



Inspiring Excellence

Differential Geometry II (MAT401)

Lecture Notes

Preface

This series of lecture notes has been prepared for aiding students who took the BRAC University course **Differential Geometry II (MAT401)** in Fall 2023 semester. These notes were typeset under the supervision of mathematician **Dr. Syed Hasibul Hassan Chowdhury**. The main goal of this typeset is to have an organized digital version of the notes, which is easier to share and handle. If you see any mistakes or typos, please send me an email at atonuroychowdhury@gmail.com

Atonu Roy Chowdhury

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- *An Introduction to Differentiable Manifolds and Riemannian Geometry*, by **William Boothby**
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- *Lectures on Differential geometry*, by **S.S Chern, W.H. Chen and K.S. Lam**
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1 Review of Multilinear Algebra

§1.1 Dual Space

Let V and W be real vector spaces. We denote by $\text{Hom}(V, W)$ the vector space of all linear maps $f : V \rightarrow W$. In particular, if we choose $W = \mathbb{R}$, we get the **dual space** V^* .

$$V^* = \text{Hom}(V, \mathbb{R}).$$

The elements of V^* are called covectors on V . In the rest of the lecture, we will assume V to be a finite dimensional vector space. Let $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ be a basis for V . Then every $\mathbf{v} \in V$ is a unique linear combination

$$\mathbf{v} = \sum_{i=1}^n v^i \mathbf{e}_i, \quad (1.1)$$

with $v^i \in \mathbb{R}$. v^i 's are called the coordinates of \mathbf{v} relative to the basis $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$. Let $\hat{\alpha}^i$ be the linear function on V that picks up the i -th coordinate of the vector, i.e.

$$\hat{\alpha}^i(\mathbf{v}) = \hat{\alpha}^i\left(\sum_{i=1}^n v^i \mathbf{e}_i\right) = v^i. \quad (1.2)$$

When \mathbf{v} is one of the basis vectors,

$$\hat{\alpha}^i(\mathbf{e}_j) = \delta^i_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \quad (1.3)$$

Proposition 1.1

The functions $\hat{\alpha}^1, \dots, \hat{\alpha}^n$ form a basis for V^* .

Proof. Suppose $f \in V^*$. Then for any $\mathbf{v} = \sum_{i=1}^n v^i \mathbf{e}_i \in V$,

$$f(\mathbf{v}) = f\left(\sum_{i=1}^n v^i \mathbf{e}_i\right) = \sum_{i=1}^n v^i f(\mathbf{e}_i) = \sum_{i=1}^n f(\mathbf{e}_i) \hat{\alpha}^i(\mathbf{v}).$$

Since this holds for any $\mathbf{v} \in V$,

$$f = \sum_{i=1}^n f(\mathbf{e}_i) \hat{\alpha}^i. \quad (1.4)$$

Therefore, $\hat{\alpha}^1, \dots, \hat{\alpha}^n$ span V^* . As for linear independence, suppose

$$\sum_{i=1}^n c_i \hat{\alpha}^i = \mathbf{0}, \quad (1.5)$$

where $\mathbf{0}$ is the function that takes all of V to 0 in \mathbb{R} . If we evaluate (1.5) at \mathbf{e}_j , we get

$$0 = \sum_{i=1}^n c_i \hat{\alpha}^i(\mathbf{e}_j) = \sum_{i=1}^n c_i \delta^i_j = c_j. \quad (1.6)$$

So $c_j = 0$, and this holds for each $j = 1, 2, \dots, n$. Therefore, $\{\hat{\alpha}^1, \dots, \hat{\alpha}^n\}$ is a linearly independent set that spans V^* , i.e. a basis. ■

Corollary 1.2

The dual space V^* of a finite dimensional vector space has the same dimension as V .

The basis $\{\hat{\alpha}^1, \dots, \hat{\alpha}^n\}$ for V^* is said to be dual to the basis $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ for V .

§1.2 Permutations

Fix a positive integer k . A permutation of the set $A = \{1, 2, \dots, k\}$ is a bijection $\sigma : A \rightarrow A$. The product of two permutations τ and σ is the composition $\tau \circ \sigma : A \rightarrow A$. The **cyclic permutation** $(a_1 a_2 \cdots a_r)$ is the permutation σ such that

$$\sigma(a_1) = a_2, \sigma(a_2) = a_3, \dots, \sigma(a_{r-1}) = a_r, \text{ and } \sigma(a_r) = 1,$$

leaving all other elements unchanged, i.e. $\sigma(j) = j$ if j is not one of the a_i 's. A cyclic permutation $(a_1 a_2 \cdots a_r)$ is also called a **cycle** of length r or an r -cycle. A **transposition** is a permutation of the form $(a b)$ that interchanges a and b , leaving all other elements of A fixed.

A permutation $\sigma : A \rightarrow A$ can be described by

$$\begin{bmatrix} 1 & 2 & \cdots & k \\ \sigma(1) & \sigma(2) & \cdots & \sigma(k) \end{bmatrix}.$$

We also write it as

$$[1 \ 2 \ \cdots \ k] \xrightarrow{\sigma} [\sigma(1) \ \sigma(2) \ \cdots \ \sigma(k)].$$

Example 1.1. Suppose $\sigma : \{1, 2, 3, 4, 5\} \rightarrow \{1, 2, 3, 4, 5\}$ is the permutation given by

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 5 & 1 & 3 \end{bmatrix}.$$

In other words, $\sigma(1) = 2$, $\sigma(2) = 4$, $\sigma(3) = 5$, $\sigma(4) = 1$, and $\sigma(5) = 3$.

$$[1 \ 2 \ 3 \ 4 \ 5] \xrightarrow{\sigma} [2 \ 4 \ 5 \ 1 \ 3].$$

Observe that the cyclic permutation $\sigma' = (1 \ 2 \ 4)$ acts as $\sigma'(1) = 2$, $\sigma'(2) = 4$ and $\sigma'(4) = 1$, keeping 3 and 5 unchanged, i.e. $\sigma'(3) = 3$ and $\sigma'(5) = 5$.

$$[1 \ 2 \ 3 \ 4 \ 5] \xrightarrow{(1 \ 2 \ 4)} [2 \ 4 \ 3 \ 1 \ 5].$$

Now the transposition $\sigma'' = (3 \ 5)$ acts as $\sigma''(3) = 5$ and $\sigma''(5) = 3$, keeping 1, 2, 4 unchanged. Therefore,

$$\begin{array}{ccccc} [1 \ 2 \ 3 \ 4 \ 5] & \xrightarrow{(1 \ 2 \ 4)} & [2 \ 4 \ 3 \ 1 \ 5] & \xrightarrow{(3 \ 5)} & [2 \ 4 \ 5 \ 1 \ 3] \\ & \searrow & & \nearrow & \\ & & (3 \ 5)(1 \ 2 \ 4) & & \end{array}$$

so that $\sigma = (3 \ 5)(1 \ 2 \ 4)$.

Let S_k be the group of permutations of the set $\{1, 2, \dots, k\}$. The order of this group is $k!$. A permutation is *even* or *odd* depending on whether it is the product of an even or an odd number of transpositions. The sign of a permutation σ is 1 if the permutation is even, and -1 otherwise. It is denoted by $\text{sgn } \sigma$. For example, in [Example 1.1](#), $\sigma = (3 \ 5)(1 \ 2 \ 4)$. Note that we can write $(1 \ 2 \ 4)$ as a product of two transpositions:

$$\begin{array}{ccccc} [1 \ 2 \ 3 \ 4 \ 5] & \xrightarrow{(1 \ 2)} & [2 \ 1 \ 3 \ 4 \ 5] & \xrightarrow{(1 \ 4)} & [2 \ 4 \ 3 \ 1 \ 5] \\ & \searrow & & \nearrow & \\ & & (1 \ 4)(1 \ 2) = (1 \ 2 \ 4) & & \end{array}$$

In other words, $\sigma = (3 \ 5)(1 \ 4)(1 \ 2)$. Hence, $\text{sgn } \sigma = -1$. One can easily check that

$$\text{sgn}(\sigma\tau) = (\text{sgn } \sigma)(\text{sgn } \tau). \quad (1.7)$$

So $\text{sgn} : S_k \rightarrow \{1, -1\}$ is a group homomorphism.

Example 1.2. Observe that the 5-cycle $(1\ 2\ 3\ 4\ 5)$ can be written as

$$(1\ 2\ 3\ 4\ 5) = (1\ 5)(1\ 4)(1\ 3)(1\ 2).$$

Indeed,

$$\begin{array}{ccccccc} [1\ 2\ 3\ 4\ 5] & \xrightarrow{(1\ 2)} & [2\ 1\ 3\ 4\ 5] & \xrightarrow{(1\ 3)} & [2\ 3\ 1\ 4\ 5] & \xrightarrow{(1\ 4)} & [2\ 3\ 4\ 1\ 5] & \xrightarrow{(1\ 5)} & [2\ 3\ 4\ 5\ 1] \\ & & & & & & \nearrow & & \\ & & & & & & (1\ 2\ 3\ 4\ 5) = (1\ 5)(1\ 4)(1\ 3)(1\ 2) & & \end{array}$$

Therefore, $\text{sgn}(1\ 2\ 3\ 4\ 5) = 1$.

An **inversion** in a permutation σ is an ordered pair $(\sigma(i), \sigma(j))$ such that $i < j$ but $\sigma(i) > \sigma(j)$. In [Example 1.1](#), $\sigma(1) = 2$, $\sigma(2) = 4$, $\sigma(3) = 5$, $\sigma(4) = 1$, and $\sigma(5) = 3$. So, the inversions in this permutation are

$$(\sigma(1), \sigma(4)), (\sigma(2), \sigma(4)), (\sigma(2), \sigma(5)), (\sigma(3), \sigma(4)), (\sigma(3), \sigma(5)).$$

Hence, there are 5 inversions associated with the permutation σ . There is an efficient way of determining the sign of a permutation.

Proposition 1.3

A permutation is even if and only if it has an even number of inversions.

Proof. Let $\sigma \in S_k$ with n inversions. We shall prove that we can multiply σ by n transpositions and get the identity permutation. This will prove that $\text{sgn } \sigma = (-1)^n$.

Suppose $\sigma(j_1) = 1$. Then for each $i < j_1$, $(\sigma(i), \sigma(j_1))$ is an inversion, and there are $j_1 - 1$ many of them. These are all the inversions with 1 in the second slot of the ordered pair of inversion. If we now multiply σ by the j_1 -cycle

$$(\sigma(1)\ 1)(\sigma(2)\ 1) \cdots (\sigma(j_1 - 1)\ 1)$$

to the left of σ , the resulting permutation σ_1 would be

$$\begin{bmatrix} 1 & 2 & 3 & \cdots & j_1 & j_1 + 1 & \cdots & k \\ 1 & \sigma(1) & \sigma(2) & \cdots & \sigma(j_1 - 1) & \sigma(j_1 + 1) & \cdots & \sigma(k) \end{bmatrix}$$

This permutation has no inversion with 1 in the second slot of the ordered pair of inversion. Suppose now that $\sigma(j_2) = 2$. Now observe that if $(\sigma_1(i), 2)$ is an inversion in σ_1 , then either $(\sigma(i), 2)$ (if $i \geq j_1 + 1$) or $\sigma(i - 1), 2$ (if $i \leq j_1 - 1$) is an inversion in σ . Therefore, the number of inversions in σ_1 ending in 2 is precisely the same as the number of inversions in σ ending in 2. So following a similar procedure as above, we can multiply σ_1 by i_2 -many transpositions to the left (i_2 is the number of transpositions ending in 2) and get

$$\begin{bmatrix} 1 & 2 & 3 & \cdots & k \\ 1 & 2 & \sigma(1) & \cdots & \sigma(k) \end{bmatrix}$$

We can continue these steps for each $j = 1, 2, \dots, k$, and the number of transpositions required to move j to its natural position is the same as the number of inversions ending in j . In the end we achieve the identity permutation. Therefore, $\text{sgn } \sigma = (-1)^n$, where n is the number of inversions. ■

§1.3 Multilinear Functions

Definition 1.1. Let V^k be the cartesian product of k -copies of a real vector space V .

$$V^k = \underbrace{V \times V \times \cdots \times V}_{k\text{-copies}}$$

A function $f : V^k \rightarrow \mathbb{R}$ is called k -linear if it is linear in each of its k arguments:

$$f(\dots, a\mathbf{v} + b\mathbf{w}, \dots) = a f(\dots, \mathbf{v}, \dots) + b f(\dots, \mathbf{w}, \dots), \quad (1.8)$$

for $a, b \in \mathbb{R}$ and $\mathbf{v}, \mathbf{w} \in V$.

Instead of 2-linear and 3-linear, it's customary to call “bilinear” and “trilinear”, respectively. A k -linear function on V is called a **k -tensor** on V . We will denote the vector space of all k -tensors on V by $L_k(V)$. The vector addition and scalar multiplication of the real vector space $L_k(V)$ is the straightforward pointwise operation.

Example 1.3. The dot product $f(\mathbf{v}, \mathbf{w}) = \mathbf{v} \cdot \mathbf{w}$ on \mathbb{R}^n is bilinear: if $\mathbf{v} = \sum_{i=1}^n v^i \mathbf{e}_i$ and $\mathbf{w} = \sum_{i=1}^n w^i \mathbf{e}_i$, then

$$f(\mathbf{v}, \mathbf{w}) = \mathbf{v} \cdot \mathbf{w} = \sum_{i=1}^n v^i w^i.$$

Example 1.4. The determinant

$$f(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n) = \det \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}$$

viewed as a function of the n column vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ is n -linear.

Definition 1.2 (Symmetric and alternating function). A k -linear function $f : V^k \rightarrow \mathbb{R}$ is **symmetric** if

$$f(\mathbf{v}_{\sigma(1)}, \mathbf{v}_{\sigma(2)}, \dots, \mathbf{v}_{\sigma(k)}) = f(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k), \quad (1.9)$$

for all permutations $\sigma \in S_k$. It is **alternating** if

$$f(\mathbf{v}_{\sigma(1)}, \mathbf{v}_{\sigma(2)}, \dots, \mathbf{v}_{\sigma(k)}) = (\text{sgn } \sigma) f(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k), \quad (1.10)$$

for all permutations $\sigma \in S_k$.

The dot product function on \mathbb{R}^n in [Theorem 1.3](#) is symmetric, and the determinant function on \mathbb{R}^n in [Theorem 1.4](#) is alternating.

We are especially interested in the vector space $A_k(V)$ of all alternating k -linear functions on a vector space V , for $k > 0$. The elements of $A_k(V)$ are called alternating k -tensors (also known as k -covectors). We define $A_0(V)$ to be \mathbb{R} . The elements of $A_0(V)$ are simply constants, which we call 0-covectors. The elements of $A_1(V)$ are simply covectors, i.e. the elements of V^* .

Permutation action on k -linear functions

If $f \in L_k(V)$ and $\sigma \in S_k$, define $\sigma f \in L_k(V)$ as follows:

$$(\sigma f)(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k) = f(\mathbf{v}_{\sigma(1)}, \mathbf{v}_{\sigma(2)}, \dots, \mathbf{v}_{\sigma(k)}). \quad (1.11)$$

Thus, f is symmetric if and only if $f = \sigma f$ for all $\sigma \in S_k$; and f is alternating if and only if $\sigma f = (\text{sgn } \sigma) f$ for all $\sigma \in S_k$. When $k = 1$, S_k only has the identity permutation. In that case, a 1-linear function or simply linear function on V is both symmetric and alternating. In particular,

$$A_1(V) = L_1(V) = V^*.$$

Lemma 1.4

If $\sigma, \tau \in S_k$ and $f \in L_k(V)$, then $\tau(\sigma f) = (\tau\sigma)f$.

Proof. For any $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in V$,

$$\begin{aligned}
 (\tau(\sigma f))(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k) &= (\sigma f)(\mathbf{v}_{\tau(1)}, \mathbf{v}_{\tau(2)}, \dots, \mathbf{v}_{\tau(k)}) \\
 &= (\sigma f)(\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_k) && [\mathbf{w}_i = \mathbf{v}_{\tau(i)}] \\
 &= f(\mathbf{w}_{\sigma(1)}, \mathbf{w}_{\sigma(2)}, \dots, \mathbf{w}_{\sigma(k)}) \\
 &= f(\mathbf{v}_{\tau(\sigma(1))}, \mathbf{v}_{\tau(\sigma(2))}, \dots, \mathbf{v}_{\tau(\sigma(k))}) \\
 &= ((\tau\sigma)f)(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k).
 \end{aligned}$$

Therefore, $\tau(\sigma f) = (\tau\sigma)f$. ■

Definition 1.3. If G is a group and X is a set, a map

$$\begin{aligned}
 G \times X &\rightarrow X \\
 (g, x) &\mapsto g \cdot x
 \end{aligned}$$

is called a **left action** of G on X if

- (i) $e \cdot x = x$, where e is the identity element in G and x is any element in X ; and
- (ii) $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$, for all $g_1, g_2 \in G$ and $x \in X$.

Similarly, a **right action** of G on X is a map

$$\begin{aligned}
 X \times G &\rightarrow X \\
 (x, g) &\mapsto x \cdot g
 \end{aligned}$$

such that

- (i) $x \cdot e = x$, for all $x \in X$; and
- (ii) $(x \cdot g_1) \cdot g_2 = x \cdot (g_1 g_2)$, for all $g_1, g_2 \in G$ and $x \in X$.

Symmetrizing and alternating operators

Given $f \in L_k(V)$, there is a way to make it a symmetric k -linear function $\mathcal{S}f$ from it:

$$(\mathcal{S}f)(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k) = \sum_{\sigma \in S_k} f(\mathbf{v}_{\sigma(1)}, \mathbf{v}_{\sigma(2)}, \dots, \mathbf{v}_{\sigma(k)}). \quad (1.12)$$

In other words,

$$\mathcal{S}f = \sum_{\sigma \in S_k} \sigma f. \quad (1.13)$$

Similarly, there is a way to make an alternating k -linear function from f :

$$\mathcal{A}f = \sum_{\sigma \in S_k} (\text{sgn } \sigma) \sigma f. \quad (1.14)$$

Proposition 1.5 (i) The k -linear function $\mathcal{S}f$ is symmetric.

(ii) The k -linear function $\mathcal{A}f$ is alternating.

Proof. (i) Let $\tau \in S_k$. Then

$$\tau(\mathcal{S}f) = \tau\left(\sum_{\sigma \in S_k} \sigma f\right). \quad (1.15)$$

The group action of S_k on $L_k(V)$ is distributive over the vector space addition. Therefore,

$$\tau(\mathcal{S}f) = \sum_{\sigma \in S_k} \tau(\sigma f) = \sum_{\sigma \in S_k} (\tau\sigma)f. \quad (1.16)$$

As σ varies over S_k , $\tau\sigma$ also varies over S_k . Therefore, $\sum_{\sigma \in S_k} (\tau\sigma)f = \mathcal{S}f$. In other words,

$$\tau(\mathcal{S}f) = \mathcal{S}f, \quad (1.17)$$

i.e. $\mathcal{S}f$ is symmetric.

(ii) Let $\tau \in S_k$. Then

$$\tau(\mathcal{A}f) = \tau\left(\sum_{\sigma \in S_k} (\text{sgn } \sigma) \sigma f\right) = \sum_{\sigma \in S_k} (\text{sgn } \sigma) \tau(\sigma f) = \sum_{\sigma \in S_k} (\text{sgn } \sigma) (\tau\sigma)f. \quad (1.18)$$

Since $(\text{sgn } \tau)^2 = 1$,

$$\begin{aligned} \tau(\mathcal{A}f) &= \sum_{\sigma \in S_k} (\text{sgn } \tau)^2 (\text{sgn } \sigma) (\tau\sigma)f \\ &= (\text{sgn } \tau) \sum_{\sigma \in S_k} (\text{sgn } \tau) (\text{sgn } \sigma) (\tau\sigma)f \\ &= (\text{sgn } \tau) \sum_{\sigma \in S_k} (\text{sgn } (\tau\sigma)) (\tau\sigma)f. \end{aligned} \quad (1.19)$$

As σ varies over S_k , $\tau\sigma$ also varies over S_k . Therefore, $\sum_{\sigma \in S_k} (\text{sgn } (\tau\sigma)) (\tau\sigma)f = \mathcal{A}f$. In other words,

$$\tau(\mathcal{A}f) = \mathcal{A}f, \quad (1.20)$$

i.e. $\mathcal{A}f$ is alternating. ■

Lemma 1.6

If $f \in A_k(V)$, then $\mathcal{A}f = (k!)f$.

Proof. Since f is alternating,

$$\mathcal{A}f = \sum_{\sigma \in S_k} (\text{sgn } \sigma) \sigma f = \sum_{\sigma \in S_k} (\text{sgn } \sigma)^2 f = \sum_{\sigma \in S_k} f = (k!)f, \quad (1.21)$$

because the order of S_k is $k!$. ■

§1.4 Tensor Product and Wedge Product

Definition 1.4 (Tensor Product). Let f be a k -linear function and g an l -linear function on a vector space V . Their tensor product $f \otimes g$ is the $(k+l)$ -linear function defined by

$$(f \otimes g)(\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{v}_{k+1}, \dots, \mathbf{v}_{k+l}) = f(\mathbf{v}_1, \dots, \mathbf{v}_k) g(\mathbf{v}_{k+1}, \dots, \mathbf{v}_{k+l}). \quad (1.22)$$

$(k+l)$ -linearity of $f \otimes g$ follows from k -linearity of f and l -linearity of g .

Lemma 1.7 (Associativity of Tensor Product)

Let $f \in L_k(V)$, $g \in L_l(V)$ and $h \in L_m(V)$. Then

$$(f \otimes g) \otimes h = f \otimes (g \otimes h).$$

Proof. For $\mathbf{v}_1, \dots, \mathbf{v}_{k+l+m}$,

$$\begin{aligned} [(f \otimes g) \otimes h](\mathbf{v}_1, \dots, \mathbf{v}_{k+l+m}) &= (f \otimes g)(\mathbf{v}_1, \dots, \mathbf{v}_{k+l}) h(\mathbf{v}_{k+l+1}, \dots, \mathbf{v}_{k+l+m}) \\ &= f(\mathbf{v}_1, \dots, \mathbf{v}_k) g(\mathbf{v}_{k+1}, \dots, \mathbf{v}_{k+l}) h(\mathbf{v}_{k+l+1}, \dots, \mathbf{v}_{k+l+m}). \end{aligned} \quad (1.23)$$

$$\begin{aligned} [f \otimes (g \otimes h)](\mathbf{v}_1, \dots, \mathbf{v}_{k+l+m}) &= f(\mathbf{v}_1, \dots, \mathbf{v}_k) (g \otimes h)(\mathbf{v}_{k+1}, \dots, \mathbf{v}_{k+l+m}) \\ &= f(\mathbf{v}_1, \dots, \mathbf{v}_k) g(\mathbf{v}_{k+1}, \dots, \mathbf{v}_{k+l}) h(\mathbf{v}_{k+l+1}, \dots, \mathbf{v}_{k+l+m}). \end{aligned} \quad (1.24)$$

Therefore, $(f \otimes g) \otimes h = f \otimes (g \otimes h)$, i.e. tensor product is associative. \blacksquare

Example 1.5. Let $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ be the standard basis for \mathbb{R}^n , and $\{\hat{\alpha}^1, \dots, \hat{\alpha}^n\}$ its dual basis. The Euclidean inner product on \mathbb{R}^n is the bilinear function

$$\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$$

defined by

$$\langle \cdot, \cdot \rangle(\mathbf{v}, \mathbf{w}) = \langle \mathbf{v}, \mathbf{w} \rangle = \sum_{i=1}^n v^i w^i,$$

for $\mathbf{v} = \sum_{i=1}^n v^i \mathbf{e}_i$ and $\mathbf{w} = \sum_{i=1}^n w^i \mathbf{e}_i$. We can express $\langle \cdot, \cdot \rangle$ in terms of tensor product as follows:

$$\langle \cdot, \cdot \rangle(\mathbf{v}, \mathbf{w}) = \sum_{i=1}^n v^i w^i = \sum_{i=1}^n \hat{\alpha}^i(\mathbf{v}) \hat{\alpha}^i(\mathbf{w}) = \sum_{i=1}^n (\hat{\alpha}^i \otimes \hat{\alpha}^i)(\mathbf{v}, \mathbf{w}).$$

Since \mathbf{v}, \mathbf{w} are arbitrary,

$$\langle \cdot, \cdot \rangle = \sum_{i=1}^n (\hat{\alpha}^i \otimes \hat{\alpha}^i). \quad (1.25)$$

If $f \in A_k(V)$ and $g \in A_l(V)$, then it's not true that $f \otimes g \in A_{k+l}(V)$, in general. We need to construct a product that is also alternating.

Definition 1.5 (Wedge Product). For $f \in A_k(V)$ and $g \in A_l(V)$, the wedge product of f and g is defined as follows:

$$f \wedge g = \frac{1}{k!l!} \mathcal{A}(f \otimes g). \quad (1.26)$$

Explicitly,

$$\begin{aligned} (f \wedge g)(\mathbf{v}_1, \dots, \mathbf{v}_{k+l}) &= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (\text{sgn } \sigma) \sigma(f \otimes g)(\mathbf{v}_1, \dots, \mathbf{v}_{k+l}) \\ &= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (\text{sgn } \sigma) (f \otimes g)(\mathbf{v}_{\sigma(1)}, \dots, \mathbf{v}_{\sigma(k+l)}) \\ &= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (\text{sgn } \sigma) f(\mathbf{v}_{\sigma(1)}, \dots, \mathbf{v}_{\sigma(k)}) g(\mathbf{v}_{\sigma(k+1)}, \dots, \mathbf{v}_{\sigma(k+l)}). \end{aligned} \quad (1.27)$$

When $k = 0$, the element $f \in A_0(V)$ is simply a constant $c \in \mathbb{R}$ as discussed earlier. In this case, the wedge product $c \wedge g$ is just scalar multiplication as is evident from (1.27).

$$\begin{aligned}
 (c \wedge g)(\mathbf{v}_1, \dots, \mathbf{v}_l) &= \frac{1}{l!} \sum_{\sigma \in S_l} (\operatorname{sgn} \sigma) c g(\mathbf{v}_{\sigma(1)}, \dots, \mathbf{v}_{\sigma(l)}) \\
 &= \frac{1}{l!} \sum_{\sigma \in S_l} (\operatorname{sgn} \sigma) c (\operatorname{sgn} \sigma) g(\mathbf{v}_1, \dots, \mathbf{v}_l) \\
 &= \frac{1}{l!} \sum_{\sigma \in S_l} c g(\mathbf{v}_1, \dots, \mathbf{v}_l) \\
 &= \frac{1}{l!} l! c g(\mathbf{v}_1, \dots, \mathbf{v}_l) \\
 &= c g(\mathbf{v}_1, \dots, \mathbf{v}_l).
 \end{aligned}$$

Thus $c \wedge g = cg$, for $c \in \mathbb{R}$ and $g \in A_l(V)$.

Example 1.6. For $f \in A_2(V)$ and $g \in A_1(V)$,

$$\begin{aligned}
 \mathcal{A}(f \otimes g)(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) &= f(\mathbf{v}_1, \mathbf{v}_2)g(\mathbf{v}_3) - f(\mathbf{v}_1, \mathbf{v}_3)g(\mathbf{v}_2) - f(\mathbf{v}_2, \mathbf{v}_1)g(\mathbf{v}_3) \\
 &\quad - f(\mathbf{v}_3, \mathbf{v}_2)g(\mathbf{v}_1) + f(\mathbf{v}_2, \mathbf{v}_3)g(\mathbf{v}_1) + f(\mathbf{v}_3, \mathbf{v}_1)g(\mathbf{v}_2).
 \end{aligned}$$

Among these 6 terms, there are 3 pairs of equal terms due to the alternating nature of f .

$$\begin{aligned}
 f(\mathbf{v}_1, \mathbf{v}_2)g(\mathbf{v}_3) &= -f(\mathbf{v}_2, \mathbf{v}_1)g(\mathbf{v}_3), \\
 f(\mathbf{v}_3, \mathbf{v}_1)g(\mathbf{v}_2) &= -f(\mathbf{v}_1, \mathbf{v}_3)g(\mathbf{v}_2), \\
 f(\mathbf{v}_2, \mathbf{v}_3)g(\mathbf{v}_1) &= -f(\mathbf{v}_3, \mathbf{v}_2)g(\mathbf{v}_1).
 \end{aligned}$$

Therefore,

$$\mathcal{A}(f \otimes g)(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = 2f(\mathbf{v}_1, \mathbf{v}_2)g(\mathbf{v}_3) + 2f(\mathbf{v}_3, \mathbf{v}_1)g(\mathbf{v}_2) + 2f(\mathbf{v}_2, \mathbf{v}_3)g(\mathbf{v}_1). \quad (1.28)$$

Hence,

$$\begin{aligned}
 (f \wedge g)(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) &= \frac{1}{2!1!} \mathcal{A}(f \otimes g)(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \\
 &= f(\mathbf{v}_1, \mathbf{v}_2)g(\mathbf{v}_3) + f(\mathbf{v}_3, \mathbf{v}_1)g(\mathbf{v}_2) + f(\mathbf{v}_2, \mathbf{v}_3)g(\mathbf{v}_1).
 \end{aligned} \quad (1.29)$$

Example 1.7 (Wedge product of 2 covectors). If $f, g \in A_1(V)$, and $\mathbf{v}_1, \mathbf{v}_2 \in V$, then

$$(f \wedge g)(\mathbf{v}_1, \mathbf{v}_2) = \frac{1}{1!1!} \mathcal{A}(f \otimes g)(\mathbf{v}_1, \mathbf{v}_2).$$

S_2 has 2 elements: the identity element e and $(1\ 2)$. Therefore,

$$(f \wedge g)(\mathbf{v}_1, \mathbf{v}_2) = \mathcal{A}(f \otimes g)(\mathbf{v}_1, \mathbf{v}_2) = f(\mathbf{v}_1)g(\mathbf{v}_2) - f(\mathbf{v}_2)g(\mathbf{v}_1).$$

Proposition 1.8 (Anticommutativity of wedge product)

The wedge product is anticommutative: if $f \in A_k(V)$ and $g \in A_l(V)$, then

$$f \wedge g = (-1)^{kl} g \wedge f.$$

Proof. Define $\tau \in S_{k+l}$ to be the following permutation:

$$\begin{bmatrix} 1 & 2 & \cdots & l & l+1 & l+2 & \cdots & l+k \\ k+1 & k+2 & \cdots & k+l & 1 & 2 & \cdots & k \end{bmatrix}.$$

In other words,

$$\tau(i) = \begin{cases} k+i & \text{if } 1 \leq i \leq l, \\ i-l & \text{if } l+1 \leq i \leq l+k. \end{cases}$$

Then for any $\sigma \in S_{k+l}$,

$$\sigma(j) = \begin{cases} \sigma(\tau(l+j)) & \text{if } 1 \leq j \leq k, \\ \sigma(\tau(j-k)) & \text{if } k+1 \leq j \leq k+l. \end{cases} \quad (1.30)$$

Now, for any $\mathbf{v}_1, \dots, \mathbf{v}_{k+l} \in V$,

$$\begin{aligned} \mathcal{A}(f \otimes g)(\mathbf{v}_1, \dots, \mathbf{v}_{k+l}) &= \sum_{\sigma \in S_{k+l}} (\text{sgn } \sigma) f(\mathbf{v}_{\sigma(1)}, \dots, \mathbf{v}_{\sigma(k)}) g(\mathbf{v}_{\sigma(k+1)}, \dots, \mathbf{v}_{\sigma(k+l)}) \\ &= \sum_{\sigma \in S_{k+l}} (\text{sgn } \sigma) f(\mathbf{v}_{\sigma(\tau(l+1))}, \dots, \mathbf{v}_{\sigma(\tau(l+k))}) g(\mathbf{v}_{\sigma(\tau(1))}, \dots, \mathbf{v}_{\sigma(\tau(l))}) \\ &= (\text{sgn } \tau) \sum_{\sigma \in S_{k+l}} (\text{sgn } \sigma) (\text{sgn } \tau) g(\mathbf{v}_{\sigma(\tau(1))}, \dots, \mathbf{v}_{\sigma(\tau(l))}) f(\mathbf{v}_{\sigma(\tau(l+1))}, \dots, \mathbf{v}_{\sigma(\tau(l+k))}) \\ &= (\text{sgn } \tau) \sum_{\sigma \in S_{k+l}} (\text{sgn } \sigma \tau) g(\mathbf{v}_{\sigma(\tau(1))}, \dots, \mathbf{v}_{\sigma(\tau(l))}) f(\mathbf{v}_{\sigma(\tau(l+1))}, \dots, \mathbf{v}_{\sigma(\tau(l+k))}). \end{aligned}$$

Again, as σ varies over S_{k+l} , $\sigma\tau$ also varies over S_{k+l} . Therefore,

$$\mathcal{A}(f \otimes g)(\mathbf{v}_1, \dots, \mathbf{v}_{k+l}) = (\text{sgn } \tau) \mathcal{A}(g \otimes f)(\mathbf{v}_1, \dots, \mathbf{v}_{k+l}). \quad (1.31)$$

Now, let us evaluate the sign of the permutation τ . Let $(\tau(i), \tau(j))$ be an inversion of τ . Then it's not possible that $1 \leq i < j \leq l$, or $l+1 \leq i < j \leq l+k$; because if we have $1 \leq i < j \leq l$ or $l+1 \leq i < j \leq l+k$, then $\tau(i) < \tau(j)$. Therefore, i must be in between 1 and l (inclusive), and j must be in between $l+1$ and $l+k$ (inclusive). So there are l options for i , and k options for j . Therefore, τ has kl many inversions. So $\text{sgn } \tau = (-1)^{kl}$. Using (1.31),

$$\mathcal{A}(f \otimes g) = (-1)^{kl} \mathcal{A}(g \otimes f). \quad (1.32)$$

Dividing by $k!l!$, we obtain

$$f \wedge g = (-1)^{kl} g \wedge f. \quad (1.33)$$

■

Corollary 1.9

If f is a k -covector on V , i.e. $f \in A_k(V)$, and k is odd, then $f \wedge f = 0$.

Proof. By anticommutativity of wedge product,

$$f \wedge f = (-1)^{k^2} f \wedge f = -f \wedge f.$$

Therefore, $f \wedge f = 0$. ■

If f is a k -covector and g is an l -covector, i.e. $f \in A_k(V)$ and $g \in A_l(V)$, then we have defined their wedge product to be the $(k+l)$ -covector

$$f \wedge g = \frac{1}{k!l!} \mathcal{A}(f \otimes g). \quad (1.34)$$

We have the following lemmas associated with the alternating operator \mathcal{A} .

Lemma 1.10

Suppose $f \in L_k(V)$ and $g \in L_l(V)$. Then

- (i) $\mathcal{A}(\mathcal{A}(f) \otimes g) = k! \mathcal{A}(f \otimes g)$.
- (ii) $\mathcal{A}(f \otimes \mathcal{A}(g)) = l! \mathcal{A}(f \otimes g)$.

Proof. (i) By definition,

$$\begin{aligned} \mathcal{A}(\mathcal{A}(f) \otimes g) &= \sum_{\sigma \in S_{k+l}} (\text{sgn } \sigma) \sigma(\mathcal{A}(f) \otimes g) \\ &= \sum_{\sigma \in S_{k+l}} (\text{sgn } \sigma) \sigma \left[\sum_{\tau \in S_k} (\text{sgn } \tau) (\tau f) \otimes g \right]. \end{aligned} \quad (1.35)$$

We can view $\tau \in S_k$ as a permutation in the following way: define $\tau' \in S_{k+l}$ as follows

$$\tau'(i) = \begin{cases} \tau(i) & \text{if } i \leq k, \\ i & \text{if } i > k. \end{cases} \quad (1.36)$$

Then for $\mathbf{v}_1, \dots, \mathbf{v}_{k+l}$, we have

$$\begin{aligned} [(\tau f) \otimes g](\mathbf{v}_1, \dots, \mathbf{v}_{k+l}) &= (\tau f)(\mathbf{v}_1, \dots, \mathbf{v}_k) g(\mathbf{v}_{k+1}, \dots, \mathbf{v}_{k+l}) \\ &= f(\mathbf{v}_{\tau(1)}, \dots, \mathbf{v}_{\tau(k)}) g(\mathbf{v}_{k+1}, \dots, \mathbf{v}_{k+l}) \\ &= f(\mathbf{v}_{\tau'(1)}, \dots, \mathbf{v}_{\tau'(k)}) g(\mathbf{v}_{\tau'(k+1)}, \dots, \mathbf{v}_{\tau'(k+l)}) \\ &= [\tau'(f \otimes g)](\mathbf{v}_1, \dots, \mathbf{v}_{k+l}). \end{aligned}$$

Therefore, $(\tau f) \otimes g = \tau'(f \otimes g)$. Furthermore, $\text{sgn } \tau = \text{sgn } \tau'$ since the inversions $(\tau'(i), \tau'(j))$ occur only when $1 \leq i < j \leq k$, so that the τ and τ' has the same number of inversions.

Let us abuse notation a bit and denote by S_k the subgroup of permutations in S_{k+l} by keeping the last l arguments fixed. This subgroup of S_{k+l} is indeed isomorphic to S_k , so we will denote both these groups by S_k . Therefore, from (1.35),

$$\begin{aligned} \mathcal{A}(\mathcal{A}(f) \otimes g) &= \sum_{\sigma \in S_{k+l}} (\text{sgn } \sigma) \sigma \left[\sum_{\tau' \in S_k \subseteq S_{k+l}} (\text{sgn } \tau') \tau'(f \otimes g) \right] \\ &= \sum_{\sigma \in S_{k+l}} \sum_{\tau' \in S_k \subseteq S_{k+l}} (\text{sgn } \sigma) (\text{sgn } \tau') \sigma \tau'(f \otimes g) \\ &= \sum_{\tau' \in S_k \subseteq S_{k+l}} \sum_{\sigma \in S_{k+l}} (\text{sgn } \sigma \text{sgn } \tau') ((\sigma \tau')(f \otimes g)). \end{aligned}$$

For a fixed τ' , as σ varies over S_{k+l} , $\sigma \tau'$ also varies over S_{k+l} . Therefore,

$$\mathcal{A}(\mathcal{A}(f) \otimes g) = \sum_{\tau' \in S_k \subseteq S_{k+l}} \mathcal{A}(f \otimes g) = k! \mathcal{A}(f \otimes g). \quad (1.37)$$

(ii) By (1.32),

$$\begin{aligned} \mathcal{A}(f \otimes \mathcal{A}(g)) &= \mathcal{A}((-1)^{kl} \mathcal{A}(g) \otimes f) \\ &= (-1)^{kl} \mathcal{A}(\mathcal{A}(g) \otimes f) \\ &= (-1)^{kl} l! \mathcal{A}(g \otimes f) \\ &= l! \mathcal{A}((-1)^{kl} g \otimes f) \\ &= l! \mathcal{A}(f \otimes g). \end{aligned} \quad (1.38)$$

■

Proposition 1.11 (Associativity of wedge product)

Let V be a real vector space and f, g, h be alternating multilinear functions on V of degree k, l, m , respectively. Then

$$(f \wedge g) \wedge h = f \wedge (g \wedge h).$$

Proof. Using the definition of wedge product,

$$\begin{aligned} (f \wedge g) \wedge h &= \frac{1}{(k+l)!m!} \mathcal{A}[(f \wedge g) \otimes h] \\ &= \frac{1}{(k+l)!m!} \mathcal{A}\left[\frac{1}{k!l!} \mathcal{A}(f \otimes g) \otimes h\right] \\ &= \frac{1}{(k+l)!k!l!m!} \mathcal{A}[\mathcal{A}(f \otimes g) \otimes h] \\ &= \frac{(k+l)!}{(k+l)!k!l!m!} \mathcal{A}[(f \otimes g) \otimes h] \\ &= \frac{1}{k!l!m!} \mathcal{A}[(f \otimes g) \otimes h]. \end{aligned}$$

On the other hand,

$$\begin{aligned} f \wedge (g \wedge h) &= \frac{1}{k!(l+m)!} \mathcal{A}[f \otimes (g \wedge h)] \\ &= \frac{1}{k!(l+m)!} \mathcal{A}\left[f \otimes \left(\frac{1}{l!m!} \mathcal{A}(g \otimes h)\right)\right] \\ &= \frac{1}{k!(l+m)!l!m!} \mathcal{A}[f \otimes \mathcal{A}(g \otimes h)] \\ &= \frac{(l+m)!}{k!(l+m)!l!m!} \mathcal{A}[f \otimes (g \otimes h)] \\ &= \frac{1}{k!l!m!} \mathcal{A}[f \otimes (g \otimes h)]. \end{aligned}$$

Since tensor product is associative (by [Lemma 1.7](#)), we conclude that

$$(f \wedge g) \wedge h = f \wedge (g \wedge h). \quad (1.39)$$

■

By associativity, we can omit the parenthesis and write univocally $f \wedge g \wedge h$ instead of $(f \wedge g) \wedge h$ or $f \wedge (g \wedge h)$.

Corollary 1.12

Under the hypothesis of [Proposition 1.11](#),

$$f \wedge g \wedge h = \frac{1}{k!l!m!} \mathcal{A}[f \otimes g \otimes h]. \quad (1.40)$$

This easily generalizes to an arbitrary number of factors: if $f_i \in A_{d_i}(V)$ for $i = 1, 2, \dots, r$, i.e. f_i is an alternating d_i -linear function on V , then

$$f_1 \wedge \dots \wedge f_r = \frac{1}{d_1! \dots d_r!} \mathcal{A}(f_1 \otimes \dots \otimes f_r). \quad (1.41)$$

Proposition 1.13

Let $\hat{\alpha}^1, \hat{\alpha}^2, \dots, \hat{\alpha}^k$ be linear functions on a real vector space V (i.e. $\hat{\alpha}^i : V \rightarrow \mathbb{R}$) and $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in V$. Then

$$\begin{aligned} (\hat{\alpha}^1 \wedge \dots \wedge \hat{\alpha}^k)(\mathbf{v}_1, \dots, \mathbf{v}_k) &= \det [\hat{\alpha}^i(\mathbf{v}_j)] \\ &= \det \begin{bmatrix} \hat{\alpha}^1(\mathbf{v}_1) & \hat{\alpha}^1(\mathbf{v}_2) & \dots & \hat{\alpha}^1(\mathbf{v}_k) \\ \hat{\alpha}^2(\mathbf{v}_1) & \hat{\alpha}^2(\mathbf{v}_2) & \dots & \hat{\alpha}^2(\mathbf{v}_k) \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\alpha}^k(\mathbf{v}_1) & \hat{\alpha}^k(\mathbf{v}_2) & \dots & \hat{\alpha}^k(\mathbf{v}_k) \end{bmatrix}. \end{aligned}$$

Proof. By 1.41,

$$(\hat{\alpha}^1 \wedge \dots \wedge \hat{\alpha}^k)(\mathbf{v}_1, \dots, \mathbf{v}_k) = \mathcal{A}(\hat{\alpha}^1 \otimes \dots \otimes \hat{\alpha}^k)(\mathbf{v}_1, \dots, \mathbf{v}_k).$$

By the definition of the action of alternating operator,

$$\mathcal{A}(\hat{\alpha}^1 \otimes \dots \otimes \hat{\alpha}^k)(\mathbf{v}_1, \dots, \mathbf{v}_k) = \sum_{\sigma \in S_k} (\text{sgn } \sigma) \hat{\alpha}^1(\mathbf{v}_{\sigma(1)}) \dots \hat{\alpha}^k(\mathbf{v}_{\sigma(k)}). \quad (1.42)$$

By the definition of determinant of a $k \times k$ matrix $A = [a_{ij}]$,

$$\det A = \sum_{\sigma \in S_k} (\text{sgn } \sigma) a_{1\sigma(1)} a_{2\sigma(2)} \dots a_{k\sigma(k)}. \quad (1.43)$$

Using (1.43) in (1.42), we get

$$\mathcal{A}(\hat{\alpha}^1 \otimes \dots \otimes \hat{\alpha}^k)(\mathbf{v}_1, \dots, \mathbf{v}_k) = \det [\hat{\alpha}^i(\mathbf{v}_j)]. \quad (1.44)$$

■

§1.5 A Basis for $A_k(V)$

Let $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be a basis for a real vector space V , and let $\{\hat{\alpha}^1, \dots, \hat{\alpha}^n\}$ be the dual basis for V^* . Introduce the multi-index notation

$$I = (i_1, i_2, \dots, i_k)$$

and write \mathbf{e}_I for $(\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \dots, \mathbf{e}_{i_k})$ and $\hat{\alpha}^I$ for $\hat{\alpha}^{i_1} \wedge \hat{\alpha}^{i_2} \wedge \dots \wedge \hat{\alpha}^{i_k}$.

A k -linear function f on V is completely determined by its values on all k -tuples $(\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \dots, \mathbf{e}_{i_k})$. If f is alternating, then f is completely determined by its values on all k -tuples $(\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \dots, \mathbf{e}_{i_k})$ with

$$1 \leq i_1 < i_2 < \dots < i_k \leq n.$$

In other words, it's sufficient to consider \mathbf{e}_I with I in ascending order.

Lemma 1.14

Suppose I and J are ascending multi-indices of length k . Then

$$\hat{\alpha}^I(\mathbf{e}_J) = \delta^I_J := \begin{cases} 1 & \text{if } I = J, \\ 0 & \text{if } I \neq J. \end{cases}$$

Proof. Suppose $I = (i_1, \dots, i_k)$ and $J = (j_1, \dots, j_k)$. Using (1.42), we get

$$\begin{aligned} \hat{\alpha}^I(\mathbf{e}_J) &= (\hat{\alpha}^{i_1} \wedge \hat{\alpha}^{i_2} \wedge \dots \wedge \hat{\alpha}^{i_k})(\mathbf{e}_{j_1}, \mathbf{e}_{j_2}, \dots, \mathbf{e}_{j_k}) \\ &= \sum_{\sigma \in S_k} (\text{sgn } \sigma) \hat{\alpha}^{i_1}(\mathbf{e}_{j_{\sigma(1)}}) \dots \hat{\alpha}^{i_k}(\mathbf{e}_{j_{\sigma(k)}}) \\ &= \sum_{\sigma \in S_k} (\text{sgn } \sigma) \delta^{i_1}_{j_{\sigma(1)}} \dots \delta^{i_k}_{j_{\sigma(k)}}. \end{aligned} \quad (1.45)$$

The terms in the sum (1.45) contribute $\text{sgn } \sigma$ if and only if

$$(i_1, i_2, \dots, i_k) = (j_{\sigma(1)}, j_{\sigma(2)}, \dots, j_{\sigma(k)});$$

otherwise they contribute 0 to the sum. Both I and J are ascending multi-indices. Permuting the elements of J no longer gives an ascending multi-index (unless the permutation σ is the identity permutation). Therefore, in (1.45), all the summands corresponding to σ being a non-identity permutation contribute 0.

$$\hat{\alpha}^I(\mathbf{e}_J) = \sum_{\sigma \in S_k} (\text{sgn } \sigma) \delta^{i_1}_{j_{\sigma(1)}} \cdots \delta^{i_k}_{j_{\sigma(k)}} = \delta^{i_1}_{j_1} \cdots \delta^{i_k}_{j_k} = \begin{cases} 1 & \text{if } I = J, \\ 0 & \text{if } I \neq J. \end{cases} \quad (1.46)$$

■

Proposition 1.15

The alternating k -linear functions $\hat{\alpha}^I$, $I = (i_1, \dots, i_k)$, with $1 \leq i_1 < \dots < i_k \leq n$ form a basis for the space $A_k(V)$ of alternating k -linear functions on V .

Proof. Let us first show linear independence. Suppose

$$\sum_I c_I \hat{\alpha}^I = \mathbf{0}, \quad (1.47)$$

$c_I \in \mathbb{R}$ with I running over ascending multi-indices of length k . Applying \mathbf{e}_J to both sides, we get

$$0 = \sum_I c_I \hat{\alpha}^I(\mathbf{e}_J) = \sum_I c_I \delta^I_J = c_J. \quad (1.48)$$

Therefore, $\{\hat{\alpha}^I \mid I \text{ is ascending multi-index of length } k\}$ is a linearly independent set. Now let us prove that this set spans $A_k(V)$. Let $f \in A_k(V)$. We claim that

$$f = \sum_I f(\mathbf{e}_I) \hat{\alpha}^I. \quad (1.49)$$

Let $g = \sum_I f(\mathbf{e}_I) \hat{\alpha}^I$. We need to prove that $f = g$. By k -linearity and alternating property, if two k -covectors agree on all \mathbf{e}_J where J is an ascending multi-index, then they are equal. Now,

$$g(\mathbf{e}_J) = \sum_I f(\mathbf{e}_I) \hat{\alpha}^I(\mathbf{e}_J) = \sum_I f(\mathbf{e}_I) \delta^I_J = f(\mathbf{e}_J). \quad (1.50)$$

Therefore, $f = g = \sum_I f(\mathbf{e}_I) \hat{\alpha}^I$. ■

Corollary 1.16

If the vector space V has dimension n , then the vector space $A_k(V)$ of k -covectors on V has dimension $\binom{n}{k}$.

Proof. An ascending multi-index $I = (i_1, i_2, \dots, i_k)$, $1 \leq i_1 < i_2 < \dots < i_k \leq n$ is obtained by choosing a k -element subset of $\{1, 2, \dots, n\}$. This can be done in $\binom{n}{k}$ ways. ■

Corollary 1.17

If $k > \dim V$, then $A_k(V) = 0$.

Proof. If $k > \dim V = n$, then in the expression

$$\hat{\alpha}^{i_1} \wedge \hat{\alpha}^{i_2} \wedge \dots \wedge \hat{\alpha}^{i_k}$$

with each $i \in \{1, 2, \dots, n\}$, there must be a repeated i_j 's, say $\hat{\alpha}^r$. Then $\hat{\alpha}^r \wedge \hat{\alpha}^r$ arises in the expression $\hat{\alpha}^{i_1} \wedge \hat{\alpha}^{i_2} \wedge \dots \wedge \hat{\alpha}^{i_k}$. But $\hat{\alpha}^r \wedge \hat{\alpha}^r = 0$ by Corollary 1.9. Hence, $\hat{\alpha}^{i_1} \wedge \hat{\alpha}^{i_2} \wedge \dots \wedge \hat{\alpha}^{i_k} = 0$. Therefore, the basis set of $A_k(V)$ is empty, meaning $A_k(V) = 0$. ■

2 Differential Forms on \mathbb{R}^n

Given an open set $U \subseteq \mathbb{R}^n$ and $p \in U$, $T_p U$ is the set of tangent vectors at $p \in U$ is identified with the point derivations of C_p^∞ (germs of smooth functions at p), i.e. a tangent vector $X_p \in T_p U$ is a map $X_p : C_p^\infty \rightarrow \mathbb{R}$ such that X_p is \mathbb{R} -linear:

$$X_p(\alpha f + g) = \alpha(X_p f) + X_p g; \quad (2.1)$$

and satisfies the Leibniz condition:

$$X_p(fg) = (X_p f)g(p) + f(p)(X_p g).$$

In contrast to the notion of point derivation, there is this notion of derivation of an algebra. If X is a C^∞ vector field on an open subset $U \subseteq \mathbb{R}^n$, i.e. $X \in \mathfrak{X}(U)$, and f is a C^∞ function on U , i.e. $f \in C^\infty(U)$, then $Xf \in C^\infty(U)$ defined by

$$(Xf)(p) = X_p f.$$

Remember that f in (2.1) and (2) is a representative of an equivalence class, the equivalence class of germs of C^∞ functions at $p \in U$. These equivalence classes constitute $C_p^\infty(U)$. It is of course an \mathbb{R} -algebra. While in (2), $f \in C^\infty(U)$, the algebra of C^∞ functions on U with no reference of p whatsoever.

From the discussion above, a C^∞ vector field X gives rise to an \mathbb{R} -linear map $C^\infty(U) \rightarrow C^\infty(U)$ by $f \mapsto Xf$ that additionally has to satisfy the following Leibniz condition:

$$X(fg) = (Xf)g + f(Xg). \quad (2.2)$$

Note that a derivation at p is not a derivation of the algebra C_p^∞ . A derivation at p is a map from $C_p^\infty \rightarrow \mathbb{R}$ that satisfies (2), while a derivation of the algebra C_p^∞ is supposed to be a map from C_p^∞ to itself obeying Leibniz condition.

§2.1 1 form

From any C^∞ function $f : U \rightarrow \mathbb{R}$, one can construct a 1-form (dual notion of C^∞ vector field) df , the restriction of which to a given point $p \in U$ yields a covector $(df)_p \in T_p^* U$, the dual space of $T_p U$, in the following way:

$$(df)_p(X_p) = X_p f. \quad (2.3)$$

Proposition 2.1

If x^1, x^2, \dots, x^n are the standard coordinates on \mathbb{R}^n , then at each point $p \in \mathbb{R}^n$,

$$\left\{ (dx^1)_p, (dx^2)_p, \dots, (dx^n)_p \right\}$$

is the basis for the cotangent space $T_p^* \mathbb{R}^n$ dual to the basis $\left\{ \frac{\partial}{\partial x^1} \Big|_p, \dots, \frac{\partial}{\partial x^n} \Big|_p \right\}$ for the tangent space $T_p \mathbb{R}^n$.

Proof. $(dx^i)_p : T_p^* \mathbb{R}^n \rightarrow \mathbb{R}$ is a linear map for each i . Now,

$$(dx^i)_p \left(\frac{\partial}{\partial x^j} \Big|_p \right) = \frac{\partial}{\partial x^j} \Big|_p (x^i) = \delta^i_j. \quad (2.4)$$

Therefore, $\{(\mathrm{d}x^1)_p, (\mathrm{d}x^2)_p, \dots, (\mathrm{d}x^n)_p\}$ is the basis of $T_p^*\mathbb{R}^n$ dual to the basis $\left\{\frac{\partial}{\partial x^1}\Big|_p, \dots, \frac{\partial}{\partial x^n}\Big|_p\right\}$ for $T_p\mathbb{R}^n$. \blacksquare

If ω is a 1-form on an open subset $U \subseteq \mathbb{R}^n$, then by [Proposition 2.1](#), there is a linear combination

$$\omega_p = \sum_{i=1}^n a_i(p) (\mathrm{d}x^i)_p, \quad (2.5)$$

for some $a_i(p) \in \mathbb{R}$. As p varies over U , the coefficients a_i become functions on U , and we may write

$$\omega = \sum_{i=1}^n a_i \mathrm{d}x^i. \quad (2.6)$$

The 1-form ω is said to be C^∞ on U if the coefficient functions a_i are all C^∞ functions on U .

Proposition 2.2 (The differential in terms of coordinates)

If $f : U \rightarrow \mathbb{R}$ is a C^∞ function on an open set $U \subseteq \mathbb{R}^n$, then

$$\mathrm{d}f = \sum_{i=1}^n \frac{\partial f}{\partial x^i} \mathrm{d}x^i.$$

Proof. By [Proposition 2.1](#), at each point $p \in U$,

$$(\mathrm{d}f)_p = \sum_{i=1}^n a_i(p) (\mathrm{d}x^i)_p, \quad (2.7)$$

for some constants $a_i(p)$ depending on p . Thus

$$\mathrm{d}f = \sum_{i=1}^n a_i \mathrm{d}x^i, \quad (2.8)$$

for some functions a_i on U . To evaluate a_j , apply both sides of (2.8) to the coordinate vector field $\frac{\partial}{\partial x^j}$:

$$\mathrm{d}f\left(\frac{\partial}{\partial x^j}\right) = \sum_{i=1}^n a_i \mathrm{d}x^i\left(\frac{\partial}{\partial x^j}\right) = \sum_{i=1}^n a_i \delta^i_j = a_j. \quad (2.9)$$

On the other hand, using $(\mathrm{d}f)_p(X_p) = X_p f = (Xf)(p)$, we get $(\mathrm{d}f)(X) = Xf$. So

$$\mathrm{d}f\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial f}{\partial x^j}. \quad (2.10)$$

Therefore, $a_j = \frac{\partial f}{\partial x^j}$. Hence,

$$\mathrm{d}f = \sum_{i=1}^n \frac{\partial f}{\partial x^i} \mathrm{d}x^i. \quad (2.11)$$

\blacksquare

(2.11) tells us that $\mathrm{d}f$ will be a C^∞ 1-form if $\frac{\partial f}{\partial x^i}$ is C^∞ on U . Hence, it is sufficient to have f as a C^∞ function on U in order to have $\mathrm{d}f$ as a C^∞ 1-form.

§2.2 Differential k -forms

A differential form ω of degree k (or a k -form) on an open subset $U \subseteq \mathbb{R}^n$ is a map that assigns to each point $p \in U$, an alternating k -linear function on the tangent space $T_p\mathbb{R}^n$, i.e.

$$\omega_p \in A_k(T_p\mathbb{R}^n).$$

By Proposition 1.15, a basis for $A_k(T_p\mathbb{R}^n)$ is

$$\left(dx^I\right)_p = \left(dx^{i_1}\right)_p \wedge \left(dx^{i_2}\right)_p \wedge \cdots \wedge \left(dx^{i_k}\right)_p,$$

where $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. Therefore, at each point $p \in U$, ω_p is a linear combination

$$\omega_p = \sum_I a_I(p) \left(dx^I\right)_p, \quad (2.12)$$

and a k -form ω on U is a linear combination

$$\omega = \sum_I a_I dx^I, \quad (2.13)$$

with function coefficients $a_I : U \rightarrow \mathbb{R}$. We say that a k -form ω is **smooth** on U if all the coefficients a_I are C^∞ functions on U .

Denote by $\Omega^k(U)$ the vector space of C^∞ k -forms on U . A 0-form on U assigns to each point $p \in U$ an element of $A_0(T_p\mathbb{R}^n) = \mathbb{R}$. Thus a 0-form on U is simply a real-valued function on U , and $\Omega^0(U) = C^\infty(U)$.

Since one can multiply a C^∞ k -form by a C^∞ function on U from the left, the set $\Omega^k(U)$ of C^∞ k -forms on U is both a real vector space and a $C^\infty(U)$ -module. With the wedge product as multiplication, the direct sum

$$\Omega^*(U) = \bigoplus_{k=0}^n \Omega^k(U)$$

becomes an algebra over \mathbb{R} as well as a module over $C^\infty(U)$. As an algebra, it is anticommutative and associative.

Example 2.1. Let x, y, z be the coordinates on \mathbb{R}^3 . The C^∞ 1-forms on \mathbb{R}^3 are

$$P(x, y, z) dx + Q(x, y, z) dy + R(x, y, z) dz,$$

where P, Q, R range over all C^∞ functions on \mathbb{R}^3 . The C^∞ 2-forms are

$$A(x, y, z) dy \wedge dz + B(x, y, z) dx \wedge dz + C(x, y, z) dx \wedge dy;$$

and the C^∞ 1-forms are

$$a(x, y, z) dx \wedge dy \wedge dz.$$

Example 2.2 (A basis for 3-covectors). Let x^1, x^2, x^3, x^4 be the standard coordinates on \mathbb{R}^4 , and $p \in \mathbb{R}^4$. A basis for $A_3(T_p\mathbb{R}^4)$ is

$$\left\{ \left(dx^1\right)_p \wedge \left(dx^2\right)_p \wedge \left(dx^3\right)_p, \left(dx^1\right)_p \wedge \left(dx^2\right)_p \wedge \left(dx^4\right)_p, \right. \\ \left. \left(dx^1\right)_p \wedge \left(dx^3\right)_p \wedge \left(dx^4\right)_p, \left(dx^2\right)_p \wedge \left(dx^3\right)_p \wedge \left(dx^4\right)_p \right\}.$$

So $\dim(A_3(T_p\mathbb{R}^n)) = 4$.

§2.3 Exterior Derivative

Before defining exterior derivative of a C^∞ k -form on an open subset $U \subseteq \mathbb{R}^n$, we first define it on 0-forms. The exterior derivative of a C^∞ function $f \in C^\infty(U)$ is its differential:

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i \in \Omega^1(U).$$

Definition 2.1 (Exterior Derivative). If $\omega = \sum_I a_I dx^I \in \omega^K(U)$, then its exterior derivative is defined as follows:

$$d\omega = \sum_I da_I \wedge dx^I = \sum_I \left(\sum_{j=1}^n \frac{\partial a_I}{\partial x^j} dx^j \right) \wedge dx^I \in \Omega^{k+1}(U). \quad (2.14)$$

Example 2.3. Let ω be the 1 form $f dx + g dy$ on \mathbb{R}^2 , where f and g are C^∞ functions on \mathbb{R}^2 . Let us write $f_x = \frac{\partial f}{\partial x}$ and $f_y = \frac{\partial f}{\partial y}$. Then

$$\begin{aligned} d\omega &= df \wedge dx + dg \wedge dy \\ &= (f_x dx + f_y dy) \wedge dx + (g_x dx + g_y dy) \wedge dy \\ &= -f_y dx \wedge dy + g_x dx \wedge dy \\ &= (g_x - f_y) dx \wedge dy. \end{aligned}$$

Definition 2.2 (Graded Algebra). An algebra A over a field \mathbb{K} is said to be **graded** if it can be written as a direct sum

$$A = \bigoplus_{k=0}^{\infty} A^k$$

of vector spaces over \mathbb{K} so that the multiplication map sends $A^k \times A^l$ to A^{k+l} .

The notation $A = \bigoplus_{k=0}^{\infty} A^k$ means that each element of A is uniquely a **finite sum**

$$a = a_{i_1} + a_{i_2} + \cdots + a_{i_m},$$

where $a_{i_j} \in A^{i_j}$.

Example 2.4. The polynomial algebra

$$\mathbb{R}[x, y] = \bigoplus_{k=0}^{\infty} A^k$$

with A^k being the vector space of homogenous polynomials of degree k in x and y . Observe that the 0 polynomial is trivially homogenous of any degree, and hence belongs to A^k for all $k \geq 0$. Multiplication of degree k homogenous polynomial with a degree l homogenous polynomial in x and y will result in a homogenous polynomial of degree $k + l$ in x and y .

Example 2.5. The algebra $\Omega^*(U)$ of C^∞ differential forms on U is also graded by the degree of differential forms. Each $\Omega^k(U)$ is a vector space. Multiplication of differential forms is defined by wedge product between them. The wedge product of a degree k differential form on U with a degree l differential form results in a degree $k + l$ differential form.

Definition 2.3 (Anti-derivation). Let $A = \bigoplus_{k=0}^{\infty} A^k$ be a graded algebra over a field \mathbb{K} . An **anti-derivation** of the graded algebra A is a \mathbb{K} -linear map $D : A \rightarrow A$ such that for $\omega \in A^k$ and $\tau \in A^l$, one has

$$D(\omega\tau) = (D\omega)\tau + (-1)^k \omega(D\tau). \quad (2.15)$$

If the antiderivation D sends $\omega \in A^k$ to $D\omega \in A^{k+m}$, we say that it is an antiderivation of degree m .

Proposition 2.3 (i) The exterior derivative $d : \Omega^*(U) \rightarrow \Omega^*(U)$ is an antiderivation of degree 1:

$$d(\omega \wedge \tau) = (d\omega) \wedge \tau + (-1)^{\deg \omega} \omega \wedge d\tau. \quad (2.16)$$

(ii) $d^2 = 0$.

(iii) If $f \in \Omega^0(U) = C^\infty(U)$ and $X \in \mathfrak{X}(U)$ (the space of C^∞ vector fields), then $(df)(X) = Xf$.

Proof. (i) Since the exterior derivative operator $d : \Omega^k(U) \rightarrow \Omega^{k+1}(U)$ is linear, it suffices to check the equality (2.16) for $\omega = f dx^I$ and $\tau = g dx^J$ with $I = (i_1, \dots, i_k)$ and $J = (j_1, \dots, j_l)$ being strictly ascending multi-indices.

$$\begin{aligned} d(\omega \wedge \tau) &= d(fg dx^I \wedge dx^J) \\ &= \sum_{i=1}^n \frac{\partial(fg)}{\partial x^i} dx^i \wedge dx^I \wedge dx^J \\ &= \sum_{i=1}^n \frac{\partial f}{\partial x^i} \cdot g dx^i \wedge dx^I \wedge dx^J + \sum_{i=1}^n f \cdot \frac{\partial g}{\partial x^i} dx^i \wedge dx^I \wedge dx^J \\ &= \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i \wedge dx^I \wedge g dx^J + \sum_{i=1}^n \frac{\partial g}{\partial x^i} dx^i \wedge f dx^I \wedge dx^J. \end{aligned} \quad (2.17)$$

Now, in the second sum in (2.17), one has to push $\frac{\partial g}{\partial x^i} dx^i$ through the k -fold wedge product dx^I and hence in the process picks out a sign $(-1)^k$. Therefore,

$$\sum_{i=1}^n \frac{\partial g}{\partial x^i} dx^i \wedge f dx^I \wedge dx^J = (-1)^k f dx^I \wedge \frac{\partial g}{\partial x^i} dx^i \wedge dx^J. \quad (2.18)$$

Now, observe that

$$d\omega = d(f dx^I) = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i \wedge dx^I, \text{ and} \quad (2.19)$$

$$d\tau = d(g dx^J) = \sum_{i=1}^n \frac{\partial g}{\partial x^i} dx^i \wedge dx^J. \quad (2.20)$$

Therefore,

$$d(\omega \wedge \tau) = d\omega \wedge \tau + (-1)^k \omega \wedge d\tau. \quad (2.21)$$

(ii) Again, by \mathbb{R} -linearity of d , it suffices to show that $d^2\omega = 0$ for $\omega = f dx^I$.

$$\begin{aligned} d^2(f dx^I) &= d\left(\sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i \wedge dx^I\right) \\ &= \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 f}{\partial x^j \partial x^i} dx^j \wedge dx^i \wedge dx^I. \end{aligned} \quad (2.22)$$

If $i = j$, then $dx^j \wedge dx^i = 0$. If $i \neq j$, then $\frac{\partial^2 f}{\partial x^j \partial x^i}$ is symmetric in i and j , but $dx^j \wedge dx^i$ is alternating in i and j . Therefore, the terms with $i \neq j$ pair up and cancel out.

(iii) Let $X = \sum_{i=1}^n a^i \frac{\partial}{\partial x^i}$. Then

$$\begin{aligned}
 (df)(X) &= \left(\sum_{j=1}^n \frac{\partial f}{\partial x^j} dx^j \right) \left(\sum_{i=1}^n a^i \frac{\partial}{\partial x^i} \right) \\
 &= \sum_{j=1}^n \sum_{i=1}^n a^i \frac{\partial f}{\partial x^j} dx^j \left(\frac{\partial}{\partial x^i} \right) \\
 &= \sum_{j=1}^n \sum_{i=1}^n a^i \frac{\partial f}{\partial x^j} \delta^j_i \\
 &= \sum_{i=1}^n a^i \frac{\partial f}{\partial x^i} = Xf.
 \end{aligned} \tag{2.23}$$

■

Proposition 2.4 (Characterization of exterior derivative)

The 3 properties of Proposition 2.3 uniquely characterize exterior derivative on an open set $U \subseteq \mathbb{R}^n$. In other words, if $D : \Omega^*(U) \rightarrow \Omega^*(U)$ is an antiderivation of degree 1 such that $D^2 = 0$ and for $f \in C^\infty(U)$ and $X \in \mathfrak{X}(U)$, $(Df)(X) = Xf$, then $D = d$.

Proof. Since every k -form on U is a sum of terms such as $f dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}$, by linearity of d , it suffices to show that $D = d$ on a k -form of this type. Applying property (iii) for $f = x^i$, one has

$$Dx^i(X) = X(x^i).$$

Writing $X = \sum_{j=1}^n a^j \frac{\partial}{\partial x^j}$, we get

$$Dx^i \left(\sum_{j=1}^n a^j \frac{\partial}{\partial x^j} \right) = \sum_{j=1}^n a^j \frac{\partial}{\partial x^j} (x^i) = a^i = dx^i \left(\sum_{j=1}^n a^j \frac{\partial}{\partial x^j} \right).$$

Therefore,

$$Dx^i = dx^i. \tag{2.24}$$

Now,

$$\begin{aligned}
 D(f dx^{i_1} \wedge \cdots \wedge dx^{i_k}) &= D(f Dx^{i_1} \wedge \cdots \wedge Dx^{i_k}) \\
 &= Df \wedge (Dx^{i_1} \wedge \cdots \wedge Dx^{i_k}) + (-1)^0 f D(Dx^{i_1} \wedge \cdots \wedge Dx^{i_k}).
 \end{aligned} \tag{2.25}$$

Now, since $df(X) = Xf = Df(X)$ for any $X \in \mathfrak{X}(U)$, $df = Df$. Furthermore, $D(Dx^{i_1}) = 0$, and

$$\begin{aligned}
 D(Dx^{i_1} \wedge \cdots \wedge Dx^{i_k}) &= D^2 x^{i_1} \wedge Dx^{i_2} \wedge \cdots \wedge Dx^{i_k} - Dx^{i_1} \wedge D(Dx^{i_2} \wedge \cdots \wedge Dx^{i_k}) \\
 &= -Dx^{i_1} \wedge D(Dx^{i_2} \wedge \cdots \wedge Dx^{i_k}).
 \end{aligned} \tag{2.26}$$

Therefore, by induction on k ,

$$D(Dx^{i_1} \wedge \cdots \wedge Dx^{i_k}) = 0. \tag{2.27}$$

Hence, from (2.25),

$$\begin{aligned}
 D(f dx^{i_1} \wedge \cdots \wedge dx^{i_k}) &= Df \wedge (Dx^{i_1} \wedge \cdots \wedge Dx^{i_k}) \\
 &= df \wedge (dx^{i_1} \wedge \cdots \wedge dx^{i_k}) \\
 &= d(f dx^{i_1} \wedge \cdots \wedge dx^{i_k}).
 \end{aligned} \tag{2.28}$$

So $D = d$ on $\Omega^*(U)$.

■

Closed Forms and Exact Forms

A k -form ω on U is **closed** if $d\omega = 0$; it's **exact** if there is a $(k-1)$ -form τ on U such that $\omega = d\tau$. Since $d^2 = 0$, every exact form is closed. But in general, a closed form may fail to be exact. We will see how non-exact closed forms capture the geometry of a manifold when we do de Rham cohomology on a manifold.

Example 2.6. Define a 1-form ω on $\mathbb{R}^2 \setminus \{0\}$ by

$$\omega = \frac{1}{x^2 + y^2} (-ydx + xdy). \quad (2.29)$$

Then ω is closed.

A collection of vector spaces $\{V^k\}_{k=0}^\infty$ with linear maps $d_k : V^k \rightarrow V^{k+1}$ such that $d_{k+1} \circ d_k = 0$ is called a **differential complex** or a **cochain complex**. For any open set $U \subseteq \mathbb{R}^n$, the exterior derivative d makes the vector space $\Omega^*(U)$ of C^∞ forms on U into a cochain complex, called the **de Rham complex** on U :

$$\Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \Omega^2(U) \xrightarrow{d} \dots$$

The closed forms are precisely the elements of the kernel of d and the exact forms are the elements of the image of d . In the language of cohomology, d is also called the coboundary operator.

§2.4 Applications to Vector Calculus

The theory of differential forms unifies many theorems in vector calculus on \mathbb{R}^3 . A vector valued function on \mathbb{R}^3 is the same as a vector field. Recall the 3 operators on scalar and vector-valued functions on \mathbb{R}^3 .

$$\{\text{scalar function}\} \xrightarrow{\text{grad}} \{\text{vector function}\} \xrightarrow{\text{curl}} \{\text{vector function}\} \xrightarrow{\text{div}} \{\text{scalar function}\}.$$

Let f be a scalar function and $\begin{bmatrix} P \\ Q \\ R \end{bmatrix}$ be a vector field on \mathbb{R}^3 , where each of P, Q, R is a scalar function on \mathbb{R}^3 . Then

$$\begin{aligned} \text{grad } f &= \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix}, \\ \text{curl} \begin{bmatrix} P \\ Q \\ R \end{bmatrix} &= \begin{bmatrix} R_y - Q_z \\ P_z - R_x \\ Q_x - P_y \end{bmatrix}, \\ \text{div} \begin{bmatrix} P \\ Q \\ R \end{bmatrix} &= P_x + Q_y + R_z. \end{aligned} \quad (2.30)$$

Then one has the following results.

Proposition 2.5

$$\text{curl}(\text{grad } f) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \quad (2.31)$$

Proposition 2.6

$$\operatorname{div} \left(\operatorname{curl} \begin{bmatrix} P \\ Q \\ R \end{bmatrix} \right) = 0. \quad (2.32)$$

Proposition 2.7

On \mathbb{R}^3 , a vector field $\mathbf{F}(x, y, z)$ is the gradient of some scalar function if and only if $\operatorname{curl} \mathbf{F} = \mathbf{0}$.

A 1-form on \mathbb{R}^3 can be written as

$$P(x, y, z) dx + Q(x, y, z) dy + R(x, y, z) dz.$$

This 1-form on \mathbb{R}^3 can be identified with the vector field $\begin{bmatrix} P \\ Q \\ R \end{bmatrix}$.

Similarly, the 2-forms on \mathbb{R}^3 given by

$$A(x, y, z) dy \wedge dz + B(x, y, z) dz \wedge dx + C(x, y, z) dx \wedge dy$$

can be identified with the vector field $\begin{bmatrix} A \\ B \\ C \end{bmatrix}$ on \mathbb{R}^3 .

In terms of these identifications, the exterior derivative of a 0-form f (scalar function) is

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz,$$

which can be identified with the vector field

$$\begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \operatorname{grad} f.$$

The exterior derivative of a 1-form on \mathbb{R}^3 is

$$\begin{aligned} & d(Pdx + Qdy + Rdz) \\ &= \frac{\partial P}{\partial y} dy \wedge dx + \frac{\partial P}{\partial z} dz \wedge dx + \frac{\partial Q}{\partial x} dx \wedge dy + \frac{\partial Q}{\partial z} dz \wedge dy + \frac{\partial R}{\partial x} dx \wedge dz + \frac{\partial R}{\partial y} dy \wedge dz \\ &= (R_y - Q_z) dy \wedge dz + (P_z - R_x) dz \wedge dx + (Q_x - P_y) dx \wedge dy, \end{aligned}$$

which corresponds to

$$\begin{bmatrix} R_y - Q_z \\ P_z - R_x \\ Q_x - P_y \end{bmatrix} = \operatorname{curl} \begin{bmatrix} P \\ Q \\ R \end{bmatrix}.$$

The exterior derivative of a 2-form is

$$\begin{aligned} & d(A dy \wedge dz + B dz \wedge dx + C dx \wedge dy) \\ &= A_x dx \wedge dy \wedge dz + B_y dy \wedge dz \wedge dx + C_z dz \wedge dx \wedge dy \\ &= (A_x + B_y + C_z) dx \wedge dy \wedge dz, \end{aligned}$$

which corresponds to

$$A_x + B_y + C_z = \operatorname{div} \begin{bmatrix} A \\ B \\ C \end{bmatrix}.$$

In summary, exterior derivative d on 0-forms is identified with **gradient**; exterior derivative d on 1-forms is identified with **curl**; exterior derivative d on 2-forms is identified with **divergence**. Using de Rham complex on U :

$$\Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \Omega^2(U) \xrightarrow{d} \Omega^3(U).$$

Using vector calculus language,

$$C^\infty(U) \xrightarrow{\text{grad}} \mathfrak{X}(U) \xrightarrow{\text{curl}} \mathfrak{X}(U) \xrightarrow{\text{div}} C^\infty(U).$$

Remark 2.1. Proposition 2.5 and Proposition 2.6 express the property $d^2 = 0$ of exterior deriva-

tive. A vector field $\begin{bmatrix} P \\ Q \\ R \end{bmatrix}$ is the gradient of a C^∞ function f if and only if the corresponding 1-form $Pdx + Qdy + Rdz$ is df . Proposition 2.7 expresses the fact that a 1-form on \mathbb{R}^3 is exact if and only if it is closed. It's worth remarking at this stage that Proposition 2.7 need not hold true on a region other than \mathbb{R}^3 , as the following well-known example from calculus suggests.

Example 2.7. Suppose $U = \mathbb{R}^3 \setminus \{z\text{-axis}\}$, and $\mathbf{F}(x, y, z)$ is the vector field

$$\mathbf{F} = \begin{bmatrix} \frac{-y}{x^2+y^2} \\ \frac{x}{x^2+y^2} \\ 0 \end{bmatrix}$$

on U . Then $\text{curl } \mathbf{F} = \mathbf{0}$. Indeed,

$$\begin{aligned} \text{curl } \mathbf{F} &= \begin{bmatrix} \frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}\left(\frac{x}{x^2+y^2}\right) \\ \frac{\partial}{\partial z}\left(\frac{-y}{x^2+y^2}\right) - \frac{\partial}{\partial x}(0) \\ \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ \frac{(x^2+y^2)-x \cdot 2x}{(x^2+y^2)^2} - \frac{-(x^2+y^2)+y \cdot 2y}{(x^2+y^2)^2} \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0}. \end{aligned}$$

But \mathbf{F} is not the gradient of a C^∞ function on U . Recall the theorem from vector calculus that the line integral of the gradient of a function along a curve gives the total change in the value of the function from the start to the end of the curve. In other words, if $\mathbf{r} : [a, b] \rightarrow \mathbb{R}^3$ is a curve and $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ is a scalar function, then

$$\int_a^b (\nabla f) \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)). \quad (2.33)$$

Then if \mathbf{F} is the gradient of a smooth scalar function, then the line integral

$$\oint_C \frac{-y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy$$

over any closed curve would become 0. Let us take the closed curve to be the unit circle: $x = \cos t$,

$y = \sin t$, $z = 0$ for $t \in [0, 2\pi]$. Then

$$\begin{aligned} & \oint_C \frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy \\ &= \int_0^{2\pi} -\sin t \, d(\cos t) + \int_0^{2\pi} \cos t \, d(\sin t) \\ &= \int_0^{2\pi} \sin^2 t \, dt + \int_0^{2\pi} \cos^2 t \, dt \\ &= 2\pi. \end{aligned}$$

Hence, although $\text{curl } \mathbf{F} = \mathbf{0}$, there is no C^∞ function f on U such that $\mathbf{F} = \text{grad } f$. In the language of differential forms, the 1-form

$$\omega = \frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy$$

is closed but not exact.

It turns out that whether [Proposition 2.7](#) is true for a region $U \subseteq \mathbb{R}^3$ depends on the topology of U . One measure of the failure of a closed k -form to be exact is the quotient vector space

$$H^k(U) = \frac{\{\text{closed } k\text{-forms on } U\}}{\{\text{exact } k\text{-forms on } U\}},$$

called the k -th de Rham cohomology of U . The generalization of [Proposition 2.7](#) to any differential form on \mathbb{R}^n is called the **Poincare lemma**:

For $k \geq 1$, every closed k -form on \mathbb{R}^n is exact.

This statement is equivalent to the vanishing of the k -th de Rham cohomology $H^k(\mathbb{R}^n)$ for $k \geq 1$.

3 Differential Forms on Manifold

§3.1 Definition and Local Expression

Let M be a smooth manifold and $p \in M$. The **cotangent space** of M at p , denoted by T_p^*M is the dual space of the tangent space T_pM . An element in T_p^*M is called a covector at p . Thus, a covector $\omega_p \in T_p^*M$ is a linear function

$$\omega_p : T_pM \rightarrow \mathbb{R}.$$

A 1-form on M is a function that assigns to each $p \in M$, a covector at p .

Definition 3.1 (Differential of a function). Let $f : M \rightarrow \mathbb{R}$ be a C^∞ function on a manifold M . Its **differential** is defined to be the 1-form df on M such that for any $p \in M$ and $X_p \in T_pM$,

$$(df)_p(X_p) = X_p f. \quad (3.1)$$

Proposition 3.1

If $f : M \rightarrow \mathbb{R}$ is a C^∞ function, then for $p \in M$ and $X_p \in T_pM$,

$$f_{*,p}(X_p) = (df)_p(X_p) \left. \frac{\partial}{\partial x} \right|_{f(p)}.$$

Proof. Since $f_{*,p}(X_p) \in T_{f(p)}\mathbb{R}$, there is a real number c such that

$$f_{*,p}(X_p) = c \left. \frac{\partial}{\partial x} \right|_{f(p)}. \quad (3.2)$$

(Here the chart chosen on \mathbb{R} is $(\mathbb{R}, \mathbb{1}_{\mathbb{R}})$ so that x is the coordinate of this chart, i.e. $x = \mathbb{1}_{\mathbb{R}}$.) To evaluate c , apply both sides of (3.2) to the function $x \in C^\infty(\mathbb{R})$. Then

$$f_{*,p}(X_p)(x) = c \left. \frac{\partial}{\partial x} \right|_{f(p)}(x) = c.$$

Therefore,

$$c = f_{*,p}(X_p)(x) = X_p(x \circ f) = X_p f = (df)_p(X_p), \quad (3.3)$$

since $x = \mathbb{1}_{\mathbb{R}}$. Therefore, substituting the value of c into (3.2),

$$f_{*,p}(X_p) = (df)_p(X_p) \left. \frac{\partial}{\partial x} \right|_{f(p)}. \quad (3.4)$$

■

Let $(U, \varphi) \equiv (U, x^1, x^2, \dots, x^n)$ be a coordinate chart on M . Here $x^i = r^i \circ \varphi$, where r^i is the i -th coordinate function of a vector in \mathbb{R}^n . Then the differentials dx^1, dx^2, \dots, dx^n are 1-forms on U .

Proposition 3.2

At each point $p \in U$, the covectors $(dx^1)_p, \dots, (dx^n)_p$ form a basis for the cotangent space T_p^*M , dual to the basis $\left\{ \left. \frac{\partial}{\partial x^1} \right|_p, \dots, \left. \frac{\partial}{\partial x^n} \right|_p \right\}$ for the tangent space T_pM .

Proof. Observe that

$$\left(dx^i\right)_p \left(\frac{\partial}{\partial x^j}\Big|_p\right) = \frac{\partial}{\partial x^j}\Big|_p \left(x^i\right) = \delta^i_j. \quad (3.5)$$

So $\{(dx^1)_p, \dots, (dx^n)_p\}$ is the dual basis to $\left\{\frac{\partial}{\partial x^1}\Big|_p, \dots, \frac{\partial}{\partial x^n}\Big|_p\right\}$. ■

Thus, every 1-form ω on U can be written as a linear combination

$$\omega = \sum_{i=1}^n a_i dx^i,$$

where a_i are functions on U . In particular, if f is a C^∞ function on M , then the 1-form df , when restricted to U , must be a linear combination

$$df = \sum_{i=1}^n a_i dx^i. \quad (3.6)$$

If we evaluate both sides of (3.6) on $\frac{\partial}{\partial x^j}$,

$$(df) \left(\frac{\partial}{\partial x^j}\right) = \sum_{i=1}^n a_i dx^i \left(\frac{\partial}{\partial x^j}\right) = \sum_{i=1}^n a_i \delta^i_j = a_j.$$

Then

$$a_j = (df) \left(\frac{\partial}{\partial x^j}\right) = \frac{\partial f}{\partial x^j}.$$

Therefore,

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i. \quad (3.7)$$

§3.2 The Cotangent Bundle

The underlying set of the **cotangent bundle** is the disjoint union of the cotangent spaces at all points of M :

$$T^*M = \bigsqcup_{p \in M} T_p^*M = \bigcup_{p \in M} \{p\} \times T_p^*M. \quad (3.8)$$

Let us give T^*M a topology in the following way: let (U, x^1, \dots, x^n) be a chart on M and $p \in U$. Then each $\omega_p \in T_p^*M$ can be written uniquely as a linear combination

$$\omega_p = \sum_{i=1}^n c_i(\omega_p) (dx^i)_p,$$

with $c_i(\omega_p) \in \mathbb{R}$. This gives rise to a bijection

$$\begin{aligned} \tilde{\varphi} : T^*U &\rightarrow \varphi(U) \times \mathbb{R}^n \\ (p, \omega_p) &\mapsto (\varphi(p), c_1(\omega_p), c_2(\omega_p), \dots, c_n(\omega_p)). \end{aligned}$$

We use this bijection $\tilde{\varphi}$ to transfer the topology of $\varphi(U) \times \mathbb{R}^n$ to T^*U : a set $A \subseteq T^*U$ is said to be open if and only if $\tilde{\varphi}(A)$ is open in $\varphi(U) \times \mathbb{R}^n$, where $\varphi(U) \times \mathbb{R}^n$ is given the subspace topology of \mathbb{R}^{2n} . Now, let $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ be the maximal atlas of M . Now, let

$$\begin{aligned} \mathcal{B} &= \bigcup_{\alpha \in I} \{A \mid A \subseteq T^*U_\alpha \text{ is open in } A \subseteq T^*U_\alpha\} \\ &= \{A \mid A \subseteq T^*U_\alpha \text{ is open in } A \subseteq T^*U_\alpha, \alpha \in I\}. \end{aligned}$$

It can be shown using the same technique of tangent bundle that \mathcal{B} forms a basis for topology. We give T^*M the topology generated by the basis \mathcal{B} . We declare $A \subseteq T^*M$ to be open if and only if there exists a subfamily $\{B_\lambda\}_\lambda \subseteq \mathcal{B}$ such that

$$A = \bigcup_\lambda B_\lambda.$$

Furthermore, T^*M has the structure of a C^∞ manifold. An atlas for T^*M is

$$\{(T^*U_\alpha, \tilde{\varphi}_\alpha)\}_{\alpha \in I}.$$

If two coordinate open sets U_α and U_β intersect, suppose $U_{\alpha\beta} = U_\alpha \cap U_\beta$. Then for $p \in U_{\alpha\beta}$, each $\omega_p \in T_p^*M$ has two basis expansions:

$$\omega_p = \sum_{i=1}^n a_i (dx^i)_p = \sum_{j=1}^n b_j (dy^j)_p. \quad (3.9)$$

(Here $(U_\alpha, x^1, \dots, x^n)$ and $(U_\beta, y^1, \dots, y^n)$ are charts.) Now applying $\frac{\partial}{\partial y^k} \Big|_p$ to both sides of (3.9),

$$b_k = \sum_{i=1}^n a_i (dx^i)_p \left(\frac{\partial}{\partial y^k} \Big|_p \right) = \sum_{i=1}^n a_i \frac{\partial x^i}{\partial y^k} \Big|_p.$$

Therefore, $\tilde{\varphi}_\beta \circ \tilde{\varphi}_\alpha^{-1} : \varphi_\alpha(U_{\alpha\beta}) \times \mathbb{R}^n \rightarrow \varphi_\beta(U_{\alpha\beta}) \times \mathbb{R}^n$ is given by

$$(\varphi_\alpha(p), a_1, \dots, a_n) \mapsto \left((\varphi_\beta \circ \varphi_\alpha^{-1})(\varphi_\alpha(p)), \sum_{i=1}^n a_i \frac{\partial x^i}{\partial y^1} \Big|_p, \dots, \sum_{i=1}^n a_i \frac{\partial x^i}{\partial y^n} \Big|_p \right).$$

$\varphi_\beta \circ \varphi_\alpha^{-1}$ is smooth, and each $\frac{\partial x^i}{\partial y^j}$ is smooth. Therefore, the transition map $\tilde{\varphi}_\beta \circ \tilde{\varphi}_\alpha^{-1}$ is smooth, making T^*M a smooth manifold.

T^*M is, in fact, a **vector bundle** of rank n over M . It has a natural projection $\pi : T^*M \rightarrow M$ given by $(p, \omega_p) \mapsto p$. In terms of cotangent bundle, a 1-form on M is simply a section of the cotangent bundle T^*M , i.e. it is a map $\omega : M \rightarrow T^*M$ such that $\pi \circ \omega = \mathbb{1}_M$. We say that a 1-form is **smooth** if it is C^∞ as a map $\omega : M \rightarrow T^*M$ between two manifolds.

§3.3 Characterization of Smooth 1-forms

By definition, a 1-form on an open set $U \subseteq M$ is C^∞ if it is C^∞ as a section of the cotangent bundle T^*M over U .

Lemma 3.3

Let $(U, \varphi) \equiv (U, x^1, \dots, x^n)$ be a chart on a manifold M . A 1-form $\omega = \sum a_i dx^i$ on U is smooth if and only if the coefficient functions a_i are all smooth on U .

Proof. This is a special case of Proposition 9.4.2 of DG1 which states that:

Let $\pi : E \rightarrow M$ be a C^∞ vector bundle and U an open subset of M . Suppose s_1, \dots, s_r is a C^∞ frame for E over U . Then a section $s = \sum_{j=1}^r c^j s_j$ of E over U is C^∞ if and only if the coefficients c^j are C^∞ functions on U .

Here we take E to be the cotangent bundle T^*M , and $\{s_i\}_{i=1}^r$ the C^∞ frame for E over U to be the coordinate 1-forms $\{(dx^i)\}_{i=1}^n$. ■

Proposition 3.4

Let ω be a 1-form on a manifold M . Then the following are equivalent:

- (i) ω is C^∞ .
- (ii) For every point $p \in M$, there is a chart (U, x^1, \dots, x^n) about p such that if $\omega = \sum_{i=1}^n a_i dx^i$ on U , then the functions a_i are C^∞ on U .
- (iii) For any chart (U, x^1, \dots, x^n) on M , if $\omega = \sum_{i=1}^n a_i dx^i$ on U , then the functions a_i are C^∞ on U .

Proof. (ii) \Rightarrow (i): By Lemma 3.3, for every point $p \in M$, there is a chart (U, x^1, \dots, x^n) about p such that ω is smooth on U . In particular, the section $\omega : M \rightarrow T^*M$ is smooth at p , for every $p \in M$. Therefore, $\omega : M \rightarrow T^*M$ is a smooth map between manifolds.

(i) \Rightarrow (iii): If $\omega : M \rightarrow T^*M$ is a smooth map between manifolds, ω is smooth at every chart of M . Therefore, by Lemma 3.3, if $\omega = \sum_{i=1}^n a_i dx^i$ on a chart (U, x^1, \dots, x^n) , each a_i is smooth on U .

(iii) \Rightarrow (ii): Obvious. ■

Proposition 3.5

A 1-form ω on a manifold M is C^∞ if and only if for every C^∞ vector field X , the function $\omega(X)$ is C^∞ on M .

Proof. (\Rightarrow) Suppose ω is a C^∞ 1-form and X is a C^∞ vector field. Let (U, x^1, \dots, x^n) be a chart on M . Then

$$\omega = \sum_{i=1}^n a_i dx^i \quad \text{and} \quad X = \sum_{j=1}^n b^j \frac{\partial}{\partial x^j}, \quad (3.10)$$

for C^∞ functions a_i and b^j on U . Then on U , one has

$$\omega(X) = \left(\sum_{i=1}^n a_i dx^i \right) \left(\sum_{j=1}^n b^j \frac{\partial}{\partial x^j} \right) = \sum_{i=1}^n \sum_{j=1}^n a_i b^j \delta^i_j = \sum_{i=1}^n a_i b^i, \quad (3.11)$$

which is a C^∞ function on U . Since U was chosen to be an arbitrary coordinate open set, $\omega(X)$ is a smooth function on all of M .

(\Leftarrow) Suppose ω is a 1-form on M such that for every C^∞ vector field X on M , the function $\omega(X)$ is smooth on M . For a given $p \in M$, choose a coordinate neighborhood $(U, \varphi) \equiv (U, x^1, \dots, x^n)$ about p . Then one has

$$\omega = \sum_{i=1}^n a_i dx^i$$

on U . Now fix an integer $j \in \{1, 2, \dots, n\}$. We can extend the C^∞ vector field $\frac{\partial}{\partial x^j}$ on U to a C^∞ vector field X on the whole of M that agrees with $\frac{\partial}{\partial x^j}$ in a neighborhood V of p (not necessarily the whole of U , but possibly a smaller neighborhood) contained in U (Proposition 11.1.4 of DG1). The extended vector field is defined in the following way: let $\sigma : M \rightarrow \mathbb{R}$ be a C^∞ bump function which is identically 1 on a neighborhood V of p and which has support contained in U . Now, define the vector field $q \mapsto X_q \in T_q M$, denoted by X , in terms of the bump function σ in the following way:

$$X_q = \begin{cases} \sigma(q) \frac{\partial}{\partial x^j} \Big|_q & \text{if } q \in U, \\ \mathbf{0} & \text{if } q \notin U. \end{cases} \quad (3.12)$$

The vector field X is smooth in the whole of M , as proved in Proposition 11.1.4 of DG1. Now, by the hypothesis, $\omega(X)$ is C^∞ on M . In particular, $\omega(X)$ is smooth on V . Therefore,

$$\omega(X) = \left(\sum_{i=1}^n a_i dx^i \right) \left(\frac{\partial}{\partial x^j} \right) = \sum_{i=1}^n a_i \delta^i_j = a_j$$

is smooth on V . We, therefore, see that the coefficient functions a_i 's appearing in $\omega = \sum_{i=1}^n a_i dx^i$ are smooth on $V \subseteq U$. It means that for a given point p , we can find a chart $(V, \tilde{x}^1, \dots, \tilde{x}^n)$, where

$$\tilde{x}^i = r^i \circ \varphi|_V,$$

such that $\omega = \sum_{i=1}^n a_i|_V d\tilde{x}^i$ on V , with each $a_i|_V$ smooth on V . Therefore, by [Proposition 3.4](#), ω is C^∞ . ■

§3.4 Pullback of 1-forms

Recall that the differential of a smooth map $F : N \rightarrow M$ at $p \in N$ is a linear map $F_{*,p} : T_p N \rightarrow T_{F(p)} M$ defined by

$$[F_{*,p}(X_p)](f) = X_p(f \circ F), \quad (3.13)$$

where $f \in C_{F(p)}^\infty(M)$. Indeed, $f \circ F \in C_p^\infty(N)$. Analogously, the **codifferential** (the dual of a differential) at $F(p) \in M$ is a linear map

$$F^{*,p} : T_{F(p)}^* M \rightarrow T_p^* N.$$

One observes that the differential $F_{*,p}$ pushes forward a tangent vector at $p \in N$ while the codifferential $F^{*,p}$ pulls back a covector from $T_{F(p)}^* M$ at $F(p) \in M$ to $T_p^* N$.

Remark 3.1. Note that a vector field, in general, cannot be pushed forward under a smooth map $F : N \rightarrow M$. Suppose $F : N \rightarrow M$ is a smooth map of manifolds. Also suppose $F(p) = F(q) = z \in M$ so that F is not injective. Now, the differentials

$$F_{*,p} : T_p N \rightarrow T_z M \text{ and } F_{*,q} : T_q N \rightarrow T_z M$$

are linear maps. Now, let $X \in \mathfrak{X}(N)$ be a C^∞ vector field on N so that X_p under $F_{*,p}$ is pushed forward to $F_{*,p}(X_p) \in T_z M$ and X_q is pushed forward to $F_{*,q}(X_q) \in T_z M$ under $F_{*,q}$. There is no reason for $F_{*,p}(X_p)$ and $F_{*,q}(X_q)$ to be the same tangent vector in $T_z M$. In other words, in general,

$$F_{*,p}(X_p) \neq F_{*,q}(X_q),$$

so that $z \mapsto F_{*,p}(X_p) := Y_z \in T_z M$ and $z \mapsto F_{*,q}(X_q) := Y'_z \in T_z M$ are distinct vector fields on M , denoted by Y and Y' , respectively. Therefore, if there were push forward of vector fields $F_* : \mathfrak{X}(N) \rightarrow \mathfrak{X}(M)$ associated with the non-injective smooth map $F : N \rightarrow M$, there is an ambiguity regarding which vector field X gets mapped to.

Furthermore, if F is not surjective, there is $z \in M$ such that $z \neq F(p)$ for any $p \in N$. In that case as well, defining the push forward vector field $F_*(X)$ at the point z is impossible. However, when $F : N \rightarrow M$ is a diffeomorphism, one can define the push forward of a vector field.

Contrary to the non-existence of push forward of a vector field associated with a generic smooth map $F : N \rightarrow M$, one can always talk about pullback of a 1-form ω on M :

$$(F^*\omega)_p(X_p) = \omega_{F(p)}(F_{*,p}(X_p)). \quad (3.14)$$

Here, $\omega \in \Omega^1(M)$, $X_p \in T_p N$, $p \in N$. Note that $(F^*\omega)_p$ is simply the image of the covector $\omega_{F(p)} \in T_{F(p)}^* M$ under the codifferential $F^{*,p} : T_{F(p)}^* M \rightarrow T_p^* N$. In other words,

$$(F^*\omega)_p = F^{*,p}(\omega_{F(p)}). \quad (3.15)$$

4 Differential k -forms

§4.1 Definition and Local Expression

We denoted by $A_k(V)$ the vector space of alternating k -tensors on V . We have also seen that if $\{\hat{\alpha}^1, \dots, \hat{\alpha}^n\}$ is a basis for 1-tensors on V , then a basis element of $A_k(V)$ is

$$\hat{\alpha}^{i_1} \wedge \hat{\alpha}^{i_2} \wedge \dots \wedge \hat{\alpha}^{i_k},$$

where $1 \leq i_1 < i_2 < \dots < i_k \leq n$. We apply this construction to the tangent space $T_p M$ of a manifold M at a point $p \in M$. The vector space $A_k(T_p M)$, usually denoted by $\Lambda^k(T_p^* M)$, is the space of all alternating k -tensors on the tangent space $T_p M$.

Definition 4.1 (Differential k -form). A **differential k -form** on a manifold M is a function ω that assigns to each point $p \in M$, a k -covector $\omega_p \in \Lambda^k(T_p^* M)$. An n -form on a manifold of dimension n is called a **top degree form**.

Example 4.1. On \mathbb{R}^n , at each point p , there is a standard basis for the tangent space $T_p \mathbb{R}^n$:

$$\left\{ \frac{\partial}{\partial r^1} \Big|_p, \frac{\partial}{\partial r^2} \Big|_p, \dots, \frac{\partial}{\partial r^n} \Big|_p \right\}.$$

Let $\{(dr^1)_p, \dots, (dr^n)_p\}$ be the dual basis of $T_p^* \mathbb{R}^n$.

$$(dr^i)_p \left(\frac{\partial}{\partial r^j} \Big|_p \right) = \delta^i_j.$$

As p varies over \mathbb{R}^n , we get differential forms dr^1, \dots, dr^n on \mathbb{R}^n . By [Proposition 1.15](#), a basis element of alternating k -tensors $\Lambda^k(T_p^* \mathbb{R}^n)$ is

$$(dr^{i_1})_p \wedge (dr^{i_2})_p \wedge \dots \wedge (dr^{i_k})_p,$$

where $1 \leq i_1 < i_2 < \dots < i_k \leq n$. If ω is a k -form on \mathbb{R}^n , then at each point $p \in \mathbb{R}^n$, ω_p is the following linear combination:

$$\omega_p = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} a_{i_1 \dots i_k} (dr^{i_1})_p \wedge (dr^{i_2})_p \wedge \dots \wedge (dr^{i_k})_p. \quad (4.1)$$

Omitting the point p , we write

$$\omega = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} a_{i_1 \dots i_k} dr^{i_1} \wedge dr^{i_2} \wedge \dots \wedge dr^{i_k}. \quad (4.2)$$

In the expression above, $a_{i_1 \dots i_k}$ are functions on \mathbb{R}^n . To simplify the notations, we use multi-indices to write (4.2) as

$$\omega = \sum_I a_I dr^I, \quad (4.3)$$

where $dr^I = dr^{i_1} \wedge dr^{i_2} \wedge \dots \wedge dr^{i_k}$, and $I = (i_1, i_2, \dots, i_k)$ is a strictly ascending multi-index.

Suppose (U, x^1, \dots, x^n) is a coordinate chart on a manifold M . We have already defined the 1-forms dx^1, \dots, dx^n on U . Since at each point $p \in U$, $\{(dx^1)_p, \dots, (dx^n)_p\}$ is a basis for T_p^*M , by [Proposition 1.15](#), a basis for $\Lambda^k(T_p^*\mathbb{R}^n)$ is

$$(dx^{i_1})_p \wedge (dx^{i_2})_p \wedge \dots \wedge (dx^{i_k})_p,$$

where $1 \leq i_1 < i_2 < \dots < i_k \leq n$. Thus, locally a k -form on U will be a linear combination

$$\omega = \sum_I a_I dx^I, \quad (4.4)$$

where $dx^I = dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_k}$, $I = (i_1, i_2, \dots, i_k)$ is a strictly ascending multi-index, and a_I are functions on U .

§4.2 The Bundle Point of View

Let V be a real vector space. Another common notation for the vector space $A_k(V)$ of alternating k -linear functionos on V is $\Lambda^k(V^*)$.

$$\begin{aligned} \Lambda^0(V^*) &= A_0(V) = \mathbb{R}, \\ \Lambda^1(V^*) &= A_1(V) = V^*, \\ \Lambda^2(V^*) &= A_2(V), \end{aligned}$$

and so on. Now, $\Lambda^k(T^*M)$ is defined to be the disjoint union of the vector spaces $\Lambda^k(T_p^*M)$ as p varies over M . So

$$\begin{aligned} \Lambda^k(T^*M) &= \bigsqcup_{p \in M} \Lambda^k(T_p^*M) = \bigsqcup_{p \in M} A_k(T_pM) \\ &= \bigcup_{p \in M} \{p\} \times A_k(T_pM), \end{aligned} \quad (4.5)$$

which is the set of all alternating k -tensors at all points of M . This set is called the k -th **exterior power** of the cotangent bundle T^*M .

If (U, φ) is a coordinate chart on M , then there is a bijection $\bar{\varphi}: \Lambda^k(T^*U) \rightarrow \varphi(U) \times \mathbb{R}^{\binom{n}{k}}$ defined as follows: a generic element of $\Lambda^k(T^*U)$ is (p, ω_p) , where $\omega_p \in \Lambda^k(T_p^*U)$. Then ω_p is a unique linear combination

$$\omega_p = \sum_I a_I(p) dx^I,$$

where I runs over the set of strictly ascending multi-indices of length k . There are $\binom{n}{k}$ many such multi-indices. If we fix a labeling of the multi-indices once and for all, then we have a $\binom{n}{k}$ -tuple $(a_I)_I$. Then we define

$$\bar{\varphi}(p, \omega_p) = (\varphi(p), (a_I)_I) \in \varphi(U) \times \mathbb{R}^{\binom{n}{k}}.$$

Thus, $\Lambda^k(T^*U)$ is in a bijective correspondence with $\varphi(U) \times \mathbb{R}^{\binom{n}{k}}$. Using this bijective correspondence, one transfers the topology of $\varphi(U) \times \mathbb{R}^{\binom{n}{k}}$ to $\Lambda^k(T^*U)$. By varying the open set U in the charts contained in the maximal atlas of M , one can obtain a basis that generates the topology on the whole of $\Lambda^k(T^*M)$.

$\Lambda^k(T^*M)$ can, in fact, be shown to be a C^∞ vector bundle of rank $\binom{n}{k}$ over M , i.e. $\pi: \Lambda^k(T^*U) \rightarrow M$ is a C^∞ vector bundle of rank $\binom{n}{k}$ over M . A differential k -form is a section of this vector bundle. We define a k -form to be C^∞ if it is C^∞ as a section of the vector bundle $\Lambda^k(T^*M)$.

Notation. If $\pi: E \rightarrow M$ is a C^∞ vector bundle, then the vector space of C^∞ sections of E is denoted by $\Gamma(E)$, or $\Gamma(M, E)$. The vector space of all C^∞ k -forms, i.e. all C^∞ sections of the bundle $\Lambda^k(T^*M)$ is usually denoted by $\Omega^k(M)$. Thus,

$$\Omega^k(M) = \Gamma(\Lambda^k(T^*M)) = \Gamma(M, \Lambda^k(T^*M)).$$

Lemma 4.1

Let (U, x^1, \dots, x^n) be a chart on a manifold M . A k -form $\omega = \sum a_I dx^I$ on U is smooth if and only if the coefficient functions a_I are all smooth on U .

Proposition 4.2 (Characterization of a smooth k -form)

Let ω be a k -form on a manifold M . The following are equivalent:

- (i) The k -form ω is C^∞ on M .
- (ii) The manifold M has an atlas such that on every chart $(U, \phi) = (U, x^1, \dots, x^n)$ in the atlas, the coefficients a_I of $\omega = \sum a_I dx^I$ relative to the coordinate frame $\{dx^I\}_I$ are all C^∞ .
- (iii) On every chart $(U, \phi) = (U, x^1, \dots, x^n)$ on M , the coefficients a_I of $\omega = \sum a_I dx^I$ relative to the coordinate frame $\{dx^I\}_I$ are all C^∞ .
- (iv) For any k smooth vector fields X_1, \dots, X_k on M , the function $\omega(X_1, \dots, X_k)$ is C^∞ on M .

Example 4.2. We defined the 0-tensors and the 0-covectors as constants, i.e. for a real vector space V , $A_0(V) = L_0(V) = \mathbb{R}$. Now, recall that

$$\Lambda^k(T^*M) = \bigcup_{p \in M} \{p\} \times \Lambda^k(T_p^*M).$$

Since $\Lambda^0(T_p^*M) = \mathbb{R}$ for every $p \in M$, one has

$$\Lambda^0(T^*M) = \bigcup_{p \in M} \{p\} \times \mathbb{R} = M \times \mathbb{R}. \quad (4.6)$$

Hence,

$$\Omega^0(M) = \Gamma(\Lambda^0(T^*M)) = \Gamma(M, M \times \mathbb{R}). \quad (4.7)$$

A C^∞ section of the 0-th exterior power of the tangent bundle T^*M is nothing but a C^∞ section of the globally trivial C^∞ vector bundle $M \times \mathbb{R}$ over M . Such a section maps $p \in M$ to a pair $(p, \sigma(p))$ with $\sigma(p) \in \mathbb{R}$. Therefore, such a section is nothing but a smooth assignment $p \mapsto \sigma(p)$, i.e. $\sigma \in C^\infty(M, \mathbb{R})$. So

$$\Omega^0(M) = \Gamma(M, M \times \mathbb{R}) = C^\infty(M, \mathbb{R}).$$

§4.3 Pullback of k -forms

Let $F : N \rightarrow M$ be a smooth map of manifolds. Recall that a 1-form $\omega \in \Omega^1(M)$ can be pulled back to $\Omega^1(N)$ via the pullback $F^* : \Omega^1(M) \rightarrow \Omega^1(N)$ defined by

$$(F^*\omega)_p(X_p) = \omega_{F(p)}(F_{*,p}(X_p)). \quad (4.8)$$

For 0-forms, i.e. functions, the pullback is defined by composition:

$$N \xrightarrow{F} M \xrightarrow{f} \mathbb{R}$$

Given $f \in C^\infty(M, \mathbb{R})$, its pullback is defined to be

$$F^*(f) = f \circ F \in C^\infty(N, \mathbb{R}), \quad (4.9)$$

so that indeed $F^* : \Omega^0(M) \rightarrow \Omega^0(N)$.

For a k -form ω on M , we define its pullback $F^*\omega$ as follows: if $p \in N$ and $X_p^1, X_p^2, \dots, X_p^k \in T_p N$ are k tangent vectors, then

$$(F^*\omega)_p(X_p^1, X_p^2, \dots, X_p^k) = \omega_{F(p)}(F_{*,p}(X_p^1), F_{*,p}(X_p^2), \dots, F_{*,p}(X_p^k)). \quad (4.10)$$

Proposition 4.3 (Linearity of pullback)

Let $F : N \rightarrow M$ be a C^∞ map. If ω, τ are k -forms on M and α is a real number, then

- (i) $F^*(\omega + \tau) = F^*\omega + F^*\tau.$
- (ii) $F^*(\alpha\omega) = \alpha F^*\omega.$

§4.4 The Wedge Product

If $\omega \in \Omega^k(M)$ and $\tau \in \Omega^l(M)$, then for any $p \in M$, $\omega_p \in \Lambda^k(T_p^*M)$ and $\tau_p \in \Lambda^l(T_p^*M)$ and $\omega_p \wedge \tau_p \in \Lambda^{k+l}(T_p^*M)$. Then we define the wedge product of ω and τ to be the $(k+l)$ -form $\omega \wedge \tau$ such that

$$(\omega \wedge \tau)_p = \omega_p \wedge \tau_p. \quad (4.11)$$

Proposition 4.4

If ω and τ are C^∞ forms on M , then so is $\omega \wedge \tau$.

Proof. Let (U, x^1, \dots, x^n) be a chart on M . On U ,

$$\omega = \sum_I a_I dx^I, \quad \tau = \sum_J b_J dx^J \quad (4.12)$$

for C^∞ functions a_I, b_J on U . Their Wedge product is

$$\begin{aligned} \omega \wedge \tau &= \left(\sum_I a_I dx^I \right) \wedge \left(\sum_J b_J dx^J \right) \\ &= \sum_{I,J} a_I b_J dx^I \wedge dx^J \dots \end{aligned} \quad (4.13)$$

In (4.13), $dx^I \wedge dx^J = 0$ if I and J have at least an index in common. If I and J are disjoint, i.e., have none of their indices to be common, then

$$dx^I \wedge dx^J = \pm dx^K, \quad (4.14)$$

where $K = I \cup J$ but reordered as an increasing sequence. Thus,

$$\omega \wedge \tau = \sum_K \left(\sum_{I \cup J = K} \pm a_I b_J \right) dx^K. \quad (4.15)$$

Since the coefficients of dx^K in (4.15) are C^∞ , by Proposition 4.2, $\omega \wedge \tau$ is C^∞ on M . ■

Proposition 4.5 (Pullback of wedge product)

If $F : N \rightarrow M$ is a C^∞ map of manifolds and ω, τ are differential forms on M , then

$$F^*(\omega \wedge \tau) = F^*(\omega) \wedge F^*(\tau). \quad (4.16)$$

We define the vector space $\Omega^*(M)$ of C^∞ differential forms on a manifold M of dimension n to be the direct sum

$$\Omega^*(M) = \bigoplus_{k=0}^n \Omega^k(M). \quad (4.17)$$

Each element of $\Omega^*(M)$ is uniquely a formal sum $\sum_{i=1}^r \omega_{k_i}$ with $\omega_{k_i} \in \Omega^{k_i}(M)$. With the wedge product, the vector space $\Omega^*(M)$ becomes a **graded algebra**, graded by the the degree of differential forms. [Proposition 4.3](#) and [Proposition 4.5](#) tells us that the pullback map $F^* : \Omega^*(M) \rightarrow \Omega^*(N)$ is a homomorphism of graded algebras¹.

¹Note that we haven't yet proved that F^* prerserves smoothness of forms, so we don't yet know that F^* maps $\Omega^k(M)$ into $\Omega^k(N)$. But we shall soon prove this in [Theorem 5.6](#), and once we do that we are all good with the notation.

5 Exterior Derivative

The basic objects in differential geometry are differential forms. Our goal will be to learn how we can differentiate and integrate differential forms on manifolds. Recall that an antiderivation on a graded algebra $A = \bigoplus_{k=0}^{\infty} A^k$ is an \mathbb{R} -linear map $D : A \rightarrow A$ such that

$$D(\omega \cdot \tau) = (D\omega) \cdot \tau + (-1)^k \omega \cdot (D\tau),$$

for $\omega \in A^k$ and $\tau \in A^l$, and \cdot is the multiplication of the graded algebra. In the graded algebra A , an element of A^k is called a **homogenous element of degree k** . The antiderivation D is of degree m if

$$\deg(D\omega) = \deg \omega + m$$

for all homogenous elements $\omega \in A$.

Now, let M be a manifold and $\Omega^*(M)$ the graded algebra of C^∞ differential forms on M . Now, we'll see that on the graded algebra $\Omega^*(M)$, there is a uniquely and intrinsically defined anti-derivation called exterior derivative.

Definition 5.1 (Exterior derivative). An **exterior derivative** on a manifold M is an \mathbb{R} -linear map

$$D : \Omega^*(M) \rightarrow \Omega^*(M)$$

such that

- (i) D is an antiderivation of degree 1,
- (ii) $D \circ D = 0$,
- (iii) if f is a C^∞ function and X is a C^∞ vector field on M , then $(Df)(X) = Xf$.

Remark 5.1. Condition (iii) in the definition above says that on 0-forms, i.e. C^∞ functions on M , an exterior derivative agrees with the differential df of a function f . We have learned earlier that in a coordinate chart (U, x^1, \dots, x^n) , the 1-form df can be expressed as

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i.$$

Hence, in the chart (U, x^1, \dots, x^n) ,

$$Df = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i.$$

We now prove the existence and uniqueness of the exterior differentiation on a manifold.

Lemma 5.1

Let $D : \Omega^*(M) \rightarrow \Omega^*(M)$ be an exterior derivative on M . If f^1, \dots, f^k are smooth functions on U , then

$$D(Df^1 \wedge Df^2 \wedge \dots \wedge Df^k) = 0.$$

Proof. We prove it by induction on k . The base case $k = 1$ follows trivially from $D \circ D = 0$. Suppose

$D(Df^1 \wedge Df^2 \wedge \cdots \wedge Df^{k-1}) = 0$. Then

$$\begin{aligned}
 & D(Df^1 \wedge Df^2 \wedge \cdots \wedge Df^k) \\
 &= D\left((Df^1 \wedge Df^2 \wedge \cdots \wedge Df^{k-1}) \wedge Df^k\right) \\
 &= D(Df^1 \wedge Df^2 \wedge \cdots \wedge Df^{k-1}) \wedge Df^k + (-1)^{k-1} (Df^1 \wedge Df^2 \wedge \cdots \wedge Df^{k-1}) \wedge D(Df^k) \\
 &= 0.
 \end{aligned} \tag{5.1}$$

Therefore, $D(Df^1 \wedge Df^2 \wedge \cdots \wedge Df^k) = 0$ for any $k \geq 1$. ■

§5.1 Exterior Derivative on a Coordinate Chart

Suppose (U, x^1, \dots, x^n) is a coordinate chart on a manifold M . Then any k -form ω on U is uniquely a linear combination

$$\omega = \sum_I a_I dx^I,$$

where $a_I \in C^\infty(U)$, and the sum runs over all strictly ascending multi-indices I of length k . The \mathbb{R} -linear map $d : \Omega^*(U) \rightarrow \Omega^*(U)$ can be constructed to be an exterior derivative on U . In fact, d acts on a homogenous element $\omega \in \Omega^k(U)$ in the following way:

$$\begin{aligned}
 d\omega &= d\left(\sum_I a_I dx^I\right) = \sum_I da_I \wedge dx^I + (-1)^0 \sum_I a_I ddx^I \\
 &= \sum_I da_I \wedge dx^I + \sum_I a_I d(dx^{i_1} \wedge \cdots \wedge dx^{i_k}) \\
 &= \sum_I da_I \wedge dx^I \\
 &= \sum_I \sum_j \frac{\partial a_I}{\partial x^j} dx^j \wedge dx^I.
 \end{aligned} \tag{5.2}$$

(5.2) suggests that $d\omega \in \Omega^{k+1}(U)$, and it can be written in the chart (U, x^1, \dots, x^n) using (5.2). This proves the existence of the exterior derivative $d : \Omega^*(U) \rightarrow \Omega^*(U)$, on an open set U of M . The uniqueness of $d : \Omega^*(U) \rightarrow \Omega^*(U)$ can be shown exactly the same way we proved it for the Euclidean case in Proposition 2.4.

Sometimes we write $d_U \omega$ instead of $d\omega$ to emphasize that it is the **unique** exterior derivative on the open set $U \subseteq M$. In other words, if (U, x^i) and (U, y^j) are two charts on M , and $\omega = \sum a_I dx^I = \sum b_J dy^J$, then

$$d_U \omega = \sum_I \sum_i \frac{\partial a_I}{\partial x^i} dx^i \wedge dx^I = \sum_J \sum_j \frac{\partial b_J}{\partial y^j} dy^j \wedge dy^J. \tag{5.3}$$

This reveals that the expression $d_U \omega$ is chart independent.

§5.2 Local Operators

An endomorphism of a vector space W (a linear transformation from W to itself) is often called an operator on W . For example, if $W = C^\infty(\mathbb{R})$, the vector space of C^∞ functions on \mathbb{R} , then $\frac{d}{dx}$ is an operator on W :

$$\frac{d}{dx} f(x) = f'(x).$$

The derivative has the desired property that the value of f' at a point p depends only on the values of f in a small neighborhood of p . More precisely, if $f = g$ on an open set $U \subseteq \mathbb{R}$, then $f' = g'$ on U . We say that the derivative is a local operator on $C^\infty(\mathbb{R})$.

Definition 5.2 (Local operator). An operator $D : \Omega^*(M) \rightarrow \Omega^*(M)$ is said to be **local** if for all $k \geq 0$, whenever a k -form $\omega \in \Omega^k(M)$ restricts to 0 on an open set U (i.e. $\omega_p = 0$ at every $p \in U$), then $D\omega \equiv 0$ on U (i.e. $(D\omega)_p = 0$ at every $p \in U$).

An equivalent definition of local operator is that for all $k \geq 0$, whenever two k -forms $\omega, \tau \in \Omega^k(M)$ agree on an open set U , then $D\omega \equiv D\tau$ on U (i.e. $(D\omega)_p = (D\tau)_p$ at every $p \in U$).

Proposition 5.2

Any antiderivation D on $\Omega^*(M)$ is a local operator.

Proof. Suppose $\omega \in \Omega^*(M)$ and $\omega \equiv 0$ on an open subset U . Let $p \in U$. It suffices to show that $(D\omega)_p = 0$. Take a bump function f at p supported in U , i.e. $\text{supp } f \subseteq U$. In particular, $f \equiv 1$ in a neighborhood V of p in U , so that $V \subset \text{supp } f \subseteq U$. Then $f\omega \equiv 0$ on M . This can be seen by noting that if $q \in U$,

$$(f\omega)_q = f(q)\omega_q = 0,$$

since $\omega_q = 0$ by hypothesis. On the other hand, if $q \notin U$, then $q \notin \text{supp } f$, so $f(q) = 0$, which yields

$$(f\omega)_q = f(q)\omega_q = 0.$$

Therefore, $f\omega \equiv 0$ on M . Applying D on $f\omega = f \wedge \omega$, we get

$$D(f\omega) = (Df) \wedge \omega + (-1)^0 f \wedge D\omega. \quad (5.4)$$

By the linearity of D , $D(f\omega) = 0$. Now, we evaluate the RHS of (5.4) at $p \in U$, and use the fact that $f(p) = 1$ and $\omega_p = 0$. As a result,

$$\begin{aligned} (Df)_p \wedge \omega_p + f(p) \wedge (D\omega)_p &= 0 \\ \implies (D\omega)_p &= 0. \end{aligned} \quad (5.5)$$

Since $p \in U$ is arbitrary, $D\omega \equiv 0$ on U . ■

Sometimes we are given a differential form τ that is defined only on an open subset U of a manifold M . We can use bump functions to extend τ to a global form $\tilde{\tau}$ on M that agrees with τ near some point.

Proposition 5.3

Suppose τ is a C^∞ differential k -form on an open subset U of M (such a differential form is called a local differential form). For any $p \in U$. There is a C^∞ global form $\tilde{\tau}$ on M (can be defined anywhere on M using its charts) that agrees with τ on a neighborhood of p contained in U .

Proof. Choose a smooth bump function f at p supported in U , i.e. $\text{supp } f \subseteq U$. In particular, $f \equiv 1$ in a neighborhood V of p in U , so that $V \subset \text{supp } f \subseteq U$. Then we define

$$\tilde{\tau}_q = \begin{cases} f(q) \tau_q & \text{if } q \in U, \\ \mathbf{0}_{\Lambda^k(T_q^*M)} & \text{if } q \notin U. \end{cases}$$

By the definition of $\tilde{\tau}$, it agrees with τ on V . By *Proposition 9.3.1(ii)* of [DG1](#), $\tilde{\tau}$ is smooth on U . Now, let $q \notin U$. We want to show that $\tilde{\tau}$ is smooth at q .

Since $\text{supp } f \subseteq U$, $q \notin U$ implies $q \in M \setminus U \subseteq M \setminus \text{supp } f$. Since $\text{supp } f$ is closed, $M \setminus \text{supp } f$ is open. Hence, we can find a coordinate chart (W, φ) about q such that $W \subseteq M \setminus \text{supp } f$. Then, for $r \in W$, $\tilde{\tau}_r = \mathbf{0}_{\Lambda^k(T_r^*M)}$. Also, $(\Lambda^k(T^*U), \bar{\varphi})$ is a chart on $\Lambda^k(T^*M)$ about $\mathbf{0}_{\Lambda^k(T_r^*M)}$.

$$(\bar{\varphi} \circ \tilde{\tau})(r) = (\varphi(r), \underbrace{0, 0, \dots, 0}_{\binom{n}{k} \text{ 0-s}}).$$

φ is smooth. Therefore, $\tilde{\tau}$ is smooth on W . In particular, $\tilde{\tau}$ is smooth at q . Since $q \notin U$ was arbitrary, $\tilde{\tau}$ is smooth at every $q \notin U$. Therefore, $\tilde{\tau}$ is smooth on all of M . ■

§5.3 Existence and Uniqueness of an Exterior Differentiation

To define an exterior derivative $d : \Omega^*(M) \rightarrow \Omega^*(M)$, let $\omega \in \Omega^k(M)$ and $p \in M$. Choose a chart (U, x^1, \dots, x^n) about p . Suppose $\omega = \sum_I a_I dx^I$ on U . Now, $d\omega$ is supposed to be a $(k+1)$ -form on M , i.e. $d\omega \in \Omega^{k+1}(M)$. Define $d\omega \in \Omega^{k+1}(M)$ such that at $p \in U$, $(d\omega)_p$ is expressed as

$$(d\omega)_p = \left(\sum_I da_I \wedge dx^I \right)_p. \quad (5.6)$$

It needs to be proven that the definition (5.6) is independent of chart. If (V, y^1, \dots, y^n) is another chart about p , and $\omega = \sum_J b_J dy^J$ on V , then on $U \cap V$,

$$\sum_I a_I d_{U \cap V} x^I = \sum_J b_J d_{U \cap V} y^J,$$

where $d_{U \cap V}$ is the unique exterior derivative $d_{U \cap V} : \Omega^*(U \cap V) \rightarrow \Omega^*(U \cap V)$. Then by the locality of exterior derivative,

$$d_{U \cap V} \left(\sum_I a_I d_{U \cap V} x^I \right) = d_{U \cap V} \left(\sum_J b_J d_{U \cap V} y^J \right). \quad (5.7)$$

Reading off the antiderivation $d_{U \cap V}$ in the chart $(U \cap V, x^1, \dots, x^n)$ using (5.6), the LHS of (5.7) can be recast into

$$\sum_I d_{U \cap V} a_I d_{U \cap V} x^I.$$

On the other hand, the antiderivation $d_{U \cap V}$ in the chart $(U \cap V, y^1, \dots, y^n)$ can be expressed using (5.6) to compute the RHS of (5.7):

$$\sum_J d_{U \cap V} b_J d_{U \cap V} y^J.$$

Therefore,

$$\sum_I d_{U \cap V} a_I d_{U \cap V} x^I = \sum_J d_{U \cap V} b_J d_{U \cap V} y^J, \quad (5.8)$$

on $U \cap V$. In particular, for $p \in U \cap V$,

$$\left(\sum_I d_{U \cap V} a_I d_{U \cap V} x^I \right)_p = \left(\sum_J d_{U \cap V} b_J d_{U \cap V} y^J \right)_p,$$

proving that the definition (5.6) is indeed chart independent. As p varies over all of M , (5.6) defines an operator

$$d : \Omega^*(M) \rightarrow \Omega^*(M).$$

It's straightforward to verify that the 3 desired conditions of exterior derivative are fulfilled by the definition (5.6).

Now we prove the uniqueness of exterior derivative. Suppose $D : \Omega^*(M) \rightarrow \Omega^*(M)$ is an exterior derivative. We will now show that D coincides with the exterior derivative defined by (5.6).

Let $\omega \in \Omega^k(M)$, and $p \in M$. Choose a chart (U, x^1, \dots, x^n) about p , and suppose $\omega = \sum_I a_I dx^I$ on U . Extend the functions a_I, x^1, \dots, x^n to C^∞ functions $\tilde{a}_I, \tilde{x}^1, \dots, \tilde{x}^n$ that agrees with a_I, x^1, \dots, x^n in a neighborhood V of p . Define

$$\tilde{\omega} = \sum_I \tilde{a}_I d\tilde{x}^I. \quad (5.9)$$

Then $\omega \equiv \tilde{\omega}$ on V . Since D is a local operator, one must have $D\omega \equiv D\tilde{\omega}$ on V . Thus,

$$(D\omega)_p = (D\tilde{\omega})_p = \left[D \left(\sum_I \tilde{a}_I d\tilde{x}^I \right) \right]_p. \quad (5.10)$$

Since D is an exterior derivative operator on Ω^*M , and d is the exterior derivative operator defined by (5.6), for $f \in C^\infty(M)$,

$$(Df)(X) = Xf = (df)(X),$$

for any C^∞ vector field X . In particular,

$$D\tilde{a}_I = d\tilde{a}_I, \text{ and } D\tilde{x}^i = d\tilde{x}^i,$$

so that $D\tilde{x}^I = d\tilde{x}^I$, for a strictly ascending multi-index I of length k . Hence, (5.10) reduces to

$$\begin{aligned} (D\omega)_p &= \left[D \left(\sum_I \tilde{a}_I d\tilde{x}^I \right) \right]_p \\ &= \left[D \left(\sum_I \tilde{a}_I D\tilde{x}^I \right) \right]_p \\ &= \left(\sum_I D\tilde{a}_I \wedge D\tilde{x}^I \right)_p \\ &= \left(\sum_I d\tilde{a}_I \wedge d\tilde{x}^I \right)_p. \end{aligned}$$

Now, since $\tilde{a}_I = a_I$ and $\tilde{x}^i = x^i$ in a neighborhood of p , we have $d\tilde{a}_I = da_I$ and $d\tilde{x}^I = dx^I$ at p . Therefore,

$$(D\omega)_p = \left(\sum_I d\tilde{a}_I \wedge d\tilde{x}^I \right)_p = \left(\sum_I da_I \wedge dx^I \right)_p = (d\omega)_p. \quad (5.11)$$

So $D = d$, and hence the exterior derivative is unique.

The restriction of a k -form to a submanifold

Let S be a regular submanifold of a manifold M , and ω is a k -form on M , i.e. $\omega \in \Omega^k(M)$. Then the restriction of ω to S is the k -form $\omega|_S$ on S defined by

$$(\omega|_S)_p(X_p^1, \dots, X_p^k) = \omega_p(X_p^1, \dots, X_p^k), \quad (5.12)$$

for $X_p^1, \dots, X_p^k \in T_pS \subseteq T_pM$. Thus, $(\omega|_S)_p$ is obtained from ω_p by restricting its domain to $T_pS \times T_pS \times \dots \times T_pS$ (k -times).

Example 5.1. If S is a smooth curve in \mathbb{R}^2 defined by the non-constant function $f(x, y) = 0$ (f could be $x^2 + y^2 - 1$, defining the unit circle in \mathbb{R}^2), then

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

is a nonzero 1-form on \mathbb{R}^2 . But since f is identically 0 on S , $(df)|_S = 0$. So a nonzero form on M can be restricted to a zero form on a submanifold S .

A form that is not identically zero is called a **nonzero form**. On the other hand, a form ω that is nowhere zero, i.e. $\omega_p \neq 0$ for all $p \in M$, is called a **nowhere vanishing form**.

Example 5.2 (A nowhere vanishing 1-form on S^1). Let S^1 be the unit circle defined by $x^2 + y^2 = 1$ in \mathbb{R}^2 . The 1-form dx restricts from \mathbb{R}^2 to a 1-form on S^1 . When restricted to S^1 , the domain of the covector $((dx)|_{S^1})_p$ is T_pS^1 instead of $T_p\mathbb{R}^2$:

$$((dx)|_{S^1})_p : T_pS^1 \rightarrow \mathbb{R}.$$

Now, from $x^2 + y^2 = 1$, one obtains

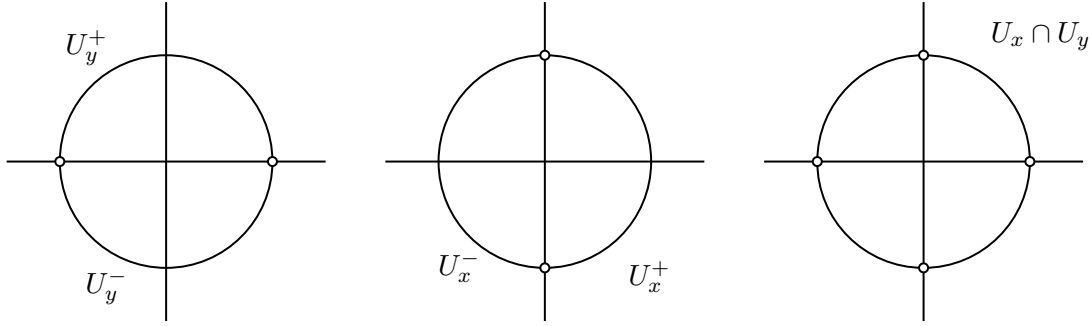
$$2x dx + 2y dy = 0. \quad (5.13)$$

At $p = (1, 0)$, (5.13) reduces to $(dx)_p = 0$. It shows that although dx is a nowhere vanishing 1-form on \mathbb{R}^2 , it vanishes at $(1, 0)$ when restricted to S^1 .

To find a nowhere vanishing 1-form on S^1 , we again take exterior derivative of both sides of the equation $x^2 + y^2 - 1 = 0$ to arrive at

$$2x dx + 2y dy = 0. \quad (5.14)$$

Let $U_x = \{(x, y) \in S^1 \mid x \neq 0\}$, and $U_y = \{(x, y) \in S^1 \mid y \neq 0\}$.



By (5.14), then one obtains on $U_x \cap U_y$,

$$\frac{dy}{x} = -\frac{dx}{y}. \quad (5.15)$$

Now we define a 1-form ω on S^1 by

$$\omega = \begin{cases} \frac{dy}{x} & \text{on } U_x, \\ -\frac{dx}{y} & \text{on } U_y. \end{cases} \quad (5.16)$$

Since $\frac{dy}{x} = -\frac{dx}{y}$ on $U_x \cap U_y$, ω is a well-defined 1-form on $S^1 = U_x \cup U_y$. To show that ω is C^∞ and nowhere vanishing, we need charts.

$$\begin{aligned} U_x^+ &= \{(x, y) \in S^1 \mid x > 0\}, U_x^- = \{(x, y) \in S^1 \mid x < 0\}, \\ U_y^+ &= \{(x, y) \in S^1 \mid y > 0\}, U_y^- = \{(x, y) \in S^1 \mid y < 0\}. \end{aligned}$$

On U_x^+ , the local coordinates are the y -coordinates, so that $(dy)_p$ is a basis for the cotangent space $T_p^*S^1$ at each $p \in U_x^+$. Now, since $\omega = \frac{dy}{x}$ on U_x^+ , ω is C^∞ and nowhere zero on U_x^+ . Similarly, $\omega = \frac{dy}{x}$ on U_x^- is also C^∞ and nowhere zero on U_x^- . One can show using similar argument that $\omega = -\frac{dx}{y}$ is C^∞ and nowhere vanishing on U_y^+ and U_y^- . Hence, ω is C^∞ and nowhere zero on S^1 .

§5.4 Exterior Differentiation Under a Pullback

Theorem 5.4

Let $F : N \rightarrow M$ be a smooth map of manifolds. If $\omega \in \Omega^k(M)$, then

$$dF^*\omega = F^*d\omega.$$

Proof. Let us first check the case when $k = 0$, i.e. when ω is a 0-form (C^∞ function). We denote this smooth function with h . For $p \in N$ and $X_p \in T_pN$,

$$(dF^*h)_p(X_p) = X_p(F^*h) = X_p(h \circ F), \quad (5.17)$$

since $(df)_p(X_p) = X_p f$ for $f \in C^\infty(M)$. On the other hand,

$$(F^*dh)_p(X_p) = (dh)_{F(p)}(F_{*,p}X_p) = (F_{*,p}X_p)(h) = X_p(h \circ F). \quad (5.18)$$

Combining (5.17) and (5.18), we get

$$(dF^*h)_p = (F^*dh)_p.$$

Since $p \in N$ is arbitrary,

$$dF^*h = F^*dh. \quad (5.19)$$

Now, consider the general case of a C^∞ k -form ω on M , i.e. $\omega \in \Omega^k(M)$. It suffices to verify that $dF^*\omega = F^*d\omega$ at an arbitrary point $p \in N$. This reduces the proof to a local computation. If (V, y^1, \dots, y^n) is a chart of M at $F(p)$, then on V ,

$$\omega = \sum_I a_I dy^I = \sum_I a_I dy^{i_1} \wedge dy^{i_2} \wedge \dots \wedge dy^{i_k},$$

for some C^∞ functions a_I on V . Now,

$$F^*\omega = \sum_I (F^*a_I) (F^*dy^{i_1}) \wedge (F^*dy^{i_2}) \wedge \dots \wedge (F^*dy^{i_k}).$$

Since $dF^*h = F^*dh$ for C^∞ function h , we have

$$\begin{aligned} F^*\omega &= \sum_I (a_I \circ F) d(F^*y^{i_1}) \wedge d(F^*y^{i_2}) \wedge \dots \wedge d(F^*y^{i_k}) \\ &= \sum_I (a_I \circ F) d(y^{i_1} \circ F) \wedge d(y^{i_2} \circ F) \wedge \dots \wedge d(y^{i_k} \circ F) \\ &= \sum_I (a_I \circ F) dF^{i_1} \wedge dF^{i_2} \wedge \dots \wedge dF^{i_k}. \end{aligned} \quad (5.20)$$

Therefore, from (5.20), one obtains

$$dF^*\omega = \sum_I d(a_I \circ F) \wedge dF^{i_1} \wedge dF^{i_2} \wedge \dots \wedge dF^{i_k}. \quad (5.21)$$

On the other hand,

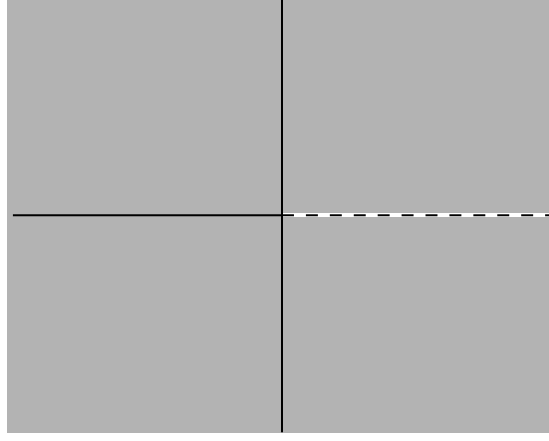
$$\begin{aligned} F^*d\omega &= F^*\left(\sum_I da_I \wedge dy^{i_1} \wedge dy^{i_2} \wedge \dots \wedge dy^{i_k}\right) \\ &= \sum_I F^*(da_I) \wedge F^*(dy^{i_1}) \wedge \dots \wedge F^*(dy^{i_k}) \\ &= \sum_I d(F^*a_I) \wedge d(F^*y^{i_1}) \wedge \dots \wedge d(F^*y^{i_k}) \\ &= \sum_I d(a_I \circ F) \wedge d(y^{i_1} \circ F) \wedge \dots \wedge d(y^{i_k} \circ F) \\ &= \sum_I d(a_I \circ F) \wedge dF^{i_1} \wedge dF^{i_2} \wedge \dots \wedge dF^{i_k}. \end{aligned} \quad (5.22)$$

Comparing (5.21) and (5.22), one obtains

$$dF^*\omega = F^*d\omega, \quad (5.23)$$

on V . In particular, (5.23) holds at $p \in N$. Since $p \in N$ is arbitrary, (5.23) holds everywhere on N . ■

Example 5.3. Let U be the open set $(0, \infty) \times (0, 2\pi)$ in the (r, θ) plane \mathbb{R}^2 , i.e. U is \mathbb{R}^2 except the non-negative x -axis.



Define $F : U \rightarrow \mathbb{R}^2$ by

$$(x, y) = F(r, \theta) = (r \cos \theta, r \sin \theta).$$

Let us compute the pullback $F^*(dx \wedge dy)$.

$$F^*dx = dF^*x = d(x \circ F) = d(r \cos \theta) = \cos \theta dr - r \sin \theta d\theta; \quad (5.24)$$

$$F^*dy = dF^*y = d(y \circ F) = d(r \sin \theta) = \sin \theta dr + r \cos \theta d\theta. \quad (5.25)$$

Therefore,

$$\begin{aligned} F^*(dx \wedge dy) &= F^*dx \wedge F^*dy \\ &= [\cos \theta dr - r \sin \theta d\theta] \wedge [\sin \theta dr + r \cos \theta d\theta] \\ &= r \cos^2 \theta dr \wedge d\theta - r \sin^2 \theta d\theta \wedge dr \\ &= r dr \wedge d\theta. \end{aligned} \quad (5.26)$$

§5.5 Pullback Preserves Smoothness of Forms

In this section, we will prove that if ω is a smooth k -form on M , and $F : N \rightarrow M$ is smooth, then $F^*\omega$ is a smooth k -form on N . For that purpose, we need a lemma first.

Lemma 5.5

Let (U, x^1, \dots, x^n) be a chart on a manifold and f^1, \dots, f^k smooth functions on U . Then

$$df^1 \wedge \dots \wedge df^k = \sum_I \frac{\partial (f^1, \dots, f^k)}{\partial (x^{i_1}, \dots, x^{i_k})} dx^{i_1} \wedge \dots \wedge dx^{i_k},$$

where $I = (i_1, \dots, i_k)$ is a strictly ascending multi-index of length k .

Proof. On U ,

$$df^1 \wedge \dots \wedge df^k = \sum_J c_J dx^{j_1} \wedge \dots \wedge dx^{j_k}, \quad (5.27)$$

for some functions c_J . By the definition of the differential,

$$df^i \left(\frac{\partial}{\partial x^j} \right) = \frac{\partial f^i}{\partial x^j}.$$

Applying both sides of (5.27) to the list of coordinate vector fields $\frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_k}}$, we get

$$\begin{aligned} \text{LHS} &= (df^1 \wedge \dots \wedge df^k) \left(\frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_k}} \right) = \det \left[\frac{\partial f^i}{\partial x^{i_j}} \right] \\ &= \frac{\partial (f^1, \dots, f^k)}{\partial (x^{i_1}, \dots, x^{i_k})}, \end{aligned} \quad (5.28)$$

by [Proposition 1.13](#). On the other hand,

$$\text{RHS} = \sum_J c_J \left(dx^{j_1} \wedge \cdots \wedge dx^{j_k} \right) \left(\frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_k}} \right) = \sum_J c_J \delta_I^J = c_I. \quad (5.29)$$

Hence, $c_I = \frac{\partial(f^1, \dots, f^k)}{\partial(x^{i_1}, \dots, x^{i_k})}$. ■

Theorem 5.6

If $F : N \rightarrow M$ is a C^∞ map of manifolds and ω is a C^∞ k -form on M , then $F^*\omega$ is a C^∞ k -form on N .

Proof. It is enough to show that every point in N has a neighborhood on which $F^*\omega$ is C^∞ . Fix $p \in N$ and choose a chart (V, y^1, \dots, y^m) on M about $F(p)$. Let $F^i = y^i \circ F$ be the i -th coordinate of the map F in this chart. By the continuity of F , there is a chart (U, x^1, \dots, x^n) on N about p such that $F(U) \subset V$. Since ω is C^∞ , on V ,

$$\omega = \sum_I a_I dy^{i_1} \wedge \cdots \wedge dy^{i_k}$$

for some C^∞ functions $a_I \in C^\infty(V)$. By properties of the pullback,

$$\begin{aligned} F^*\omega &= F^* \left(\sum_I a_I dy^{i_1} \wedge \cdots \wedge dy^{i_k} \right) \\ &= \sum_I (F^*a_I) F^*(dy^{i_1}) \wedge \cdots \wedge F^*(dy^{i_k}) \\ &= \sum_I (a_I \circ F) dF^*y^{i_1} \wedge \cdots \wedge dF^*y^{i_k} \\ &= \sum_I (a_I \circ F) dF^{i_1} \wedge \cdots \wedge dF^{i_k} \\ &= \sum_{I,J} (a_I \circ F) \frac{\partial(F^{i_1}, \dots, F^{i_k})}{\partial(x^{j_1}, \dots, x^{j_k})} dx^J. \end{aligned} \quad (5.30)$$

Since the $a_I \circ F$ and $\frac{\partial(F^{i_1}, \dots, F^{i_k})}{\partial(x^{j_1}, \dots, x^{j_k})}$ are all C^∞ , $F^*\omega$ is C^∞ on U . In particular, $F^*\omega$ is C^∞ at p . Since $p \in N$ is arbitrary, $F^*\omega$ is C^∞ on the whole of N . ■

Remark 5.2. [Theorem 5.6](#) tells us that F^* is indeed a map from $\Omega^k(M)$ to $\Omega^k(N)$. So we can think of it as a map between the graded algebras:

$$F^* : \Omega^*(M) \rightarrow F^*(N).$$

Previously we were writing this without really verifying that F^* preserves the smoothness of forms. Now, by [Proposition 4.3](#) and [Proposition 4.5](#), $F^* : \Omega^*(M) \rightarrow \Omega^*(N)$ is a homomorphism of graded algebras. This gives rise to a contravariant functor from the category **Man** of manifolds and smooth maps to the category **GrAlg** of graded algebras:

$$\mathcal{F} : \mathbf{Man} \rightarrow \mathbf{GrAlg}.$$

\mathcal{F} takes an object of **Man**, a manifold M , to the graded algebra $\Omega^*(M)$; and it makes an arrow of **Man**, a smooth map $F : N \rightarrow M$, to the graded algebra homomorphism $F^* : \Omega^*(M) \rightarrow \Omega^*(N)$. Since \mathcal{F} reverses the direction of arrows, it is a contravariant functor.

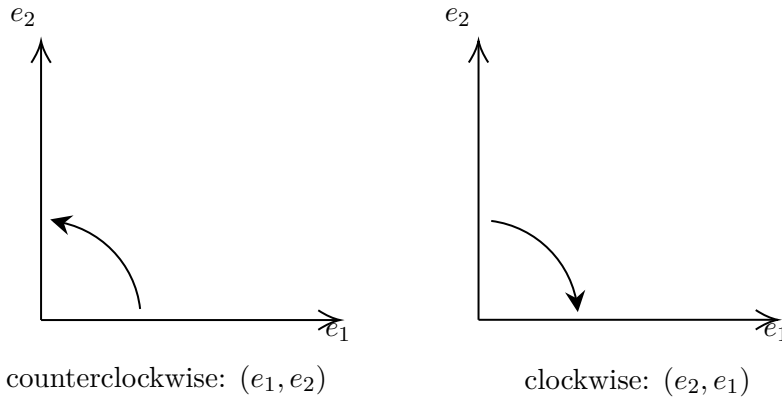
6 Orientation

§6.1 Orientations on a Vector Space

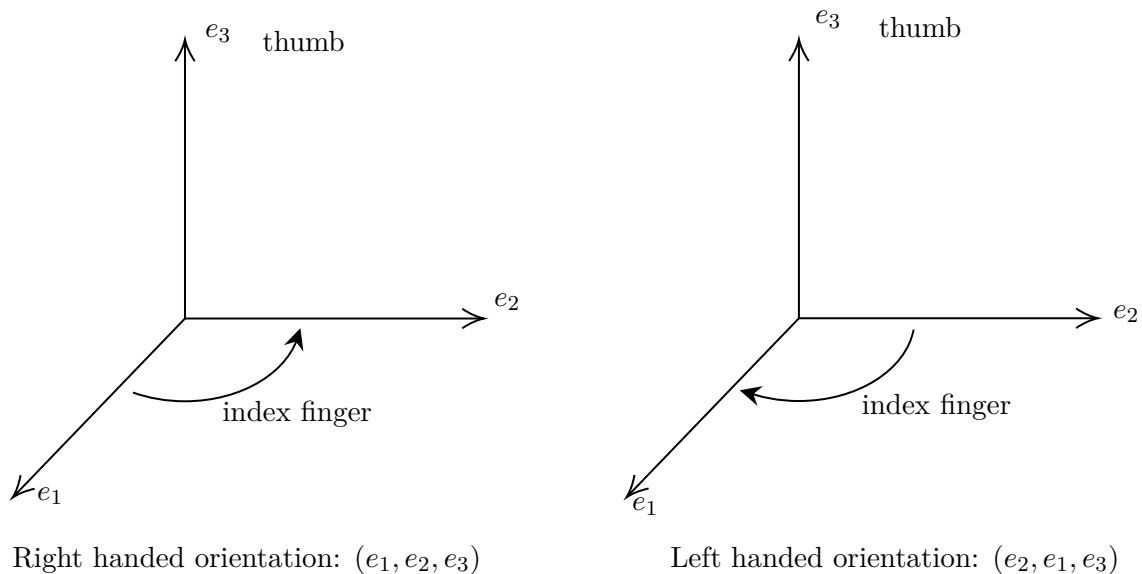
On \mathbb{R} , an orientation is one of the two possible directions:



On \mathbb{R}^2 , an orientation is either counterclockwise or clockwise:



On \mathbb{R}^3 , an orientation is either right handed or left handed:



Now, we want to define an orientation on \mathbb{R}^4 , or more generally on \mathbb{R}^n . We do it through ordered basis for \mathbb{R}^n . Let e_1, \dots, e_n be the standard basis for \mathbb{R}^n . For \mathbb{R}^1 , an orientation is given by e_1 , or $-e_1$. For \mathbb{R}^2 , counterclockwise orientation is (e_1, e_2) , and clockwise orientation is (e_2, e_1) . For \mathbb{R}^3 , the right handed orientation is (e_1, e_2, e_3) , and the left handed orientation is (e_2, e_1, e_3) .

For any two ordered bases (u_1, u_2) and (v_1, v_2) of \mathbb{R}^2 , there is a unique non-singular 2×2 matrix $A = [a_{ij}]$ such that

$$u_j = \sum_{i=1}^2 v_i a_{ij}. \quad (6.1)$$

A is called the change of basis matrix from (v_1, v_2) to (u_1, u_2) . In matrix notation, (6.1) can be written as

$$\begin{bmatrix} u_1 & u_2 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 \end{bmatrix} A. \quad (6.2)$$

We say that two ordered bases are **equivalent** if the change of basis matrix A has positive determinant. Then one can check that this is indeed an equivalence relation on the set of all ordered bases of \mathbb{R}^2 . It, therefore, partitions the ordered bases into two equivalence classes. Each equivalence class is called an **orientation** on \mathbb{R}^2 .

The equivalence class containing (e_1, e_2) is the counterclockwise orientation, and the equivalence class containing (e_2, e_1) is called clockwise orientation. Indeed,

$$\begin{bmatrix} e_1 & e_2 \end{bmatrix} = \begin{bmatrix} e_2 & e_1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

and

$$\det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1.$$

Similarly, for \mathbb{R}^3 , the ordered bases (e_1, e_2, e_3) and (e_2, e_1, e_3) don't belong to the same equivalence class:

$$\begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} = \begin{bmatrix} e_2 & e_1 & e_3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and

$$\det \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = -1.$$

The general case for an n -dimensional vector space V is as follows:

Definition 6.1. Two ordered bases $u = [u_1 \ \cdots \ u_n]$ and $v = [v_1 \ \cdots \ v_n]$ of an n -dimensional vector space V are said to be **equivalent** if

$$u = vA,$$

for an $n \times n$ matrix A with $\det A > 0$. An **orientation** on V is an equivalence class of ordered bases.

The 0-dimensional vector space $\{\mathbf{0}\}$ is a special case as its basis is the empty set \emptyset . We define an orientation on $\{\mathbf{0}\}$ to be one of the two numbers ± 1 .

§6.2 Orientations and n -covectors

Instead of using an ordered basis, we can also use an n -covector to specify an orientation on an n -dimensional vector space V . This is based on the fact that the vector space $\Lambda^n(V^*)$ of n -covectors on V is 1-dimensional (so that it has 2 orientations).

Lemma 6.1

Let u_1, \dots, u_n and v_1, \dots, v_n be vectors in a vector space V . Suppose

$$u_j = \sum_{i=1}^n v_i a_{ij},$$

for a matrix $A = [a_{ij}]$ of real numbers. If ω is an n -covector on V , then

$$\omega(u_1, \dots, u_n) = \det A \omega(v_1, \dots, v_n).$$

Proof. By hypothesis, $u_j = \sum_{i=1}^n v_i a_{ij}$. Using linearity of ω , one arrives at

$$\begin{aligned}\omega(u_1, u_2, \dots, u_n) &= \omega\left(\sum_{i_1=1}^n v_{i_1} a_{i_1 1}, \sum_{i_2=1}^n v_{i_2} a_{i_2 2}, \dots, \sum_{i_n=1}^n v_{i_n} a_{i_n n}\right) \\ &= \sum_{i_1, i_2, \dots, i_n=1}^n a_{i_1 1} a_{i_2 2} \cdots a_{i_n n} \omega(v_{i_1}, v_{i_2}, \dots, v_{i_n}).\end{aligned}\quad (6.3)$$

For $\omega(v_{i_1}, v_{i_2}, \dots, v_{i_n})$ to be nonzero, i_1, \dots, i_n must all be different as ω is alternating. Therefore, i_1, \dots, i_n can be thought of as a permutation σ of $\{1, 2, \dots, n\}$ that takes each j to i_j . From the alternating property of ω , one has

$$\omega(v_{\sigma(1)}, \dots, v_{\sigma(n)}) = (\operatorname{sgn} \sigma) \omega(v_1, \dots, v_n). \quad (6.4)$$

Therefore, using (6.3),

$$\begin{aligned}\omega(u_1, u_2, \dots, u_n) &= \sum_{\sigma \in S_n} a_{\sigma(1)1} a_{\sigma(2)2} \cdots a_{\sigma(n)n} (\operatorname{sgn} \sigma) \omega(v_1, \dots, v_n) \\ &= (\det A) \omega(v_1, \dots, v_n).\end{aligned}\quad (6.5)$$

■

Corollary 6.2

If u_1, \dots, u_n and v_1, \dots, v_n are ordered bases of a vector space V , then

$$\begin{aligned}\omega(u_1, u_2, \dots, u_n) \text{ and } \omega(v_1, \dots, v_n) &\text{ have the same sign} \\ \iff \det A > 0 \\ \iff u_1, \dots, u_n \text{ and } v_1, \dots, v_n &\text{ are equivalent ordered bases.}\end{aligned}$$

We say that the n -covector represents the orientation (v_1, \dots, v_n) if $\omega(v_1, \dots, v_n) > 0$. By Corollary 6.2, this notion is well-defined, i.e. independent of the choice of ordered basis v_1, \dots, v_n from the same equivalence class.

Remark 6.1. $\Lambda^n(V^*) \cong \mathbb{R}$, so that the set of nonzero n -covectors can be identified with $\mathbb{R} \setminus \{0\}$, which has 2 connected components. Two nonzero n -covectors ω and ω' on V are in the same component if and only if $\omega = a\omega'$ for some real number $a > 0$. Thus, each connected component of $\Lambda^n(V^*) \setminus \{0\}$ represents an orientation on V .

Example 6.1. Let e_1, e_2 be the standard basis of \mathbb{R}^2 , and α^1, α^2 its dual basis. Then the 2-covector $\alpha^1 \wedge \alpha^2$ represents the counterclockwise orientation on \mathbb{R}^2 , since

$$(\alpha^1 \wedge \alpha^2)(e_1, e_2) = 1 > 0.$$

Example 6.2. Let $\frac{\partial}{\partial x}\big|_p, \frac{\partial}{\partial y}\big|_p$ be the standard basis for the tangent space $T_p\mathbb{R}^2$, and $(dx)_p, (dy)_p$ be the dual basis, i.e. for the basis of $T_p^*\mathbb{R}^2$. Then $(dx)_p \wedge (dy)_p$ represents the counterclockwise orientation on $T_p\mathbb{R}^2$.

We define an equivalence relation on the nonzero n -covectors on the n -dimensional vector space V as follows:

$$\omega \sim \omega' \iff \omega = a\omega' \text{ for some } a > 0.$$

Then an orientation on V is also given by an equivalence class of nonzero n -covectors on V .

§6.3 Orientations on a Manifold

Every vector space of dimension n has two orientations, corresponding to the two equivalence classes of ordered bases or the two equivalence classes of nonzero n -covectors. To orient a manifold M , we orient the tangent space at each point $p \in M$ in a coherent way so that the orientation doesn't change abruptly in a neighborhood of a point. The simplest way to guarantee this is to require that the n -form (or top degree form) on M specifying the orientation at each point be C^∞ . We also want the n -form to be nowhere vanishing.

Definition 6.2. A manifold M of dimension n is **orientable** if it has a C^∞ nowhere vanishing n -form. If ω is a nowhere vanishing C^∞ n -form on M , then at each point $p \in M$, the n -covector ω_p picks out an equivalence class of ordered bases for the tangent space $T_p M$.

Example 6.3. The Euclidean space \mathbb{R}^n is orientable as a manifold, because it has the nowhere vanishing n -form $dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$.

If ω and ω' are both C^∞ nowhere vanishing n -forms on a manifold M of dimension n , then $\omega = f\omega'$ for a C^∞ nowhere vanishing function f on M . On a connected manifold M , such a function f is either everywhere positive or everywhere negative. Thus, the C^∞ nowhere-vanishing n -forms on a connected manifold M can be partitioned into 2 equivalence class:

$$\omega \sim \omega' \iff \omega = f\omega' \text{ with } f > 0. \quad (6.6)$$

We call either equivalence class an orientation on the connected manifold M . Thus, by definition, a connected manifold has exactly 2 orientations. If the manifold M is not connected, then each connected component of M has one of the 2 possible orientations. We call a C^∞ nowhere-vanishing n -form on M that specifies an orientation of M an **orientation form**. An **oriented manifold** is a pair $(M, [\omega])$, where M is a manifold of dimension n and $[\omega]$ is an orientation on M , i.e. the equivalence class of nowhere vanishing C^∞ n -forms containing ω .

Remark 6.2 (Orientations on a 0-dimensional manifold). A zero dimensional manifold is a point, and by definition is always orientable. Its two orientations are represented by the numbers ± 1 .

Definition 6.3. A diffeomorphism $F : (N, [\omega_N]) \rightarrow (M, [\omega_M])$ of oriented manifolds is said to be **orientation preserving** if $[F^*\omega_M] = [\omega_N]$. It's **orientation reversing** if $[F^*\omega_M] = [-\omega_N]$.

Proposition 6.3

Let U and V be open subsets of \mathbb{R}^n . A diffeomorphism $F : U \rightarrow V$ is orientation-preserving if and only if the Jacobian determinant $\det \left[\frac{\partial F^i}{\partial x^j} \right]$ is everywhere positive on U .

Proof. Let (x^1, \dots, x^n) and y^1, \dots, y^n be standard coordinates on $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^n$, respectively.

$$\begin{aligned} F^* (dy^1 \wedge \cdots \wedge dy^n) &= F^* (dy^1) \wedge \cdots \wedge F^* (dy^n) \\ &= d(F^* y^1) \wedge \cdots \wedge d(F^* y^n) \\ &= d(y^1 \circ F) \wedge \cdots \wedge d(y^n \circ F) \\ &= dF^1 \wedge \cdots \wedge dF^n \\ &= \det \left[\frac{\partial F^i}{\partial x^j} \right] dx^1 \wedge \cdots \wedge dx^n, \end{aligned} \quad (6.7)$$

where the last equality follows from Lemma 5.5. Now, F is orientation preserving if and only if

$$F^* (dy^1 \wedge \cdots \wedge dy^n) \sim dx^1 \wedge \cdots \wedge dx^n, \quad (6.8)$$

where \sim is defined as (6.6). Using (6.7), we can conclude that (6.8) holds if and only if $\det \left[\frac{\partial F^i}{\partial x^j} \right]$ is everywhere positive on U . ■

§6.4 Orientation and Atlases

Definition 6.4 (Oriented atlas). An atlas on M is said to be **oriented** if for any two overlapping charts (U, x^1, \dots, x^n) and (V, y^1, \dots, y^n) of the atlas, the Jacobian determinant $\det \left[\frac{\partial y^i}{\partial x^j} \right]$ is everywhere positive on $U \cap V$.

Proposition 6.4

A manifold M of dimension n has a C^∞ nowhere vanishing n -form ω if and only if it has an oriented atlas.

Proof. (\Leftarrow) Suppose we are given an oriented atlas

$$\left\{ (U_\alpha, x_\alpha^1, x_\alpha^2, \dots, x_\alpha^n) \right\}_{\alpha \in A}.$$

Suppose $\{\rho_\alpha\}_{\alpha \in A}$ is a C^∞ partition of unity subordinate to the open cover $\{U_\alpha\}_{\alpha \in A}$. Define

$$\omega = \sum \rho_\alpha dx_\alpha^1 \wedge \dots \wedge dx_\alpha^n. \quad (6.9)$$

Since $\{\text{supp } \rho_\alpha\}$ is locally finite by definition of partition of unity, for any $p \in M$, there is an open neighborhood U_p of p that intersects only finitely many of the sets $\text{supp } \rho_\alpha$. Thus, (6.9) is a finite sum on U_p . This actually shows that ω is defined and C^∞ at every point of M .

Let (U, x^1, \dots, x^n) be one of the charts about p in the oriented atlas. On $U_\alpha \cap U$, by Lemma 5.5,

$$dx_\alpha^1 \wedge \dots \wedge dx_\alpha^n = \det \left[\frac{\partial x_\alpha^i}{\partial x^j} \right] dx^1 \wedge \dots \wedge dx^n. \quad (6.10)$$

By hypothesis, $\det \left[\frac{\partial x_\alpha^i}{\partial x^j} \right] > 0$ as the atlas is oriented. Then on $U_p \cap U$,

$$\omega = \sum \rho_\alpha dx_\alpha^1 \wedge \dots \wedge dx_\alpha^n = \left(\sum \rho_\alpha \det \left[\frac{\partial x_\alpha^i}{\partial x^j} \right] \right) dx^1 \wedge \dots \wedge dx^n. \quad (6.11)$$

The sum in (6.11) is a finite sum, since U_p intersects only finitely many of the sets $\text{supp } \rho_\alpha$. Now, it's easy to see that the finite number in the parenthesis is actually positive at p . Indeed, $\det \left[\frac{\partial x_\alpha^i}{\partial x^j} \right] > 0$ at p , since the atlas is oriented. Furthermore, $\rho_\alpha(p) > 0$ for at least one $\alpha \in A$. Hence.

$$\omega_p = (\text{positive number}) \times (dx^1 \wedge \dots \wedge dx^n)_p \neq 0.$$

As p is an arbitrary point of M , the n -form ω is nowhere vanishing on M .

(\Rightarrow) Suppose ω is a C^∞ nowhere vanishing n -form on M . Given an atlas on M , we will use ω to modify the atlas so that it becomes oriented. Without loss of generality, assume that all the open sets of the atlas are connected.

On a chart (U, x^1, \dots, x^n) ,

$$\omega = f dx^1 \wedge \dots \wedge dx^n \quad (6.12)$$

for a C^∞ function f on U . Since ω is nowhere-vanishing and f is continuous, f is either everywhere positive or everywhere negative on U . If $f > 0$, we leave the chart as it is; if $f < 0$, we replace the chart by $(U, -x^1, x^2, \dots, x^n)$. After all the charts have been checked and replaced if necessary, we have that on every chart (V, y^1, \dots, y^n)

$$\omega = h dy^1 \wedge \dots \wedge dy^n \quad (6.13)$$

with $h > 0$. This can be seen to be an oriented atlas, since if (U, x^1, \dots, x^n) and (V, y^1, \dots, y^n) are two charts, then on $U \cap V$

$$\omega = f \, dx^1 \wedge \dots \wedge dx^n = h \, dy^1 \wedge \dots \wedge dy^n, \quad (6.14)$$

with $f, h > 0$. From (6.14),

$$dy^1 \wedge \dots \wedge dy^n = \frac{f}{h} dx^1 \wedge \dots \wedge dx^n. \quad (6.15)$$

By Lemma 5.5,

$$dy^1 \wedge \dots \wedge dy^n = \det \left[\frac{\partial y^i}{\partial x^j} \right] dx^1 \wedge \dots \wedge dx^n. \quad (6.16)$$

Comparing (6.15) and (6.16),

$$\det \left[\frac{\partial y^i}{\partial x^j} \right] = \frac{f}{h} > 0 \quad (6.17)$$

on $U \cap V$. Hence, the modified atlas is oriented. ■