Representation Theory (MAT440)

Lecture Notes

Preface

This series of lecture notes has been prepared for aiding students who took the BRAC University course **Representation Theory (MAT440)** in Summer 2024 semester. These notes were typeset under the supervision of mathematician **Dr. Syed Hasibul Hassan Chowdhury**. The main goal of this typeset is to have an organized digital version of the notes, which is easier to share and handle. If you see any mistakes or typos, please send me an email at atonuroychowdhury@gmail.com

Atonu Roy Chowdhury

References:

- *Representation Theory: A First Course*, by **Joe Harris and William Fulton**
- *Representations of Finite and Compact Groups*, by **Barry Simon**
- *Introduction to Representation Theory*, by **Pavel Etingof et al.**

Contents

1 Representation of Finite Groups

§1.1 Definitions

Definition 1.1 (Representation)**.** A **representation** of a finite group G on a finite dimensional complex vector space V is a homomorphism $\rho : G \to \mathrm{GL}(V)$ of G to the group of invertible linear transformations on V. We often say that such a homomorphism gives V the structure of a G-module. The dimension of V is sometimes called the **degree** of the representation ρ. We also sometimes call V itself a representation of G.

Definition 1.2. A **map** φ between two representations V and W of G is a linear map $\varphi : V \to W$ such that the following diagram commutes for every $g \in G$:

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "\rho(g)"'] & W \arrow[d, "\sigma(g)"] \\ V \arrow[r, "\varphi"'] & W \end{tikzcd}$$

In other words, $\varphi \circ \rho(g) = \sigma(g) \circ \varphi$. Here, $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$ are two group homomorphisms in question. We distinguish such a linear map $\varphi : V \to W$ between two representations of G from an ordinary linear map between vector spaces by calling it a **G-linear map**.

One can then define G-module structure on $\operatorname{Ker}\varphi$ and $\operatorname{im}\varphi$ by restricting the group homomorphisms $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$, namely,

$$\rho_1 : G \to \mathrm{GL}\left(\operatorname{Ker}\varphi\right) \text{ and } \sigma_1 : G \to \mathrm{GL}\left(\operatorname{im}\varphi\right).$$

Suppose $\mathbf{v} \in \operatorname{Ker}\varphi$. Then $\rho\left(g\right)\left(\mathbf{v}\right) \in \operatorname{Ker}\varphi$, because

$$\varphi\left(\rho\left(g\right)\left(\mathbf{v}\right)\right) = \sigma\left(g\right)\left(\varphi\left(\mathbf{v}\right)\right) = \sigma\left(g\right)\left(\mathbf{0}\right) = \mathbf{0}. \tag{1.1}$$

Also, let $\mathbf{w} \in \operatorname{im}\varphi$. Then $\mathbf{w} = \varphi(\mathbf{v})$ for some $\mathbf{v} \in V$. Then $\sigma\left(g\right)\left(\mathbf{w}\right) \in \operatorname{im}\varphi$, because

$$\sigma\left(g\right)\varphi(\mathbf{v}) = \varphi\left(\rho\left(g\right)\left(\mathbf{v}\right)\right) \in \operatorname{im}\varphi. \tag{1.2}$$

One can also give the quotient vector space $W/\operatorname{im}\varphi = \operatorname{Coker}\varphi$ a G-module structure by introducing the group homomorphism $\sigma_2 : G \to \mathrm{GL}(\operatorname{Coker}\varphi)$. Given $\mathbf{w} + \operatorname{im}\varphi \in \operatorname{Coker}\varphi$ and $g \in G$, one defines

$$\sigma_2\left(g\right)\left(\mathbf{w} + \operatorname{im}\varphi \in \operatorname{Coker}\varphi\right) = \sigma\left(\mathbf{w}\right) + \operatorname{im}\varphi \in \operatorname{Coker}\varphi. \tag{1.3}$$

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Definition 1.3 (Subrepresentation)**.** Suppose one is given a representation V of G with the help of the group homomorphism $\rho : G \to \mathrm{GL}(V)$ and $W \subset V$ be a vector subspace. One calls W **invariant** under the action of G if for all $g \in G$ and all $\mathbf{w} \in W$, one has $\rho(g)\mathbf{w} \in W$.

A **subrepresentation** of a representation V of G is a vector subspace W of V that is invariant under the action of G. A representation V of G is called **irreducible** if there is no proper nonzero invariant subspace W of V, i.e., there is no invariant subsapce $W \subset V$ such that $W \neq \{\mathbf{0}\}$ and $W \neq V$.

§1.2 Linear algebra revisited

Definition 1.4 (Tensor product). The **tensor product** of two complex vector spaces V and W is another complex vector space $V \otimes W$ equipped with a bilinear map $\theta : V \times W \to V \otimes W$ that is *universal*: for any bilinear map $\beta : V \times W \to U$ to a complex vector space U, there exists a unique linear map $\alpha : V \otimes W \to U$ such that the following diagram commutes:

$$\begin{tikzcd} V \times W \arrow[r, "\theta"] \arrow[d, "\beta"'] & V \otimes W \arrow[dl, dashed, "\exists!\,\alpha"] \\ U & \end{tikzcd}$$

In other words, $\beta = \alpha \circ \theta$.

If we want the ground field $\mathbb{C}$ to be mentioned, we write the tensor product by $V \otimes_{\mathbb{C}} W$. If $\{\mathbf{e}_i\}$ and $\{\mathbf{f}_j\}$ are bases of V and W, respectively, $\{\mathbf{e}_i \otimes \mathbf{f}_j\}$ form a basis for $V \otimes W$. Similarly, one can form the tensor product $V_1 \otimes \cdots \otimes V_n$ of n vector spaces, with the universal (in the above sense) multilinear map

$$\begin{aligned} \theta : V_1 \times \cdots \times V_n &\to V_1 \otimes \cdots \otimes V_n \\ (\mathbf{v}_1, \ldots, \mathbf{v}_n) &\mapsto \mathbf{v}_1 \otimes \cdots \otimes \mathbf{v}_n. \end{aligned} \tag{1.4}$$

In particular, one can construct

$$V^{\otimes n} = \underbrace{V \otimes \cdots \otimes V}_{n\text{-copies}},$$

for a fixed complex vector space V. If $\{\mathbf{e}_i \mid i = 1, 2, \ldots, m\}$ is a basis for V, then the set

$$\{\mathbf{e}_{i_1} \otimes \mathbf{e}_{i_2} \otimes \cdots \otimes \mathbf{e}_{i_n} \mid i_1, \ldots, i_n \in \{1, 2, \ldots, m\}\} \tag{1.5}$$

is a basis for $V^{\otimes n}$. It follows that $\dim V^{\otimes n} = m^n$.

Let $\mathfrak{S}_n$ be the symmetric group on the set $\{1, 2, \ldots, n\}$. It is a finite group of order $n!$ that consists of all the permutations (i.e. bijections) on the set $\{1, 2, \ldots, n\}$. An alternating multilinear map $\beta : V \times \cdots \times V \to U$ satisfies

$$\beta\left(\mathbf{v}_{\sigma(1)}, \ldots, \mathbf{v}_{\sigma(n)}\right) = \operatorname{sgn}\sigma\beta\left(\mathbf{v}_1, \ldots, \mathbf{v}_n\right), \tag{1.6}$$

for every $\sigma \in \mathfrak{S}_n$.

Definition 1.5 (Exterior power). The **exterior power** of a complex vector spaces V is another complex vector space $\Lambda^n V$ equipped with an alternating multilinear map

$$\begin{aligned} \kappa : V \times \cdots \times V &\to \Lambda^n V \\ (\mathbf{v}_1, \ldots, \mathbf{v}_n) &\mapsto \mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_n, \end{aligned}$$

that is *universal*: for any alternating multilinear map $\beta : V \times \cdots \times V \to U$ to a complex vector space U, there exists a unique linear map $\alpha : \Lambda^n V \to U$ such that the following diagram commutes:

$$\begin{tikzcd} V \times \cdots \times V \arrow[r, "\kappa"] \arrow[d, "\beta"'] & \Lambda^n V \arrow[dl, dashed, "\exists!\,\alpha"] \\ U & \end{tikzcd}$$

In other words, $\beta = \alpha \circ \kappa$.

If $\{\mathbf{e}_i \mid i = 1, 2, \ldots, m\}$ is a basis for V, then the set

$$\{\mathbf{e}_{i_1} \wedge \mathbf{e}_{i_2} \wedge \cdots \wedge \mathbf{e}_{i_n} \mid 1 \le i_1 < i_2 < \cdots < i_n \le m\} \tag{1.7}$$

is a basis for $\Lambda^n V$. It follows that $\dim \Lambda^n V = \binom{m}{n}$.

A symmetric multilinear map $\beta : V \times \cdots \times V \to U$ satisfies

$$\beta\left(\mathbf{v}_{\sigma(1)}, \ldots, \mathbf{v}_{\sigma(n)}\right) = \beta\left(\mathbf{v}_1, \ldots, \mathbf{v}_n\right), \tag{1.8}$$

for every $\sigma \in \mathfrak{S}_n$.

Definition 1.6 (Symmetric power)**.** The **symmetric power** of a complex vector spaces V is another complex vector space $\operatorname{Sym}^n V$ equipped with an symmetric multilinear map

$$\begin{aligned} \delta : V \times \cdots \times V &\to \operatorname{Sym}^n V \\ (\mathbf{v}_1, \ldots, \mathbf{v}_n) &\mapsto \mathbf{v}_1 \odot \cdots \odot \mathbf{v}_n, \end{aligned}$$

that is *universal*: for any symmetric multilinear map $\beta : V \times \cdots \times V \to U$ to a complex vector space U, there exists a unique linear map $\alpha : \operatorname{Sym}^n V \to U$ such that the following diagram commutes:

$$\begin{tikzcd} V \times \cdots \times V \arrow[r, "\delta"] \arrow[d, "\beta"'] & \operatorname{Sym}^n V \arrow[dl, dashed, "\exists!\, \alpha"] \\ U \end{tikzcd}$$

In other words, $\beta = \alpha \circ \kappa$.

If $\{\mathbf{e}_i \mid i = 1, 2, \ldots, m\}$ is a basis for V, then the set

$$\{\mathbf{e}_{i_1} \odot \mathbf{e}_{i_2} \odot \cdots \odot \mathbf{e}_{i_n} \mid 1 \le i_1 \le i_2 \le \cdots \le i_n \le m\} \tag{1.9}$$

is a basis for $\operatorname{Sym}^n V$. It follows that $\dim \Lambda^n V = \binom{m+n-1}{n}$.

§1.3 New representations from old ones

If V and W are representations of G, then so are the direct sum $V \oplus W$ and the tensor product $V \otimes W$. More explicitly, suppose $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$ are the relevant group homomorphisms. Then, one defines $\rho \oplus \sigma : G \to \mathrm{GL}(V \oplus W)$ by

$$(\rho \oplus \sigma)(g)(\mathbf{v} \oplus \mathbf{w}) = \rho(g)\mathbf{v} \oplus \sigma(g)\mathbf{w}, \tag{1.10}$$

for $g \in G$. Similarly, one can define the group homomorphism $\rho \otimes \rho : G \to \mathrm{GL}(V \otimes V)$ by

$$(\rho \otimes \sigma)(g)(\mathbf{v} \otimes \mathbf{w}) = \rho(g)\mathbf{v} \otimes \sigma(g)\mathbf{w} \tag{1.11}$$

for $g \in G$.

For a representation V of G, the nth tensor power $V^{\otimes n}$ is again a representation of G:

$$(\rho \otimes \rho \otimes \cdots \otimes \rho)(g)(\mathbf{v}_1 \otimes \mathbf{v}_2 \otimes \cdots \otimes \mathbf{v}_n) = \rho(g)\mathbf{v}_1 \otimes \rho(g)\mathbf{v}_2 \otimes \cdots \otimes \rho(g)\mathbf{v}_n, \tag{1.12}$$

for $g \in G$. The exterior power $\Lambda^n(V)$ and the symmetric power $\operatorname{Sym}^n(V)$ are subrepresentations of $V^{\otimes n}$. Given the group homomorphism $\rho : G \to \mathrm{GL}(V)$, we defined the nth tensor power representation

$\rho^{\otimes n}: G \to \mathrm{GL}\left(V^{\otimes n}\right)$ by (1.12). Now, the exterior power representation $\Lambda^n \rho : G \to \mathrm{GL}(\Lambda^n V)$, being a subrepresentation of $V^{\otimes n}$, can be defined as follows:

$$(\Lambda^n \rho)(g)(\mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \cdots \wedge \mathbf{v}_n) = \rho(g)\mathbf{v}_1 \wedge \rho(g)\mathbf{v}_2 \wedge \cdots \wedge \rho(g)\mathbf{v}_n. \tag{1.13}$$

One can now write down the group homomorphism $\mathrm{Sym}^n \rho : G \to \mathrm{GL}(\mathrm{Sym}^n V)$ associated with the subrepresentation $\mathrm{Sym}^n V$ of the representation $V^{\otimes n}$ of G:

$$(\mathrm{Sym}^n \rho)(g)(\mathbf{v}_1 \odot \mathbf{v}_2 \odot \cdots \odot \mathbf{v}_n) = \rho(g)\mathbf{v}_1 \odot \rho(g)\mathbf{v}_2 \odot \cdots \odot \rho(g)\mathbf{v}_n. \tag{1.14}$$

Now, let us define $\rho^* : G \to \mathrm{GL}(V^*)$, given $\rho : G \to \mathrm{GL}(V)$. Suppose $\{\mathbf{e}_i\}_{i=1}^m$ and $\{\widehat{\alpha}^i\}_{i=1}^m$ are bases of V and V^*, respectively. Here, $V^* = \mathrm{Hom}(V, \mathbb{C})$, the dual vector space of linear functionals on V. Any linear functional $\widehat{\omega} \in V^*$ can be written as

$$\widehat{\omega} = \sum_{i=1}^m \omega_i \widehat{\alpha}^i \tag{1.15}$$

Also, any vector $\mathbf{v} \in V$ can be written as

$$\mathbf{v} = \sum_{i=1}^m v^i \mathbf{e}_i. \tag{1.16}$$

In a given basis $\{\mathbf{e}_i\}_{i=1}^m$ of V and its dual basis $\{\widehat{\alpha}^i\}_{i=1}^m$ of V^*, $\omega \in V^*$ can be coordinated as a column vector $\begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_m \end{bmatrix}$, whereas a vector $\mathbf{v} \in V$ can be coordinated as $\begin{bmatrix} v^1 \\ v^2 \\ \vdots \\ v^m \end{bmatrix}$. We will simply denote the column vector $\begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_m \end{bmatrix}$ by $\widehat{o}mega$, and the column vector $\begin{bmatrix} v^1 \\ v^2 \\ \vdots \\ v^m \end{bmatrix}$ by $\mathbf{v}$. We then write the dual pairing

$$\langle \widehat{\omega}, \mathbf{v} \rangle = \widehat{\omega}(\mathbf{v}) = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_m \end{bmatrix}^T \begin{bmatrix} v^1 \\ v^2 \\ \vdots \\ v^m \end{bmatrix} = \widehat{\omega}^T \mathbf{v}. \tag{1.17}$$

Now, we want the dual representation V^* of V to satisfy

$$\langle \rho^* g(\widehat{\omega}), \rho(g)\mathbf{v} \rangle = \langle \widehat{\omega}, \mathbf{v} \rangle \tag{1.18}$$

for $g \in G$, $\mathbf{v} \in V$ and $\widehat{\omega} \in V^*$. Now, we claim that $\rho^* : V^* \to V^*$ defined by

$$\rho^*(g)(\widehat{\omega}) = \left[\rho\left(g^{-1}\right)\right]^T \widehat{\omega} \tag{1.19}$$

satisfies (1.18). Indeed,

$$\begin{aligned} \langle \rho^* g(\widehat{\omega}), \rho(g)\mathbf{v} \rangle &= \rho^* g(\widehat{\omega})\left(\rho(g)\mathbf{v}\right) \\ &= \left[\rho\left(g^{-1}\right)\right]^T \widehat{\omega}\left[\rho(g)\mathbf{v}\right] \\ &= \widehat{\omega}\left(\rho\left(g^{-1}\right)\rho(g)\mathbf{v}\right) \\ &= \widehat{\omega}\left(\mathbf{v}\right) = \langle \widehat{\omega}, \mathbf{v} \rangle . \end{aligned}$$

Here we used the following definition of transpose: given a linear map $f : V \to W$, its transpose map $f^T : W^* \to V^*$ is defined as $f^T\left(\widehat{\omega}\right)\left(\mathbf{v}\right) = \widehat{\omega}\left(f\left(\mathbf{v}\right)\right)$. In light of this, we can also write (1.19) as

$$\rho^*(g)\left(\widehat{\omega}\right)\left(\mathbf{v}\right) = \left[\rho\left(g^{-1}\right)\right]^T \widehat{\omega}\left(\mathbf{v}\right) = \widehat{\omega}\left(\rho\left(g^{-1}\right)\mathbf{v}\right). \tag{1.20}$$

Now, if V and W are representations of G, then so is $\mathrm{Hom}(V, W)$. In order to see this, we shall use the fact that

$$\mathrm{Hom}\,(V, W) \cong V^* \otimes W. \tag{1.21}$$

Note here that both V and W are finite dimensional complex vector spaces. Consider the group homomorphisms $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$. Now, the group homomorphism associated with dual representation on V^* of G is given by $\rho^* : G \to GL(V^*)$. Note that for $\widehat{\omega} \in V^*$, one has $\widehat{\omega}\,(\mathbf{e}_i) = \omega_i$, and for $\mathbf{v} \in V$, $\widehat{\alpha}^i\,(\mathbf{v}) = v^i$, where $\{\mathbf{e}_i\}_{i=1}^m$ is a basis for V and $\{\widehat{\alpha}^i\}_{i=1}^m$ is the dual basis for V^*. Note that $\widehat{\alpha}^i\,(\mathbf{e}_j) = \delta^i{}_j$.

Given $\varphi \in \mathrm{Hom}(V, W)$, define $\widetilde{g} : \mathrm{Hom}(V, W) \to V^* \otimes W$ by

$$\widetilde{g}\,(\varphi) = \sum_{i=1}^{m} \widehat{\alpha}^i \otimes \varphi\,(\mathbf{e}_i)\,. \tag{1.22}$$

On the other hand, define $\widetilde{f} : V^* \otimes W \to \mathrm{Hom}(V, W)$ by

$$\widetilde{f}\,(\widehat{\kappa} \otimes \mathbf{w})\,(\mathbf{v}) = \widehat{\kappa}\,(\mathbf{v})\,\mathbf{w}, \tag{1.23}$$

where $\widehat{\kappa} \in V^*$, $\mathbf{v} \in V$, $\mathbf{w} \in W$. Then observe that $\widetilde{f}$ and $\widetilde{g}$ are inverses of each other. In fact,

$$\begin{aligned}
\widetilde{f}\,(\widetilde{g}\,(\varphi))\,(\mathbf{v}) &= \widetilde{f}\left(\sum_{i=1}^{m} \widehat{\alpha}^i \otimes \varphi\,(\mathbf{e}_i)\right)(\mathbf{v}) \\
&= \sum_{i=1}^{m} \widetilde{f}\left(\widehat{\alpha}^i \otimes \varphi\,(\mathbf{e}_i)\right)(\mathbf{v}) \\
&= \sum_{i=1}^{m} \widehat{\alpha}^i\,(\mathbf{v})\,\varphi\,(\mathbf{e}_i) \\
&= \sum_{i=1}^{m} v^i \varphi\,(\mathbf{e}_i) \\
&= \varphi\left(\sum_{i=1}^{m} v^i \mathbf{e}_i\right) \\
&= \varphi\,(\mathbf{v})\,.
\end{aligned}$$

Therefore,

$$\widetilde{f} \circ \widetilde{g} = \mathbb{1}_{\mathrm{Hom}(V,W)}\,. \tag{1.24}$$

Now, for a given $\widehat{\kappa} \otimes \mathbf{w} \in V^* \otimes W$,

$$\begin{aligned}
\widetilde{g}\left(\widetilde{f}\,(\widehat{\kappa} \otimes \mathbf{w})\right) &= \sum_{i=1}^{m} \widehat{\alpha}^i \otimes \widetilde{f}\,(\widehat{\kappa} \otimes \mathbf{w})\,(\mathbf{e}_i) \\
&= \sum_{i=1}^{m} \widehat{\alpha}^i \otimes \widehat{\kappa}\,(\mathbf{e}_i)\,\mathbf{w} \\
&= \sum_{i=1}^{m} \widehat{\kappa}\,(\mathbf{e}_i)\,\widehat{\alpha}^i \otimes \mathbf{w} \\
&= \sum_{i=1}^{m} \kappa_i \widehat{\alpha}^i \otimes \mathbf{w} \\
&= \widehat{\kappa} \otimes \mathbf{w}.
\end{aligned}$$

Therefore,

$$\widetilde{g} \circ \widetilde{f} = \mathbb{1}_{V^* \otimes W}\,. \tag{1.25}$$

(1.24) and (1.25) together imply that $\mathrm{Hom}\,(V, W) \cong V^* \otimes W$. We now define the representation of G on $\mathrm{Hom}(V, W)$ via the representation of G on $V^* \otimes W$. In fact, G acts on $V^* \otimes W$ via the map

$\rho^* \otimes \sigma : G \to \mathrm{GL}\,(V^* \otimes W)$, so that $(\rho^* \otimes \sigma)\, g\, (\widehat{\kappa} \otimes \mathbf{w}) \in V^* \otimes W$. Then via $\widetilde{f} : V^* \otimes W \to \mathrm{Hom}(V, W)$, one has $\widetilde{f}\,((\rho^* \otimes \sigma)\, g\, (\widehat{\kappa} \otimes \mathbf{w})) \in \mathrm{Hom}(V, W)$. This is, by definition, the representation of G on $\mathrm{Hom}(V, W)$. In other words, $\gamma : G \to \mathrm{GL}\,(\mathrm{Hom}(V, W))$ is defined by

$$\begin{aligned} \gamma\,(g) \left(\widetilde{f}\,(\widehat{\kappa} \otimes \mathbf{w})\right)(\mathbf{v}) &= \widetilde{f}\,((\rho^* \otimes \sigma)\, g\, (\widehat{\kappa} \otimes \mathbf{w}))\,(\mathbf{v}) \\ &= \widetilde{f}\,(\rho^*\,(g)\,\widehat{\kappa} \otimes \sigma(g)\mathbf{w})\,(\mathbf{v}) \\ &= (\rho^*\,(g)\,\widehat{\kappa})\,(\mathbf{v})\,\sigma(g)\mathbf{w} \\ &= \widehat{\kappa}\left(\rho\left(g^{-1}\right)\mathbf{v}\right)\sigma(g)\mathbf{w} \\ &= \sigma\,(g)\left(\widehat{\kappa}\left(\rho\left(g^{-1}\right)\mathbf{v}\right)\mathbf{w}\right). \end{aligned} \tag{1.26}$$

Now, let us write $\widetilde{f}\,(\widehat{\kappa} \otimes \mathbf{w}) = \varphi \in \mathrm{Hom}(V, W)$. So we have

$$\varphi\,(\mathbf{v}) = \widetilde{f}\,(\widehat{\kappa} \otimes \mathbf{w})\,(\mathbf{v}) = \widehat{\kappa}\,(\mathbf{v})\,\mathbf{w}. \tag{1.27}$$

As a result,

$$\varphi\left(\rho\left(g^{-1}\right)\mathbf{v}\right) = \widehat{\kappa}\left(\rho\left(g^{-1}\right)\mathbf{v}\right)\mathbf{w}. \tag{1.28}$$

(1.26) and (1.28) together imply that

$$(\gamma\,(g)\,\varphi)\,(\mathbf{v}) = \sigma\,(g)\left(\varphi\left(\rho\left(g^{-1}\right)\mathbf{v}\right)\right). \tag{1.29}$$

(1.29) can be expressed by means of the commutativity of the following diagram:

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "\rho(g)"'] & W \arrow[d, "\sigma(g)"] \\ V \arrow[r, "\gamma(g)\varphi"'] & W \end{tikzcd}$$

> **Definition 1.7** (Hermitian inner product)**.** If V is a complex vector space, then a **Hermitian inner product** is a positive definite sesquilinear map $H : V \times V \to \mathbb{C}$ that satisfies the following:
>
> (i) $H(a\mathbf{u} + b\mathbf{v}, \mathbf{w}) = \bar{a}H(\mathbf{u}, \mathbf{w}) + \bar{b}H(\mathbf{v}, \mathbf{w})$ and $H(\mathbf{w}, a\mathbf{u} + b\mathbf{v}) = aH(\mathbf{w}, \mathbf{u}) + bH(\mathbf{w}, \mathbf{v})$ for all $a, b \in \mathbb{C}$, $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$.
>
> (ii) $H(\mathbf{u}, \mathbf{v}) = \overline{H(\mathbf{v}, \mathbf{u})}$, for all $\mathbf{u}, \mathbf{v} \in V$.
>
> (iii) $H(\mathbf{u}, \mathbf{u}) > 0$, for every $\mathbf{u} \in V \setminus \{\mathbf{0}\}$ (positive definite).

If $W \subset V$ is a vector subspace of a complex vector space with a Hermitian inner product, we define the following subspace:

$$W^{\perp} = \{\mathbf{v} \in V \mid H(\mathbf{v}, \mathbf{w}) = 0, \text{ for all } \mathbf{w} \in W\}. \tag{1.30}$$

If V is a finite dimensional complex vector space, then we can write $V = W \oplus W^{\perp}$, i.e. $W^{\perp}$ is the the orthogonal complement of W. We also say that $W^{\perp}$ is the complementary subspace of V.

> **Definition 1.8.** A Hermitian inner product H on a finite dimensional representation V of a finite group G ($\rho : G \to \mathrm{GL}(V)$) is said to be **preserved under group action** if
>
> $$H(\rho(g)\mathbf{u}, \rho(g)\mathbf{w}) = H(\mathbf{u}, \mathbf{w}) \tag{1.31}$$
>
> for all $g \in G$ and $\mathbf{u}, \mathbf{w} \in V$. H is then called a G-**invariant** Hermitian inner product.

If H is a G-invariant Hermitian inner product on a finite dimensional representation V of a finite group G, then we have

$$\begin{aligned} H\left(\rho\left(g\right)\mathbf{v},\mathbf{w}\right) &= H\left(\rho\left(g\right)\mathbf{v},\rho\left(g\right)\rho\left(g^{-1}\right)\mathbf{w}\right) \\ &= H\left(\mathbf{v},\rho\left(g^{-1}\right)\mathbf{w}\right). \end{aligned} \tag{1.32}$$

Lemma 1.1

If $H : V \times V \to \mathbb{C}$ is a G-invariant Hermitian inner product on a finite dimensional representation V of a finite group G and $W \subset V$ is a subrepresentation, then $W^{\perp}$ is a G-invariant complement to W.

Proof. Since we are dealing with finite dimensional complex vector spaces, $W^{\perp}$ is a complement to W. It, therefore, suffices to show that $W^{\perp}$ is G-invariant.

Suppose $g \in G$, $\mathbf{u} \in W^{\perp}$, and $\mathbf{w} \in W$. Let us denote the group homomorphism associated with the finite dimensional complex representation by $\rho : G \to \mathrm{GL}(V)$. Since the Hermitian inner product $H : V \times V \to \mathbb{C}$ is G-invariant, one has

$$H(\rho(g)\mathbf{u},\mathbf{w}) = H(\mathbf{u},\rho(g^{-1})\mathbf{w}). \tag{1.33}$$

Since W is a subrepresentation of V, one must have $\rho(g^{-1})\mathbf{w} \in W$ for any $g \in G$ and $\mathbf{w} \in W$. Hence, $H(\mathbf{u},\rho(g^{-1})\mathbf{w}) = 0$ in (1.33) leads to

$$H(\rho(g)\mathbf{u},\mathbf{w}) = 0 \tag{1.34}$$

This is true for all $\mathbf{w} \in W$. Therefore, from the definition of $W^{\perp}$, one then must have $\rho(g)\mathbf{u} \in W^{\perp}$ for any $g \in G$, which then implies that the subspace $W^{\perp}$ is G-invariant. ■

Proposition 1.2

If V is a complex representation of a finite group G, then there is a G-invariant Hermitian inner product on V.

Proof. Pick a Hermitian inner product $H_0 : V \times V \to \mathbb{C}$ on the finite dimensional complex vector space V with respect to which a given basis of V is orthonormal, i.e., choose a basis $\{\mathbf{e}_i\}_{i=1}^m$ of V and define $H_0(\mathbf{e}_i,\mathbf{e}_j) = \delta_{ij}$ and extend H_0 to all of $V \times V$ sesquilinearly. Given $\mathbf{v} = \sum_{i=1}^m v^i\mathbf{e}_i$ and $\mathbf{w} = \sum_{j=1}^m w^j\mathbf{e}_j$, we then have

$$H_0(\mathbf{v},\mathbf{w}) = H_0\left(\sum_{i=1}^m v^i\mathbf{e}_i, \sum_{j=1}^m w^j\mathbf{e}_j\right) = \sum_{i=1}^m \overline{v^i}w^i. \tag{1.35}$$

Then define a new Hermitian inner product $H_1 : V \times V \to \mathbb{C}$ by averaging over all of G via representation $\rho : G \to \mathrm{GL}(V)$:

$$H_1(\mathbf{v},\mathbf{w}) = \frac{1}{|G|}\sum_{g\in G} H_0(\rho(g)\mathbf{v},\rho(g)\mathbf{w}). \tag{1.36}$$

Using the Hermitian inner product properties of H_0, one can verify that H_1 is also a Hermitian inner product on V. Additionally,

$$\begin{aligned} H_1(\rho(h)\mathbf{v},\rho(h)\mathbf{w}) &= \frac{1}{|G|}\sum_{g\in G} H_0(\rho(g)\rho(h)\mathbf{v},\rho(g)\rho(h)\mathbf{w}) \\ &= \frac{1}{|G|}\sum_{g\in G} H_0(\rho(gh)\mathbf{v},\rho(gh)\mathbf{w}) \\ &= \frac{1}{|G|}\sum_{g'\in G} H_0(\rho(g')\mathbf{v},\rho(g')\mathbf{w}) \quad (\text{where } g' = gh) \\ &= H_1\left(\mathbf{v},\mathbf{w}\right). \end{aligned} \tag{1.37}$$

Then (1.37) implies that the Hermitian inner product $H_1 : V \times V \to \mathbb{C}$ defined by (1.36) on V is G-invariant. ■

Corollary 1.3

If W is a subrepresentation of a finite dimensional complex representation V of a finite group G, then there exists a complementary invariant subspace $W^\perp$ of V so that $V = W \oplus W^\perp$.

Proof. Given that V is a complex representation of a finite group G, there is a G-invariant Hermitian inner product on V by Proposition 1.2. Now, if W is a subrepresentation of V, then by Lemma 1.1, the complementary subspace $W^\perp$ is G-invariant, i.e., $V = W \oplus W^\perp$. ■