = \psi(\operatorname{im} m) = [i_{\operatorname{im} m}] = [m].$$

Therefore,

$$\psi \circ \varphi = \mathbb{1}_{\mathrm{Sub}_{\mathbf{Sets}}(X)/\equiv}. \tag{6.11}$$

Combining (6.10) and (6.11), we get that $\varphi : \mathrm{Sub}_{\mathbf{Sets}}(X)/{\equiv} \to \mathcal{P}(X)$ is an isomorphism, with inverse $\psi : \mathcal{P}(X) \to \mathrm{Sub}_{\mathbf{Sets}}(X)/{\equiv}$. ■

Suppose m, m' are subobjects of $X \in \mathrm{Ob}(\mathcal{C})$, i.e. $m : M \rightarrowtail X$ and $m' : M' \rightarrowtail X$ are monic. Also, let $f : m \to m'$ be an arrow in $\mathrm{Sub}_{\mathcal{C}}(X)$. Then the following diagram commutes:

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[dr, "m"'] & M' \arrow[d, "m'"] \\
& X
\end{tikzcd}$$

By Proposition 3.5, since m is monic, so is f. Hence, $f \in \mathrm{Sub}_{\mathcal{C}}(M')$.

Given a monic arrow $m : M \rightarrowtail X$ in $\mathcal{C}$, one can construct a functor called **composition functor** $m_* : \mathrm{Sub}_{\mathcal{C}}(M) \to \mathrm{Sub}_{\mathcal{C}}(X)$ by

$$m_*(f) = m \circ f, \tag{6.12}$$

at the level of objects; and at the level of arrows, m_* is defined trivially:

$$m_*(a) = a, \tag{6.13}$$

as can be seen by looking at the following commutative diagram.

$$\begin{tikzcd}
N \arrow[rr, "a"] \arrow[dr, "f"] \arrow[ddr, bend right, "m_*(f)=m\circ f"'] & & P \arrow[dl, "g"'] \arrow[ddl, bend left, "m_*(g)=m\circ g"] \\
& M \arrow[d, "m"] & \\
& X &
\end{tikzcd}$$

If $f : N \rightarrowtail M$ is monic, then $m \circ f : N \to X$ is also monic by Proposition 3.5. Similarly, $m \circ g : P \to X$ is a monomorphism. Now, $a : f \to g$ is an arrow in $\mathrm{Sub}_{\mathcal{C}}(M)$, i.e. $a : N \to P$ is an arrow in $\mathcal{C}$ such that $f = g \circ a$. $m_*(a)$ is supposed to be an arrow from $m_*(f) = m \circ f$ to $m_*(g) = m \circ g$, i.e. $m_*(a) : N \to P$ is an arrow in $\mathcal{C}$ such that

$$m \circ f = m \circ g \circ m_*(a).$$

Choosing $m_*(a) = a$ clearly satisfies our requirement, since $f = g \circ a$.

§6.2 Pullback

Definition 6.2 (Pullback)**.** In any category $\mathcal{C}$, given arrows f, g with $\mathrm{cod}\, f = \mathrm{cod}\, g$,

$$\begin{tikzcd}
& B \arrow[d, "g"] \\
A \arrow[r, "f"'] & C
\end{tikzcd}$$

the pullback of f and g consists of arrows

$$\begin{tikzcd}
P \arrow[r, "p_2"] \arrow[d, "p_1"'] & B \\
A &
\end{tikzcd}$$

such that $f \circ p_1 = g \circ p_2$ with a certain universal property.

$$\begin{tikzcd}
P \arrow[r, "p_2"] \arrow[d, "p_1"'] & B \arrow[d, "g"] \\
A \arrow[r, "f"'] & C
\end{tikzcd}$$

The universal property is as follows: given any $z_1 : Z \to A$ and $z_2 : Z \to B$ with $f \circ z_1 = g \circ z_2$, there is a **unique** arrow $u : Z \to P$ such that $z_1 = p_1 \circ u$ and $z_2 = p_2 \circ u$. In other words, the diagram below commutes:

$$\begin{tikzcd}
Z \arrow[rrd, bend left, "z_2"] \arrow[rdd, bend right, "z_1"'] \arrow[rd, dashed, "u"] & & \\
& P \arrow[r, "p_2"] \arrow[d, "p_1"'] & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}$$

There is a product-style notation that is often used:

$$\begin{tikzcd}
Z \arrow[rrd, bend left, "z_2"] \arrow[rdd, bend right, "z_1"'] \arrow[rd, dashed, "{(z_1,z_2)}"] & & \\
& A \times_C B \arrow[r, "p_2"] \arrow[d, "p_1"'] & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}$$

Pullbacks are clearly unique up to isomorphism as they are given by a universal mapping property.

Example 6.1. If $f : X \to Y$ is a monomorphism, then the following is a pullback diagram:

$$\begin{tikzcd}
X \arrow[r, "\mathbb{1}_X"] \arrow[d, "\mathbb{1}_X"'] & X \arrow[d, "f"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd} \tag{6.14}$$

Clearly, $f \circ \mathbb{1}_X = f \circ \mathbb{1}_X$. Now, suppose there is another object Z with arrows $z_1 : Z \to X$ and $z_2 : Z \to X$ such that $f \circ z_1 = f \circ z_2$.

$$\begin{tikzcd}
Z \arrow[rrd, bend left, "z_2"] \arrow[rdd, bend right, "z_1"'] \arrow[rd, dashed, "z"] & & \\
& X \arrow[r, "\mathbb{1}_X"] \arrow[d, "\mathbb{1}_X"'] & X \arrow[d, "f"] \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}$$

Since f is monic, we must have $z_1 = z_2$. Therefore, the unique arrow $z : Z \to X$ is equal to $z_1 = z_2$. It clearly makes the above diagram commutative. Furthermore, it is unique. If there is another $z' : Z \to X$ commuting the above diagram, then we would have

$$z = z_1 = \mathbb{1}_X \circ z' = z'. \tag{6.15}$$

Therefore, (6.14) is a pullback diagram.

Example 6.2. If $e : X \to Y$ is a split epi with right inverse $s : Y \to X$, i.e. $e \circ s = \mathbb{1}_Y$, and e is **not** an isomorphism, then the following is **not** a pullback diagram:

$$\begin{tikzcd} Y \arrow[r, "\mathbb{1}_Y"] \arrow[d, "s"'] & Y \arrow[d, "\mathbb{1}_Y"] \\ X \arrow[r, "e"'] & Y \end{tikzcd} \tag{6.16}$$

If it were a pullback diagram, then for any object Z with arrows $z_1 : Z \to X$ and $z_2 : Z \to Y$ such that $e \circ z_1 = \mathbb{1}_Y \circ z_2$, there would've existed a unique $z : Z \to Y$ such that $s \circ z = z_1$ and $\mathbb{1}_Y \circ z = z_2$.

$$\begin{tikzcd} Z \arrow[dr, dashed, "z"] \arrow[drr, bend left, "z_2"] \arrow[ddr, bend right, "z_1"'] & & \\ & Y \arrow[r, "\mathbb{1}_Y"] \arrow[d, "s"'] & Y \arrow[d, "\mathbb{1}_Y"] \\ & X \arrow[r, "e"'] & Y \end{tikzcd}$$

Let us now choose $Z = X$, $z_1 = \mathbb{1}_X$ and $z_2 = e$. It clearly satisfies $e \circ z_1 = \mathbb{1}_Y \circ z_2$. So there is a unique $z : X \to X$ such that the following diagram commutes.

$$\begin{tikzcd} X \arrow[dr, dashed, "z"] \arrow[drr, bend left, "e"] \arrow[ddr, bend right, "\mathbb{1}_X"'] & & \\ & Y \arrow[r, "\mathbb{1}_Y"] \arrow[d, "s"'] & Y \arrow[d, "\mathbb{1}_Y"] \\ & X \arrow[r, "e"'] & Y \end{tikzcd}$$

In other words,

$$s \circ z = z_1 = \mathbb{1}_X \quad \text{and} \quad \mathbb{1}_Y \circ z = e. \tag{6.17}$$

The second equation $\mathbb{1}_Y \circ z = e$ gives us that $z = e$. Plugging it into the first equation, we get

$$s \circ e = \mathbb{1}_X\,. \tag{6.18}$$

This clearly violates the hypothesis that e is not an isomorphism. Therefore, (6.16) is **not** a pullback diagram.

Proposition 6.2

In a category with products and equalizers, given a corner of arrows

$$\begin{tikzcd}
 & B \arrow[d, "g"] \\
A \arrow[r, "f"'] & C
\end{tikzcd}$$

Consider the diagram

$$\begin{tikzcd}
E \arrow[rr, "p_2"] \arrow[dr, "e"] \arrow[dd, "p_1"'] & & B \arrow[dd, "g"] \\
 & A \times B \arrow[ur, "\pi_2"'] \arrow[dl, "\pi_1"] & \\
A \arrow[rr, "f"'] & & C
\end{tikzcd}$$

in which e is an equalizer of $f \circ \pi_1$ and $g \circ \pi_2$ and $\pi_1 \circ e = p_1$, $\pi_2 \circ e = p_2$. Then E, p_1, p_2 is a pullback of f and g. Conversely, if E, p_1, p_2 is a pullback of f and g are given as such a pullback, then the arrow

$$e = \langle p_1, p_2 \rangle : E \to A \times B$$

is an equalizer of $f \circ \pi_1$ and $g \circ \pi_2$.

Proof. Suppose that $e : E \to A \times B$ is an equalizer diagram of the parallel arrows $f \circ \pi_1, g \circ \pi_2 : A \times B \to C$.

$$\begin{tikzcd}
E \arrow[r, "e"] & A \times B \arrow[r, shift left, "f\circ\pi_1"] \arrow[r, shift right, "g\circ\pi_2"'] & C
\end{tikzcd}$$

Also, $\pi_1 \circ e = p_1$ and $\pi_2 \circ e = p_2$. We need to show that

$$\begin{tikzcd}
E \arrow[r, "p_2"] \arrow[d, "p_1"'] & B \\
A &
\end{tikzcd}$$

satisfies the universal mapping property of pullback. In other words, we need to show that $f \circ p_1 = g \circ p_2$ holds, and given any object Z and arrows $z_1 : Z \to A$ and $z_2 : Z \to B$ satisfying $f \circ z_1 = g \circ z_2$, there exists a **unique** $u : Z \to E$ such that $p_1 \circ u = z_1$ and $p_2 \circ u = z_2$.

$$\begin{tikzcd}
Z \arrow[drrr, bend left, "z_2"] \arrow[ddrr, bend left, "\langle z_1, z_2 \rangle"] \arrow[dddr, bend right, "z_1"'] \arrow[dr, dashed, "\exists!\, u"'] & & & \\
 & E \arrow[rr, "p_2"] \arrow[dr, "e"] \arrow[dd, "p_1"'] & & B \arrow[dd, "g"] \\
 & & A \times B \arrow[ur, "\pi_2"'] \arrow[dl, "\pi_1"] & \\
 & A \arrow[rr, "f"'] & & C
\end{tikzcd}$$

Since $e : E \to A \times B$ is an equalizer of $f \circ \pi_1, g \circ \pi_2 : A \times B \to C$, we have

$$f \circ \pi_1 \circ e = g \circ \pi_2 \circ e. \tag{6.19}$$

We have $\pi_1 \circ e = p_1$ and $\pi_2 \circ e = p_2$. Therefore,

$$f \circ p_1 = g \circ p_2. \tag{6.20}$$

Since $A \xleftarrow{\pi_1} A \times B \xrightarrow{\pi_2} B$ is a product diagram, for $A \xleftarrow{z_1} A \times B \xrightarrow{z_2} B$, there exists a **unique** $\langle z_1, z_2 \rangle : Z \to A \times B$ such that

$$\pi_1 \circ \langle z_1, z_2 \rangle = z_1 \text{ and } \pi_2 \circ \langle z_1, z_2 \rangle = z_2. \tag{6.21}$$

According to the hypothesis, $f \circ z_1 = g \circ z_2$. Therefore,

$$(f \circ \pi_1) \circ \langle z_1, z_2 \rangle = f \circ z_1 = g \circ z_2 = (g \circ \pi_2) \circ \langle z_1, z_2 \rangle . \tag{6.22}$$

Therefore, $(f \circ \pi_1) \circ \langle z_1, z_2 \rangle = (g \circ \pi_2) \circ \langle z_1, z_2 \rangle$.

$$\begin{tikzcd}
E \arrow[r, "e"] & A \times B \arrow[r, shift left, "f\circ\pi_1"] \arrow[r, shift right, "g\circ\pi_2"'] & C \\
Z \arrow[u, dashed, "u"] \arrow[ur, "\langle z_1,z_2\rangle"'] & &
\end{tikzcd}$$

By the UMP of equalizer, there is a **unique** $u : Z \to E$ such that

$$e \circ u = \langle z_1, z_2 \rangle . \tag{6.23}$$

Now,

$$p_1 \circ u = \pi_1 \circ e \circ u = \pi_1 \circ \langle z_1, z_2 \rangle = z_1, \tag{6.24}$$

$$p_2 \circ u = \pi_2 \circ e \circ u = \pi_2 \circ \langle z_1, z_2 \rangle = z_2. \tag{6.25}$$

Therefore, there exists an arrow $u : Z \to E$ such that $p_1 \circ u = z_1$ and $p_2 \circ u = z_2$. Now we are left to show the uniqueness of u. Suppose there is another arrow $v : Z \to E$ satisfying $p_1 \circ v = z_1$ and $p_2 \circ v = z_2$. Then we have

$$z_1 = p_1 \circ v = \pi_1 \circ e \circ v \text{ and } z_2 = p_2 \circ v = \pi_2 \circ e \circ v. \tag{6.26}$$

$$\begin{tikzcd}
& Z \arrow[dl, "z_1"'] \arrow[d, shift right, "\langle z_1,z_2\rangle"'] \arrow[d, shift left, "e\circ v"] \arrow[dr, "z_2"] & \\
A & A \times B \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & B
\end{tikzcd}$$

Therefore, $e \circ v = \langle z_1, z_2 \rangle$ by the UMP of the product $A \times B$. Then again, by (6.23), $e \circ u = \langle z_1, z_2 \rangle$. As a result,

$$e \circ v = e \circ u. \tag{6.27}$$

$$\begin{tikzcd}
Z \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & E \arrow[r, "e"] & A \times B
\end{tikzcd}$$

But since e is an equalizer, it is monic. Therefore, $u = v$, and hence u is unique. So

$$\begin{tikzcd}
E \arrow[r, "p_2"] \arrow[d, "p_1"'] & B \\
A &
\end{tikzcd}$$

is a pullback of f and g.

Conversely, suppose E, p_1, p_2 is a pullback of f and g, and $e = \langle p_1, p_2 \rangle : E \to A \times B$. Then we need to show that e is an equalizer of $f \circ \pi_1$ and $g \circ \pi_2$.

$$\begin{tikzcd} E \arrow[r, "e=\langle p_1,p_2\rangle"] & A \times B \arrow[r, shift left, "f\circ\pi_1"] \arrow[r, shift right, "g\circ\pi_2"'] & C \end{tikzcd}$$

Since E, p_1, p_2 is a pullback of f and g, we have $f \circ p_1 = g \circ p_2$. So we have

$$(f \circ \pi_1) \circ e = f \circ \pi_1 \circ \langle p_1, p_2 \rangle = f \circ p_1, \tag{6.28}$$

$$(g \circ \pi_2) \circ e = g \circ \pi_2 \circ \langle p_1, p_2 \rangle = g \circ p_2. \tag{6.29}$$

Therefore, $(f \circ \pi_1) \circ e = (g \circ \pi_2) \circ e$. Now suppose there is another arrow $z : Z \to A \times B$ such that $(f \circ \pi_1) \circ z = (g \circ \pi_2) \circ z$. Then we need to show the existence of a **unique** $u : Z \to E$ such that $z = e \circ u$, i.e. the following diagram commutes:

$$\begin{tikzcd} E \arrow[r, "e"] & A \times B \arrow[r, shift left, "f\circ\pi_1"] \arrow[r, shift right, "g\circ\pi_2"'] & C \\ Z \arrow[u, dashed, "u"] \arrow[ur, "z"'] & & \end{tikzcd}$$

Let $\pi_1 \circ z = z_1$ and $\pi_2 \circ z = z_2$. From $(f \circ \pi_1) \circ z = (g \circ \pi_2) \circ z$, we get

$$(f \circ \pi_1) \circ z = (g \circ \pi_2) \circ z \implies f \circ z_1 = g \circ z_2. \tag{6.30}$$

But since E, p_1, p_2 is a pullback of f and g, and $z_1 : Z \to A$ and $z_2 : Z \to B$ satisfies $f \circ z_1 = g \circ z_2$, there exists a **unique** $u : Z \to E$ such that $p_1 \circ u = z_1$ and $p_2 \circ u = z_2$, commuting the following diagram:

$$\begin{tikzcd} Z \arrow[rrrd, bend left, "z_2"] \arrow[dr, dashed, "u"] \arrow[ddrr, bend left, "\langle z_1,z_2\rangle=z"] \arrow[dddr, bend right, "z_1"'] & & & \\ & E \arrow[rr, "p_2"] \arrow[dr, "e"] \arrow[dd, "p_1"'] & & B \arrow[dd, "g"] \\ & & A \times B \arrow[ur, "\pi_2"'] \arrow[dl, "\pi_1"] & \\ & A \arrow[rr, "f"'] & & C \end{tikzcd}$$

Then we have

$$\pi_1 \circ (e \circ u) = \pi_1 \circ \langle p_1, p_2 \rangle \circ u = p_1 \circ u = z_1 \tag{6.31}$$

$$\pi_2 \circ (e \circ u) = \pi_2 \circ \langle p_1, p_2 \rangle \circ u = p_2 \circ u = z_2. \tag{6.32}$$

Therefore,

$$e \circ u = \langle z_1, z_2 \rangle = z. \tag{6.33}$$

So indeed, there exists an arrow $u : Z \to E$ such that $z = e \circ u$. Now we need to show the uniqueness of u. Suppose there is another arrow $v : Z \to E$ such that $e \circ v = z$. Then we have

$$z_1 = \pi_1 \circ z = \pi_1 \circ e \circ v = p_1 \circ v \text{ and } z_2 = \pi_2 \circ z = \pi_2 \circ e \circ v = p_2 \circ v. \tag{6.34}$$

So $p_1 \circ v = z_1$ and $p_2 \circ v = z_2$. But $u : Z \to E$ is the **unique** arrow satisfying $p_1 \circ u = z_1$ and $p_2 \circ u = z_2$. Therefore, $u = v$, proving the uniqueness of u subjected to $z = e \circ u$. Therefore, $e = \langle p_1, p_2 \rangle : E \to A \times B$ is ineed an equalizer of $f \circ \pi_1$ and $g \circ \pi_2$. ■

Corollary 6.3
If a category $\mathcal{C}$ has binary products and equalizers, then it has pullbakcs.

Lemma 6.4 (Two pullbacks lemma)
Consider the commutative diagram below in a category $\mathcal{C}$ with pullbacks:

$$\begin{tikzcd}
F \arrow[r, "f'"] \arrow[d, "h''"'] & E \arrow[r, "g'"] \arrow[d, "h'"] & D \arrow[d, "h"] \\
A \arrow[r, "f"'] & B \arrow[r, "g"'] & C
\end{tikzcd}$$

1. If the two squares are pullbacks, then so is the outer rectangle. Thus
$$A \times_B (B \times_C D) \cong A \times_C D.$$
2. If the right square and the outer rectangle are pullbacks, so is the left square.

Proof. 1. Suppose both the squares are pullbacks. One needs to show that the outer rectangle is a pullback. The outer rectangle is

$$\begin{tikzcd}
F \arrow[r, "g' \circ f'"] \arrow[d, "h''"'] & D \arrow[d, "h"] \\
A \arrow[r, "g \circ f"'] & C
\end{tikzcd}$$

Clearly this square is commutative, since
$$g \circ f \circ h'' = g \circ h' \circ f' = h \circ g' \circ f'. \tag{6.35}$$

Now suppose there is an object Z and arrows $z_1 : Z \to A$ and $z_2 : Z \to D$ such that $g \circ f \circ z_1 = h \circ z_2$. We need to show the existence of a **unique** $v : Z \to F$ such that $h'' \circ v = z_1$ and $g' \circ f' \circ v = z_2$.

$$\begin{tikzcd}
Z \arrow[rrrd, bend left, "z_2"] \arrow[rdd, bend right, "z_1"'] \arrow[rd, dashed, "v"] \arrow[rrd, dashed, "u"] & & & \\
& F \arrow[r, "f'"'] \arrow[d, "h''"'] & E \arrow[r, "g'"] \arrow[d, "h'"] & D \arrow[d, "h"] \\
& A \arrow[r, "f"'] & B \arrow[r, "g"'] & C
\end{tikzcd}$$

$g \circ f \circ z_1 = h \circ z_2$ gives us that
$$g \circ (f \circ z_1) = h \circ z_2. \tag{6.36}$$

Since the right square is a pullback for g and h, there is a **unique** $u : Z \to E$ such that
$$f \circ z_1 = h' \circ u \text{ and } z_2 = g' \circ u. \tag{6.37}$$

Now, $f \circ z_1 = h' \circ u$, and the left square is a pullback of f and h'. Therefore, there exists a **unique** $v : Z \to F$ such that
$$z_1 = h'' \circ v \text{ and } u = f' \circ v. \tag{6.38}$$

Now, we have $h'' \circ v = z_1$, and

$$g' \circ f' \circ v = g' \circ u = z_2. \tag{6.39}$$

Therefore, there **exists** $v : Z \to F$ such that $h'' \circ v = z_1$ and $g' \circ f' \circ v = z_2$. We need to show the uniqueness of v. Suppose there is another $v' : Z \to F$ such that

$$h'' \circ v' = z_1 \text{ and } g' \circ f' \circ v' = z_2. \tag{6.40}$$

$$\begin{tikzcd}
Z \arrow[rrrd, bend left, "z_2"] \arrow[rd, "v'"] \arrow[rrd, "u'=f'\circ v'"] \arrow[rdd, bend right, "z_1"'] & & & \\
& F \arrow[r, "f'"'] \arrow[d, "h''"'] & E \arrow[r, "g'"] \arrow[d, "h'"] & D \arrow[d, "h"] \\
& A \arrow[r, "f"'] & B \arrow[r, "g"'] & C
\end{tikzcd}$$

Now, suppose $u' = f' \circ v'$. Then,

$$g' \circ u' = g' \circ f' \circ v' = z_2, \text{ and} \tag{6.41}$$
$$h' \circ u' = h' \circ f' \circ v' = f \circ h'' \circ v' = f \circ z_1. \tag{6.42}$$

But $u : Z \to E$ is the **unique** arrow such that $f \circ z_1 = h' \circ u$ and $z_2 = g' \circ u$. Therefore, u and u' must coincide.

$$u = u' = f' \circ v'. \tag{6.43}$$

Furthermore, $v : Z \to F$ is the **unique** arrow such that $z_1 = h'' \circ v$ and $u = f' \circ v$. By (6.40) and (6.43), $v' : Z \to F$ also satisfies $h'' \circ v' = z_1$ and $f' \circ v' = u$. Therefore, $v = v'$, proving the uniqueness of v. Therefore, the outer rectangle is also a pullback diagram.

2. Suppose the right square and the outer rectangle are pullbacks. We need to show that the left square is also a pullback diagram. The left square clearly commutes, since the given diagram is a commutative diagram. Now, given another object Z and arrows $z_1 : Z \to A$ and $z_2 : Z \to E$ such that $f \circ z_1 = h' \circ z_2$. We need to show the existence of a **unique** $u : Z \to F$ such that $z_1 = h'' \circ u$ and $z_2 = f' \circ u$.

$$\begin{tikzcd}
Z \arrow[rrrd, bend left, "g'\circ z_2"] \arrow[rrd, bend left, "z_2"] \arrow[rd, dashed, "u"] \arrow[rdd, bend right, "z_1"'] & & & \\
& F \arrow[r, "f'"'] \arrow[d, "h''"'] & E \arrow[r, "g'"] \arrow[d, "h'"] & D \arrow[d, "h"] \\
& A \arrow[r, "f"'] & B \arrow[r, "g"'] & C
\end{tikzcd}$$

The outer rectangle is a pullback, and

$$h \circ (g' \circ z_2) = g \circ h' \circ z_2 = (g \circ f) \circ z_1. \tag{6.44}$$

Therefore, there exists a **unique** $u : Z \to F$ such that

$$h'' \circ u = z_1 \text{ and } g' \circ f' \circ u = g' \circ z_2. \tag{6.45}$$

$$\begin{tikzcd}
Z \arrow[dr, shift left, "z_2"] \arrow[dr, shift right, "f'\circ u"'] \arrow[drr, bend left, "g'\circ z_2"] \arrow[ddr, bend right, "f\circ z_1"'] & & \\
& E \arrow[r, "g'"] \arrow[d, "h'"] & D \arrow[d, "h"] \\
& B \arrow[r, "g"'] & C
\end{tikzcd}$$

We have $g \circ f \circ z_1 = h \circ g' \circ z_2$. Therefore, there exists a **unique** $v : Z \to E$ such that

$$h' \circ v = f \circ z_1 \text{ and } g' \circ v = g' \circ z_2. \tag{6.46}$$

Clearly, taking z_2 in place of v works, since $f \circ z_1$. Therefore, $v = z_2$, by the uniqueness of v. Furthermore,

$$h' \circ f' \circ u = f \circ h'' \circ u = f \circ z_1, g' \circ f' \circ u = g' \circ z_2. \tag{6.47}$$

Therefore, by the uniqueness of v, $v = f' \circ u$. Hence,

$$f' \circ u = z_2. \tag{6.48}$$

So, there exists $u : Z \to F$ such that $h'' \circ u = z_1$ and $f' \circ u = z_2$. Now we need to show the uniqueness of u. Let's say there is another $u' : Z \to F$ satisfying $h'' \circ u' = z_1$ and $f' \circ u' = z_2$.

$$\begin{tikzcd}
Z \arrow[dr, shift left, "u"] \arrow[dr, shift right, "u'"'] \arrow[drr, bend left, "z_2"] \arrow[drrr, bend left=40, "g'\circ z_2"] \arrow[ddr, bend right, "z_1"'] & & & \\
& F \arrow[r, "f'"'] \arrow[d, "h''"'] & E \arrow[r, "g'"] \arrow[d, "h'"] & D \arrow[d, "h"] \\
& A \arrow[r, "f"'] & B \arrow[r, "g"'] & C
\end{tikzcd}$$

Now, $h'' \circ u' = z_1$, and

$$g' \circ f' \circ u' = g' \circ z_2. \tag{6.49}$$

But $u : Z \to F$ is the **unique** arrow satisfying $h'' \circ u = z_1$ and $g' \circ f' \circ u = g' \circ z_2$. Therefore, $u = u'$, proving the uniqueness of u. Hence, the left square is a pullback.

■

Remark 6.1. In practice, we often see a slightly different form of Two pullbacks lemma. In the following commutative diagram,

$$\begin{tikzcd}
F \arrow[r, "h''"] \arrow[d, "f'"'] & A \arrow[d, "f"] \\
E \arrow[r, "h'"] \arrow[d, "g'"'] & B \arrow[d, "g"] \\
D \arrow[r, "h"'] & C
\end{tikzcd}$$

1. if the two squares are pullbacks, then so is the outer rectangle.
2. if the bottom square and the outer rectangle is pullback, so is the top square.

This is equivalent to the version we just proved. This diagram can be obtained by rotating our diagram 90 degrees clockwise, and then reflecting about the diagonal.

Remark 6.2. In the Two pullbacks lemma, if the left square and the outer rectangle are pullbacks, then the right square need not be a pullback. The following is a counterexample: let $e : X \to Y$ be a split epi with right inverse $s : Y \to X$, i.e. $e \circ s = \mathbb{1}_Y$. Furthermore, suppose e is **not** an isomorphism. Then consider the following commutative diagram:

$$\begin{tikzcd}
Y \arrow[r, "\mathbb{1}_Y"] \arrow[d, "\mathbb{1}_Y"'] & Y \arrow[r, "\mathbb{1}_Y"] \arrow[d, "s"] & Y \arrow[d, "\mathbb{1}_Y"] \\
Y \arrow[r, "s"'] & X \arrow[r, "e"'] & Y
\end{tikzcd}$$

This diagram commutes, since $e \circ s = \mathbb{1}_Y$. Furthermore, s has a left inverse, so it is monic. Therefore, the left square is a pullback diagram, by Example 6.1. Also, the outer rectangle is

$$\begin{tikzcd}
Y \arrow[r, "\mathbb{1}_Y"] \arrow[d, "\mathbb{1}_Y"'] & Y \arrow[d, "\mathbb{1}_Y"] \\
Y \arrow[r, "e \circ s = \mathbb{1}_Y"'] & Y
\end{tikzcd}$$

This is also a pullback diagram by Example 6.1, since $\mathbb{1}_Y$ is monic. However, we assumed e is **not** an isomorphism. So by Example 6.2, the right square is **not** a pullback diagram. Therefore, the right square can fail to be a pullback even if the left square and the outer rectangle are pullbacks.

Corollary 6.5

The pullback of a commutative triangle is a commutative triangle. Specifically, given a commutative triangle as on the right end of the following "prism diagram":

$$\begin{tikzcd}
A' \arrow[rr, "h_\alpha"] \arrow[dd, "\alpha'"'] \arrow[dr, dashed, "\gamma'"] & & A \arrow[dr, "\gamma"] \arrow[dd, "\alpha"' near start] & \\
& B' \arrow[rr, crossing over, "h_\beta" near start] \arrow[dl, "\beta'"] & & B \arrow[dl, "\beta"] \\
C' \arrow[rr, "h"'] & & C &
\end{tikzcd} \tag{6.50}$$

for any $h : C' \to C$, if one forms the pullbacks α' and β' as on the left end, i.e. if $C' \xleftarrow{\alpha'} A' \xrightarrow{h_\alpha} A$ is a pullback of $C' \xrightarrow{h} C \xleftarrow{\alpha} A$ and $C' \xleftarrow{\beta'} B' \xrightarrow{h_\beta} B$ is a pullback of $C' \xrightarrow{h} C \xleftarrow{\beta} B$, then there exists a unique $\gamma' : A' \to C'$ making the left end a commutative triangle, and the upper face a commutative rectangle, and indeed a pullback .

Proof. Since $C' \xleftarrow{\alpha'} A' \xrightarrow{h_\alpha} A$ is a pullback of $C' \xrightarrow{h} C \xleftarrow{\alpha} A$, we have the following commutative pullback square:

$$\begin{tikzcd}
A' \arrow[r, "h_\alpha"] \arrow[d, "\alpha'"'] & A \arrow[d, "\alpha"] \\
C' \arrow[r, "h"'] & C
\end{tikzcd}$$

Therefore,

$$h \circ \alpha' = \alpha \circ h_\alpha = \beta \circ \gamma \circ h_\alpha. \tag{6.51}$$

Furthermore, $C' \xleftarrow{\beta'} B' \xrightarrow{h_\beta} B$ is a pullback of $C' \xrightarrow{h} C \xleftarrow{\beta} B$, and $\alpha' : A' \to C'$ and $\gamma \circ h_\alpha : A' \to C$ are arrows such that $h \circ \alpha' = \beta \circ \gamma \circ h_\alpha$.

$$\begin{tikzcd}
A' \arrow[rrd, bend left, "\gamma\circ h_\alpha"] \arrow[rd, dashed, "\gamma'"] \arrow[rdd, bend right, "\alpha'"'] & & \\
& B' \arrow[r, "h_\beta"'] \arrow[d, "\beta'"] & B \arrow[d, "\beta"] \\
& C' \arrow[r, "h"'] & C
\end{tikzcd}$$

Therefore, there exists a **unique** $\gamma' : A' \to B'$ such that

$$\alpha' = \beta' \circ \gamma' \text{ and } \gamma \circ h_\alpha = h_\beta \circ \gamma'. \tag{6.52}$$

So γ' is the unique arrow making the left end triangle and the upper face of the prism diagram commutative. Now, the prism diagram is equivalent to the following commutative diagram:

$$\begin{tikzcd}
A' \arrow[r, "h_\alpha"] \arrow[d, "\gamma'"] \arrow[dd, bend right=50, "\alpha'=\beta'\circ\gamma'"'] & A \arrow[d, "\gamma"'] \arrow[dd, bend left=50, "\alpha=\beta\circ\gamma"] \\
B' \arrow[r, "h_\beta"'] \arrow[d, "\beta'"] & B \arrow[d, "\beta"'] \\
C' \arrow[r, "h"'] & C
\end{tikzcd}$$

The outer rectangle and the bottom square are pullbacks. Therefore, by Two pullbacks lemma, the top square is also a pullback. ■

Proposition 6.6

Pullback is a functor, i.e. for fixed $h : C' \to C$ in a category $\mathcal{C}$ with pullbacks, there is a functor $h^* : \mathcal{C}/C \to \mathcal{C}/C'$ defined by

$$\left(A \xrightarrow{\alpha} C\right) \mapsto \left(C' \times_C A \xrightarrow{\alpha'} C'\right),$$

where α' is the pullback of α along h. The effect of the functor h^* on an arrow $\gamma : \alpha \to \beta$ in $\mathcal{C}/C$ is given by the pullback diagram of a commutative triangle (6.50): $h^*(\gamma) = \gamma'$, whose unique existence is guaranted by Corollary 6.5.

Proof. Given an object $\alpha \in \mathrm{Ob}(\mathcal{C}/C)$, we must check that $h^*(\mathbb{1}_\alpha) = \mathbb{1}_{h^*(\alpha)}$. Given $\alpha : A \to C$ in $\mathcal{C}$, it is an object in $\mathcal{C}/C$. The arrow $\mathbb{1}_\alpha$ in $\mathcal{C}/C$ is the arrow $\mathbb{1}_A$ in $\mathcal{C}$. Then the following is a commutative triangle in $\mathcal{C}$.

$$\begin{tikzcd}
A \arrow[dr, "\mathbb{1}_A"] \arrow[dd, "\alpha"'] & \\
& A \arrow[dl, "\alpha"] \\
C &
\end{tikzcd}$$

Consider the pullback of this commutative triangle along $h : C' \to C$.

$$\begin{tikzcd}
C' \times_C A = A' \arrow[rr, "h_\alpha"] \arrow[dr, dashed, "h^*(\mathbb{1}_\alpha)"] \arrow[dd, "\alpha'"'] & & A \arrow[dr, "\mathbb{1}_A"] \arrow[dd, "\alpha"' near start] & \\
& A' \arrow[rr, "h_\alpha" near start] \arrow[dl, "\alpha'"] & & A \arrow[dl, "\alpha"] \\
C' \arrow[rr, "h"'] & & C &
\end{tikzcd}$$

By Corollary 6.5 $h^*(\mathbb{1}_\alpha)$ is the unique arrow $A' \to A'$ such that the left end triangle and the upper face of the prism are commutative. In other words,

$$\alpha' = \alpha' \circ h^*(\mathbb{1}_\alpha) \text{ and } h_\alpha \circ h^*(\mathbb{1}_\alpha) = \mathbb{1}_A \circ h_\alpha. \tag{6.53}$$

One can easily check that $\mathbb{1}_{A'}$ clearly satisfies this requirement.

$$\begin{tikzcd}
A' \arrow[r, "\mathbb{1}_{A'}"] \arrow[d, "\alpha'"'] & A' \arrow[dl, "\alpha'"] \\
C' &
\end{tikzcd}
\qquad
\begin{tikzcd}
A' \arrow[r, "h_\alpha"] \arrow[d, "\mathbb{1}_{A'}"'] & A \arrow[d, "\mathbb{1}_A"] \\
A' \arrow[r, "h_\alpha"'] & A
\end{tikzcd}$$

Indeed, $\alpha' = \alpha' \circ \mathbb{1}_{A'}$, and

$$h_\alpha \circ \mathbb{1}_{A'} = h_\alpha = \mathbb{1}_A \circ h_\alpha. \tag{6.54}$$

Therefore, by the uniqueness of $h^*(\mathbb{1}_\alpha)$, $h^*(\mathbb{1}_\alpha) = \mathbb{1}_{A'}$, which is the idenitity arrow of $\alpha' = h^*(\alpha)$ in the slice category $\mathcal{C}/C'$. So

$$h^*(\mathbb{1}_\alpha) = \mathbb{1}_{\alpha'} = \mathbb{1}_{h^*(\alpha)}\,. \tag{6.55}$$

Let $\gamma_1 : \alpha_1 \to \alpha_2, \gamma_2 : \alpha_2 \to \alpha_3$ be arrows in $\mathcal{C}/C$. We have to verify that $h^*(\gamma_2 \circ \gamma_1) = h^*(\gamma_2) \circ h^*(\gamma_1)$. In other words, we need to check that $(\gamma_2 \circ \gamma_1)' = \gamma_2' \circ \gamma_1'$.

Suppose $\alpha_1 : A \to C$, $\alpha_2 : B \to C$, $\alpha_3 : D \to C$ in $\mathcal{C}$. Then the following is a commutative diagram in $\mathcal{C}$.

$$\begin{tikzcd}
A \arrow[dr, "\gamma_1"] \arrow[dd, "\alpha_1"'] & \\
& B \arrow[dl, "\alpha_2"'] \arrow[dd, "\gamma_2"] \\
C & \\
& D \arrow[ul, "\alpha_3"]
\end{tikzcd}$$

γ_1' and γ_2' are the **unique** arrows such that the left triangles and upper faces of the following prism diagrams commute.

$$\begin{tikzcd}
A' \arrow[rr, "h_{\alpha_1}"] \arrow[dr, dashed, "\gamma_1'"] \arrow[dd, "\alpha_1'"'] & & A \arrow[dr, "\gamma_1"] \arrow[dd, "\alpha_1"'] & \\
& B' \arrow[rr, "h_{\alpha_2}"] \arrow[dl, "\alpha_2'"] & & B \arrow[dl, "\alpha_2"] \\
C' \arrow[rr, "h"'] & & C &
\end{tikzcd}
\qquad
\begin{tikzcd}
B' \arrow[rr, "h_{\alpha_2}"] \arrow[dr, dashed, "\gamma_2'"] \arrow[dd, "\alpha_2'"'] & & B \arrow[dr, "\gamma_2"] \arrow[dd, "\alpha_2"'] & \\
& D' \arrow[rr, "h_{\alpha_3}"] \arrow[dl, "\alpha_3'"] & & D \arrow[dl, "\alpha_3"] \\
C' \arrow[rr, "h"'] & & C &
\end{tikzcd}$$

In other words,

$$\alpha_1' = \alpha_2' \circ \gamma_1' \text{ and } \gamma_1 \circ h_{\alpha_1} = h_{\alpha_2} \circ \gamma_1'; \tag{6.56}$$

$$\alpha_2' = \alpha_3' \circ \gamma_2' \text{ and } \gamma_2 \circ h_{\alpha_2} = h_{\alpha_3} \circ \gamma_2'. \tag{6.57}$$

Combining these two prism diagrams, we get a commutative "cubic diagram":

$$\begin{tikzcd}
A' \arrow[rr, "h_{\alpha_1}"] \arrow[dr, "\gamma_1'"] \arrow[dd, "\alpha_1'"'] & & A \arrow[dr, "\gamma_1"] \arrow[dd, "\alpha_1"'] & \\
& B' \arrow[rr, "h_{\alpha_2}"'] \arrow[dl, "\alpha_2'"] \arrow[dd, "\gamma_2'"] & & B \arrow[dl, "\alpha_2"] \arrow[dd, "\gamma_2"] \\
C' \arrow[rr, "h"'] & & C & \\
& D' \arrow[ul, "\alpha_3'"] \arrow[rr, "h_{\alpha_3}"'] & & D \arrow[ul, "\alpha_3"]
\end{tikzcd}$$

Now consider the following commutative triangle and its pullback:

$$\begin{tikzcd}
A \arrow[dr, "\gamma_2 \circ \gamma_1"] \arrow[dd, "\alpha_1"'] & \\
& D \arrow[dl, "\alpha_3"] \\
C &
\end{tikzcd}$$

$(\gamma_2 \circ \gamma_1)'$ is the **unique** arrow such that left triangle and upper face of the following prism diagram commute:

$$\begin{tikzcd}
A' \arrow[rr, "h_{\alpha_1}"] \arrow[dr, dashed, "(\gamma_2 \circ \gamma_1)'"] \arrow[dd, "\alpha_1'"'] & & A \arrow[dr, "\gamma_2 \circ \gamma_1"] \arrow[dd, "\alpha_1"'] & \\
& D' \arrow[rr, "h_{\alpha_3}"] \arrow[dl, "\alpha_3'"] & & D \arrow[dl, "\alpha_2"] \\
C' \arrow[rr, "h"'] & & C &
\end{tikzcd}$$

In other words,

$$\alpha_1' = \alpha_3' \circ (\gamma_2 \circ \gamma_1)' \text{ and } (\gamma_2 \circ \gamma_1) \circ h_{\alpha_1} = h_{\alpha_3} \circ (\gamma_2 \circ \gamma_1)'. \tag{6.58}$$

One can easily check that $\gamma_2' \circ \gamma_1'$ clearly satisfies this requirement.

$$\begin{tikzcd}
A' \arrow[r, "\gamma_2'\circ\gamma_1'"] \arrow[d, "\alpha_1'"'] & D' \arrow[dl, "\alpha_3'"] \\
C' &
\end{tikzcd}
\qquad
\begin{tikzcd}
A' \arrow[r, "h_{\alpha_1}"] \arrow[d, "\gamma_2'\circ\gamma_1'"'] & A \arrow[d, "\gamma_2\circ\gamma_1"] \\
D' \arrow[r, "h_{\alpha_3}"'] & D
\end{tikzcd}$$

Using (6.56) and (6.57), we get

$$\alpha_3' \circ (\gamma_2' \circ \gamma_1') = (\alpha_3' \circ \gamma_2') \circ \gamma_1' = \alpha_2' \circ \gamma_1' = \alpha_1'. \tag{6.59}$$

Again, applying (6.56) and (6.57), we get

$$\begin{aligned} h_{\alpha_3} \circ (\gamma_2' \circ \gamma_1') &= (h_{\alpha_3} \circ \gamma_2') \circ \gamma_1' = (\gamma_2 \circ h_{\alpha_2}) \circ \gamma_1' \\ &= \gamma_2 \circ (h_{\alpha_2} \circ \gamma_1') = \gamma_2 \circ (\gamma_1 \circ h_{\alpha_1}) \\ &= (\gamma_2 \circ \gamma_1) \circ h_{\alpha_1}. \end{aligned} \tag{6.60}$$

Therefore, $\gamma_2' \circ \gamma_1'$ satisfies (6.58). Since $(\gamma_2 \circ \gamma_1)'$ is **unique**, we must have

$$(\gamma_2 \circ \gamma_1)' = \gamma_2' \circ \gamma_1'. \tag{6.61}$$

Hence, h^* is indeed a functor. ■

Example 6.3. Let I be an index set, and consider an I-indexed family of sets

$$(A_i)_{i \in I}\,. \tag{6.62}$$

Given a function $\alpha : J \to I$, there is a J-indexed family

$$\left(A_{\alpha(j)}\right)_{j \in J}, \tag{6.63}$$

obtained by "reindexing along α". Let us describe this reindexing using pullback. For each set A_i in the family (6.62), take the constant i-valued function $p_i : A_i \to I$ with all of A_i mapped to $i \in I$, and consider the induced map on the coproduct

$$p = [p_i] : \coprod_{i \in I} A_i \to I.$$

It is the **unique** function $\coprod_{i \in I} A_i \to I$ such that each of the following triangles commute:

$$\begin{tikzcd}
 & I \\
A_k \arrow[ur, "p_k"] \arrow[r, "\iota_k"'] & \coprod_{i \in I} A_i \arrow[u, "p"']
\end{tikzcd}$$

Therefore, given $(a, k) \in \coprod_{i \in I} A_i$, with $a \in A_k$,

$$p\,(a, k) = p\,(\iota_k\,(a)) = p_k\,(a) = k.$$

In a similar manner, for each $A_{\alpha(j)}$ in the family (6.63), we can form the constant j-valued function $q_j : A_{\alpha(j)} \to J$ with all of $A_{\alpha(j)}$ mapped to $j \in J$, and consider the induced map on the coproduct

$$q = [q_j] : \coprod_{j \in J} A_{\alpha(j)} \to J.$$

It is the **unique** function $\coprod_{j \in J} A_{\alpha(j)} \to J$ such that each of the following triangles commute for all $l \in J$:

$$\begin{tikzcd}
 & J \\
A_{\alpha(l)} \arrow[ur, "q_l"] \arrow[r, "\iota'_l"'] & \coprod_{j\in J} A_{\alpha(j)} \arrow[u, "q"']
\end{tikzcd}$$

Therefore, given $(a,l)\in\coprod_{j\in J}A_{\alpha(j)}$, with $a\in A_{\alpha(l)}$,

$$q\,(a,l)=q\,(\iota'_l\,(a))=q_l\,(a)=l.$$

The reindexed family $\left(A_{\alpha(j)}\right)_{j\in J}$ can be obtained by taking the pullback along the function $\alpha : J\to I$ as can be seen from the following diagram:

$$\begin{tikzcd}
\coprod_{j\in J} A_{\alpha(j)} \arrow[r, "\alpha_p"] \arrow[d, "q"'] & \coprod_{i\in I} A_i \arrow[d, "p"] \\
J \arrow[r, "\alpha"'] & I
\end{tikzcd} \tag{6.64}$$

where $\alpha_p : \coprod_{j\in J}A_{\alpha(j)}\to\coprod_{i\in I}A_i$ is defined as follows: given $(a,l)\in\coprod_{j\in J}A_{\alpha(j)}$, with $a\in A_{\alpha(l)}$,

$$\alpha_p\,(a,l)=(a,\alpha\,(l))\in\coprod_{i\in I}A_i.$$

It's easy to see that the square above commutes, i.e. $p\circ\alpha_p=\alpha\circ q$. Given $(a,l)\in\coprod_{j\in J}A_{\alpha(j)}$, with $a\in A_{\alpha(l)}$, one has

$$\begin{aligned}(p\circ\alpha_p)\,(a,l)&=p\,(a,\alpha\,(l))=\alpha\,(l)\,,\\ (\alpha\circ q)\,(a,l)&=\alpha\,(l)\,.\end{aligned}$$

Therefore, $p\circ\alpha_p=\alpha\circ q$. The fact the the UMP of pullback is satisfied for the above square can be checked easily. Suppose there is another set X and functions $f : X\to J$ and $g : X\to\coprod_{i\in I}A_i$ such that $p\circ g=\alpha\circ f$. We need to show the existence of a **unique** $u : X\to\coprod_{j\in J}A_{\alpha(j)}$ such that $f=q\circ u$ and $g=\alpha_p\circ u$, i.e. the following diagram commutes:

$$\begin{tikzcd}
X \arrow[dr, dashed, "u"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "f"'] & & \\
 & \coprod_{j\in J} A_{\alpha(j)} \arrow[r, "\alpha_p"] \arrow[d, "q"'] & \coprod_{i\in I} A_i \arrow[d, "p"] \\
 & J \arrow[r, "\alpha"'] & I
\end{tikzcd}$$

Given $x\in X$, $g\,(x)\in\coprod_{i\in I}A_i$, so $g\,(x)=(a,i)$ for some $i\in I$ and $a\in A_i$. Furthermore, let $f\,(x)=j\in J$. Then $p\circ g=\alpha\circ f$ gives us

$$\alpha\,(j)=\alpha\,(f\,(x))=p\,(g\,(x))=p\,(a,i)=i. \tag{6.65}$$

Therefore, $g\,(x)=(a,\alpha\,(j))$, where $j=f\,(x)$. Now we define a function $u : X\to\coprod_{j\in J}A_{\alpha(j)}$ as follows:

$$u\,(x)=(a,j)\in\coprod_{j\in J}A_{\alpha(j)}, \tag{6.66}$$

where $j = f(x)$ and $(a, \alpha(j)) = g(x)$. In terms of the inclusions of coproduct,

$$\iota_{\alpha(j)}(a) = (a, \alpha(j)) \in \coprod_{i \in I} A_i. \tag{6.67}$$

So $a = \iota^{-1}_{\alpha(f(x))}(g(x))$. In other words,

$$u(x) = \left(\iota^{-1}_{\alpha(f(x))}(g(x)), f(x)\right). \tag{6.68}$$

Then

$$(q \circ u)(x) = q(a, j) = j = f(x), \tag{6.69}$$

$$(\alpha_p \circ u)(x) = \alpha_p(a, j) = (a, \alpha(j)) = g(x). \tag{6.70}$$

So there exists a function $u : X \to \coprod_{j \in J} A_{\alpha(j)}$ such that $f = q \circ u$ and $g = \alpha_p \circ u$. We need to show the uniqueness of u. Suppose there is another function $v : X \to \coprod_{j \in J} A_{\alpha(j)}$ satisfying $f = q \circ v$ and $g = \alpha_p \circ v$. Given $x \in X$, suppose $v(x) = (a, j)$ for some $j \in J$ and $a \in A_{\alpha(j)}$. $g = \alpha_p \circ v$ gives us

$$g(x) = \alpha_p(v(x)) = \alpha_p(a, j) = (a, \alpha(j)). \tag{6.71}$$

Furthermore, $f = q \circ v$ gives us

$$f(x) = q(v(x)) = q(a, j) = j. \tag{6.72}$$

$(a, \alpha(j)) = g(x)$ gives us $a = \iota^{-1}_{\alpha(j)}(g(x))$. Therefore,

$$v(x) = (a, j) = \left(\iota^{-1}_{\alpha(f(x))}(g(x)), f(x)\right), \tag{6.73}$$

which matches precisely with the expression of u, from (6.68). Therefore, u is unique, and hence (6.64) is a pullback diagram. Hence,

$$J \times_I \left(\coprod_{i \in I} A_i\right) \cong \coprod_{j \in J} A_{\alpha(j)}. \tag{6.74}$$

Proposition 6.7

A category has finite products and equalizers if and only if it has pullbacks and a terminal object.

Proof. (⇒) If a category has all finite products, then it has nullary product as well. A nullary product is precisely a terminal object, as we observed in Section 4.5. Also, if a category admits all finite products, it definitely admits all binary products. A category admitting binary products and equalizers have pullbacks, by Corollary 6.3.

(⇐) Suppose we are in a category with pullbacks and a terminal object 1. In order to show that the category admits finite products, it suffices to show that the category admits binary products, since any finite product can be constructed from binary products. Given objects A and B, there are unique arrows $u_A : A \to 1$ and $u_B : B \to 1$. Now, consider the pullback of $A \xrightarrow{u_A} 1 \xleftarrow{u_B} B$:

$$\begin{tikzcd}
A \times_1 B \arrow[r, "p_2"] \arrow[d, "p_1"'] & B \arrow[d, "u_B"] \\
A \arrow[r, "u_A"'] & 1
\end{tikzcd}$$

We claim that $A \times_1 B$ is a product of A and B. In order to show that, we need to verify the UMP of product. Suppose we are given any arrows $x_1 : X \to A$ and $x_2 : X \to B$.

$$\begin{tikzcd} & X \arrow[dl, "x_1"'] \arrow[d, dashed, "u"] \arrow[dr, "x_2"] & \\ A & A \times_1 B \arrow[l, "p_1"] \arrow[r, "p_2"'] & B \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[dr, dashed, "u"] \arrow[drr, bend left, "x_2"] \arrow[ddr, bend right, "x_1"'] & & \\ & A \times_1 B \arrow[r, "p_2"'] \arrow[d, "p_1"] & B \arrow[d, "u_B"] \\ & A \arrow[r, "u_A"'] & 1 \end{tikzcd}$$

Since 1 is a terminal object, there is a unique arrow $X \to 1$. $u_A \circ x_1, u_B \circ x_2$ are arrows from X to 1. Therefore, $u_A \circ x_1 = u_B \circ x_2$. From the UMP of product, there is a **unique** arrow $u : X \to A \times_1 B$ such that $x_1 = p_1 \circ u$ and $x_2 = p_2 \circ u$ (which is precisely what the UMP of product says). Therefore, $A \times_1 B$ is a product of A and B, i.e.

$$A \times_1 B \cong A \times B. \tag{6.75}$$

Now, we have to prove the existence of equalizers. Suppose we are given two parallel arrows $f, g : A \to B$. Since we already proved that the category admits binary products, we can form the arrows $\langle f, g\rangle : A \to B \times B$ and $\Delta = \langle \mathbb{1}_B, \mathbb{1}_B\rangle : B \to B \times B$. They are the **unique** arrows such that the diagrams below commute:

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[d, dashed, "\langle f,g\rangle"] \arrow[dr, "g"] & \\ B & B \times B \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & B \end{tikzcd} \qquad \begin{tikzcd} & B \arrow[dl, "\mathbb{1}_B"'] \arrow[d, dashed, "\langle \mathbb{1}_B,\mathbb{1}_B\rangle"] \arrow[dr, "\mathbb{1}_B"] & \\ B & B \times B \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & B \end{tikzcd}$$

Now, let us form the pullback of $A \xrightarrow{\langle f,g\rangle} B \times B \xleftarrow{\Delta} B$:

$$\begin{tikzcd} E \arrow[r, "h"] \arrow[d, "e"'] & B \arrow[d, "\Delta=\langle \mathbb{1}_B,\mathbb{1}_B\rangle"] \\ A \arrow[r, "\langle f,g\rangle"'] & B \times B \end{tikzcd} \tag{6.76}$$

We claim that $e : E \to A$ is an equalizer of the parallel arrows $f, g : A \to B$.

$$\begin{tikzcd} E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

First, let us verify that $f \circ e = g \circ e$. Since (6.76) is a pullback diagram, it's a commutative square. Therefore,

$$\langle f, g\rangle \circ e = \langle \mathbb{1}_B, \mathbb{1}_B\rangle \circ h. \tag{6.77}$$

Using Proposition 3.10, $\langle f, g\rangle \circ e = \langle f \circ e, g \circ e\rangle$ and $\langle \mathbb{1}_B, \mathbb{1}_B\rangle \circ h = \langle h, h\rangle$. As a result,

$$\langle f \circ e, g \circ e\rangle = \langle h, h\rangle\,. \tag{6.78}$$

Composing these arrows with π_1 and π_2, we get

$$\pi_1 \circ \langle f \circ e, g \circ e\rangle = \pi_1 \circ \langle h, h\rangle \implies f \circ e = h, \tag{6.79}$$

$$\pi_2 \circ \langle f \circ e, g \circ e\rangle = \pi_2 \circ \langle h, h\rangle \implies g \circ e = h. \tag{6.80}$$

Therefore, $f \circ e = g \circ e$. Now, suppose there is another $a : Z \to A$ such that $f \circ a = g \circ a$. We need to show that there exists a **unique** arrow $u : Z \to E$ such that $e \circ u = a$, i.e. the following diagram commutes:

$$\begin{tikzcd}
E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\
Z \arrow[u, dashed, "u"] \arrow[ur, "a"'] & &
\end{tikzcd}$$

Consider the following diagram:

$$\begin{tikzcd}
Z \arrow[drr, bend left, "f\circ a=g\circ a"] \arrow[dr, dashed, "u", "\exists!"'] \arrow[ddr, bend right, "a"'] & & \\
& E \arrow[r, "h"] \arrow[d, "e"'] & B \arrow[d, "\Delta=\langle \mathbb{1}_B,\mathbb{1}_B\rangle"] \\
& A \arrow[r, "\langle f,g\rangle"'] & B\times B
\end{tikzcd}$$

Since $f\circ a = g\circ a$, we have

$$\langle f,g\rangle \circ a = \langle f\circ a, g\circ a\rangle = \langle \mathbb{1}_B \circ(f\circ a), \mathbb{1}_B \circ(f\circ a)\rangle = \langle \mathbb{1}_B, \mathbb{1}_B\rangle \circ (f\circ a). \tag{6.81}$$

Therefore, $\langle f,g\rangle \circ a = \Delta \circ (f\circ a)$. By the UMP of pullbacks, there exists a **unique** arrow $u : Z \to E$ such that $e\circ u = a$ and $h\circ u = f\circ a$. So there indeed exists an arrow $u : Z \to E$ satisfying $e\circ u = a$. Now we need to verify the uniqueness of u.

Suppose there is another $v : Z \to E$ satisfying $e\circ v = a$. Now, consider the following diagram:

$$\begin{tikzcd}
Z \arrow[drr, bend left, "f\circ a=g\circ a"] \arrow[dr, shift left, "u"] \arrow[dr, shift right, "v"'] \arrow[ddr, bend right, "a"'] & & \\
& E \arrow[r, "h"] \arrow[d, "e"'] & B \arrow[d, "\Delta=\langle \mathbb{1}_B,\mathbb{1}_B\rangle"] \\
& A \arrow[r, "\langle f,g\rangle"'] & B\times B
\end{tikzcd}$$

By (6.79) and (6.80), $h = f\circ e = g\circ e$. Now,

$$h\circ v = (f\circ e)\circ v = f\circ (e\circ v) = f\circ a. \tag{6.82}$$

Therefore, $v : Z \to E$ is an arrow that satisfies $e\circ v = a$ and $h\circ v = f\circ a$. But u is the **unique** arrow such that $e\circ u = a$ and $h\circ u = f\circ a$. Therefore, $u = v$, and hence u is unique. So $e : E \to A$ satisfies the UMP of equalizer. ■

Remark 6.3. Although $u : Z \to E$ is the unique arrow such that $e \circ u = a$ and $h \circ u = f \circ a$, it doesn't immediately satisfy the uniqueness property required in the UMP of equalizer. In the equalizer diagram,

$$\begin{tikzcd}
E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\
Z \arrow[u, dashed, "u"] \arrow[ur, "a"'] & &
\end{tikzcd}$$

we needed to show the existence of a **unique** $u : Z \to E$ such that the triangle above commutes, i.e. $e \circ u = a$. u is the unique arrow such that $e \circ u = a$ and $h \circ u = f \circ a$. But we need uniqueness with respect to the condition $e \circ u = a$ only. We cannot invoke the argument regarding uniqueness of u in the pullback diagram to conclude that u is unique in the equalizer diagram, since we are relaxing the condition a little bit. In principle, there could exist another arrow v for which $e \circ v = a$ but $h \circ v \neq f \circ a$. This does not violate the uniqueness property of u. That is why we needed to check the uniqueness again.

7 Limits and Colimits

§7.1 Limits

Definition 7.1. Let $\mathcal{J}$ be a small category and $\mathcal{C}$ be any category. A **diagram** of type J in $\mathcal{C}$ is a functor

$$D : \mathcal{J} \to \mathcal{C}.$$

We write the objects in the "index category" $\mathcal{J}$ with lowercase $i, j, \ldots$, and the values of the functor $D : \mathcal{J} \to \mathcal{C}$ in the form $D_i, D_j, \ldots$ being objects in $\mathcal{C}$.

Definition 7.2 (Cone)**.** A **cone** to a diagram D consists of

- an object C in $\mathcal{C}$, and
- a family of arrows in $\mathcal{C}$: $c_j : C \to D_j$ one for each object j in $\mathcal{J}$.

such that for each arrow $\alpha : i \to j$ in $\mathcal{J}$, $c_j = D_\alpha \circ c_i$, i.e. the following diagram commutes:

$$\begin{tikzcd} & C \arrow[dl, "c_i"'] \arrow[dr, "c_j"] & \\ D_i \arrow[rr, "D_\alpha"'] & & D_j \end{tikzcd}$$

Such a cone is denoted by (C, c_j). A morphism of cones

$$\vartheta : (C, c_j) \to \left(C', c_j'\right)$$

is an arrow $\vartheta : C \to C'$ in $\mathcal{C}$ making each triangle of the following form commute: (for each $j \in \operatorname{Ob}(\mathcal{J})$):

$$\begin{tikzcd} C \arrow[r, "\vartheta"] \arrow[dr, "c_j"'] & C' \arrow[d, "c_j'"] \\ & D_j \end{tikzcd}$$

i.e. $c_j = c_j' \circ \vartheta$ for all $j \in \operatorname{Ob}(\mathcal{J})$. Thus we have a category **Cone** (D) of all cones to a diagram D.

Given arrows $\vartheta : (C, c_j) \to \left(C', c_j'\right)$ and $\varrho : \left(C', c_j'\right) \to \left(C'', c_j''\right)$ between cones, one can form the composite

$$\varrho \circ \vartheta : (C, c_j) \to \left(C'', c_j''\right).$$

Then $\vartheta : C \to C'$ and $\varrho : C' \to C$ are arrows in $\mathcal{C}$ such that the following triangles commute for each $j \in \operatorname{Ob}(\mathcal{J})$:

$$\begin{tikzcd} C \arrow[r, "\vartheta"] \arrow[dr, "c_j"'] & C' \arrow[d, "c_j'"] \\ & D_j \end{tikzcd} \qquad \begin{tikzcd} C' \arrow[r, "\varrho"] \arrow[dr, "c_j'"'] & C'' \arrow[d, "c_j''"] \\ & D_j \end{tikzcd}$$

Combining these two triangles, we get the following commutative diagram, for each $j \in \operatorname{Ob}(\mathcal{J})$:

$$\begin{tikzcd}
C \arrow[r, "\vartheta"'] \arrow[rr, bend left, "\varrho\circ\vartheta"] \arrow[dr, "c_j"'] & C' \arrow[r, "\varrho"'] \arrow[d, "c_j'"] & C'' \arrow[dl, "c_j''"] \\
& D_j &
\end{tikzcd}$$

Therefore, the following triangle commutes for each $j \in \operatorname{Ob}(\mathcal{J})$:

$$\begin{tikzcd}
C \arrow[r, "\varrho\circ\vartheta"] \arrow[dr, "c_j"'] & C'' \arrow[d, "c_j''"] \\
& D_j
\end{tikzcd}$$

So $\varrho \circ \vartheta$ is indeed an arrow from (C, c_j) to $\left(C'', c_j''\right)$ in **Cone** (D). The associativity of composition is clear, since the arrow in **Cone** (D) are actually arrows in the category $\mathcal{C}$, and associativity of composition holds in $\mathcal{C}$.

Given any object (C, c_j) in **Cone** (D), its identity arrow $\mathbb{1}_{(C,c_j)}$ is the arrow $\mathbb{1}_C$ in $\mathcal{C}$. Indeed, the following triangle commutes for each $j \in \operatorname{Ob}(\mathcal{J})$:

$$\begin{tikzcd}
C \arrow[r, "\mathbb{1}_C"] \arrow[dr, "c_j"'] & C \arrow[d, "c_j"] \\
& D_j
\end{tikzcd}$$

Therefore, $\mathbb{1}_C$ is indeed an arrow (C, c_j) to itself. Given any arrow $\vartheta : (C, c_j) \to \left(C', c_j'\right)$ in **Cone**(D), since $\vartheta : C \to C'$ is an arroow in $\mathcal{C}$, $\vartheta \circ \mathbb{1}_C = \vartheta$ in $\mathcal{C}$. Therefore,

$$\vartheta \circ \mathbb{1}_{(C,c_j)} = \vartheta$$

in **Cone**(D). Similarly, $\mathbb{1}_{C'} \circ \vartheta = \vartheta$ in $\mathcal{C}$. Therefore,

$$\mathbb{1}_{\left(C',c_j'\right)} \circ \vartheta = \vartheta$$

in **Cone**(D). Hence, $\mathbb{1}_{(C,c_j)} = \mathbb{1}_C$ is indeed the identity arrow of the object (C, c_j) in **Cone** (D). Therefore, **Cone**(D) is a category.

We can imagine a cone to be cone over a diagram D with its apex in C.

> **Definition 7.3** (Limit). A **limit** for a diagram $D : \mathcal{J} \to \mathcal{C}$ is a terminal object in **Cone** (D). A **finite limit** is a limit for a diagram on a finite index category $\mathcal{J}$.

We often denote a limit in the form

$$p_i : \varprojlim_j D_j \to D_i.$$

The limit of a diagram itself is a cone $\left(\varprojlim_j D_j, p_j\right)$. The limit of a diagram has the following UMP: Given any cone (C, c_j) to D, there is a **unique** arrow $u : C \to \varprojlim_j D_j$ such that

$$p_j \circ u = c_j, \tag{7.1}$$

for all j. In other words, the following diagram commutes for each i, j:

$$\begin{tikzcd}
 & C \arrow[d, dashed, "u"] \arrow[ddl, bend right, "c_i"'] \arrow[ddr, bend left, "c_j"] & \\
 & \varprojlim_j D_j \arrow[dl, "p_i"'] \arrow[dr, "p_j"] & \\
D_i \arrow[rr, "D_\alpha"'] & & D_j
\end{tikzcd}$$

Thus the limiting cone $\left(\varprojlim_j D_j, p_j\right)$ can be thought of as the "closest" one to the diagram D, and indeed the other cones come from it just by composing with an arrow at the vertex, namely $u : C \to \varprojlim_j D_j$.

Example 7.1. 1. Take $\mathcal{J} = \{1, 2\}$, the discrete category with two objects and no nonidentity arrows. A diagram $D : \mathcal{J} \to \mathcal{C}$ is a pair of objects D_1 and D_2 in $\mathcal{C}$. A cone to D is an object C of $\mathcal{C}$ equipped with arrows

$$D_1 \xleftarrow{c_1} C \xrightarrow{c_2} D_2$$

A limit of D is a terminal such cone, that is product of D_1 and D_2 in $\mathcal{C}$:

$$D_1 \xleftarrow{\pi_1} D_1 \times D_2 \xrightarrow{\pi_2} D_2$$

Therefore,

$$\varprojlim_j D_j \cong D_1 \times D_2.$$

2. Take $\mathcal{J}$ to be the following category

$$\begin{tikzcd}
1 \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & 2
\end{tikzcd}$$

A diagram of type $\mathcal{J}$ looks like

$$\begin{tikzcd}
D_1 \arrow[r, shift left, "D_\alpha"] \arrow[r, shift right, "D_\beta"'] & D_2
\end{tikzcd}$$

A cone is an object C along with a pair of arrows $c_1 : C \to D_1$ and $c_2 : C \to D_2$ such that the following diagram commutes:

$$\begin{tikzcd}
D_1 \arrow[r, shift left, "D_\alpha"] \arrow[r, shift right, "D_\beta"'] & D_2 \\
C \arrow[u, "c_1"] \arrow[ur, "c_2"'] &
\end{tikzcd}$$

In other words,

$$D_\alpha \circ c_1 = c_2 = D_\beta \circ c_1. \tag{7.2}$$

A limit for D is, therefore, an equalizer $C \xrightarrow{c_1} D_1$ of the parallel arrows $D_\alpha, D_\beta : D_1 \to D_2$. If there is any other cone

$$\begin{tikzcd}
D_1 \arrow[r, shift left, "D_\alpha"] \arrow[r, shift right, "D_\beta"'] & D_2 \\
C' \arrow[u, "c_1'"] \arrow[ur, "c_2'"'] &
\end{tikzcd}$$

satisfying $D_\alpha \circ c_1' = c_2' = D_\beta \circ c_1'$, then from the UMP of limit, we know that there exists a **unique** arrow $u : C' \to C$ such that $c_1 \circ u = c_1'$ and $c_2 \circ u = c_2'$.

$$\begin{tikzcd}
D_1 \arrow[rr, shift left, "D_\alpha"] \arrow[rr, shift right, "D_\beta"'] & & D_2 \\
& C \arrow[ul, "c_1"] \arrow[ur, "c_2"'] & \\
& C' \arrow[uul, bend left, "c_1'"] \arrow[uur, bend right, "c_2'"'] \arrow[u, dashed, "u", "\exists!"'] &
\end{tikzcd}$$

The diagram is equivalent to the following equalizer diagram:

$$\begin{tikzcd}
C \arrow[r, "c_1"] & D_1 \arrow[r, shift left, "D_\alpha"] \arrow[r, shift right, "D_\beta"'] & D_2 \\
C' \arrow[u, dashed, "u"] \arrow[ur, "c_1'"'] & &
\end{tikzcd}$$

Therefore, the equalizer is the "nearest" cone to the underlying diagram, i.e. a limit.

3. Now, $\mathcal{J}$ can be any small category, even the empty category. When $\mathcal{J}$ is the empty category, it has no objects and no arrows. In this case, the cone consists of the apex (an object of $\mathcal{C}$) only. Therefore, the category **Cone** (D) is actually the category $\mathcal{C}$. The limit for this diagram is the terminal object of **Cone** (D), which coincides with the terminal object 1 of $\mathcal{C}$. Therefore, the limit of the diagram $D : \mathcal{J} \to \mathcal{C}$ when $\mathcal{J}$ is empty is 1. We write this as

$$\varprojlim_{j} D_j \cong 1.$$

4. Let $\mathcal{J}$ be the following category:

$$\begin{tikzcd}
& 2 \arrow[d, "\beta"] \\
3 \arrow[r, "\alpha"'] & 1
\end{tikzcd}$$

So a diagram $D : \mathcal{J} \to \mathcal{C}$ of type $\mathcal{J}$ consists of the objects and arrows in $\mathcal{C}$ as below:

$$\begin{tikzcd}
& D_2 \arrow[d, "D_\beta"] \\
D_3 \arrow[r, "D_\alpha"'] & D_1
\end{tikzcd}$$

A cone to this diagram D is given by the following commutative diagram:

$$\begin{tikzcd}
C \arrow[r, "c_2"] \arrow[d, "c_3"'] \arrow[dr, "c_1"] & D_2 \arrow[d, "D_\beta"] \\
D_3 \arrow[r, "D_\alpha"'] & D_1
\end{tikzcd}$$

Each of the triangles above commute, so that

$$D_\alpha \circ c_3 = c_1 = D_\alpha \circ c_2. \tag{7.3}$$

Now, if C is the nearest cone to D, then from the UMP of a limit, we know that given any other cone $\left(C', c_j'\right)$, there is a **unique** arrow $u : C' \to C$ such that $c_j' = c_j \circ u$ for all j, i.e. the following diagram commutes:

$$\begin{tikzcd}
C' \arrow[rd, dashed, "u", "\exists!"'] \arrow[rrd, bend left, "c_2'"] \arrow[rrdd, bend left=20, "c_1'"] \arrow[rdd, bend right, "c_3'"'] & & \\
& C \arrow[r, "c_2"'] \arrow[d, "c_3"'] \arrow[rd, "c_1"] & D_2 \arrow[d, "D_\beta"] \\
& D_3 \arrow[r, "D_\alpha"'] & D_1
\end{tikzcd}$$

This is precisely the UMP of pullback. Therefore,

$$\varprojlim_j D_j \cong D_3 \times_{D_1} D_2. \tag{7.4}$$

Definition 7.4. A category $\mathcal{C}$ is said to have all finite limits if every finite diagram $D : \mathcal{J} \to \mathcal{C}$ has a limit in $\mathcal{C}$.

Proposition 7.1

A category has all finite limits if and only if it has finite products and equalizers, i.e. it has pullbacks and a terminal object.

Proof. ($\Rightarrow$) If a category has all finite limits, then it has all finite products and equalizers, since finite products and equalizers can be seen as finite limits, by Example 7.1.

($\Leftarrow$) Let a category $\mathcal{C}$ admit finite products and equalizers. We have to prove that it admits all finite limits. Let $D : \mathcal{J} \to \mathcal{C}$ be a finite diagram, i.e. the index category $\mathcal{J}$ is finite. Let $\mathcal{J}_0$ and $\mathcal{J}_1$ denote the object set and arrow set of $\mathcal{J}$, respectively, which are both finite sets. For $i \in \mathcal{J}_0$, D_i is an object in $\mathcal{C}$. Since $\mathcal{C}$ admits finite products, we can form the following products:

$$\prod_{i\in\mathcal{J}_0} D_i \quad \text{and} \quad \prod_{\beta\in\mathcal{J}_1} D_{\operatorname{cod}\beta}.$$

Suppose the following are the relevant projections:

$$\pi_j : \prod_{i\in\mathcal{J}_0} D_i \to D_j \quad \text{and} \quad p_\alpha : \prod_{\beta\in\mathcal{J}_1} D_{\operatorname{cod}\beta} \to D_{\operatorname{cod}\alpha}, \tag{7.5}$$

for $j \in \mathcal{J}_0$ and $\alpha \in \mathcal{J}_1$. Now, consider the α-indexed family of arrows

$$\pi_{\operatorname{cod}\alpha} : \prod_{i\in\mathcal{J}_0} D_i \to D_{\operatorname{cod}\alpha},$$

for $\alpha \in \mathcal{J}_1$. By the UMP of product, there is a **unique** arrow $\varphi : \prod_{i\in\mathcal{J}_0} D_i \to \prod_{\beta\in\mathcal{J}_1} D_{\operatorname{cod}\beta}$ such that the following triangle commutes for each $\alpha \in \mathcal{J}_1$:

$$\begin{tikzcd}
& \prod\limits_{i\in\mathcal{J}_0} D_i \arrow[ld, "\pi_{\operatorname{cod}\alpha}"'] \arrow[d, dashed, "\varphi"] \\
D_{\operatorname{cod}\alpha} & \prod\limits_{\beta\in\mathcal{J}_1} D_{\operatorname{cod}\beta} \arrow[l, "p_\alpha"]
\end{tikzcd}$$

In other words, $\varphi : \prod_{i\in\mathcal{J}_0} D_i \to \prod_{\beta\in\mathcal{J}_1} D_{\operatorname{cod}\beta}$ is the **unique** arrow such that

$$\pi_{\operatorname{cod}\alpha} = p_\alpha \circ \varphi \tag{7.6}$$

for each $\alpha \in \mathcal{J}_1$. Recall that for $\alpha : j \to k$ in $\mathcal{J}$, $D_\alpha : D_j \to D_k$ is an arrow in $\mathcal{C}$, i.e.

$$D_\alpha : D_{\mathrm{dom}\,\alpha} \to D_{\mathrm{cod}\,\alpha}. \tag{7.7}$$

Now, form the composite arrow $D_\alpha \circ \pi_{\mathrm{dom}\,\alpha} : \prod_{i\in\mathcal{J}_0} D_i \to D_{\mathrm{cod}\,\alpha}$. We, therefore, have an α-indexed family of arrows

$$D_\alpha \circ \pi_{\mathrm{dom}\,\alpha} : \prod_{i\in\mathcal{J}_0} D_i \to D_{\mathrm{cod}\,\alpha},$$

for $\alpha \in \mathcal{J}_1$. By the UMP of product, there is a **unique** arrow $\psi : \prod_{i\in\mathcal{J}_0} D_i \to \prod_{\beta\in\mathcal{J}_1} D_{\mathrm{cod}\,\beta}$ such that the following triangle commutes for each $\alpha \in \mathcal{J}_1$:

$$\begin{tikzcd}
 & \prod\limits_{i\in\mathcal{J}_0} D_i \arrow[dl, "D_\alpha\circ\pi_{\mathrm{dom}\,\alpha}"'] \arrow[d, dashed, "\psi"] \\
D_{\mathrm{cod}\,\alpha} & \prod\limits_{\beta\in\mathcal{J}_1} D_{\mathrm{cod}\,\beta} \arrow[l, "p_\alpha"]
\end{tikzcd}$$

In other words, $\psi : \prod_{i\in\mathcal{J}_0} D_i \to \prod_{\beta\in\mathcal{J}_1} D_{\mathrm{cod}\,\beta}$ is the **unique** arrow such that

$$D_\alpha \circ \pi_{\mathrm{dom}\,\alpha} = p_\alpha \circ \psi. \tag{7.8}$$

Let us combine the above commutative diagrams in the following diagram:

$$\begin{tikzcd}
 & D_{\mathrm{cod}\,\alpha} \\
\prod\limits_{i\in\mathcal{J}_0} D_i \arrow[ur, "\pi_{\mathrm{dom}\,\alpha}"] \arrow[r, shift left, "\varphi"] \arrow[r, shift right, "\psi"'] \arrow[d, "\pi_{\mathrm{dom}\,\alpha}"'] & \prod\limits_{\beta\in\mathcal{J}_1} D_{\mathrm{cod}\,\beta} \arrow[u, "p_\alpha"'] \arrow[d, "p_\alpha"] \\
D_{\mathrm{dom}_\alpha} \arrow[r, "D_\alpha"'] & D_{\mathrm{cod}\,\alpha}
\end{tikzcd}$$

(Note that this is **NOT** technically a commutative diagram. The upper triangle commutes with respect to φ, the lower square commutes with respect to ψ.) So we have two parallel arrows

$$\begin{tikzcd}
\prod\limits_{i\in\mathcal{J}_0} D_i \arrow[r, shift left, "\varphi"] \arrow[r, shift right, "\psi"'] & \prod\limits_{\beta\in\mathcal{J}_1} D_{\mathrm{cod}\,\beta}
\end{tikzcd}$$

Since $\mathcal{C}$ has all equalizers, take the equalizer $e : E \to \prod_{i\in\mathcal{J}_0} D_i$ of φ and ψ.

$$\begin{tikzcd}
E \arrow[r, "e"] & \prod\limits_{i\in\mathcal{J}_0} D_i \arrow[r, shift left, "\varphi"] \arrow[r, shift right, "\psi"'] & \prod\limits_{\beta\in\mathcal{J}_1} D_{\mathrm{cod}\,\beta}
\end{tikzcd}$$

For $j \in \mathcal{J}_0$, let $\mu_j = \pi_j \circ e : E \to D_j$. We claim that (E, μ_j) is a limiting cone to the diagram $D : \mathcal{J} \to \mathcal{C}$. Let us first see that (E, μ_j) is a cone to the diagram D.

$$\begin{tikzcd}
 & E \arrow[dl, "\mu_j=\pi_j\circ e"'] \arrow[dr, "\mu_k=\pi_k\circ e"] & \\
D_j \arrow[rr, "\alpha"'] & & D_k
\end{tikzcd}$$

Since $e : E \to \prod_{i\in\mathcal{J}_0} D_i$ is an equalizer of of φ and ψ, $\varphi \circ e = \psi \circ e$. Given an arrow $\alpha : j \to k$ in $\mathcal{J}$, we have

$$\begin{aligned} \mu_k &= \pi_k \circ e = \pi_{\operatorname{cod}\alpha} \circ e \\ &= p_\alpha \circ \varphi \circ e = p_\alpha \circ \psi \circ e \\ &= D_\alpha \circ \pi_{\operatorname{dom}\alpha} \circ e = D_\alpha \circ \pi_j \circ e \\ &= D_\alpha \circ \mu_j. \end{aligned} \tag{7.9}$$

Hence, (E, μ_j) is a cone to the diagram D. We now need to verify the UMP of limit. Suppose $F, \tau_j : F \to D_j$ is another cone to the diagram D. Then for any arrow $\alpha : j \to k$ in $\mathcal{J}$, the following diagram commutes:

$$\begin{tikzcd} & F \arrow[dl, "\tau_j"'] \arrow[dr, "\tau_k"] & \\ D_j \arrow[rr, "D_\alpha"'] & & D_k \end{tikzcd}$$

In other words, $D_\alpha \circ \tau_j = \tau_k$ for all $\alpha \in \mathcal{J}_1$, i.e.

$$D_\alpha \circ \tau_{\operatorname{dom}\alpha} = \tau_{\operatorname{cod}\alpha}. \tag{7.10}$$

Consider the j-indexed family of arrows $\tau_j : F \to D_j$, there is a **unique** arrow $h : F \to \prod_{i\in\mathcal{J}_0} D_i$ such that the following diagram commutes for all $j \in \mathcal{J}_0$:

$$\begin{tikzcd} & F \arrow[dl, "\tau_j"'] \arrow[d, dashed, "h"] \\ D_j & \prod\limits_{i\in\mathcal{J}_0} D_i \arrow[l, "\pi_j"] \end{tikzcd}$$

In other words, $\pi_j \circ h = \tau_j$, for all $j \in \mathcal{J}_0$. Now, given arrows $\tau_{\operatorname{cod}\alpha} = \pi_{\operatorname{cod}\alpha} \circ h : F \to D_{\operatorname{cod}\alpha}$, there is a **unique** arrow $g : F \to \prod_{\beta\in\mathcal{J}_1} D_{\operatorname{cod}\beta}$ such that the following diagram commutes for all $\alpha \in \mathcal{J}_1$:

$$\begin{tikzcd} & F \arrow[dl, "\tau_{\operatorname{cod}\alpha}=\pi_{\operatorname{cod}\alpha}\circ h"'] \arrow[d, dashed, "g"] \\ D_{\operatorname{cod}\alpha} & \prod\limits_{\beta\in\mathcal{J}_1} D_{\operatorname{cod}\beta} \arrow[l, "p_\alpha"] \end{tikzcd}$$

In other words, $p_\alpha \circ g = \tau_{\operatorname{cod}\alpha}$, for all $\alpha \in \mathcal{J}_1$. We now claim that $g = \varphi \circ h = \psi \circ h$.

$$p_\alpha \circ \varphi \circ h = \pi_{\operatorname{cod}\alpha} \circ h = \tau_{\operatorname{cod}\alpha}, \tag{7.11}$$

$$p_\alpha \circ \psi \circ h = D_\alpha \circ \pi_{\operatorname{dom}\alpha} \circ h = D_\alpha \circ \tau_{\operatorname{dom}\alpha} = \tau_{\operatorname{cod}\alpha}. \tag{7.12}$$

Therefore, by the uniqueness of g, $g = \varphi \circ h = \psi \circ h$. Since $e : E \to \prod_{i\in\mathcal{J}_0} D_i$ is an equalizer of φ and ψ, and $h : F \to \prod_{i\in\mathcal{J}_0} D_i$ is another arrow satisfying $\varphi \circ h = \psi \circ h$, there is a **unique** arrow $u : F \to E$ such that the following diagram commutes:

$$\begin{tikzcd} E \arrow[r, "e"] & \prod\limits_{i\in\mathcal{J}_0} D_i \arrow[r, shift left, "\varphi"] \arrow[r, shift right, "\psi"'] & \prod\limits_{\beta\in\mathcal{J}_1} D_{\operatorname{cod}\beta} \\ F \arrow[u, dashed, "u"] \arrow[ur, "h"'] & & \end{tikzcd}$$

In other words, $e \circ u = h$.

$$\begin{tikzcd}
F \arrow[r, "u"] \arrow[d, "\tau_j"'] & E \arrow[ld, "\mu_j"] \\
D_j &
\end{tikzcd}$$

$$\mu_j \circ u = \pi_j \circ e \circ u = \pi_j \circ h = \tau_j, \tag{7.13}$$

for all $j \in \mathcal{J}_0$. Therefore, u is indeed an arrow from the cone (F, τ_j) to (E, μ_j). We now need to show the uniqueness of u. Suppose there is another arrow $v : F \to E$ such that $\mu_j \circ v = \tau_j$. Then we have

$$\tau_j = \mu_j \circ v = \pi_j \circ (e \circ v). \tag{7.14}$$

for all $j \in \mathcal{J}_0$. But h is the **unique** arrow such that $\pi_j \circ h = \tau_j$ for all $j \in \mathcal{J}_0$. Therefore, by the uniqueness of h, $h = e \circ v$. Moreover, u is the **unique** arrow such that $e \circ u = h$. So $u = v$, proving the uniqueness of u. Hence, there is a **unique** arrow from the cone (F, τ_j) to (E, μ_j). So (E, μ_j) is a limiting cone. ■

Note that there is no real use of the finiteness of the index category, except for forming the products

$$\prod_{i \in \mathcal{J}_0} D_i \text{ and } \prod_{\beta \in \mathcal{J}_1} D_{\operatorname{cod} \beta}.$$

So we have the following corollary.

Corollary 7.2

A category has all limits of some cardinality if and only if it has all equalizers and all products of that cardinality, where the category $\mathcal{C}$ is said to have limits (resp. products) of a cardinality κ if and only if $\mathcal{C}$ has a limit for every diagram $D : \mathcal{J} \to \mathcal{C}$ where $\operatorname{card}(\mathcal{J}_1) \leq \kappa$.

§7.2 Preservation of Limits

Definition 7.5. A functor $F : \mathcal{C} \to \mathcal{D}$ is said to **preserve limits** of type $\mathcal{J}$ if, whenever $p_j : \varprojlim_k D_k \to D_j$ is a limit for a diagram $D : \mathcal{J} \to \mathcal{C}$; the cone

$$F(p_j) : F\left(\varprojlim_k D_k\right) \to F(D_j)$$

is then a limit for the diagram $F \circ D : \mathcal{J} \to \mathcal{D}$. In other words,

$$F\left(\varprojlim_k D_k\right) \cong \varprojlim_k FD_k.$$

(Here, $FD_i = (FD)_i = F(D_i)$) A functor that preserves all limits is said to be **continuous**.

We have seen earlier that given a category $\mathcal{C}$ admitting products of a given cardinality and all equalizers, it admits all limits of the same cardinality for any diagram $D : \mathcal{J} \to \mathcal{C}$. Now, take a functor $F : \mathcal{C} \to \mathcal{D}$ that preserves products of the given cardinality and also preserve equalizers. One can immediately form a diagram $F \circ D : \mathcal{J} \to \mathcal{D}$. Now, construct a limit $(E, \mu_j : E \to D_j)$ of the diagram $D : \mathcal{J} \to \mathcal{C}$ following the way outlined in the proof of Proposition 7.1, where the apex E of the limiting cone is obtained from the equalizer of a pair of parallel arrows between two products. Now use the property of the functor $F : \mathcal{C} \to \mathcal{D}$ that is preserves products and equalizers to construct a limit of the diagram

$F \circ D : \mathcal{J} \to \mathcal{D}$ in the category $\mathcal{D}$. Then the pertinent limit is $(F(E), F(\mu_j) : F(E) \to F(D_j))$. It means that one has

$$F\left(\varprojlim_k D_k\right) \cong \varprojlim_k FD_k.$$

The above definition is precisely the definition of preservtion of a limit under the functor $F : \mathcal{C} \to \mathcal{D}$. One observes that in order to verify that a functor preserves limit of a certain cardinality, it suffices to show that it preserves products of that cardinality and all equalizers.

For instance, let $\mathcal{C}$ be a (locally small) category with all small limits (i.e. the index category is small). The representable functors will preserves those limits. In other words, for a given object C in $\mathcal{C}$, the functor

$$\mathrm{Hom}_{\mathcal{C}}(C, -) : \mathcal{C} \to \mathbf{Sets}$$

will preserve limits. In fact, a stronger result is true. $\mathrm{Hom}_{\mathcal{C}}(C, -)$ preserves all limits that exist.

Theorem 7.3

The covariant representable functors $\mathrm{Hom}_{\mathcal{C}}(X, -)$ preserve limits that exist. In other words, if $\mu_j : \varprojlim_i D_i \to D_j$ is a limit for the diagram $D : \mathcal{J} \to \mathcal{C}$, then the cone

$$\mathrm{Hom}_{\mathcal{C}}(X, -)(\mu_j) : \mathrm{Hom}_{\mathcal{C}}(X, -)\left(\varprojlim_i D_i\right) \to \mathrm{Hom}_{\mathcal{C}}(X, -)(D_j)$$

is a limit for the diagram

$$\mathrm{Hom}_{\mathcal{C}}(X, -) \circ D : \mathcal{J} \to \mathbf{Sets}.$$

Proof. Given $\alpha : j \to k$ in $\mathcal{J}$, the following is a commutative diagram in $\mathcal{C}$.

$$\begin{tikzcd}
& \varprojlim_i D_i \arrow[dl, "\mu_j"'] \arrow[dr, "\mu_k"] & \\
D_j \arrow[rr, "D_\alpha"'] & & D_k
\end{tikzcd}$$

Under the action of the functor $\mathrm{Hom}_{\mathcal{C}}(X, -)$, this commutative diagram goes to the following commutative diagram in **Sets**.

$$\begin{tikzcd}
& \mathrm{Hom}_{\mathcal{C}}\left(X, \varprojlim_i D_i\right) \arrow[dl, "(\mu_j)_*"'] \arrow[dr, "(\mu_k)_*"] & \\
\mathrm{Hom}_{\mathcal{C}}(X, D_j) \arrow[rr, "(D_\alpha)_*"'] & & \mathrm{Hom}_{\mathcal{C}}(X, D_k)
\end{tikzcd}$$

Therefore, $\left(\mathrm{Hom}_{\mathcal{C}}\left(X, \varprojlim_i D_i\right), (\mu_j)_*\right)$ is a cone to the diagram

$$\mathrm{Hom}_{\mathcal{C}}(X, -) \circ D : \mathcal{J} \to \mathbf{Sets}.$$

We now need to show that it is a limiting cone. For that purpose, suppose (C, p_j) is another cone to this diagram in **Sets**. We need to show the existence of a **unique** arrow $u : (C, p_j) \to \left(\mathrm{Hom}_{\mathcal{C}}\left(X, \varprojlim_i D_i\right), (\mu_j)_*\right)$.

$$\begin{tikzcd}
 & C \arrow[d, dashed, "u"] \arrow[ddl, bend right, "p_j"'] \arrow[ddr, bend left, "p_k"] & \\
 & \operatorname{Hom}_{\mathcal{C}}\left(X, \varprojlim_i D_i\right) \arrow[dl, "(\mu_j)_*"] \arrow[dr, "(\mu_k)_*"'] & \\
\operatorname{Hom}_{\mathcal{C}}(X, D_j) \arrow[rr, "(D_\alpha)_*"'] & & \operatorname{Hom}_{\mathcal{C}}(X, D_k)
\end{tikzcd}$$

Since (C, p_j) is a cone, we have

$$(D_\alpha)_* \circ p_j = p_k \tag{7.15}$$

for every arrow $\alpha : j \to k$ in $\mathcal{J}$. Therefore, given $c \in C$,

$$p_k(c) = ((D_\alpha)_* \circ p_j)(c) = (D_\alpha)_*(p_j(c)) = D_\alpha \circ p_j(c), \tag{7.16}$$

where $p_j(c) : X \to D_j$ and $p_k(c) : X \to D_k$ are arrows in $\mathcal{C}$. In other words, the following diagram commutes for all arrows $\alpha : j \to k$ in $\mathcal{J}$:

$$\begin{tikzcd}
 & X \arrow[dl, "p_j(c)"'] \arrow[dr, "p_k(c)"] & \\
D_j \arrow[rr, "D_\alpha"'] & & D_k
\end{tikzcd}$$

Therefore, we have a cone $(X, p_j(c))$ in $\mathcal{C}$ to the diagram $D : \mathcal{J} \to \mathcal{C}$. Since $\mu_j : \varprojlim_i D_i \to D_j$ is a limit for the diagram $D : \mathcal{J} \to \mathcal{C}$, there is a **unique** arrow $u(c) : (X, p_j(c)) \to \left(\varprojlim_i D_i, \mu_j\right)$.

$$\begin{tikzcd}
 & X \arrow[d, dashed, "u(c)"] \arrow[ddl, bend right, "p_j(c)"'] \arrow[ddr, bend left, "p_k(c)"] & \\
 & \varprojlim_i D_i \arrow[dl, "\mu_j"] \arrow[dr, "\mu_k"'] & \\
D_j \arrow[rr, "D_\alpha"'] & & D_k
\end{tikzcd}$$

In other words, there exists a **unique** arrow $u(c) : X \to \varprojlim_i D_i$ such that

$$\mu_j \circ u(c) = p_j(c) \tag{7.17}$$

for all $j \in \operatorname{Ob}(\mathcal{J})$. So we define the function

$$\begin{aligned} u : C &\longrightarrow \operatorname{Hom}_{\mathcal{C}}\left(X, \varprojlim_i D_i\right) \\ c &\longmapsto u(c). \end{aligned} \tag{7.18}$$

Then for all $c \in C$,

$$p_j(c) = \mu_j \circ u(c) = ((\mu_j)_* \circ u)(c). \tag{7.19}$$

In other words,

$$p_j = (\mu_j)_* \circ u \tag{7.20}$$

for all $j \in \mathrm{Ob}(\mathcal{J})$. So u is indeed an arrow

$$u : (C, p_j) \to \left(\mathrm{Hom}_{\mathcal{C}}\left(X, \varprojlim_i D_i\right), (\mu_j)_*\right).$$

We now need to show the uniqueness of u. Suppose there is another

$$v : (C, p_j) \to \left(\mathrm{Hom}_{\mathcal{C}}\left(X, \varprojlim_i D_i\right), (\mu_j)_*\right).$$

So we have the following commutative diagram in **Sets**:

$$\begin{tikzcd}
 & C \arrow[d, shift right, "u"'] \arrow[d, shift left, "v"] \arrow[ddl, bend right, "p_j"'] \arrow[ddr, bend left, "p_k"] & \\
 & \mathrm{Hom}_{\mathcal{C}}\left(X, \varprojlim_i D_i\right) \arrow[dl, "(\mu_j)_*"] \arrow[dr, "(\mu_k)_*"'] & \\
\mathrm{Hom}_{\mathcal{C}}(X, D_j) \arrow[rr, "(D_\alpha)_*"'] & & \mathrm{Hom}_{\mathcal{C}}(X, D_k)
\end{tikzcd}$$

Then for all $j \in \mathrm{Ob}(\mathcal{J})$,

$$(\mu_j)_* \circ u = p_j = (\mu_j)_* \circ v. \tag{7.21}$$

Then for all $c \in C$,

$$p_j(c) = ((\mu_j)_* \circ v)(c) = \mu_j \circ v(c), \tag{7.22}$$

which is true for all $j \in \mathrm{Ob}(\mathcal{J})$. But $u(c)$ was the **unique** arrow $u(c) : X \to \varprojlim_i D_i$ such that $p_j(c) = \mu_j \circ u(c)$. So $v(c) = u(c)$, by the uniqueness of $u(c)$. Therefore, u is unique, and hence

$$(\mu_j)_* : \mathrm{Hom}_{\mathcal{C}}\left(X, \varprojlim_i D_i\right) \to \mathrm{Hom}_{\mathcal{C}}(X, D_j)$$

is a limit for the diagram

$$\mathrm{Hom}_{\mathcal{C}}(X, -) \circ D : \mathcal{J} \to \textbf{Sets}.$$

■

Corollary 7.4

Let $\mathcal{C}$ be a category that admits all limits. Then the representable functors $\mathrm{Hom}_{\mathcal{C}}(C, -)$ preserve all limits.

Definition 7.6 (Contravariant functor). A **contravariant functor** $F : \mathcal{C} \to \mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$ is an assignment of $\mathrm{Ob}(\mathcal{C})$ to $\mathrm{Ob}(\mathcal{D})$ and a mapping of arrows in $\mathcal{C}$ to $\mathcal{D}$:

$$\mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F(B), F(A)),$$

where $F(A)$ and $F(B)$ are the assigned objects of $\mathcal{D}$ under F. In other words, given an arrow $f : A \to B$ in $\mathcal{C}$, $F(f) : F(B) \to F(A)$ is an arrow in $\mathcal{D}$, such that the following hold:

1. $F(\mathbb{1}_A) = \mathbb{1}_{F(A)}$, for all $A \in \mathrm{Ob}(\mathcal{C})$.
2. $F(g \circ f) = F(f) \circ F(g)$ for arrows $f : A \to B$ and $g : B \to C$ in $\mathcal{C}$.

The idea of a contravariant functor can be pictured as follows:

$$\begin{tikzcd}
& \mathcal{C} & & & \mathcal{D} \\
A \arrow[r, "f"] \arrow[dr, "g\circ f"'] & B \arrow[d, "g"] & & F(A) & F(B) \arrow[l, "F(f)"'] \\
& C & {} \arrow[r, "F"] & {} & F(C) \arrow[u, "F(g)"'] \arrow[ul, "F(g\circ f)"]
\end{tikzcd}$$

A contravariant functor $F : \mathcal{C} \to \mathcal{D}$ is in one-to-one correspondence with a (covariant) functor $F' : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$. F' is defined as follows: $\mathrm{Ob}\,(\mathcal{C})$ is the same as $\mathrm{Ob}\,(\mathcal{C}^{\mathrm{op}})$, so given an object $X \in \mathrm{Ob}\,(\mathcal{C}^{\mathrm{op}})$,

$$F'\,(X) = F\,(X)\,. \tag{7.23}$$

Given an arrow $f^{\mathrm{op}} : Y \to X$ in $\mathcal{C}^{\mathrm{op}}$, there is an arrow $f : X \to Y$ in $\mathcal{C}$, and we define

$$F'\,(f^{\mathrm{op}}) = F\,(f)\,. \tag{7.24}$$

Since $F : \mathcal{C} \to \mathcal{D}$ is a contravariant functor, $F\,(f)$ is an arrow from $F\,(Y)$ to $F\,(X)$, i.e.

$$F'\,(f^{\mathrm{op}}) : F'\,(Y) \to F'\,(X)\,.$$

Let $g : Y \to Z$ be an arrow in $\mathcal{C}$ so that f and g are composable in $\mathcal{C}$.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & Y \arrow[r, "g"] & Z
\end{tikzcd} \qquad \text{in } \mathcal{C} \qquad \begin{tikzcd}
X & Y \arrow[l, "f^{\mathrm{op}}"'] & Z \arrow[l, "g^{\mathrm{op}}"'] \arrow[ll, bend left, "(g\circ f)^{\mathrm{op}}=f^{\mathrm{op}}\circ g^{\mathrm{op}}"]
\end{tikzcd} \qquad \text{in } \mathcal{C}^{\mathrm{op}}$$

Since F is a contravariant functor,

$$\begin{aligned} F'\,(f^{\mathrm{op}} \circ g^{\mathrm{op}}) &= F'\,((g \circ f)^{\mathrm{op}}) = F\,(g \circ f) \\ &= F\,(f) \circ F\,(g) = F'\,(f^{\mathrm{op}}) \circ F'\,(g^{\mathrm{op}})\,. \end{aligned} \tag{7.25}$$

Besides, since the identity arrows in both $\mathcal{C}$ and $\mathcal{C}^{\mathrm{op}}$ are the same, one has

$$F'\,(\mathbb{1}_X) = F\,(\mathbb{1}_X) = \mathbb{1}_{F(X)} = \mathbb{1}_{F'(X)}\,. \tag{7.26}$$

So $F' : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ is indeed a covariant functor.

A typical example of a contravariant functor is given by

$$\mathrm{Hom}_{\mathcal{C}}\,(-, C) : \mathcal{C} \to \mathbf{Sets}$$

associated with an object $C \in \mathrm{Ob}\,(\mathcal{C})$, which is defined as follows: given an arrow $f : X \to Y$ in $\mathcal{C}$, the function $\mathrm{Hom}_{\mathcal{C}}\,(f, C) = f^* : \mathrm{Hom}_{\mathcal{C}}\,(Y, C) \to \mathrm{Hom}_{\mathcal{C}}\,(X, C)$ is defined as

$$f^*\,(g) = g \circ f, \tag{7.27}$$

for all $g \in \mathrm{Hom}_{\mathcal{C}}\,(Y, C)$.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & Y \arrow[r, "g"] & C
\end{tikzcd}$$

The associated covariant functor with this contravariant functor is

$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}\,(C, -) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Sets}.$$

§7.3 Colimits

Colimit is the dual construction of limit. As before, we define a cocone to a diagram $D : \mathcal{J} \to \mathcal{C}$ and we form the category of cocones. A colimit for the diagram D is then defined to be an initial object in the category of cocones.

Definition 7.7 (Cocone)**.** A **cocone** to a diagram D consists of

- an object C in $\mathcal{C}$, and
- a family of arrows in $\mathcal{C}$: $c_j : D_j \to C$ one for each object j in $\mathcal{J}$.

such that for each arrow $\alpha : i \to j$ in $\mathcal{J}$, $c_j \circ D_\alpha = c_i$, i.e. the following diagram commutes:

$$\begin{tikzcd}
D_i \arrow[rr, "D_\alpha"] \arrow[dr, "c_i"'] & & D_j \arrow[dl, "c_j"] \\
& C &
\end{tikzcd}$$

Such a cone is denoted by (C, c_j). A morphism of cones

$$\vartheta : (C, c_j) \to \left(C', c'_j\right)$$

is an arrow $\vartheta : C \to C'$ in $\mathcal{C}$ making each triangle of the following form commute: (for each $j \in \mathrm{Ob}(\mathcal{J})$)

$$\begin{tikzcd}
C \arrow[r, "\vartheta"] \arrow[dr, "c_j"'] & C' \arrow[d, "c'_j"] \\
& D_j
\end{tikzcd}$$

i.e. $c'_j = \vartheta \circ c_j$ for all $j \in \mathrm{Ob}(\mathcal{J})$. Thus we have a category **Cocone** (D) of all cocones to a diagram D.

Definition 7.8 (Coimit)**.** A **colimit** for a diagram $D : \mathcal{J} \to \mathcal{C}$ is a terminal object in **Cocone** (D). A **finite colimit** is a colimit for a diagram on a finite index category $\mathcal{J}$.

We often denote a colimit in the form

$$p_i : D_i \to \varinjlim_j D_j.$$

The colimit of a diagram itself is a cocone $\left(\varinjlim_j D_j, p_j\right)$. The colimit of a diagram has the following UMP: Given any cocone (C, c_j) to D, there is a **unique** arrow $u : \varinjlim_j D_j \to C$ such that

$$u \circ p_j = c_j, \tag{7.28}$$

for all j. In other words, the following diagram commutes for each i, j:

$$\begin{tikzcd}
& C & \\
& \varinjlim_j D_j \arrow[u, dashed, "u"'] & \\
D_i \arrow[uur, bend left, "c_i"] \arrow[ur, "p_i"] \arrow[rr, "D_\alpha"'] & & D_j \arrow[uul, bend right, "c_j"'] \arrow[ul, "p_j"']
\end{tikzcd}$$

Thus the colimiting cone $\left(\varinjlim_j D_j, p_j\right)$ can be thought of as the "furthest" one to the diagram D, or the "universal repelling object".

Lemma 7.5

Let (L, μ_j) be a limit to the diagram $D : \mathcal{J} \to \mathcal{C}$. Then $\left(L, \mu_j^{\text{op}}\right)$ is a colimit to the opposite diagram $D^{\text{op}} : \mathcal{J}^{\text{op}} \to \mathcal{C}^{\text{op}}$, defined as

$$\begin{aligned} D^{\text{op}}(i) &= D_i, \\ D^{\text{op}}(\alpha^{\text{op}}) &= D(\alpha)^{\text{op}} = D_\alpha^{\text{op}}. \end{aligned}$$

Proof. Since (L, μ_j) is a cone to the diagram D in $\mathcal{C}$, we have $\mu_j = D_\alpha \circ \mu_i$ for each arrow $\alpha : i \to j$ in $\mathcal{J}$, i.e. the following diagram commutes in $\mathcal{C}^{\text{op}}$:

$$\begin{tikzcd} & L \arrow[dl, "\mu_i"'] \arrow[dr, "\mu_j"] & \\ D_i \arrow[rr, "D_\alpha"'] & & D_j \end{tikzcd}$$

Then reversing all the arrows, we get the following commutative diagram in $\mathcal{C}^{\text{op}}$:

$$\begin{tikzcd} & L & \\ D_i \arrow[ur, "\mu_i^{\text{op}}"] & & D_j \arrow[ul, "\mu_j^{\text{op}}"'] \arrow[ll, "D_\alpha^{\text{op}}"] \end{tikzcd}$$

Therefore, for every arrow $\alpha^{\text{op}} : j \to i$ in $\mathcal{J}^{\text{op}}$, $\mu_j^{\text{op}} = \mu_i^{\text{op}} \circ D_\alpha^{\text{op}}$. So $\left(L, \mu_j^{\text{op}}\right)$ is indeed a cocone to the diagram $D^{\text{op}} : \mathcal{J}^{\text{op}} \to \mathcal{C}^{\text{op}}$ in $\mathcal{C}^{\text{op}}$.

Note that we only used the fact that (L, μ_j) is a cone. So this actually proves that if (C, p_j) is a cone to the diagram $D : \mathcal{J} \to \mathcal{C}$, then $\left(C, p_j^{\text{op}}\right)$ is a cocone to the opposite diagram $D^{\text{op}} : \mathcal{J}^{\text{op}} \to \mathcal{C}^{\text{op}}$. Dually, if $\left(C, p_j^{\text{op}}\right)$ is a cocone to the diagram $D^{\text{op}} : \mathcal{J}^{\text{op}} \to \mathcal{C}^{\text{op}}$, then (C, p_j) is a cone to the diagram $D : \mathcal{J} \to \mathcal{C}$.

Let us now verify that $\left(L, \mu_j^{\text{op}}\right)$ is a colimit. Suppose we have another cocone $\left(C, p_j^{\text{op}}\right)$ to the diagram $D^{\text{op}} : \mathcal{J}^{\text{op}} \to \mathcal{C}^{\text{op}}$. Then (C, p_j) is a cone to the diagram $D : \mathcal{J} \to \mathcal{C}$. Since (L, μ_j) is a limiting cone to the diagram D, there is a **unique** arrow $u : C \to L$ such that $p_j = \mu_j \circ u$ for all $j \in \operatorname{Ob} \mathcal{J}$. In other words, the following diagram commutes for each i, j:

$$\begin{tikzcd} & C \arrow[d, dashed, "u"] \arrow[ddl, bend right, "p_i"'] \arrow[ddr, bend left, "p_j"] & \\ & L \arrow[dl, "\mu_i"'] \arrow[dr, "\mu_j"] & \\ D_i \arrow[rr, "D_\alpha"'] & & D_j \end{tikzcd}$$

Reversing all the arrows, we get the following commutative diagram in $\mathcal{C}^{\text{op}}$:

$$\begin{tikzcd}
 & C & \\
 & L \arrow[u, dashed, "u^{\mathrm{op}}"] & \\
D_i \arrow[ur, "\mu_i^{\mathrm{op}}"] \arrow[uur, bend left, "p_i^{\mathrm{op}}"] & & D_j \arrow[ul, "\mu_j^{\mathrm{op}}"'] \arrow[uul, bend right, "p_j^{\mathrm{op}}"'] \arrow[ll, "D_\alpha^{\mathrm{op}}"]
\end{tikzcd}$$

So, given another cocone $\left(C, p_j^{\mathrm{op}}\right)$, there is an arrow $u^{\mathrm{op}} : L \to C$ such that $p_j^{\mathrm{op}} = u^{\mathrm{op}} \circ \mu_j^{\mathrm{op}}$ for all $j \in \mathrm{Ob}\left(\mathcal{J}^{\mathrm{op}}\right)$. The uniqueness of u follows from the fact that u is unique. Indeed, if there existed another arrow $v^{\mathrm{op}} : L \to C$ satisfying $p_j^{\mathrm{op}} = v^{\mathrm{op}} \circ \mu_j^{\mathrm{op}}$ (in $\mathcal{C}^{\mathrm{op}}$) for all $j \in \mathrm{Ob}\left(\mathcal{J}^{\mathrm{op}}\right)$, then v satisfies $p_j = \mu_j \circ v$ (in $\mathcal{C}$) for all $j \in \mathrm{Ob}\,\mathcal{J}$. Therefore, by the uniqueness of u, $u = v$, and hence $v^{\mathrm{op}} = u^{\mathrm{op}}$. Hence, given a cocone $\left(C, p_j^{\mathrm{op}}\right)$, there is a **unique** arrow $u^{\mathrm{op}} : \left(L, \mu_j^{\mathrm{op}}\right) \to \left(C, p_j^{\mathrm{op}}\right)$ in the cocone category **Cocone**(D^{op}). Therefore, $\left(L, \mu_j^{\mathrm{op}}\right)$ is an initial object in **Cocone**(D^{op}), i.e. a colimit to the diagram D^{op}. ■

Corollary 7.6

The contravariant representable functors $\mathrm{Hom}_{\mathcal{C}}(-,X)$ take colimits to limits. In other words, if $\mu_j : D_j \to \varinjlim_i D_i$ is a colimit for the diagram $D : \mathcal{J} \to \mathcal{C}$, then the cocone

$$\mathrm{Hom}_{\mathcal{C}}(-,X)(\mu_j) : \mathrm{Hom}_{\mathcal{C}}(-,X)\left(\varinjlim_i D_i\right) \to \mathrm{Hom}_{\mathcal{C}}(-,X)(D_j)$$

is a limit for the diagram

$$\mathrm{Hom}_{\mathcal{C}}(-,X) \circ D : \mathcal{J} \to \mathbf{Sets}.$$

Proof. Given the colimiting cocone

$$\begin{tikzcd}
D_i \arrow[rr, "D_\alpha"] \arrow[dr, "\mu_i"'] & & D_j \arrow[dl, "\mu_j"] \\
 & \varinjlim_i D_i &
\end{tikzcd}$$

let us examine where the functor $\mathrm{Hom}_{\mathcal{C}}(-,X)$ takes this cocone to.

$$\mathrm{Hom}_{\mathcal{C}}(-,X)\left(\begin{tikzcd}
D_i \arrow[rr, "D_\alpha"] \arrow[dr, "\mu_i"'] & & D_j \arrow[dl, "\mu_j"] \\
 & \varinjlim_i D_i &
\end{tikzcd}\right) = \begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(D_i, X) & & \mathrm{Hom}_{\mathcal{C}}(D_j, X) \arrow[ll, "\mathrm{Hom}_{\mathcal{C}}(D_\alpha, X)"'] \\
 & \mathrm{Hom}_{\mathcal{C}}\left(\varinjlim_i D_i, X\right) \arrow[ul, "\mathrm{Hom}_{\mathcal{C}}(\mu_i, X)"] \arrow[ur, "\mathrm{Hom}_{\mathcal{C}}(\mu_j, X)"'] &
\end{tikzcd} \tag{7.29}$$

Since the contravariant functor $\mathrm{Hom}_{\mathcal{C}}(-,X)$ is in one-to-one correspondence with the covariant functor $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X,-) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Sets}$, the diagram on the RHS of (7.29) is isomorphic to the follwoing diagram:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, D_i) & & \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, D_j) \arrow[ll, "\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}\left(X, D_\alpha^{\mathrm{op}}\right)"'] \\
 & \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}\left(X, \varinjlim_i D_i\right) \arrow[ul, "\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}\left(X, \mu_i^{\mathrm{op}}\right)"] \arrow[ur, "\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}\left(X, \mu_j^{\mathrm{op}}\right)"'] &
\end{tikzcd} \tag{7.30}$$

Furthermore, since $\left(\varinjlim_i D_i, \mu_j\right)$ is a colimit to the diagram $D : \mathcal{J} \to \mathcal{C}$, by the dual statement of Lemma 7.5, $\left(\varinjlim_i D_i, \mu_j^{\mathrm{op}}\right)$ is a limit to the diagram $D^{\mathrm{op}} : \mathcal{J}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$. Furthermore, by Theorem 7.3, the covariant hom-functors preserve limits. Therefore, the diagram in (7.30) is a limiting cone. In other words, the diagrams in (7.29) are also limiting cones. Hence, $\mathrm{Hom}_{\mathcal{C}}(-, X)$ takes colimits to limits. ■

Example 7.2 (Pushout)**.** Pushout is the dual construction of pushback. Let us consider pushout in **Sets**. Suppose we have two functions:

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "f"'] & C \\
B &
\end{tikzcd}$$

We can construct the pushout of f and g as follows: Start with the coproduct (disjoint union) of B and C.

$$\begin{tikzcd}
B \arrow[r, "i_B"] & B + C & C \arrow[l, "i_C"']
\end{tikzcd}$$

The identity those elements $b \in B$ with the elements $c \in C$ whenever there is an element $a \in A$ with

$$f(a) = b \text{ and } g(a) = c,$$

i.e. we take the equivalence relation $\sim$ on $B + C$ generated by the conditions

$$(f(a), 0) \sim (g(a), 1),$$

for each $a \in A$. Finally quotient out $B + C$ by $\sim$ to obtain the pushout

$$B +_A C \quad \cong \quad (B + C) / \sim . \tag{7.31}$$

We obtain $B +_A C$ as the following coequalizer:

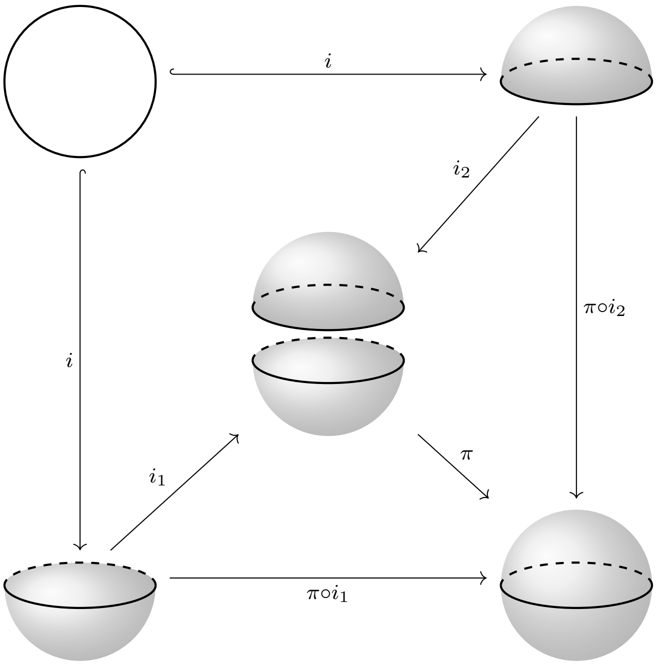
$$\begin{tikzcd}
A \arrow[r, shift left, "i_B \circ f"] \arrow[r, shift right, "i_C \circ g"'] & B + C \arrow[r, "\pi"] & B +_A C
\end{tikzcd}$$

It's the dual construction of the pullback, as the pullback was obtained as an equalizer between two parallel arrows.

$$\begin{tikzcd}
A \arrow[rr, "g"] \arrow[dd, "f"'] & & C \arrow[dl, "i_C"'] \arrow[dd, "\pi \circ i_C"] \\
& B + C \arrow[dr, "\pi"] & \\
B \arrow[ur, "i_B"] \arrow[rr, "\pi \circ i_B"'] & & B +_A C
\end{tikzcd}$$

Example 7.3 (Pushout in **Top**)**.** Pushouts in **Top** are similarly formed from coproducts and coequalizers, which can be made first in **Sets** and then topologized as sum (topological sum) and quotient spaces. For example, pushout can be used to construct S^2 from D^2. Let D^2 be the 2-disk and S^1 be the 1-dimensional sphere (i.e. circle) with the inclusion $i : S^1 \to D^2$ as the boundary of the disk. Then the 2-sphere S^2 is the pushout

$$\begin{tikzcd}
S^1 \arrow[r, hook, "i"] \arrow[d, hook, "i"'] & D^2 \arrow[d] \\
D^2 \arrow[r] & S^2
\end{tikzcd}$$



Here π is the quotient map that identifies the boundaries of the 2-disks.

Example 7.4 (Direct limit of groups)**.** Suppose we are given a sequence

$$G_0 \xrightarrow{g_0} G_1 \xrightarrow{g_1} G_2 \xrightarrow{g_2} G_3 \xrightarrow{g_3} \cdots$$

of groups and group homomorphsms, and we want a colimiting group G_∞ with group homomorphisms $u_n : G_n \to G_\infty$ satisfying $u_{n+1} \circ g_n = u_n$ for all $n \in \mathbb{N}$. Moreover, G_∞ should be "universal" with this property. We are looking at the colimit of the diagram $D : \omega \to$ **Groups**, where the index category is the ordinal numbers $\omega = (\mathbb{N}, \leq)$ regarded as a poset category

$$0 \longrightarrow 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots$$

So the diagram in **Groups** looks like

$$G_0 \xrightarrow{g_0} G_1 \xrightarrow{g_1} G_2 \xrightarrow{g_2} G_3 \xrightarrow{g_3} \cdots \tag{7.32}$$

It is a diagram of type ω in the category **Groups**. The colimiting group is the colimit of this diagram D, which is an initial object in the category **Cocone** (D).

$$\varinjlim_{n\in\omega} G_n \cong G_\infty.$$

This group always exists and can be constructed as follows. Begin with the coproduct (disjoint union) of the sets G_n, for $n \in \omega$:

$$\bigsqcup_{n\in\omega} G_n$$

is regarded as the union of all ordered pairs (x_n, n) for $x_n \in G_n$, as n varies over ω. We want to put an equivalence relation $\sim$ on $\bigsqcup_{n\in\omega} G_n$. For $i \leq j$, we define $\widetilde{g}_{i,j}$ to be the composite:

$$\begin{tikzcd}
G_i \arrow[r, "g_i"] \arrow[rrrr, bend right, "\widetilde{g}_{i,j}=g_{j-1}\circ\cdots\circ g_{i+1}\circ g_i"'] & G_{i+1} \arrow[r, "g_{i+1}"] & G_{i+2} \arrow[r, "g_{i+2}"] & \cdots \arrow[r, "g_{j-1}"] & G_j
\end{tikzcd}$$

In particular, $\widetilde{g}_{i,i+1} = g_i$, and $\widetilde{g}_{i,i} = \mathbb{1}_{G_i}$. We define $(x_n, n) \sim (y_m, m)$ if and only if there exists some $k \geq m, n$ such that

$$\widetilde{g}_{n,k}\left(x_n\right) = \widetilde{g}_{m,k}\left(y_m\right). \tag{7.33}$$

Since $\widetilde{g}_{n,n+1} = g_n$, one has $(x_n, n) \sim (g_n(x_n), n+1)$ for each $x_n \in G_n$. One can easily verify that $\sim$ is an equivalence relation.

- (Reflexivity) Given $(x_n, n) \in \bigsqcup_{i\in\omega} G_i$, $(x_n, n) \sim (x_n, n)$, because $\widetilde{g}_{n,n}\left(x_n\right) = \widetilde{g}_{n,n}\left(x_n\right)$.
- (Symmetry) If $(x_n, n) \sim (y_m, m)$, there exists some $k \geq m, n$ such that $\widetilde{g}_{n,k}\left(x_n\right) = \widetilde{g}_{m,k}\left(y_m\right)$. So there is some $k \geq n, m$ such that $\widetilde{g}_{m,k}\left(y_m\right) = \widetilde{g}_{n,k}\left(x_n\right)$. Therefore, $(y_m, m) \sim (x_n, n)$.
- (Transitivity) Suppose $(x_n, n) \sim (y_m, m)$ and $(y_m, m) \sim (z_l, l)$. Then there exists some $k_1 \geq m, n$ and $k_2 \geq n, l$ such that

  $$\widetilde{g}_{n,k_1}\left(x_n\right) = \widetilde{g}_{m,k_1}\left(y_m\right) \text{ and } \widetilde{g}_{m,k_2}\left(y_m\right) = \widetilde{g}_{l,k_2}\left(z_l\right). \tag{7.34}$$

  Without loss of generality, suppose $k_2 \geq k_1$ (if not, we can always choose a larger k_2). Then

  $$\widetilde{g}_{k_1,k_2} \circ \widetilde{g}_{n,k_1} = \left(g_{k_2-1} \circ \cdots \circ g_{k_1+1} \circ g_{k_1}\right) \circ \left(g_{k_1-1} \circ \cdots \circ g_{n+1} \circ g_n\right) = g_{n,k_2}. \tag{7.35}$$

  Similarly, $g_{m,k_2} = \widetilde{g}_{k_1,k_2} \circ \widetilde{g}_{m,k_1}$. Then

  $$\begin{aligned} \widetilde{g}_{l,k_2}\left(z_l\right) &= \widetilde{g}_{m,k_2}\left(y_m\right) \\ &= \widetilde{g}_{k_1,k_2}\left(\widetilde{g}_{m,k_1}\left(y_m\right)\right) \\ &= \widetilde{g}_{k_1,k_2}\left(\widetilde{g}_{n,k_1}\left(x_n\right)\right) \\ &= \widetilde{g}_{n,k_2}\left(x_n\right). \end{aligned} \tag{7.36}$$

  Therefore, $(x_n, n) \sim (z_l, l)$.

So $\sim$ is an equivalence relation. Then we define

$$G_\infty = \bigsqcup_{n\in\omega} G_n / \sim. \tag{7.37}$$

An equivalence class in G_∞ is denoted as $[x_n]$. The group operation on G_∞ is defined as follows:

$$[x] \cdot [y] = \left[x' \cdot y'\right], \tag{7.38}$$

with $x \sim x'$ and $y \sim y'$ for $x', y' \in G_n$ for sufficiently large n. If $x_i \in G_i$ and $y_j \in G_j$, then choosing $n = \max\{i, j\}$ is sufficient. So

$$[x_i] \cdot [y_j] = \left[\widetilde{g}_{i,n}\left(x_i\right) \cdot \widetilde{g}_{j,n}\left(y_j\right)\right]. \tag{7.39}$$

The well-definedness of the operation (7.39) can be checked easily. Suppose $[x_i] = [a_k]$ and $[y_j] = [b_l]$, for some $a_k \in G_k$ and $b_l \in G_l$. Let $n = \max\{i, j\}$ and $m = \max\{k, l\}$. Then

$$[x_i] \cdot [y_j] = \left[\widetilde{g}_{i,n}\left(x_i\right) \cdot \widetilde{g}_{j,n}\left(y_j\right)\right], \tag{7.40}$$

$$[a_k] \cdot [b_l] = \left[\widetilde{g}_{k,m}\left(a_k\right) \cdot \widetilde{g}_{l,m}\left(b_l\right)\right]. \tag{7.41}$$

Since $x_i \sim a_k$, there exists some $q \geq i, k$ such that

$$\widetilde{g}_{i,q}\left(x_i\right) = \widetilde{g}_{k,q}\left(a_k\right). \tag{7.42}$$

Similarly, since $y_j \sim b_l$, there exists some $r \geq j, l$ such that

$$\widetilde{g}_{j,r}\left(y_j\right) = \widetilde{g}_{l,r}\left(b_l\right). \tag{7.43}$$

Let $s = \max\{q, r\}$. Then $s \geq q \geq i, k$ and $s \geq r \geq j, l$. So $s \geq i, j$ and $s \geq k, l$. As a result, $s \geq n = \max\{i, j\}$ and $s \geq m = \max\{k, l\}$. Now,

$$\begin{aligned}
\widetilde{g}_{n,s}\left(\widetilde{g}_{i,n}\left(x_i\right) \cdot \widetilde{g}_{j,n}\left(y_j\right)\right) &= \widetilde{g}_{n,s}\left(\widetilde{g}_{i,n}\left(x_i\right)\right) \cdot \widetilde{g}_{n,s}\left(\widetilde{g}_{j,n}\left(y_j\right)\right) \\
&= \widetilde{g}_{i,s}\left(x_i\right) \cdot \widetilde{g}_{j,s}\left(y_j\right) \\
&= \widetilde{g}_{q,s}\left(\widetilde{g}_{i,q}\left(x_i\right)\right) \cdot \widetilde{g}_{r,s}\left(\widetilde{g}_{j,r}\left(y_j\right)\right) \\
&= \widetilde{g}_{q,s}\left(\widetilde{g}_{k,q}\left(a_k\right)\right) \cdot \widetilde{g}_{r,s}\left(\widetilde{g}_{l,r}\left(b_l\right)\right) \\
&= \widetilde{g}_{k,s}\left(a_k\right) \cdot \widetilde{g}_{l,s}\left(b_l\right) \\
&= \widetilde{g}_{m,s}\left(\widetilde{g}_{k,m}\left(a_k\right)\right) \cdot \widetilde{g}_{m,s}\left(\widetilde{g}_{l,m}\left(b_l\right)\right) \\
&= \widetilde{g}_{m,s}\left(\widetilde{g}_{k,m}\left(a_k\right) \cdot \widetilde{g}_{l,m}\left(b_l\right)\right).
\end{aligned}$$

Therefore, $\widetilde{g}_{i,n}\left(x_i\right) \cdot \widetilde{g}_{j,n}\left(y_j\right) \sim \widetilde{g}_{k,m}\left(a_k\right) \cdot \widetilde{g}_{l,m}\left(b_l\right)$, and hence

$$\left[\widetilde{g}_{i,n}\left(x_i\right) \cdot \widetilde{g}_{j,n}\left(y_j\right)\right] = \left[\widetilde{g}_{k,m}\left(a_k\right) \cdot \widetilde{g}_{l,m}\left(b_l\right)\right]. \tag{7.44}$$

Therefore, the operation defined in (7.39) is well-defined.

If e_i denotes the unit element of the group G_i, then $e_i \sim e_j$ for any i, j since group homomorphisms take unit element to unit element. Then the equivalence class $[e_i]$ is the identity element of G_∞. Indeed, for any $x_n \in G_n$,

$$\begin{aligned}
[x_n] \cdot [e_i] = [x_n] \cdot [e_n] = [x_n \cdot e_n] = [x_n], \\
[e_i] \cdot [x_n] = [e_n] \cdot [x_n] = [e_n \cdot x_n] = [x_n].
\end{aligned}$$

So $[e_i]$ is the identity element of G_∞. The inverse of $[x_n]$ is

$$[x_n]^{-1} = \left[x_n^{-1}\right], \tag{7.45}$$

because

$$[x_n] \cdot \left[x_n^{-1}\right] = \left[x_n \cdot x_n^{-1}\right] = [e_n] \text{ and } \left[x_n^{-1}\right] \cdot [x_n] = \left[x_n^{-1} \cdot x_n\right] = [e_n].$$

So we have defined the group G_∞. Now take the homomorphisms

$$\begin{aligned}
u_n : G_n &\to G_\infty \\
x_n &\mapsto [x_n].
\end{aligned} \tag{7.46}$$

$$\begin{tikzcd}
G_n \arrow[rr, "g_n"] \arrow[dr, "u_n"'] & & G_{n+1} \arrow[dl, "u_{n+1}"] \\
& G_\infty &
\end{tikzcd}$$

For $x_n \in G_n$, $x_n \sim g_n(x_n)$. So $[x_n] = [g_n(x_n)]$. Therefore,

$$\left(u_{n+1} \circ g_n\right)(x_n) = [g_n(x_n)] = [x_n] = u_n(x_n). \tag{7.47}$$

Hence, $u_{n+1} \circ g_n = u_n$ for each n. So $(G_\infty, u_n : G_n \to G_\infty)$ is indeed a cocone. The universality of G_∞ along with the homomorphisms $u_n : G_n \to G_\infty$ results from the fact that this construction is essentially a colimit in **Sets** with an induced group structure. Suppose there is a group H with homomorphisms $h_n : G_n \to H$ such that $h_{n+1} \circ g_n = h_n$ for each n. We need to show the existence of a **unique** $h : G_\infty \to H$ such that $h_n = h \circ u_n$ for each n.

$$\begin{tikzcd}
G_n \arrow[rr, "g_n"] \arrow[dr, "u_n"] \arrow[ddr, bend right, "h_n"'] & & G_{n+1} \arrow[dl, "u_{n+1}"'] \arrow[ddl, bend left, "h_{n+1}"] \\
& G_\infty \arrow[d, dashed, "h"] & \\
& H &
\end{tikzcd} \tag{7.48}$$

Let us define $h : G_\infty \to H$ as follows: given $[x_n] \in G_\infty$ for some $x_n \in G_n$, we define

$$h\left([x_n]\right) = h_n\left(x_n\right). \tag{7.49}$$

We have to show that h is well-defined. Suppose $x_n \sim y_m$ for some $y_m \in G_m$. Then there exists $k \geq n, m$ such that $\widetilde{g}_{n,k}\left(x_n\right) = \widetilde{g}_{m,k}\left(y_m\right)$. Now,

$$\begin{aligned} h_n &= h_{n+1} \circ g_n \\ &= h_{n+2} \circ g_{n+1} \circ g_n \\ &= h_{n+3} \circ g_{n+2} \circ g_{n+1} \circ g_n \\ &= \cdots \\ &= h_k \circ g_{k-1} \circ \cdots \circ g_{n+1} \circ g_n = h_k \circ \widetilde{g}_{n,k}. \end{aligned} \tag{7.50}$$

$$\begin{tikzcd}
 & & H & & \\
G_n \arrow[urr, "h_n"] \arrow[r, "g_n"'] \arrow[rrrr, bend right, "\widetilde{g}_{n,k}=g_{k-1}\circ\cdots\circ g_{n+1}\circ g_n"'] & G_{n+1} \arrow[ur, "h_{n+1}"'] \arrow[r, "g_{n+1}"'] & G_{n+2} \arrow[u, "h_{n+2}"'] \arrow[r, "g_{n+2}"'] & \cdots \arrow[r, "g_{k-1}"'] & G_k \arrow[ull, "h_k"']
\end{tikzcd}$$

Similarly, $h_m = h_k \circ \widetilde{g}_{m,k}$. Therefore,

$$\begin{aligned} h\left([x_n]\right) &= h_n\left(x_n\right) \\ &= h_k\left(\widetilde{g}_{n,k}\left(x_n\right)\right) \\ &= h_k\left(\widetilde{g}_{m,k}\left(y_m\right)\right) \\ &= h_m\left(y_m\right) = h\left([y_m]\right). \end{aligned} \tag{7.51}$$

So h is well-defined. Now, for any $x_n \in G_n$,

$$\left(h \circ u_n\right)\left(x_n\right) = h\left([x_n]\right) = h_n\left(x_n\right). \tag{7.52}$$

Therefore, $h \circ u_n = h_n$ for all $n \in \omega$. So there indeed exists a homomorphism $h : G_\infty \to H$ making the diagram (7.48) commute. Now we have to show the uniqueness.

$$\begin{tikzcd}
G_n \arrow[rr, "g_n"] \arrow[dr, "u_n"] \arrow[ddr, bend right, "h_n"'] & & G_{n+1} \arrow[dl, "u_{n+1}"'] \arrow[ddl, bend left, "h_{n+1}"] \\
 & G_\infty \arrow[d, shift right, "h"'] \arrow[d, shift left, "h'"] & \\
 & H &
\end{tikzcd}$$

Suppose there is another homomorphism $h' : G_\infty \to H$ such that $h' \circ u_n = h_n$ for all $n \in \omega$. Now, given $[x_n] \in G_\infty$ with $x_n \in G_n$, we have

$$h'\left([x_n]\right) = \left(h' \circ u_n\right)\left(x_n\right) = h_n\left(x_n\right) = h\left([x_n]\right). \tag{7.53}$$

Therefore, $h' = h$ proving the uniqueness of h. Hence, $(G_\infty, u_n : G_n \to G_\infty)$ is a colimit to the diagram (7.32).

8 Exponentials

§8.1 Exponential in a Category

Exponentials can be thought of as the categorical notion of a "function space". As with everything, we shall first consider the category **Sets** to motivate ourselves about the definition of exponentials. In **Sets**, we use the notation C^B to denote the set of all functions from B to C, i.e.

$$C^B = \mathrm{Hom}_{\mathbf{Sets}}(B, C).$$

Consider a function of sets:

$$f(-,-) : A \times B \to C.$$

Now, if we hold $a \in A$ fixed, we have a function

$$f(a,-) : B \to C,$$

so that $f(a,-) \in C^B$. If we now vary a over A, then we get a function

$$\begin{aligned} \widetilde{f} : A &\to C^B \\ a &\mapsto f(a,-). \end{aligned}$$

In other words, $\widetilde{f}(a) = f(a,-)$. This function is uniquely determined by

$$\widetilde{f}(a)(b) = f(a,-)(b) = f(a,b).$$

Furthermore, any map $g : A \to C^B$ is uniquely of the form $\widetilde{f}$, for some $f : A \times B \to C$. This function f is defined as follows:

$$f(a,b) = g(a)(b).$$

So we have an isomorphism of Hom-sets

$$\mathrm{Hom}_{\mathbf{Sets}}(A \times B, C) \cong \mathrm{Hom}_{\mathbf{Sets}}\left(A, C^B\right). \tag{8.1}$$

In other words, there is a bijection $\mathrm{Hom}_{\mathbf{Sets}}(A \times B, C) \to \mathrm{Hom}_{\mathbf{Sets}}\left(A, C^B\right)$ taking f to $\widetilde{f}$. This bijection is mediated by an operation of evaluation. In **Sets**, this evaluation function is

$$\begin{aligned} \mathrm{eval} : C^B \times B &\to C \\ (g,b) &\mapsto g(b). \end{aligned}$$

In other words, $\mathrm{eval}(g,b) = g(b)$. This evaluation function has the following UMP: given any set A and any function $f : A \times B \to C$, there is a **unique** function $\widetilde{f} : A \to C^B$ such that

$$\mathrm{eval} \circ \left(\widetilde{f} \times \mathbb{1}_B\right) = f. \tag{8.2}$$

That is, for any $a \in A$ and $b \in B$,

$$f(a,b) = \left(\mathrm{eval} \circ \left(\widetilde{f} \times \mathbb{1}_B\right)\right)(a,b) = \mathrm{eval}\left(\widetilde{f}(a), b\right).$$

$$\begin{tikzcd} C^B & & C^B \times B \arrow[r, "\mathrm{eval}"] & C \\ A \arrow[u, "\widetilde{f}"] & & A \times B \arrow[u, "\widetilde{f} \times \mathbb{1}_B"] \arrow[ur, "f"'] & \end{tikzcd}$$

Now we can extend this to any arbitrary category that admits binary products.

Definition 8.1 (Exponential)**.** Let $\mathcal{C}$ be a category admitting binary products. An **exponential** of objects B and C consists of an object C^B and an arrow $\epsilon : C^B \times B \to C$ such that for any object A and arrow $f : A \times B \to C$, there is a unique arrow $\widetilde{f} : A \to C^B$ such that

$$\epsilon \circ \left(\widetilde{f} \times \mathbb{1}_B\right) = f. \tag{8.3}$$

$$\begin{tikzcd}
C^B & & C^B \times B \arrow[r, "\epsilon"] & C \\
A \arrow[u, dashed, "\widetilde{f}"'] & & A \times B \arrow[u, "\widetilde{f}\times\mathbb{1}_B"] \arrow[ur, "f"'] &
\end{tikzcd}$$

The arrow $\epsilon : C^B \times B \to C$ is called **evaluation**, and the unique arrow $\widetilde{f} : A \to C^B$ is called the (exponential) **transpose** of $f : A \times B \to C$. Given any arrow $g : A \to C^B$, we define

$$\overline{g} = \epsilon \circ (g \times \mathbb{1}_B) : A \times B \to C,$$

and also call $\overline{g}$ the **transpose** of g. Let $\varphi : \mathrm{Hom}_{\mathcal{C}}(A \times B, C) \to \mathrm{Hom}_{\mathcal{C}}\left(A, C^B\right)$ be the function defined as

$$\varphi(f) = \widetilde{f}. \tag{8.4}$$

Furthermore, we define $\psi : \mathrm{Hom}_{\mathcal{C}}\left(A, C^B\right) \to \mathrm{Hom}_{\mathcal{C}}(A \times B, C)$ as

$$\psi(g) = \overline{g} = \epsilon \circ (g \times \mathbb{1}_B). \tag{8.5}$$

Then given $g : A \to C^B$, $\varphi(\psi(g)) = \widetilde{\overline{g}}$, which is the **unique** arrow $A \to C^B$ such that the following diagram commutes:

$$\begin{tikzcd}
C^B \times B \arrow[r, "\epsilon"] & C \\
A \times B \arrow[u, "\widetilde{\overline{g}}\times\mathbb{1}_B"] \arrow[ur, "\overline{g}"'] &
\end{tikzcd}$$

In other words,

$$\overline{g} = \epsilon \circ \left(\widetilde{\overline{g}} \times \mathbb{1}_B\right). \tag{8.6}$$

But the definition of $\overline{g}$ is $\overline{g} = \epsilon \circ (g \times \mathbb{1}_B)$. Therefore, by the uniqueness of $\widetilde{\overline{g}}$, we get

$$\widetilde{\overline{g}} = g, \tag{8.7}$$

for any $g \in \mathrm{Hom}_{\mathcal{C}}\left(A, C^B\right)$. Therefore,

$$\varphi \circ \psi = \mathbb{1}_{\mathrm{Hom}_{\mathcal{C}}(A, C^B)}. \tag{8.8}$$

Furthermore, given $f : A \times B \to C$, $\psi(\varphi(f)) = \overline{\widetilde{f}}$, which is, by definition,

$$\overline{\widetilde{f}} = \epsilon \circ \left(\widetilde{f} \times \mathbb{1}_B\right). \tag{8.9}$$

Using the definition of $\widetilde{f}$, the following diagram commutes:

$$\begin{tikzcd}
C^B \times B \arrow[r, "\epsilon"] & C \\
A \times B \arrow[u, "\widetilde{f}\times\mathbb{1}_B"] \arrow[ur, "f"'] &
\end{tikzcd}$$

So $\epsilon \circ \left(\widetilde{f} \times \mathbb{1}_B\right) = f$, and hence

$$\overline{\widetilde{f}} = f, \tag{8.10}$$

for any $f \in \mathrm{Hom}_{\mathcal{C}}(A \times B, C)$. Therefore,

$$\psi \circ \varphi = \mathbb{1}_{\mathrm{Hom}_{\mathcal{C}}(A \times B, C)} \, . \tag{8.11}$$

Therefore, φ is an isomorphism (in **Sets**), i.e.

$$\mathrm{Hom}_{\mathcal{C}}(A \times B, C) \cong \mathrm{Hom}_{\mathcal{C}}\left(A, C^B\right). \tag{8.12}$$

§8.2 Cartesian Closed Categories

> **Definition 8.2.** A category is called **cartesian closed**, if it has all finite products and exponentials.

Example 8.1. We have already seen that **Sets** is a cartesian closed category. The category $\mathbf{Sets}_{\mathrm{fin}}$ of finite sets is also a cartesian closed category (CCC). If M and N are finite sets, then the set N^M of all functions from M to N is also a finite set. In fact, it has cardinality

$$\left|N^M\right| = |N|^{|M|}.$$

Example 8.2. Consider the category **Pos** of posets and monotone functions. This is a cartesian closed category. Given posets P and Q, the cartesian product of P and Q is also a poset, and is partially ordered by

$$(p, q) \leqslant (p', q') \iff p \leq_P p' \text{ and } q \leq_Q q'. \tag{8.13}$$

Then $\leqslant$ is a partial order on $P \times Q$, because

- $(p, q) \leqslant (p, q)$, since $p \leq_P p$ and $q \leq_Q q$.
- If $(p, q) \leqslant (p', q')$ and $(p', q') \leqslant (p'', q'')$, then $p \leq_P p'$ and $p' \leq p''$; and $q \leq_Q q'$ and $q' \leq_Q q''$. By the transitivity of $\leq_P$ and $\leq_Q$, we get $p \leq p''$ and $q \leq q''$. Therefore, $(p, q) \leqslant (p'', q'')$.
- If $(p, q) \leqslant (p', q')$ and $(p', q') \leqslant (p, q)$, then $p \leq_P p'$ and $p' \leq p$; and $q \leq_Q q'$ and $q' \leq_Q q$. By the antisymmetry of $\leq_P$ and $\leq_Q$, $p = p'$ and $q = q'$. Therefore, $(p, q) = (p', q')$.

So $P \times Q$ is a poset. We define the projections $\pi_1 : P \times Q \to P$ and $\pi_2 : P \times Q \to Q$ as

$$\pi_1(p, q) = p \text{ and } \pi_2(p, q) = q. \tag{8.14}$$

$$\begin{tikzcd} P & P \times Q \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"'] & Q \end{tikzcd}$$

π_1 and π_2 are clearly monotone, since given $(p, q) \leqslant (p', q')$, we have $p \leq_P p'$ and $q \leq_Q q'$. So

$$(p, q) \leqslant (p', q') \implies \pi_1(p, q) \leq_P \pi_1(p', q').$$

Therefore, π_1 is monotone. Similarly, π_2 is also monotone. Now suppose that given another poset X, there are monotone functions $x_1 : X \to P$ and $x_2 : X \to Q$. We need to show the existence of a **unique** monotone function $h : X \to P \times Q$ such that the following diagram commutes:

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[d, dashed, "h"] \arrow[dr, "g"] & \\ P & P \times Q \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & Q \end{tikzcd} \tag{8.15}$$

In other words, $\pi_1 \circ h = f$ and $\pi_2 \circ h = g$. Let us define

$$h(x) = (f(x), g(x)), \tag{8.16}$$

for $x \in X$. If $x \leq_X x'$, then $f(x) \leq_P f(x')$ since f is monotone. Similarly, $g(x) \leq_Q g(x')$ since g is monotone. Therefore,

$$h(x) = (f(x), g(x)) \leqslant (f(x'), g(x')) = h(x').$$

Therefore, h is monotone. Now,

$$(\pi_1 \circ h)(x) = \pi_1(f(x), g(x)) = f(x),$$

so $\pi_1 \circ h = f$. Similarly, $\pi_2 \circ h = g$. Therefore, there indeed exists a monotone function $h : X \to P \times Q$ such that (8.15) commutes. We now need to show the uniqueness of h. Suppose there is another monotone function $h' : X \to P \times Q$ making (8.15) commute.

$$\begin{tikzcd}
 & X \arrow[dl, "f"'] \arrow[d, shift right, "h"'] \arrow[d, shift left, "h'"] \arrow[dr, "g"] & \\
P & P \times Q \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & Q
\end{tikzcd}$$

Let $h'(x) = (p, q)$, for a given $x \in X$. Since $\pi_1 \circ h' = f$, we have

$$p = (\pi_1 \circ h')(x) = f(x).$$

Similarly, $q = g(x)$. Therefore,

$$h'(x) = (f(x), g(x)) = h(x), \tag{8.17}$$

and this holds for any $x \in X$. So $h = h'$, proving the uniqueness of h. Therefore,

$$\begin{tikzcd}
P & P \times Q \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & Q
\end{tikzcd}$$

is a product diagram in **Pos**. In other words, **Pos** admits finite products (any singleton $\{*\}$ is a terminal object in **Pos**). Let us now show that **Pos** admits exponentials. Given posets P and Q, we define

$$Q^P = \mathrm{Hom}_{\mathbf{Pos}}(P, Q) = \{f : P \to Q \mid f \text{ is monotone }\}. \tag{8.18}$$

We put a partial order $\preceq$ on Q^P as follows:

$$f \preceq g \iff f(p) \leq_Q g(p) \text{ for any } p \in P. \tag{8.19}$$

$\preceq$ is indeed a partial order, because

- $f(p) \leq_Q f(p)$ for all $p \in P$. Therefore, $f \preceq f$.
- If $f \preceq g$ and $g \preceq h$, then $f(p) \leq_Q g(p)$ and $g(p) \leq_Q h(p)$ for any $p \in P$. By the transitivity of $\leq_Q$, $f(p) \leq_Q h(p)$ for all $p \in P$. Therefore, $f \preceq h$.
- If $f \preceq g$ and $g \preceq f$, then $f(p) \leq_Q g(p)$ and $g(p) \leq_Q f(p)$ for any $p \in P$. By the antisymmetry of $\leq_Q$, $f(p) = g(p)$ for all $p \in P$. Therefore, $f = g$.

So Q^P is a poset. Let us define the evaluation function $\epsilon : Q^P \times P \to Q$ by

$$\epsilon(f, p) = f(p). \tag{8.20}$$

Let us verify that ϵ is a monotone function. Suppose $(f, p) \leqslant (g, p')$ in $Q^P \times P$. Then $f \preceq g$ in Q^P and $p \leq_P p'$. f is monotone, so $f(p) \leq_Q f(p')$. Since $f \preceq g$, $f(p) \leq_Q g(p)$ for any $p \in P$. Therefore,

$$\epsilon(f, p) = f(p) \leq_Q f(p') \leq_Q g(p') = \epsilon(g, p'). \tag{8.21}$$

So ϵ is monotone. Now, given any monotone function $f : X \times P \to Q$, the transpose $\widetilde{f} : X \to Q^P$ is defined as

$$\widetilde{f}(x) = f(x, -) : P \to Q. \tag{8.22}$$

We need to verify that $\widetilde{f}$ is monotone as well. Given $x \leq_X x'$, for any $p \in P$, $(x, p) \leqslant (x', p)$ in $X \times P$. Since f is monotone,

$$f(x, p) \leq_Q f(x', p),$$

for any $p \in P$. Therefore, $f(x, -) \preceq f(x', -)$ in Q^P, i.e. $\widetilde{f}(x) \preceq \widetilde{f}(x')$ in Q^P, proving the monotonicity of $\widetilde{f}$.

$$\begin{tikzcd}
Q^P & & Q^P \times P \arrow[r, "\epsilon"] & Q \\
X \arrow[u, dashed, "\widetilde{f}"'] & & X \times P \arrow[u, "\widetilde{f}\times\mathbb{1}_P"] \arrow[ur, "f"'] &
\end{tikzcd}$$

Then clearly we have $\epsilon \circ \left(\widetilde{f} \times \mathbb{1}_B\right) = f$, because given any $(x, p) \in X \times P$,

$$\left(\epsilon \circ \left(\widetilde{f} \times \mathbb{1}_P\right)\right)(x, p) = \epsilon\left(\widetilde{f}(x), p\right) = \widetilde{f}(x)(p) = f(x, p). \tag{8.23}$$

Now we have to show the uniqueness of $\widetilde{f}$. Suppose $g : X \to Q^P$ is another monotone function such that $\epsilon \circ (g \times \mathbb{1}_B) = f$.

$$\begin{tikzcd}
Q^P \times P \arrow[r, "\epsilon"] & Q \\
X \times P \arrow[u, shift right, "g\times\mathbb{1}_P"] \arrow[u, shift left, "\widetilde{f}\times\mathbb{1}_P"'] \arrow[ur, "f"'] &
\end{tikzcd}$$

Then for any $x \in X$ and $p \in P$,

$$f(x, p) = (\epsilon \circ (g \times \mathbb{1}_B))(x, p) = \epsilon(g(x), p) = g(x)(p). \tag{8.24}$$

Since this equality holds for any $p \in P$, we can conclude that $g(x) = f(x, -)$. Therefore, $g = \widetilde{f}$, proving the uniqueness of $\widetilde{f}$. Therefore, Q^P is indeed an exponential of the objects P and Q in **Pos**.

Proposition 8.1

In a cartesian closed category (CCC) $\mathcal{C}$, exponentiation by a fixed object A is a functor

$$(-)^A : \mathcal{C} \to \mathcal{C}.$$

$(-)^A$ takes an object B to the exponential B^A; and given an arrow $\beta : B \to C$ in $\mathcal{C}$, $(-)^A$ takes it to

$$\beta^A := \widetilde{\beta \circ \epsilon_B} : B^A \to C^A,$$

where $\epsilon_B : B^A \times A \to B$ is the evaluation arrow.

Proof. $\beta^A : B^A \to C^A$ is the **unique** arrow such that the triangle below commutes:

$$\begin{tikzcd}
C^A & & C^A \times A \arrow[r, "\epsilon_C"] & C \\
B^A \arrow[u, dashed, "\beta^A"'] & & B^A \times A \arrow[u, "\beta^A\times\mathbb{1}_A"] \arrow[ur, "\beta\circ\epsilon_B"'] &
\end{tikzcd}$$

We can also rewrite the triangle as a commutative square:

$$\begin{tikzcd}
C^A \times A \arrow[r, "\epsilon_C"] & C \\
B^A \times A \arrow[u, "\beta^A \times \mathbb{1}_A"] \arrow[r, "\epsilon_B"'] & B \arrow[u, "\beta"']
\end{tikzcd}$$

In other words, $\beta^A : B^A \to C^A$ is the **unique** arrow such that

$$\epsilon_C \circ \left(\beta^A \times \mathbb{1}_A\right) = \beta \circ \epsilon_B. \tag{8.25}$$

Let us now verify the functorial properties of $(-)^A$. Given the identity arrow $\mathbb{1}_B : B \to B$, $(\mathbb{1}_B)^A$ is the **unique** arrow $B^A \to B^A$ such that the following square commutes:

$$\begin{tikzcd}
B^A \times A \arrow[r, "\epsilon_B"] & B \\
B^A \times A \arrow[u, "(\mathbb{1}_B)^A \times \mathbb{1}_A"] \arrow[r, "\epsilon_B"'] & B \arrow[u, "\mathbb{1}_B"']
\end{tikzcd}$$

In other words, $(\mathbb{1}_B)^A$ is the **unique** arrow satisfying

$$\epsilon_B \circ \left((\mathbb{1}_B)^A \times \mathbb{1}_A\right) = \mathbb{1}_B \circ \epsilon_B. \tag{8.26}$$

Since $\mathcal{C}$ is a category admitting binary products, $\times : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a functor, as verified in § 3.5.ii. Therefore,

$$\mathbb{1}_{B^A \times A} = \mathbb{1}_{B^A} \times \mathbb{1}_A\,. \tag{8.27}$$

So, the following diagram commutes:

$$\begin{tikzcd}
B^A \times A \arrow[r, "\epsilon_B"] & B \\
B^A \times A \arrow[u, "\mathbb{1}_{B^A \times A} = \mathbb{1}_{B^A} \times \mathbb{1}_A"] \arrow[r, "\epsilon_B"'] & B \arrow[u, "\mathbb{1}_B"']
\end{tikzcd}$$

Writing this commutativity in an equation, we get

$$\epsilon_B \circ (\mathbb{1}_{B^A} \times \mathbb{1}_A) = \mathbb{1}_B \circ \epsilon_B. \tag{8.28}$$

But $(\mathbb{1}_B)^A$ is the **unique** arrow such that (8.26) holds. In (8.28), we found that $\mathbb{1}_{B^A}$ also satisfies the same commutativity. Therefore, by the uniqueness, we get

$$(\mathbb{1}_B)^A = \mathbb{1}_{B^A}\,. \tag{8.29}$$

Now, suppose we have arrows $\beta : B \to C$ and $\gamma : C \to D$. We need to show that $\gamma^A \circ \beta^A = (\gamma \circ \beta)^A$. β^A, γ^A, $(\gamma \circ \beta)^A$ are the **unique** arrows making each of the following squares commute:

$$\begin{tikzcd}
C^A \times A \arrow[r, "\epsilon_C"] & C \\
B^A \times A \arrow[u, "\beta^A \times \mathbb{1}_A"] \arrow[r, "\epsilon_B"'] & B \arrow[u, "\beta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
D^A \times A \arrow[r, "\epsilon_D"] & C \\
C^A \times A \arrow[u, "\gamma^A \times \mathbb{1}_A"] \arrow[r, "\epsilon_C"'] & B \arrow[u, "\gamma"']
\end{tikzcd}
\qquad
\begin{tikzcd}
D^A \times A \arrow[r, "\epsilon_D"] & C \\
B^A \times A \arrow[u, "(\gamma \circ \beta)^A \times \mathbb{1}_A"] \arrow[r, "\epsilon_B"'] & B \arrow[u, "\gamma \circ \beta"']
\end{tikzcd}$$

In other words, $(\gamma \circ \beta)^A$ is the **unique** arrow $B^A \to D^A$ such that

$$\epsilon_D \circ \left((\gamma \circ \beta)^A \times \mathbb{1}_A \right) = \gamma \circ \beta \circ \epsilon_B. \tag{8.30}$$

Combining the two commutative squares for β^A and γ^A, we get the following commutative diagram:

$$\begin{tikzcd}
D^A \times A \arrow[r, "\epsilon_D"] & D \\
C^A \times A \arrow[u, "\gamma^A \times \mathbb{1}_A"] \arrow[r, "\epsilon_C"] & C \arrow[u, "\gamma"'] \\
B^A \times A \arrow[u, "\beta^A \times \mathbb{1}_A"] \arrow[r, "\epsilon_B"'] & B \arrow[u, "\beta"']
\end{tikzcd}$$

So the following square commutes:

$$\begin{tikzcd}
D^A \times A \arrow[r, "\epsilon_D"] & D \\
B^A \times A \arrow[u, "(\gamma^A \times \mathbb{1}_A) \circ (\beta^A \times \mathbb{1}_A)"] \arrow[r, "\epsilon_B"] & B \arrow[u, "\gamma \circ \beta"']
\end{tikzcd} \tag{8.31}$$

Again, since $\mathcal{C}$ is a category admitting binary products, $\times : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a functor (see § 3.5.ii). Therefore,

$$\left(\gamma^A \times \mathbb{1}_A\right) \circ \left(\beta^A \times \mathbb{1}_A\right) = \left(\gamma^A \circ \beta^A\right) \times (\mathbb{1}_A \circ \mathbb{1}_A) = \left(\gamma^A \circ \beta^A\right) \times \mathbb{1}_A\,. \tag{8.32}$$

Now, using the commutativity of (8.31), we get

$$\epsilon_D \circ \left(\left(\gamma^A \circ \beta^A\right) \times \mathbb{1}_A \right) = \gamma \circ \beta \circ \epsilon_B. \tag{8.33}$$

But $(\gamma \circ \beta)^A$ is the **unique** arrow such that (8.30) holds. In (8.33), we found that $\gamma^A \circ \beta^A$ also satisfies the same commutativity. Therefore, by the uniqueness, we get

$$(\gamma \circ \beta)^A = \gamma^A \circ \beta^A. \tag{8.34}$$

Therefore, $(-)^A : \mathcal{C} \to \mathcal{C}$ is a functor, if $\mathcal{C}$ is a CCC. ■

9 Naturality

§9.1 The Category of Categories

Let us start by discussing the category **Cat** of categories and functors. This category has finite coproducts **0** (the empty category) and $\mathcal{C} + \mathcal{D}$; and finite products **1** (the category with only one object and its identity arrow) and $\mathcal{C} \times \mathcal{D}$. We can also construct equalizers in **Cat** as follows: suppose we are given two parallel functors $F, G : \mathcal{C} \to \mathcal{D}$. Then we define the category $\mathcal{E}$ and functor E,

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "E"] & \mathcal{C} \arrow[r, shift left, "F"] \arrow[r, shift right, "G"'] & \mathcal{D}
\end{tikzcd}$$

as follows:

$$\begin{aligned}
\mathcal{E}_0 &= \{C \in \mathcal{C}_0 \mid F(C) = G(C)\} \\
\mathcal{E}_1 &= \{f \in \mathcal{C}_\infty \mid F(f) = G(f)\},
\end{aligned} \tag{9.1}$$

and let $E : \mathcal{E} \to \mathcal{C}$ be the inclusion. Then this is an equalizer, which can be easily checked. The category $\mathcal{E}$ is a subcategory of $\mathcal{C}$, i.e. $E : \mathcal{E} \to \mathcal{C}$ is a monomorphism (since it is an equalizer).

Definition 9.1. A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be

- **injective on objects** if the object part $F_0 : \mathcal{C}_0 \to \mathcal{D}_0$ is injective;
- **surjective on objects** if the object part $F_0 : \mathcal{C}_0 \to \mathcal{D}_0$ is surjective;
- **injective on arrows** if the arrow part $F_1 : \mathcal{C}_1 \to \mathcal{D}_1$ is injective;
- **surjective on arrows** if the arrow part $F_1 : \mathcal{C}_1 \to \mathcal{D}_1$ is surjective;
- F is **faithful** if for all $A, B \in \mathcal{C}_0$, the map

  $$\begin{aligned}
  F_{A,B} : \operatorname{Hom}_{\mathcal{C}}(A, B) &\to \operatorname{Hom}_{\mathcal{D}}(F(A), F(B)) \\
  f &\mapsto F(f)
  \end{aligned}$$

  is injective.
- F is **full** if $F_{A,B}$ is surjective for all $A, B \in \mathcal{C}_0$.

Example 9.1. A faithful functor need not be injective on arrows. Consider the coproduct of categories $\mathcal{C} + \mathcal{C}$. The objects of this category are of the form $(C, 0)$ or $(D, 1)$, for $C, D \in \mathcal{C}_0$. If $f : C \to D$ is an arrow in $\mathcal{C}$, then

$$(f, 0) : (C, 0) \to (D, 0) \;\text{ and }\; (f, 1) : (C, 1) \to (D, 1)$$

are arrows in $\mathcal{C} + \mathcal{C}$. The inclusion functors $i_1, i_2 : \mathcal{C} \to \mathcal{C} + \mathcal{C}$ are defined as

$$i_1(C) = (C, 0), \;\; i_1(f) = (f, 0); \tag{9.2}$$

$$i_2(C) = (C, 1), \;\; i_2(f) = (f, 1). \tag{9.3}$$

Then the "codiagonal functor" $\nabla : \mathcal{C} + \mathcal{C} \to \mathcal{C}$ is the **unique** functor such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "i_1"] \arrow[dr, "\mathbb{1}_{\mathcal{C}}"'] & \mathcal{C} + \mathcal{C} \arrow[d, "\nabla"] & \mathcal{C} \arrow[l, "i_2"'] \arrow[dl, "\mathbb{1}_{\mathcal{C}}"] \\
& \mathcal{C} &
\end{tikzcd}$$

This codiagonal functor ∇ is faithful, but not injective on arrows. If $f : C \to D$ is an arrow in $\mathcal{C}$, then $(f, 0) : (C, 0) \to (D, 1)$ is an arrow in $\mathcal{C} + \mathcal{C}$, so is $(f, 1) : (C, 1) \to (D, 1)$. Then

$$\nabla (f, 0) = \nabla (i_1 (f)) = \mathbb{1}_{\mathcal{C}} (f) = f,$$
$$\nabla (f, 1) = \nabla (i_2 (f)) = \mathbb{1}_{\mathcal{C}} (f) = f.$$

Therefore, ∇ is not injective on arrows. But it is faithful. Given two objects (C, i) and (D, j) in $\mathcal{C}+\mathcal{C}$, with $i, j \in \{0, 1\}$ and $C, D \in \mathcal{C}_0$, if $i \neq j$, then

$$\mathrm{Hom}_{\mathcal{C}+\mathcal{C}} ((C, i), (D, j)) = \varnothing.$$

Therefore, the function

$$\nabla_{(C,i),(D,j)} : \mathrm{Hom}_{\mathcal{C}+\mathcal{C}} ((C, i), (D, j)) \to \mathrm{Hom}_{\mathcal{C}} (C, D)$$

is vacuously injective. On the other hand, if $i = j$, then the function

$$\nabla_{(C,i),(D,j)} : \mathrm{Hom}_{\mathcal{C}+\mathcal{C}} ((C, i), (D, i)) \to \mathrm{Hom}_{\mathcal{C}} (C, D)$$

is a bijection, so it's clearly injective. Therefore, ∇ is a faithful functor, but not injective on arrows.

A **full subcategory** $\mathcal{U} \rightarrowtail \mathcal{C}$ consists of some objects of $\mathcal{C}$, and *all* the arrows between them. For example, $\mathbf{Sets}_{\mathrm{fin}}$ is a full subcategory of **Sets**. The inclusion functor $\mathbf{Sets}_{\mathrm{fin}} \rightarrowtail \mathbf{Sets}$ is full and faithful. On the other hand, the forgetful functor $\mathbf{Groups} \to \mathbf{Sets}$ is faithful but not full. There is another forgetful functor for groups: $\mathbf{Groups} \to \mathbf{Cat}$. This is both full and faithful, since a functor between two groups is exactly the same thing as group homomorphism.

Example 9.2. Let $\mathcal{C}$ be a (locally small) category, so that we have the representable functors

$$\mathrm{Hom}_{\mathcal{C}} (C, -) : \mathcal{C} \to \mathbf{Sets},$$

for all objects $C \in \mathcal{C}_0$. This functor is faithful if and only if for all objects X, Y, the function

$$F_{X,Y} : \mathrm{Hom}_{\mathcal{C}} (X, Y) \to \mathrm{Hom}_{\mathbf{Sets}} (\mathrm{Hom}_{\mathcal{C}} (C, X), \mathrm{Hom}_{\mathcal{C}} (C, Y)) \tag{9.4}$$

is injective (here $F = \mathrm{Hom}_{\mathcal{C}} (C, -)$). $F_{X,Y}$ is injective if and only if for $f, g : X \to Y$ with $f \neq g$, $F_{X,Y} (f) \neq F_{X,Y} (g)$, i.e. $F (f) \neq F (g)$. But $F (f)$ and $F (g)$ are both functions from $\mathrm{Hom}_{\mathcal{C}} (C, X)$ to $\mathrm{Hom}_{\mathcal{C}} (C, Y)$. Two functions are unequal if and only if there exists some input where the functions don't agree. Therefore, $F (f) \neq F (g)$ if and only if there is some $x \in \mathrm{Hom}_{\mathcal{C}} (C, X)$ such that $F (f) (x) \neq F (g) (x)$. But since $F = \mathrm{Hom}_{\mathcal{C}} (C, -)$,

$$F (f) (x) = \mathrm{Hom}_{\mathcal{C}} (C, f) (x) = (f_*) (x) = f \circ x. \tag{9.5}$$

Similarly, $F (g) (x) = g \circ x$. Therefore, $\mathrm{Hom}_{\mathcal{C}} (C, -)$ is faithful if and only if for all objects X, Y and arrows $f, g : X \to Y$ with $f \neq g$, there exists an arrow $x : C \to X$ such that $f \circ x \neq g \circ x$.

Example 9.3. Let G be a group in a (locally small) category $\mathcal{C}$. Then the contravariant representable functor $\mathrm{Hom}_{\mathcal{C}} (-, G)$ has a group structure, so that we have a functor

$$\mathrm{Hom}_{\mathcal{C}} (-, G) : \mathcal{C} \to \mathbf{Groups}.$$

For example, if $\mathcal{C} = \mathbf{Sets}$, then for each set X, we can define the group operation on $\mathrm{Hom}_{\mathbf{Sets}} (X, G)$ pointwise:

- Suppose $f, g \in \mathrm{Hom}_{\mathbf{Sets}} (X, G)$. Then we define

  $$(f \star g) (x) = f (x) *_G g (x), \tag{9.6}$$

  where $*_G$ is the group operation on G.

- Let $e \in G$ be the identity element. Then the identity element of $\mathrm{Hom}_{\mathbf{Sets}}(X,G)$ is the function $u : X \to G$ that sends all $x \in X$ to $e \in G$.
- The inverse of $f \in \mathrm{Hom}_{\mathbf{Sets}}(X,G)$ is the function $g : X \to G$ such that
$$g(x) = f(x)^{-1}. \tag{9.7}$$

Then $\mathrm{Hom}_{\mathbf{Sets}}(X,G)$ is a group under $\star$. We then have an isomorphism
$$\mathrm{Hom}_{\mathbf{Sets}}(X,G) \cong \prod_{x\in X} G. \tag{9.8}$$
Indeed, the elements of the product $\prod_{x\in X} G_x$ are functions
$$\varphi : X \to \bigcup_{x\in X} G_x$$
such that $\varphi(x) \in G_x$ for every $x \in X$. In our case, $G_x = G$ for all $x \in X$. Therefore, the elements of the product $\prod_{x\in X} G$ are all the functions $\varphi : X \to G$, so that the isomorphism in (9.8) holds.

Furthermore, given any function $h : X \to Y$, $h^* : \mathrm{Hom}_{\mathbf{Sets}}(Y,G) \to \mathrm{Hom}_{\mathbf{Sets}}(X,G)$ is a group homomorphism. Indeed, given any $f, g : Y \to G$,
$$\begin{aligned}[h^*(f \star_Y g)](x) &= [(f \star_Y g) \circ h](x) \\ &= (f \star_Y g)(h(x)) \\ &= f(h(x)) *_G g(h(x)) \\ &= [f \circ h \star_X g \circ h](x) \\ &= [h^*(f) \star_X h^*(g)](x).\end{aligned}$$
Therefore,
$$h^*(f \star_Y g) = h^*(f) \star_X h^*(g), \tag{9.9}$$
i.e. h^* is a group homomorphism. So we say that the contravariant functor $\mathrm{Hom}_{\mathbf{Sets}}(-,G)$ "captures" the group structure of G.

The group structure is not particularly a special structure. We can do the above for pretty much any algebraic structure. For instance, $\mathbb{R}$ has a ring structure, and using that ring structure, we can form the ring $\mathrm{Hom}_{\mathbf{Top}}(X,\mathbb{R})$ of all the continuous functions from the topological space X to $\mathbb{R}$. $\mathrm{Hom}_{\mathbf{Top}}(X,\mathbb{R})$ is also written $\mathcal{C}(X)$. The ring structure of $\mathrm{Hom}_{\mathbf{Top}}(X,\mathbb{R})$ is inherited from that of $\mathbb{R}$ (the addition and multiplication are defined pointwise, in a similar manner as before). In this case as well, given a continuous function $h : X \to Y$, the function
$$h^* : \mathrm{Hom}_{\mathbf{Top}}(Y,\mathbb{R}) \to \mathrm{Hom}_{\mathbf{Top}}(X,\mathbb{R})$$
is a ring homomorphism.

However, not all algebraic structures are preserved. For instance, $\mathbb{R}$ has the structure of a field, but $\mathrm{Hom}_{\mathbf{Top}}(X,\mathbb{R})$ is not a field. We can consider $X = \mathbb{R}$, and the continuous function $f(x) = x$. Then $f \neq 0$, but f has no multiplicative inverse in $\mathrm{Hom}_{\mathbf{Top}}(X,\mathbb{R})$.

§9.2 Naturality

We have already defined natural transformation in Definition 3.10. Let us recall the definition once again.

> **Definition 9.2** (Natural Transformation)**.** Let F and G be two functors between the categories $\mathcal{C}$ and $\mathcal{D}$. Then a **natural transformation** $\eta : F \Rightarrow G$ is a family of arrows
> $$\{\eta_X : F(X) \to G(X)\}_{X\in\mathcal{C}_0}$$
> in $\mathcal{D}$ such that for every arrow $f : X \to Y$ in $\mathcal{C}$, one must have $\eta_Y \circ F(f) = G(f) \circ \eta_X$. In other words, the diagram below commutes:

$$\begin{tikzcd}
F(X) \arrow[r, "F(f)"] \arrow[d, "\eta_X"'] & F(Y) \arrow[d, "\eta_Y"] \\
G(X) \arrow[r, "G(f)"'] & G(Y)
\end{tikzcd}$$

Given such a natural transformation $\eta : F \Rightarrow G$, the arrow η_X is called the component of η at X.

Example 9.4. Consider the free monoid $M(X)$ on a set X, and let U : **Sets** $\to$ **Sets** be the functor that takes a set X to the underlying set of its free monoid, i.e. $U(X) = |M(X)|$. Let us see the action of U on functions. Suppose $f : X \to Y$ is a function between sets. Let $i_X : X \to |M(X)|$ and $i_Y : Y \to |M(Y)|$ be the insertion of generators. Then we have a function $i_Y \circ f : X \to |M(Y)|$. By the UMP of a free monoid, there exists a **unique** monoid homomorphism $\widetilde{f} : M(X) \to M(Y)$ such that the following diagram commutes in **Sets**:

$$\begin{tikzcd}
{|M(X)|} \arrow[r, "|\widetilde{f}|"] & {|M(Y)|} \\
X \arrow[u, "i_X"] \arrow[r, "f"'] & Y \arrow[u, "i_Y"']
\end{tikzcd}$$

We define $U(f) = \left|\widetilde{f}\right|$. Now we have a natural transformation $\eta : \mathbb{1}_{\mathbf{Sets}} \to U$, whose components are $\eta_X = i_X$. Given any function $f : X \to Y$, the following diagram commutes, so that η is a natural transformation.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\eta_X"'] & Y \arrow[d, "\eta_Y"] \\
U(X) = {|M(X)|} \arrow[r, "U(f)=\left|\widetilde{f}\right|"'] & U(Y) = {|M(Y)|}
\end{tikzcd}$$

Example 9.5. Let $\mathcal{C}$ be a category with products. In § 3.5.ii, we have seen a functor

$$\times : \mathcal{C} \times \mathcal{C} \to \mathcal{C}.$$

Now we define a new functor

$$\begin{aligned}
\overline{\times} : \mathcal{C} \times \mathcal{C} &\to \mathcal{C}, \\
(A, B) &\mapsto B \times A, \\
(f, g) &\mapsto g \times f.
\end{aligned} \tag{9.10}$$

Then we define a "twist" natural transformation $t : \times \Rightarrow \overline{\times}$ as follows: given the product diagrams

$$\begin{tikzcd}
A & A \times B \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & B \\
B & B \times A \arrow[l, "p_1"'] \arrow[r, "p_2"] & A,
\end{tikzcd}$$

we take $t_{(A,B)} : A \times B \to B \times A$ to be the **unique** arrow such that the following diagram commutes:

$$\begin{tikzcd}
& A \times B \arrow[dl, "\pi_2"'] \arrow[d, dashed, "t_{(A,B)}"] \arrow[dr, "\pi_1"] & \\
B & B \times A \arrow[l, "p_1"] \arrow[r, "p_2"'] & A
\end{tikzcd}$$

In other words,

$$p_1 \circ t_{(A,B)} = \pi_2 \text{ and } p_2 \circ t_{(A,B)} = \pi_1. \tag{9.11}$$

In order to show that $t : \times \Rightarrow \overline{\times}$ is a natural transformation, we need to show the commutativity of the following diagram, given arrows $\alpha : A \to A'$ and $\beta : B \to B'$ in $\mathcal{C}$:

$$\begin{tikzcd}
A \times B \arrow[r, "\alpha\times\beta"] \arrow[d, "t_{(A,B)}"'] & A' \times B' \arrow[d, "t_{(A',B')}"] \\
B \times A \arrow[r, "\beta\times\alpha"'] & B' \times A'
\end{tikzcd}$$

We have two arrows $\beta \circ \pi_2 : A \times B \to B'$ and $\alpha \circ \pi_1 : A \times B \to A'$. So there is a **unique** arrow $u : A \times B \to B' \times A'$ such that the following diagram commutes:

$$\begin{tikzcd}
& A \times B \arrow[dl, "\beta\circ\pi_2"'] \arrow[d, dashed, "u"] \arrow[dr, "\alpha\circ\pi_1"] & \\
B' & B' \times A' \arrow[l, "p_1'"] \arrow[r, "p_2'"'] & A'
\end{tikzcd}$$

In other words,

$$p_1' \circ u = \beta \circ \pi_2 \text{ and } p_2' \circ u = \alpha \circ \pi_1. \tag{9.12}$$

Now, in light of the following commutative diagrams,

$$\begin{tikzcd}
& A \times B \arrow[dl, "\pi_2"'] \arrow[d, dashed, "t_{(A,B)}"] \arrow[dr, "\pi_1"] & \\
B \arrow[d, "\beta"'] & B \times A \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, dashed, "\beta\times\alpha"] & A \arrow[d, "\alpha"] \\
B' & B' \times A' \arrow[l, "p_1'"] \arrow[r, "p_2'"'] & A'
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[d, "\alpha"'] & A \times B \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, dashed, "\alpha\times\beta"] & B \arrow[d, "\beta"] \\
A' & A' \times B' \arrow[l, "\pi_1'"'] \arrow[r, "\pi_2'"] \arrow[d, dashed, "t_{(A',B')}"] & B' \\
& B' \times A' \arrow[ul, "p_2'"] \arrow[ur, "p_1'"'] &
\end{tikzcd}$$

we can see that both $(\beta \times \alpha) \circ t_{(A,B)}$ and $t_{(A',B')} \circ (\alpha \times \beta)$ fits the requirement of (9.12). Therefore, by the uniqueness of u,

$$(\beta \times \alpha) \circ t_{(A,B)} = u = t_{(A',B')} \circ (\alpha \times \beta)\,, \tag{9.13}$$

and hence $t : \times \Rightarrow \overline{\times}$ is a natural transformation.

Definition 9.3 (Functor Category). The **functor category** $\mathrm{Fun}\,(\mathcal{C}, \mathcal{D})$ has

- **Objects:** functors $F : \mathcal{C} \to \mathcal{D}$
- **Arrows:** natural transformations $\vartheta : F \Rightarrow G$

For each object F, the natural transformation $\mathbb{1}_F$ has components

$$(\mathbb{1}_F)_C = \mathbb{1}_{F(C)} : F\,(C) \to F\,(C)\,.$$

The composite natural transformation of

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=60, "F", ""{name=U, below}] \arrow[r, "G"{name=M}] \arrow[r, bend right=60, "H"', ""{name=D, above}] & \mathcal{D}
\arrow[Rightarrow, from=U, to=M, "\eta"] \arrow[Rightarrow, from=M, to=D, "\vartheta"]
\end{tikzcd}$$

has components

$$(\vartheta \circ \eta)_C = \vartheta_C \circ \eta_C .$$

Definition 9.4 (Natural isomorphism)**.** A **natural isomorphism** is a natural transformation $\vartheta : F \Rightarrow G$ which is an isomorphism in the functor category $\mathrm{Fun}\,(\mathcal{C}, \mathcal{D})$.

Lemma 9.1

A natural transformation $\vartheta : F \Rightarrow G$ is a natural isomorphism if and only if each componenet $\vartheta_C : F\,(C) \to G\,(C)$ is an isomorphism.

Proof. Suppose ϑ is a natural isomorphism, i.e. an isomorphism in $\mathrm{Fun}\,(\mathcal{C}, \mathcal{D})$. Then there is another natural transformation $\eta : G \Rightarrow F$ such that $\vartheta \circ \eta = \mathbb{1}_G$ and $\eta \circ \vartheta = \mathbb{1}_F$. Then for each $C \in \mathcal{C}_0$,

$$\vartheta_C \circ \eta_C = (\vartheta \circ \eta)_C = (\mathbb{1}_G)_C = \mathbb{1}_{G(C)} . \tag{9.14}$$

Similarly, $\eta_C \circ \vartheta_C = \mathbb{1}_{F(C)}$. Therefore, $\vartheta_C : F\,(C) \to G\,(C)$ is an isomorphism.

Conversely, suppose $\vartheta_C : F\,(C) \to G\,(C)$ is an isomorphism for each $C \in \mathcal{C}_0$. Then we define another natural transformation $\eta : G \Rightarrow F$, whose components are $\eta_C = \vartheta_C^{-1}$. One can easily check that η is a natural transformation. For that purpose, we need to show the commutativity of the following square given any arrow $f : X \to Y$ in $\mathcal{C}$:

$$\begin{tikzcd}
F(X) \arrow[r, "F(f)"] & F(Y) \\
G(X) \arrow[u, "\vartheta_X^{-1}"] \arrow[r, "G(f)"'] & G(Y) \arrow[u, "\vartheta_Y^{-1}"']
\end{tikzcd} \tag{9.15}$$

Since ϑ is a natural transformation, the following square commutes:

$$\begin{tikzcd}
F(X) \arrow[r, "F(f)"] \arrow[d, "\vartheta_X"'] & F(Y) \arrow[d, "\vartheta_Y"] \\
G(X) \arrow[r, "G(f)"'] & G(Y)
\end{tikzcd}$$

In other words,

$$\vartheta_Y \circ F\,(f) = G\,(f) \circ \vartheta_X . \tag{9.16}$$

Composing with ϑ_Y^{-1} to the left and ϑ_X^{-1} to the right, we get

$$F\,(f) \circ \vartheta_X^{-1} = \vartheta_Y^{-1} \circ G\,(f) . \tag{9.17}$$

Therefore, (9.15) commutes, so η is a natural transformation. Now, for any $C \in \mathcal{C}_0$,

$$(\eta \circ \vartheta)_C = \eta_C \circ \vartheta_C = \mathbb{1}_{F(C)} = (\mathbb{1}_F)_C . \tag{9.18}$$

Therefore, $\eta \circ \vartheta = \mathbb{1}_F$. Similarly, $\vartheta \circ \eta = \mathbb{1}_G$. So ϑ is an isomorphism in the functor category $\mathrm{Fun}\,(\mathcal{C}, \mathcal{D})$. ∎

Example 9.6. Consider the category $\mathbf{Vect}_{\mathbb{R}}$ of real vector spaces and linear transformations $f : V \to W$. Every vector space has a dual space

$$V^* = \mathrm{Hom}_{\mathbf{Vect}_{\mathbb{R}}}(V, \mathbb{R}) \tag{9.19}$$

of linear transformations $\varphi : V \to \mathbb{R}$. Every linear transformation $f : V \to W$ gives rise to a dual linear transformation

$$f^* : W^* \to V^*,$$

defined by precomposition, $f^*(\varphi) = \varphi \circ f$ for $\varphi : W \to \mathbb{R}$. Therefore, $(-)^* = \mathrm{Hom}_{\mathbf{Vect}_{\mathbb{R}}}(-, \mathbb{R})$ is just the contravariant representable functor endowed with the vector space structure.

There is a canonical linear transformation from each vector space to its double dual

$$\begin{aligned} \eta_V : V &\to V^{**} \\ x &\mapsto (\mathrm{ev}_x : V^* \to \mathbb{R}), \end{aligned}$$

where $\mathrm{ev}_x(\varphi) = \varphi(x)$ for every $\varphi : V \to \mathbb{R}$. This map η_V is the component of a natural transformation

$$\eta : \mathbb{1}_{\mathbf{Vect}_{\mathbb{R}}} \Rightarrow (-)^{**},$$

since the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\eta_V"'] & W \arrow[d, "\eta_W"] \\ V^{**} \arrow[r, "f^{**}"'] & W^{**} \end{tikzcd} \tag{9.20}$$

Indeed, given any $x \in V$,

$$(f^{**} \circ \eta_V)(x) = f^{**}(\mathrm{ev}_x) = \mathrm{ev}_x \circ f^*, \text{ and } (\eta_W \circ f)(x) = \mathrm{ev}_{f(x)}. \tag{9.21}$$

Both $\mathrm{ev}_{f(x)}$ and $\mathrm{ev}_x \circ f^*$ are elements of W^{**}, i.e. they are linear maps $W^* \to \mathbb{R}$. Now, given any $\psi \in W^*$, $\psi : W \to \mathbb{R}$ is a linear map. Now,

$$(\mathrm{ev}_x \circ f^*)(\psi) = \mathrm{ev}_x(f^*(\psi)) = \mathrm{ev}_x(\psi \circ f) = \psi(f(x)). \tag{9.22}$$

On the other hand,

$$\mathrm{ev}_{f(x)}(\psi) = \psi(f(x)). \tag{9.23}$$

Therefore, $\mathrm{ev}_x \circ f^* = \mathrm{ev}_{f(x)}$ for every $x \in V$. As a result,

$$f^{**} \circ \eta_V = \eta_W \circ f. \tag{9.24}$$

Therefore, (9.20) commutes, so that $\eta : \mathbb{1}_{\mathbf{Vect}_{\mathbb{R}}} \Rightarrow (-)^{**}$ is a natural transformation. When V is finite dimensional, η_V is an isomorphism, so

$$\eta : \mathbb{1}_{\mathbf{Vect}_{\mathbb{R}}^{\mathrm{fin}}} \Rightarrow (-)^{**}$$

is a natural isomorphism, where $\mathbf{Vect}_{\mathbb{R}}^{\mathrm{fin}}$ is the category of finite dimensional $\mathbb{R}$-vector spaces and linear transformations.

§9.3 Exponentials of Categories

In this section, we shall prove that the category **Cat** of (small) categories and functors is a cartesian closed category. Given categories $\mathcal{C}$ and $\mathcal{D}$, we take $\mathcal{D}^{\mathcal{C}} = \mathrm{Fun}(\mathcal{C}, \mathcal{D})$, for which we need to show the required UMP of exponentials. Before going into the proofs, we need the following lemma:

Lemma 9.2 (Bifunctor Lemma)

Given categories $\mathcal{A}, \mathcal{B}, \mathcal{C}$, if $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ is a functor, then

1. F is a functor in each argument: $F(A, -) : \mathcal{B} \to \mathcal{C}$ and $F(-, B) : \mathcal{A} \to \mathcal{C}$ are functors for all $A \in \mathcal{A}_0$ and $B \in \mathcal{B}_0$;
2. F satisfies the following "interchange law": Given $\alpha : A \to A'$ in $\mathcal{A}$ and $\beta : B \to B'$ in $\mathcal{B}$, the following diagram commutes in $\mathcal{C}$:

$$\begin{tikzcd}
F(A,B) \arrow[r, "F(A,\beta)"] \arrow[d, "F(\alpha,B)"'] & F(A,B') \arrow[d, "F(\alpha,B')"] \\
F(A',B) \arrow[r, "F(A',\beta)"'] & F(A',B')
\end{tikzcd}$$

Conversely, if we have functors $F(A, -) : \mathcal{B} \to \mathcal{C}$ and $F(-, B) : \mathcal{A} \to \mathcal{C}$ for all $A \in \mathcal{A}_0$ and $B \in \mathcal{B}_0$, such that

$$F(A', -)(\beta) \circ F(-, B)(\alpha) = F(-, B')(\alpha) \circ F(A, -)(\beta),$$

for arrows $\alpha : A \to A'$ in $\mathcal{A}$ and $\beta : B \to B'$ in $\mathcal{B}$, and

$$F(A, -)(B) = F(-, B)(A)$$

for all $A \in \mathcal{A}_0$ and $B \in \mathcal{B}_0$, then $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ is a functor.

Proof. (**⇒**) If $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ is a functor, then for arrows $\beta : B \to B'$, $\beta' : B' \to B''$ in $\mathcal{B}$,

$$\begin{aligned}
F(A, -)(\beta' \circ \beta) &= F(\mathbb{1}_A, \beta' \circ \beta) \\
&= F(\mathbb{1}_A, \beta') \circ F(\mathbb{1}_A, \beta) \\
&= F(A, -)(\beta') \circ F(A, -)(\beta).
\end{aligned} \tag{9.25}$$

Furthermore, given any object $B \in \mathcal{B}_0$,

$$\begin{aligned}
F(A, -)(\mathbb{1}_B) = F(\mathbb{1}_A, \mathbb{1}_B) &= F\left(\mathbb{1}_{(A,B)}\right) \\
&= \mathbb{1}_{F(A,B)} = \mathbb{1}_{F(A,-)(B)}.
\end{aligned} \tag{9.26}$$

Therefore, $F(A, -) : \mathcal{B} \to \mathcal{C}$ is a functor for every $A \in \mathcal{A}_0$. Similarly, $F(-, B) : \mathcal{A} \to \mathcal{C}$ is also a functor for every $B \in \mathcal{B}_0$.

Now, suppose $\alpha : A \to A'$ is an arrow in $\mathcal{A}$ and $\beta : B \to B'$ is an arrow in $\mathcal{B}$. Then $(\alpha, \beta) : (A, B) \to (A', B')$ is an arrow in $\mathcal{A} \times \mathcal{B}$. We then have the following commutative diagram in $\mathcal{A} \times \mathcal{B}$:

$$\begin{tikzcd}
(A,B) \arrow[r, "(\mathbb{1}_A,\beta)"] \arrow[d, "(\alpha,\mathbb{1}_B)"'] \arrow[dr, "(\alpha,\beta)" description] & (A,B') \arrow[d, "(\alpha,\mathbb{1}_{B'})"] \\
(A',B) \arrow[r, "(\mathbb{1}_{A'},\beta)"'] & (A',B')
\end{tikzcd}$$

Since $F : \mathcal{A} \to \mathcal{B} \to \mathcal{C}$ is a functor, we have

$$\begin{aligned}
F(\alpha, \beta) &= F((\alpha, \mathbb{1}_{B'}) \circ (\mathbb{1}_A, \beta)) \\
&= F(\alpha, \mathbb{1}_{B'}) \circ F(\mathbb{1}_A, \beta) \\
&= F(-, B')(\alpha) \circ F(A, -)(\beta).
\end{aligned} \tag{9.27}$$

Similarly,

$$\begin{aligned} F\left(\alpha,\beta\right) &= F\left(\left(\mathbb{1}_{A'},\beta\right)\circ\left(\alpha,\mathbb{1}_B\right)\right)\\ &= F\left(\mathbb{1}_{A'},\beta\right)\circ F\left(\alpha,\mathbb{1}_B\right)\\ &= F\left(A'-\right)\left(\beta\right)\circ F\left(-,B\right)\left(\alpha\right). \end{aligned} \tag{9.28}$$

In other words, the following square commutes in $\mathcal{C}$:

$$\begin{tikzcd} F(A,B) \arrow[r, "F(\mathbb{1}_A{,}\beta)"] \arrow[d, "F(\alpha{,}\mathbb{1}_B)"'] \arrow[dr, "F(\alpha{,}\beta)" description] & F(A,B') \arrow[d, "F(\alpha{,}\mathbb{1}_{B'})"] \\ F(A',B) \arrow[r, "F(\mathbb{1}_{A'}{,}\beta)"'] & F(A',B') \end{tikzcd}$$

Therefore,

$$F\left(-,B'\right)\left(\alpha\right)\circ F\left(A,-\right)\left(\beta\right) = F\left(A'-\right)\left(\beta\right)\circ F\left(-,B\right)\left(\alpha\right), \tag{9.29}$$

i.e. the interchange law holds.

(⇐) We define the (proposed) functor $F : \mathcal{A}\times\mathcal{B}\to\mathcal{C}$ as follows:

$$F\left(A,B\right) = F\left(A,-\right)\left(B\right) = F\left(-,B\right)\left(A\right) \tag{9.30}$$

$$F\left(\alpha,\beta\right) = F\left(A',-\right)\left(\beta\right)\circ F\left(-,B\right)\left(\alpha\right) = F\left(-,B'\right)\left(\alpha\right)\circ F\left(A,-\right)\left(\beta\right), \tag{9.31}$$

for $(A,B)\in(\mathcal{A}\times\mathcal{B})_0$ and arrow $(\alpha,\beta):(A,B)\to(A',B')$. Let us now check the functorial properties. Suppose $\mathbb{1}_{(A,B)} = (\mathbb{1}_A,\mathbb{1}_B)$ is the identity arrow of the object $(A,B)\in(\mathcal{A}\times\mathcal{B})_0$. Then,

$$\begin{aligned} F\left(\mathbb{1}_{(A,B)}\right) &= F\left(\mathbb{1}_A,\mathbb{1}_B\right) = F\left(A,-\right)\left(\mathbb{1}_B\right)\circ F\left(-,B\right)\left(\mathbb{1}_A\right)\\ &= \mathbb{1}_{F(A,-)(B)}\circ\mathbb{1}_{F(-,B)(A)}\\ &= \mathbb{1}_{F(A,B)}\circ\mathbb{1}_{F(A,B)}\\ &= \mathbb{1}_{F(A,B)}\,. \end{aligned} \tag{9.32}$$

Now, suppose the following is a composition of arrows in $\mathcal{A}\times\mathcal{B}$:

$$\begin{tikzcd} (A,B) \arrow[r, "(\alpha{,}\beta)"] \arrow[rr, bend right, "(\alpha'{,}\beta')\circ(\alpha{,}\beta)=(\alpha'\circ\alpha{,}\beta'\circ\beta)"'] & (A',B') \arrow[r, "(\alpha'{,}\beta')"] & (A'',B'') \end{tikzcd}$$

Now, using the following commutative diagram (commutativity is guaranteed by the interchange law (9.31)),

$$\begin{tikzcd} F(A,B) \arrow[r, "F(A{,}-)(\beta'\circ\beta)"] \arrow[d, "F(-{,}B)(\alpha'\circ\alpha)"'] \arrow[dr, "F(\alpha{,}\beta)" description] & F(A,B'') \arrow[d, "F(-{,}B'')(\alpha'\circ\alpha)"] \\ F(A'',B) \arrow[r, "F(A''{,}-)(\beta'\circ\beta)"'] & F(A'',B'') \end{tikzcd}$$

$$\begin{aligned} F\left(\left(\alpha',\beta'\right)\circ\left(\alpha,\beta\right)\right) &= F\left(\alpha'\circ\alpha,\beta'\circ\beta\right)\\ &= F\left(A'',-\right)\left(\beta'\circ\beta\right)\circ F\left(-,B\right)\left(\alpha'\circ\alpha\right)\\ &= F\left(A'',-\right)\left(\beta'\right)\circ F\left(A'',-\right)\left(\beta\right)\circ F\left(-,B\right)\left(\alpha'\right)\circ F\left(-,B\right)\left(\alpha\right). \end{aligned} \tag{9.33}$$

Using interchange law once again, we get

$$\begin{tikzcd}
F(A', B) \arrow[r, "F(A',-)(\beta)"] \arrow[d, "F(-,B)(\alpha')"'] & F(A', B') \arrow[d, "F(-,B')(\alpha')"] \\
F(A'', B) \arrow[r, "F(A'',-)(\beta)"'] & F(A'', B')
\end{tikzcd}$$

$$F\left(A'', -\right)\left(\beta\right) \circ F\left(-, B\right)\left(\alpha'\right) = F\left(-, B'\right)\left(\alpha'\right) \circ F\left(A', -\right)\left(\beta\right). \tag{9.34}$$

Plugging (9.34) into (9.33), we get

$$F\left(\left(\alpha', \beta'\right) \circ \left(\alpha, \beta\right)\right) = F\left(A'', -\right)\left(\beta'\right) \circ F\left(-, B'\right)\left(\alpha'\right) \circ F\left(A', -\right)\left(\beta\right) \circ F\left(-, B\right)\left(\alpha\right). \tag{9.35}$$

Now we shall use interchange law once again.

$$\begin{tikzcd}
F(A, B) \arrow[r, "F(A,-)(\beta)"] \arrow[d, "F(-,B)(\alpha)"'] \arrow[dr, "F(\alpha,\beta)"] & F(A, B') \arrow[d, "F(-,B')(\alpha)"] \\
F(A', B) \arrow[r, "F(A',-)(\beta)"'] & F(A', B')
\end{tikzcd}
\qquad
\begin{tikzcd}
F(A', B') \arrow[r, "F(A',-)(\beta')"] \arrow[d, "F(-,B')(\alpha')"'] \arrow[dr, "F(\alpha',\beta')"] & F(A', B'') \arrow[d, "F(-,B'')(\alpha')"] \\
F(A'', B') \arrow[r, "F(A'',-)(\beta')"'] & F(A'', B'')
\end{tikzcd}$$

Therefore,

$$F\left(\alpha, \beta\right) = F\left(A', -\right)\left(\beta\right) \circ F\left(-, B\right)\left(\alpha\right), \tag{9.36}$$

$$F\left(\alpha', \beta'\right) = F\left(A'', -\right)\left(\beta'\right) \circ F\left(-, B'\right)\left(\alpha'\right). \tag{9.37}$$

Now plugging (9.36) and (9.37) into (9.35), we get

$$F\left(\left(\alpha', \beta'\right) \circ \left(\alpha, \beta\right)\right) = F(\alpha', \beta') \circ F(\alpha, \beta). \tag{9.38}$$

Therefore, $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ is a functor. ■

Theorem 9.3

Cat is cartesian closed, with the exponential

$$\mathcal{D}^{\mathcal{C}} = \mathrm{Fun}\left(\mathcal{C}, \mathcal{D}\right).$$

Proof. We need to show that the evaluation arrow

$$\epsilon : \mathrm{Fun}\left(\mathcal{C}, \mathcal{D}\right) \times \mathcal{C} \to \mathcal{D}$$

is a functor. The action of ϵ on objects is clear. It takes an object $F \in \mathrm{Fun}\left(\mathcal{C}, \mathcal{D}\right)_0$, i.e. a functor $F : \mathcal{C} \to \mathcal{D}$, and an object $C \in \mathcal{C}_0$, and yields $F\left(C\right) \in \mathcal{D}_0$ as output. In other words,

$$\epsilon\left(F, C\right) = F\left(C\right). \tag{9.39}$$

The action of ϵ on arrows is defined as follows: given an arrow in $\mathrm{Fun}\left(\mathcal{C}, \mathcal{D}\right)$, i.e. a natural transformation $\vartheta : F \Rightarrow G$ between functors $F, G : \mathcal{C} \to \mathcal{D}$, and an arrow $f : C \to C'$ in $\mathcal{C}$, we define

$$\epsilon\left(\vartheta, f\right) = \vartheta_{C'} \circ F\left(f\right) = G\left(f\right) \circ \vartheta_C. \tag{9.40}$$

These two arrows are equal, since ϑ is a natural transformation, so that the following diagram commutes:

$$\begin{tikzcd}
F(C) \arrow[r, "F(f)"] \arrow[d, "\vartheta_C"'] & F(C') \arrow[d, "\vartheta_{C'}"] \\
G(C) \arrow[r, "G(f)"'] & G(C')
\end{tikzcd}$$

We shall use Bifunctor Lemma to prove that ϵ is a functor. First, we need to show that ϵ is functorial in each argument. If we fix a functor $F : \mathcal{C} \to \mathcal{D}$, then

$$\epsilon\left(F, -\right) : \mathcal{C} \to \mathcal{D}$$

takes an object $C \in \mathcal{C}_0$ to $F\left(C\right) \in \mathcal{D}_0$. Furthermore, it takes an arrow $f \in \mathcal{C}_1$

$$\epsilon\left(F, -\right)\left(f\right) = \epsilon\left(\mathbb{1}_F, f\right) = \mathbb{1}_{F(C')} \circ F\left(f\right) = F\left(f\right). \tag{9.41}$$

So $\epsilon\left(F, -\right)$ is just F itself. Therefore, $\epsilon\left(F, -\right)$ is a functor.

On the other hand, if we fix $C \in \mathcal{C}_0$, then

$$\epsilon\left(-, C\right) : \mathrm{Fun}\left(\mathcal{C}, \mathcal{D}\right) \to \mathcal{D}$$

takes an object $F \in \mathrm{Fun}\left(\mathcal{C}, \mathcal{D}\right)_0$, i.e. a functor $F : \mathcal{C} \to \mathcal{D}$, to $F\left(C\right)$. Furthermore, it takes an arrow $\vartheta : F \Rightarrow G$ to

$$\epsilon\left(-, C\right)\left(\vartheta\right) = \epsilon\left(\vartheta, \mathbb{1}_C\right) = G\left(\mathbb{1}_C\right) \circ \vartheta_C = \mathbb{1}_{G(C)} \circ \vartheta_C = \vartheta_C. \tag{9.42}$$

So $\epsilon\left(-, C\right)\left(\vartheta\right)$ is just the component of ϑ at C. As a result, $\epsilon\left(-, C\right)$ is also a functor.

Now we need to verify interchange law. Let $\vartheta : F \Rightarrow G$ be a natural transformation, and $f : C \to C'$ be an arrow in $\mathcal{C}$. We need to verify that

$$\epsilon\left(G, -\right)\left(f\right) \circ \epsilon\left(-, C\right)\left(\vartheta\right) = \epsilon\left(-, C'\right)\left(\vartheta\right) \circ \epsilon\left(F, -\right)\left(f\right).$$

In other words, we need to check that the following diagram commutes:

$$\begin{tikzcd}
\epsilon(F, C) \arrow[r, "\epsilon(F,-)(f)"] \arrow[d, "\epsilon(-,C)(\vartheta)"'] & \epsilon(F, C') \arrow[d, "\epsilon(-,C')(\vartheta)"] \\
\epsilon(G, C) \arrow[r, "\epsilon(G,-)(f)"'] & \epsilon(G, C')
\end{tikzcd}$$

The above diagram can be rewritten as the following diagram:

$$\begin{tikzcd}
F(C) \arrow[r, "F(f)"] \arrow[d, "\vartheta_C"'] & F(C') \arrow[d, "\vartheta_{C'}"] \\
G(C) \arrow[r, "G(f)"'] & G(C')
\end{tikzcd} \tag{9.43}$$

(9.43) clearly commutes since ϑ is a natural transformation. Therefore, interchnage law holds, and hence $\epsilon : \mathrm{Fun}\left(\mathcal{C}, \mathcal{D}\right) \times \mathcal{C} \to \mathcal{D}$ is a functor.

Given any category $\mathcal{X}$ and a functor $F : \mathcal{X} \times \mathcal{C} \to \mathcal{D}$, we need to show the existence of a **unique** functor $\widetilde{F} : \mathcal{X} \to \mathrm{Fun}\left(\mathcal{C}, \mathcal{D}\right)$ such that

$$F = \epsilon \circ \left(\widetilde{F} \times \mathbb{1}_{\mathcal{C}}\right).$$

i.e. the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \times \mathcal{C} \arrow[r, "\epsilon"] & \mathcal{D} \\
\mathcal{X} \times \mathcal{C} \arrow[u, "\widetilde{F} \times \mathbb{1}_{\mathcal{C}}"] \arrow[ur, "F"']
\end{tikzcd}$$

Given $X \in \mathcal{X}_0$, $F(X,-) : \mathcal{C} \to \mathcal{D}$ is a functor, by the forward direction of Bifunctor Lemma. So we define

$$\widetilde{F}(X) = F(X,-) \in \operatorname{Fun}(\mathcal{C},\mathcal{D})_0 . \tag{9.44}$$

Now, given an arrow $f : X \to Y$ in $\mathcal{X}$, we define a natural transformation $\eta : F(X,-) \Rightarrow F(Y,-)$ as follows: the component of η at $C \in \mathcal{C}_0$ is the arrow $F(f, \mathbb{1}_C)$, i.e.

$$\eta_C = F(-,C)(f) = F(f, \mathbb{1}_C) . \tag{9.45}$$

η is indeed a natural transformation from $F(X,-)$ to $F(Y,-)$, since the following diagram commutes (commutativity is guaranteed by interchange law, since F is a bifunctor) for any arrow $\alpha : C \to C'$ in $\mathcal{C}$:

$$\begin{tikzcd}
F(X,C) \arrow[r, "F(X,-)(\alpha)"] \arrow[d, "\eta_C=F(-,C)(f)"'] & F(X,C') \arrow[d, "\eta_{C'}=F(-,C')(f)"] \\
F(Y,C) \arrow[r, "F(Y,-)(\alpha)"'] & F(Y,C')
\end{tikzcd}$$

So we define

$$\widetilde{F}(f) = (\eta : F(X,-) \Rightarrow F(Y,-)) \in \operatorname{Fun}(\mathcal{C},\mathcal{D})_1 . \tag{9.46}$$

In other words,

$$\left(\widetilde{F}(f)\right)_C = F(-,C)(f) .$$

This is clearly functorial since $F(-,C) : \mathcal{X} \to \mathcal{D}$ is a functor. So we have constructed the functor $\widetilde{F}$. Now, given any $X \in \mathcal{X}_0$ and $C \in \mathcal{C}_0$,

$$\left[\epsilon \circ \left(\widetilde{F} \times \mathbb{1}_{\mathcal{C}}\right)\right](X,C) = \epsilon\left(\widetilde{F}(X), C\right) = F(X,-)(C) = F(X,C) . \tag{9.47}$$

Now, given any $f : X \to Y$ in $\mathcal{X}$ and $\alpha : C \to C'$ in $\mathcal{C}$,

$$\left[\epsilon \circ \left(\widetilde{F} \times \mathbb{1}_{\mathcal{C}}\right)\right](f,\alpha) = \epsilon\left(\widetilde{F}(f), \alpha\right) . \tag{9.48}$$

Now, using (9.40), $\epsilon(\vartheta, g) = \vartheta_{C_1} \circ G(f)$, where $\vartheta : G \Rightarrow H$ is a natural transformation, and $g : C_2 \to C_1$ is an arrow in $\mathcal{C}$. Therefore,

$$\epsilon\left(\widetilde{F}(f), \alpha\right) = \left(\widetilde{F}(f)\right)_C \circ F(Y,-)(\alpha) = F(-,C')(f) \circ F(X,-)(\alpha) = F(f,\alpha) , \tag{9.49}$$

where the last equality follows from the interchange law of the bifunctor F.

$$\begin{tikzcd}
F(X,C) \arrow[r, "F(X,-)(\alpha)"] \arrow[d, "F(-,C)(f)"'] \arrow[dr, "F(f,\alpha)" description] & F(X,C') \arrow[d, "F(-,C')(f)"] \\
F(Y,C) \arrow[r, "F(Y,-)(\alpha)"'] & F(Y,C')
\end{tikzcd}$$

Therefore,

$$\epsilon \circ \left(\widetilde{F} \times \mathbb{1}_{\mathcal{C}}\right) = F, \tag{9.50}$$

on both objects and arrows. Now we need to show the uniqueness of $\widetilde{F}$. Suppose there is another functor $G : \mathcal{X} \to \operatorname{Fun}(\mathcal{C},\mathcal{D})$ such that $\epsilon \circ (G \times \mathbb{1}_{\mathcal{C}}) = F$, i.e. the following diagram commutes

$$\begin{tikzcd}
\operatorname{Fun}(\mathcal{C},\mathcal{D}) \times \mathcal{C} \arrow[r, "\epsilon"] & \mathcal{D} \\
\mathcal{X} \times \mathcal{C} \arrow[u, "G \times \mathbb{1}_{\mathcal{C}}"] \arrow[ur, "F"']
\end{tikzcd}$$

Then $F = \epsilon \circ (G \times \mathbb{1}_{\mathcal{C}})$. We shall now look at the exponential transpose of $\epsilon \circ \widetilde{(G \times \mathbb{1}_{\mathcal{C}})}$. Write $H := \epsilon \circ (G \times \mathbb{1}_{\mathcal{C}}) : \mathcal{X} \times \mathcal{C} \to \mathcal{D}$. Then $\widetilde{H} : \mathcal{X} \to \mathrm{Fun}\,(\mathcal{C}, \mathcal{D})$ is an arrow (defined as above) such that the following diagram commutes

$$\begin{tikzcd}
\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \times \mathcal{C} \arrow[r, "\epsilon"] & \mathcal{D} \\
\mathcal{X} \times \mathcal{C} \arrow[u, "\widetilde{H} \times \mathbb{1}_{\mathcal{C}}"] \arrow[ur, "H"'] &
\end{tikzcd}$$

Now, for any $X \in \mathcal{X}_0$, using (9.44),

$$\widetilde{H}\,(X) = H\,(X, -) = [\epsilon \circ (G \times \mathbb{1}_{\mathcal{C}})]\,(X, -) = \epsilon\,(G\,(X)\,, -) = G\,(X)\,. \tag{9.51}$$

For any arrow $f : X \to Y$ in $\mathcal{X}$, $\widetilde{H}\,(f) : \widetilde{H}\,(X) \Rightarrow \widetilde{H}\,(Y)$ is a natural transformation. Since $\widetilde{H} = G$ at the object level, $\widetilde{H}\,(f) : G\,(X) \Rightarrow G\,(Y)$ is a natural transformation, with components

$$\left(\widetilde{H}\,(f)\right)_C = H\,(-, C)\,(f) = [\epsilon \circ (G \times \mathbb{1}_{\mathcal{C}})]\,(f, \mathbb{1}_C) = \epsilon\,(G\,(f)\,, \mathbb{1}_C)\,. \tag{9.52}$$

Now using (9.40),

$$\epsilon\,(G\,(f)\,, \mathbb{1}_C) = G\,(Y)\,(\mathbb{1}_C) \circ G\,(f)_C = \mathbb{1}_{G(Y)(C)} \circ (G\,(f))_C = (G\,(f))_C \tag{9.53}$$

Therefore,

$$\left(\widetilde{H}\,(f)\right)_C = G\,(f)_C \tag{9.54}$$

for all $C \in \mathcal{C}_0$. Therefore, $\widetilde{H}\,(f) = G\,(f)$ for all $f \in \mathcal{X}_1$. Hence, $\widetilde{H} = G$ both at the object and arrow level. But since $H := \epsilon \circ (G \times \mathbb{1}_{\mathcal{C}}) = F$,

$$G = \widetilde{H} = \widetilde{F}. \tag{9.55}$$

Therefore, $\widetilde{F} = G$, so that $\widetilde{F}$ is unique. So the UMP of exponential is satisfied, and thus **Cat** is a CCC. ■