Category Theory (MAT434)

Lecture Notes

Preface

This series of lecture notes has been prepared for aiding students who took the BRAC University course **Category Theory (MAT434)** in Summer 2023 semester. These notes were typeset under the supervision of mathematician **Dr. Syed Hasibul Hassan Chowdhury**. The main goal of this typeset is to have an organized digital version of the notes, which is easier to share and handle. If you see any mistakes or typos, please send me an email at atonuroychowdhury@gmail.com

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References:

- *Category Theory*, by **Steve Awodey**.
- *Category Theory for Scientists*, by **David Spivak**.
- *Categories for the Working Mathematician*, by **Saunders Mac Lane**.
- *Basic Category Theory*, by **Tom Leinster**.

Contents

1 Categories

§1.1 Definition of a Category

Category theory arises from the idea of a system of "functions" among some objects.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g\circ f"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

A category consists of objects $A, B, C, \ldots$ and arrows $f : A \to B, g : B \to C, \ldots$ that are closed under composition and satisfy certain conditions typical of composition of functions. Before formally defining what a category is, let us begin our discussion with the setting where the objects are sets and arrows are functions between sets.

Let f be a function from a set A to another set B. This is mathematically expressed as $f : A \to B$. Explicitly, it refers to the fact that f is defined for all of A, and all the values of f are contained in B. In other words, $\mathrm{range}\,(f) \subseteq B$.

Now suppose we have another function $g : B \to C$. Then there is a unique function $g \circ f : A \to C$, given by

$$(g \circ f)(a) = g(f(a)), \qquad \text{for } a \in A. \tag{1.1}$$

This unique function is called the composite of g and f, or g after f.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g\circ f"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

Now, this operation $\circ$ of composition of functions is associative. In other words, the two arrows from A to D in the following diagram are the same:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] \arrow[rrr, bend left=50, "(h\circ g)\circ f"] \arrow[rrr, bend right=50, "h\circ(g\circ f)"'] & B \arrow[r, "g"] \arrow[rr, bend left, "h\circ g"] & C \arrow[r, "h"] & D \end{tikzcd}$$

Given $f : A \to B$, $g : B \to C$ and $h : C \to D$, one has unique compositions $h \circ g : B \to D$ and $g \circ f : A \to C$. These two composed functions can be further composed with f (from the left) and with h (from the right), respectively, to yield a unique function

$$(h \circ g) \circ f = h \circ (g \circ f), \tag{1.2}$$

from A to D as demanded by the associativity law. Using the definition of composition of functions, one verifies that this is indeed the case:

$$\begin{aligned} ((h \circ g) \circ f)(a) &= (h \circ g)(f(a)) = h(g(f(a))), \\ (h \circ (g \circ f))(a) &= h((g \circ f)(a)) = h(g(f(a))). \end{aligned}$$

Therefore, $(h \circ g) \circ f = h \circ (g \circ f)$.

Finally, note that for every set A, there is an identity function $\mathbb{1}_A : A \to A$ given by

$$\mathbb{1}_A (a) = a. \tag{1.3}$$

These identity functions act as units for composition, i.e. given $f : A \to B$, we have

$$\begin{aligned}
(f \circ \mathbb{1}_A)(a) &= f(\mathbb{1}_A(a)) = f(a), \\
(\mathbb{1}_B \circ f)(a) &= \mathbb{1}_B(f(a)) = f(a),
\end{aligned}$$

for each $a \in A$. Therefore,

$$f \circ \mathbb{1}_A = \mathbb{1}_B \circ f = f. \tag{1.4}$$

The equality above is equivalent to the following commutative diagram:

$$\begin{tikzcd}
A \arrow[r, "\mathbb{1}_A"] \arrow[dr, "f\circ\mathbb{1}_A"'] & A \arrow[d, "f"'] \arrow[dr, "\mathbb{1}_B\circ f"] & \\
& B \arrow[r, "\mathbb{1}_B"'] & B
\end{tikzcd}$$

We have the following abstract version of sets and functions between sets called a **category**.

Definition 1.1 (Category)**.** A **category** $\mathcal{C}$ consists of the following data:

- **Objects:** $A, B, C, \ldots$. The collection of objects of $\mathcal{C}$ is denoted by $\mathrm{Ob}(\mathcal{C})$.
- **Arrows:** $f, g, h, \ldots$. Given two objects A and B, the set of arrows from A to B is denoted by $\mathrm{Hom}_{\mathcal{C}}(A, B)$.
- For each arrow f, there are given objects $\mathrm{dom}(f), \mathrm{cod}(f)$ called the **domain** and **codomain** of f. We write $f : A \to B$ to indicate that $A = \mathrm{dom}(f)$ and $B = \mathrm{cod}(f)$.
- Given arrows $f : A \to B$ and $g : B \to C$, i.e. with $\mathrm{cod}(f) = \mathrm{dom}(g)$, there is a unique arrow $g \circ f : A \to C$, i.e. $g \circ f \in \mathrm{Hom}_{\mathcal{C}}(A, C)$ called the **composite** of f and g. This fact can be rephrased as the following: given $A, B, C \in \mathrm{Ob}(\mathcal{C})$, there is a function

$$\circ : \mathrm{Hom}_{\mathcal{C}}(B, C) \times \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(A, C), \tag{1.5}$$

  with $(g, f) \mapsto g \circ f$. The well-definedness of $\circ$ is synonymous to claiming that $g \circ f \in \mathrm{Hom}_{\mathcal{C}}(A, C)$ is unique for given $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$ and $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$.
- For each $A \in \mathrm{Ob}(\mathcal{C})$, there exists an unique arrow $\mathbb{1}_A \in \mathrm{Hom}_{\mathcal{C}}(A, A)$.

The above data are required to satisfy the following laws:

- **Associativity:** For any $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$, $h \in \mathrm{Hom}_{\mathcal{C}}(C, D)$ with $A, B, C, D \in \mathrm{Ob}(\mathcal{C})$,

$$(h \circ g) \circ f = h \circ (g \circ f), \tag{1.6}$$

- **Unit:** For any $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ with $A, B \in \mathrm{Ob}(\mathcal{C})$,

$$f \circ \mathbb{1}_A = \mathbb{1}_B \circ f = f. \tag{1.7}$$

Remark 1.1. Suppose we have the following commutative diagram:

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[dr, "h\circ f=h\circ g"'] & B \arrow[d, "h"] \\ & C \end{tikzcd}$$

Commutativity of this diagram doesn't violate the uniqueness of the composition $\circ$. It just means that the map $\circ$ in (1.5) is a many-to-one function.

§1.2 Examples of Categories

- Sets and functions between sets. This category is called **Sets**.
- Groups and group homomorphisms
- Vector spaces and linear mappings between them
- Graphs and graph isomorphisms
- The set of real numbers $\mathbb{R}$ as an object, and continuous functions $f : \mathbb{R} \to \mathbb{R}$ as arrows
- Open subsets $U \subseteq \mathbb{R}$ and continuous functions $f : U \to V \subseteq \mathbb{R}$ defined on them
- Differentiable manifolds and smooth (C^∞) mappings
- Posets and monotone functions.

Let us discuss the last category at length.

Definition 1.2. A **partially ordered set** or **poset** is a set A equipped with a binary relation (a subset of $A \times A$) $a \leq_A b$ (in other words, $(a, b) \in R \subset A \times A$) such that the following conditions hold for all $a, b, c \in A$:

(i) **Reflexivity:** $a \leq_A a$.

(ii) **Transitivity:** if $a \leq_A b$ and $b \leq_A c$, then $a \leq_A c$.

(iii) **Antisymmetry:** if $a \leq_A b$ and $b \leq_A a$, then $a = b$.

Remark 1.2. The antisymmetry condition tells us that if both $a \leq_A b$ and $b \leq_A a$ hold, then a and b cannot be distinct. Contrapositively, for distinct a and b in A, not both $a \leq_A b$ and $b \leq_A a$ hold true. Also, note that if $(A, \leq_A)$ is a partially ordered set, there can be elements $a, b \in A$ such that neither (a, b) nor (b, a) is in R. If it happens that given any $a, b \in A$, either (a, b) or (b, a) is in R, i.e. either $a \leq_A b$ or $b \leq_A a$, then we call A a **totally ordered set**.

Example 1.1. $(\mathbb{R}, \leq)$, the set of real numbers with the usual ordering $\leq$ is a totally ordered set.

Now we define an arrow from a poset $(A, \leq_A)$ to another poset $(B, \leq_B)$ to be a function $m : A \to B$ that is **monotone**, in the sense that for all $a, a' \in A$,

$$\text{whenever } a \leq_A a', \text{ one has } m(a) \leq_B m(a').$$

We need to verify that under this definition of arrows, we have a category. First of all, we must have $\mathbb{1}_A : A \to A$, defined by $\mathbb{1}_A(a) = a$ for each $a \in A$, to be monotone. Indeed, if $a \leq a'$ in A, then we automatically have $\mathbb{1}_A(a) \leq \mathbb{1}_A(a')$. Therefore, $\mathbb{1}_A$ is monotone.

Given monotone functions $f : A \to B$ bewteen posets $(A, \leq_A)$ and $(B, \leq_B)$, and $g : B \to C$ bewteen posets $(B, \leq_B)$ and $(C, \leq_C)$, we need to verify that the composition $g \circ f : A \to C$ is also monotone. Indeed, given $a \leq_A a'$, since f is monotone, we have

$$f(a) \leq_B f(a') . \tag{1.8}$$

Since g is monotone, this gives us

$$g(f(a)) \leq_C g(f(a')) . \tag{1.9}$$

In other words, $(g \circ f)(a) \leq_C (g \circ f)(a')$ given $a \leq_A a'$. Therefore, $g \circ f : A \to C$ is monotone.

The category thus formed is called the category of posets and monotone functions, and is denoted by **Pos**.

Finite Categories

A finite category consists of finitely many objects and finitely many arrows between them.

- The category **1** looks as follows:

$$\begin{tikzcd} * \arrow[loop right] \end{tikzcd}$$

  It has one object $*$ and its identity arrow.

- The category **2** looks as follows:

$$\begin{tikzcd} \star \arrow[loop left] \arrow[r] & * \arrow[loop right] \end{tikzcd}$$

  It has two objects $\star$ and $*$, their identity arrows, and exactly one arrow $\star \to *$.

- The categort **3** looks as follows:

$$\begin{tikzcd} \star \arrow[loop left] \arrow[r] \arrow[dr] & * \arrow[loop right] \arrow[d] \\ & \bullet \arrow[loop below] \end{tikzcd}$$

  It has three objects $\star, *, \bullet$, their respective identity arrows, and the other arrows are $\star \to *$, $* \to \bullet$, and $\star \to \bullet$ (which is the composition of the previous two arrows).

- The category **0** looks as follows:

  It has no objects or arrows.

§1.3 Functor

Definition 1.3 (Functor). A **functor** $F : \mathcal{C} \to \mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$ is an assignment of $\mathrm{Ob}(\mathcal{C})$ to $\mathrm{Ob}(\mathcal{D})$ and a mapping of arrows in $\mathcal{C}$ to arrows in $\mathcal{D}$, i.e. for any $A, B \in \mathrm{Ob}(\mathcal{C})$, a mapping

$$\mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F(A), F(B)),$$

where $F(A), F(B) \in \mathrm{Ob}(\mathcal{D})$ are the assigned objects of $\mathcal{D}$ under F. In other words, for given $A, B \in \mathrm{Ob}(\mathcal{C})$ and an arrow $f : A \to B$, one has $F(A), F(B) \in \mathrm{Ob}(\mathcal{D})$ and an arrow $F(f) : F(A) \to F(B)$ such that the following hold:

(a) $F(\mathbb{1}_A) = \mathbb{1}_{F(A)}$.

(b) $F(g \circ f) = F(g) \circ F(f)$.

> In other words, F preserves domains and codomains, identity arrows and composition.

$$
\begin{tikzcd}
\mathcal{C} \arrow[dd, "F"'] & A \arrow[r, "f"] \arrow[dr, "g\circ f"'] & B \arrow[d, "g"] \\
 & & C \\
\mathcal{D} & F(A) \arrow[r, "F(f)"] \arrow[dr, "F(g\circ f)=F(g)\circ F(f)"'] & F(B) \arrow[d, "F(g)"] \\
 & & F(C)
\end{tikzcd}
$$

Now, one can see that functors compose in the expected way and that every category $\mathcal{C}$ has a distinguished functor called the identity functor $\mathbb{1}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$. Thus we have a category, namely **Cat**, the category of all categories and functors between them.

Preorder

A **preorder** is a set P equipped with a binary relation $\leq$ that is both reflexive and transitive: $a \leq a$; and if $a \leq b$ and $b \leq c$, then $a \leq c$ for any $a, b, c \in P$. Any preorder $(P, \leq)$ can be regarded as a category by taking the objects of the category to be the elements of P and taking a unique arrow

$$a \to b \text{ if and only if } a \leq b \text{ in } \ (P, \leq)\,. \tag{1.10}$$

Remark 1.3. Reflexivity and transitivity property ensures that the preorder $(P, \leq)$ is indeed a category. Note that the above condition implies that there is at most one arrow from an object of $(P, \leq)$ to another. In the other direction, any category with at most one arrow from an object to another determines a preorder simply by defininig a binary relation $\leq$ on the objects by (1.10).

Remark 1.4 (On the similarities between a poset and a preorder)**.** A poset $(P, \leq)$ is evidently a preorder with the additional condition of antisymmetry. Hence, a poset is also a category. Examples of poset include the power set $\mathscr{P}(X)$ of a given set X under the usual subset relation: $U \subseteq V$ between the subsets U, V of X.

There can be preorders that are not posets. For instance, $(\mathbb{Z}, |)$ is a preorder on the set of integers, where "$|$" is the usual divides binary relation: given $a, b \in \mathbb{Z}$, we have $a \mid b$ (read a divides b) if and only if $b = ca$ for some $c \in \mathbb{Z}$. It is clearly reflexive and transitive. Note that $a \mid b$ and $b \mid a$ imply $a = \pm b$ which is not the same as $a = b$. Hence, "$|$" is not antisymmetric. Therefore, $(\mathbb{Z}, |)$ is a preorder that is not a poset.

§1.4 Monoid

> **Definition 1.4** (Monoid)**.** A **monoid** is a set M equipped with a binary operation $\cdot : M \times M \to M$ and a distinguished "unit" element $u \in M$ such that for each $x, y, z \in M$,
>
> $$x \cdot (y \cdot z) = (x \cdot y) \cdot z \text{ and } u \cdot x = x \cdot u = x. \tag{1.11}$$

Equivalently, a monoid is a category with just one object. The arrows of the category are the elements of the monoid. In particular, the identity arrow on the object is the unit element u. Composition of arrows is the binary operation $x \cdot y$ of the monoid.

For example, $\mathbb{N}$ (we are adopting the convention that $0 \in \mathbb{N}$), $\mathbb{Q}$, $\mathbb{R}$ with addition and 0 as the unit element. Also, $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$ with multiplication and 1 as unit are monoids. For any set X, the set of functions from X to itself, written as

$$\mathrm{Hom}_{\mathbf{Sets}}\left(X, X\right),$$

is a monoid under the operation of composition. Here **Sets** is the category of sets and functions between sets. More generally, for any object $C \in \mathrm{Ob}\left(\mathcal{C}\right)$ in a category $\mathcal{C}$, the set of arrows from C to itself, written as

$$\mathrm{Hom}_{\mathcal{C}}\left(C, C\right),$$

is a monoid under the composition of arrows in $\mathcal{C}$.

Since monoids are structured sets (sets equipped with a binary operation fulfilling associativity, unitality etc.), there is a category **Mon** whose objects are monoids and arrows are functions that preserve the monoid structure, namely monoid homomorphisms. In detail, a **monoid homomorphism** from a monoid $(M, \cdot_M)$ to a monoid $(N, \cdot_N)$ is a function $f : M \to N$ such that for all $m, n \in M$,

$$h\left(m \cdot_M n\right) = h\left(m\right) \cdot_N h\left(n\right) \text{ and } h\left(u_M\right) = u_N. \tag{1.12}$$

Here u_M and u_N are unit elements of M and N, respectively.

§1.4.i Isomorphisms

Definition 1.5. In any category $\mathcal{C}$, an arrow $f : A \to B$ is called an **isomorphism** if there is an arrow $g : B \to A$ such that

$$g \circ f = \mathbb{1}_A \text{ and } f \circ g = \mathbb{1}_B\,. \tag{1.13}$$

Suppose there is another arrow $\widetilde{g} : B \to A$ with

$$\widetilde{g} \circ f = \mathbb{1}_A \text{ and } f \circ \widetilde{g} = \mathbb{1}_B\,. \tag{1.14}$$

Then we have

$$g = g \circ \mathbb{1}_B = g \circ (f \circ \widetilde{g}) = (g \circ f) \circ \widetilde{g} = \mathbb{1}_A \circ \widetilde{g} = \widetilde{g}. \tag{1.15}$$

Hence, if an arrow $g : B \to A$ exists satisfying (1.13), then it is unique. Such unique arrow $g : B \to A$ is called the inverse of $f : A \to B$, and we write $g = f^{-1}$. When such an arrow $f : A \to B$ exists, we say that A is isomorphic to B, written $A \cong B$.

Definition 1.6 (Group). A **group** G is a monoid with an inverse g^{-1} for every element $g \in G$. Thus G is a category with one object in which every arrow is an isomorphism.

The natural numbers $\mathbb{N}$ do not form a group either under addition or multiplication. But the integers $\mathbb{Z}$ form a group under addition. So do the positive rationals $\mathbb{Q}^+$ under multiplication. For any set X, we have the group $\mathrm{Aut}\left(X\right)$ of all the automorphisms of X, i.e. isomorphisms $f : X \to X$. A **group of permutations** is a subgroup $G \subseteq \mathrm{Aut}\left(X\right)$ for some X. Thus the set G must satisfy the following:

1. The identity function $\mathbb{1}_X$ on X is in G.
2. If $g, g' \in G$, then $g \circ g' \in G$.
3. If $g \in G$, $g^{-1} \in G$.

We now have the following theorem due to Arthur Cayley.

Theorem 1.1 (Cayley's theorem)

Every group G is isomorphic to a group of permutations.

Sketch of proof. First, define the Cayley representation $\overline{G}$ of G to be the following group of permutations on a set: the set is G itself, and for each $g \in G$, one has the permutation $\overline{g} : G \to G$ defined as

$$\overline{g}(h) = g \cdot h \text{ for each } h \in G. \tag{1.16}$$

Indeed, $\overline{g}$ has an inverse $\overline{g}^{-1} = \overline{g^{-1}}$:

$$\overline{g}^{-1}(h) = g^{-1}h. \tag{1.17}$$

One, thus, verifies that $\overline{g} : G \to G$ is indeed an isomorphism, and hence a permutation on G.

Now define homomorphisms $i : G \to \overline{G}$ by $i(g) = \overline{g}$, and $j : \overline{G} \to G$ by $j(\overline{g}) = \overline{g}(u) = g$, with u being the identity element of the group G.

Observe that $i \circ j = \mathbb{1}_{\overline{G}}$ and $j \circ i = \mathbb{1}_G$. Indeed, for $g \in G$ and $\overline{g} \in \overline{G}$,

$$(j \circ i)(g) = j(i(g)) = j(\overline{g}) = g,$$
$$(i \circ j)(\overline{g}) = i(j(\overline{g})) = i(g) = \overline{g},$$

establishing that $i : G \to \overline{G}$ is an isomorphism. ∎

Remark 1.5. There are two different types of isomorphisms involved in this proof. For each $g \in G$, one defines an isomorphism $\overline{g} : G \to G$. This is an isomorphism in the category **Sets**. Later, we defined an isomorphism $i : G \to \overline{G}$, which is an isomorphism in the categorty **Groups** of groups and group homomorphisms.

Remark 1.6. The group $\overline{G}$ is the group of permutations (automorphisms) on the group G which is a subgroup of the automorphism group on G itself. This subgroup has the same unit element of that of the automorphism group on G, i.e. $\mathbb{1}_G$, the identity function on the group G. Note that this is also the unit of the group $\overline{G}$ which is not the same as $\mathbb{1}_{\overline{G}}$. This identity function $\mathbb{1}_{\overline{G}}$ on $\overline{G}$ was used to establish the required isomorphism in Cayley's theorem.

Cayley's theorem can be generalized to prove that any category not "too big" (which has the collection of objects to be a set) is isomorphic to a category in which the objects are sets and the arrows are functions between those sets. In other words, any not "too big" category is isomorphic to a subcategory of **Sets**.

Theorem 1.2 (Generalized Cayley's Theorem)

Every category $\mathcal{C}$ with a set of arrows is isomorphic to one in which the objects are sets and the arrows are functions.

Sketch of proof. Define the Cayley representation $\overline{\mathcal{C}}$ of $\mathcal{C}$ to be the following concrete category (i.e. a category where objects are sets and arrows are functions between them):

- **Objects:** objects are sets of the form

$$\overline{C} = \{f : X \to C\} = \{f \in \mathrm{Hom}_{\mathcal{C}}(X, C) \mid X \in \mathrm{Ob}(\mathcal{C})\}. \tag{1.18}$$

  In other words, the object $\overline{C}$ is the set of functions whose codomains are C.

- **Arrows:** arrows are functions

$$\overline{g} : \overline{C} \to \overline{D} \tag{1.19}$$

  for $g : C \to D$ in $\mathcal{C}$, defined for any $f : X \to C$ in $\overline{C}$ by $\overline{g}(f) = g \circ f$.

$$\begin{tikzcd} X \arrow[d, "f"'] \arrow[dr, "\overline{g}(f)=g\circ f"] & \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

Then we define a functor $\mathcal{F}:\mathcal{C}\to\overline{\mathcal{C}}$ that takes the object $C\in\mathrm{Ob}\,(\mathcal{C})$ to the object $\overline{C}\in\mathrm{Ob}\left(\overline{\mathcal{C}}\right)$, and takes the arrow $g:C\to D$ in $\mathcal{C}$ to the arrow $\overline{g}:\overline{C}\to\overline{D}$ in $\overline{\mathcal{C}}$. In other words,

$$\mathcal{F}\,(C)=\overline{C}\text{ and }\mathcal{F}\,(g)=\overline{g}. \tag{1.20}$$

This is a functor since $\mathcal{F}\,(\mathbb{1}_C)=\mathbb{1}_{\overline{C}}$ and $\mathcal{F}\,(g\circ h)=\mathcal{F}\,(g)\circ\mathcal{F}\,(h)$ for composable arrows g,h in $\mathcal{C}$.

We then define another functor $\mathcal{G}:\overline{\mathcal{C}}\to\mathcal{C}$ that takes the object $\overline{C}$ to the codomains of the functions that are in $\overline{C}$, i.e. $\mathcal{G}\left(\overline{C}\right)=C$, and takes the arrow $\overline{g}:\overline{C}\to\overline{D}$ to the arrow $\overline{g}\left(\mathbb{1}_{\mathcal{G}\left(\overline{C}\right)}\right)=\overline{g}\,(\mathbb{1}_C)=g\circ\mathbb{1}_C=g:C\to D$ in $\mathcal{C}$. In other words,

$$\mathcal{G}\left(\overline{C}\right)=C\text{ and }\mathcal{G}\,(\overline{g})=g. \tag{1.21}$$

This is a functor since $\mathcal{G}\left(\mathbb{1}_{\overline{C}}\right)=\mathbb{1}_C$ and $\mathcal{G}\left(\overline{g}\circ\overline{h}\right)=\mathcal{G}\,(\overline{g})\circ\mathcal{G}\left(\overline{h}\right)$ for composable arrows $\overline{g},\overline{h}$ in $\overline{\mathcal{C}}$. Then one can verify that

$$\mathcal{G}\circ\mathcal{F}=\mathbb{1}_{\mathcal{C}}\text{ and }\mathcal{F}\circ\mathcal{G}=\mathbb{1}_{\overline{\mathcal{C}}}. \tag{1.22}$$

Therefore, $\mathcal{C}$ and $\overline{\mathcal{C}}$ are isomorphic. ■

§1.5 Construction on Categories

1. The product of 2 categories $\mathcal{C}$ and $\mathcal{D}$, written as $\mathcal{C}\times\mathcal{D}$ has objects of the form (C,D) for $C\in\mathrm{Ob}\,(\mathcal{C})$ and $D\in\mathrm{Ob}\,(\mathcal{D})$, and arrows of the form

$$(f,g):(C,D)\to\left(C',D'\right),$$

with $C,C'\in\mathrm{Ob}\,(\mathcal{C})$, $D,D'\in\mathrm{Ob}\,(\mathcal{D})$, $f\in\mathrm{Hom}_{\mathcal{C}}\,(C,C')$ and $g\in\mathrm{Hom}_{\mathcal{D}}\,(D,D')$.

Composition and units are defined componentwise, i.e.

$$\left(f',g'\right)\circ(f,g)=\left(f'\circ f,g'\circ g\right)\text{ and }\mathbb{1}_{(C,D)}=(\mathbb{1}_C,\mathbb{1}_D), \tag{1.23}$$

with $C,C',C''\in\mathrm{Ob}\,(\mathcal{C})$ and $D,D',D''\in\mathrm{Ob}\,(\mathcal{D})$ and

$$\begin{tikzcd} C \arrow[loop left, "\mathbb{1}_C"] \arrow[r, "f"] \arrow[rr, bend right, "f'\circ f"'] & C' \arrow[r, "f'"] & C'' \end{tikzcd} \qquad \begin{tikzcd} D \arrow[loop above, "\mathbb{1}_D"] \arrow[r, "g"] \arrow[rr, bend right, "g'\circ g"'] & D' \arrow[r, "g'"] & D'' \end{tikzcd}$$

Then in $\mathcal{C}\times\mathcal{D}$, we have

$$\begin{tikzcd} (C,D) \arrow[loop left, "\mathbb{1}_{(C,D)}=(\mathbb{1}_C,\mathbb{1}_D)"] \arrow[r, "(f,g)"] \arrow[rr, bend right, "(f',g')\circ(f,g)=(f'\circ f,g'\circ g)"'] & (C',D') \arrow[r, "(f',g')"] & (C'',D'') \end{tikzcd}$$

Then there are two **projection functors**:

$$\mathcal{C}\xleftarrow{\pi_1}\mathcal{C}\times\mathcal{D}\xrightarrow{\pi_2}\mathcal{D}$$

Given $(C,D)\in\mathrm{Ob}\,(\mathcal{C}\times\mathcal{D})$ and $(f,g):(C,D)\to(C',D,)$,

$$\pi_1\,(C,D)=C,\;\;\pi_1\,(f,g)=f. \tag{1.24}$$

Similarly,

$$\pi_2\,(C,D)=D,\;\;\pi_2\,(f,g)=g. \tag{1.25}$$

2. The opposite category $\mathcal{C}^{\mathrm{op}}$ has objects that are in a one-to-one correspondence with the objects of $\mathcal{C}$. Let $C^* \in \mathrm{Ob}(\mathcal{C}^{\mathrm{op}})$ be the object in $\mathcal{C}^{\mathrm{op}}$ that corresponds to $C \in \mathrm{Ob}(\mathcal{C})$. Then an arrow $f : C \to D$ in $\mathcal{C}$ corresponds to an arrow $f^* : D^* \to C^*$. With this notation, one can define composition and units in $\mathcal{C}^{\mathrm{op}}$ in terms of the corresponding operations in $\mathcal{C}$, namely

$$\mathbb{1}_{C^*} = (\mathbb{1}_C)^* . \tag{1.26}$$

For $f : C \to D$, $g : D \to E$ in $\mathcal{C}$, we have $f^* : D^* \to C^*$ and $g^* : E^* \to D^*$ in $\mathcal{C}^{\mathrm{op}}$. Then their composition is defined as

$$f^* \circ g^* = (g \circ f)^* . \tag{1.27}$$

$$\begin{tikzcd} C \arrow[loop left, "\mathbb{1}_C"] \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & D \arrow[r, "g"] & E \end{tikzcd} \quad \overset{\text{Duality}}{\longleftrightarrow} \quad \begin{tikzcd} C^* \arrow[loop above, "\mathbb{1}_{C^*}=(\mathbb{1}_C)^*"] & D^* \arrow[l, "f^*"'] & E^* \arrow[l, "g^*"'] \arrow[ll, bend left, "f^*\circ g^*=(g\circ f)^*"] \end{tikzcd}$$

3. The slice category $\mathcal{C}/C$ of a category $\mathcal{C}$ over an object $C \in \mathrm{Ob}(\mathcal{C})$ has
   - **Objects:** all arrows f in $\mathcal{C}$ such that $\mathrm{cod}(f) = C$. In other words, all arrows $f \in \mathrm{Hom}_{\mathcal{C}}(X, C)$ with some $X \in \mathrm{Ob}(\mathcal{C})$.
   - **Arrows:** an arrow a from $f : X \to C$ to $f' : X' \to C$ is precisely an arrow in $\mathrm{Hom}_{\mathcal{C}}(X, X')$ such that $f' \circ a = f$. In othe words, the following diagram commutes:

$$\begin{tikzcd} X \arrow[rr, "a"] \arrow[dr, "f"'] & & X' \arrow[dl, "f'"] \\ & C & \end{tikzcd}$$

Now suppose $f, g, h \in \mathrm{Ob}\,\mathcal{C}/C$ and $a \in \mathrm{Hom}_{\mathcal{C}/C}(f, g), b \in \mathrm{Hom}_{\mathcal{C}/C}(g, h)$. Then there are objects $X, X', X'' \in \mathrm{Ob}(\mathcal{C})$ such that the two triangles in the following diagram commute:

$$\begin{tikzcd} X \arrow[r, "a"'] \arrow[rr, bend left, "b\circ a"] \arrow[dr, "f"'] & X' \arrow[r, "b"'] \arrow[d, "g"'] & X'' \arrow[dl, "h"] \\ & C & \end{tikzcd}$$

In other words, $g \circ a = f$ and $h \circ b = g$, so that one obtains

$$f = g \circ a = (h \circ b) \circ a = h \circ (b \circ a) . \tag{1.28}$$

Therefore, we have the following commutative diagram:

$$\begin{tikzcd} X \arrow[rr, "b\circ a"] \arrow[dr, "f"'] & & X'' \arrow[dl, "h"] \\ & C & \end{tikzcd}$$

Hence, $b \circ a \in \mathrm{Hom}_{\mathcal{C}/C}(f, h)$, using the definition of arrows in $\mathcal{C}/C$. For a given $f \in \mathrm{Ob}(\mathcal{C}/C)$, $\mathbb{1}_f$ is precisely the identity arrow on $\mathrm{dom}(f)$ in the category $\mathcal{C}$, which is evident from the following commutative diagram:

$$\begin{tikzcd} \mathrm{dom}(f) \arrow[rr, "\mathbb{1}_{\mathrm{dom}(f)}"] \arrow[dr, "f"'] & & \mathrm{dom}(f) \arrow[dl, "f"] \\ & C & \end{tikzcd}$$

If $g : C \to D$ is any arrow in $\mathcal{C}$, then there is a functor called the **composition functor**:

$$g_* : \mathcal{C}/C \to \mathcal{C}/D,$$

defined on $\mathrm{Ob}(\mathcal{C}/C)$ as

$$g_*(f) = g \circ f. \tag{1.29}$$

$$\begin{tikzcd} X \arrow[d, "f"'] \arrow[dr, "g\circ f"] & \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

Commutativity of the above diagram dictates that $g \circ f \in \mathrm{Ob}\,(\mathcal{C}/D)$. Now suppose $f, f' \in \mathrm{Ob}\,(\mathcal{C}/C)$, and consider $a \in \mathrm{Hom}_{\mathcal{C}/C}\,(f, f')$ so that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "f"'] \arrow[dr, "g\circ f"] & X' \arrow[dl, "f'"'] \arrow[d, "g\circ f'"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

$$g \circ f = g \circ \left(f' \circ a\right) = \left(g \circ f'\right) \circ a, \tag{1.30}$$

so the diagram indeed commutes. So we have the following commutative diagram:

$$\begin{tikzcd} X \arrow[rr, "a"] \arrow[dr, "g\circ f"'] & & X' \arrow[dl, "g\circ f'"] \\ & D & \end{tikzcd}$$

The commutativity of this diagram dictates that $g_* \,(a) = a$. In fact, the whole construction above is a functor $\mathcal{C}/(-) : \mathcal{C} \to \mathbf{Cat}$.

$$\boxed{\begin{array}{c} \mathcal{C} \\ \\ C \xrightarrow{g} D \end{array}} \xrightarrow{\mathcal{C}/(-)} \boxed{\begin{array}{c} \mathbf{Cat} \\ \\ \mathcal{C}/C \xrightarrow{g_*} \mathcal{C}/D \end{array}}$$

4. The coslice category $C/\mathcal{C}$ of a category $\mathcal{C}$ under an object $C \in \mathrm{Ob}\,(\mathcal{C})$ has as objects all arrows f of $\mathcal{C}$ such that $\mathrm{dom}\,(f) = C$. An arrow in $\mathrm{Hom}_{C/\mathcal{C}}\,(f, f')$ is an arrow $h \in \mathrm{Hom}_{\mathcal{C}}\,(X, X')$ (where $X = \mathrm{cod}\,(X)$ and $X' = \mathrm{cod}\,(f')$) such that the diagram below commutes:

$$\begin{tikzcd} & C \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ X \arrow[rr, "h"'] & & X' \end{tikzcd}$$

In other words,

$$h \circ f = f'. \tag{1.31}$$

Question. How can the coslice category be defined in terms of the slice category and the opposite construction?

Example 1.2. The category $\mathbf{Sets}_*$ of pointed sets consists of sets A with a distinguished element $a \in A$, and arrows $f : (A, a) \to (B, b)$ are functions $f : A \to B$ that preserves the distinguished elements $f\,(a) = b$. Now, $\mathbf{Sets}_*$ is isomorphic to the coslice categroy $1/\mathbf{Sets}$ of sets under any singleton $1 = \{\star\}$.

$$\mathbf{Sets}_* \cong 1/\mathbf{Sets}. \tag{1.32}$$

Indeed, functions $\overline{a} : 1 \to A$ are uniquely determined by $\overline{a}\,(\star) = a \in A$, and are objects in $1/\mathbf{Sets}$. Now we define a functor $\mathcal{F} : \mathbf{Sets}_* \to 1/\mathbf{Sets}$ by

$$\mathcal{F}\,(A, a) = \overline{a} \text{ and } \mathcal{F}\,(f) = f. \tag{1.33}$$

Then we define $\mathcal{G} : 1/\mathbf{Sets} \to \mathbf{Sets}_*$ by

$$\mathcal{G}\,(\overline{a}) = (A, \overline{a}\,(*)) \text{ and } \mathcal{G}\,(f) = f. \tag{1.34}$$

One can easily verify that $\mathcal{G}$ and $\mathcal{F}$ are functors, and

$$\mathcal{G} \circ \mathcal{F} = \mathbb{1}_{\mathbf{Sets}_*} \text{ and } \mathcal{F} \circ \mathcal{G} = \mathbb{1}_{1/\mathbf{Sets}}\,. \tag{1.35}$$

Therefore, $1/\mathbf{Sets}$ and $\mathbf{Sets}_*$ are isomorphic categories.

2 Free Categories

§2.1 Free Monoid

Start with an "alphabet" A of "letters" $\mathtt{a}, \mathtt{b}, \mathtt{c}, \dots$, i.e. a set

$$A = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \dots\}. \tag{2.1}$$

A word over A is a finite sequence of letters:

$$\mathtt{thisword}, \mathtt{categoriesarefun}, \mathtt{asdfghjkl}, \dots$$

We write "-" for empty word. The **Kleene closure** of A is defined to be the set

$$A^* = \{\text{words over } A\}. \tag{2.2}$$

Define a binary operation $*$ on A^* by $w * w' = ww'$ for words $w, w' \in A^*$. Thus, the binary operation $*$ on A^* is just concatenation. The operation can easily be seen to be associative, and the empty word "-" is a unit. Therefore, A^* is a monoid– called the **free monoid** on the set A.

The number of letters in a word is called its **length**. The elements $\mathtt{a} \in A$ can be regarded as words of length 1. One has a function $i : A \to A^*$ defined by $i(\mathtt{a}) = \mathtt{a}$, and called the " insertion of generators". The elements of A generate the free monoid, in the sense that every $w \in A^*$ can be written as a $*$ products of elements of A, i.e.,

$$w = \mathtt{a}_1 * \mathtt{a}_2 * \cdots * \mathtt{a}_\mathtt{n},$$

for some $\mathtt{a}_1, \dots, \mathtt{a}_\mathtt{n} \in A$.
A monoid M is **freely generated** by a subset A of M, if the following conditions hold:

(a) Every element $m \in M$ can be written as a product of elements of A:

$$m = a_1 \cdot_M a_2 \cdot_M \cdots \cdot_M a_n\,, \quad \text{where } a_i \in A.$$

(b) No "nontrivial" relations hold in M. In other words, if

$$a_1 \cdot_M \cdots \cdot_M a_n = a'_1 \cdot_M \cdots \cdot_M a'_k,$$

for $a_i, a'_j \in A$, then this is required by the axioms of monoids.

The second condition of the definition of a free monoid is made more precise in the following way:
First, every monoid N has an underlying set $|N|$, and every monoid homomorphism $f : N \to M$ has an underlying function $|f| : |N| \to |M|$. This way, one has a functor **Mon** $\to$ **Sets**. This functor is called the **forgetful functor**.

The free monoid $M(A)$ on a set A is by definition "the" monoid with the following universal mapping property or UMP:

> **Universal mapping property (UMP) of $M(A)$:**
> There is a function $i : A \to |M(A)|$; and given any monoid N and any function $f : A \to |N|$, there is a **unique** monoid homomorphism $\overline{f} : M(A) \to N$ such that $\left|\overline{f}\right| \circ i = f$, as indicated in the following diagram:

$$\text{in } \mathbf{Mon}: \qquad M(A) \xrightarrow{\overline{f}} N$$

$$\text{in } \mathbf{Sets}: \qquad \begin{tikzcd} {|M(A)|} \arrow[r, "|\overline{f}|"] & {|N|} \\ A \arrow[u, "i"] \arrow[ur, "f"'] & \end{tikzcd}$$

i is called the insertion of generators.

Proposition 2.1

A^* has the UMP of the free monoid on A.

Proof. Given any monoid N and any function $f : A \to |N|$, define $\overline{f} : A^* \to N$ by $\overline{f}(\text{-}) = u_N$, and

$$\overline{f}(\mathtt{a}_1\mathtt{a}_2 \cdots \mathtt{a}_n) = \overline{f}(\mathtt{a}_1 * \mathtt{a}_2 * \cdots * \mathtt{a}_n) := f(\mathtt{a}_1) \cdot_N \cdots \cdot_N f(\mathtt{a}_n). \tag{2.3}$$

$\overline{f} : A^* \to N$ is clearly a monoid homomorphism, with $\overline{f}(\mathtt{a}) = f(\mathtt{a})$, so that

$$\left(|\overline{f}| \circ i\right)(a) = f(a). \tag{2.4}$$

Therefore, the following diagram commutes:

$$\begin{tikzcd} {|A^*|} \arrow[r, "|\overline{f}|"] & {|N|} \\ A \arrow[u, "i"] \arrow[ur, "f"'] & \end{tikzcd}$$

This proves the existence of $\overline{f} : A^* \to N$ with the required universal mapping property. Let us now prove the uniqueness. Suppose there is another monoid homomorphism $g : A^* \to N$ so that $g(\mathtt{a}) = f(\mathtt{a})$, which in turn will give us the commutative diagram exhibiting UMP. Therefore, for all $\mathtt{a}_1, \ldots, \mathtt{a}_n \in A$,

$$\begin{aligned} g(\mathtt{a}_1\mathtt{a}_2 \cdots \mathtt{a}_n) &= g(\mathtt{a}_1 * \mathtt{a}_2 * \cdots * \mathtt{a}_n) \\ &= g(\mathtt{a}_1) \cdot_N \cdots \cdot_N g(\mathtt{a}_n) \\ &= f(\mathtt{a}_1) \cdot_N \cdots \cdot_N f(\mathtt{a}_n) \\ &= \overline{f}(\mathtt{a}_1\mathtt{a}_2 \cdots \mathtt{a}_n). \end{aligned}$$

Therefore, $g = \overline{f}$, proving the uniqueness of $\overline{f} : A^* \to N$. ■

Remark 2.1. Existence of a monoid homomorphism $\overline{f} : M(A) \to N$ implies that if there is an additional equality (sometimes referred to as "noise") besides the ones imposed by associativity law and unitality law in $M(A)$, then the additional equality is transported to the monoid N. But N is **any** monoid to which the free monoid $M(A)$ is supposed to be mapped to via the monoid homomorphism $\overline{f} : M(A) \to N$. Hence, the existence of monoid homomorphism $\overline{f} : M(A) \to N$ is equivalent to the second condition of absense of any "noise" in $M(A)$.

Proposition 2.2

Given monoids M and N with functions $i : A \to |M|$ and $j : A \to |N|$, each with the UMP of the free monoid on A, there is a unique monoid isomorphism $h : M \xrightarrow{\cong} N$ such that

$$|h| \circ i = j \text{ and } \left|h^{-1}\right| \circ j = i.$$

Proof. From $j : A \to |N|$ and the UMP of M, one has $\overline{j} : M \to N$ with $\left|\overline{j}\right| \circ i = j$.

$$\text{in } \mathbf{Mon}: \qquad \begin{tikzcd} M \arrow[r, dashed, "\overline{j}"] & N \end{tikzcd}$$

$$\text{in } \mathbf{Sets}: \qquad \begin{tikzcd} {|M|} \arrow[r, "\left|\overline{j}\right|"] & {|N|} \\ A \arrow[u, "i"] \arrow[ur, "j"'] & \end{tikzcd}$$

From $i : A \to |M|$ and the UMP of N, one has $\overline{i} : N \to M$ with $\left|\overline{i}\right| \circ j = i$.

$$\text{in } \mathbf{Mon}: \qquad \begin{tikzcd} N \arrow[r, dashed, "\overline{i}"] & M \end{tikzcd}$$

$$\text{in } \mathbf{Sets}: \qquad \begin{tikzcd} {|N|} \arrow[r, "\left|\overline{i}\right|"] & {|M|} \\ A \arrow[u, "j"] \arrow[ur, "i"'] & \end{tikzcd}$$

Combining these two, we get the following commutative diagram:

$$\text{in } \mathbf{Mon}: \qquad \begin{tikzcd} M \arrow[r, dashed, "\overline{j}"'] \arrow[rr, bend left, "\overline{i} \circ \overline{j}"] & N \arrow[r, dashed, "\overline{i}"'] & M \end{tikzcd}$$

$$\text{in } \mathbf{Sets}: \qquad \begin{tikzcd} {|M|} \arrow[r, "\left|\overline{j}\right|"] & {|N|} \arrow[r, "\left|\overline{i}\right|"] & {|M|} \\ & A \arrow[ul, "i"] \arrow[u, "j"] \arrow[ur, "i"'] & \end{tikzcd}$$

From $i : A \to |M|$ and the UMP of M, we have the existence of a unique homomorphism $f : M \to M$ such that $|f| \circ i = i$. From the above commutative diagram, we get that $f = \overline{i} \circ \overline{j}$ satisfies $|f| \circ i = i$. Furthermore, $f = \mathbb{1}_M : M \to M$ also satisfies $|f| \circ i = i$. Therefore,

$$\overline{i} \circ \overline{j} = \mathbb{1}_M \,. \tag{2.5}$$

Similarly, exchanging M and N, we get

$$\overline{j} \circ \overline{i} = \mathbb{1}_N \,. \tag{2.6}$$

Now, $\overline{j} : M \to N$ is the required monoid isomorphism h, i.e. $\overline{j} = h$ and $\overline{i} = h^{-1}$, so that we have $|h| \circ i = j$ and $\left|h^{-1}\right| \circ j = i$. ■

In light of Proposition 2.1 and Proposition 2.2, we can say that if $M(A)$ has the UMP of a free monoid on A, then $M(A)$ is isomorphic to A^*.

§2.2 Free Category

Just as a monoid has an underlying set, a category has an underlying graph. A directed graph consists of vertices and edges, each of which has a "source" and a "target" vertex. Figure 2.1 is an example of a graph.

$$\begin{tikzcd}
A \arrow[r, "z"] \arrow[dr, "u"] & B \\
C \arrow[u, "x"] & D \arrow[u, "y"']
\end{tikzcd}$$

Figure 2.1: A graph

Definition 2.1. A **(directed) graph** consists of two sets: a set E of edges, and a set V of vertices, and two functions $s : E \to V$ (called source) adn $t : E \to V$ (called target). We denote a directed graph G by a quadruple (V, E, s, t). A **path** in a graph G is a finite sequence of edges $e_1, \ldots, e_n$ such that $t\,(e_i) = s\,(e_{i+1})$ for each $i = 1, \ldots, n-1$.

Suppose we have a path $e_1, \ldots, e_n$ in G.

$$v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} v_2 \xrightarrow{e_3} \cdots \xrightarrow{e_n} v_n$$

Put $\mathrm{dom}\,(e_n \cdots e_1) = s\,(e_1)$ and $\mathrm{cod}\,(e_n \cdots e_1) = t\,(e_n)$, and define composition by concatenation:

$$e_n \cdots e_1 \circ e'_m \cdots e'_1 = e_n \cdots e_1 e'_m \cdots e'_1, \tag{2.7}$$

where $\mathrm{dom}\,(e_n \cdots e_1) = \mathrm{cod}\,(e'_m \cdots e'_1)$.

For each vertex v, we have an "empty path" denoted by $\mathbb{1}_v$ which is to be the identity arrow at v. With all of these terminologies at our disposal, we see that every graph G generates a category $\mathcal{C}\,(G)$ called the **free category** on G. It is defined by taking vertices of G as objects and paths in G as arrows. For example, take the graph given in Figure 2.1 with 4 vertices A, B, C, D. The corresponding free category on G is given by:

$G:$

$$\begin{tikzcd}
A \arrow[r, "z"] \arrow[dr, "u"] & B \\
C \arrow[u, "x"] & D \arrow[u, "y"']
\end{tikzcd}$$

$\mathcal{C}(G)$

$$\begin{tikzcd}
C \arrow[r, "x"] \arrow[rr, bend left=50, "z \circ x"] \arrow[rr, bend right=30, "y \circ u \circ x"'] \arrow[rrr, bend right=50, "u \circ x"'] & A \arrow[r, "z"] \arrow[r, bend left, "y \circ u"] \arrow[rr, bend right=20, "u"'] & B & D \arrow[l, "y"']
\end{tikzcd}$$

Definition 2.2 (Graph Homomorphism)**.** Let $G \equiv (V, E, s, t)$ and $G' \equiv (V', E', s', t')$ be two graphs. A **graph homomorphism** f from G to G', denoted by $f : G \to G'$ consists of two functions $f_0 : V \to V'$ and $f_1 : E \to E'$ such that the following diagrams commute:

$$\begin{tikzcd}
E \arrow[r, "f_1"] \arrow[d, "s"'] & E' \arrow[d, "s'"] \\
V \arrow[r, "f_0"'] & V'
\end{tikzcd}
\qquad
\begin{tikzcd}
E \arrow[r, "f_1"] \arrow[d, "t"'] & E' \arrow[d, "t'"] \\
V \arrow[r, "f_0"'] & V'
\end{tikzcd}$$

Remark 2.2. Note that if G has only one vertex, then $\mathcal{C}\,(G)$ is just the free monoid on the set of edges of G. If, on the other hand, G has only vertices with no edges, then $\mathcal{C}\,(G)$ is the discrete category on the set of vertices of G.

Let us now see that $\mathcal{C}(G)$ has a UMP (universal mapping property). Define a "forgetful functor" $\mathcal{U} : \mathbf{Cat} \to \mathbf{Graphs}$ in the following way: the underlying graph of a (small) category has the collection of arrows as the set of edges E and the collection of objects as the set of vertices V, with $s = \mathrm{dom}$ and $t = \mathrm{cod}$.

Also, observe that we can describe a category $\mathcal{C}$ with a diagram as below:

$$\begin{tikzcd} C_2 \arrow[r, "\circ"] & C_1 \arrow[r, shift left=2, "\mathrm{cod}"] \arrow[r, shift right=2, "\mathrm{dom}"'] & C_0 \arrow[l, "i" description] \end{tikzcd},$$

where C_0 is the collection of objects of $\mathcal{C}_1$, C_1 is the collection of arrows, i is the identity arrow operation, and C_2 is the collection

$$C_2 = \{(f, g) \in C_1 \times C_1 \mid \mathrm{cod}\, f = \mathrm{dom}\, g\}.$$

Then a functor $F : \mathcal{C} \to \mathcal{D}$ from $\mathcal{C}$ to another category $\mathcal{D}$ (with D_2, D_1, D_0 as given above) is a pair of assignments $F_0 : C_0 \to D_0$ and $F_1 : C_1 \to D_1$ such that each similarly labeled square in the following diagram commutes:

$$\begin{tikzcd} C_2 \arrow[r, "\circ"] \arrow[d, "F_2"'] & C_1 \arrow[r, shift left=2, "\mathrm{cod}"] \arrow[r, shift right=2, "\mathrm{dom}"'] \arrow[d, "F_1"'] & C_0 \arrow[l, "i" description] \arrow[d, "F_0"] \\ D_2 \arrow[r, "\circ"] & D_1 \arrow[r, shift left=2, "\mathrm{cod}"] \arrow[r, shift right=2, "\mathrm{dom}"'] & D_0 \arrow[l, "i" description] \end{tikzcd},$$

where $F_2(f, g) = (F_1(f), F_1(g))$. Commutativity of the first square tells us that

$$F_1(g \circ f) = F_1(g) \circ F_1(f). \tag{2.8}$$

Commutativity of the second square is reminiscent of graph homomorphism if one removes the identity arrow operation. Note that the underlying graph of a category $\mathcal{C}$ can be depicted as

$$\begin{tikzcd} C_1 \arrow[r, shift left, "\mathrm{cod}"] \arrow[r, shift right, "\mathrm{dom}"'] & C_0 \end{tikzcd}.$$

Therefore, at the level of objects, the forgetful functor

$$\mathcal{U} : \mathbf{Cat} \to \mathbf{Graphs} \tag{2.9}$$

sends the category (object of **Cat**)

$$\begin{tikzcd} C_2 \arrow[r, "\circ"] & C_1 \arrow[r, shift left=2, "\mathrm{cod}"] \arrow[r, shift right=2, "\mathrm{dom}"'] & C_0 \arrow[l, "i" description] \end{tikzcd}$$

to the underlying graph (objecr of **Graphs**)

$$\begin{tikzcd} C_1 \arrow[r, shift left, "\mathrm{cod}"] \arrow[r, shift right, "\mathrm{dom}"'] & C_0 \end{tikzcd}.$$

And functors (arrows of **Cat**) under $\mathcal{U}$ are sent to graph homomorphisms (arrows of **Graphs**), i.e.

$$\begin{tikzcd} C_2 \arrow[r, "\circ"] \arrow[d, "F_2"'] & C_1 \arrow[r, shift left=2, "\mathrm{cod}"] \arrow[r, shift right=2, "\mathrm{dom}"'] \arrow[d, "F_1"'] & C_0 \arrow[l, "i" description] \arrow[d, "F_0"] \\ D_2 \arrow[r, "\circ"] & D_1 \arrow[r, shift left=2, "\mathrm{cod}"] \arrow[r, shift right=2, "\mathrm{dom}"'] & D_0 \arrow[l, "i" description] \end{tikzcd},$$

is sent to

$$
\begin{tikzcd}
C_1 \arrow[r, shift left, "\mathrm{cod}"] \arrow[r, shift right, "\mathrm{dom}"'] \arrow[d, "F_1"'] & C_0 \arrow[d, "F_0"] \\
D_1 \arrow[r, shift left, "\mathrm{cod}"] \arrow[r, shift right, "\mathrm{dom}"'] & D_0
\end{tikzcd}
$$

under $\mathcal{U}$. Given a category $\mathcal{C}$, its underlying graph is denoted by $|\mathcal{C}| = \mathcal{U}(\mathcal{C})$, where $\mathcal{U}$ is the forgetful functor $\mathcal{U} : \mathbf{Cat} \to \mathbf{Graphs}$. The **free category** $\mathcal{C}(G)$ on a graph G has the following universal mapping property (UMP).

Universal mapping property (UMP) of $\mathcal{C}(G)$:

There is a graph homomorphism $i : G \to |\mathcal{C}(G)|$; and given any category $\mathcal{D}$ and any graph homomorphism $h : G \to |\mathcal{D}|$, there is a **unique** functor $\overline{h} : \mathcal{C}(G) \to \mathcal{D}$ such that $\left|\overline{h}\right| \circ i = h$, as indicated in the following diagram:

$$
\text{in } \mathbf{Cat}: \quad \begin{tikzcd} \mathcal{C}(G) \arrow[r, dashed, "\overline{f}"] & N \end{tikzcd}
$$

$$
\text{in } \mathbf{Graphs}: \quad \begin{tikzcd} \left|\mathcal{C}(G)\right| \arrow[r, "\left|\overline{h}\right|"] & \left|\mathcal{D}\right| \\ G \arrow[u, "i"] \arrow[ur, "h"'] & \end{tikzcd}
$$

The free category on a graph with just one vertex is just a free monoid on the set oof edges. The free category on a graph with two vertices and one edge between them is the finite category $\mathbf{2}$. On the other hand, the free category on a graph of the form

$$
\begin{tikzcd} A \arrow[r, shift left, "e"] & B \arrow[l, shift left, "f"] \end{tikzcd}
$$

has (in addition to the identity arrows) the infinitely many arrows:

$$
e, f, ef, fe, efe, fef, efef, fefe, \dots. \tag{2.10}
$$

3 Abstract Characterizations

§3.1 Epis and Monos

In **Sets**, a function $f : A \to B$ is called

- **injective** if $f(a) = f(a')$ implies $a = a'$ for all $a, a' \in A$,
- **surjective** if for each $b \in B$, there exists $a \in A$ such that $f(a) = b$.

We have the following abstract characterizations of these properties:

Definition 3.1 (Monomorphism and Epimorphism)**.** In any category $\mathcal{C}$, an arrow $f : A \to B$ is called a **monomorphism** if given any $g, h : C \to A$, $f \circ g = f \circ h$ implies $g = h$.

$$C \underset{h}{\overset{g}{\rightrightarrows}} A \xrightarrow{f} B$$

An arrow $f : A \to B$ is called an **epimorphism** if given any $i, j : B \to D$, $i \circ f = j \circ f$ implies $i = j$.

$$A \xrightarrow{f} B \underset{j}{\overset{i}{\rightrightarrows}} D$$

We often write $f : A \rightarrowtail B$ if f is a monomorphism and $f : A \twoheadrightarrow B$ if f is an epimorphism.

Proposition 3.1
A function $f : A \to B$ between sets is a monomorphism if and only if it is injective.

Proof. **(⇒)** Suppose $f : A \rightarrowtail B$. We need to show that f is injective. Suppose $f(a) = f(a')$ for some $a, a' \in A$. Consider functions $g, h : \{*\} \to A$ as $g(*) = a$ and $h(*) = a'$. Then

$$(f \circ g)(*) = f(g(*)) = f(a) = f(a') = f(h(*)) = (f \circ h)(*) . \tag{3.1}$$

So $f \circ g = f \circ h$. Since f is a monomorphism, we must have $g = h$. Therefore, $a = a'$, proving the injectivity of f.

(⇐) Conversely, let $f : A \to B$ be injective. Let $g, h : C \to A$ such that $f \circ g = f \circ h$. Then for any $c \in C$,

$$f(g(c)) = (f \circ g)(c) = (f \circ h)(c) = f(h(c)) . \tag{3.2}$$

So $f(g(c)) = f(h(c))$. Since f is injective, this implies $g(c) = h(c)$. This is true for any $c \in C$. Therefore, $g = h$. In other words, $f \circ g = f \circ h$ implies $g = h$. Hence, f is a monomorphism. ■

Proposition 3.2
A function $f : A \to B$ between sets is an epimorphism if and only if it is surjective.

Proof. **(⇒)** Suppose $f : A \twoheadrightarrow B$. We need to show that f is surjective. We construct two functions $g_1, g_2 : B \to \{0, 1\}$ as follows:

$$g_1(b) = 0 \text{ and } g_2(b) = \begin{cases} 0 & \text{if } b \in \operatorname{im} f \\ 1 & \text{otherwise.} \end{cases} \tag{3.3}$$

$$A \xrightarrow{f} B \overset{g_1}{\underset{g_2}{\rightrightarrows}} \{0,1\}$$

Then for each $a \in A$, $g_1(f(a)) = g_2(f(a)) = 0$. So $g_1 \circ f = g_2 \circ f$. Since f is an epimorphism, this gives us $g_1 = g_2$. In other words,

$$g_2(b) = 0 \text{ for each } b \in B. \tag{3.4}$$

Therefore, all $b \in B$ are in $\operatorname{im} f$, proving the surjectivity of f.

(⇐) Conversely, let $f : A \to B$ be surjective. Let $g_1, g_2 : B \to C$ such that $g_1 \circ f = g_2 \circ f$.

$$A \xrightarrow{f} B \overset{g_1}{\underset{g_2}{\rightrightarrows}} \{0,1\}$$

Since f is surjective, for any $b \in B$, there exists $a \in A$ such that $f(a) = b$. Then we have

$$g_1(b) = g_1(f(a)) = (g_1 \circ f)(a) = (g_2 \circ f)(a) = g_2(f(a)) = g_2(b). \tag{3.5}$$

So $g_1(b) = g_2(b)$, and this is true for any $b \in B$. Therefore, $g_1 = g_2$ and hence f is an epimorphism. ■

Monomorphisms are often obbreviated as monos or monic, epimorphisms are often obbreviated as epis or epic.

Proposition 3.3

A monoid homomorphism $h : M \to N$ is monic if and only if the underlying function $|h| : |M| \to |N|$ is monic (i.e. injective by Proposition 3.1).

Proof. **(⇒)** Suppose h is monic. We need to show that $|h|$ is injective. Let $|h|(m_1) = |h|(m_2)$ for $m_1, m_2 \in |M|$. Take two functions $x, y : 1 \to |M|$, where $1 = \{*\}$ is any singleton set, defined by $x(*) = m_1$ and $y(*) = m_2$. Let $M(1)$ be the free monoid generated by 1. By the UMP of a free monoid, there are unique monoid homomorphisms $\overline{x} : M(1) \to M$ and $\overline{y} : M(1) \to M$ such that the following diagrams commute:

$$\begin{tikzcd} {|M(1)|} \arrow[rr, "|\overline{x}|"] & & {|M|} \\ & 1 \arrow[ul, "i"] \arrow[ur, "x"'] & \end{tikzcd} \qquad \begin{tikzcd} {|M(1)|} \arrow[rr, "|\overline{y}|"] & & {|M|} \\ & 1 \arrow[ul, "i"] \arrow[ur, "y"'] & \end{tikzcd}$$

In other words, $|\overline{x}| \circ i = x$ and $|\overline{y}| \circ i = y$. Furthermore, by the UMP of $M(1)$ for the functions $|h| \circ x, |h| \circ y : 1 \to N$, there are unique monoid homomorphisms $f : M(1) \to N$ and $g : M(1) \to N$ such that the following diagrams commute:

$$\begin{tikzcd} {|M(1)|} \arrow[rr, "|f|"] & & {|N|} \\ & 1 \arrow[ul, "i"] \arrow[ur, "|h|\circ x"'] & \end{tikzcd} \qquad \begin{tikzcd} {|M(1)|} \arrow[rr, "|g|"] & & {|N|} \\ & 1 \arrow[ul, "i"] \arrow[ur, "|h|\circ y"'] & \end{tikzcd}$$

But since the following diagrams commute,

$$\begin{tikzcd} {|M(1)|} \arrow[rr, "|\overline{x}|"'] \arrow[rrrr, bend left, "|h|\circ|\overline{x}|=|h\circ\overline{x}|"] & & {|M|} \arrow[rr, "|h|"'] & & {|N|} \\ & 1 \arrow[ul, "i"] \arrow[ur, "x"'] \arrow[urrr, bend right, "|h|\circ x"'] & & & \end{tikzcd} \qquad \begin{tikzcd} {|M(1)|} \arrow[rr, "|\overline{y}|"'] \arrow[rrrr, bend left, "|h|\circ|\overline{y}|=|h\circ\overline{y}|"] & & {|M|} \arrow[rr, "|h|"'] & & {|N|} \\ & 1 \arrow[ul, "i"] \arrow[ur, "y"'] \arrow[urrr, bend right, "|h|\circ y"'] & & & \end{tikzcd}$$

the uniqueness of f and g guarantees that

$$f = h \circ \overline{x} \text{ and } g = h \circ \overline{y}. \tag{3.6}$$

However, $|h| \circ x$ and $|h| \circ y$ are equal, since

$$(|h| \circ x)(*) = |h|(m_1) = |h|(m_2) = (|h| \circ y)(*). \tag{3.7}$$

Therefore, $f = g$, and hence,

$$h \circ \overline{x} = h \circ \overline{y}. \tag{3.8}$$

Since h is monic, $\overline{x} = \overline{y}$. Then we have

$$x = |\overline{x}| \circ i = |\overline{y}| \circ i = y. \tag{3.9}$$

Hence, $m_1 = x(*) = y(*) = m_2$, and thus $|h|$ is injective.

($\Leftarrow$) Conversely, suppose $|h| : |M| \to |N|$ is monic. Let $f, g : X \to M$ be monoid homomorphisms such that $h \circ f = h \circ g$.

$$\begin{tikzcd} X \arrow[r, shift left, "g"] \arrow[r, shift right, "f"'] & M \arrow[r, "h"] & N. \end{tikzcd}$$

In **Sets**, we then have $|h| \circ |f| = |h| \circ |g|$.

$$\begin{tikzcd} |X| \arrow[r, shift left, "|g|"] \arrow[r, shift right, "|f|"'] & |M| \arrow[r, "|h|"] & |N|. \end{tikzcd}$$

Since $|h|$ is monic, we must have $|f| = |g|$. Therefore, $f = g$, and hence $h : M \to N$ is monic. ■

Example 3.1. In the category **Mon** of monoids and monoid homomorphisms, there is a monic homomorphism $\overline{\mathbb{N}} \rightarrowtail \overline{\mathbb{Z}}$, where $\overline{\mathbb{N}} = (\mathbb{N}, +, 0)$ is the additive monoid of natural numbers, and $\overline{\mathbb{Z}} = (\mathbb{Z}, +, 0)$ is the additive monoid of integers. This map given by the inclusion $\mathbb{N} \subset \mathbb{Z}$ of sets is monic in **Mon**, by Proposition 3.3, since it is an injective homomorphism. We shall now show that this is also epic in **Mon**.

$$\begin{tikzcd} \overline{\mathbb{N}} \arrow[r, "i"] & \overline{\mathbb{Z}} \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \overline{M} \end{tikzcd}$$

Given any monoid $\overline{M} = (M, *, u)$, with the underlying set M, let $f, g : \overline{\mathbb{Z}} \to \overline{M}$ be monoid homomorphisms such that $f \circ i = g \circ i$. To prove that i is epic, we need to show that $f = g$.

Since $i : \overline{\mathbb{N}} \to \overline{\mathbb{Z}}$ is the inclusion, $f \circ i = g \circ i$ gives us that $f\big|_{\overline{\mathbb{N}}} = g\big|_{\overline{\mathbb{N}}}$, i.e. $f(n) = g(n)$ for each $n \in \overline{\mathbb{N}}$. We need to show that $f(-n) = g(-n)$.

$$\begin{aligned} f(-n) &= f(-n) * u = f(-n) * g(0) = f(-n) * g(n + (-n)) \\ &= f(-n) * [g(n) * g(-n)] = [f(-n) * g(n)] * g(-n) \\ &= [f(-n) * f(n)] * g(-n) = f(-n + n) * g(-n) \\ &= u * g(-n) = g(-n). \end{aligned} \tag{3.10}$$

Therefore, $f = g$, and hence i is epic.

Proposition 3.4

Every isomorphism is both monic and epic.

Proof. Let $m : B \to C$ be an isomorphism. Then there exists $e : C \to B$ such that $m \circ e = \mathbb{1}_C$ and $e \circ m = \mathbb{1}_B$. So the two triangles in the following diagram commute:

$$\begin{tikzcd}
A \arrow[r, shift left, "x"] \arrow[r, shift right, "y"'] & B \arrow[r, "m"] \arrow[dr, "e\circ m=\mathbb{1}_B"'] & C \arrow[d, "e"] \arrow[dr, "m\circ e=\mathbb{1}_C"] & & \\
& & B \arrow[r, "m"'] & C \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & C
\end{tikzcd}$$

Given any arrows $x, y : A \to B$ such that $m \circ x = m \circ y$, we have

$$\begin{aligned}
m \circ x = m \circ y &\implies e \circ (m \circ x) = e \circ (m \circ y) \\
&\implies (e \circ m) \circ x = (e \circ m) \circ y \\
&\implies \mathbb{1}_B \circ x = \mathbb{1}_B \circ y \\
&\implies x = y. \qquad (3.11)
\end{aligned}$$

Therefore, m is monic. Now, given arrows $f, g : C \to D$ with $f \circ m = g \circ m$, we have

$$\begin{aligned}
f \circ m = g \circ m &\implies (f \circ m) \circ e = (g \circ m) \circ e \\
&\implies f \circ (m \circ e) = g \circ (m \circ e) \\
&\implies f \circ \mathbb{1}_C = g \circ \mathbb{1}_C \\
&\implies f = g. \qquad (3.12)
\end{aligned}$$

Therefore, m is epic. ■

Remark 3.1. In **Sets**, the converse of Proposition 3.4 also holds: every monic and epic is an isomorphism. But this is not true, in general. The counterexample is provided in the context of the category **Mon** in Example 3.1. There we saw that the inclusion homomorphism $i : (\mathbb{N}, +, 0) \to (\mathbb{Z}, +, 0)$ is both monic and epic. But this arrow in **Mon** is not an iso, i.e. its inverse does not exist. In particular, there does not exist an arrow $j : \mathbb{Z} \to \mathbb{N}$ such that $i \circ j = \mathbb{1}_{\mathbb{Z}}$ and $j \circ i = \mathbb{1}_{\mathbb{N}}$.

§3.2 Sections and Retractions

We have seen in Proposition 3.4 that any iso is both monic and epic. More generally, if an arrow $f : A \to B$ has a left inverse $g : B \to A$, i.e. $g \circ f = \mathbb{1}_A$, then f must be monic and g epic.

$$\begin{tikzcd}
C \arrow[r, shift left, "x"] \arrow[r, shift right, "y"'] & A \arrow[r, "f"] \arrow[dr, "g\circ f=\mathbb{1}_A"'] & B \arrow[d, "g"] & \\
& & A \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & D
\end{tikzcd}$$

Given arrows $x, y : C \to A$ with $f \circ x = f \circ y$,

$$\begin{aligned}
f \circ x = f \circ y &\implies g \circ (f \circ x) = g \circ (f \circ y) \\
&\implies (g \circ f) \circ x = (g \circ f) \circ y \\
&\implies \mathbb{1}_A \circ x = \mathbb{1}_A \circ y \\
&\implies x = y. \qquad (3.13)
\end{aligned}$$

Therefore, $f : A \to B$ is monic. Now, given $p, q : A \to D$ with $p \circ g = q \circ g$,

$$\begin{aligned}
p \circ g = q \circ g &\implies (p \circ g) \circ f = (q \circ g) \circ f \\
&\implies p \circ (g \circ f) = q \circ (g \circ f) \\
&\implies p \circ \mathbb{1}_A = q \circ \mathbb{1}_A \\
&\implies p = q. \qquad (3.14)
\end{aligned}$$

Therefore, $g : B \to A$ is epic.

Definition 3.2. A **split mono** is an arrow with a left inverse. A **split epi** is an arrow with a right inverse. Given arrows $e : X \to A$ and $s : A \to X$ with $e \circ s = \mathbb{1}_A$, the arrow s is called a **section** or **splitting** of e, and e is called a **retraction** of s. The object A is called a retract of X. Therefore, section is monic and retraction is epic.

Remark 3.2. Since functors preserve identities, they also preserve split epis and split monos. Functors do not preserve epi, in general. Counterexample is the forgetful functor $\mathcal{U} : \mathbf{Mon} \to \mathbf{Sets}$, which does not preserve the epi $i : \overline{\mathbb{N}} \to \overline{\mathbb{Z}}$, i.e. $\mathcal{U}(i) : \mathbb{N} \to \mathbb{Z}$ is not an epi in **Sets** as it's not surjective.

In **Sets**, every mono splits except those of the form $\varnothing \rightarrowtail A$. The condition that every epi splits is the categorical version of the axiom of choice. Let us see this fact in detail. Consider an epi $e : E \twoheadrightarrow X$. We then have the following family of nonempty sets

$$E_x = e^{-1}(\{x\}) \quad \text{for } x \in X.$$

Each E_x is nonempty since e is surjective. Now, for each $x \in X$, one chooses an element $s(x)$ from $E_x \subset E$, and thus define the function $s : X \to E$. By construction, $e(s(x)) = x$ for each $x \in X$, i.e. $e \circ s = \mathbb{1}_X$. One, thus finds that the choice function $s : X \to E$ associated with the family $(E_x)_{x \in X}$ is exactly a splitting of e.

Let us do the reverse construction now. Given a family of nonempty subsets $(E_x)_{x \in X}$, take

$$E = \{(x, y) \mid x \in X, y \in E_x\} = \bigcup_{x \in X} (\{x\} \times E_x), \tag{3.15}$$

and define the epi $e : E \twoheadrightarrow X$ by $(x, y) \mapsto x$. A splitting $s : X \to E$ hence determines a choice function for the family $(E_x)_{x \in X}$,

Remark 3.3. A notion related to the existence of "choice function" is that of being projective. An object P is said to be **projective** if for any $e : E \twoheadrightarrow X$ and arrow $f : P \to X$, there is some (not necessarily unique) arrow $\overline{f} : P \to E$ such that $e \circ \overline{f} = f$, as indicated in the following commutative diagram:

$$\begin{tikzcd} & E \arrow[d, two heads, "e"] \\ P \arrow[ur, "\overline{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

One says that f **lifts across** e.

Let P be projective and let $e : E \twoheadrightarrow P$ be an epi. Since P is projective, given $\mathbb{1}_P : P \to P$, there exists $s : P \to E$ such that the following diagram commutes:

$$\begin{tikzcd} & E \arrow[d, two heads, "e"] \\ P \arrow[ur, dashed, "s"] \arrow[r, "\mathbb{1}_P"'] & P \end{tikzcd}$$

yielding $s \circ e = \mathbb{1}_P$. In other words, the epi $e : E \twoheadrightarrow P$ splits.

§3.3 Initial and Terminal Objects

Definition 3.3. In any category $\mathcal{C}$, an object

- 0 is **initial** if for any object C, there is a unique arrow $0 \to C$,
- 1 is **terminal** if for any object C, there is a unique arrow $C \to 1$.

Proposition 3.5

Initial and terminal objects are unique up to isomorphism.

Proof. Suppose both 0 and $0'$ are initial objects in some category $\mathcal{C}$. Since 0 is initial, there is a unique arrow $u : 0 \to 0'$. Again, since $0'$ is initial, there is a unique arrow $v : 0' \to 0$. Then we can form the composition $v \circ u : 0 \to 0$. But since 0 is an initial object, there is a unique arrow $0 \to 0$, which has to be the identity arrow $\mathbb{1}_0$. Therefore, $v \circ u = \mathbb{1}_0$.

$$\begin{tikzcd}
0 \arrow[r, "u"] \arrow[dr, "v\circ u=\mathbb{1}_0"'] & 0' \arrow[d, "v"] \arrow[dr, "u\circ v=\mathbb{1}_{0'}"] & \\
 & 0 \arrow[r, "u"] & 0'
\end{tikzcd}$$

Similarly, $u \circ v : 0' \to 0'$ is an arrow from $0'$ to itself. But since $0'$ is an initial object, there is a unique arrow $0' \to 0'$, which has to be the identity arrow $\mathbb{1}_{0'}$. Therefore, $u \circ v = \mathbb{1}_{0'}$. So 0 and $0'$ are isomorphic via a unique isomorphism $u : 0 \to 0'$.

Let 1 and $1'$ be terminal objects of $\mathcal{C}$. Since $1'$ is terminal, there is a unique arrow $u : 1 \to 1'$. Again, since 1 is terminal, there is a unique arrow $v : 1' \to 1$. Then we can form the composition $v \circ u : 1 \to 1$. But since 1 is a terminal object, there is a unique arrow $1 \to 1$, which has to be the identity arrow $\mathbb{1}_1$. Therefore, $v \circ u = \mathbb{1}_1$.

$$\begin{tikzcd}
1 \arrow[r, "u"] \arrow[dr, "v\circ u=\mathbb{1}_1"'] & 1' \arrow[d, "v"] \arrow[dr, "u\circ v=\mathbb{1}_{1'}"] & \\
 & 1 \arrow[r, "u"] & 1'
\end{tikzcd}$$

Similarly, $u \circ v : 1' \to 1'$ is an arrow from $1'$ to itself. But since $1'$ is a terminal object, there is a unique arrow $1' \to 1'$, which has to be the identity arrow $\mathbb{1}_{1'}$. Therefore, $u \circ v = \mathbb{1}_{1'}$. So 1 and $1'$ are isomorphic via a unique isomorphism $u : 1 \to 1'$. ■

Example 3.2. 1. In **Sets**, the empty set $\varnothing$ is initial, and sny singleton set is terminal. Indeed, for any set B, there is a unique function from $\varnothing$ to B, called the empty function. When $B = \varnothing$, the empty function from $\varnothing$ to $\varnothing$ is the required identity arrow on the object $\varnothing \in \mathrm{Ob}\,(\mathbf{Sets})$.

There is also a unique function from any set B to a singleton set $\{*\}$, $f : B \to \{*\}$, given by $f(b) = *$ for every $b \in B$. It is unique in the sense that there can't be any other function from the same set B to the singleton set $\{*\}$. In other words, $\mathrm{Hom}_{\mathbf{Sets}}(B, \{*\})$ for a given set $B \in \mathrm{Ob}\,(\mathbf{Sets})$ is a singleton set.

2. In $\mathbf{Vect}_{\mathbb{K}}$, the category of vector spaces over the field $\mathbb{K}$ and linear transformations, the 0-dimensional vector space $\{\mathbf{0}\}$ consisting of the zero-vector (additive identity) only is both the initial and terminal objects.

Given a $\mathbb{K}$-vector space V, there is only one linear transformation from $\{\mathbf{0}\}$ to V, namely the one that takes $\mathbf{0}$ to $\mathbf{0}_V$, the additive identity of the $\mathbb{K}$-vector space V. Also, given a $\mathbb{K}$-vector space V, there is only one linear transformation from V to $\{\mathbf{0}\}$ that takes all of V to $\mathbf{0}$. In other words, for a given $V \in \mathrm{Ob}\,(\mathbf{Vect}_{\mathbb{K}})$, both $\mathrm{Hom}_{\mathbf{Vect}_{\mathbb{K}}}(\{\mathbf{0}\}, V)$ and $\mathrm{Hom}_{\mathbf{Vect}_{\mathbb{K}}}(V, \{\mathbf{0}\})$ are singleton sets.

§3.3.i The Category of Boolean Algebras

Definition 3.4 (Boolean algebra)**.** A **Boolean algebra** is a poset $(B, \leqslant)$ together with distinguished elements $0, 1$, binary operations $a \vee b$ (read a join b) and $a \wedge b$ (read a meet b) and a unary operation $\neg b$ (read b complement) such that the following conditions are satisfied for all $a, b, c \in B$:

1. $0 \leqslant a$ and $a \leqslant 1$.
2. $a \leqslant c$ and $b \leqslant c$ if and only if $a \vee b \leqslant c$.
3. $c \leqslant a$ and $c \leqslant b$ if and only if $c \leqslant a \wedge b$.
4. $a \leqslant \neg b$ if and only if $a \wedge b = 0$.
5. $\neg(\neg a) = a$.

A typical example of a Boolean algebra is the power set $\mathscr{P}(X)$ of a set X, ordered by inclusion $A \subseteq B$, with the distinguished elements $0 = \varnothing$ and $1 = X$. The binary operations join and meet are given by union and intersection of subsets, respectively, and the unary operation complementation is $\neg A = X \setminus A$.

The union and intersection of sets satisfy $A \cap B \subseteq A \subseteq A \cup B$ and $A \cap B \subseteq B \subseteq A \cup B$. The same is satisfied in any Boolean algebra.

Proposition 3.6
In a Boolean algebra B, $a \wedge b \leqslant a \leqslant a \vee b$ and $a \wedge b \leqslant b \leqslant a \vee b$ for any $a, b \in B$.

Proof. Since $(B, \leqslant)$ is a poset, by the reflexivity property, $a \vee b \leqslant a \vee b$. Then using property 2 from the definition of Boolean algebra, we get

$$a \leqslant a \vee b \text{ and } b \leqslant a \vee b. \tag{3.16}$$

Again, $a \wedge b \leqslant a \wedge b$ by reflexivity. Then using property 3 from the definition of Boolean algebra, we get

$$a \wedge b \leqslant a \text{ and } a \wedge b \leqslant b. \tag{3.17}$$

Therefore, combining (3.16) and (3.17), we get $a \wedge b \leqslant a \leqslant a \vee b$ and $a \wedge b \leqslant b \leqslant a \vee b$. ■

A special case of Boolean algebras is the 2-element Boolean algebra $\mathbf{2} = \{0, 1\}$, considered as $\mathscr{P}(\{*\})$, the power set pf the singleton set $\{*\}$ which consists of only the empty set $\varnothing$ and $\{*\}$.

The category of Boolean algebras is denoted by **BA**, where objects are Boolean algebras and arrows are Boolean homomorphisms, that is functions $h : B \to B'$ that, in addition to being monotone, satisfy

- $h(0_B) = 0_{B'}$, $h(1_B) = 1_{B'}$;
- $h(a \vee_B b) = h(a) \vee_{B'} h(b)$;
- $h(a \wedge_B b) = h(a) \wedge_{B'} h(b)$;
- $h(\neg_B a) = \neg_{B'} h(a)$,

for any $a, b \in B$. In this category, $\mathbf{2}$ is the initial object. There is this 1-element Boolean algebra $\mathbf{1} = \{0\}$, which is regarded as the power set of $\varnothing$, i.e. $\mathbf{1} \equiv \mathscr{P}(\varnothing) = \{\varnothing\}$. In this situation, the distinguished elements 0 and 1 conicide. $\mathbf{1}$ acts as the terminal object of **BA**. In other words, given a Boolean algebra B, there is exactly one boolean homomorphism from $h : \mathbf{2} \to B$, given by

$$h(0) = 0_B \text{ and } h(1) = 1_B.$$

Furthermore, there is exactly one boolean homomorphism $f : B \to \mathbf{1}$ given by

$$f(b) = 0$$

for any $b \in B$.

Definition 3.5 (Filter)**.** A **filter** in a Boolean algebra B is a nonempty subset $F \subseteq B$ that is closed upward and under meets, i.e.

- $a \in F$ and $a \leqslant b$ implies $b \in F$,
- $a, b \in F$ implies $a \wedge b \in F$.

Definition 3.6 (Ultrafilter)**.** A filter F is called **maximal** if $F \subset F'$ implies $F' = B$, i.e. the only strictly larger filter is the "improper" filter B itself. A maximal filter is called an **ultrafilter**.

One can easily verify that a filter F is an ultrafilter if and only if for every element $b \in B$, either $b \in F$ or $\neg b \in F$ and not both.

§3.4 Generalized Elements

In **Sets**, we have the notion of an element of a set. However, in a general category, we don't have the notion of an element of an object, because the objects need not be sets. That's why we need to generalize the concept of an element to any category. One idea can be that for a given object X, we can consider all the arrows from the terminal object to X. This idea works in **Sets**, since any set X is determined uniquely by the set of all the functions from $1 = \{*\}$ to X. In other words,

$$X \cong \mathrm{Hom}_{\mathbf{Sets}}(1, X). \tag{3.18}$$

Any $x \in X$ uniquely corresponds to a function $\overline{x} : 1 \to X$ given by $\overline{x}(*) = x$. Conversely, any function $f : 1 \to X$ uniquely corresponds to an element $f(*)$ of X. Thus, we get the one-to-one correspondence (i.e. an isomorphism in **Sets**) $X \cong \mathrm{Hom}_{\mathbf{Sets}}(1, X)$.

The arrows $1 \to X$ are called **global elements** of X. However, this idea of taking all the arrows from the terminal object does not always work in other categories. For instance, in $\mathbf{Vect}_{\mathbb{K}}$, the terminal object is $\{\mathbf{0}\}$, which is also the initial object. So given a $\mathbb{K}$-vector space V, there is only one linear transformation from $\{\mathbf{0}\}$ to V, $x : \{\mathbf{0}\} \to V$, given by

$$x(\mathbf{0}) = \mathbf{0}_V.$$

Given two $\mathbb{K}$-vector spaces V and W, and two linear transformations $f, g : V \to W$, we always have $f \circ x = g \circ x$, since

$$\begin{aligned}(f \circ x)(\mathbf{0}) &= f(x(\mathbf{0})) = f(\mathbf{0}_V) = \mathbf{0}_W, \\ (g \circ x)(\mathbf{0}) &= g(x(\mathbf{0})) = g(\mathbf{0}_V) = \mathbf{0}_W.\end{aligned}$$

No matter how different f and g are, composing them with x yields the same arrow. This is not what we want. Ideally, we want generalized elements to have the following property:

$$f = g \text{ if and only if } f \circ x = g \circ x \text{ for every generalized element } x \text{ of } \operatorname{dom} f.$$

The motivation for requiring this condition comes from sets and functions. We call two functions $f, g : X \to Y$ the same if and only if $f(x) = g(x)$ for every $x \in X$.

For this, we draw motivations from the proof of Generalized Cayley's Theorem. In the sketch of the proof, we showed that a "not too big" category is isomorphic to a concrete category (i.e. a category where objects are sets and arrows are functions between them). We've seen that under the isomorphism, the object C of the "not too big" category $\mathcal{C}$ corresponds to the set of arrows whose codomains are C. Keeping this in mind, we define the generalized elements of C to be the arrows whose codomainds are C.

> **Definition 3.7** (Generalized Element)**.** Let $\mathcal{C}$ be a category, and $A \in \mathrm{Ob}(\mathcal{C})$. The arrows whose codomains are A are called **generalized elements** of A. In other words, any arbitrary arrow $f : X \to A$ (with arbitrary domain X) is a generalized element of A.

Then these generalized elements satisfy the required property that $f = g$ if and only if $f \circ x = g \circ x$ for every generalized element x of $\mathrm{dom}\, f$ $(= \mathrm{dom}\, g)$. Let $f, g : A \to B$ be two arrows. If $f = g$, then obviously $f \circ x = g \circ x$. Conversely, suppose $f \circ x = g \circ x$ for every generalized element x of A. Then if we choose $x = \mathbb{1}_A$ (which is an arrow with codomain A, so it's a generalized element of A), we get

$$f \circ \mathbb{1}_A = g \circ \mathbb{1}_A \implies f = g. \tag{3.19}$$

Furthermore, we defined monomorphisms (generalizations of injective functions) as follows: $f : A \to B$ is a monomorphism if for any $x, y : C \to A$, $f \circ x = f \circ y$ implies $x = y$.

$$\begin{tikzcd} C \arrow[r, shift left, "y"] \arrow[r, shift right, "x"'] & A \arrow[r, "f"] & B \end{tikzcd}$$

In other words, $f : A \to B$ is a monomorphism if it is "injective on generalized elements of A".

§3.5 Products

Let us begin by considering products of sets. Given sets A and B, the cartesian product of A and B is the set of ordered pairs

$$A \times B = \{(a, b) \mid a \in A, b \in B\}.$$

Observe that there are 2 coordinate projections

$$A \xleftarrow{\pi_1} A \times B \xrightarrow{\pi_2} B$$

with $\pi_1 (a, b) = a$ and $\pi_2 (a, b) = b$. Indeed, given any element $c \in A \times B$, one has $c = (\pi_1 c, \pi_2 c)$. This situation is captured by the following commutative diagram:

$$\begin{tikzcd} & 1 \arrow[dl, "a"'] \arrow[d, dashed, "{(a,b)}"] \arrow[dr, "b"] & \\ A & A \times B \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & B \end{tikzcd}$$

$$(a, b)(*) = (a(*), b(*)).$$

$$(\pi_1 \circ (a, b))(*) = \pi_1 (a(*), b(*)) = a(*).$$

$$(\pi_2 \circ (a, b))(*) = \pi_2 (a(*), b(*)) = b(*).$$

Here, $a \in \mathrm{Hom}_{\mathbf{Sets}} (1, A), b \in \mathrm{Hom}_{\mathbf{Sets}} (1, B)$ are global elements or constants of A and B, respectively. Replacing global elements by generalized elements, one has the following definition.

> **Definition 3.8** (Product)**.** In any category $\mathcal{C}$, a product diagram for the objects A and B consists of an object P and arrows
>
> $$A \xleftarrow{p_1} P \xrightarrow{p_2} B$$
>
> satisfying the following UMP: given any diagram of the form
>
> $$A \xleftarrow{x_1} X \xrightarrow{x_2} B$$
>
> there exists a unique morphism $u : X \to P$ such that the following diagram commutes:

$$\begin{tikzcd}
& X \arrow[dl, "x_1"'] \arrow[d, dashed, "\exists!\, u"] \arrow[dr, "x_2"] & \\
A & P \arrow[l, "p_1"'] \arrow[r, "p_2"] & B
\end{tikzcd}$$

i.e. $x_1 = p_1 \circ u$ and $x_2 = p_2 \circ u$.

Proposition 3.7

Products are unique up to isomorphism.

Proof. Suppose one has two product diagrams as below:

$$A \xleftarrow{p_1} P \xrightarrow{p_2} B$$
$$A \xleftarrow{q_1} Q \xrightarrow{q_2} B$$

where P, Q are both taken to be products of A and B. Since Q is a product, there is a unique $i : P \to Q$ such that $q_1 \circ i = p_1$ and $q_2 \circ i = p_2$. Again, since P is a product, there is a unique $j : Q \to P$ such that $p_1 \circ j = q_1$ and $p_2 \circ j = q_2$. In other words, the following diagram commutes:

$$\begin{tikzcd}
& P \arrow[dl, "p_1"'] \arrow[d, dashed, "i"] \arrow[dr, "p_2"] & \\
A & Q \arrow[l, "q_1"'] \arrow[r, "q_2"] \arrow[d, dashed, "j"] & B \\
& P \arrow[ul, "p_1"] \arrow[ur, "p_2"'] &
\end{tikzcd}$$

Then we have

$$p_1 = q_1 \circ i = p_1 \circ (j \circ i) \text{ and } p_2 = q_2 \circ i = p_2 \circ (j \circ i)\,. \tag{3.20}$$

Since P is a product, there is a unique $f : P \to P$ such that the following diagram commutes:

$$\begin{tikzcd}
& P \arrow[dl, "p_1"'] \arrow[d, dashed, "f"] \arrow[dr, "p_2"] & \\
A & P \arrow[l, "p_1"] \arrow[r, "p_2"'] & B
\end{tikzcd}$$

$f = \mathbb{1}_P$ makes this diagram commutative, since $p_1 = p_1 \circ \mathbb{1}_P$ and $p_2 = p_2 \circ \mathbb{1}_P$. Therefore, the uniqueness of f guarantees that $f = \mathbb{1}_P$. Furthermore, (3.20) tells us that $f = j \circ i$ also makes the above diagram commutative. Therefore,

$$f = \mathbb{1}_P = j \circ i. \tag{3.21}$$

Similarly, using the following commutative diagram,

$$\begin{tikzcd}
& Q \arrow[dl, "q_1"'] \arrow[d, dashed, "j"] \arrow[dr, "q_2"] & \\
A & P \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, dashed, "i"] & B \\
& Q \arrow[ul, "q_1"] \arrow[ur, "q_2"'] &
\end{tikzcd}$$

we have

$$q_1 = p_1 \circ j = q_1 \circ (i \circ j) \text{ and } q_2 = p_2 \circ i = q_2 \circ (i \circ j)\,. \tag{3.22}$$

Since Q is a product, there is a unique $g : Q \to Q$ such that the following diagram commutes:

$$\begin{tikzcd}
 & Q \arrow[dl, "q_1"'] \arrow[d, dashed, "g"] \arrow[dr, "q_2"] & \\
A & Q \arrow[l, "q_1"] \arrow[r, "q_2"'] & B
\end{tikzcd}$$

$g = \mathbb{1}_Q$ makes this diagram commutative, since $q_1 = q_1 \circ \mathbb{1}_Q$ and $q_2 = q_2 \circ \mathbb{1}_Q$. Therefore, the uniqueness of g guarantees that $g = \mathbb{1}_Q$. Furthermore, (3.22) tells us that $g = i \circ j$ also makes the above diagram commutative. Therefore,

$$g = \mathbb{1}_Q = i \circ j. \tag{3.23}$$

Therefore, we have $i \circ j = \mathbb{1}_Q$ and $j \circ i = \mathbb{1}_P$. Hence, i is an isomorphism and $P \cong Q$. ■

If A and B have a product, we write

$$A \xleftarrow{p_1} A \times B \xrightarrow{p_2} B$$

for one such product. Then given X, x_1, x_2 as below

$$A \xleftarrow{x_1} X \xrightarrow{x_2} B$$

there is a unique arrow $u : X \to A \times B$ such that $x_i = p_i \circ u$ for $i = 1, 2$. We denote $u = \langle x_1, x_2 \rangle$. Hence, $p_1 \circ \langle x_1, x_2 \rangle = x_1$ and $p_2 \circ \langle x_1, x_2 \rangle = x_2$. Now, however, a pair of objects may have different products in a category $\mathcal{C}$. For example, given a product $A \times B, p_1, p_2$ and an isomorphism $h : A \times B \to Q$, one finds that $Q, p_1 \circ h^{-1}, p_2 \circ h^{-1}$ is also a product of A and B.

$$\begin{tikzcd}
 & & X \arrow[d, "u"'] \arrow[ddll, bend right, "x_1"'] \arrow[ddrr, bend left, "x_2"] \arrow[dd, bend left, "\widetilde{u}"] & & \\
 & & Q \arrow[dll, "p_1 \circ h^{-1}"'] \arrow[d, "h^{-1}"'] \arrow[drr, "p_2 \circ h^{-1}"] & & \\
A & & A \times B \arrow[ll, "p_1"] \arrow[rr, "p_2"'] & & B
\end{tikzcd}$$

Now, given a diagram $A \xleftarrow{x_1} X \xrightarrow{x_2} B$, there exists unique $\widetilde{u} : X \to A \times B$ such that $p_1 \circ \widetilde{u} = x_1$ and $p_2 \circ \widetilde{u} = x_2$. Define $u : X \to Q$ as $u = h \circ \widetilde{u}$. Then $h^{-1} \circ u = \widetilde{u}$. So

$$x_1 = p_1 \circ \widetilde{u} = \left(p_1 \circ h^{-1}\right) \circ u \text{ and } x_2 = p_2 \circ \widetilde{u} = \left(p_2 \circ h^{-1}\right) \circ u,$$

and the diagraam above commutes. Furthermore, u is unique. Indeed, if there is another $v : X \to Q$ such that $x_1 = \left(p_1 \circ h^{-1}\right) \circ v$ and $x_2 = \left(p_2 \circ h^{-1}\right) \circ v$, then by the uniqueness of $\widetilde{u}$,

$$\widetilde{u} = h^{-1} \circ v \implies v = h \circ \widetilde{u} = u.$$

So u is unique. This proves that $Q, p_1 \circ h^{-1}, p_2 \circ h^{-1}$ fulfills the universal property of a product diagram.

Example 3.3. 1. Products of "structured sets" like monoids or groups can often be constructed as products of the underlying sets with componentwise operations: if G and H are groups, then $G \times H$ can be constructed by taking the underlying set of $G \times H$ to be the set $\{\langle g, h\rangle \mid g \in G, h \in H\}$ and define the binary operation in $G \times H$ by

$$\langle g, h\rangle *_{G\times H} \langle g', h'\rangle = \langle g *_G g', h *_H h'\rangle ,$$

where the group G is given by the triple $(G, *_G, u_G)$ and the group H is given by the triple $(H, *_H, u_H)$. The unit of $G \times H$ is given by

$$u_{G\times H} = \langle u_G, u_H\rangle .$$

The inverse of $\langle g, h\rangle$ in $G \times H$ is

$$\langle g, h\rangle^{-1} = \langle g^{-1}, h^{-1}\rangle .$$

The projection homomorphism $G \times H \to G$ (or H) are the evident ones

$$\langle g, h\rangle \mapsto g \text{ (or } h) .$$

2. Let $(P, \leqslant)$ be a poset and consider a product $p \times q$ of elements $p, q \in P$. We must have projections $p \times q \to p$ and $p \times q \to q$. In a poset, these arrows indicate

$$p \times q \leqslant p \text{ and } p \times q \leqslant q$$

. Furthermore, we need $p \times q$ to fulfill the UMP of product: if for a given element $x \in P$, $x \to p$ and $x \to q$ (i.e. $x \leqslant p$ and $x \leqslant q$ in the poset), then there exists a unique arrow $x \to p \times q$ (i.e. $x \leqslant p \times q$). The product $p \times q$ is easily recognized to be the greatest lower bound $p \wedge q$.

Remark 3.4. Note that the projection arrows $p_1 : A \times B \to A$ and $p_2 : A \times B \to B$ need not be epic. In **Sets**, let A be a non-empty set. Consider the product of $\varnothing$ and A. Then the projection

$$p_2 : \varnothing \times A \to A$$

is not surjective, since the domain is empty set but the codomain is nonempty. Therefore, the projection p_2 is not an epimorphism in **Sets**.

§3.5.i Product Topological Space

Let us show that the product of two topological spaces X and Y is a product $X \times Y$ in **Top**, the category of topological spaces and continuous maps. Let $\mathcal{O}(X)$ be the collection of open sets in X. Suppose we have two topological space X and Y and the product space $X \times Y$ along with the projections

$$X \xleftarrow{p_1} X \times Y \xrightarrow{p_2} Y$$

Recall that $\mathcal{O}(X \times Y)$ is generated by basic open sets of the form $U \times V$, where $U \in \mathcal{O}(X)$ and $V \in \mathcal{O}(Y)$, so that every $W \in \mathcal{O}(X \times Y)$ is a union of such basic open sets. We have the following observations:

- Clearly $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ are continuous: for $U \in \mathcal{O}(X)$ and $V \in \mathcal{O}(Y)$,

$$p_1^{-1}(U) = U \times Y \in \mathcal{O}(X \times Y),$$
$$p_2^{-1}(V) = X \times V \in \mathcal{O}(X \times Y).$$

  So $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ are continuous.

- Given any cotinuous $f_1 : Z \to X$ and $f_2 : Z \to Y$, let $f : Z \to X \times Y$ be defined as

$$f(z) = (f_1(z), f_2(z)).$$

We need to verify that f is continuous. Given $W \in \mathcal{O}(X \times Y)$, W can be expressed as

$$W = \bigcup_\alpha (U_\alpha \times V_\alpha),$$

for $U_\alpha \in \mathcal{O}(X)$ and $V_\alpha \in \mathcal{O}(Y)$. Then

$$f^{-1}(W) = \bigcup_\alpha f^{-1}(U_\alpha \times V_\alpha).$$

Therefore, it suffices to show that $f^{-1}(U \times V) \in \mathcal{O}(Z)$ for $U \in \mathcal{O}(X)$ and $V \in \mathcal{O}(Y)$.

$$\begin{aligned} f^{-1}(U \times V) &= f^{-1}((U \times Y) \cap (X \times V)) \\ &= f^{-1}(U \times Y) \cap f^{-1}(X \times V) \\ &= f^{-1}\left(p_1^{-1}(U)\right) \cap f^{-1}\left(p_2^{-1}(V)\right) \\ &= (p_1 \circ f)^{-1}(U) \cap (p_2 \circ f)^{-1}(V) \\ &= f_1^{-1}(U) \cap f_2^{-1}(V). \end{aligned}$$

Both $f_1^{-1}(U)$ and $f_2^{-1}(V)$ are open in Z since f_1, f_2 are continuous. Hence, f is continuous.

- Therefore, given $f_1 : Z \to X$ and $f_2 : Z \to Y$ continuous, there exists a continuous $f : Z \to X \times Y$ such that $p_1 \circ f = f_1$ and $p_2 \circ f = f_2$. Now, suppose there exists another continuous $g : Z \to X \times Y$ such that $p_1 \circ g = f_1$ and $p_2 \circ g = f_2$. Let $g(z) = (g_1(z), g_2(z))$. Then

$$f_1(z) = (p_1 \circ g)(z) = p_1(g_1(z), g_2(z)) = g_1(z), \tag{3.24}$$

for every $z \in Z$. Similarly, $f_2(z) = g_2(z)$. Therefore, $g = f$, proving the uniqueness of f.

So $X \times Y$ is a product of X and Y in **Top**.

§3.5.ii Category with Products

Let $\mathcal{C}$ be a category that has a product diagram for every pair of objects. Such a category is said to have a binary product. Suppose we have objects and arrows as indicated below:

$$\begin{tikzcd} A \arrow[d, "f"'] & A \times A' \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, dashed, "f \times f'"] \arrow[dl, "f \circ p_1"'] \arrow[dr, "f' \circ p_2"] & A' \arrow[d, "f'"] \\ B & B \times B' \arrow[l, "q_1"] \arrow[r, "q_2"'] & B' \end{tikzcd} \tag{3.25}$$

Then there exists a unique arrow from the product $A \times A'$ to $B \times B'$ as follows:

$$f \times f' = \langle f \circ p_1, f' \circ p_2 \rangle. \tag{3.26}$$

In other words, $f \times f'$ is the unique arrow from $A \times A'$ to $B \times B'$ which makes the above diagram (3.25) commutative. One can, therefore, construct a functor $\times : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$[1] as follows:

$$\times(A, A') = A \times A' \text{ and } \times(f, f') = f \times f'. \tag{3.27}$$

Let us now verify that $\times$ is indeed a functor. For an object (A, A') in $\mathcal{C} \times \mathcal{C}$, let $\mathbb{1}_{(A,A')} = (\mathbb{1}_A, \mathbb{1}_{A'})$ be its identity arrow. We need to show that $\times\left(\mathbb{1}_{(A,A')}\right) = \mathbb{1}_A \times \mathbb{1}_{A'} = \mathbb{1}_{A \times A'}$.

[1] $\mathcal{C} \times \mathcal{C}$ is the product category as discussed in Section 1.5. An object in $\mathcal{C} \times \mathcal{C}$ is of the form (A, A') with $A, A' \in \mathrm{Ob}(\mathcal{C})$. Given arrows $f : A \to B$ and $f' : A' \to B'$ in $\mathcal{C}$, one has $(f, f') : (A, A') \to (B, B')$ as an arrow in $\mathcal{C} \times \mathcal{C}$ between objects (A, A') and (B, B').

$$\begin{tikzcd}
A \arrow[d, "\mathbb{1}_A"'] & A \times A' \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[dl, "\mathbb{1}_A \circ p_1"'] \arrow[d, dashed, "\mathbb{1}_A \times \mathbb{1}_{A'}"] \arrow[dr, "\mathbb{1}_{A'} \circ p_2"] & A' \arrow[d, "\mathbb{1}_{A'}"] \\
A & A \times A' \arrow[l, "p_1"] \arrow[r, "p_2"'] & A'
\end{tikzcd}$$

$\mathbb{1}_A \times \mathbb{1}_{A'}$ is the **unique** arrow $A \times A' \to A \times A'$ that makes this diagram commute. If we choose $\mathbb{1}_{A\times A'}$ in place of $\mathbb{1}_A \times \mathbb{1}_{A'}$, then also the diagram commutes:

$$\begin{tikzcd}
A \arrow[d, "\mathbb{1}_A"'] & A \times A' \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[dl, "\mathbb{1}_A \circ p_1"'] \arrow[d, dashed, "\mathbb{1}_{A\times A'}"] \arrow[dr, "\mathbb{1}_{A'} \circ p_2"] & A' \arrow[d, "\mathbb{1}_{A'}"] \\
A & A \times A' \arrow[l, "p_1"] \arrow[r, "p_2"'] & A'
\end{tikzcd}$$

since

$$p_1 \circ \mathbb{1}_{A\times A'} = p_1 = \mathbb{1}_A \circ p_1 \text{ and } p_2 \circ \mathbb{1}_{A\times A'} = p_2 = \mathbb{1}_{A'} \circ p_2.$$

Therefore, by the uniqueness, $\mathbb{1}_A \times \mathbb{1}_{A'} = \mathbb{1}_{A\times A'}$. Hence,

$$\times\left(\mathbb{1}_{(A,A')}\right) = \times(\mathbb{1}_A, \mathbb{1}_{A'}) = \mathbb{1}_A \times \mathbb{1}_{A'} = \mathbb{1}_{A\times A'}\,. \tag{3.28}$$

Let $(f, f') : (A, A') \to (B, B')$ and $(g, g') : (B, B') \to (C, C')$ be arrows in $\mathcal{C} \times \mathcal{C}$. Then $(g, g') \circ (f, f') = (g \circ f, g' \circ f')$. Consider the following diagram:

$$\begin{tikzcd}
A \arrow[d, "f"'] & A \times A' \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[dl, "f \circ p_1"'] \arrow[d, dashed, "f\times f'"] \arrow[dr, "f' \circ p_2"] & A' \arrow[d, "f'"] \\
B \arrow[d, "g"'] & B \times B' \arrow[l, "q_1"'] \arrow[r, "q_2"] \arrow[dl, "g \circ q_1"'] \arrow[d, dashed, "g\times g'"] \arrow[dr, "g' \circ q_2"] & B' \arrow[d, "g'"] \\
C & C \times C' \arrow[l, "r_1"] \arrow[r, "r_2"'] & C'
\end{tikzcd}$$

In this diagram, $f \times f' : A \times A' \to B \times B'$ and $g \times g'$ are the **unique** maps such that the diagram above commutes. In other words,

$$q_1 \circ f \times f' = f \circ p_1,\ q_2 \circ f \times f' = f' \circ p_2, \text{ and} \tag{3.29}$$

$$r_1 \circ g \times g' = g \circ q_1,\ r_2 \circ g \times g' = g' \circ q_2. \tag{3.30}$$

$g \circ f : A \to C$ and $g' \circ f' : A' \to C'$. Then $(g \circ f) \times (g' \circ f')$ is the **unique** map from $A \times A'$ to $C \times C'$ such that the following diagram commutes:

$$\begin{tikzcd}
A \arrow[d, "g\circ f"'] & A \times A' \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[dl, "g\circ f\circ p_1"'] \arrow[d, dashed, "(g\circ f)\times(g'\circ f')"] \arrow[dr, "g'\circ f'\circ p_2"] & A' \arrow[d, "g'\circ f'"] \\
C & C \times C' \arrow[l, "r_1"] \arrow[r, "r_2"'] & C'
\end{tikzcd}$$

In other words,

$$r_1 \circ (g \circ f) \times (g' \circ f') = g \circ f \circ p_1 \text{ and } r_2 \circ (g \circ f) \times (g' \circ f') = g' \circ f' \circ p_2. \tag{3.31}$$

If we take $(g \times g') \circ (f \times f')$ in place of $(g \circ f) \times (g' \circ f')$, then also the diagram commutes, since

$$r_1 \circ (g \times g') \circ (f \times f') = g \circ q_1 \circ (f \times f') = g \circ f \circ p_1, \tag{3.32}$$

$$r_2 \circ (g \times g') \circ (f \times f') = g' \circ q_2 \circ (f \times f') = g' \circ f' \circ p_2. \tag{3.33}$$

$$\begin{tikzcd}
A \arrow[d, "g\circ f"'] & A\times A' \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[dl, "g\circ f\circ p_1"'] \arrow[dr, "g'\circ f'\circ p_2"] \arrow[d, dashed, "(g\times g')\circ(f\times f')"] & A' \arrow[d, "g'\circ f'"] \\
C & C\times C' \arrow[l, "r_1"] \arrow[r, "r_2"'] & C'
\end{tikzcd}$$

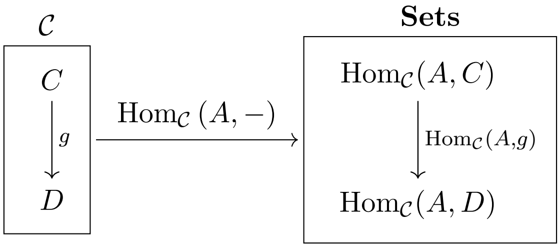
Therefore, by the uniqueness, we have $(g\times g')\circ(f\times f') = (g\circ f)\times(g'\circ f')$. Then

$$\times\left(\left(g,g'\right)\circ\left(f,f'\right)\right) = \times\left(g\circ f, g'\circ f'\right) = \left(g\circ f\right)\times\left(g'\circ f'\right) = \left(g\times g'\right)\circ\left(f\times f'\right) = \times\left(g,g'\right)\circ\times\left(f,f'\right). \tag{3.34}$$

Therefore, $\times : \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ is indeed a functor.

§3.6 Hom-sets

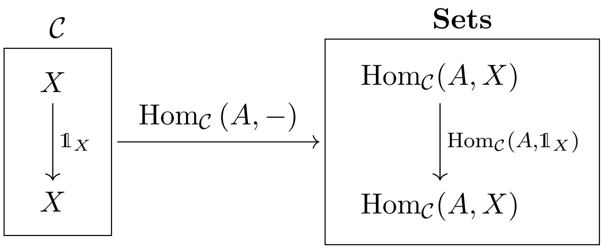
We are dealing with locally small categories, where given any pair of objects A and B, the collection $\mathrm{Hom}_{\mathcal{C}}\left(A,B\right)$ of all the arrows from A to B forms a set. We call such a set of arrows **Hom-set**. Now, fix $A\in\mathrm{Ob}\left(\mathcal{C}\right)$ once and for all. Then we consider a functor $\mathrm{Hom}_{\mathcal{C}}\left(A,-\right) : \mathcal{C}\to\mathbf{Sets}$ as follows:



with the function $\mathrm{Hom}_{\mathcal{C}}(A,g) : \mathrm{Hom}_{\mathcal{C}}(A,C)\to\mathrm{Hom}_{\mathcal{C}}(A,D)$ defined by

$$\mathrm{Hom}_{\mathcal{C}}(A,g)\,(f) = g\circ f,$$

for each $f\in\mathrm{Hom}_{\mathcal{C}}(A,C)$. $\mathrm{Hom}_{\mathcal{C}}\left(A,-\right)$ is called the **representable functor** of A. Let us now show that $\mathrm{Hom}_{\mathcal{C}}\left(A,\mathbb{1}_X\right) = \mathbb{1}_{\mathrm{Hom}_{\mathcal{C}}(A,X)}$.



Indeed, let $x\in\mathrm{Hom}_{\mathcal{C}}\left(A,X\right)$. Then

$$\mathrm{Hom}_{\mathcal{C}}\left(A,\mathbb{1}_X\right)(x) = \mathbb{1}_X\circ x = x = \mathbb{1}_{\mathrm{Hom}_{\mathcal{C}}(A,X)}(x), \tag{3.35}$$

for any $x\in\mathrm{Hom}_{\mathcal{C}}\left(A,X\right)$. Therefore, $\mathrm{Hom}_{\mathcal{C}}\left(A,\mathbb{1}_X\right) = \mathbb{1}_{\mathrm{Hom}_{\mathcal{C}}(A,X)}$.

Now, take two composable arrows $g : Y\to Z$ and $f : X\to Y$ in $\mathcal{C}$. We want to show that

$$\mathrm{Hom}_{\mathcal{C}}\left(A,g\circ f\right) = \mathrm{Hom}_{\mathcal{C}}\left(A,g\right)\circ\mathrm{Hom}_{\mathcal{C}}\left(A,f\right) : \mathrm{Hom}_{\mathcal{C}}\left(A,X\right)\to\mathrm{Hom}_{\mathcal{C}}\left(A,Z\right).$$

Let's take any $x\in\mathrm{Hom}_{\mathcal{C}}\left(A,X\right)$. Then,

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C}}\left(A,g\circ f\right)(x) &= g\circ f\circ x = g\circ\left(\mathrm{Hom}_{\mathcal{C}}\left(A,f\right)(x)\right) \\
&= \mathrm{Hom}_{\mathcal{C}}\left(A,g\right)\left(\mathrm{Hom}_{\mathcal{C}}\left(A,f\right)(x)\right) \\
&= \left[\mathrm{Hom}_{\mathcal{C}}\left(A,g\right)\circ\mathrm{Hom}_{\mathcal{C}}\left(A,f\right)\right](x),
\end{aligned} \tag{3.36}$$

which is true for any $x \in \mathrm{Hom}_{\mathcal{C}}(A, X)$. Therefore,

$$\mathrm{Hom}_{\mathcal{C}}(A, g \circ f) = \mathrm{Hom}_{\mathcal{C}}(A, g) \circ \mathrm{Hom}_{\mathcal{C}}(A, f). \tag{3.37}$$

Hence, $\mathrm{Hom}_{\mathcal{C}}(A, -) : \mathcal{C} \to \mathbf{Sets}$ is indeed a functor.

We will now present an alternative characterization of product using Hom-sets. For any object P, a pair of arrows $p_1 : P \to A$ and $p_2 : P \to B$ determine an element (p_1, p_2) of the set

$$\mathrm{Hom}_{\mathcal{C}}(P, A) \times \mathrm{Hom}_{\mathcal{C}}(P, B).$$

Given any arrow $x : X \to P$, composing with p_1 and p_2 gives a pair of arrows $x_1 = p_1 \circ x : X \to A$ and $x_2 = p_2 \circ x : X \to B$ as indicated in the following diagram:

$$\begin{tikzcd} & X \arrow[dl, "x_1 = p_1 \circ x"'] \arrow[d, "x"] \arrow[dr, "x_2 = p_2 \circ x"] & \\ A & P \arrow[l, "p_1"] \arrow[r, "p_2"'] & B \end{tikzcd}$$

In this way, one has a function

$$\vartheta_X \equiv (\mathrm{Hom}_{\mathcal{C}}(X, p_1), \mathrm{Hom}_{\mathcal{C}}(X, p_2)) : \mathrm{Hom}_{\mathcal{C}}(X, P) \to \mathrm{Hom}_{\mathcal{C}}(X, A) \times \mathrm{Hom}_{\mathcal{C}}(X, B).$$

Given any $x : X \to P$, ϑ maps it to

$$(\mathrm{Hom}_{\mathcal{C}}(X, p_1)(x), \mathrm{Hom}_{\mathcal{C}}(X, p_2)(x)) = (p_1 \circ x, p_2 \circ x).$$

In other words,

$$\vartheta(x) = (p_1 \circ x, p_2 \circ x) = (x_1, x_2). \tag{3.38}$$

Proposition 3.8

A diagram of the form $A \xleftarrow{p_1} P \xrightarrow{p_2} B$ is a product diagram if and only if for every object X, the canonical function ϑ_X defined by (3.38) is an isomorphism,

$$\vartheta_X : \mathrm{Hom}_{\mathcal{C}}(X, P) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{C}}(X, A) \times \mathrm{Hom}_{\mathcal{C}}(X, B).$$

Proof. **(⇒)** Suppose P is a product. Then given any object X of $\mathcal{C}$ and a pair of arrows $x_1 : X \to A$ and $x_2 : X \to B$, there is a unique arrow $x : X \to P$ such that the following diagram commutes:

$$\begin{tikzcd} & X \arrow[dl, "x_1"'] \arrow[d, dashed, "x"] \arrow[dr, "x_2"] & \\ A & P \arrow[l, "p_1"] \arrow[r, "p_2"'] & B \end{tikzcd}$$

In other words, $p_1 \circ x = x_1$ and $p_2 \circ x = x_2$. Therefore, for any object X, and $x_1 \in \mathrm{Hom}_{\mathcal{C}}(X, A)$, $x_2 \in \mathrm{Hom}_{\mathcal{C}}(X, B)$, there is a unique $x \in \mathrm{Hom}_{\mathcal{C}}(X, P)$ which makes the above diagram commutative. We, therefore, have a function

$$\wp_X : \mathrm{Hom}_{\mathcal{C}}(X, A) \times \mathrm{Hom}_{\mathcal{C}}(X, B) \to \mathrm{Hom}_{\mathcal{C}}(X, P)$$

defined by

$$\wp_X(x_1, x_2) = x, \tag{3.39}$$

where x is the **unique** arrow from X to P such that $p_1 \circ x = x_1$ and $p_2 \circ x = x_2$. In particular,

$$\wp_X(p_1 \circ x, p_2 \circ x) = x. \tag{3.40}$$

Then for any $(x_1, x_2) \in \mathrm{Hom}_{\mathcal{C}}(X, A) \times \mathrm{Hom}_{\mathcal{C}}(X, B)$,

$$(\vartheta_X \circ \wp_X)(x_1, x_2) = \vartheta_X(x) = (p_1 \circ x, p_2 \circ x) = (x_1, x_2),$$

and for any $x \in \mathrm{Hom}_{\mathcal{C}}(X, P)$,

$$(\wp_X \circ \vartheta_X)(x) = \wp_X(p_1 \circ x, p_2 \circ x) = x.$$

Therefore,

$$\vartheta_X \circ \wp_X = \mathbb{1}_{\mathrm{Hom}_{\mathcal{C}}(X,A) \times \mathrm{Hom}_{\mathcal{C}}(X,B)} \quad \text{and} \quad \wp_X \circ \vartheta_X = \mathbb{1}_{\mathrm{Hom}_{\mathcal{C}}(X,P)}. \tag{3.41}$$

Therefore, ϑ_X is an isomorphism for any object X.

(⇐) Conversely, suppose, for every object X, the canonical function

$$\vartheta_X : \mathrm{Hom}_{\mathcal{C}}(X, P) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{C}}(X, A) \times \mathrm{Hom}_{\mathcal{C}}(X, B)$$

is an isomorphism. Given any $x_1 \in \mathrm{Hom}_{\mathcal{C}}(X, A)$ and $x_2 \in \mathrm{Hom}_{\mathcal{C}}(X, B)$, since ϑ_X is an isomorphism (i.e. a bijection since we are in **Sets**), there exists a **unique** $x \in \mathrm{Hom}_{\mathcal{C}}(X, P)$ such that $\vartheta_X(x) = (x_1, x_2)$. Using (3.38), we get

$$(x_1, x_2) = \vartheta_X(x) = (p_1 \circ x, p_2 \circ x). \tag{3.42}$$

Therefore, given any $A \xleftarrow{x_1} X \xrightarrow{x_2} B$, there is a unique $x : X \to P$ such that $p_1 \circ x = x_1$ and $p_2 \circ x = x_2$. In other words, the following diagram commutes:

$$\begin{tikzcd} & X \arrow[dl, "x_1"'] \arrow[d, dashed, "x"] \arrow[dr, "x_2"] & \\ A & P \arrow[l, "p_1"] \arrow[r, "p_2"'] & B \end{tikzcd}$$

Therefore, P is a product of A and B. ■

Definition 3.9. Let $\mathcal{C}$ and $\mathcal{D}$ be categories with binary products. A functor $F : \mathcal{C} \to \mathcal{D}$ is said to **preserve binary products** if it takes every product diagram

$$A \xleftarrow{p_1} A \times B \xrightarrow{p_2} B \qquad \text{in } \mathcal{C}$$

to a product diagram

$$F(A) \xleftarrow{F(p_1)} F(A \times B) \xrightarrow{F(p_2)} F(B) \qquad \text{in } \mathcal{D}.$$

The latter is a product diagram in $\mathcal{D}$ if and only if $F(A \times B) \cong F(A) \times F(B)$. In other words, if and only if the canonical arrow

$$\langle F(p_1), F(p_2) \rangle : F(A \times B) \xrightarrow{\cong} F(A) \times F(B)$$

is an isomorphism. Note that $\langle F(p_1), F(p_2) \rangle$ is the **unique** arrow from $F(A \times B)$ to $F(A) \times F(B)$ such that the diagram below commutes:

$$\begin{tikzcd} & F(A \times B) \arrow[dl, "F(p_1)"'] \arrow[d, dashed, "\langle F(p_1),F(p_2)\rangle"] \arrow[dr, "F(p_2)"] & \\ F(A) & F(A) \times F(B) \arrow[l, "\widetilde{p}_1"] \arrow[r, "\widetilde{p}_2"'] & F(B) \end{tikzcd}$$

Therefore, if $\mathcal{C}$ and $\mathcal{D}$ are categories with binary products, then a functor $F : \mathcal{C} \to \mathcal{D}$ preserves binary products if and only if $F(A \times B) \cong F(A) \times F(B)$ in $\mathcal{D}$ for any $A, B \in \mathrm{Ob}(\mathcal{C})$. For example, the forgetful functor $\mathcal{U} : \textbf{Mon} \to \textbf{Sets}$ preserves binary products.

Corollary 3.9
For any object X in a category $\mathcal{C}$ with products, the covariant representable functor

$$\mathrm{Hom}_{\mathcal{C}}\left(X,-\right):\mathcal{C}\to\mathbf{Sets}$$

preserves products.

Proof. For $A, B \in \mathrm{Ob}\left(\mathcal{C}\right)$, Proposition 3.8 says that there is a canonical isomorphism

$$\mathrm{Hom}_{\mathcal{C}}\left(X, A\times B\right)\cong\mathrm{Hom}_{\mathcal{C}}\left(X, A\right)\times\mathrm{Hom}_{\mathcal{C}}\left(X, B\right),$$

as $A\times B$ is a product of A and B. In other words,

$$\mathrm{Hom}_{\mathcal{C}}\left(X,-\right)\left(A\times B\right)\cong\mathrm{Hom}_{\mathcal{C}}\left(X,-\right)\left(A\right)\times\mathrm{Hom}_{\mathcal{C}}\left(X,-\right)\left(B\right). \tag{3.43}$$

Hence, $\mathrm{Hom}_{\mathcal{C}}\left(X,-\right):\mathcal{C}\to\mathbf{Sets}$ is indeed a binary product preserving functor. ■

4 Duality

§4.1 Duality Principle

The **Elementary Theory of an Abstract Category (ETAC)** consists of certain statements which involves letters $A, B, C, \ldots$ for objects and $f, g, h, \ldots$ for arrows. For example, the statement $f : A \to B$ can be phrased as "A is the domain of f, and B is the codomain of f." A sentence is a statement with all variables quantified (quantifiers are "for all A", "for all f", "there exists an A", "there exists an f" etc.), and none being free. For example,

For all f, there exists A and B such that $f : A \to B$

is a sentence. Axioms of ETAC are examples of sentences that are true in every category.

To each category $\mathcal{C}$, we also associate the **opposite category** $\mathcal{C}^{\text{op}}$. The objects of $\mathcal{C}^{\text{op}}$ are the same as those of $\mathcal{C}$, i.e. given $A \in \text{Ob}(\mathcal{C})$, one also has $A \in \text{Ob}(\mathcal{C}^{\text{op}})$.

$$\text{Ob}(\mathcal{C}) = \text{Ob}(\mathcal{C}^{\text{op}}).$$

The arrows of $\mathcal{C}^{\text{op}}$ are arrows f^{op} in the reverse direction, and hence are in 1-1 correspondence

$$f \mapsto f^{\text{op}}$$

with the arrows f of $\mathcal{C}$. In other words, if $f : A \to B$ is an arrow in $\mathcal{C}$, then the corresponding arrow in $\mathcal{C}^{\text{op}}$ is given by $f^{\text{op}} : B \to A$ so that

$$\text{dom}(f^{\text{op}}) = \text{cod}(f) \text{ and } \text{cod}(f^{\text{op}}) = \text{dom}(f).$$

Now, given arrows

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

in $\mathcal{C}$, one has

$$\begin{tikzcd} A & B \arrow[l, "f^{\text{op}}"'] & C \arrow[l, "g^{\text{op}}"'] \arrow[ll, bend left, "f^{\text{op}}\circ g^{\text{op}}=(g\circ f)^{\text{op}}"] \end{tikzcd}$$

in $\mathcal{C}^{\text{op}}$ so that $f^{\text{op}} \circ g^{\text{op}} = (g \circ f)^{\text{op}}$ is defined in $\mathcal{C}^{\text{op}}$ exactly when the composite $g \circ f$ is defined in $\mathcal{C}$, i.e. (g, f) is a composable pair (g after f) in $\mathcal{C}$ if and only if $(f^{\text{op}}, g^{\text{op}})$ is a composable pair (f^{op} after g^{op}) in $\mathcal{C}^{\text{op}}$, and $f^{\text{op}} \circ g^{\text{op}}$ is precisely the opposite arrow of the composite arrow $g \circ f$ in $\mathcal{C}$.

Now, the dual of any statement Σ of ETAC is formed by making the following replacements throughout in Σ: "domain" by "codomain", "codomain" by "domain", "h is the composite of f with g" by "h is the composite of g with f". As a result, arrows and composites are reversed in the dual statement Σ^*. While forming the dual sentence, logic / quantifiers (and, or, for all, there exists etc.) remain unchanged. Some examples of statements Σ and their dual statements Σ^* are listed below:

Statement Σ	Dual Statement Σ^*
$f : A \to B$	$f : B \to A$
$A = \text{dom}(f)$	$A = \text{cod}(f)$
$i = \mathbb{1}_A$	$i = \mathbb{1}_A$
$h = g \circ f$	$h = f \circ g$
T is a terminal object	T is an initial object

A sentence Σ is true in $\mathcal{C}^{\text{op}}$ if and only if its dual statement Σ^* is true in $\mathcal{C}$ (arrows in $\mathcal{C}^{\text{op}}$ are read without the op prefix).

Remark 4.1. Note that the dual of the dual is the original statement ($\Sigma^{**} = \Sigma$). There could be broadly two types of statements or sentences. A statement can be consequences of the axioms of ETAC, for example, "a terminal object of a category, if it exists, is unique up to isomorphism." The other type of statements do not follow from the axioms of ETAC. For example, "$a = \mathrm{dom}\, f$", or $h = g \circ f$. For the latter type of statements Σ in $\mathcal{C}$, the dual statements Σ^* refer to opposite category $\mathcal{C}^{\mathrm{op}}$. For the former type of statements, i.e. for the statements that are consequences of the axioms of ETAC, Σ and its dual Σ^* refer to the category $\mathcal{C}$. This fact is captured by the **duality principle**.

Duality Principle:

If a statement Σ of ETAC is a consequence of the axioms, so is the dual statement Σ^*.

For example, take Σ to be "a terminal object, if it exists, is unique up to isomorphism." We have the dual Σ^* that reads "an initial object, if it exists, is unique up to isomorphism." This dual statement Σ^* applies to the same category $\mathcal{C}$.

§4.2 Coproducts

Definition 4.1. A diagram $A \xrightarrow{q_1} Q \xleftarrow{q_2} B$ is called a coproduct of A and B if for any object Z and $A \xrightarrow{z_1} Z \xleftarrow{z_2} B$, there is a **unique** arrow $u : Q \to Z$ such that the following diagram commutes:

$$\begin{tikzcd}
 & Z & \\
A \arrow[ur, "z_1"] \arrow[r, "q_1"'] & Q \arrow[u, dashed, "\exists!\, u"] & B \arrow[ul, "z_2"'] \arrow[l, "q_2"]
\end{tikzcd}$$

i.e. $u \circ q_1 = z_1$ and $u \circ q_2 = z_2$.

We usually write $A \xrightarrow{i_1} A + B \xleftarrow{i_2} B$ for the coproduct, and $[f, g]$ for the uniquely determined arrow $[f, g] : A + B \to Z$ as in the following diagram:

$$\begin{tikzcd}
 & Z & \\
A \arrow[ur, "f"] \arrow[r, "i_1"'] & A + B \arrow[u, "{[f,g]}"] & B \arrow[ul, "g"'] \arrow[l, "i_2"]
\end{tikzcd}$$

Remark 4.2. Similar to Remark 3.4, the "co-projections" $i_1 : A \to A + B$ and $i_2 : B \to A + B$ need not be monomorphisms. A coproduct of two objects is a product of the same two objects in the opposite category, as can be seen by reversing the arrows above and using the commutative diagram of a product. Therefore, given a nonempty set A, the product $\varnothing \times A$ in **Sets** is the coproduct $\varnothing + A$ in $\mathbf{Sets}^{\mathrm{op}}$. In other words,

$$\varnothing +_{\mathbf{Sets}^{\mathrm{op}}} A = \varnothing \times_{\mathbf{Sets}} A. \tag{4.1}$$

Then the coprojection

$$i_2 : A \to \varnothing +_{\mathbf{Sets}^{\mathrm{op}}} A$$

is not a monomorphism in $\mathbf{Sets}^{\mathrm{op}}$, since the opposite arrow $p_2 : \varnothing \times_{\mathbf{Sets}} A \to A$ is not an epimorphism.

The 'co-projections" $i_1 : A \to A + B$ and $i_2 : B \to A + B$ are often called coproduct injections.

Example 4.1. In **Sets**, the coproduct $A + B$ of two sets is their disjoint union which can be constreucted as

$$A + B = \{(a, 1) \mid a \in A\} \cup \{(b, 2) \mid b \in B\} = (A \times \{1\}) \cup (B \times \{2\}), \tag{4.2}$$

with evident coproduct injections $i_1(a) = (a, 1)$ and $i_2(b) = (b, 2)$. Given any functions $f : A \to Z$ and $g : B \to Z$,

$$\begin{tikzcd}
& Z & \\
A \arrow[ur, "f"] \arrow[r, "i_1"'] & A + B \arrow[u, dashed, "{[f,g]}"] & B \arrow[ul, "g"'] \arrow[l, "i_2"]
\end{tikzcd}$$

we define $[f, g] : A + B \to Z$ as follows:

$$[f, g](x, \delta) = \begin{cases} f(x) & \text{if } \delta = 1, \\ g(x) & \text{if } \delta = 2. \end{cases} \tag{4.3}$$

Uniqueness of $[f, g]$ defined in (4.3) follows by noting that if for $h : A + B \to Z$ the above diagram commutes, then one must have $h \circ i_1 = f$ and $h \circ i_2 = g$. This leads to

$$f(a) = (h \circ i_1)(a) = h(i_1(a)) = h(a, 1), \text{ and} \tag{4.4}$$
$$g(b) = (h \circ i_2)(b) = h(i_2(b)) = h(b, 2). \tag{4.5}$$

Therefore, $h(x, \delta) = [f, g](x, \delta)$, for any $(x, \delta) \in A + B$. So $[f, g] = h$ proving the uniqueness of $[f, g]$.

In **Sets**, if A is a finite set with cardinality n, then

$$A \cong \underbrace{1 + 1 + \cdots + 1}_{n\text{-times}}, \tag{4.6}$$

where 1 stands for a singleton set. Indeed, if $A = \{a_1, a_2, \ldots, a_n\}$, then

$$\{a_1\} + \{a_2\} + \cdots \{a_n\} = \{(a_1, 1), (a_2, 2), \ldots, (a_n, n)\} \cong \{a_1, a_2, \ldots, a_n\}. \tag{4.7}$$

Proposition 4.1

Coproducts are unique up to isomorphism.

We don't need to prove it again. We can just use the fact that products are unique up to isomorphism (Proposition 3.7), and then invoke duality principle.

Example 4.2. If $M(A)$ and $M(B)$ are free monoids on sets A and B, then in the category **Mon**, we can construct their coproduct as

$$M(A) + M(B) \cong M(A + B).$$

In other words, the coproduct of the free monoids on A and B is the free monoid on the coproduct $A + B$ of A and B in **Sets**.

We shall prove that $M(A + B)$ satisfies the UMP of coproduct of $M(A)$ and $M(B)$. Let $\mathcal{U} : \textbf{Mon} \to \textbf{Sets}$ be the forgetful functor, and let

$$\eta_A : A \to \mathcal{U}(M(A)),$$
$$\eta_B : B \to \mathcal{U}(M(B)),$$
$$\eta_{A+B} : A + B \to \mathcal{U}(M(A + B))$$

be the respective insertions of generators. Let

$$A \xrightarrow{i_1} A+B \xleftarrow{i_2} B$$

be the coproduct diagram for A and B. Consider the following diagram in **Sets**:

$$\begin{tikzcd}
\mathcal{U}(M(A)) \arrow[r, dashed, "\mathcal{U}(j_1)"] & \mathcal{U}(M(A+B)) & \mathcal{U}(M(B)) \arrow[l, dashed, "\mathcal{U}(j_2)"'] \\
A \arrow[u, "\eta_A"] \arrow[r, "i_1"'] & A+B \arrow[u, "\eta_{A+B}"] & B \arrow[u, "\eta_B"'] \arrow[l, "i_2"]
\end{tikzcd}$$

Given the monoid $M(A+B)$ and the function $\eta_{A+B} \circ i_1 : A \to \mathcal{U}\left(M\left(A+B\right)\right)$, by the UMP of free monoid on the set A, there is a **unique** monoid homomorphism $j_1 : M\left(A\right) \to M\left(A+B\right)$ such that

$$\eta_{A+B} \circ i_1 = \mathcal{U}\left(j_1\right) \circ \eta_A. \tag{4.8}$$

Similarly, using the UMP of free monoid on the set B and the function $\eta_{A+B} \circ i_2 : B \to \mathcal{U}\left(M\left(A+B\right)\right)$, there is a **unique** monoid homomorphism $j_2 : M\left(B\right) \to M\left(A+B\right)$ such that

$$\eta_{A+B} \circ i_2 = \mathcal{U}\left(j_2\right) \circ \eta_B. \tag{4.9}$$

In other words, the diagram above commutes in **Sets**. We now claim that

$$M(A) \xrightarrow{j_1} M(A+B) \xleftarrow{j_2} M(B)$$

is a coproduct diagram. Let N be any monoid and let $f : M\left(A\right) \to N$ and $g : M\left(B\right) \to N$ be monoid homomorphisms.

$$\begin{tikzcd}
& \mathcal{U}(N) & \\
\mathcal{U}(M(A)) \arrow[ur, "\mathcal{U}(f)"] & & \mathcal{U}(M(B)) \arrow[ul, "\mathcal{U}(g)"'] \\
A \arrow[u, "\eta_A"] \arrow[r, "i_1"'] & A+B \arrow[uu, dashed, "h"] & B \arrow[u, "\eta_B"'] \arrow[l, "j_1"]
\end{tikzcd}$$

Since $A+B$ is the coproduct of A and B in **Sets**, there exists a **unique** $h : A+B \to \mathcal{U}\left(N\right)$ such that

$$h \circ i_1 = \mathcal{U}(f) \circ \eta_A \text{ and } h \circ i_2 = \mathcal{U}(g) \circ \eta_B. \tag{4.10}$$

$$\begin{tikzcd}
& \mathcal{U}(N) & \\
\mathcal{U}(M(A)) \arrow[ur, "\mathcal{U}(f)"] \arrow[r, "\mathcal{U}(j_1)"] & \mathcal{U}(M(A+B)) \arrow[u, dashed] & \mathcal{U}(M(B)) \arrow[ul, "\mathcal{U}(g)"'] \arrow[l, "\mathcal{U}(j_2)"'] \\
A \arrow[u, "\eta_A"] \arrow[r, "i_1"'] & A+B \arrow[u, "\eta_{A+B}"] & B \arrow[u, "\eta_B"'] \arrow[l, "j_1"]
\end{tikzcd}$$

Now, given the monoid N and the function $h : A+B \to \mathcal{U}\left(N\right)$, by the UMP of free monoid on the set $A+B$, there exists a **unique** monoid homomorphism $\overline{h} : M\left(A+B\right) \to N$ such that

$$h = \mathcal{U}\left(\overline{h}\right) \circ \eta_{A+B}. \tag{4.11}$$

In other words, the following diagram commutes

$$\begin{tikzcd}
\mathcal{U}(M(A+B)) \arrow[r, "\mathcal{U}(\overline{h})"] & \mathcal{U}(N) \\
A+B \arrow[u, "\eta_{A+B}"] \arrow[ur, "h"'] &
\end{tikzcd}$$

So we get the following diagram:

$$\begin{tikzcd}
& \mathcal{U}(N) & \\
\mathcal{U}(M(A)) \arrow[ur, "\mathcal{U}(f)"] \arrow[r, "\mathcal{U}(j_1)"] & \mathcal{U}(M(A+B)) \arrow[u, "\mathcal{U}(\overline{h})"] & \mathcal{U}(M(B)) \arrow[ul, "\mathcal{U}(g)"'] \arrow[l, "\mathcal{U}(j_2)"'] \\
A \arrow[u, "\eta_A"] \arrow[r, "i_1"'] & A+B \arrow[u, "\eta_{A+B}"] & B \arrow[u, "\eta_B"'] \arrow[l, "j_1"]
\end{tikzcd}$$

In this diagram, using (4.11) and (4.10), we have

$$\mathcal{U}\left(\overline{h}\right) \circ \eta_{A+B} \circ i_1 = h \circ i_1 = \mathcal{U}\left(f\right) \circ \eta_A. \tag{4.12}$$

Again, using (4.8),

$$\mathcal{U}\left(\overline{h}\right) \circ \eta_{A+B} \circ i_1 = \mathcal{U}\left(\overline{h}\right) \circ \mathcal{U}\left(j_1\right) \circ \eta_A = \mathcal{U}\left(\overline{h} \circ j_1\right) \circ \eta_A. \tag{4.13}$$

However, for the function $\mathcal{U}\left(\overline{h}\right) \circ \eta_{A+B} \circ i_1 : A \to \mathcal{U}\left(N\right)$, using the UMP of free monoid on the set A, there exists a **unique** monoid homomorphism $u : M\left(A\right) \to N$ such that

$$\mathcal{U}\left(\overline{h}\right) \circ \eta_{A+B} \circ i_1 = \mathcal{U}\left(u\right) \circ \eta_A. \tag{4.14}$$

In other words, there is a **unique** monoid homomorphism $u : M(A) \to N$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{U}(M(A)) \arrow[r, "\mathcal{U}(u)"] & \mathcal{U}(N) \\
A \arrow[u, "\eta_A"] \arrow[ur, "\mathcal{U}(\overline{h})\circ\eta_{A+B}\circ i_1"'] &
\end{tikzcd}$$

But from (4.12) and (4.13), we get that taking $u = f$ and $u = \overline{h} \circ j_1$ makes the above diagram commute. Therefore, we must have

$$f = \overline{h} \circ j_1. \tag{4.15}$$

Similarly,

$$g = \overline{h} \circ j_2. \tag{4.16}$$

Therefore, given the following diagram commutes:

$$\begin{tikzcd}
& N & \\
M(A) \arrow[ur, "f"] \arrow[r, "j_1"'] & M(A+B) \arrow[u, "\overline{h}"] & M(B) \arrow[ul, "g"'] \arrow[l, "j_2"]
\end{tikzcd}$$

Furthermore, we need to show that $\overline{h}$ is unique. Suppose there exists another monoid homomorphism $v : M(A+B) \to N$ such that $f = v \circ j_1$ and $g = v \circ j_2$. Then using (4.8), we get

$$\mathcal{U}(f) \circ \eta_A = \mathcal{U}(v) \circ \mathcal{U}(j_1) \circ \eta_A = \mathcal{U}(v) \circ \eta_{A+B} \circ i_1. \tag{4.17}$$

Similarly, using (4.9),

$$\mathcal{U}(g) \circ \eta_B = \mathcal{U}(v) \circ \mathcal{U}(j_2) \circ \eta_B = \mathcal{U}(v) \circ \eta_{A+B} \circ i_2. \tag{4.18}$$

However, $h : A+B \to \mathcal{U}(N)$ is the **unique** map such that

$$h \circ i_1 = \mathcal{U}(f) \circ \eta_A \text{ and } h \circ i_2 = \mathcal{U}(g) \circ \eta_B.$$

The uniqueness of h along with (4.17) and (4.18) implies that

$$h = \mathcal{U}(v) \circ \eta_{A+B}. \tag{4.19}$$

On the other hand, $\overline{h}$ is the **unique** monoid homomorphism $M(A+B) \to N$ such that $h = \mathcal{U}\left(\overline{h}\right) \circ \eta_{A+B}$. Therefore, from (4.19), we must have $\overline{h} = v$. Hence, $\overline{h}$ is the **unique** monoid homomorphism $M(A+B) \to N$ such that $f = \overline{h} \circ j_1$ and $g = \overline{h} \circ j_2$. So $M(A+B)$ satisfies the UMP of coproduct of $M(A)$ and $M(B)$. Therefore,

$$M(A) + M(B) \cong M(A+B). \tag{4.20}$$

Example 4.3. Two monoids A, B have a coproduct of the form

$$A + B = M(|A| + |B|) / \sim.$$

Here, $|A| + |B|$ is the disjoint union of the underlying sets $|A|$ and $|B|$ of the monoids A and B, respectively. One then forms the free monoid $M(|A| + |B|)$ on the set $|A|+|B|$, which is the collection of words over $|A|+|B|$. Now given two words $v, w \in M(|A| + |B|)$, one declares them to be equivalent as follows:

$$\begin{aligned} \cdots x u_A y \cdots &\sim \cdots x y \cdots, \\ \cdots x u_B y \cdots &\sim \cdots x y \cdots, \\ \cdots a \cdot_A a' \cdots &\sim \cdots a a' \cdots, \\ \cdots b \cdot_B b' \cdots &\sim \cdots b b' \cdots. \end{aligned} \tag{4.21}$$

One, thus, forms quotient of the free monoid $M(|A| + |B|)$ subjected to the equivalence relation $\sim$ provided in (4.21). Two equivalence classes $[x \ldots y], [x' \ldots y'] \in M(|A| + |B|) / \sim$ have the straightforward multiplication:

$$[x \ldots y] \cdot \left[x' \ldots y'\right] = \left[x \ldots y x' \ldots y'\right]. \tag{4.22}$$

The unit in $M(|A| + |B|) / \sim$ is the equivalence class [-] consisting of the empty word. Clearly,

$$[\text{-}] = [u_A] = [u_B].$$

The collection of all the equivalence classes thus formed is defined as the coproduct of A and B:

$$A + B = M(|A| + |B|) / \sim. \tag{4.23}$$

The coproduct injections $i_A : A \to A+B$ and $i_B : B \to A+B$ are simply

$$i_A(a) = [a], \; i_B(b) = [b], \tag{4.24}$$

which can easily be verified to be monoid homomorphism. The UMP of $A+B$ should ensure the existence of a **unique** monoid homomorphism $[f, g] : A+B \to N$ for a given monoid N and monoid homomorphisms $f : A \to N$ and $g : B \to N$ such that the following diagram commutes:

$$\begin{tikzcd} & N & \\ A \arrow[ur, "f"] \arrow[r, "i_A"'] & A+B \arrow[u, dashed, "{[f,g]}"] & B \arrow[ul, "g"'] \arrow[l, "i_B"] \end{tikzcd}$$

Let us now construct $[f, g]$. From the given monoid homomorphisms $f : A \to N$ and $g : B \to N$, one obtains the functions $|f| : |A| \to |N|$ and $|g| : |B| \to |N|$. Now, using the UMP of the coproduct $|A| + |B|$, one knows that there exists a **unique** function $h = [|f|, |g|] : |A| + |B| \to |N|$ such that the following diagram commutes:

$$\begin{tikzcd} & |N| & \\ |A| \arrow[ur, "|f|"] \arrow[r, "i_1"'] & |A|+|B| \arrow[u, dashed, "h"] & |B| \arrow[ul, "|g|"'] \arrow[l, "i_2"] \end{tikzcd}$$

where $i_1 : |A| \to |A| + |B|$ and $i_2 : |B| \to |A| + |B|$ are coproduct injections of the coproduct $|A| + |B|$ in **Sets**. Now, consider the free monoid $M(|A| + |B|)$ on the set $|A| + |B|$. Invoking the UMP of free monoid, given the monoid N and the function $h : |A| + |B| \to |N|$, there exists a **unique** monoid homomorphism $\overline{h} : M(|A| + |B|) \to N$ such that the following diagram commutes in **Sets**:

$$\begin{tikzcd} {|M(|A|+|B|)|} \arrow[r, "\overline{h}"] & {|N|} \\ {|A|+|B|} \arrow[u, "i"] \arrow[ur, "h"'] & \end{tikzcd}$$

where $i : |A| + |B| \to |M(|A| + |B|)|$ is the insertion of generators. We now need to verify that $\overline{h}$ "respects the equivalence relation $\sim$", i.e. $v \sim w$ implies $\overline{h}(v) = \overline{h}(w)$.

Given $a \in A$,

$$h(a) = h(i_1(a)) = f(a). \tag{4.25}$$

Since $i(a) = a$, so

$$\overline{h}(a) = \overline{h}(i(a)) = h(a) = f(a). \tag{4.26}$$

Similarly, for $b \in B$,

$$\overline{h}(b) = g(b). \tag{4.27}$$

So $\overline{h}(u_A) = \overline{h}(u_B) = u_N$. As a result,

$$\overline{h}(\cdots x u_A y \cdots) = \cdots \overline{h}(x)\,\overline{h}(u_A)\,\overline{h}(y) \cdots = \cdots \overline{h}(x)\,\overline{h}(y) \cdots = \overline{h}(\cdots xy \cdots).$$

Similarly,

$$\overline{h}(\cdots x u_B y \cdots) = \cdots \overline{h}(x)\,\overline{h}(u_B)\,\overline{h}(y) \cdots = \cdots \overline{h}(x)\,\overline{h}(y) \cdots = \overline{h}(\cdots xy \cdots).$$

Since f is a homomorphism,

$$\begin{aligned} \overline{h}\left(\cdots a \cdot_A a' \cdots\right) &= \cdots \overline{h}\left(a \cdot_A a'\right) \cdots = \cdots f\left(a \cdot_A a'\right) \cdots \\ &= \cdots f(a)\, f\left(a'\right) \cdots = \cdots \overline{h}(a)\,\overline{h}\left(a'\right) \cdots \\ = \overline{h}\left(\cdots aa' \cdots\right). \end{aligned}$$

Similarly, using the fact that g is a homomorphism,

$$\begin{aligned} \overline{h}\left(\cdots b \cdot_B b' \cdots\right) &= \cdots \overline{h}\left(b \cdot_B b'\right) \cdots = \cdots g\left(b \cdot_B b'\right) \cdots \\ &= \cdots g(b)\, g\left(b'\right) \cdots = \cdots \overline{h}(b)\,\overline{h}\left(b'\right) \cdots \\ = \overline{h}\left(\cdots bb' \cdots\right). \end{aligned}$$

Therefore, $v \sim w$ implies $\overline{h}(v) = \overline{h}(w)$. So we have a monoid homomorphism $\widetilde{h} : M(|A| + |B|) / \sim \to N$ defined by

$$\widetilde{h}[w] = \overline{h}(w). \tag{4.28}$$

This is well-defined since $\overline{h}$ is constant on the equivalence classes. Therefore, the following diagram commutes:

$$\begin{tikzcd} M(|A|+|B|) \arrow[r, two heads, "\pi"] \arrow[dr, "\overline{h}"'] & M(|A|+|B|)/sim \arrow[d, "\widetilde{h}"] \\ & N \end{tikzcd}$$

Now, this $\widetilde{h} : M(|A| + |B|) / \sim \to N$ is our desired $[f, g]$. Indeed,

$$\left(\widetilde{h} \circ i_A\right)(a) = \widetilde{h}([a]) = \overline{h}(a) = f(a). \tag{4.29}$$

$$\left(\widetilde{h} \circ i_B\right)(b) = \widetilde{h}([b]) = \overline{h}(b) = g(b). \tag{4.30}$$

Therefore, $\widetilde{h} \circ i_A = f$ and $\widetilde{h} \circ i_B = g$. In other words, the following diagram commutes:

$$\begin{tikzcd} & N & \\ A \arrow[ur, "f"] \arrow[r, "i_A"'] & M(|A|+|B|)/\sim \arrow[u, "\widetilde{h}"] & B \arrow[ul, "g"'] \arrow[l, "i_B"] \end{tikzcd}$$

Furthermore, $\widetilde{h}$ is unique. If there exists another monoid homomorphism $u : M(|A| + |B|) / \sim \to N$ such that $u \circ i_A = f$ and $u \circ i_B = g$, then we must have

$$\widetilde{h}([a]) = f(a) = (u \circ i_A)(a) = u([a]), \tag{4.31}$$

and similarly,

$$\widetilde{h}([b]) = g(b) = (u \circ i_B)(b) = u([b]), \tag{4.32}$$

for any $a \in A$ and $b \in B$. So u agrees with $\widetilde{h}$ on the equivalence classes of elements of A and B. Since any element of $M(|A| + |B|) / \sim$ can be written as a finite product of equivalence classes of elements of A and B, we can conclude that u and $\widetilde{h}$ agrees everywhere in $M(|A| + |B|) / \sim$. Therefore, $\widetilde{h}$ is indeed the **unique** arrow in **Mon** such that $\widetilde{h} \circ i_A = f$ and $\widetilde{h} \circ i_B = g$. Therefore, $M(|A| + |B|) / \sim = A + B$, and $\widetilde{h} = [f, g]$.

Abuse of Notation. In Example 4.3, we did not distinguish between $a \in |A|$ and $(a, 1) \in |A| + |B|$; and between $b \in B$ and $(b, 2) \in |A| + |B|$. This can be justified by assuming without loss of generality that $|A|$ and $|B|$ are disjoint. Indeed, if they are not disjoint, we can just replace them by $|A| \times \{1\}$ and $|B| \times \{2\}$.

Example 4.4. Coproduct in groups is called the free product. Suppose A and B are groups. The free product $A * B$ consists of words of the form

$$a_1 b_1 a_2 b_2 \cdots,$$

where $a_i \in A$ and $b_j \in B$. For example,

$$(a_1 b_1 a_2 b_2) = (a_1 b_1) * (a_2 b_2) \text{ and } (a_2 b_2 a_1 b_1) = (a_2 b_2) * (a_1 b_1)$$

are both words in $A * B$, but they are unequal highlighting the non-abelian nature of the free product of $A * B$, even when A and B are both abelian groups. When A and B are both abelian groups, one defines their direct sum as the quotient

$$A \oplus B = A * B / \sim,$$

where the equivalence relation $\sim$ is defined as

$$(a_1 b_1 a_2 b_2 \cdots a_n b_n \cdots) \sim (a_1 a_2 \cdots a_n \cdots b_1 b_2 \cdots b_n \cdots) . \tag{4.33}$$

On the RHS, all the a_i's are flushed to the left. Now, since A and B are both abelian, so is $A \oplus B$:

$$(a_1 b_1) * (a_2 b_2) = a_1 b_1 a_2 b_2 = a_1 a_2 b_1 b_2 = a_2 a_1 b_2 b_1 = a_2 b_2 a_1 b_1 = (a_2 b_2) * (a_1 b_1) .$$

Thus, using (4.33), one can identify a word in $A \oplus B$ by a pair

$$(a_1 + a_2 + \cdots + a_n + \cdots , b_1 + b_2 + \cdots + b_n + \cdots) =\equiv (a, b)$$

with $a \in A$ and $b \in B$. But

$$|A \times B| = \{(a, b) \mid a \in A, b \in B\} .$$

Given two abelian groups A and B, their coproduct is defined as their direct sum, which has their product set as the underlying set as we have just seen. Hence, $|A + B| = |A \times B|$, for A, B abelian groups. Then the coproduct injections $i_A : A \to A + B$ and $i_B : B \to A + B$ are defined as

$$i_A(a) = (a, 0_B) \text{ and } i_B(b) = (0_A, b) . \tag{4.34}$$

Given any group homomorphisms $A \xrightarrow{f} X \xleftarrow{g} B$ for any abelian group X, we construct $[f, g] : A + B \to X$ as follows:

$$[f, g](a, b) = f(a) +_X g(b) . \tag{4.35}$$

Then,

$$\begin{aligned} ([f, g] \circ i_A)(a) &= [f, g](a, 0_B) = f(a) + g(0_B) = f(a) , \\ ([f, g] \circ i_B)(b) &= [f, g](0_A, b) = f(0_A) + g(b) = g(b) . \end{aligned}$$

Therefore, $[f, g] \circ i_A = f$ and $[f, g] \circ i_B = g$. In other words, the following diagram commutes:

$$\begin{tikzcd} & X & \\ A \arrow[ur, "f"] \arrow[r, "i_A"'] & A + B \arrow[u, "{[f,g]}"] & B \arrow[ul, "g"'] \arrow[l, "i_B"] \end{tikzcd}$$

Furthermore, $[f, g]$ is the **unique** homomorphism $A + B \to X$ such that the diagram above commutes. If there exists another homomorphism $h : A + B \to X$ such that $h \circ i_A = f$ and $h \circ i_B = g$, then for any $a \in A$ and $b \in B$,

$$h(a, 0_B) = h(i_A(a)) = f(a) \text{ and } h(0_A, b) = h(i_B(b)) = g(b) .$$

Therefore,

$$h(a, b) = h(a, 0_B) + h(0_A, b) = f(a) + g(b) = [f, g](a, b) . \tag{4.36}$$

So $[f, g] = h$, and hence $[f, g]$ is unique. Therefore, $A + B$ is indeed the coproduct of A and B.

We have just seen that $|A + B| = |A \times B|$ for A, B abelian groups. In fact, a stronger result holds in the category **Ab** of abelian groups.

Proposition 4.2

In the category **Ab** of abelian groups, there is a canonical isomorphism between the binary product and coproduct, i.e. $A + B \cong A \times B$.

Proof. The goal is to define an arrow $\vartheta : A+B \to A\times B$. In order to do so, one needs first an arrow $A \to A\times B$ (which is determined by an arrow $A\to A$ and an arrow $A\to B$) and another arrow $B\to A\times B$ (which is determined by an arrow $B\to A$ and an arrow $B \to B$). Therefore, we need 4 arrows altogether, which we choose as follows:

$$\mathbb{1}_A : A\to A\,,\ \mathbf{0}_B : A\to B\,,\ \mathbf{0}_A : B\to A\,,\ \mathbb{1}_B : B\to B,$$

where $\mathbf{0}_B : A\to B$ maps all of A to the identity $0_B\in B$, and $\mathbf{0}_A : B\to A$ maps all of B to the identity $0_A\in A$.

$$\begin{tikzcd}
& A \arrow[dl, "\mathbb{1}_A"'] \arrow[d, dashed, "\langle \mathbb{1}_A{,}\mathbf{0}_B\rangle"] \arrow[dr, "\mathbf{0}_B"] & \\
A & A\times B \arrow[l, "p_1"] \arrow[r, "p_2"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
& B \arrow[dl, "\mathbf{0}_A"'] \arrow[d, dashed, "\langle \mathbf{0}_A{,}\mathbb{1}_B\rangle"] \arrow[dr, "\mathbb{1}_B"] & \\
A & A\times B \arrow[l, "p_1"] \arrow[r, "p_2"'] & B
\end{tikzcd}$$

$$\begin{tikzcd}
& A\times B & \\
A \arrow[ur, "\langle \mathbf{0}_A{,}\mathbb{1}_B\rangle"] \arrow[r, "i_A"'] & A+B \arrow[u, dashed, "{[\langle \mathbb{1}_A{,}\mathbf{0}_B\rangle{,}\langle \mathbf{0}_A{,}\mathbb{1}_B\rangle]}"] & B \arrow[l, "i_B"] \arrow[ul, "\langle \mathbb{1}_A{,}\mathbf{0}_B\rangle"']
\end{tikzcd}$$

Using the commutative diagrams above, we define

$$\vartheta = \left[\langle \mathbb{1}_A, \mathbf{0}_B\rangle\,, \langle \mathbf{0}_A, \mathbb{1}_B\rangle\right] : A+B\to A\times B. \tag{4.37}$$

Given $(a,b)\in A+B$, one obtains

$$\begin{aligned}
\vartheta\,(a,b) &= \left[\langle \mathbb{1}_A, \mathbf{0}_B\rangle\,, \langle \mathbf{0}_A, \mathbb{1}_B\rangle\right](a,b)\\
&= \langle \mathbb{1}_A, \mathbf{0}_B\rangle\,(a) + \langle \mathbf{0}_A, \mathbb{1}_B\rangle\,(b)\\
&= \left(\mathbb{1}_A\,(a)\,, \mathbf{0}_B\,(b)\right) + \left(\mathbf{0}_A\,(b)\,, \mathbb{1}_B\,(b)\right)\\
&= (a, 0_B) + (0_A, b)\\
&= (a,b)\,.
\end{aligned} \tag{4.38}$$

Therefore, ϑ is a bijective group homomorphism, and hence an isomorphism. ■