



Inspiring Excellence

## **Differential Geometry II (MAT401)**

**Lecture Notes**

# Preface

This series of lecture notes has been prepared for aiding students who took the BRAC University course **Differential Geometry II (MAT401)** in Fall 2023 semester. These notes were typeset under the supervision of mathematician **Dr. Syed Hasibul Hassan Chowdhury**. The main goal of this typeset is to have an organized digital version of the notes, which is easier to share and handle. If you see any mistakes or typos, please send me an email at [atonuroychowdhury@gmail.com](mailto:atonuroychowdhury@gmail.com)

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## References:

- *An Introduction to Manifolds*, by **Loring W. Tu**
- *An Introduction to Differentiable Manifolds and Riemannian Geometry*, by **William Boothby**
- *Introduction to Smooth Manifolds*, by **John M. Lee**
- *Lectures on Differential geometry*, by **S.S Chern, W.H. Chen and K.S. Lam**
- *Geometry of Differential Forms*, by **Shigeyuki Morita**
- *From Calculus to Cohomology: De Rham Cohomology and Characteristic Classes*, by **Ib Madsen and Jxrgen Tornehave**.

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# 1 Review of Multilinear Algebra

## §1.1 Dual Space

Let  $V$  and  $W$  be real vector spaces. We denote by  $\text{Hom}(V, W)$  the vector space of all linear maps  $f : V \rightarrow W$ . In particular, if we choose  $W = \mathbb{R}$ , we get the **dual space**  $V^*$ .

$$V^* = \text{Hom}(V, \mathbb{R}).$$

The elements of  $V^*$  are called covectors on  $V$ . In the rest of the lecture, we will assume  $V$  to be a finite dimensional vector space. Let  $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$  be a basis for  $V$ . Then every  $\mathbf{v} \in V$  is a unique linear combination

$$\mathbf{v} = \sum_{i=1}^n v^i \mathbf{e}_i, \quad (1.1)$$

with  $v^i \in \mathbb{R}$ .  $v^i$ 's are called the coordinates of  $\mathbf{v}$  relative to the basis  $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ . Let  $\hat{\alpha}^i$  be the linear function on  $V$  that picks up the  $i$ -th coordinate of the vector, i.e.

$$\hat{\alpha}^i(\mathbf{v}) = \hat{\alpha}^i\left(\sum_{i=1}^n v^i \mathbf{e}_i\right) = v^i. \quad (1.2)$$

When  $\mathbf{v}$  is one of the basis vectors,

$$\hat{\alpha}^i(\mathbf{e}_j) = \delta^i_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \quad (1.3)$$

### Proposition 1.1

The functions  $\hat{\alpha}^1, \dots, \hat{\alpha}^n$  form a basis for  $V^*$ .

*Proof.* Suppose  $f \in V^*$ . Then for any  $\mathbf{v} = \sum_{i=1}^n v^i \mathbf{e}_i \in V$ ,

$$f(\mathbf{v}) = f\left(\sum_{i=1}^n v^i \mathbf{e}_i\right) = \sum_{i=1}^n v^i f(\mathbf{e}_i) = \sum_{i=1}^n f(\mathbf{e}_i) \hat{\alpha}^i(\mathbf{v}).$$

Since this holds for any  $\mathbf{v} \in V$ ,

$$f = \sum_{i=1}^n f(\mathbf{e}_i) \hat{\alpha}^i. \quad (1.4)$$

Therefore,  $\hat{\alpha}^1, \dots, \hat{\alpha}^n$  span  $V^*$ . As for linear independence, suppose

$$\sum_{i=1}^n c_i \hat{\alpha}^i = \mathbf{0}, \quad (1.5)$$

where  $\mathbf{0}$  is the function that takes all of  $V$  to 0 in  $\mathbb{R}$ . If we evaluate (1.5) at  $\mathbf{e}_j$ , we get

$$0 = \sum_{i=1}^n c_i \hat{\alpha}^i(\mathbf{e}_j) = \sum_{i=1}^n c_i \delta^i_j = c_j. \quad (1.6)$$

So  $c_j = 0$ , and this holds for each  $j = 1, 2, \dots, n$ . Therefore,  $\{\hat{\alpha}^1, \dots, \hat{\alpha}^n\}$  is a linearly independent set that spans  $V^*$ , i.e. a basis. ■

### Corollary 1.2

The dual space  $V^*$  of a finite dimensional vector space has the same dimension as  $V$ .

The basis  $\{\hat{\alpha}^1, \dots, \hat{\alpha}^n\}$  for  $V^*$  is said to be dual to the basis  $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$  for  $V$ .

## §1.2 Permutations

Fix a positive integer  $k$ . A permutation of the set  $A = \{1, 2, \dots, k\}$  is a bijection  $\sigma : A \rightarrow A$ . The product of two permutations  $\tau$  and  $\sigma$  is the composition  $\tau \circ \sigma : A \rightarrow A$ . The **cyclic permutation**  $(a_1 a_2 \cdots a_r)$  is the permutation  $\sigma$  such that

$$\sigma(a_1) = a_2, \sigma(a_2) = a_3, \dots, \sigma(a_{r-1}) = a_r, \text{ and } \sigma(a_r) = 1,$$

leaving all other elements unchanged, i.e.  $\sigma(j) = j$  if  $j$  is not one of the  $a_i$ 's. A cyclic permutation  $(a_1 a_2 \cdots a_r)$  is also called a **cycle** of length  $r$  or an  $r$ -cycle. A **transposition** is a permutation of the form  $(a b)$  that interchanges  $a$  and  $b$ , leaving all other elements of  $A$  fixed.

A permutation  $\sigma : A \rightarrow A$  can be described by

$$\begin{bmatrix} 1 & 2 & \cdots & k \\ \sigma(1) & \sigma(2) & \cdots & \sigma(k) \end{bmatrix}.$$

We also write it as

$$[1 \ 2 \ \cdots \ k] \xrightarrow{\sigma} [\sigma(1) \ \sigma(2) \ \cdots \ \sigma(k)].$$

**Example 1.1.** Suppose  $\sigma : \{1, 2, 3, 4, 5\} \rightarrow \{1, 2, 3, 4, 5\}$  is the permutation given by

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 5 & 1 & 3 \end{bmatrix}.$$

In other words,  $\sigma(1) = 2$ ,  $\sigma(2) = 4$ ,  $\sigma(3) = 5$ ,  $\sigma(4) = 1$ , and  $\sigma(5) = 3$ .

$$[1 \ 2 \ 3 \ 4 \ 5] \xrightarrow{\sigma} [2 \ 4 \ 5 \ 1 \ 3].$$

Observe that the cyclic permutation  $\sigma' = (1 \ 2 \ 4)$  acts as  $\sigma'(1) = 2$ ,  $\sigma'(2) = 4$  and  $\sigma'(4) = 1$ , keeping 3 and 5 unchanged, i.e.  $\sigma'(3) = 3$  and  $\sigma'(5) = 5$ .

$$[1 \ 2 \ 3 \ 4 \ 5] \xrightarrow{(1 \ 2 \ 4)} [2 \ 4 \ 3 \ 1 \ 5].$$

Now the transposition  $\sigma'' = (3 \ 5)$  acts as  $\sigma''(3) = 5$  and  $\sigma''(5) = 3$ , keeping 1, 2, 4 unchanged. Therefore,

$$\begin{array}{ccccc} [1 \ 2 \ 3 \ 4 \ 5] & \xrightarrow{(1 \ 2 \ 4)} & [2 \ 4 \ 3 \ 1 \ 5] & \xrightarrow{(3 \ 5)} & [2 \ 4 \ 5 \ 1 \ 3] \\ & \searrow & & \nearrow & \\ & & (3 \ 5)(1 \ 2 \ 4) & & \end{array}$$

so that  $\sigma = (3 \ 5)(1 \ 2 \ 4)$ .

Let  $S_k$  be the group of permutations of the set  $\{1, 2, \dots, k\}$ . The order of this group is  $k!$ . A permutation is *even* or *odd* depending on whether it is the product of an even or an odd number of transpositions. The sign of a permutation  $\sigma$  is 1 if the permutation is even, and  $-1$  otherwise. It is denoted by  $\text{sgn } \sigma$ . For example, in [Example 1.1](#),  $\sigma = (3 \ 5)(1 \ 2 \ 4)$ . Note that we can write  $(1 \ 2 \ 4)$  as a product of two transpositions:

$$\begin{array}{ccccc} [1 \ 2 \ 3 \ 4 \ 5] & \xrightarrow{(1 \ 2)} & [2 \ 1 \ 3 \ 4 \ 5] & \xrightarrow{(1 \ 4)} & [2 \ 4 \ 3 \ 1 \ 5] \\ & \searrow & & \nearrow & \\ & & (1 \ 4)(1 \ 2) = (1 \ 2 \ 4) & & \end{array}$$

In other words,  $\sigma = (3 \ 5)(1 \ 4)(1 \ 2)$ . Hence,  $\text{sgn } \sigma = -1$ . One can easily check that

$$\text{sgn } (\sigma\tau) = (\text{sgn } \sigma)(\text{sgn } \tau). \quad (1.7)$$

So  $\text{sgn} : S_k \rightarrow \{1, -1\}$  is a group homomorphism.

**Example 1.2.** Observe that the 5-cycle  $(1\ 2\ 3\ 4\ 5)$  can be written as

$$(1\ 2\ 3\ 4\ 5) = (1\ 5)(1\ 4)(1\ 3)(1\ 2).$$

Indeed,

$$\begin{array}{ccccccc} [1\ 2\ 3\ 4\ 5] & \xrightarrow{(1\ 2)} & [2\ 1\ 3\ 4\ 5] & \xrightarrow{(1\ 3)} & [2\ 3\ 1\ 4\ 5] & \xrightarrow{(1\ 4)} & [2\ 3\ 4\ 1\ 5] & \xrightarrow{(1\ 5)} & [2\ 3\ 4\ 5\ 1] \\ & & & & & & \nearrow & & \\ & & & & & & (1\ 2\ 3\ 4\ 5) = (1\ 5)(1\ 4)(1\ 3)(1\ 2) & & \end{array}$$

Therefore,  $\text{sgn}(1\ 2\ 3\ 4\ 5) = 1$ .

An **inversion** in a permutation  $\sigma$  is an ordered pair  $(\sigma(i), \sigma(j))$  such that  $i < j$  but  $\sigma(i) > \sigma(j)$ . In [Example 1.1](#),  $\sigma(1) = 2$ ,  $\sigma(2) = 4$ ,  $\sigma(3) = 5$ ,  $\sigma(4) = 1$ , and  $\sigma(5) = 3$ . So, the inversions in this permutation are

$$(\sigma(1), \sigma(4)), (\sigma(2), \sigma(4)), (\sigma(2), \sigma(5)), (\sigma(3), \sigma(4)), (\sigma(3), \sigma(5)).$$

Hence, there are 5 inversions associated with the permutation  $\sigma$ . There is an efficient way of determining the sign of a permutation.

### Proposition 1.3

A permutation is even if and only if it has an even number of inversions.

*Proof.* Let  $\sigma \in S_k$  with  $n$  inversions. We shall prove that we can multiply  $\sigma$  by  $n$  transpositions and get the identity permutation. This will prove that  $\text{sgn } \sigma = (-1)^n$ .

Suppose  $\sigma(j_1) = 1$ . Then for each  $i < j_1$ ,  $(\sigma(i), \sigma(j_1))$  is an inversion, and there are  $j_1 - 1$  many of them. These are all the inversions with 1 in the second slot of the ordered pair of inversion. If we now multiply  $\sigma$  by the  $j_1$ -cycle

$$(\sigma(1)\ 1)(\sigma(2)\ 1) \cdots (\sigma(j_1 - 1)\ 1)$$

to the left of  $\sigma$ , the resulting permutation  $\sigma_1$  would be

$$\begin{bmatrix} 1 & 2 & 3 & \cdots & j_1 & j_1 + 1 & \cdots & k \\ 1 & \sigma(1) & \sigma(2) & \cdots & \sigma(j_1 - 1) & \sigma(j_1 + 1) & \cdots & \sigma(k) \end{bmatrix}$$

This permutation has no inversion with 1 in the second slot of the ordered pair of inversion. Suppose now that  $\sigma(j_2) = 2$ . Now observe that if  $(\sigma_1(i), 2)$  is an inversion in  $\sigma_1$ , then either  $(\sigma(i), 2)$  (if  $i \geq j_1 + 1$ ) or  $\sigma(i - 1), 2$  (if  $i \leq j_1 - 1$ ) is an inversion in  $\sigma$ . Therefore, the number of inversions in  $\sigma_1$  ending in 2 is precisely the same as the number of inversions in  $\sigma$  ending in 2. So following a similar procedure as above, we can multiply  $\sigma_1$  by  $i_2$ -many transpositions to the left ( $i_2$  is the number of transpositions ending in 2) and get

$$\begin{bmatrix} 1 & 2 & 3 & \cdots & k \\ 1 & 2 & \sigma(1) & \cdots & \sigma(k) \end{bmatrix}$$

We can continue these steps for each  $j = 1, 2, \dots, k$ , and the number of transpositions required to move  $j$  to its natural position is the same as the number of inversions ending in  $j$ . In the end we achieve the identity permutation. Therefore,  $\text{sgn } \sigma = (-1)^n$ , where  $n$  is the number of inversions. ■

### §1.3 Multilinear Functions

**Definition 1.1.** Let  $V^k$  be the cartesian product of  $k$ -copies of a real vector space  $V$ .

$$V^k = \underbrace{V \times V \times \cdots \times V}_{k\text{-copies}}$$

A function  $f : V^k \rightarrow \mathbb{R}$  is called  $k$ -linear if it is linear in each of its  $k$  arguments:

$$f(\dots, a\mathbf{v} + b\mathbf{w}, \dots) = a f(\dots, \mathbf{v}, \dots) + b f(\dots, \mathbf{w}, \dots), \quad (1.8)$$

for  $a, b \in \mathbb{R}$  and  $\mathbf{v}, \mathbf{w} \in V$ .

Instead of 2-linear and 3-linear, it's customary to call “bilinear” and “trilinear”, respectively. A  $k$ -linear function on  $V$  is called a  **$k$ -tensor** on  $V$ . We will denote the vector space of all  $k$ -tensors on  $V$  by  $L_k(V)$ . The vector addition and scalar multiplication of the real vector space  $L_k(V)$  is the straightforward pointwise operation.

**Example 1.3.** The dot product  $f(\mathbf{v}, \mathbf{w}) = \mathbf{v} \cdot \mathbf{w}$  on  $\mathbb{R}^n$  is bilinear: if  $\mathbf{v} = \sum_{i=1}^n v^i \mathbf{e}_i$  and  $\mathbf{w} = \sum_{i=1}^n w^i \mathbf{e}_i$ , then

$$f(\mathbf{v}, \mathbf{w}) = \mathbf{v} \cdot \mathbf{w} = \sum_{i=1}^n v^i w^i.$$

**Example 1.4.** The determinant

$$f(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n) = \det \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}$$

viewed as a function of the  $n$  column vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  is  $n$ -linear.

**Definition 1.2** (Symmetric and alternating function). A  $k$ -linear function  $f : V^k \rightarrow \mathbb{R}$  is **symmetric** if

$$f(\mathbf{v}_{\sigma(1)}, \mathbf{v}_{\sigma(2)}, \dots, \mathbf{v}_{\sigma(k)}) = f(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k), \quad (1.9)$$

for all permutations  $\sigma \in S_k$ . It is **alternating** if

$$f(\mathbf{v}_{\sigma(1)}, \mathbf{v}_{\sigma(2)}, \dots, \mathbf{v}_{\sigma(k)}) = (\text{sgn } \sigma) f(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k), \quad (1.10)$$

for all permutations  $\sigma \in S_k$ .

The dot product function on  $\mathbb{R}^n$  in [Theorem 1.3](#) is symmetric, and the determinant function on  $\mathbb{R}^n$  in [Theorem 1.4](#) is alternating.

We are especially interested in the vector space  $A_k(V)$  of all alternating  $k$ -linear functions on a vector space  $V$ , for  $k > 0$ . The elements of  $A_k(V)$  are called alternating  $k$ -tensors (also known as  $k$ -covectors). We define  $A_0(V)$  to be  $\mathbb{R}$ . The elements of  $A_0(V)$  are simply constants, which we call 0-covectors. The elements of  $A_1(V)$  are simply covectors, i.e. the elements of  $V^*$ .

#### Permutation action on $k$ -linear functions

If  $f \in L_k(V)$  and  $\sigma \in S_k$ , define  $\sigma f \in L_k(V)$  as follows:

$$(\sigma f)(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k) = f(\mathbf{v}_{\sigma(1)}, \mathbf{v}_{\sigma(2)}, \dots, \mathbf{v}_{\sigma(k)}). \quad (1.11)$$

Thus,  $f$  is symmetric if and only if  $f = \sigma f$  for all  $\sigma \in S_k$ ; and  $f$  is alternating if and only if  $\sigma f = (\text{sgn } \sigma) f$  for all  $\sigma \in S_k$ . When  $k = 1$ ,  $S_k$  only has the identity permutation. In that case, a 1-linear function or simply linear function on  $V$  is both symmetric and alternating. In particular,

$$A_1(V) = L_1(V) = V^*.$$



**Lemma 1.4**

If  $\sigma, \tau \in S_k$  and  $f \in L_k(V)$ , then  $\tau(\sigma f) = (\tau\sigma)f$ .

*Proof.* For any  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in V$ ,

$$\begin{aligned}
 (\tau(\sigma f))(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k) &= (\sigma f)(\mathbf{v}_{\tau(1)}, \mathbf{v}_{\tau(2)}, \dots, \mathbf{v}_{\tau(k)}) \\
 &= (\sigma f)(\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_k) && [\mathbf{w}_i = \mathbf{v}_{\tau(i)}] \\
 &= f(\mathbf{w}_{\sigma(1)}, \mathbf{w}_{\sigma(2)}, \dots, \mathbf{w}_{\sigma(k)}) \\
 &= f(\mathbf{v}_{\tau(\sigma(1))}, \mathbf{v}_{\tau(\sigma(2))}, \dots, \mathbf{v}_{\tau(\sigma(k))}) \\
 &= ((\tau\sigma)f)(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k).
 \end{aligned}$$

Therefore,  $\tau(\sigma f) = (\tau\sigma)f$ . ■

**Definition 1.3.** If  $G$  is a group and  $X$  is a set, a map

$$\begin{aligned}
 G \times X &\rightarrow X \\
 (g, x) &\mapsto g \cdot x
 \end{aligned}$$

is called a **left action** of  $G$  on  $X$  if

- (i)  $e \cdot x = x$ , where  $e$  is the identity element in  $G$  and  $x$  is any element in  $X$ ; and
- (ii)  $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$ , for all  $g_1, g_2 \in G$  and  $x \in X$ .

Similarly, a **right action** of  $G$  on  $X$  is a map

$$\begin{aligned}
 X \times G &\rightarrow X \\
 (x, g) &\mapsto x \cdot g
 \end{aligned}$$

such that

- (i)  $x \cdot e = x$ , for all  $x \in X$ ; and
- (ii)  $(x \cdot g_1) \cdot g_2 = x \cdot (g_1 g_2)$ , for all  $g_1, g_2 \in G$  and  $x \in X$ .

**Symmetrizing and alternating operators**

Given  $f \in L_k(V)$ , there is a way to make it a symmetric  $k$ -linear function  $\mathcal{S}f$  from it:

$$(\mathcal{S}f)(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k) = \sum_{\sigma \in S_k} f(\mathbf{v}_{\sigma(1)}, \mathbf{v}_{\sigma(2)}, \dots, \mathbf{v}_{\sigma(k)}). \quad (1.12)$$

In other words,

$$\mathcal{S}f = \sum_{\sigma \in S_k} \sigma f. \quad (1.13)$$

Similarly, there is a way to make an alternating  $k$ -linear function from  $f$ :

$$\mathcal{A}f = \sum_{\sigma \in S_k} (\text{sgn } \sigma) \sigma f. \quad (1.14)$$

**Proposition 1.5** (i) The  $k$ -linear function  $\mathcal{S}f$  is symmetric.

(ii) The  $k$ -linear function  $\mathcal{A}f$  is alternating.

*Proof.* (i) Let  $\tau \in S_k$ . Then

$$\tau(\mathcal{S}f) = \tau\left(\sum_{\sigma \in S_k} \sigma f\right). \quad (1.15)$$

The group action of  $S_k$  on  $L_k(V)$  is distributive over the vector space addition. Therefore,

$$\tau(\mathcal{S}f) = \sum_{\sigma \in S_k} \tau(\sigma f) = \sum_{\sigma \in S_k} (\tau\sigma)f. \quad (1.16)$$

As  $\sigma$  varies over  $S_k$ ,  $\tau\sigma$  also varies over  $S_k$ . Therefore,  $\sum_{\sigma \in S_k} (\tau\sigma)f = \mathcal{S}f$ . In other words,

$$\tau(\mathcal{S}f) = \mathcal{S}f, \quad (1.17)$$

i.e.  $\mathcal{S}f$  is symmetric.

(ii) Let  $\tau \in S_k$ . Then

$$\tau(\mathcal{A}f) = \tau\left(\sum_{\sigma \in S_k} (\text{sgn } \sigma) \sigma f\right) = \sum_{\sigma \in S_k} (\text{sgn } \sigma) \tau(\sigma f) = \sum_{\sigma \in S_k} (\text{sgn } \sigma) (\tau\sigma)f. \quad (1.18)$$

Since  $(\text{sgn } \tau)^2 = 1$ ,

$$\begin{aligned} \tau(\mathcal{A}f) &= \sum_{\sigma \in S_k} (\text{sgn } \tau)^2 (\text{sgn } \sigma) (\tau\sigma)f \\ &= (\text{sgn } \tau) \sum_{\sigma \in S_k} (\text{sgn } \tau) (\text{sgn } \sigma) (\tau\sigma)f \\ &= (\text{sgn } \tau) \sum_{\sigma \in S_k} (\text{sgn } (\tau\sigma)) (\tau\sigma)f. \end{aligned} \quad (1.19)$$

As  $\sigma$  varies over  $S_k$ ,  $\tau\sigma$  also varies over  $S_k$ . Therefore,  $\sum_{\sigma \in S_k} (\text{sgn } (\tau\sigma)) (\tau\sigma)f = \mathcal{A}f$ . In other words,

$$\tau(\mathcal{A}f) = \mathcal{A}f, \quad (1.20)$$

i.e.  $\mathcal{A}f$  is alternating. ■

### Lemma 1.6

If  $f \in A_k(V)$ , then  $\mathcal{A}f = (k!)f$ .

*Proof.* Since  $f$  is alternating,

$$\mathcal{A}f = \sum_{\sigma \in S_k} (\text{sgn } \sigma) \sigma f = \sum_{\sigma \in S_k} (\text{sgn } \sigma)^2 f = \sum_{\sigma \in S_k} f = (k!)f, \quad (1.21)$$

because the order of  $S_k$  is  $k!$ . ■

## §1.4 Tensor Product and Wedge Product

**Definition 1.4 (Tensor Product).** Let  $f$  be a  $k$ -linear function and  $g$  an  $l$ -linear function on a vector space  $V$ . Their tensor product  $f \otimes g$  is the  $(k+l)$ -linear function defined by

$$(f \otimes g)(\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{v}_{k+1}, \dots, \mathbf{v}_{k+l}) = f(\mathbf{v}_1, \dots, \mathbf{v}_k) g(\mathbf{v}_{k+1}, \dots, \mathbf{v}_{k+l}). \quad (1.22)$$

$(k+l)$ -linearity of  $f \otimes g$  follows from  $k$ -linearity of  $f$  and  $l$ -linearity of  $g$ .

**Lemma 1.7** (Associativity of Tensor Product)

Let  $f \in L_k(V)$ ,  $g \in L_l(V)$  and  $h \in L_m(V)$ . Then

$$(f \otimes g) \otimes h = f \otimes (g \otimes h).$$

*Proof.* For  $\mathbf{v}_1, \dots, \mathbf{v}_{k+l+m}$ ,

$$\begin{aligned} [(f \otimes g) \otimes h](\mathbf{v}_1, \dots, \mathbf{v}_{k+l+m}) &= (f \otimes g)(\mathbf{v}_1, \dots, \mathbf{v}_{k+l}) h(\mathbf{v}_{k+l+1}, \dots, \mathbf{v}_{k+l+m}) \\ &= f(\mathbf{v}_1, \dots, \mathbf{v}_k) g(\mathbf{v}_{k+1}, \dots, \mathbf{v}_{k+l}) h(\mathbf{v}_{k+l+1}, \dots, \mathbf{v}_{k+l+m}). \end{aligned} \quad (1.23)$$

$$\begin{aligned} [f \otimes (g \otimes h)](\mathbf{v}_1, \dots, \mathbf{v}_{k+l+m}) &= f(\mathbf{v}_1, \dots, \mathbf{v}_k) (g \otimes h)(\mathbf{v}_{k+1}, \dots, \mathbf{v}_{k+l+m}) \\ &= f(\mathbf{v}_1, \dots, \mathbf{v}_k) g(\mathbf{v}_{k+1}, \dots, \mathbf{v}_{k+l}) h(\mathbf{v}_{k+l+1}, \dots, \mathbf{v}_{k+l+m}). \end{aligned} \quad (1.24)$$

Therefore,  $(f \otimes g) \otimes h = f \otimes (g \otimes h)$ , i.e. tensor product is associative.  $\blacksquare$

**Example 1.5.** Let  $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$  be the standard basis for  $\mathbb{R}^n$ , and  $\{\hat{\alpha}^1, \dots, \hat{\alpha}^n\}$  its dual basis. The Euclidean inner product on  $\mathbb{R}^n$  is the bilinear function

$$\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$$

defined by

$$\langle \cdot, \cdot \rangle(\mathbf{v}, \mathbf{w}) = \langle \mathbf{v}, \mathbf{w} \rangle = \sum_{i=1}^n v^i w^i,$$

for  $\mathbf{v} = \sum_{i=1}^n v^i \mathbf{e}_i$  and  $\mathbf{w} = \sum_{i=1}^n w^i \mathbf{e}_i$ . We can express  $\langle \cdot, \cdot \rangle$  in terms of tensor product as follows:

$$\langle \cdot, \cdot \rangle(\mathbf{v}, \mathbf{w}) = \sum_{i=1}^n v^i w^i = \sum_{i=1}^n \hat{\alpha}^i(\mathbf{v}) \hat{\alpha}^i(\mathbf{w}) = \sum_{i=1}^n (\hat{\alpha}^i \otimes \hat{\alpha}^i)(\mathbf{v}, \mathbf{w}).$$

Since  $\mathbf{v}, \mathbf{w}$  are arbitrary,

$$\langle \cdot, \cdot \rangle = \sum_{i=1}^n (\hat{\alpha}^i \otimes \hat{\alpha}^i). \quad (1.25)$$

If  $f \in A_k(V)$  and  $g \in A_l(V)$ , then it's not true that  $f \otimes g \in A_{k+l}(V)$ , in general. We need to construct a product that is also alternating.

**Definition 1.5** (Wedge Product). For  $f \in A_k(V)$  and  $g \in A_l(V)$ , the wedge product of  $f$  and  $g$  is defined as follows:

$$f \wedge g = \frac{1}{k!l!} \mathcal{A}(f \otimes g). \quad (1.26)$$

Explicitly,

$$\begin{aligned} (f \wedge g)(\mathbf{v}_1, \dots, \mathbf{v}_{k+l}) &= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (\text{sgn } \sigma) \sigma(f \otimes g)(\mathbf{v}_1, \dots, \mathbf{v}_{k+l}) \\ &= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (\text{sgn } \sigma) (f \otimes g)(\mathbf{v}_{\sigma(1)}, \dots, \mathbf{v}_{\sigma(k+l)}) \\ &= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (\text{sgn } \sigma) f(\mathbf{v}_{\sigma(1)}, \dots, \mathbf{v}_{\sigma(k)}) g(\mathbf{v}_{\sigma(k+1)}, \dots, \mathbf{v}_{\sigma(k+l)}). \end{aligned} \quad (1.27)$$

When  $k = 0$ , the element  $f \in A_0(V)$  is simply a constant  $c \in \mathbb{R}$  as discussed earlier. In this case, the wedge product  $c \wedge g$  is just scalar multiplication as is evident from (1.27).

$$\begin{aligned}
 (c \wedge g)(\mathbf{v}_1, \dots, \mathbf{v}_l) &= \frac{1}{l!} \sum_{\sigma \in S_l} (\operatorname{sgn} \sigma) c g(\mathbf{v}_{\sigma(1)}, \dots, \mathbf{v}_{\sigma(l)}) \\
 &= \frac{1}{l!} \sum_{\sigma \in S_l} (\operatorname{sgn} \sigma) c (\operatorname{sgn} \sigma) g(\mathbf{v}_1, \dots, \mathbf{v}_l) \\
 &= \frac{1}{l!} \sum_{\sigma \in S_l} c g(\mathbf{v}_1, \dots, \mathbf{v}_l) \\
 &= \frac{1}{l!} l! c g(\mathbf{v}_1, \dots, \mathbf{v}_l) \\
 &= c g(\mathbf{v}_1, \dots, \mathbf{v}_l).
 \end{aligned}$$

Thus  $c \wedge g = cg$ , for  $c \in \mathbb{R}$  and  $g \in A_l(V)$ .

**Example 1.6.** For  $f \in A_2(V)$  and  $g \in A_1(V)$ ,

$$\begin{aligned}
 \mathcal{A}(f \otimes g)(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) &= f(\mathbf{v}_1, \mathbf{v}_2)g(\mathbf{v}_3) - f(\mathbf{v}_1, \mathbf{v}_3)g(\mathbf{v}_2) - f(\mathbf{v}_2, \mathbf{v}_1)g(\mathbf{v}_3) \\
 &\quad - f(\mathbf{v}_3, \mathbf{v}_2)g(\mathbf{v}_1) + f(\mathbf{v}_2, \mathbf{v}_3)g(\mathbf{v}_1) + f(\mathbf{v}_3, \mathbf{v}_1)g(\mathbf{v}_2).
 \end{aligned}$$

Among these 6 terms, there are 3 pairs of equal terms due to the alternating nature of  $f$ .

$$\begin{aligned}
 f(\mathbf{v}_1, \mathbf{v}_2)g(\mathbf{v}_3) &= -f(\mathbf{v}_2, \mathbf{v}_1)g(\mathbf{v}_3), \\
 f(\mathbf{v}_3, \mathbf{v}_1)g(\mathbf{v}_2) &= -f(\mathbf{v}_1, \mathbf{v}_3)g(\mathbf{v}_2), \\
 f(\mathbf{v}_2, \mathbf{v}_3)g(\mathbf{v}_1) &= -f(\mathbf{v}_3, \mathbf{v}_2)g(\mathbf{v}_1).
 \end{aligned}$$

Therefore,

$$\mathcal{A}(f \otimes g)(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = 2f(\mathbf{v}_1, \mathbf{v}_2)g(\mathbf{v}_3) + 2f(\mathbf{v}_3, \mathbf{v}_1)g(\mathbf{v}_2) + 2f(\mathbf{v}_2, \mathbf{v}_3)g(\mathbf{v}_1). \quad (1.28)$$

Hence,

$$\begin{aligned}
 (f \wedge g)(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) &= \frac{1}{2!1!} \mathcal{A}(f \otimes g)(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \\
 &= f(\mathbf{v}_1, \mathbf{v}_2)g(\mathbf{v}_3) + f(\mathbf{v}_3, \mathbf{v}_1)g(\mathbf{v}_2) + f(\mathbf{v}_2, \mathbf{v}_3)g(\mathbf{v}_1).
 \end{aligned} \quad (1.29)$$

**Example 1.7** (Wedge product of 2 covectors). If  $f, g \in A_1(V)$ , and  $\mathbf{v}_1, \mathbf{v}_2 \in V$ , then

$$(f \wedge g)(\mathbf{v}_1, \mathbf{v}_2) = \frac{1}{1!1!} \mathcal{A}(f \otimes g)(\mathbf{v}_1, \mathbf{v}_2).$$

$S_2$  has 2 elements: the identity element  $e$  and  $(1\ 2)$ . Therefore,

$$(f \wedge g)(\mathbf{v}_1, \mathbf{v}_2) = \mathcal{A}(f \otimes g)(\mathbf{v}_1, \mathbf{v}_2) = f(\mathbf{v}_1)g(\mathbf{v}_2) - f(\mathbf{v}_2)g(\mathbf{v}_1).$$

### Proposition 1.8 (Anticommutativity of wedge product)

The wedge product is anticommutative: if  $f \in A_k(V)$  and  $g \in A_l(V)$ , then

$$f \wedge g = (-1)^{kl} g \wedge f.$$

*Proof.* Define  $\tau \in S_{k+l}$  to be the following permutation:

$$\begin{bmatrix} 1 & 2 & \cdots & l & l+1 & l+2 & \cdots & l+k \\ k+1 & k+2 & \cdots & k+l & 1 & 2 & \cdots & k \end{bmatrix}.$$

In other words,

$$\tau(i) = \begin{cases} k+i & \text{if } 1 \leq i \leq l, \\ i-l & \text{if } l+1 \leq i \leq l+k. \end{cases}$$

Then for any  $\sigma \in S_{k+l}$ ,

$$\sigma(j) = \begin{cases} \sigma(\tau(l+j)) & \text{if } 1 \leq j \leq k, \\ \sigma(\tau(j-k)) & \text{if } k+1 \leq j \leq k+l. \end{cases} \quad (1.30)$$

Now, for any  $\mathbf{v}_1, \dots, \mathbf{v}_{k+l} \in V$ ,

$$\begin{aligned} \mathcal{A}(f \otimes g)(\mathbf{v}_1, \dots, \mathbf{v}_{k+l}) &= \sum_{\sigma \in S_{k+l}} (\text{sgn } \sigma) f(\mathbf{v}_{\sigma(1)}, \dots, \mathbf{v}_{\sigma(k)}) g(\mathbf{v}_{\sigma(k+1)}, \dots, \mathbf{v}_{\sigma(k+l)}) \\ &= \sum_{\sigma \in S_{k+l}} (\text{sgn } \sigma) f(\mathbf{v}_{\sigma(\tau(l+1))}, \dots, \mathbf{v}_{\sigma(\tau(l+k))}) g(\mathbf{v}_{\sigma(\tau(1))}, \dots, \mathbf{v}_{\sigma(\tau(l))}) \\ &= (\text{sgn } \tau) \sum_{\sigma \in S_{k+l}} (\text{sgn } \sigma) (\text{sgn } \tau) g(\mathbf{v}_{\sigma(\tau(1))}, \dots, \mathbf{v}_{\sigma(\tau(l))}) f(\mathbf{v}_{\sigma(\tau(l+1))}, \dots, \mathbf{v}_{\sigma(\tau(l+k))}) \\ &= (\text{sgn } \tau) \sum_{\sigma \in S_{k+l}} (\text{sgn } \sigma \tau) g(\mathbf{v}_{\sigma(\tau(1))}, \dots, \mathbf{v}_{\sigma(\tau(l))}) f(\mathbf{v}_{\sigma(\tau(l+1))}, \dots, \mathbf{v}_{\sigma(\tau(l+k))}). \end{aligned}$$

Again, as  $\sigma$  varies over  $S_{k+l}$ ,  $\sigma\tau$  also varies over  $S_{k+l}$ . Therefore,

$$\mathcal{A}(f \otimes g)(\mathbf{v}_1, \dots, \mathbf{v}_{k+l}) = (\text{sgn } \tau) \mathcal{A}(g \otimes f)(\mathbf{v}_1, \dots, \mathbf{v}_{k+l}). \quad (1.31)$$

Now, let us evaluate the sign of the permutation  $\tau$ . Let  $(\tau(i), \tau(j))$  be an inversion of  $\tau$ . Then it's not possible that  $1 \leq i < j \leq l$ , or  $l+1 \leq i < j \leq l+k$ ; because if we have  $1 \leq i < j \leq l$  or  $l+1 \leq i < j \leq l+k$ , then  $\tau(i) < \tau(j)$ . Therefore,  $i$  must be in between 1 and  $l$  (inclusive), and  $j$  must be in between  $l+1$  and  $l+k$  (inclusive). So there are  $l$  options for  $i$ , and  $k$  options for  $j$ . Therefore,  $\tau$  has  $kl$  many inversions. So  $\text{sgn } \tau = (-1)^{kl}$ . Using (1.31),

$$\mathcal{A}(f \otimes g) = (-1)^{kl} \mathcal{A}(g \otimes f). \quad (1.32)$$

Dividing by  $k!l!$ , we obtain

$$f \wedge g = (-1)^{kl} g \wedge f. \quad (1.33)$$

■

### Corollary 1.9

If  $f$  is a  $k$ -covector on  $V$ , i.e.  $f \in A_k(V)$ , and  $k$  is odd, then  $f \wedge f = 0$ .

*Proof.* By anticommutativity of wedge product,

$$f \wedge f = (-1)^{k^2} f \wedge f = -f \wedge f.$$

Therefore,  $f \wedge f = 0$ . ■

If  $f$  is a  $k$ -covector and  $g$  is an  $l$ -covector, i.e.  $f \in A_k(V)$  and  $g \in A_l(V)$ , then we have defined their wedge product to be the  $(k+l)$ -covector

$$f \wedge g = \frac{1}{k!l!} \mathcal{A}(f \otimes g). \quad (1.34)$$

We have the following lemmas associated with the alternating operator  $\mathcal{A}$ .

**Lemma 1.10**

Suppose  $f \in L_k(V)$  and  $g \in L_l(V)$ . Then

- (i)  $\mathcal{A}(\mathcal{A}(f) \otimes g) = k! \mathcal{A}(f \otimes g)$ .
- (ii)  $\mathcal{A}(f \otimes \mathcal{A}(g)) = l! \mathcal{A}(f \otimes g)$ .

*Proof.* (i) By definition,

$$\begin{aligned} \mathcal{A}(\mathcal{A}(f) \otimes g) &= \sum_{\sigma \in S_{k+l}} (\text{sgn } \sigma) \sigma(\mathcal{A}(f) \otimes g) \\ &= \sum_{\sigma \in S_{k+l}} (\text{sgn } \sigma) \sigma \left[ \sum_{\tau \in S_k} (\text{sgn } \tau) (\tau f) \otimes g \right]. \end{aligned} \quad (1.35)$$

We can view  $\tau \in S_k$  as a permutation in the following way: define  $\tau' \in S_{k+l}$  as follows

$$\tau'(i) = \begin{cases} \tau(i) & \text{if } i \leq k, \\ i & \text{if } i > k. \end{cases} \quad (1.36)$$

Then for  $\mathbf{v}_1, \dots, \mathbf{v}_{k+l}$ , we have

$$\begin{aligned} [(\tau f) \otimes g](\mathbf{v}_1, \dots, \mathbf{v}_{k+l}) &= (\tau f)(\mathbf{v}_1, \dots, \mathbf{v}_k) g(\mathbf{v}_{k+1}, \dots, \mathbf{v}_{k+l}) \\ &= f(\mathbf{v}_{\tau(1)}, \dots, \mathbf{v}_{\tau(k)}) g(\mathbf{v}_{k+1}, \dots, \mathbf{v}_{k+l}) \\ &= f(\mathbf{v}_{\tau'(1)}, \dots, \mathbf{v}_{\tau'(k)}) g(\mathbf{v}_{\tau'(k+1)}, \dots, \mathbf{v}_{\tau'(k+l)}) \\ &= [\tau'(f \otimes g)](\mathbf{v}_1, \dots, \mathbf{v}_{k+l}). \end{aligned}$$

Therefore,  $(\tau f) \otimes g = \tau'(f \otimes g)$ . Furthermore,  $\text{sgn } \tau = \text{sgn } \tau'$  since the inversions  $(\tau'(i), \tau'(j))$  occur only when  $1 \leq i < j \leq k$ , so that the  $\tau$  and  $\tau'$  has the same number of inversions.

Let us abuse notation a bit and denote by  $S_k$  the subgroup of permutations in  $S_{k+l}$  by keeping the last  $l$  arguments fixed. This subgroup of  $S_{k+l}$  is indeed isomorphic to  $S_k$ , so we will denote both these groups by  $S_k$ . Therefore, from (1.35),

$$\begin{aligned} \mathcal{A}(\mathcal{A}(f) \otimes g) &= \sum_{\sigma \in S_{k+l}} (\text{sgn } \sigma) \sigma \left[ \sum_{\tau' \in S_k \subseteq S_{k+l}} (\text{sgn } \tau') \tau'(f \otimes g) \right] \\ &= \sum_{\sigma \in S_{k+l}} \sum_{\tau' \in S_k \subseteq S_{k+l}} (\text{sgn } \sigma) (\text{sgn } \tau') \sigma \tau'(f \otimes g) \\ &= \sum_{\tau' \in S_k \subseteq S_{k+l}} \sum_{\sigma \in S_{k+l}} (\text{sgn } \sigma \text{sgn } \tau') ((\sigma \tau')(f \otimes g)). \end{aligned}$$

For a fixed  $\tau'$ , as  $\sigma$  varies over  $S_{k+l}$ ,  $\sigma \tau'$  also varies over  $S_{k+l}$ . Therefore,

$$\mathcal{A}(\mathcal{A}(f) \otimes g) = \sum_{\tau' \in S_k \subseteq S_{k+l}} \mathcal{A}(f \otimes g) = k! \mathcal{A}(f \otimes g). \quad (1.37)$$

(ii) By (1.32),

$$\begin{aligned} \mathcal{A}(f \otimes \mathcal{A}(g)) &= \mathcal{A}((-1)^{kl} \mathcal{A}(g) \otimes f) \\ &= (-1)^{kl} \mathcal{A}(\mathcal{A}(g) \otimes f) \\ &= (-1)^{kl} l! \mathcal{A}(g \otimes f) \\ &= l! \mathcal{A}((-1)^{kl} g \otimes f) \\ &= l! \mathcal{A}(f \otimes g). \end{aligned} \quad (1.38)$$

■

**Proposition 1.11** (Associativity of wedge product)

Let  $V$  be a real vector space and  $f, g, h$  be alternating multilinear functions on  $V$  of degree  $k, l, m$ , respectively. Then

$$(f \wedge g) \wedge h = f \wedge (g \wedge h).$$

*Proof.* Using the definition of wedge product,

$$\begin{aligned} (f \wedge g) \wedge h &= \frac{1}{(k+l)!m!} \mathcal{A}[(f \wedge g) \otimes h] \\ &= \frac{1}{(k+l)!m!} \mathcal{A}\left[\frac{1}{k!l!} \mathcal{A}(f \otimes g) \otimes h\right] \\ &= \frac{1}{(k+l)!k!l!m!} \mathcal{A}[\mathcal{A}(f \otimes g) \otimes h] \\ &= \frac{(k+l)!}{(k+l)!k!l!m!} \mathcal{A}[(f \otimes g) \otimes h] \\ &= \frac{1}{k!l!m!} \mathcal{A}[(f \otimes g) \otimes h]. \end{aligned}$$

On the other hand,

$$\begin{aligned} f \wedge (g \wedge h) &= \frac{1}{k!(l+m)!} \mathcal{A}[f \otimes (g \wedge h)] \\ &= \frac{1}{k!(l+m)!} \mathcal{A}\left[f \otimes \left(\frac{1}{l!m!} \mathcal{A}(g \otimes h)\right)\right] \\ &= \frac{1}{k!(l+m)!l!m!} \mathcal{A}[f \otimes \mathcal{A}(g \otimes h)] \\ &= \frac{(l+m)!}{k!(l+m)!l!m!} \mathcal{A}[f \otimes (g \otimes h)] \\ &= \frac{1}{k!l!m!} \mathcal{A}[f \otimes (g \otimes h)]. \end{aligned}$$

Since tensor product is associative (by [Lemma 1.7](#)), we conclude that

$$(f \wedge g) \wedge h = f \wedge (g \wedge h). \quad (1.39)$$

■

By associativity, we can omit the parenthesis and write univocally  $f \wedge g \wedge h$  instead of  $(f \wedge g) \wedge h$  or  $f \wedge (g \wedge h)$ .

**Corollary 1.12**

Under the hypothesis of [Proposition 1.11](#),

$$f \wedge g \wedge h = \frac{1}{k!l!m!} \mathcal{A}[f \otimes g \otimes h]. \quad (1.40)$$

This easily generalizes to an arbitrary number of factors: if  $f_i \in A_{d_i}(V)$  for  $i = 1, 2, \dots, r$ , i.e.  $f_i$  is an alternating  $d_i$ -linear function on  $V$ , then

$$f_1 \wedge \dots \wedge f_r = \frac{1}{d_1! \dots d_r!} \mathcal{A}(f_1 \otimes \dots \otimes f_r). \quad (1.41)$$

**Proposition 1.13**

Let  $\hat{\alpha}^1, \hat{\alpha}^2, \dots, \hat{\alpha}^k$  be linear functions on a real vector space  $V$  (i.e.  $\hat{\alpha}^i : V \rightarrow \mathbb{R}$ ) and  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in V$ . Then

$$\begin{aligned} (\hat{\alpha}^1 \wedge \dots \wedge \hat{\alpha}^k)(\mathbf{v}_1, \dots, \mathbf{v}_k) &= \det [\hat{\alpha}^i(\mathbf{v}_j)] \\ &= \det \begin{bmatrix} \hat{\alpha}^1(\mathbf{v}_1) & \hat{\alpha}^1(\mathbf{v}_2) & \dots & \hat{\alpha}^1(\mathbf{v}_k) \\ \hat{\alpha}^2(\mathbf{v}_1) & \hat{\alpha}^2(\mathbf{v}_2) & \dots & \hat{\alpha}^2(\mathbf{v}_k) \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\alpha}^k(\mathbf{v}_1) & \hat{\alpha}^k(\mathbf{v}_2) & \dots & \hat{\alpha}^k(\mathbf{v}_k) \end{bmatrix}. \end{aligned}$$

*Proof.* By 1.41,

$$(\hat{\alpha}^1 \wedge \dots \wedge \hat{\alpha}^k)(\mathbf{v}_1, \dots, \mathbf{v}_k) = \mathcal{A}(\hat{\alpha}^1 \otimes \dots \otimes \hat{\alpha}^k)(\mathbf{v}_1, \dots, \mathbf{v}_k).$$

By the definition of the action of alternating operator,

$$\mathcal{A}(\hat{\alpha}^1 \otimes \dots \otimes \hat{\alpha}^k)(\mathbf{v}_1, \dots, \mathbf{v}_k) = \sum_{\sigma \in S_k} (\text{sgn } \sigma) \hat{\alpha}^1(\mathbf{v}_{\sigma(1)}) \dots \hat{\alpha}^k(\mathbf{v}_{\sigma(k)}). \quad (1.42)$$

By the definition of determinant of a  $k \times k$  matrix  $A = [a_{ij}]$ ,

$$\det A = \sum_{\sigma \in S_k} (\text{sgn } \sigma) a_{1\sigma(1)} a_{2\sigma(2)} \dots a_{k\sigma(k)}. \quad (1.43)$$

Using (1.43) in (1.42), we get

$$\mathcal{A}(\hat{\alpha}^1 \otimes \dots \otimes \hat{\alpha}^k)(\mathbf{v}_1, \dots, \mathbf{v}_k) = \det [\hat{\alpha}^i(\mathbf{v}_j)]. \quad (1.44)$$

■

**§1.5 A Basis for  $A_k(V)$** 

Let  $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$  be a basis for a real vector space  $V$ , and let  $\{\hat{\alpha}^1, \dots, \hat{\alpha}^n\}$  be the dual basis for  $V^*$ . Introduce the multi-index notation

$$I = (i_1, i_2, \dots, i_k)$$

and write  $\mathbf{e}_I$  for  $(\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \dots, \mathbf{e}_{i_k})$  and  $\hat{\alpha}^I$  for  $\hat{\alpha}^{i_1} \wedge \hat{\alpha}^{i_2} \wedge \dots \wedge \hat{\alpha}^{i_k}$ .

A  $k$ -linear function  $f$  on  $V$  is completely determined by its values on all  $k$ -tuples  $(\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \dots, \mathbf{e}_{i_k})$ . If  $f$  is alternating, then  $f$  is completely determined by its values on all  $k$ -tuples  $(\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \dots, \mathbf{e}_{i_k})$  with

$$1 \leq i_1 < i_2 < \dots < i_k \leq n.$$

In other words, it's sufficient to consider  $\mathbf{e}_I$  with  $I$  in ascending order.

**Lemma 1.14**

Suppose  $I$  and  $J$  are ascending multi-indices of length  $k$ . Then

$$\hat{\alpha}^I(\mathbf{e}_J) = \delta^I_J := \begin{cases} 1 & \text{if } I = J, \\ 0 & \text{if } I \neq J. \end{cases}$$

*Proof.* Suppose  $I = (i_1, \dots, i_k)$  and  $J = (j_1, \dots, j_k)$ . Using (1.42), we get

$$\begin{aligned} \hat{\alpha}^I(\mathbf{e}_J) &= (\hat{\alpha}^{i_1} \wedge \hat{\alpha}^{i_2} \wedge \dots \wedge \hat{\alpha}^{i_k})(\mathbf{e}_{j_1}, \mathbf{e}_{j_2}, \dots, \mathbf{e}_{j_k}) \\ &= \sum_{\sigma \in S_k} (\text{sgn } \sigma) \hat{\alpha}^{i_1}(\mathbf{e}_{j_{\sigma(1)}}) \dots \hat{\alpha}^{i_k}(\mathbf{e}_{j_{\sigma(k)}}) \\ &= \sum_{\sigma \in S_k} (\text{sgn } \sigma) \delta^{i_1}_{j_{\sigma(1)}} \dots \delta^{i_k}_{j_{\sigma(k)}}. \end{aligned} \quad (1.45)$$



The terms in the sum (1.45) contribute  $\text{sgn } \sigma$  if and only if

$$(i_1, i_2, \dots, i_k) = (j_{\sigma(1)}, j_{\sigma(2)}, \dots, j_{\sigma(k)});$$

otherwise they contribute 0 to the sum. Both  $I$  and  $J$  are ascending multi-indices. Permuting the elements of  $J$  no longer gives an ascending multi-index (unless the permutation  $\sigma$  is the identity permutation). Therefore, in (1.45), all the summands corresponding to  $\sigma$  being a non-identity permutation contribute 0.

$$\hat{\alpha}^I(\mathbf{e}_J) = \sum_{\sigma \in S_k} (\text{sgn } \sigma) \delta^{i_1}_{j_{\sigma(1)}} \cdots \delta^{i_k}_{j_{\sigma(k)}} = \delta^{i_1}_{j_1} \cdots \delta^{i_k}_{j_k} = \begin{cases} 1 & \text{if } I = J, \\ 0 & \text{if } I \neq J. \end{cases} \quad (1.46)$$

■

### Proposition 1.15

The alternating  $k$ -linear functions  $\hat{\alpha}^I$ ,  $I = (i_1, \dots, i_k)$ , with  $1 \leq i_1 < \dots < i_k \leq n$  form a basis for the space  $A_k(V)$  of alternating  $k$ -linear functions on  $V$ .

*Proof.* Let us first show linear independence. Suppose

$$\sum_I c_I \hat{\alpha}^I = \mathbf{0}, \quad (1.47)$$

$c_I \in \mathbb{R}$  with  $I$  running over ascending multi-indices of length  $k$ . Applying  $\mathbf{e}_J$  to both sides, we get

$$0 = \sum_I c_I \hat{\alpha}^I(\mathbf{e}_J) = \sum_I c_I \delta^I_J = c_J. \quad (1.48)$$

Therefore,  $\{\hat{\alpha}^I \mid I \text{ is ascending multi-index of length } k\}$  is a linearly independent set. Now let us prove that this set spans  $A_k(V)$ . Let  $f \in A_k(V)$ . We claim that

$$f = \sum_I f(\mathbf{e}_I) \hat{\alpha}^I. \quad (1.49)$$

Let  $g = \sum_I f(\mathbf{e}_I) \hat{\alpha}^I$ . We need to prove that  $f = g$ . By  $k$ -linearity and alternating property, if two  $k$ -covectors agree on all  $\mathbf{e}_J$  where  $J$  is an ascending multi-index, then they are equal. Now,

$$g(\mathbf{e}_J) = \sum_I f(\mathbf{e}_I) \hat{\alpha}^I(\mathbf{e}_J) = \sum_I f(\mathbf{e}_I) \delta^I_J = f(\mathbf{e}_J). \quad (1.50)$$

Therefore,  $f = g = \sum_I f(\mathbf{e}_I) \hat{\alpha}^I$ . ■

### Corollary 1.16

If the vector space  $V$  has dimension  $n$ , then the vector space  $A_k(V)$  of  $k$ -covectors on  $V$  has dimension  $\binom{n}{k}$ .

*Proof.* An ascending multi-index  $I = (i_1, i_2, \dots, i_k)$ ,  $1 \leq i_1 < i_2 < \dots < i_k \leq n$  is obtained by choosing a  $k$ -element subset of  $\{1, 2, \dots, n\}$ . This can be done in  $\binom{n}{k}$  ways. ■

### Corollary 1.17

If  $k > \dim V$ , then  $A_k(V) = 0$ .

*Proof.* If  $k > \dim V = n$ , then in the expression

$$\hat{\alpha}^{i_1} \wedge \hat{\alpha}^{i_2} \wedge \dots \wedge \hat{\alpha}^{i_k}$$

with each  $i \in \{1, 2, \dots, n\}$ , there must be a repeated  $i_j$ 's, say  $\hat{\alpha}^r$ . Then  $\hat{\alpha}^r \wedge \hat{\alpha}^r$  arises in the expression  $\hat{\alpha}^{i_1} \wedge \hat{\alpha}^{i_2} \wedge \dots \wedge \hat{\alpha}^{i_k}$ . But  $\hat{\alpha}^r \wedge \hat{\alpha}^r = 0$  by Corollary 1.9. Hence,  $\hat{\alpha}^{i_1} \wedge \hat{\alpha}^{i_2} \wedge \dots \wedge \hat{\alpha}^{i_k} = 0$ . Therefore, the basis set of  $A_k(V)$  is empty, meaning  $A_k(V) = 0$ . ■

# 2 Differential Forms on $\mathbb{R}^n$

Given an open set  $U \subseteq \mathbb{R}^n$  and  $p \in U$ ,  $T_p U$  is the set of tangent vectors at  $p \in U$  is identified with the point derivations of  $C_p^\infty$  (germs of smooth functions at  $p$ ), i.e. a tangent vector  $X_p \in T_p U$  is a map  $X_p : C_p^\infty \rightarrow \mathbb{R}$  such that  $X_p$  is  $\mathbb{R}$ -linear:

$$X_p(\alpha f + g) = \alpha(X_p f) + X_p g; \quad (2.1)$$

and satisfies the Leibniz condition:

$$X_p(fg) = (X_p f)g(p) + f(p)(X_p g).$$

In contrast to the notion of point derivation, there is this notion of derivation of aa algebra. If  $X$  is a  $C^\infty$  vector field on an open subset  $U \subseteq \mathbb{R}^n$ , i.e.  $X \in \mathfrak{X}(U)$ , and  $f$  is a  $C^\infty$  function on  $U$ , i.e.  $f \in C^\infty(U)$ , then  $Xf \in C^\infty(U)$  defined by

$$(Xf)(p) = X_p f.$$

Remember that  $f$  in (2.1) and (2) is a representative of an equivalence class, the equivalence class of germs of  $C^\infty$  functions at  $p \in U$ . These equivalence classes constitute  $C_p^\infty(U)$ . It is of course an  $\mathbb{R}$ -algebra. While in (2),  $f \in C^\infty(U)$ , the algebra of  $C^\infty$  functions on  $U$  with no reference of  $p$  whatsoever.

From the discussion above, a  $C^\infty$  vector field  $X$  gives rise to an  $\mathbb{R}$ -linear map  $C^\infty(U) \rightarrow C^\infty(U)$  by  $f \mapsto Xf$  that additionally has to satisfy the following Leibniz condition:

$$X(fg) = (Xf)g + f(Xg). \quad (2.2)$$

Note that a derivation at  $p$  is not a derivation of the algebra  $C_p^\infty$ . A derivation at  $p$  is a map from  $C_p^\infty \rightarrow \mathbb{R}$  that satisfies (2), while a derivation of the algebra  $C_p^\infty$  is supposed to be a map from  $C_p^\infty$  to itself obeying Leibniz condition.

## §2.1 1 form

From any  $C^\infty$  function  $f : U \rightarrow \mathbb{R}$ , one can construct a 1-form (dual notion of  $C^\infty$  vector field)  $df$ , the restriction of which to a given point  $p \in U$  yields a covector  $(df)_p \in T_p^* U$ , the dual space of  $T_p U$ , in the following way:

$$(df)_p(X_p) = X_p f. \quad (2.3)$$

### Proposition 2.1

If  $x^1, x^2, \dots, x^n$  are the standard coordinates on  $\mathbb{R}^n$ , then at each point  $p \in \mathbb{R}^n$ ,

$$\left\{ (dx^1)_p, (dx^2)_p, \dots, (dx^n)_p \right\}$$

is the basis for the cotangent space  $T_p^* \mathbb{R}^n$  dual to the basis  $\left\{ \frac{\partial}{\partial x^1} \Big|_p, \dots, \frac{\partial}{\partial x^n} \Big|_p \right\}$  for the tangent space  $T_p \mathbb{R}^n$ .

*Proof.*  $(dx^i)_p : T_p^* \mathbb{R}^n \rightarrow \mathbb{R}$  is a linear map for each  $i$ . Now,

$$(dx^i)_p \left( \frac{\partial}{\partial x^j} \Big|_p \right) = \frac{\partial}{\partial x^j} \Big|_p (x^i) = \delta^i_j. \quad (2.4)$$

Therefore,  $\{(\mathrm{d}x^1)_p, (\mathrm{d}x^2)_p, \dots, (\mathrm{d}x^n)_p\}$  is the basis of  $T_p^*\mathbb{R}^n$  dual to the basis  $\left\{\frac{\partial}{\partial x^1}\Big|_p, \dots, \frac{\partial}{\partial x^n}\Big|_p\right\}$  for  $T_p\mathbb{R}^n$ .  $\blacksquare$

If  $\omega$  is a 1-form on an open subset  $U \subseteq \mathbb{R}^n$ , then by [Proposition 2.1](#), there is a linear combination

$$\omega_p = \sum_{i=1}^n a_i(p) (\mathrm{d}x^i)_p, \quad (2.5)$$

for some  $a_i(p) \in \mathbb{R}$ . As  $p$  varies over  $U$ , the coefficients  $a_i$  become functions on  $U$ , and we may write

$$\omega = \sum_{i=1}^n a_i \mathrm{d}x^i. \quad (2.6)$$

The 1-form  $\omega$  is said to be  $C^\infty$  on  $U$  if the coefficient functions  $a_i$  are all  $C^\infty$  functions on  $U$ .

**Proposition 2.2 (The differential in terms of coordinates)**

If  $f : U \rightarrow \mathbb{R}$  is a  $C^\infty$  function on an open set  $U \subseteq \mathbb{R}^n$ , then

$$\mathrm{d}f = \sum_{i=1}^n \frac{\partial f}{\partial x^i} \mathrm{d}x^i.$$

*Proof.* By [Proposition 2.1](#), at each point  $p \in U$ ,

$$(\mathrm{d}f)_p = \sum_{i=1}^n a_i(p) (\mathrm{d}x^i)_p, \quad (2.7)$$

for some constants  $a_i(p)$  depending on  $p$ . Thus

$$\mathrm{d}f = \sum_{i=1}^n a_i \mathrm{d}x^i, \quad (2.8)$$

for some functions  $a_i$  on  $U$ . To evaluate  $a_j$ , apply both sides of (2.8) to the coordinate vector field  $\frac{\partial}{\partial x^j}$ :

$$\mathrm{d}f\left(\frac{\partial}{\partial x^j}\right) = \sum_{i=1}^n a_i \mathrm{d}x^i\left(\frac{\partial}{\partial x^j}\right) = \sum_{i=1}^n a_i \delta^i_j = a_j. \quad (2.9)$$

On the other hand, using  $(\mathrm{d}f)_p(X_p) = X_p f = (Xf)(p)$ , we get  $(\mathrm{d}f)(X) = Xf$ . So

$$\mathrm{d}f\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial f}{\partial x^j}. \quad (2.10)$$

Therefore,  $a_j = \frac{\partial f}{\partial x^j}$ . Hence,

$$\mathrm{d}f = \sum_{i=1}^n \frac{\partial f}{\partial x^i} \mathrm{d}x^i. \quad (2.11)$$

$\blacksquare$

(2.11) tells us that  $\mathrm{d}f$  will be a  $C^\infty$  1-form if  $\frac{\partial f}{\partial x^i}$  is  $C^\infty$  on  $U$ . Hence, it is sufficient to have  $f$  as a  $C^\infty$  function on  $U$  in order to have  $\mathrm{d}f$  as a  $C^\infty$  1-form.

## §2.2 Differential $k$ -forms

A differential form  $\omega$  of degree  $k$  (or a  $k$ -form) on an open subset  $U \subseteq \mathbb{R}^n$  is a map that assigns to each point  $p \in U$ , an alternating  $k$ -linear function on the tangent space  $T_p\mathbb{R}^n$ , i.e.

$$\omega_p \in A_k(T_p\mathbb{R}^n).$$

By Proposition 1.15, a basis for  $A_k(T_p\mathbb{R}^n)$  is

$$\left(dx^I\right)_p = \left(dx^{i_1}\right)_p \wedge \left(dx^{i_2}\right)_p \wedge \cdots \wedge \left(dx^{i_k}\right)_p,$$

where  $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ . Therefore, at each point  $p \in U$ ,  $\omega_p$  is a linear combination

$$\omega_p = \sum_I a_I(p) \left(dx^I\right)_p, \quad (2.12)$$

and a  $k$ -form  $\omega$  on  $U$  is a linear combination

$$\omega = \sum_I a_I dx^I, \quad (2.13)$$

with function coefficients  $a_I : U \rightarrow \mathbb{R}$ . We say that a  $k$ -form  $\omega$  is **smooth** on  $U$  if all the coefficients  $a_I$  are  $C^\infty$  functions on  $U$ .

Denote by  $\Omega^k(U)$  the vector space of  $C^\infty$   $k$ -forms on  $U$ . A 0-form on  $U$  assigns to each point  $p \in U$  an element of  $A_0(T_p\mathbb{R}^n) = \mathbb{R}$ . Thus a 0-form on  $U$  is simply a real-valued function on  $U$ , and  $\Omega^0(U) = C^\infty(U)$ .

Since one can multiply a  $C^\infty$   $k$ -form by a  $C^\infty$  function on  $U$  from the left, the set  $\Omega^k(U)$  of  $C^\infty$   $k$ -forms on  $U$  is both a real vector space and a  $C^\infty(U)$ -module. With the wedge product as multiplication, the direct sum

$$\Omega^*(U) = \bigoplus_{k=0}^n \Omega^k(U)$$

becomes an algebra over  $\mathbb{R}$  as well as a module over  $C^\infty(U)$ . As an algebra, it is anticommutative and associative.

**Example 2.1.** Let  $x, y, z$  be the coordinates on  $\mathbb{R}^3$ . The  $C^\infty$  1-forms on  $\mathbb{R}^3$  are

$$P(x, y, z) dx + Q(x, y, z) dy + R(x, y, z) dz,$$

where  $P, Q, R$  range over all  $C^\infty$  functions on  $\mathbb{R}^3$ . The  $C^\infty$  2-forms are

$$A(x, y, z) dy \wedge dz + B(x, y, z) dx \wedge dz + C(x, y, z) dx \wedge dy;$$

and the  $C^\infty$  1-forms are

$$a(x, y, z) dx \wedge dy \wedge dz.$$

**Example 2.2** (A basis for 3-covectors). Let  $x^1, x^2, x^3, x^4$  be the standard coordinates on  $\mathbb{R}^4$ , and  $p \in \mathbb{R}^4$ . A basis for  $A_3(T_p\mathbb{R}^4)$  is

$$\left\{ \left(dx^1\right)_p \wedge \left(dx^2\right)_p \wedge \left(dx^3\right)_p, \left(dx^1\right)_p \wedge \left(dx^2\right)_p \wedge \left(dx^4\right)_p, \right. \\ \left. \left(dx^1\right)_p \wedge \left(dx^3\right)_p \wedge \left(dx^4\right)_p, \left(dx^2\right)_p \wedge \left(dx^3\right)_p \wedge \left(dx^4\right)_p \right\}.$$

So  $\dim(A_3(T_p\mathbb{R}^n)) = 4$ .

### §2.3 Exterior Derivative

Before defining exterior derivative of a  $C^\infty$   $k$ -form on an open subset  $U \subseteq \mathbb{R}^n$ , we first define it on 0-forms. The exterior derivative of a  $C^\infty$  function  $f \in C^\infty(U)$  is its differential:

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i \in \Omega^1(U).$$

**Definition 2.1** (Exterior Derivative). If  $\omega = \sum_I a_I dx^I \in \omega^K(U)$ , then its exterior derivative is defined as follows:

$$d\omega = \sum_I da_I \wedge dx^I = \sum_I \left( \sum_{j=1}^n \frac{\partial a_I}{\partial x^j} dx^j \right) \wedge dx^I \in \Omega^{k+1}(U). \quad (2.14)$$

**Example 2.3.** Let  $\omega$  be the 1 form  $f dx + g dy$  on  $\mathbb{R}^2$ , where  $f$  and  $g$  are  $C^\infty$  functions on  $\mathbb{R}^2$ . Let us write  $f_x = \frac{\partial f}{\partial x}$  and  $f_y = \frac{\partial f}{\partial y}$ . Then

$$\begin{aligned} d\omega &= df \wedge dx + dg \wedge dy \\ &= (f_x dx + f_y dy) \wedge dx + (g_x dx + g_y dy) \wedge dy \\ &= -f_y dx \wedge dy + g_x dx \wedge dy \\ &= (g_x - f_y) dx \wedge dy. \end{aligned}$$

**Definition 2.2** (Graded Algebra). An algebra  $A$  over a field  $\mathbb{K}$  is said to be **graded** if it can be written as a direct sum

$$A = \bigoplus_{k=0}^{\infty} A^k$$

of vector spaces over  $\mathbb{K}$  so that the multiplication map sends  $A^k \times A^l$  to  $A^{k+l}$ .

The notation  $A = \bigoplus_{k=0}^{\infty} A^k$  means that each element of  $A$  is uniquely a **finite sum**

$$a = a_{i_1} + a_{i_2} + \cdots + a_{i_m},$$

where  $a_{i_j} \in A^{i_j}$ .

**Example 2.4.** The polynomial algebra

$$\mathbb{R}[x, y] = \bigoplus_{k=0}^{\infty} A^k$$

with  $A^k$  being the vector space of homogenous polynomials of degree  $k$  in  $x$  and  $y$ . Observe that the 0 polynomial is trivially homogenous of any degree, and hence belongs to  $A^k$  for all  $k \geq 0$ . Multiplication of degree  $k$  homogenous polynomial with a degree  $l$  homogenous polynomial in  $x$  and  $y$  will result in a homogenous polynomial of degree  $k + l$  in  $x$  and  $y$ .

**Example 2.5.** The algebra  $\Omega^*(U)$  of  $C^\infty$  differential forms on  $U$  is also graded by the degree of differential forms. Each  $\Omega^k(U)$  is a vector space. Multiplication of differential forms is defined by wedge product between them. The wedge product of a degree  $k$  differential form on  $U$  with a degree  $l$  differential form results in a degree  $k + l$  differential form.

**Definition 2.3** (Anti-derivation). Let  $A = \bigoplus_{k=0}^{\infty} A^k$  be a graded algebra over a field  $\mathbb{K}$ . An **anti-derivation** of the graded algebra  $A$  is a  $\mathbb{K}$ -linear map  $D : A \rightarrow A$  such that for  $\omega \in A^k$  and  $\tau \in A^l$ , one has

$$D(\omega\tau) = (D\omega)\tau + (-1)^k \omega(D\tau). \quad (2.15)$$

If the antiderivation  $D$  sends  $\omega \in A^k$  to  $D\omega \in A^{k+m}$ , we say that it is an antiderivation of degree  $m$ .

**Proposition 2.3** (i) The exterior derivative  $d : \Omega^*(U) \rightarrow \Omega^*(U)$  is an antiderivation of degree 1:

$$d(\omega \wedge \tau) = (d\omega) \wedge \tau + (-1)^{\deg \omega} \omega \wedge d\tau. \quad (2.16)$$

(ii)  $d^2 = 0$ .

(iii) If  $f \in \Omega^0(U) = C^\infty(U)$  and  $X \in \mathfrak{X}(U)$  (the space of  $C^\infty$  vector fields), then  $(df)(X) = Xf$ .

*Proof.* (i) Since the exterior derivative operator  $d : \Omega^k(U) \rightarrow \Omega^{k+1}(U)$  is linear, it suffices to check the equality (2.16) for  $\omega = f dx^I$  and  $\tau = g dx^J$  with  $I = (i_1, \dots, i_k)$  and  $J = (j_1, \dots, j_l)$  being strictly ascending multi-indices.

$$\begin{aligned} d(\omega \wedge \tau) &= d(fg dx^I \wedge dx^J) \\ &= \sum_{i=1}^n \frac{\partial(fg)}{\partial x^i} dx^i \wedge dx^I \wedge dx^J \\ &= \sum_{i=1}^n \frac{\partial f}{\partial x^i} \cdot g dx^i \wedge dx^I \wedge dx^J + \sum_{i=1}^n f \cdot \frac{\partial g}{\partial x^i} dx^i \wedge dx^I \wedge dx^J \\ &= \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i \wedge dx^I \wedge g dx^J + \sum_{i=1}^n \frac{\partial g}{\partial x^i} dx^i \wedge f dx^I \wedge dx^J. \end{aligned} \quad (2.17)$$

Now, in the second sum in (2.17), one has to push  $\frac{\partial g}{\partial x^i} dx^i$  through the  $k$ -fold wedge product  $dx^I$  and hence in the process picks out a sign  $(-1)^k$ . Therefore,

$$\sum_{i=1}^n \frac{\partial g}{\partial x^i} dx^i \wedge f dx^I \wedge dx^J = (-1)^k f dx^I \wedge \frac{\partial g}{\partial x^i} dx^i \wedge dx^J. \quad (2.18)$$

Now, observe that

$$d\omega = d(f dx^I) = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i \wedge dx^I, \text{ and} \quad (2.19)$$

$$d\tau = d(g dx^J) = \sum_{i=1}^n \frac{\partial g}{\partial x^i} dx^i \wedge dx^J. \quad (2.20)$$

Therefore,

$$d(\omega \wedge \tau) = d\omega \wedge \tau + (-1)^k \omega \wedge d\tau. \quad (2.21)$$

(ii) Again, by  $\mathbb{R}$ -linearity of  $d$ , it suffices to show that  $d^2\omega = 0$  for  $\omega = f dx^I$ .

$$\begin{aligned} d^2(f dx^I) &= d\left(\sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i \wedge dx^I\right) \\ &= \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 f}{\partial x^j \partial x^i} dx^j \wedge dx^i \wedge dx^I. \end{aligned} \quad (2.22)$$

If  $i = j$ , then  $dx^j \wedge dx^i = 0$ . If  $i \neq j$ , then  $\frac{\partial^2 f}{\partial x^j \partial x^i}$  is symmetric in  $i$  and  $j$ , but  $dx^j \wedge dx^i$  is alternating in  $i$  and  $j$ . Therefore, the terms with  $i \neq j$  pair up and cancel out.

(iii) Let  $X = \sum_{i=1}^n a^i \frac{\partial}{\partial x^i}$ . Then

$$\begin{aligned}
 (df)(X) &= \left( \sum_{j=1}^n \frac{\partial f}{\partial x^j} dx^j \right) \left( \sum_{i=1}^n a^i \frac{\partial}{\partial x^i} \right) \\
 &= \sum_{j=1}^n \sum_{i=1}^n a^i \frac{\partial f}{\partial x^j} dx^j \left( \frac{\partial}{\partial x^i} \right) \\
 &= \sum_{j=1}^n \sum_{i=1}^n a^i \frac{\partial f}{\partial x^j} \delta^j_i \\
 &= \sum_{i=1}^n a^i \frac{\partial f}{\partial x^i} = Xf.
 \end{aligned} \tag{2.23}$$

■

### Proposition 2.4 (Characterization of exterior derivative)

The 3 properties of Proposition 2.3 uniquely characterize exterior derivative on an open set  $U \subseteq \mathbb{R}^n$ . In other words, if  $D : \Omega^*(U) \rightarrow \Omega^*(U)$  is an antiderivation of degree 1 such that  $D^2 = 0$  and for  $f \in C^\infty(U)$  and  $X \in \mathfrak{X}(U)$ ,  $(Df)(X) = Xf$ , then  $D = d$ .

*Proof.* Since every  $k$ -form on  $U$  is a sum of terms such as  $f dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}$ , by linearity of  $d$ , it suffices to show that  $D = d$  on a  $k$ -form of this type. Applying property (iii) for  $f = x^i$ , one has

$$Dx^i(X) = X(x^i).$$

Writing  $X = \sum_{j=1}^n a^j \frac{\partial}{\partial x^j}$ , we get

$$Dx^i \left( \sum_{j=1}^n a^j \frac{\partial}{\partial x^j} \right) = \sum_{j=1}^n a^j \frac{\partial}{\partial x^j} (x^i) = a^i = dx^i \left( \sum_{j=1}^n a^j \frac{\partial}{\partial x^j} \right).$$

Therefore,

$$Dx^i = dx^i. \tag{2.24}$$

Now,

$$\begin{aligned}
 D(f dx^{i_1} \wedge \cdots \wedge dx^{i_k}) &= D(f Dx^{i_1} \wedge \cdots \wedge Dx^{i_k}) \\
 &= Df \wedge (Dx^{i_1} \wedge \cdots \wedge Dx^{i_k}) + (-1)^0 f D(Dx^{i_1} \wedge \cdots \wedge Dx^{i_k}).
 \end{aligned} \tag{2.25}$$

Now, since  $df(X) = Xf = Df(X)$  for any  $X \in \mathfrak{X}(U)$ ,  $df = Df$ . Furthermore,  $D(Dx^{i_1}) = 0$ , and

$$\begin{aligned}
 D(Dx^{i_1} \wedge \cdots \wedge Dx^{i_k}) &= D^2 x^{i_1} \wedge Dx^{i_2} \wedge \cdots \wedge Dx^{i_k} - Dx^{i_1} \wedge D(Dx^{i_2} \wedge \cdots \wedge Dx^{i_k}) \\
 &= -Dx^{i_1} \wedge D(Dx^{i_2} \wedge \cdots \wedge Dx^{i_k}).
 \end{aligned} \tag{2.26}$$

Therefore, by induction on  $k$ ,

$$D(Dx^{i_1} \wedge \cdots \wedge Dx^{i_k}) = 0. \tag{2.27}$$

Hence, from (2.25),

$$\begin{aligned}
 D(f dx^{i_1} \wedge \cdots \wedge dx^{i_k}) &= Df \wedge (Dx^{i_1} \wedge \cdots \wedge Dx^{i_k}) \\
 &= df \wedge (dx^{i_1} \wedge \cdots \wedge dx^{i_k}) \\
 &= d(f dx^{i_1} \wedge \cdots \wedge dx^{i_k}).
 \end{aligned} \tag{2.28}$$

So  $D = d$  on  $\Omega^*(U)$ .

■

### Closed Forms and Exact Forms

A  $k$ -form  $\omega$  on  $U$  is **closed** if  $d\omega = 0$ ; it's **exact** if there is a  $(k-1)$ -form  $\tau$  on  $U$  such that  $\omega = d\tau$ . Since  $d^2 = 0$ , every exact form is closed. But in general, a closed form may fail to be exact. We will see how non-exact closed forms capture the geometry of a manifold when we do de Rham cohomology on a manifold.

**Example 2.6.** Define a 1-form  $\omega$  on  $\mathbb{R}^2 \setminus \{0\}$  by

$$\omega = \frac{1}{x^2 + y^2} (-ydx + xdy). \quad (2.29)$$

Then  $\omega$  is closed.

A collection of vector spaces  $\{V^k\}_{k=0}^\infty$  with linear maps  $d_k : V^k \rightarrow V^{k+1}$  such that  $d_{k+1} \circ d_k = 0$  is called a **differential complex** or a **cochain complex**. For any open set  $U \subseteq \mathbb{R}^n$ , the exterior derivative  $d$  makes the vector space  $\Omega^*(U)$  of  $C^\infty$  forms on  $U$  into a cochain complex, called the **de Rham complex** on  $U$ :

$$\Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \Omega^2(U) \xrightarrow{d} \dots$$

The closed forms are precisely the elements of the kernel of  $d$  and the exact forms are the elements of the image of  $d$ . In the language of cohomology,  $d$  is also called the coboundary operator.

## §2.4 Applications to Vector Calculus

The theory of differential forms unifies many theorems in vector calculus on  $\mathbb{R}^3$ . A vector valued function on  $\mathbb{R}^3$  is the same as a vector field. Recall the 3 operators on scalar and vector-valued functions on  $\mathbb{R}^3$ .

$$\{\text{scalar function}\} \xrightarrow{\text{grad}} \{\text{vector function}\} \xrightarrow{\text{curl}} \{\text{vector function}\} \xrightarrow{\text{div}} \{\text{scalar function}\}.$$

Let  $f$  be a scalar function and  $\begin{bmatrix} P \\ Q \\ R \end{bmatrix}$  be a vector field on  $\mathbb{R}^3$ , where each of  $P, Q, R$  is a scalar function on  $\mathbb{R}^3$ . Then

$$\begin{aligned} \text{grad } f &= \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix}, \\ \text{curl} \begin{bmatrix} P \\ Q \\ R \end{bmatrix} &= \begin{bmatrix} R_y - Q_z \\ P_z - R_x \\ Q_x - P_y \end{bmatrix}, \\ \text{div} \begin{bmatrix} P \\ Q \\ R \end{bmatrix} &= P_x + Q_y + R_z. \end{aligned} \quad (2.30)$$

Then one has the following results.

### Proposition 2.5

$$\text{curl}(\text{grad } f) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \quad (2.31)$$



**Proposition 2.6**

$$\operatorname{div} \left( \operatorname{curl} \begin{bmatrix} P \\ Q \\ R \end{bmatrix} \right) = 0. \quad (2.32)$$

**Proposition 2.7**

On  $\mathbb{R}^3$ , a vector field  $\mathbf{F}(x, y, z)$  is the gradient of some scalar function if and only if  $\operatorname{curl} \mathbf{F} = \mathbf{0}$ .

A 1-form on  $\mathbb{R}^3$  can be written as

$$P(x, y, z) dx + Q(x, y, z) dy + R(x, y, z) dz.$$

This 1-form on  $\mathbb{R}^3$  can be identified with the vector field  $\begin{bmatrix} P \\ Q \\ R \end{bmatrix}$ .

Similarly, the 2-forms on  $\mathbb{R}^3$  given by

$$A(x, y, z) dy \wedge dz + B(x, y, z) dz \wedge dx + C(x, y, z) dx \wedge dy$$

can be identified with the vector field  $\begin{bmatrix} A \\ B \\ C \end{bmatrix}$  on  $\mathbb{R}^3$ .

In terms of these identifications, the exterior derivative of a 0-form  $f$  (scalar function) is

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz,$$

which can be identified with the vector field

$$\begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \operatorname{grad} f.$$

The exterior derivative of a 1-form on  $\mathbb{R}^3$  is

$$\begin{aligned} & d(Pdx + Qdy + Rdz) \\ &= \frac{\partial P}{\partial y} dy \wedge dx + \frac{\partial P}{\partial z} dz \wedge dx + \frac{\partial Q}{\partial x} dx \wedge dy + \frac{\partial Q}{\partial z} dz \wedge dy + \frac{\partial R}{\partial x} dx \wedge dz + \frac{\partial R}{\partial y} dy \wedge dz \\ &= (R_y - Q_z) dy \wedge dz + (P_z - R_x) dz \wedge dx + (Q_x - P_y) dx \wedge dy, \end{aligned}$$

which corresponds to

$$\begin{bmatrix} R_y - Q_z \\ P_z - R_x \\ Q_x - P_y \end{bmatrix} = \operatorname{curl} \begin{bmatrix} P \\ Q \\ R \end{bmatrix}.$$

The exterior derivative of a 2-form is

$$\begin{aligned} & d(A dy \wedge dz + B dz \wedge dx + C dx \wedge dy) \\ &= A_x dx \wedge dy \wedge dz + B_y dy \wedge dz \wedge dx + C_z dz \wedge dx \wedge dy \\ &= (A_x + B_y + C_z) dx \wedge dy \wedge dz, \end{aligned}$$

which corresponds to

$$A_x + B_y + C_z = \operatorname{div} \begin{bmatrix} A \\ B \\ C \end{bmatrix}.$$

In summary, exterior derivative  $d$  on 0-forms is identified with **gradient**; exterior derivative  $d$  on 1-forms is identified with **curl**; exterior derivative  $d$  on 2-forms is identified with **divergence**. Using de Rham complex on  $U$ :

$$\Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \Omega^2(U) \xrightarrow{d} \Omega^3(U).$$

Using vector calculus language,

$$C^\infty(U) \xrightarrow{\text{grad}} \mathfrak{X}(U) \xrightarrow{\text{curl}} \mathfrak{X}(U) \xrightarrow{\text{div}} C^\infty(U).$$

**Remark 2.1.** Proposition 2.5 and Proposition 2.6 express the property  $d^2 = 0$  of exterior derivative. A vector field  $\begin{bmatrix} P \\ Q \\ R \end{bmatrix}$  is the gradient of a  $C^\infty$  function  $f$  if and only if the corresponding 1-form  $Pdx + Qdy + Rdz$  is  $df$ . Proposition 2.7 expresses the fact that a 1-form on  $\mathbb{R}^3$  is exact if and only if it is closed. It's worth remarking at this stage that Proposition 2.7 need not hold true on a region other than  $\mathbb{R}^3$ , as the following well-known example from calculus suggests.

**Example 2.7.** Suppose  $U = \mathbb{R}^3 \setminus \{z\text{-axis}\}$ , and  $\mathbf{F}(x, y, z)$  is the vector field

$$\mathbf{F} = \begin{bmatrix} \frac{-y}{x^2+y^2} \\ \frac{x}{x^2+y^2} \\ 0 \end{bmatrix}$$

on  $U$ . Then  $\text{curl } \mathbf{F} = \mathbf{0}$ . Indeed,

$$\begin{aligned} \text{curl } \mathbf{F} &= \begin{bmatrix} \frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}\left(\frac{x}{x^2+y^2}\right) \\ \frac{\partial}{\partial z}\left(\frac{-y}{x^2+y^2}\right) - \frac{\partial}{\partial x}(0) \\ \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ \frac{(x^2+y^2) - x \cdot 2x}{(x^2+y^2)^2} - \frac{-(x^2+y^2) + y \cdot 2y}{(x^2+y^2)^2} \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0}. \end{aligned}$$

But  $\mathbf{F}$  is not the gradient of a  $C^\infty$  function on  $U$ . Recall the theorem from vector calculus that the line integral of the gradient of a function along a curve gives the total change in the value of the function from the start to the end of the curve. In other words, if  $\mathbf{r} : [a, b] \rightarrow \mathbb{R}^3$  is a curve and  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$  is a scalar function, then

$$\int_a^b (\nabla f) \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)). \quad (2.33)$$

Then if  $\mathbf{F}$  is the gradient of a smooth scalar function, then the line integral

$$\oint_C \frac{-y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy$$

over any closed curve would become 0. Let us take the closed curve to be the unit circle:  $x = \cos t$ ,

$y = \sin t$ ,  $z = 0$  for  $t \in [0, 2\pi]$ . Then

$$\begin{aligned} & \oint_C \frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy \\ &= \int_0^{2\pi} -\sin t \, d(\cos t) + \int_0^{2\pi} \cos t \, d(\sin t) \\ &= \int_0^{2\pi} \sin^2 t \, dt + \int_0^{2\pi} \cos^2 t \, dt \\ &= 2\pi. \end{aligned}$$

Hence, although  $\text{curl } \mathbf{F} = \mathbf{0}$ , there is no  $C^\infty$  function  $f$  on  $U$  such that  $\mathbf{F} = \text{grad } f$ . In the language of differential forms, the 1-form

$$\omega = \frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy$$

is closed but not exact.

It turns out that whether [Proposition 2.7](#) is true for a region  $U \subseteq \mathbb{R}^3$  depends on the topology of  $U$ . One measure of the failure of a closed  $k$ -form to be exact is the quotient vector space

$$H^k(U) = \frac{\{\text{closed } k\text{-forms on } U\}}{\{\text{exact } k\text{-forms on } U\}},$$

called the  $k$ -th de Rham cohomology of  $U$ . The generalization of [Proposition 2.7](#) to any differential form on  $\mathbb{R}^n$  is called the **Poincaré lemma**:

For  $k \geq 1$ , every closed  $k$ -form on  $\mathbb{R}^n$  is exact.

This statement is equivalent to the vanishing of the  $k$ -th de Rham cohomology  $H^k(\mathbb{R}^n)$  for  $k \geq 1$ .

# 3 Differential Forms on Manifold

## §3.1 Definition and Local Expression

Let  $M$  be a smooth manifold and  $p \in M$ . The **cotangent space** of  $M$  at  $p$ , denoted by  $T_p^*M$  is the dual space of the tangent space  $T_pM$ . An element in  $T_p^*M$  is called a covector at  $p$ . Thus, a covector  $\omega_p \in T_p^*M$  is a linear function

$$\omega_p : T_pM \rightarrow \mathbb{R}.$$

A 1-form on  $M$  is a function that assigns to each  $p \in M$ , a covector at  $p$ .

**Definition 3.1** (Differential of a function). Let  $f : M \rightarrow \mathbb{R}$  be a  $C^\infty$  function on a manifold  $M$ . Its **differential** is defined to be the 1-form  $df$  on  $M$  such that for any  $p \in M$  and  $X_p \in T_pM$ ,

$$(df)_p(X_p) = X_p f. \quad (3.1)$$

### Proposition 3.1

If  $f : M \rightarrow \mathbb{R}$  is a  $C^\infty$  function, then for  $p \in M$  and  $X_p \in T_pM$ ,

$$f_{*,p}(X_p) = (df)_p(X_p) \frac{\partial}{\partial x} \Big|_{f(p)}.$$

*Proof.* Since  $f_{*,p}(X_p) \in T_{f(p)}\mathbb{R}$ , there is a real number  $c$  such that

$$f_{*,p}(X_p) = c \frac{\partial}{\partial x} \Big|_{f(p)}. \quad (3.2)$$

(Here the chart chosen on  $\mathbb{R}$  is  $(\mathbb{R}, \mathbb{1}_{\mathbb{R}})$  so that  $x$  is the coordinate of this chart, i.e.  $x = \mathbb{1}_{\mathbb{R}}$ .) To evaluate  $c$ , apply both sides of (3.2) to the function  $x \in C^\infty(\mathbb{R})$ . Then

$$f_{*,p}(X_p)(x) = c \frac{\partial}{\partial x} \Big|_{f(p)}(x) = c.$$

Therefore,

$$c = f_{*,p}(X_p)(x) = X_p(x \circ f) = X_p f = (df)_p(X_p), \quad (3.3)$$

since  $x = \mathbb{1}_{\mathbb{R}}$ . Therefore, substituting the value of  $c$  into (3.2),

$$f_{*,p}(X_p) = (df)_p(X_p) \frac{\partial}{\partial x} \Big|_{f(p)}. \quad (3.4)$$

■

Let  $(U, \varphi) \equiv (U, x^1, x^2, \dots, x^n)$  be a coordinate chart on  $M$ . Here  $x^i = r^i \circ \varphi$ , where  $r^i$  is the  $i$ -th coordinate function of a vector in  $\mathbb{R}^n$ . Then the differentials  $dx^1, dx^2, \dots, dx^n$  are 1-forms on  $U$ .

### Proposition 3.2

At each point  $p \in U$ , the covectors  $(dx^1)_p, \dots, (dx^n)_p$  form a basis for the cotangent space  $T_p^*M$ , dual to the basis  $\left\{ \frac{\partial}{\partial x^1} \Big|_p, \dots, \frac{\partial}{\partial x^n} \Big|_p \right\}$  for the tangent space  $T_pM$ .

*Proof.* Observe that

$$\left(dx^i\right)_p \left(\frac{\partial}{\partial x^j}\Big|_p\right) = \frac{\partial}{\partial x^j}\Big|_p \left(x^i\right) = \delta^i_j. \quad (3.5)$$

So  $\{(dx^1)_p, \dots, (dx^n)_p\}$  is the dual basis to  $\left\{\frac{\partial}{\partial x^1}\Big|_p, \dots, \frac{\partial}{\partial x^n}\Big|_p\right\}$ . ■

Thus, every 1-form  $\omega$  on  $U$  can be written as a linear combination

$$\omega = \sum_{i=1}^n a_i dx^i,$$

where  $a_i$  are functions on  $U$ . In particular, if  $f$  is a  $C^\infty$  function on  $M$ , then the 1-form  $df$ , when restricted to  $U$ , must be a linear combination

$$df = \sum_{i=1}^n a_i dx^i. \quad (3.6)$$

If we evaluate both sides of (3.6) on  $\frac{\partial}{\partial x^j}$ ,

$$(df) \left(\frac{\partial}{\partial x^j}\right) = \sum_{i=1}^n a_i dx^i \left(\frac{\partial}{\partial x^j}\right) = \sum_{i=1}^n a_i \delta^i_j = a_j.$$

Then

$$a_j = (df) \left(\frac{\partial}{\partial x^j}\right) = \frac{\partial f}{\partial x^j}.$$

Therefore,

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i. \quad (3.7)$$

## §3.2 The Cotangent Bundle

The underlying set of the **cotangent bundle** is the disjoint union of the cotangent spaces at all points of  $M$ :

$$T^*M = \bigsqcup_{p \in M} T_p^*M = \bigcup_{p \in M} \{p\} \times T_p^*M. \quad (3.8)$$

Let us give  $T^*M$  a topology in the following way: let  $(U, x^1, \dots, x^n)$  be a chart on  $M$  and  $p \in U$ . Then each  $\omega_p \in T_p^*M$  can be written uniquely as a linear combination

$$\omega_p = \sum_{i=1}^n c_i(\omega_p) (dx^i)_p,$$

with  $c_i(\omega_p) \in \mathbb{R}$ . This gives rise to a bijection

$$\begin{aligned} \tilde{\varphi} : T^*U &\rightarrow \varphi(U) \times \mathbb{R}^n \\ (p, \omega_p) &\mapsto (\varphi(p), c_1(\omega_p), c_2(\omega_p), \dots, c_n(\omega_p)). \end{aligned}$$

We use this bijection  $\tilde{\varphi}$  to transfer the topology of  $\varphi(U) \times \mathbb{R}^n$  to  $T^*U$ : a set  $A \subseteq T^*U$  is said to be open if and only if  $\tilde{\varphi}(A)$  is open in  $\varphi(U) \times \mathbb{R}^n$ , where  $\varphi(U) \times \mathbb{R}^n$  is given the subspace topology of  $\mathbb{R}^{2n}$ . Now, let  $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$  be the maximal atlas of  $M$ . Now, let

$$\begin{aligned} \mathcal{B} &= \bigcup_{\alpha \in I} \{A \mid A \subseteq T^*U_\alpha \text{ is open in } A \subseteq T^*U_\alpha\} \\ &= \{A \mid A \subseteq T^*U_\alpha \text{ is open in } A \subseteq T^*U_\alpha, \alpha \in I\}. \end{aligned}$$

It can be shown using the same technique of tangent bundle that  $\mathcal{B}$  forms a basis for topology. We give  $T^*M$  the topology generated by the basis  $\mathcal{B}$ . We declare  $A \subseteq T^*M$  to be open if and only if there exists a subfamily  $\{B_\lambda\}_\lambda \subseteq \mathcal{B}$  such that

$$A = \bigcup_\lambda B_\lambda.$$

Furthermore,  $T^*M$  has the structure of a  $C^\infty$  manifold. An atlas for  $T^*M$  is

$$\{(T^*U_\alpha, \tilde{\varphi}_\alpha)\}_{\alpha \in I}.$$

If two coordinate open sets  $U_\alpha$  and  $U_\beta$  intersect, suppose  $U_{\alpha\beta} = U_\alpha \cap U_\beta$ . Then for  $p \in U_{\alpha\beta}$ , each  $\omega_p \in T_p^*M$  has two basis expansions:

$$\omega_p = \sum_{i=1}^n a_i (dx^i)_p = \sum_{j=1}^n b_j (dy^j)_p. \quad (3.9)$$

(Here  $(U_\alpha, x^1, \dots, x^n)$  and  $(U_\beta, y^1, \dots, y^n)$  are charts.) Now applying  $\left. \frac{\partial}{\partial y^k} \right|_p$  to both sides of (3.9),

$$b_k = \sum_{i=1}^n a_i (dx^i)_p \left( \left. \frac{\partial}{\partial y^k} \right|_p \right) = \sum_{i=1}^n a_i \left. \frac{\partial x^i}{\partial y^k} \right|_p.$$

Therefore,  $\tilde{\varphi}_\beta \circ \tilde{\varphi}_\alpha^{-1} : \varphi_\alpha(U_{\alpha\beta}) \times \mathbb{R}^n \rightarrow \varphi_\beta(U_{\alpha\beta}) \times \mathbb{R}^n$  is given by

$$(\varphi_\alpha(p), a_1, \dots, a_n) \mapsto \left( (\varphi_\beta \circ \varphi_\alpha^{-1})(\varphi_\alpha(p)), \sum_{i=1}^n a_i \left. \frac{\partial x^i}{\partial y^1} \right|_p, \dots, \sum_{i=1}^n a_i \left. \frac{\partial x^i}{\partial y^n} \right|_p \right).$$

$\varphi_\beta \circ \varphi_\alpha^{-1}$  is smooth, and each  $\left. \frac{\partial x^i}{\partial y^j} \right|_p$  is smooth. Therefore, the transition map  $\tilde{\varphi}_\beta \circ \tilde{\varphi}_\alpha^{-1}$  is smooth, making  $T^*M$  a smooth manifold.

$T^*M$  is, in fact, a **vector bundle** of rank  $n$  over  $M$ . It has a natural projection  $\pi : T^*M \rightarrow M$  given by  $(p, \omega_p) \mapsto p$ . In terms of cotangent bundle, a 1-form on  $M$  is simply a section of the cotangent bundle  $T^*M$ , i.e. it is a map  $\omega : M \rightarrow T^*M$  such that  $\pi \circ \omega = \mathbb{1}_M$ . We say that a 1-form is **smooth** if it is  $C^\infty$  as a map  $\omega : M \rightarrow T^*M$  between two manifolds.

### §3.3 Characterization of Smooth 1-forms

By definition, a 1-form on an open set  $U \subseteq M$  is  $C^\infty$  if it is  $C^\infty$  as a section of the cotangent bundle  $T^*M$  over  $U$ .

#### Lemma 3.3

Let  $(U, \varphi) \equiv (U, x^1, \dots, x^n)$  be a chart on a manifold  $M$ . A 1-form  $\omega = \sum a_i dx^i$  on  $U$  is smooth if and only if the coefficient functions  $a_i$  are all smooth on  $U$ .

*Proof.* This is a special case of Proposition 9.4.2 of DG1 which states that:

Let  $\pi : E \rightarrow M$  be a  $C^\infty$  vector bundle and  $U$  an open subset of  $M$ . Suppose  $s_1, \dots, s_r$  is a  $C^\infty$  frame for  $E$  over  $U$ . Then a section  $s = \sum_{j=1}^r c^j s_j$  of  $E$  over  $U$  is  $C^\infty$  if and only if the coefficients  $c^j$  are  $C^\infty$  functions on  $U$ .

Here we take  $E$  to be the cotangent bundle  $T^*M$ , and  $\{s_i\}_{i=1}^r$  the  $C^\infty$  frame for  $E$  over  $U$  to be the coordinate 1-forms  $\{(dx^i)\}_{i=1}^n$ . ■

**Proposition 3.4**

Let  $\omega$  be a 1-form on a manifold  $M$ . Then the following are equivalent:

- (i)  $\omega$  is  $C^\infty$ .
- (ii) For every point  $p \in M$ , there is a chart  $(U, x^1, \dots, x^n)$  about  $p$  such that if  $\omega = \sum_{i=1}^n a_i dx^i$  on  $U$ , then the functions  $a_i$  are  $C^\infty$  on  $U$ .
- (iii) For any chart  $(U, x^1, \dots, x^n)$  on  $M$ , if  $\omega = \sum_{i=1}^n a_i dx^i$  on  $U$ , then the functions  $a_i$  are  $C^\infty$  on  $U$ .

*Proof.* (ii) $\Rightarrow$ (i): By Lemma 3.3, for every point  $p \in M$ , there is a chart  $(U, x^1, \dots, x^n)$  about  $p$  such that  $\omega$  is smooth on  $U$ . In particular, the section  $\omega : M \rightarrow T^*M$  is smooth at  $p$ , for every  $p \in M$ . Therefore,  $\omega : M \rightarrow T^*M$  is a smooth map between manifolds.

(i) $\Rightarrow$ (iii): If  $\omega : M \rightarrow T^*M$  is a smooth map between manifolds,  $\omega$  is smooth at every chart of  $M$ . Therefore, by Lemma 3.3, if  $\omega = \sum_{i=1}^n a_i dx^i$  on a chart  $(U, x^1, \dots, x^n)$ , each  $a_i$  is smooth on  $U$ .

(iii) $\Rightarrow$ (ii): Obvious. ■

**Proposition 3.5**

A 1-form  $\omega$  on a manifold  $M$  is  $C^\infty$  if and only if for every  $C^\infty$  vector field  $X$ , the function  $\omega(X)$  is  $C^\infty$  on  $M$ .

*Proof.* ( $\Rightarrow$ ) Suppose  $\omega$  is a  $C^\infty$  1-form and  $X$  is a  $C^\infty$  vector field. Let  $(U, x^1, \dots, x^n)$  be a chart on  $M$ . Then

$$\omega = \sum_{i=1}^n a_i dx^i \quad \text{and} \quad X = \sum_{j=1}^n b^j \frac{\partial}{\partial x^j}, \quad (3.10)$$

for  $C^\infty$  functions  $a_i$  and  $b^j$  on  $U$ . Then on  $U$ , one has

$$\omega(X) = \left( \sum_{i=1}^n a_i dx^i \right) \left( \sum_{j=1}^n b^j \frac{\partial}{\partial x^j} \right) = \sum_{i=1}^n \sum_{j=1}^n a_i b^j \delta^i_j = \sum_{i=1}^n a_i b^i, \quad (3.11)$$

which is a  $C^\infty$  function on  $U$ . Since  $U$  was chosen to be an arbitrary coordinate open set,  $\omega(X)$  is a smooth function on all of  $M$ .

( $\Leftarrow$ ) Suppose  $\omega$  is a 1-form on  $M$  such that for every  $C^\infty$  vector field  $X$  on  $M$ , the function  $\omega(X)$  is smooth on  $M$ . For a given  $p \in M$ , choose a coordinate neighborhood  $(U, \varphi) \equiv (U, x^1, \dots, x^n)$  about  $p$ . Then one has

$$\omega = \sum_{i=1}^n a_i dx^i$$

on  $U$ . Now fix an integer  $j \in \{1, 2, \dots, n\}$ . We can extend the  $C^\infty$  vector field  $\frac{\partial}{\partial x^j}$  on  $U$  to a  $C^\infty$  vector field  $X$  on the whole of  $M$  that agrees with  $\frac{\partial}{\partial x^j}$  in a neighborhood  $V$  of  $p$  (not necessarily the whole of  $U$ , but possibly a smaller neighborhood) contained in  $U$  (Proposition 11.1.4 of DG1). The extended vector field is defined in the following way: let  $\sigma : M \rightarrow \mathbb{R}$  be a  $C^\infty$  bump function which is identically 1 on a neighborhood  $V$  of  $p$  and which has support contained in  $U$ . Now, define the vector field  $q \mapsto X_q \in T_q M$ , denoted by  $X$ , in terms of the bump function  $\sigma$  in the following way:

$$X_q = \begin{cases} \sigma(q) \frac{\partial}{\partial x^j} \Big|_q & \text{if } q \in U, \\ \mathbf{0} & \text{if } q \notin U. \end{cases} \quad (3.12)$$

The vector field  $X$  is smooth in the whole of  $M$ , as proved in Proposition 11.1.4 of DG1. Now, by the hypothesis,  $\omega(X)$  is  $C^\infty$  on  $M$ . In particular,  $\omega(X)$  is smooth on  $V$ . Therefore,

$$\omega(X) = \left( \sum_{i=1}^n a_i dx^i \right) \left( \frac{\partial}{\partial x^j} \right) = \sum_{i=1}^n a_i \delta^i_j = a_j$$

is smooth on  $V$ . We, therefore, see that the coefficient functions  $a_i$ 's appearing in  $\omega = \sum_{i=1}^n a_i dx^i$  are smooth on  $V \subseteq U$ . It means that for a given point  $p$ , we can find a chart  $(V, \tilde{x}^1, \dots, \tilde{x}^n)$ , where

$$\tilde{x}^i = r^i \circ \varphi|_V,$$

such that  $\omega = \sum_{i=1}^n a_i|_V d\tilde{x}^i$  on  $V$ , with each  $a_i|_V$  smooth on  $V$ . Therefore, by [Proposition 3.4](#),  $\omega$  is  $C^\infty$ . ■

### §3.4 Pullback of 1-forms

Recall that the differential of a smooth map  $F : N \rightarrow M$  at  $p \in N$  is a linear map  $F_{*,p} : T_p N \rightarrow T_{F(p)} M$  defined by

$$[F_{*,p}(X_p)](f) = X_p(f \circ F), \quad (3.13)$$

where  $f \in C_{F(p)}^\infty(M)$ . Indeed,  $f \circ F \in C_p^\infty(N)$ . Analogously, the **codifferential** (the dual of a differential) at  $F(p) \in M$  is a linear map

$$F^{*,p} : T_{F(p)}^* M \rightarrow T_p^* N.$$

One observes that the differential  $F_{*,p}$  pushes forward a tangent vector at  $p \in N$  while the codifferential  $F^{*,p}$  pulls back a covector from  $T_{F(p)}^* M$  at  $F(p) \in M$  to  $T_p^* N$ .

**Remark 3.1.** Note that a vector field, in general, cannot be pushed forward under a smooth map  $F : N \rightarrow M$ . Suppose  $F : N \rightarrow M$  is a smooth map of manifolds. Also suppose  $F(p) = F(q) = z \in M$  so that  $F$  is not injective. Now, the differentials

$$F_{*,p} : T_p N \rightarrow T_z M \text{ and } F_{*,q} : T_q N \rightarrow T_z M$$

are linear maps. Now, let  $X \in \mathfrak{X}(N)$  be a  $C^\infty$  vector field on  $N$  so that  $X_p$  under  $F_{*,p}$  is pushed forward to  $F_{*,p}(X_p) \in T_z M$  and  $X_q$  is pushed forward to  $F_{*,q}(X_q) \in T_z M$  under  $F_{*,q}$ . There is no reason for  $F_{*,p}(X_p)$  and  $F_{*,q}(X_q)$  to be the same tangent vector in  $T_z M$ . In other words, in general,

$$F_{*,p}(X_p) \neq F_{*,q}(X_q),$$

so that  $z \mapsto F_{*,p}(X_p) := Y_z \in T_z M$  and  $z \mapsto F_{*,q}(X_q) := Y'_z \in T_z M$  are distinct vector fields on  $M$ , denoted by  $Y$  and  $Y'$ , respectively. Therefore, if there were push forward of vector fields  $F_* : \mathfrak{X}(N) \rightarrow \mathfrak{X}(M)$  associated with the non-injective smooth map  $F : N \rightarrow M$ , there is an ambiguity regarding which vector field  $X$  gets mapped to.

Furthermore, if  $F$  is not surjective, there is  $z \in M$  such that  $z \neq F(p)$  for any  $p \in N$ . In that case as well, defining the push forward vector field  $F_*(X)$  at the point  $z$  is impossible. However, when  $F : N \rightarrow M$  is a diffeomorphism, one can define the push forward of a vector field.

Contrary to the non-existence of push forward of a vector field associated with a generic smooth map  $F : N \rightarrow M$ , one can always talk about pullback of a 1-form  $\omega$  on  $M$ :

$$(F^*\omega)_p(X_p) = \omega_{F(p)}(F_{*,p}(X_p)). \quad (3.14)$$

Here,  $\omega \in \Omega^1(M)$ ,  $X_p \in T_p N$ ,  $p \in N$ . Note that  $(F^*\omega)_p$  is simply the image of the covector  $\omega_{F(p)} \in T_{F(p)}^* M$  under the codifferential  $F^{*,p} : T_{F(p)}^* M \rightarrow T_p^* N$ . In other words,

$$(F^*\omega)_p = F^{*,p}(\omega_{F(p)}). \quad (3.15)$$



# 4 Differential $k$ -forms

## §4.1 Definition and Local Expression

We denoted by  $A_k(V)$  the vector space of alternating  $k$ -tensors on  $V$ . We have also seen that if  $\{\hat{\alpha}^1, \dots, \hat{\alpha}^n\}$  is a basis for 1-tensors on  $V$ , then a basis element of  $A_k(V)$  is

$$\hat{\alpha}^{i_1} \wedge \hat{\alpha}^{i_2} \wedge \dots \wedge \hat{\alpha}^{i_k},$$

where  $1 \leq i_1 < i_2 < \dots < i_k \leq n$ . We apply this construction to the tangent space  $T_p M$  of a manifold  $M$  at a point  $p \in M$ . The vector space  $A_k(T_p M)$ , usually denoted by  $\Lambda^k(T_p^* M)$ , is the space of all alternating  $k$ -tensors on the tangent space  $T_p M$ .

**Definition 4.1** (Differential  $k$ -form). A **differential  $k$ -form** on a manifold  $M$  is a function  $\omega$  that assigns to each point  $p \in M$ , a  $k$ -covector  $\omega_p \in \Lambda^k(T_p^* M)$ . An  $n$ -form on a manifold of dimension  $n$  is called a **top degree form**.

**Example 4.1.** On  $\mathbb{R}^n$ , at each point  $p$ , there is a standard basis for the tangent space  $T_p \mathbb{R}^n$ :

$$\left\{ \frac{\partial}{\partial r^1} \Big|_p, \frac{\partial}{\partial r^2} \Big|_p, \dots, \frac{\partial}{\partial r^n} \Big|_p \right\}.$$

Let  $\{(dr^1)_p, \dots, (dr^n)_p\}$  be the dual basis of  $T_p^* \mathbb{R}^n$ .

$$(dr^i)_p \left( \frac{\partial}{\partial r^j} \Big|_p \right) = \delta^i_j.$$

As  $p$  varies over  $\mathbb{R}^n$ , we get differential forms  $dr^1, \dots, dr^n$  on  $\mathbb{R}^n$ . By [Proposition 1.15](#), a basis element of alternating  $k$ -tensors  $\Lambda^k(T_p^* \mathbb{R}^n)$  is

$$(dr^{i_1})_p \wedge (dr^{i_2})_p \wedge \dots \wedge (dr^{i_k})_p,$$

where  $1 \leq i_1 < i_2 < \dots < i_k \leq n$ . If  $\omega$  is a  $k$ -form on  $\mathbb{R}^n$ , then at each point  $p \in \mathbb{R}^n$ ,  $\omega_p$  is the following linear combination:

$$\omega_p = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} a_{i_1 \dots i_k} (dr^{i_1})_p \wedge (dr^{i_2})_p \wedge \dots \wedge (dr^{i_k})_p. \quad (4.1)$$

Omitting the point  $p$ , we write

$$\omega = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} a_{i_1 \dots i_k} dr^{i_1} \wedge dr^{i_2} \wedge \dots \wedge dr^{i_k}. \quad (4.2)$$

In the expression above,  $a_{i_1 \dots i_k}$  are functions on  $\mathbb{R}^n$ . To simplify the notations, we use multi-indices to write (4.2) as

$$\omega = \sum_I a_I dr^I, \quad (4.3)$$

where  $dr^I = dr^{i_1} \wedge dr^{i_2} \wedge \dots \wedge dr^{i_k}$ , and  $I = (i_1, i_2, \dots, i_k)$  is a strictly ascending multi-index.

Suppose  $(U, x^1, \dots, x^n)$  is a coordinate chart on a manifold  $M$ . We have already defined the 1-forms  $dx^1, \dots, dx^n$  on  $U$ . Since at each point  $p \in U$ ,  $\{(dx^1)_p, \dots, (dx^n)_p\}$  is a basis for  $T_p^*M$ , by [Proposition 1.15](#), a basis for  $\Lambda^k(T_p^*\mathbb{R}^n)$  is

$$(dx^{i_1})_p \wedge (dx^{i_2})_p \wedge \dots \wedge (dx^{i_k})_p,$$

where  $1 \leq i_1 < i_2 < \dots < i_k \leq n$ . Thus, locally a  $k$ -form on  $U$  will be a linear combination

$$\omega = \sum_I a_I dx^I, \quad (4.4)$$

where  $dx^I = dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_k}$ ,  $I = (i_1, i_2, \dots, i_k)$  is a strictly ascending multi-index, and  $a_I$  are functions on  $U$ .

## §4.2 The Bundle Point of View

Let  $V$  be a real vector space. Another common notation for the vector space  $A_k(V)$  of alternating  $k$ -linear functionos on  $V$  is  $\Lambda^k(V^*)$ .

$$\begin{aligned} \Lambda^0(V^*) &= A_0(V) = \mathbb{R}, \\ \Lambda^1(V^*) &= A_1(V) = V^*, \\ \Lambda^2(V^*) &= A_2(V), \end{aligned}$$

and so on. Now,  $\Lambda^k(T^*M)$  is defined to be the disjoint union of the vector spaces  $\Lambda^k(T_p^*M)$  as  $p$  varies over  $M$ . So

$$\begin{aligned} \Lambda^k(T^*M) &= \bigsqcup_{p \in M} \Lambda^k(T_p^*M) = \bigsqcup_{p \in M} A_k(T_pM) \\ &= \bigcup_{p \in M} \{p\} \times A_k(T_pM), \end{aligned} \quad (4.5)$$

which is the set of all alternating  $k$ -tensors at all points of  $M$ . This set is called the  $k$ -th **exterior power** of the cotangent bundle  $T^*M$ .

If  $(U, \varphi)$  is a coordinate chart on  $M$ , then there is a bijection  $\bar{\varphi}: \Lambda^k(T^*U) \rightarrow \varphi(U) \times \mathbb{R}^{\binom{n}{k}}$  defined as follows: a generic element of  $\Lambda^k(T^*U)$  is  $(p, \omega_p)$ , where  $\omega_p \in \Lambda^k(T_p^*U)$ . Then  $\omega_p$  is a unique linear combination

$$\omega_p = \sum_I a_I(p) dx^I,$$

where  $I$  runs over the set of strictly ascending multi-indices of length  $k$ . There are  $\binom{n}{k}$  many such multi-indices. If we fix a labeling of the multi-indices once and for all, then we have a  $\binom{n}{k}$ -tuple  $(a_I)_I$ . Then we define

$$\bar{\varphi}(p, \omega_p) = (\varphi(p), (a_I)_I) \in \varphi(U) \times \mathbb{R}^{\binom{n}{k}}.$$

Thus,  $\Lambda^k(T^*U)$  is in a bijective correspondence with  $\varphi(U) \times \mathbb{R}^{\binom{n}{k}}$ . Using this bijective correspondence, one transfers the topology of  $\varphi(U) \times \mathbb{R}^{\binom{n}{k}}$  to  $\Lambda^k(T^*U)$ . By varying the open set  $U$  in the charts contained in the maximal atlas of  $M$ , one can obtain a basis that generates the topology on the whole of  $\Lambda^k(T^*M)$ .

$\Lambda^k(T^*M)$  can, in fact, be shown to be a  $C^\infty$  vector bundle of rank  $\binom{n}{k}$  over  $M$ , i.e.  $\pi: \Lambda^k(T^*U) \rightarrow M$  is a  $C^\infty$  vector bundle of rank  $\binom{n}{k}$  over  $M$ . A differential  $k$ -form is a section of this vector bundle. We define a  $k$ -form to be  $C^\infty$  if it is  $C^\infty$  as a section of the vector bundle  $\Lambda^k(T^*M)$ .

**Notation.** If  $\pi: E \rightarrow M$  is a  $C^\infty$  vector bundle, then the vector space of  $C^\infty$  sections of  $E$  is denoted by  $\Gamma(E)$ , or  $\Gamma(M, E)$ . The vector space of all  $C^\infty$   $k$ -forms, i.e. all  $C^\infty$  sections of the bundle  $\Lambda^k(T^*M)$  is usually denoted by  $\Omega^k(M)$ . Thus,

$$\Omega^k(M) = \Gamma(\Lambda^k(T^*M)) = \Gamma(M, \Lambda^k(T^*M)).$$

**Lemma 4.1**

Let  $(U, x^1, \dots, x^n)$  be a chart on a manifold  $M$ . A  $k$ -form  $\omega = \sum a_I dx^I$  on  $U$  is smooth if and only if the coefficient functions  $a_I$  are all smooth on  $U$ .

**Proposition 4.2** (Characterization of a smooth  $k$ -form)

Let  $\omega$  be a  $k$ -form on a manifold  $M$ . The following are equivalent:

- (i) The  $k$ -form  $\omega$  is  $C^\infty$  on  $M$ .
- (ii) The manifold  $M$  has an atlas such that on every chart  $(U, \phi) = (U, x^1, \dots, x^n)$  in the atlas, the coefficients  $a_I$  of  $\omega = \sum a_I dx^I$  relative to the coordinate frame  $\{dx^I\}_I$  are all  $C^\infty$ .
- (iii) On every chart  $(U, \phi) = (U, x^1, \dots, x^n)$  on  $M$ , the coefficients  $a_I$  of  $\omega = \sum a_I dx^I$  relative to the coordinate frame  $\{dx^I\}_I$  are all  $C^\infty$ .
- (iv) For any  $k$  smooth vector fields  $X_1, \dots, X_k$  on  $M$ , the function  $\omega(X_1, \dots, X_k)$  is  $C^\infty$  on  $M$ .

**Example 4.2.** We defined the 0-tensors and the 0-covectors as constants, i.e. for a real vector space  $V$ ,  $A_0(V) = L_0(V) = \mathbb{R}$ . Now, recall that

$$\Lambda^k(T^*M) = \bigcup_{p \in M} \{p\} \times \Lambda^k(T_p^*M).$$

Since  $\Lambda^0(T_p^*M) = \mathbb{R}$  for every  $p \in M$ , one has

$$\Lambda^0(T^*M) = \bigcup_{p \in M} \{p\} \times \mathbb{R} = M \times \mathbb{R}. \quad (4.6)$$

Hence,

$$\Omega^0(M) = \Gamma(\Lambda^0(T^*M)) = \Gamma(M, M \times \mathbb{R}). \quad (4.7)$$

A  $C^\infty$  section of the 0-th exterior power of the tangent bundle  $T^*M$  is nothing but a  $C^\infty$  section of the globally trivial  $C^\infty$  vector bundle  $M \times \mathbb{R}$  over  $M$ . Such a section maps  $p \in M$  to a pair  $(p, \sigma(p))$  with  $\sigma(p) \in \mathbb{R}$ . Therefore, such a section is nothing but a smooth assignment  $p \mapsto \sigma(p)$ , i.e.  $\sigma \in C^\infty(M, \mathbb{R})$ . So

$$\Omega^0(M) = \Gamma(M, M \times \mathbb{R}) = C^\infty(M, \mathbb{R}).$$

**§4.3 Pullback of  $k$ -forms**

Let  $F : N \rightarrow M$  be a smooth map of manifolds. Recall that a 1-form  $\omega \in \Omega^1(M)$  can be pulled back to  $\Omega^1(N)$  via the pullback  $F^* : \Omega^1(M) \rightarrow \Omega^1(N)$  defined by

$$(F^*\omega)_p(X_p) = \omega_{F(p)}(F_{*,p}(X_p)). \quad (4.8)$$

For 0-forms, i.e. functions, the pullback is defined by composition:

$$N \xrightarrow{F} M \xrightarrow{f} \mathbb{R}$$

Given  $f \in C^\infty(M, \mathbb{R})$ , its pullback is defined to be

$$F^*(f) = f \circ F \in C^\infty(N, \mathbb{R}), \quad (4.9)$$

so that indeed  $F^* : \Omega^0(M) \rightarrow \Omega^0(N)$ .

For a  $k$ -form  $\omega$  on  $M$ , we define its pullback  $F^*\omega$  as follows: if  $p \in N$  and  $X_p^1, X_p^2, \dots, X_p^k \in T_p N$  are  $k$  tangent vectors, then

$$(F^*\omega)_p(X_p^1, X_p^2, \dots, X_p^k) = \omega_{F(p)}(F_{*,p}(X_p^1), F_{*,p}(X_p^2), \dots, F_{*,p}(X_p^k)). \quad (4.10)$$

**Proposition 4.3** (Linearity of pullback)

Let  $F : N \rightarrow M$  be a  $C^\infty$  map. If  $\omega, \tau$  are  $k$ -forms on  $M$  and  $\alpha$  is a real number, then

- (i)  $F^*(\omega + \tau) = F^*\omega + F^*\tau.$
- (ii)  $F^*(\alpha\omega) = \alpha F^*\omega.$

**§4.4 The Wedge Product**

If  $\omega \in \Omega^k(M)$  and  $\tau \in \Omega^l(M)$ , then for any  $p \in M$ ,  $\omega_p \in \Lambda^k(T_p^*M)$  and  $\tau_p \in \Lambda^l(T_p^*M)$  and  $\omega_p \wedge \tau_p \in \Lambda^{k+l}(T_p^*M)$ . Then we define the wedge product of  $\omega$  and  $\tau$  to be the  $(k+l)$ -form  $\omega \wedge \tau$  such that

$$(\omega \wedge \tau)_p = \omega_p \wedge \tau_p. \quad (4.11)$$

**Proposition 4.4**

If  $\omega$  and  $\tau$  are  $C^\infty$  forms on  $M$ , then so is  $\omega \wedge \tau$ .

*Proof.* Let  $(U, x^1, \dots, x^n)$  be a chart on  $M$ . On  $U$ ,

$$\omega = \sum_I a_I dx^I, \quad \tau = \sum_J b_J dx^J \quad (4.12)$$

for  $C^\infty$  functions  $a_I, b_J$  on  $U$ . Their Wedge product is

$$\begin{aligned} \omega \wedge \tau &= \left( \sum_I a_I dx^I \right) \wedge \left( \sum_J b_J dx^J \right) \\ &= \sum_{I,J} a_I b_J dx^I \wedge dx^J \dots \end{aligned} \quad (4.13)$$

In (4.13),  $dx^I \wedge dx^J = 0$  if  $I$  and  $J$  have at least an index in common. If  $I$  and  $J$  are disjoint, i.e., have none of their indices to be common, then

$$dx^I \wedge dx^J = \pm dx^K, \quad (4.14)$$

where  $K = I \cup J$  but reordered as an increasing sequence. Thus,

$$\omega \wedge \tau = \sum_K \left( \sum_{I \cup J = K} \pm a_I b_J \right) dx^K. \quad (4.15)$$

Since the coefficients of  $dx^K$  in (4.15) are  $C^\infty$ , by Proposition 4.2,  $\omega \wedge \tau$  is  $C^\infty$  on  $M$ . ■

**Proposition 4.5** (Pullback of wedge product)

If  $F : N \rightarrow M$  is a  $C^\infty$  map of manifolds and  $\omega, \tau$  are differential forms on  $M$ , then

$$F^*(\omega \wedge \tau) = F^*(\omega) \wedge F^*(\tau). \quad (4.16)$$

We define the vector space  $\Omega^*(M)$  of  $C^\infty$  differential forms on a manifold  $M$  of dimension  $n$  to be the direct sum

$$\Omega^*(M) = \bigoplus_{k=0}^n \Omega^k(M). \quad (4.17)$$

Each element of  $\Omega^*(M)$  is uniquely a formal sum  $\sum_{i=1}^r \omega_{k_i}$  with  $\omega_{k_i} \in \Omega^{k_i}(M)$ . With the wedge product, the vector space  $\Omega^*(M)$  becomes a **graded algebra**, graded by the the degree of differential forms. [Proposition 4.3](#) and [Proposition 4.5](#) tells us that the pullback map  $F^* : \Omega^*(M) \rightarrow \Omega^*(N)$  is a homomorphism of graded algebras<sup>1</sup>.

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<sup>1</sup>Note that we haven't yet proved that  $F^*$  prerserves smoothness of forms, so we don't yet know that  $F^*$  maps  $\Omega^k(M)$  into  $\Omega^k(N)$ . But we shall soon prove this in [Theorem 5.6](#), and once we do that we are all good with the notation.

# 5 Exterior Derivative

The basic objects in differential geometry are differential forms. Our goal will be to learn how we can differentiate and integrate differential forms on manifolds. Recall that an antiderivation on a graded algebra  $A = \bigoplus_{k=0}^{\infty} A^k$  is an  $\mathbb{R}$ -linear map  $D : A \rightarrow A$  such that

$$D(\omega \cdot \tau) = (D\omega) \cdot \tau + (-1)^k \omega \cdot (D\tau),$$

for  $\omega \in A^k$  and  $\tau \in A^l$ , and  $\cdot$  is the multiplication of the graded algebra. In the graded algebra  $A$ , an element of  $A^k$  is called a **homogenous element of degree  $k$** . The antiderivation  $D$  is of degree  $m$  if

$$\deg(D\omega) = \deg \omega + m$$

for all homogenous elements  $\omega \in A$ .

Now, let  $M$  be a manifold and  $\Omega^*(M)$  the graded algebra of  $C^\infty$  differential forms on  $M$ . Now, we'll see that on the graded algebra  $\Omega^*(M)$ , there is a uniquely and intrinsically defined anti-derivation called exterior derivative.

**Definition 5.1** (Exterior derivative). An **exterior derivative** on a manifold  $M$  is an  $\mathbb{R}$ -linear map

$$D : \Omega^*(M) \rightarrow \Omega^*(M)$$

such that

- (i)  $D$  is an antiderivation of degree 1,
- (ii)  $D \circ D = 0$ ,
- (iii) if  $f$  is a  $C^\infty$  function and  $X$  is a  $C^\infty$  vector field on  $M$ , then  $(Df)(X) = Xf$ .

**Remark 5.1.** Condition (iii) in the definition above says that on 0-forms, i.e.  $C^\infty$  functions on  $M$ , an exterior derivative agrees with the differential  $df$  of a function  $f$ . We have learned earlier that in a coordinate chart  $(U, x^1, \dots, x^n)$ , the 1-form  $df$  can be expressed as

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i.$$

Hence, in the chart  $(U, x^1, \dots, x^n)$ ,

$$Df = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i.$$

We now prove the existence and uniqueness of the exterior differentiation on a manifold.

## Lemma 5.1

Let  $D : \Omega^*(M) \rightarrow \Omega^*(M)$  be an exterior derivative on  $M$ . If  $f^1, \dots, f^k$  are smooth functions on  $U$ , then

$$D(Df^1 \wedge Df^2 \wedge \dots \wedge Df^k) = 0.$$

*Proof.* We prove it by induction on  $k$ . The base case  $k = 1$  follows trivially from  $D \circ D = 0$ . Suppose

$D(Df^1 \wedge Df^2 \wedge \cdots \wedge Df^{k-1}) = 0$ . Then

$$\begin{aligned}
 & D(Df^1 \wedge Df^2 \wedge \cdots \wedge Df^k) \\
 &= D\left((Df^1 \wedge Df^2 \wedge \cdots \wedge Df^{k-1}) \wedge Df^k\right) \\
 &= D(Df^1 \wedge Df^2 \wedge \cdots \wedge Df^{k-1}) \wedge Df^k + (-1)^{k-1} (Df^1 \wedge Df^2 \wedge \cdots \wedge Df^{k-1}) \wedge D(Df^k) \\
 &= 0.
 \end{aligned} \tag{5.1}$$

Therefore,  $D(Df^1 \wedge Df^2 \wedge \cdots \wedge Df^k) = 0$  for any  $k \geq 1$ . ■

## §5.1 Exterior Derivative on a Coordinate Chart

Suppose  $(U, x^1, \dots, x^n)$  is a coordinate chart on a manifold  $M$ . Then any  $k$ -form  $\omega$  on  $U$  is uniquely a linear combination

$$\omega = \sum_I a_I dx^I,$$

where  $a_I \in C^\infty(U)$ , and the sum runs over all strictly ascending multi-indices  $I$  of length  $k$ . The  $\mathbb{R}$ -linear map  $d : \Omega^*(U) \rightarrow \Omega^*(U)$  can be constructed to be an exterior derivative on  $U$ . In fact,  $d$  acts on a homogenous element  $\omega \in \Omega^k(U)$  in the following way:

$$\begin{aligned}
 d\omega &= d\left(\sum_I a_I dx^I\right) = \sum_I da_I \wedge dx^I + (-1)^0 \sum_I a_I ddx^I \\
 &= \sum_I da_I \wedge dx^I + \sum_I a_I d(dx^{i_1} \wedge \cdots \wedge dx^{i_k}) \\
 &= \sum_I da_I \wedge dx^I \\
 &= \sum_I \sum_j \frac{\partial a_I}{\partial x^j} dx^j \wedge dx^I.
 \end{aligned} \tag{5.2}$$

(5.2) suggests that  $d\omega \in \Omega^{k+1}(U)$ , and it can be written in the chart  $(U, x^1, \dots, x^n)$  using (5.2). This proves the existence of the exterior derivative  $d : \Omega^*(U) \rightarrow \Omega^*(U)$ , on an open set  $U$  of  $M$ . The uniqueness of  $d : \Omega^*(U) \rightarrow \Omega^*(U)$  can be shown exactly the same way we proved it for the Euclidean case in Proposition 2.4.

Sometimes we write  $d_U \omega$  instead of  $d\omega$  to emphasize that it is the **unique** exterior derivative on the open set  $U \subseteq M$ . In other words, if  $(U, x^i)$  and  $(U, y^j)$  are two charts on  $M$ , and  $\omega = \sum a_I dx^I = \sum b_J dy^J$ , then

$$d_U \omega = \sum_I \sum_i \frac{\partial a_I}{\partial x^i} dx^i \wedge dx^I = \sum_J \sum_j \frac{\partial b_J}{\partial y^j} dy^j \wedge dy^J. \tag{5.3}$$

This reveals that the expression  $d_U \omega$  is chart independent.

## §5.2 Local Operators

An endomorphism of a vector space  $W$  (a linear transformation from  $W$  to itself) is often called an operator on  $W$ . For example, if  $W = C^\infty(\mathbb{R})$ , the vector space of  $C^\infty$  functions on  $\mathbb{R}$ , then  $\frac{d}{dx}$  is an operator on  $W$ :

$$\frac{d}{dx} f(x) = f'(x).$$

The derivative has the desired property that the value of  $f'$  at a point  $p$  depends only on the values of  $f$  in a small neighborhood of  $p$ . More precisely, if  $f = g$  on an open set  $U \subseteq \mathbb{R}$ , then  $f' = g'$  on  $U$ . We say that the derivative is a local operator on  $C^\infty(\mathbb{R})$ .

**Definition 5.2** (Local operator). An operator  $D : \Omega^*(M) \rightarrow \Omega^*(M)$  is said to be **local** if for all  $k \geq 0$ , whenever a  $k$ -form  $\omega \in \Omega^k(M)$  restricts to 0 on an open set  $U$  (i.e.  $\omega_p = 0$  at every  $p \in U$ ), then  $D\omega \equiv 0$  on  $U$  (i.e.  $(D\omega)_p = 0$  at every  $p \in U$ ).

An equivalent definition of local operator is that for all  $k \geq 0$ , whenever two  $k$ -forms  $\omega, \tau \in \Omega^k(M)$  agree on an open set  $U$ , then  $D\omega \equiv D\tau$  on  $U$  (i.e.  $(D\omega)_p = (D\tau)_p$  at every  $p \in U$ ).

### Proposition 5.2

Any antiderivation  $D$  on  $\Omega^*(M)$  is a local operator.

*Proof.* Suppose  $\omega \in \Omega^*(M)$  and  $\omega \equiv 0$  on an open subset  $U$ . Let  $p \in U$ . It suffices to show that  $(D\omega)_p = 0$ . Take a bump function  $f$  at  $p$  supported in  $U$ , i.e.  $\text{supp } f \subseteq U$ . In particular,  $f \equiv 1$  in a neighborhood  $V$  of  $p$  in  $U$ , so that  $V \subset \text{supp } f \subseteq U$ . Then  $f\omega \equiv 0$  on  $M$ . This can be seen by noting that if  $q \in U$ ,

$$(f\omega)_q = f(q)\omega_q = 0,$$

since  $\omega_q = 0$  by hypothesis. On the other hand, if  $q \notin U$ , then  $q \notin \text{supp } f$ , so  $f(q) = 0$ , which yields

$$(f\omega)_q = f(q)\omega_q = 0.$$

Therefore,  $f\omega \equiv 0$  on  $M$ . Applying  $D$  on  $f\omega = f \wedge \omega$ , we get

$$D(f\omega) = (Df) \wedge \omega + (-1)^0 f \wedge D\omega. \quad (5.4)$$

By the linearity of  $D$ ,  $D(f\omega) = 0$ . Now, we evaluate the RHS of (5.4) at  $p \in U$ , and use the fact that  $f(p) = 1$  and  $\omega_p = 0$ . As a result,

$$\begin{aligned} (Df)_p \wedge \omega_p + f(p) \wedge (D\omega)_p &= 0 \\ \implies (D\omega)_p &= 0. \end{aligned} \quad (5.5)$$

Since  $p \in U$  is arbitrary,  $D\omega \equiv 0$  on  $U$ . ■

Sometimes we are given a differential form  $\tau$  that is defined only on an open subset  $U$  of a manifold  $M$ . We can use bump functions to extend  $\tau$  to a global form  $\tilde{\tau}$  on  $M$  that agrees with  $\tau$  near some point.

### Proposition 5.3

Suppose  $\tau$  is a  $C^\infty$  differential  $k$ -form on an open subset  $U$  of  $M$  (such a differential form is called a local differential form). For any  $p \in U$ . There is a  $C^\infty$  global form  $\tilde{\tau}$  on  $M$  (can be defined anywhere on  $M$  using its charts) that agrees with  $\tau$  on a neighborhood of  $p$  contained in  $U$ .

*Proof.* Choose a smooth bump function  $f$  at  $p$  supported in  $U$ , i.e.  $\text{supp } f \subseteq U$ . In particular,  $f \equiv 1$  in a neighborhood  $V$  of  $p$  in  $U$ , so that  $V \subset \text{supp } f \subseteq U$ . Then we define

$$\tilde{\tau}_q = \begin{cases} f(q) \tau_q & \text{if } q \in U, \\ \mathbf{0}_{\Lambda^k(T_q^*M)} & \text{if } q \notin U. \end{cases}$$

By the definition of  $\tilde{\tau}$ , it agrees with  $\tau$  on  $V$ . By *Proposition 9.3.1(ii)* of [DG1](#),  $\tilde{\tau}$  is smooth on  $U$ . Now, let  $q \notin U$ . We want to show that  $\tilde{\tau}$  is smooth at  $q$ .

Since  $\text{supp } f \subseteq U$ ,  $q \notin U$  implies  $q \in M \setminus U \subseteq M \setminus \text{supp } f$ . Since  $\text{supp } f$  is closed,  $M \setminus \text{supp } f$  is open. Hence, we can find a coordinate chart  $(W, \varphi)$  about  $q$  such that  $W \subseteq M \setminus \text{supp } f$ . Then, for  $r \in W$ ,  $\tilde{\tau}_r = \mathbf{0}_{\Lambda^k(T_r^*M)}$ . Also,  $(\Lambda^k(T^*U), \bar{\varphi})$  is a chart on  $\Lambda^k(T^*M)$  about  $\mathbf{0}_{\Lambda^k(T_r^*M)}$ .

$$(\bar{\varphi} \circ \tilde{\tau})(r) = (\varphi(r), \underbrace{0, 0, \dots, 0}_{\binom{n}{k} \text{ 0-s}}).$$

$\varphi$  is smooth. Therefore,  $\tilde{\tau}$  is smooth on  $W$ . In particular,  $\tilde{\tau}$  is smooth at  $q$ . Since  $q \notin U$  was arbitrary,  $\tilde{\tau}$  is smooth at every  $q \notin U$ . Therefore,  $\tilde{\tau}$  is smooth on all of  $M$ . ■



### §5.3 Existence and Uniqueness of an Exterior Differentiation

To define an exterior derivative  $d : \Omega^*(M) \rightarrow \Omega^*(M)$ , let  $\omega \in \Omega^k(M)$  and  $p \in M$ . Choose a chart  $(U, x^1, \dots, x^n)$  about  $p$ . Suppose  $\omega = \sum_I a_I dx^I$  on  $U$ . Now,  $d\omega$  is supposed to be a  $(k+1)$ -form on  $M$ , i.e.  $d\omega \in \Omega^{k+1}(M)$ . Define  $d\omega \in \Omega^{k+1}(M)$  such that at  $p \in U$ ,  $(d\omega)_p$  is expressed as

$$(d\omega)_p = \left( \sum_I da_I \wedge dx^I \right)_p. \quad (5.6)$$

It needs to be proven that the definition (5.6) is independent of chart. If  $(V, y^1, \dots, y^n)$  is another chart about  $p$ , and  $\omega = \sum_J b_J dy^J$  on  $V$ , then on  $U \cap V$ ,

$$\sum_I a_I d_{U \cap V} x^I = \sum_J b_J d_{U \cap V} y^J,$$

where  $d_{U \cap V}$  is the unique exterior derivative  $d_{U \cap V} : \Omega^*(U \cap V) \rightarrow \Omega^*(U \cap V)$ . Then by the locality of exterior derivative,

$$d_{U \cap V} \left( \sum_I a_I d_{U \cap V} x^I \right) = d_{U \cap V} \left( \sum_J b_J d_{U \cap V} y^J \right). \quad (5.7)$$

Reading off the antiderivation  $d_{U \cap V}$  in the chart  $(U \cap V, x^1, \dots, x^n)$  using (5.6), the LHS of (5.7) can be recast into

$$\sum_I d_{U \cap V} a_I d_{U \cap V} x^I.$$

On the other hand, the antiderivation  $d_{U \cap V}$  in the chart  $(U \cap V, y^1, \dots, y^n)$  can be expressed using (5.6) to compute the RHS of (5.7):

$$\sum_J d_{U \cap V} b_J d_{U \cap V} y^J.$$

Therefore,

$$\sum_I d_{U \cap V} a_I d_{U \cap V} x^I = \sum_J d_{U \cap V} b_J d_{U \cap V} y^J, \quad (5.8)$$

on  $U \cap V$ . In particular, for  $p \in U \cap V$ ,

$$\left( \sum_I d_{U \cap V} a_I d_{U \cap V} x^I \right)_p = \left( \sum_J d_{U \cap V} b_J d_{U \cap V} y^J \right)_p,$$

proving that the definition (5.6) is indeed chart independent. As  $p$  varies over all of  $M$ , (5.6) defines an operator

$$d : \Omega^*(M) \rightarrow \Omega^*(M).$$

It's straightforward to verify that the 3 desired conditions of exterior derivative are fulfilled by the definition (5.6).

Now we prove the uniqueness of exterior derivative. Suppose  $D : \Omega^*(M) \rightarrow \Omega^*(M)$  is an exterior derivative. We will now show that  $D$  coincides with the exterior derivative defined by (5.6).

Let  $\omega \in \Omega^k(M)$ , and  $p \in M$ . Choose a chart  $(U, x^1, \dots, x^n)$  about  $p$ , and suppose  $\omega = \sum_I a_I dx^I$  on  $U$ . Extend the functions  $a_I, x^1, \dots, x^n$  to  $C^\infty$  functions  $\tilde{a}_I, \tilde{x}^1, \dots, \tilde{x}^n$  that agrees with  $a_I, x^1, \dots, x^n$  in a neighborhood  $V$  of  $p$ . Define

$$\tilde{\omega} = \sum_I \tilde{a}_I d\tilde{x}^I. \quad (5.9)$$

Then  $\omega \equiv \tilde{\omega}$  on  $V$ . Since  $D$  is a local operator, one must have  $D\omega \equiv D\tilde{\omega}$  on  $V$ . Thus,

$$(D\omega)_p = (D\tilde{\omega})_p = \left[ D \left( \sum_I \tilde{a}_I d\tilde{x}^I \right) \right]_p. \quad (5.10)$$

Since  $D$  is an exterior derivative operator on  $\Omega^*M$ , and  $d$  is the exterior derivative operator defined by (5.6), for  $f \in C^\infty(M)$ ,

$$(Df)(X) = Xf = (df)(X),$$

for any  $C^\infty$  vector field  $X$ . In particular,

$$D\tilde{a}_I = d\tilde{a}_I, \text{ and } D\tilde{x}^i = d\tilde{x}^i,$$

so that  $D\tilde{x}^I = d\tilde{x}^I$ , for a strictly ascending multi-index  $I$  of length  $k$ . Hence, (5.10) reduces to

$$\begin{aligned} (D\omega)_p &= \left[ D \left( \sum_I \tilde{a}_I d\tilde{x}^I \right) \right]_p \\ &= \left[ D \left( \sum_I \tilde{a}_I D\tilde{x}^I \right) \right]_p \\ &= \left( \sum_I D\tilde{a}_I \wedge D\tilde{x}^I \right)_p \\ &= \left( \sum_I d\tilde{a}_I \wedge d\tilde{x}^I \right)_p. \end{aligned}$$

Now, since  $\tilde{a}_I = a_I$  and  $\tilde{x}^i = x^i$  in a neighborhood of  $p$ , we have  $d\tilde{a}_I = da_I$  and  $d\tilde{x}^I = dx^I$  at  $p$ . Therefore,

$$(D\omega)_p = \left( \sum_I d\tilde{a}_I \wedge d\tilde{x}^I \right)_p = \left( \sum_I da_I \wedge dx^I \right)_p = (d\omega)_p. \quad (5.11)$$

So  $D = d$ , and hence the exterior derivative is unique.

### The restriction of a $k$ -form to a submanifold

Let  $S$  be a regular submanifold of a manifold  $M$ , and  $\omega$  is a  $k$ -form on  $M$ , i.e.  $\omega \in \Omega^k(M)$ . Then the restriction of  $\omega$  to  $S$  is the  $k$ -form  $\omega|_S$  on  $S$  defined by

$$(\omega|_S)_p(X_p^1, \dots, X_p^k) = \omega_p(X_p^1, \dots, X_p^k), \quad (5.12)$$

for  $X_p^1, \dots, X_p^k \in T_pS \subseteq T_pM$ . Thus,  $(\omega|_S)_p$  is obtained from  $\omega_p$  by restricting its domain to  $T_pS \times \dots \times T_pS$  ( $k$ -times).

**Example 5.1.** If  $S$  is a smooth curve in  $\mathbb{R}^2$  defined by the non-constant function  $f(x, y) = 0$  ( $f$  could be  $x^2 + y^2 - 1$ , defining the unit circle in  $\mathbb{R}^2$ ), then

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

is a nonzero 1-form on  $\mathbb{R}^2$ . But since  $f$  is identically 0 on  $S$ ,  $(df)|_S = 0$ . So a nonzero form on  $M$  can be restricted to a zero form on a submanifold  $S$ .

A form that is not identically zero is called a **nonzero form**. On the other hand, a form  $\omega$  that is nowhere zero, i.e.  $\omega_p \neq 0$  for all  $p \in M$ , is called a **nowhere vanishing form**.

**Example 5.2** (A nowhere vanishing 1-form on  $S^1$ ). Let  $S^1$  be the unit circle defined by  $x^2 + y^2 = 1$  in  $\mathbb{R}^2$ . The 1-form  $dx$  restricts from  $\mathbb{R}^2$  to a 1-form on  $S^1$ . When restricted to  $S^1$ , the domain of the covector  $((dx)|_{S^1})_p$  is  $T_pS^1$  instead of  $T_p\mathbb{R}^2$ :

$$((dx)|_{S^1})_p : T_pS^1 \rightarrow \mathbb{R}.$$

Now, from  $x^2 + y^2 = 1$ , one obtains

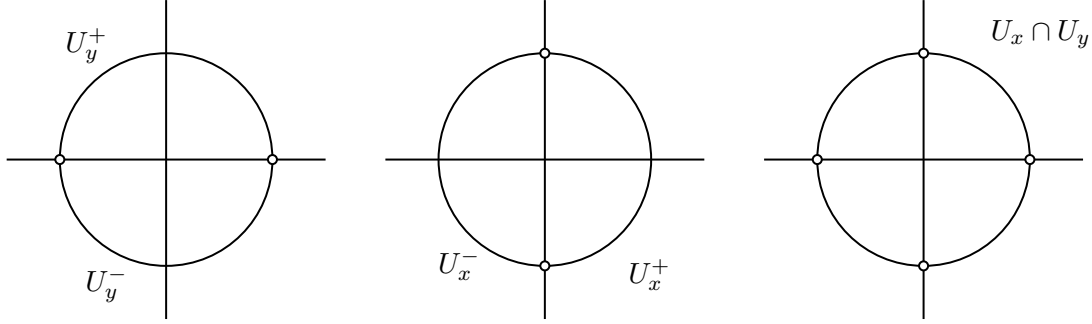
$$2x dx + 2y dy = 0. \quad (5.13)$$

At  $p = (1, 0)$ , (5.13) reduces to  $(dx)_p = 0$ . It shows that although  $dx$  is a nowhere vanishing 1-form on  $\mathbb{R}^2$ , it vanishes at  $(1, 0)$  when restricted to  $S^1$ .

To find a nowhere vanishing 1-form on  $S^1$ , we again take exterior derivative of both sides of the equation  $x^2 + y^2 - 1 = 0$  to arrive at

$$2x dx + 2y dy = 0. \quad (5.14)$$

Let  $U_x = \{(x, y) \in S^1 \mid x \neq 0\}$ , and  $U_y = \{(x, y) \in S^1 \mid y \neq 0\}$ .



By (5.14), then one obtains on  $U_x \cap U_y$ ,

$$\frac{dy}{x} = -\frac{dx}{y}. \quad (5.15)$$

Now we define a 1-form  $\omega$  on  $S^1$  by

$$\omega = \begin{cases} \frac{dy}{x} & \text{on } U_x, \\ -\frac{dx}{y} & \text{on } U_y. \end{cases} \quad (5.16)$$

Since  $\frac{dy}{x} = -\frac{dx}{y}$  on  $U_x \cap U_y$ ,  $\omega$  is a well-defined 1-form on  $S^1 = U_x \cup U_y$ . To show that  $\omega$  is  $C^\infty$  and nowhere vanishing, we need charts.

$$\begin{aligned} U_x^+ &= \{(x, y) \in S^1 \mid x > 0\}, U_x^- = \{(x, y) \in S^1 \mid x < 0\}, \\ U_y^+ &= \{(x, y) \in S^1 \mid y > 0\}, U_y^- = \{(x, y) \in S^1 \mid y < 0\}. \end{aligned}$$

On  $U_x^+$ , the local coordinates are the  $y$ -coordinates, so that  $(dy)_p$  is a basis for the cotangent space  $T_p^*S^1$  at each  $p \in U_x^+$ . Now, since  $\omega = \frac{dy}{x}$  on  $U_x^+$ ,  $\omega$  is  $C^\infty$  and nowhere zero on  $U_x^+$ . Similarly,  $\omega = \frac{dy}{x}$  on  $U_x^-$  is also  $C^\infty$  and nowhere zero on  $U_x^-$ . One can show using similar argument that  $\omega = -\frac{dx}{y}$  is  $C^\infty$  and nowhere vanishing on  $U_y^+$  and  $U_y^-$ . Hence,  $\omega$  is  $C^\infty$  and nowhere zero on  $S^1$ .

## §5.4 Exterior Differentiation Under a Pullback

### Theorem 5.4

Let  $F : N \rightarrow M$  be a smooth map of manifolds. If  $\omega \in \Omega^k(M)$ , then

$$dF^*\omega = F^*d\omega.$$

*Proof.* Let us first check the case when  $k = 0$ , i.e. when  $\omega$  is a 0-form ( $C^\infty$  function). We denote this smooth function with  $h$ . For  $p \in N$  and  $X_p \in T_pN$ ,

$$(dF^*h)_p(X_p) = X_p(F^*h) = X_p(h \circ F), \quad (5.17)$$

since  $(df)_p(X_p) = X_p f$  for  $f \in C^\infty(M)$ . On the other hand,

$$(F^*dh)_p(X_p) = (dh)_{F(p)}(F_{*,p}X_p) = (F_{*,p}X_p)(h) = X_p(h \circ F). \quad (5.18)$$

Combining (5.17) and (5.18), we get

$$(dF^*h)_p = (F^*dh)_p.$$

Since  $p \in N$  is arbitrary,

$$dF^*h = F^*dh. \quad (5.19)$$

Now, consider the general case of a  $C^\infty$   $k$ -form  $\omega$  on  $M$ , i.e.  $\omega \in \Omega^k(M)$ . It suffices to verify that  $dF^*\omega = F^*d\omega$  at an arbitrary point  $p \in N$ . This reduces the proof to a local computation. If  $(V, y^1, \dots, y^n)$  is a chart of  $M$  at  $F(p)$ , then on  $V$ ,

$$\omega = \sum_I a_I dy^I = \sum_I a_I dy^{i_1} \wedge dy^{i_2} \wedge \dots \wedge dy^{i_k},$$

for some  $C^\infty$  functions  $a_I$  on  $V$ . Now,

$$F^*\omega = \sum_I (F^*a_I) (F^*dy^{i_1}) \wedge (F^*dy^{i_2}) \wedge \dots \wedge (F^*dy^{i_k}).$$

Since  $dF^*h = F^*dh$  for  $C^\infty$  function  $h$ , we have

$$\begin{aligned} F^*\omega &= \sum_I (a_I \circ F) d(F^*y^{i_1}) \wedge d(F^*y^{i_2}) \wedge \dots \wedge d(F^*y^{i_k}) \\ &= \sum_I (a_I \circ F) d(y^{i_1} \circ F) \wedge d(y^{i_2} \circ F) \wedge \dots \wedge d(y^{i_k} \circ F) \\ &= \sum_I (a_I \circ F) dF^{i_1} \wedge dF^{i_2} \wedge \dots \wedge dF^{i_k}. \end{aligned} \quad (5.20)$$

Therefore, from (5.20), one obtains

$$dF^*\omega = \sum_I d(a_I \circ F) \wedge dF^{i_1} \wedge dF^{i_2} \wedge \dots \wedge dF^{i_k}. \quad (5.21)$$

On the other hand,

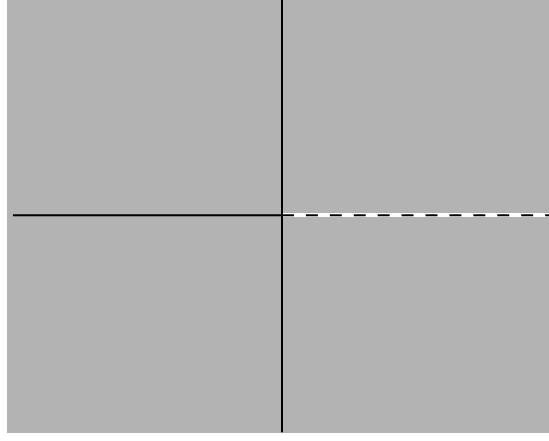
$$\begin{aligned} F^*d\omega &= F^*\left(\sum_I da_I \wedge dy^{i_1} \wedge dy^{i_2} \wedge \dots \wedge dy^{i_k}\right) \\ &= \sum_I F^*(da_I) \wedge F^*(dy^{i_1}) \wedge \dots \wedge F^*(dy^{i_k}) \\ &= \sum_I d(F^*a_I) \wedge d(F^*y^{i_1}) \wedge \dots \wedge d(F^*y^{i_k}) \\ &= \sum_I d(a_I \circ F) \wedge d(y^{i_1} \circ F) \wedge \dots \wedge d(y^{i_k} \circ F) \\ &= \sum_I d(a_I \circ F) \wedge dF^{i_1} \wedge dF^{i_2} \wedge \dots \wedge dF^{i_k}. \end{aligned} \quad (5.22)$$

Comparing (5.21) and (5.22), one obtains

$$dF^*\omega = F^*d\omega, \quad (5.23)$$

on  $V$ . In particular, (5.23) holds at  $p \in N$ . Since  $p \in N$  is arbitrary, (5.23) holds everywhere on  $N$ . ■

**Example 5.3.** Let  $U$  be the open set  $(0, \infty) \times (0, 2\pi)$  in the  $(r, \theta)$  plane  $\mathbb{R}^2$ , i.e.  $U$  is  $\mathbb{R}^2$  except the non-negative  $x$ -axis.



Define  $F : U \rightarrow \mathbb{R}^2$  by

$$(x, y) = F(r, \theta) = (r \cos \theta, r \sin \theta).$$

Let us compute the pullback  $F^*(dx \wedge dy)$ .

$$F^*dx = dF^*x = d(x \circ F) = d(r \cos \theta) = \cos \theta dr - r \sin \theta d\theta; \quad (5.24)$$

$$F^*dy = dF^*y = d(y \circ F) = d(r \sin \theta) = \sin \theta dr + r \cos \theta d\theta. \quad (5.25)$$

Therefore,

$$\begin{aligned} F^*(dx \wedge dy) &= F^*dx \wedge F^*dy \\ &= [\cos \theta dr - r \sin \theta d\theta] \wedge [\sin \theta dr + r \cos \theta d\theta] \\ &= r \cos^2 \theta dr \wedge d\theta - r \sin^2 \theta d\theta \wedge dr \\ &= r dr \wedge d\theta. \end{aligned} \quad (5.26)$$

## §5.5 Pullback Preserves Smoothness of Forms

In this section, we will prove that if  $\omega$  is a smooth  $k$ -form on  $M$ , and  $F : N \rightarrow M$  is smooth, then  $F^*\omega$  is a smooth  $k$ -form on  $N$ . For that purpose, we need a lemma first.

### Lemma 5.5

Let  $(U, x^1, \dots, x^n)$  be a chart on a manifold and  $f^1, \dots, f^k$  smooth functions on  $U$ . Then

$$df^1 \wedge \dots \wedge df^k = \sum_I \frac{\partial (f^1, \dots, f^k)}{\partial (x^{i_1}, \dots, x^{i_k})} dx^{i_1} \wedge \dots \wedge dx^{i_k},$$

where  $I = (i_1, \dots, i_k)$  is a strictly ascending multi-index of length  $k$ .

*Proof.* On  $U$ ,

$$df^1 \wedge \dots \wedge df^k = \sum_J c_J dx^{j_1} \wedge \dots \wedge dx^{j_k}, \quad (5.27)$$

for some functions  $c_J$ . By the definition of the differential,

$$df^i \left( \frac{\partial}{\partial x^j} \right) = \frac{\partial f^i}{\partial x^j}.$$

Applying both sides of (5.27) to the list of coordinate vector fields  $\frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_k}}$ , we get

$$\begin{aligned} \text{LHS} &= (df^1 \wedge \dots \wedge df^k) \left( \frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_k}} \right) = \det \left[ \frac{\partial f^i}{\partial x^{i_j}} \right] \\ &= \frac{\partial (f^1, \dots, f^k)}{\partial (x^{i_1}, \dots, x^{i_k})}, \end{aligned} \quad (5.28)$$

by [Proposition 1.13](#). On the other hand,

$$\text{RHS} = \sum_J c_J \left( dx^{j_1} \wedge \cdots \wedge dx^{j_k} \right) \left( \frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_k}} \right) = \sum_J c_J \delta_I^J = c_I. \quad (5.29)$$

Hence,  $c_I = \frac{\partial(f^1, \dots, f^k)}{\partial(x^{i_1}, \dots, x^{i_k})}$ . ■

### Theorem 5.6

If  $F : N \rightarrow M$  is a  $C^\infty$  map of manifolds and  $\omega$  is a  $C^\infty$   $k$ -form on  $M$ , then  $F^*\omega$  is a  $C^\infty$   $k$ -form on  $N$ .

*Proof.* It is enough to show that every point in  $N$  has a neighborhood on which  $F^*\omega$  is  $C^\infty$ . Fix  $p \in N$  and choose a chart  $(V, y^1, \dots, y^m)$  on  $M$  about  $F(p)$ . Let  $F^i = y^i \circ F$  be the  $i$ -th coordinate of the map  $F$  in this chart. By the continuity of  $F$ , there is a chart  $(U, x^1, \dots, x^n)$  on  $N$  about  $p$  such that  $F(U) \subset V$ . Since  $\omega$  is  $C^\infty$ , on  $V$ ,

$$\omega = \sum_I a_I dy^{i_1} \wedge \cdots \wedge dy^{i_k}$$

for some  $C^\infty$  functions  $a_I \in C^\infty(V)$ . By properties of the pullback,

$$\begin{aligned} F^*\omega &= F^* \left( \sum_I a_I dy^{i_1} \wedge \cdots \wedge dy^{i_k} \right) \\ &= \sum_I (F^*a_I) F^* (dy^{i_1}) \wedge \cdots \wedge F^* (dy^{i_k}) \\ &= \sum_I (a_I \circ F) dF^*y^{i_1} \wedge \cdots \wedge dF^*y^{i_k} \\ &= \sum_I (a_I \circ F) dF^{i_1} \wedge \cdots \wedge dF^{i_k} \\ &= \sum_{I,J} (a_I \circ F) \frac{\partial(F^{i_1}, \dots, F^{i_k})}{\partial(x^{j_1}, \dots, x^{j_k})} dx^J. \end{aligned} \quad (5.30)$$

Since the  $a_I \circ F$  and  $\frac{\partial(F^{i_1}, \dots, F^{i_k})}{\partial(x^{j_1}, \dots, x^{j_k})}$  are all  $C^\infty$ ,  $F^*\omega$  is  $C^\infty$  on  $U$ . In particular,  $F^*\omega$  is  $C^\infty$  at  $p$ . Since  $p \in N$  is arbitrary,  $F^*\omega$  is  $C^\infty$  on the whole of  $N$ . ■

### Theorem 5.7

If  $F : N \rightarrow M$  and  $G : M \rightarrow K$  are smooth maps between manifolds, then

$$(G \circ F)^* = F^* \circ G^* : \Omega^*(K) \rightarrow \Omega^*(N). \quad (5.31)$$

Furthermore, if  $\mathbb{1}_M$  is the identity map on  $M$ ,

$$(\mathbb{1}_M)^* = \mathbb{1}_{\Omega^*(M)}. \quad (5.32)$$

*Proof.* Suppose  $\mathbb{1}_M$  is the identity map on  $M$ . Take any  $\omega \in \Omega^k(M)$ . At any  $p \in M$ , for any  $X_p^1, \dots, X_p^k \in T_p M$ ,

$$\begin{aligned} ((\mathbb{1}_M)^* \omega)_p (X_p^1, \dots, X_p^k) &= \omega_{\mathbb{1}_M(p)} \left( (\mathbb{1}_M)_{*,p} X_p^1, \dots, (\mathbb{1}_M)_{*,p} X_p^k \right) \\ &= \omega_p \left( \mathbb{1}_{T_p M} X_p^1, \dots, \mathbb{1}_{T_p M} X_p^k \right) \\ &= \omega_p (X_p^1, \dots, X_p^k), \end{aligned} \quad (5.33)$$

since  $(\mathbb{1}_M)_{*,p} = \mathbb{1}_{T_p M}$  by *Remark 6.1.2* of [DG1](#). Therefore,  $((\mathbb{1}_M)^* \omega)_p = \omega_p$ . Since  $p \in M$  is arbitrary,  $(\mathbb{1}_M)^* \omega = \omega$ .

Now suppose  $F : N \rightarrow M$  and  $G : M \rightarrow K$  are smooth maps between manifolds. Take any  $\omega \in \Omega^k(K)$ . At any  $p \in N$ , for any  $X_p^1, \dots, X_p^k \in T_p N$ ,

$$\begin{aligned} ((G \circ F)^* \omega)_p (X_p^1, \dots, X_p^k) &= \omega_{G(F(p))} ((G \circ F)_{*,p} X_p^1, \dots, (G \circ F)_{*,p} X_p^k) \\ &= \omega_{G(F(p))} (G_{*,F(p)} (F_{*,p} X_p^1), \dots, G_{*,F(p)} (F_{*,p} X_p^k)), \end{aligned} \quad (5.34)$$

since  $(G \circ F)_{*,p} = G_{*,F(p)} \circ F_{*,p}$  by *Theorem 6.1.1* of [DG1](#). Now on the other hand,

$$\begin{aligned} ((F^* \circ G^*) \omega)_p (X_p^1, \dots, X_p^k) &= (F^* (G^* \omega))_p (X_p^1, \dots, X_p^k) \\ &= (G^* \omega)_{F(p)} (F_{*,p} X_p^1, \dots, F_{*,p} X_p^k) \\ &= \omega_{G(F(p))} (G_{*,F(p)} (F_{*,p} X_p^1), \dots, G_{*,F(p)} (F_{*,p} X_p^k)). \end{aligned} \quad (5.35)$$

Therefore,

$$((G \circ F)^* \omega)_p (X_p^1, \dots, X_p^k) = ((F^* \circ G^*) \omega)_p (X_p^1, \dots, X_p^k).$$

So we have  $((G \circ F)^* \omega)_p = ((F^* \circ G^*) \omega)_p$ . Since  $p \in N$  is arbitrary,

$$(G \circ F)^* \omega = (F^* \circ G^*) \omega. \quad (5.36)$$

■

**Remark 5.2.** [Theorem 5.6](#) tells us that  $F^*$  is indeed a map from  $\Omega^k(M)$  to  $\Omega^k(N)$ . So we can think of it as a map between the graded algebras:

$$F^* : \Omega^*(M) \rightarrow F^*(N).$$

Previously we were writing this without really verifying that  $F^*$  preserves the smoothness of forms. Now, by [Proposition 4.3](#) and [Proposition 4.5](#),  $F^* : \Omega^*(M) \rightarrow \Omega^*(N)$  is a homomorphism of graded algebras. This gives rise to a contravariant functor from the category **Man** of manifolds and smooth maps to the category **GrAlg** of graded algebras:

$$\mathcal{F} : \mathbf{Man} \rightarrow \mathbf{GrAlg}.$$

$\mathcal{F}$  takes an object of **Man**, a manifold  $M$ , to the graded algebra  $\Omega^*(M)$ ; and it makes an arrow of **Man**, a smooth map  $F : N \rightarrow M$ , to the graded algebra homomorphism  $F^* : \Omega^*(M) \rightarrow F^*(N)$ . Since  $\mathcal{F}$  reverses the direction of arrows, [Theorem 5.7](#) ensures that it is a contravariant functor.

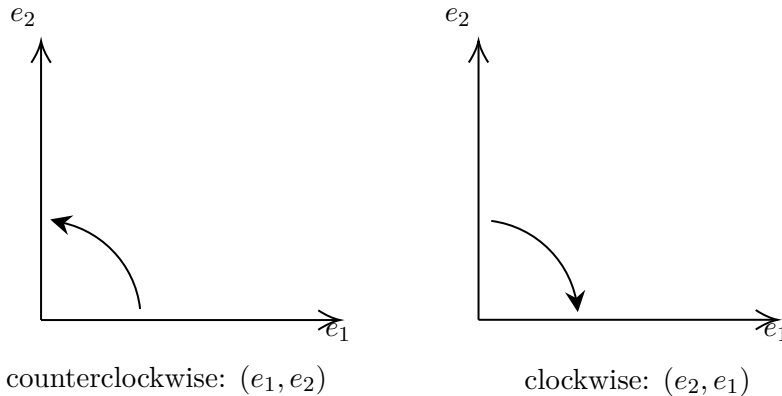
# 6 Orientation

## §6.1 Orientations on a Vector Space

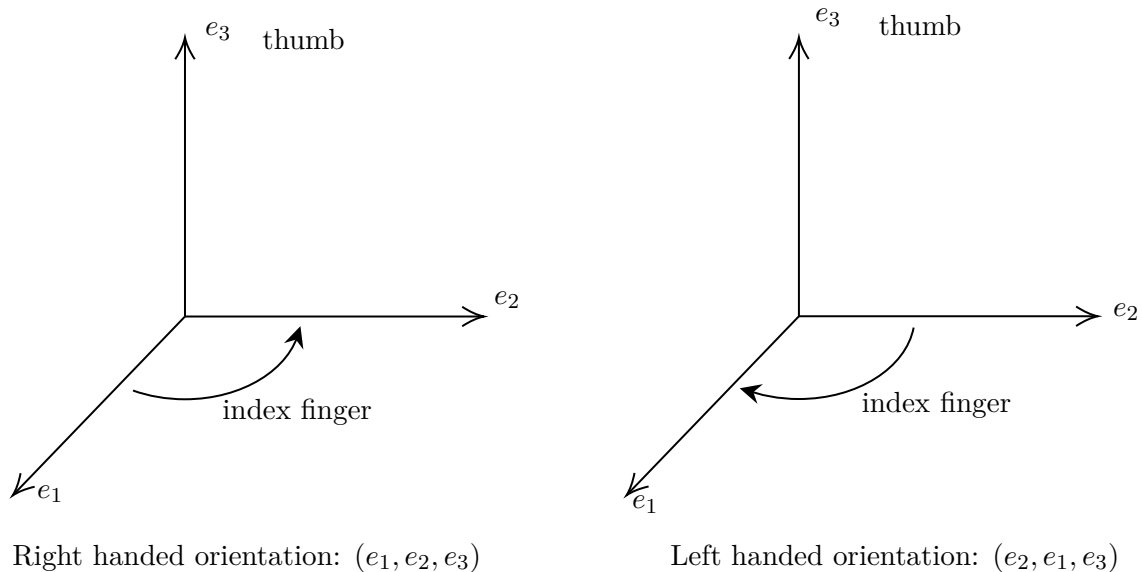
On  $\mathbb{R}$ , an orientation is one of the two possible directions:



On  $\mathbb{R}^2$ , an orientation is either counterclockwise or clockwise:



On  $\mathbb{R}^3$ , an orientation is either right handed or left handed:



Now, we want to define an orientation on  $\mathbb{R}^4$ , or more generally on  $\mathbb{R}^n$ . We do it through ordered basis for  $\mathbb{R}^n$ . Let  $e_1, \dots, e_n$  be the standard basis for  $\mathbb{R}^n$ . For  $\mathbb{R}^1$ , an orientation is given by  $e_1$ , or  $-e_1$ . For  $\mathbb{R}^2$ , counterclockwise orientation is  $(e_1, e_2)$ , and clockwise orientation is  $(e_2, e_1)$ . For  $\mathbb{R}^3$ , the right handed orientation is  $(e_1, e_2, e_3)$ , and the left handed orientation is  $(e_2, e_1, e_3)$ .

For any two ordered bases  $(u_1, u_2)$  and  $(v_1, v_2)$  of  $\mathbb{R}^2$ , there is a unique non-singular  $2 \times 2$  matrix  $A = [a_{ij}]$  such that

$$u_j = \sum_{i=1}^2 v_i a_{ij}. \quad (6.1)$$

$A$  is called the change of basis matrix from  $(v_1, v_2)$  to  $(u_1, u_2)$ . In matrix notation, (6.1) can be written as

$$\begin{bmatrix} u_1 & u_2 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 \end{bmatrix} A. \quad (6.2)$$



We say that two ordered bases are **equivalent** if the change of basis matrix  $A$  has positive determinant. Then one can check that this is indeed an equivalence relation on the set of all ordered bases of  $\mathbb{R}^2$ . It, therefore, partitions the ordered bases into two equivalence classes. Each equivalence class is called an **orientation** on  $\mathbb{R}^2$ .

The equivalence class containing  $(e_1, e_2)$  is the counterclockwise orientation, and the equivalence class containing  $(e_2, e_1)$  is called clockwise orientation. Indeed,

$$\begin{bmatrix} e_1 & e_2 \end{bmatrix} = \begin{bmatrix} e_2 & e_1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

and

$$\det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1.$$

Similarly, for  $\mathbb{R}^3$ , the ordered bases  $(e_1, e_2, e_3)$  and  $(e_2, e_1, e_3)$  don't belong to the same equivalence class:

$$\begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} = \begin{bmatrix} e_2 & e_1 & e_3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and

$$\det \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = -1.$$

The general case for an  $n$ -dimensional vector space  $V$  is as follows:

**Definition 6.1.** Two ordered bases  $u = [u_1 \ \cdots \ u_n]$  and  $v = [v_1 \ \cdots \ v_n]$  of an  $n$ -dimensional vector space  $V$  are said to be **equivalent** if

$$u = vA,$$

for an  $n \times n$  matrix  $A$  with  $\det A > 0$ . An **orientation** on  $V$  is an equivalence class of ordered bases.

The 0-dimensional vector space  $\{\mathbf{0}\}$  is a special case as its basis is the empty set  $\emptyset$ . We define an orientation on  $\{\mathbf{0}\}$  to be one of the two numbers  $\pm 1$ .

## §6.2 Orientations and $n$ -covectors

Instead of using an ordered basis, we can also use an  $n$ -covector to specify an orientation on an  $n$ -dimensional vector space  $V$ . This is based on the fact that the vector space  $\Lambda^n(V^*)$  of  $n$ -covectors on  $V$  is 1-dimensional (so that it has 2 orientations).

### Lemma 6.1

Let  $u_1, \dots, u_n$  and  $v_1, \dots, v_n$  be vectors in a vector space  $V$ . Suppose

$$u_j = \sum_{i=1}^n v_i a_{ij},$$

for a matrix  $A = [a_{ij}]$  of real numbers. If  $\omega$  is an  $n$ -covector on  $V$ , then

$$\omega(u_1, \dots, u_n) = \det A \omega(v_1, \dots, v_n).$$

*Proof.* By hypothesis,  $u_j = \sum_{i=1}^n v_i a_{ij}$ . Using linearity of  $\omega$ , one arrives at

$$\begin{aligned}\omega(u_1, u_2, \dots, u_n) &= \omega\left(\sum_{i_1=1}^n v_{i_1} a_{i_1 1}, \sum_{i_2=1}^n v_{i_2} a_{i_2 2}, \dots, \sum_{i_n=1}^n v_{i_n} a_{i_n n}\right) \\ &= \sum_{i_1, i_2, \dots, i_n=1}^n a_{i_1 1} a_{i_2 2} \cdots a_{i_n n} \omega(v_{i_1}, v_{i_2}, \dots, v_{i_n}).\end{aligned}\quad (6.3)$$

For  $\omega(v_{i_1}, v_{i_2}, \dots, v_{i_n})$  to be nonzero,  $i_1, \dots, i_n$  must all be different as  $\omega$  is alternating. Therefore,  $i_1, \dots, i_n$  can be thought of as a permutation  $\sigma$  of  $\{1, 2, \dots, n\}$  that takes each  $j$  to  $i_j$ . From the alternating property of  $\omega$ , one has

$$\omega(v_{\sigma(1)}, \dots, v_{\sigma(n)}) = (\operatorname{sgn} \sigma) \omega(v_1, \dots, v_n). \quad (6.4)$$

Therefore, using (6.3),

$$\begin{aligned}\omega(u_1, u_2, \dots, u_n) &= \sum_{\sigma \in S_n} a_{\sigma(1)1} a_{\sigma(2)2} \cdots a_{\sigma(n)n} (\operatorname{sgn} \sigma) \omega(v_1, \dots, v_n) \\ &= (\det A) \omega(v_1, \dots, v_n).\end{aligned}\quad (6.5)$$

■

### Corollary 6.2

If  $u_1, \dots, u_n$  and  $v_1, \dots, v_n$  are ordered bases of a vector space  $V$ , then

$$\begin{aligned}&\omega(u_1, u_2, \dots, u_n) \text{ and } \omega(v_1, \dots, v_n) \text{ have the same sign} \\ \iff &\det A > 0 \\ \iff &u_1, \dots, u_n \text{ and } v_1, \dots, v_n \text{ are equivalent ordered bases.}\end{aligned}$$

We say that the  $n$ -covector represents the orientation  $(v_1, \dots, v_n)$  if  $\omega(v_1, \dots, v_n) > 0$ . By Corollary 6.2, this notion is well-defined, i.e. independent of the choice of ordered basis  $v_1, \dots, v_n$  from the same equivalence class.

**Remark 6.1.**  $\Lambda^n(V^*) \cong \mathbb{R}$ , so that the set of nonzero  $n$ -covectors can be identified with  $\mathbb{R} \setminus \{0\}$ , which has 2 connected components. Two nonzero  $n$ -covectors  $\omega$  and  $\omega'$  on  $V$  are in the same component if and only if  $\omega = a\omega'$  for some real number  $a > 0$ . Thus, each connected component of  $\Lambda^n(V^*) \setminus \{0\}$  represents an orientation on  $V$ .

**Example 6.1.** Let  $e_1, e_2$  be the standard basis of  $\mathbb{R}^2$ , and  $\alpha^1, \alpha^2$  its dual basis. Then the 2-covector  $\alpha^1 \wedge \alpha^2$  represents the counterclockwise orientation on  $\mathbb{R}^2$ , since

$$(\alpha^1 \wedge \alpha^2)(e_1, e_2) = 1 > 0.$$

**Example 6.2.** Let  $\frac{\partial}{\partial x}\big|_p, \frac{\partial}{\partial y}\big|_p$  be the standard basis for the tangent space  $T_p\mathbb{R}^2$ , and  $(dx)_p, (dy)_p$  be the dual basis, i.e. for the basis of  $T_p^*\mathbb{R}^2$ . Then  $(dx)_p \wedge (dy)_p$  represents the counterclockwise orientation on  $T_p\mathbb{R}^2$ .

We define an equivalence relation on the nonzero  $n$ -covectors on the  $n$ -dimensional vector space  $V$  as follows:

$$\omega \sim \omega' \iff \omega = a\omega' \text{ for some } a > 0.$$

Then an orientation on  $V$  is also given by an equivalence class of nonzero  $n$ -covectors on  $V$ .

### §6.3 Orientations on a Manifold

Every vector space of dimension  $n$  has two orientations, corresponding to the two equivalence classes of ordered bases or the two equivalence classes of nonzero  $n$ -covectors. To orient a manifold  $M$ , we orient the tangent space at each point  $p \in M$  in a coherent way so that the orientation doesn't change abruptly in a neighborhood of a point. The simplest way to guarantee this is to require that the  $n$ -form (or top degree form) on  $M$  specifying the orientation at each point be  $C^\infty$ . We also want the  $n$ -form to be nowhere vanishing.

**Definition 6.2.** A manifold  $M$  of dimension  $n$  is **orientable** if it has a  $C^\infty$  nowhere vanishing  $n$ -form. If  $\omega$  is a nowhere vanishing  $C^\infty$   $n$ -form on  $M$ , then at each point  $p \in M$ , the  $n$ -covector  $\omega_p$  picks out an equivalence class of ordered bases for the tangent space  $T_p M$ .

**Example 6.3.** The Euclidean space  $\mathbb{R}^n$  is orientable as a manifold, because it has the nowhere vanishing  $n$ -form  $dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$ .

If  $\omega$  and  $\omega'$  are both  $C^\infty$  nowhere vanishing  $n$ -forms on a manifold  $M$  of dimension  $n$ , then  $\omega = f\omega'$  for a  $C^\infty$  nowhere vanishing function  $f$  on  $M$ . On a connected manifold  $M$ , such a function  $f$  is either everywhere positive or everywhere negative. Thus, the  $C^\infty$  nowhere-vanishing  $n$ -forms on a connected manifold  $M$  can be partitioned into 2 equivalence class:

$$\omega \sim \omega' \iff \omega = f\omega' \text{ with } f > 0. \quad (6.6)$$

We call either equivalence class an orientation on the connected manifold  $M$ . Thus, by definition, a connected manifold has exactly 2 orientations. If the manifold  $M$  is not connected, then each connected component of  $M$  has one of the 2 possible orientations. We call a  $C^\infty$  nowhere-vanishing  $n$ -form on  $M$  that specifies an orientation of  $M$  an **orientation form**. An **oriented manifold** is a pair  $(M, [\omega])$ , where  $M$  is a manifold of dimension  $n$  and  $[\omega]$  is an orientation on  $M$ , i.e. the equivalence class of nowhere vanishing  $C^\infty$   $n$ -forms containing  $\omega$ .

**Remark 6.2 (Orientations on a 0-dimensional manifold).** A zero dimensional manifold is a point, and by definition is always orientable. Its two orientations are represented by the numbers  $\pm 1$ .

**Definition 6.3.** A diffeomorphism  $F : (N, [\omega_N]) \rightarrow (M, [\omega_M])$  of oriented manifolds is said to be **orientation preserving** if  $[F^*\omega_M] = [\omega_N]$ . It's **orientation reversing** if  $[F^*\omega_M] = [-\omega_N]$ .

#### Proposition 6.3

Let  $U$  and  $V$  be open subsets of  $\mathbb{R}^n$ . A diffeomorphism  $F : U \rightarrow V$  is orientation-preserving if and only if the Jacobian determinant  $\det \left[ \frac{\partial F^i}{\partial x^j} \right]$  is everywhere positive on  $U$ .

*Proof.* Let  $(x^1, \dots, x^n)$  and  $y^1, \dots, y^n$  be standard coordinates on  $U \subset \mathbb{R}^n$  and  $V \subset \mathbb{R}^n$ , respectively.

$$\begin{aligned} F^* (dy^1 \wedge \cdots \wedge dy^n) &= F^* (dy^1) \wedge \cdots \wedge F^* (dy^n) \\ &= d(F^* y^1) \wedge \cdots \wedge d(F^* y^n) \\ &= d(y^1 \circ F) \wedge \cdots \wedge d(y^n \circ F) \\ &= dF^1 \wedge \cdots \wedge dF^n \\ &= \det \left[ \frac{\partial F^i}{\partial x^j} \right] dx^1 \wedge \cdots \wedge dx^n, \end{aligned} \quad (6.7)$$

where the last equality follows from Lemma 5.5. Now,  $F$  is orientation preserving if and only if

$$F^* (dy^1 \wedge \cdots \wedge dy^n) \sim dx^1 \wedge \cdots \wedge dx^n, \quad (6.8)$$

where  $\sim$  is defined as (6.6). Using (6.7), we can conclude that (6.8) holds if and only if  $\det \left[ \frac{\partial F^i}{\partial x^j} \right]$  is everywhere positive on  $U$ . ■

## §6.4 Orientation and Atlases

**Definition 6.4** (Oriented atlas). An atlas on  $M$  is said to be **oriented** if for any two overlapping charts  $(U, x^1, \dots, x^n)$  and  $(V, y^1, \dots, y^n)$  of the atlas, the Jacobian determinant  $\det \left[ \frac{\partial y^i}{\partial x^j} \right]$  is everywhere positive on  $U \cap V$ .

### Proposition 6.4

A manifold  $M$  of dimension  $n$  has a  $C^\infty$  nowhere vanishing  $n$ -form  $\omega$  if and only if it has an oriented atlas.

*Proof.* ( $\Leftarrow$ ) Suppose we are given an oriented atlas

$$\left\{ (U_\alpha, x_\alpha^1, x_\alpha^2, \dots, x_\alpha^n) \right\}_{\alpha \in A}.$$

Suppose  $\{\rho_\alpha\}_{\alpha \in A}$  is a  $C^\infty$  partition of unity subordinate to the open cover  $\{U_\alpha\}_{\alpha \in A}$ . Define

$$\omega = \sum \rho_\alpha dx_\alpha^1 \wedge \dots \wedge dx_\alpha^n. \quad (6.9)$$

Since  $\{\text{supp } \rho_\alpha\}$  is locally finite by definition of partition of unity, for any  $p \in M$ , there is an open neighborhood  $U_p$  of  $p$  that intersects only finitely many of the sets  $\text{supp } \rho_\alpha$ . Thus, (6.9) is a finite sum on  $U_p$ . This actually shows that  $\omega$  is defined and  $C^\infty$  at every point of  $M$ .

Let  $(U, x^1, \dots, x^n)$  be one of the charts about  $p$  in the oriented atlas. On  $U_\alpha \cap U$ , by Lemma 5.5,

$$dx_\alpha^1 \wedge \dots \wedge dx_\alpha^n = \det \left[ \frac{\partial x_\alpha^i}{\partial x^j} \right] dx^1 \wedge \dots \wedge dx^n. \quad (6.10)$$

By hypothesis,  $\det \left[ \frac{\partial x_\alpha^i}{\partial x^j} \right] > 0$  as the atlas is oriented. Then on  $U_p \cap U$ ,

$$\omega = \sum \rho_\alpha dx_\alpha^1 \wedge \dots \wedge dx_\alpha^n = \left( \sum \rho_\alpha \det \left[ \frac{\partial x_\alpha^i}{\partial x^j} \right] \right) dx^1 \wedge \dots \wedge dx^n. \quad (6.11)$$

The sum in (6.11) is a finite sum, since  $U_p$  intersects only finitely many of the sets  $\text{supp } \rho_\alpha$ . Now, it's easy to see that the finite number in the parenthesis is actually positive at  $p$ . Indeed,  $\det \left[ \frac{\partial x_\alpha^i}{\partial x^j} \right] > 0$  at  $p$ , since the atlas is oriented. Furthermore,  $\rho_\alpha(p) > 0$  for at least one  $\alpha \in A$ . Hence.

$$\omega_p = (\text{positive number}) \times (dx^1 \wedge \dots \wedge dx^n)_p \neq 0.$$

As  $p$  is an arbitrary point of  $M$ , the  $n$ -form  $\omega$  is nowhere vanishing on  $M$ .

( $\Rightarrow$ ) Suppose  $\omega$  is a  $C^\infty$  nowhere vanishing  $n$ -form on  $M$ . Given an atlas on  $M$ , we will use  $\omega$  to modify the atlas so that it becomes oriented. Without loss of generality, assume that all the open sets of the atlas are connected.

On a chart  $(U, x^1, \dots, x^n)$ ,

$$\omega = f dx^1 \wedge \dots \wedge dx^n \quad (6.12)$$

for a  $C^\infty$  function  $f$  on  $U$ . Since  $\omega$  is nowhere-vanishing and  $f$  is continuous,  $f$  is either everywhere positive or everywhere negative on  $U$ . If  $f > 0$ , we leave the chart as it is; if  $f < 0$ , we replace the chart by  $(U, -x^1, x^2, \dots, x^n)$ . After all the charts have been checked and replaced if necessary, we have that on every chart  $(V, y^1, \dots, y^n)$

$$\omega = h dy^1 \wedge \dots \wedge dy^n \quad (6.13)$$

with  $h > 0$ . This can be seen to be an oriented atlas, since if  $(U, x^1, \dots, x^n)$  and  $(V, y^1, \dots, y^n)$  are two charts, then on  $U \cap V$

$$\omega = f dx^1 \wedge \dots \wedge dx^n = h dy^1 \wedge \dots \wedge dy^n, \quad (6.14)$$

with  $f, h > 0$ . From (6.14),

$$dy^1 \wedge \dots \wedge dy^n = \frac{f}{h} dx^1 \wedge \dots \wedge dx^n. \quad (6.15)$$

By Lemma 5.5,

$$dy^1 \wedge \dots \wedge dy^n = \det \left[ \frac{\partial y^i}{\partial x^j} \right] dx^1 \wedge \dots \wedge dx^n. \quad (6.16)$$

Comparing (6.15) and (6.16),

$$\det \left[ \frac{\partial y^i}{\partial x^j} \right] = \frac{f}{h} > 0 \quad (6.17)$$

on  $U \cap V$ . Hence, the modified atlas is oriented. ■

**Example 6.4** (Non-orientability of the open Möbius band). Let  $R$  be the rectangle

$$R = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, \text{ and } -1 < y < 1\}.$$

We define an equivalence relation  $\sim$  on  $R$  as follows:

$$(0, y) \sim (1, -y), \quad (6.18)$$

for  $y \in (-1, 1)$ . Then  $M = R/\sim$  is the open Möbius band. We want to show that  $M$  is not orientable.



Consider the following open sets on  $M$ :

$$\begin{aligned} U &= \{[x, y] \in M \mid 0 < x < 1\}, \\ V &= \left\{ [x, y] \in M \mid x \neq \frac{1}{2} \right\}. \end{aligned} \quad (6.19)$$

(Here  $[x, y]$  represents the equivalence class containing the point  $(x, y) \in R$ ) Then we can define homeomorphisms  $\varphi : U \rightarrow \varphi(U) \subset \mathbb{R}^2$  and  $\psi : V \rightarrow \psi(V) \subset \mathbb{R}^2$ :

$$\begin{aligned} \varphi([x, y]) &= (x, y), \\ \psi([x, y]) &= \begin{cases} (x, y) & \text{if } x < \frac{1}{2}, \\ (x - 1, -y) & \text{if } x > \frac{1}{2}. \end{cases} \end{aligned} \quad (6.20)$$

Then  $\{(U, \varphi), (V, \psi)\}$  forms an atlas on  $M$ . Consider  $(U, \varphi) \equiv (U, x^1, x^2)$  and  $(V, \psi) = (V, y^1, y^2)$ .

Assume for the sake of contradiction that  $M$  is orientable. Then there is a nowhere vanishing 2-form  $\omega$  on  $M$ . Then on  $U$ ,

$$\omega = f dx^1 \wedge dx^2, \quad (6.21)$$

for a  $C^\infty$  nowhere vanishing function  $f$  on  $U$ . Since  $U$  is connected,  $f$  is either positive, or negative. Similarly, on  $V$ ,

$$\omega = g dy^1 \wedge dy^2, \quad (6.22)$$

for a  $C^\infty$  nowhere vanishing function  $g$  on  $V$ . Since  $V$  is connected,  $g$  is either positive, or negative. On  $U \cap V$ ,

$$\omega = g \, dy^1 \wedge dy^2 = g \det \left[ \frac{\partial y^i}{\partial x^i} \right] dx^1 \wedge dx^2, \quad (6.23)$$

using Lemma 5.5. Comparing (6.21) and (6.23), we get

$$f = g \det \left[ \frac{\partial y^i}{\partial x^i} \right] \quad (6.24)$$

on  $U \cap V$ . Since  $f$  and  $g$  are either positive everywhere on  $U \cap V$  or negative everywhere on  $U \cap V$ ,

$$\det \left[ \frac{\partial y^i}{\partial x^i} \right] = \frac{f}{g} \quad (6.25)$$

is also either positive everywhere on  $U \cap V$  or negative everywhere on  $U \cap V$ . Let us now compute  $\det \left[ \frac{\partial y^i}{\partial x^i} \right]$ .

$$\frac{\partial y^i}{\partial x^j} = \frac{\partial (r^i \circ \psi)}{\partial x^j} = \frac{\partial ((r^i \circ \psi) \circ \varphi^{-1})}{\partial r^j} = \frac{\partial (\psi \circ \varphi^{-1})^i}{\partial r^j}, \quad (6.26)$$

where  $r^j$  are coordinates of  $\mathbb{R}^2$ . Let  $A, B, C$  be the following open rectangles in  $\mathbb{R}^2$ :

$$A = \left(0, \frac{1}{2}\right) \times (-1, 1), \quad B = \left(\frac{1}{2}, 1\right) \times (-1, 1), \quad C = \left(-\frac{1}{2}, 0\right) \times (-1, 1).$$

Then  $\varphi(U \cap V) = A \cup B$ ,  $\psi(U \cap V) = A \cup C$ .  $\psi \circ \varphi^{-1} : A \cup B \rightarrow A \cup C$  is then

$$(\psi \circ \varphi^{-1})(r^1, r^2) = \begin{cases} (r^1, r^2) & \text{if } (r^1, r^2) \in A, \\ (r^1 - 1, -r^2) & \text{if } (r^1, r^2) \in B. \end{cases} \quad (6.27)$$

So its Jacobian determinant  $\det \left[ \frac{\partial (\psi \circ \varphi^{-1})^i}{\partial r^j} \right]$  is

$$\begin{aligned} \frac{\partial (\psi \circ \varphi^{-1})^i}{\partial r^j} &= \begin{cases} \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \text{on } A, \\ \det \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} & \text{on } B. \end{cases} \\ &= \begin{cases} 1 & \text{on } A, \\ -1 & \text{on } B. \end{cases} \end{aligned} \quad (6.28)$$

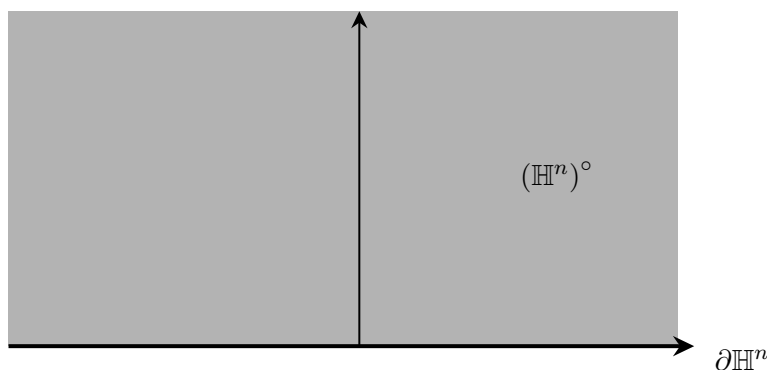
So  $\frac{\partial y^i}{\partial x^j}$  is 1 on  $\varphi^{-1}(A) \subseteq U \cap V$ , and  $-1$  on  $\varphi^{-1}(B) \subseteq U \cap V$ . But we have previously shown that  $\det \left[ \frac{\partial y^i}{\partial x^i} \right] = \frac{f}{g}$  is either positive everywhere on  $U \cap V$  or negative everywhere on  $U \cap V$ . Thus we arrive at a contradiction! Hence, no nowhere vanishing 2-form on the open Möbius band  $M$  exists.

# 7 Manifolds with Boundary

The prototype of a manifold with boundary is the closed upper half plane

$$\mathbb{H}^n = \left\{ (x^1, \dots, x^n) \in \mathbb{R}^n \mid x^n \geq 0 \right\},$$

with the subspace topology inherited from  $\mathbb{R}^n$ .



$(\mathbb{H}^n)^\circ = \{(x^1, \dots, x^n) \in \mathbb{R}^n \mid x^n > 0\}$  is called the interior of  $\mathbb{H}^n$ ; and  $\partial\mathbb{H}^n = \{(x^1, \dots, x^n) \in \mathbb{R}^n \mid x^n = 0\}$  is called the boundary of  $\mathbb{H}^n$ .

## §7.1 Invariance of Domain

**Definition 7.1.** Let  $S \subset \mathbb{R}^n$  be an arbitrary subset (not necessarily open). A function  $f : S \rightarrow \mathbb{R}^m$  is **smooth** at a point  $p \in S$  if there exists a neighborhood  $U$  of  $p$  in  $\mathbb{R}^n$ , and a  $C^\infty$  function  $\tilde{f} : U \rightarrow \mathbb{R}^m$  such that  $\tilde{f} = f$  on  $U \cap S$ . The function  $f : S \rightarrow \mathbb{R}^m$  is said to be smooth on  $S$  if it is smooth at each point  $p \in S$ .

### Lemma 7.1

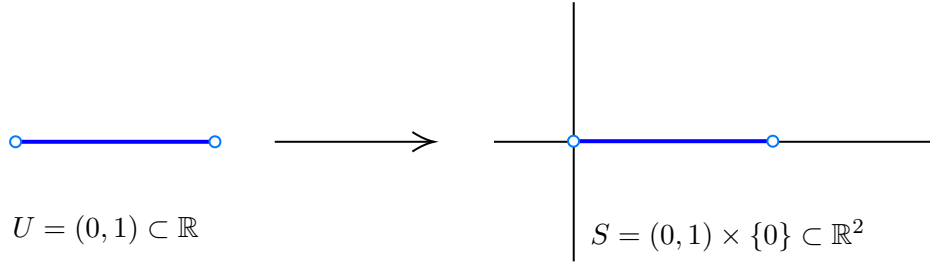
A function  $f : S \rightarrow \mathbb{R}^m$  with  $S \subset \mathbb{R}^n$  is  $C^\infty$  if and only if there exists an open set  $U \subseteq \mathbb{R}^n$  containing  $S$  and a  $C^\infty$  function  $\tilde{f} : U \rightarrow \mathbb{R}^m$  such that  $\tilde{f}|_S = f$ .

**Remark 7.1.** With the definition above, it now makes sense to speak about an arbitrary set  $S \subset \mathbb{R}^n$  being diffeomorphic to some set  $T \subset \mathbb{R}^m$ . This will be the case if and only if there are smooth maps (in the sense above)  $f : S \rightarrow T \subset \mathbb{R}^m$  and  $g : T \rightarrow S \subset \mathbb{R}^n$  that are inverses to each other.

### Theorem 7.2

Let  $U \subseteq \mathbb{R}^n$  be an open subset,  $S \subset \mathbb{R}^n$  an arbitrary subset, and  $f : U \rightarrow S$  a diffeomorphism. Then  $S$  is open in  $\mathbb{R}^n$ .

The diffeomorphism between  $U$  and  $S$  forces  $S$  to be open in  $\mathbb{R}^n$ . Given that  $f : U \rightarrow S$  is a diffeomorphism, we only know that an open subset of  $U$  is mapped to an open subset of  $S$  under  $f$ . Since  $U$  is open in itself,  $f(U) = S$  is also open in  $S$ . We can't immediately conclude that  $f(U) = S$  is open in  $\mathbb{R}^n$ . Besides, it's crucial that both  $U$  and  $S$  are subsets of the same Euclidean space  $\mathbb{R}^n$ . For example, there is a diffeomorphism between the open interval  $(0, 1) \subset \mathbb{R}$  and the open segment  $S = (0, 1) \times \{0\}$  in  $\mathbb{R}^2$ . But  $S$  is not open in  $\mathbb{R}^2$ .



*Proof of Theorem 7.2.* Let  $f(p) \in S$  be an arbitrary point in  $S$ , with  $p \in U$ . Note that any point in  $S$  can be reached this way as  $f$  is onto. Since  $f : U \rightarrow S$  is a diffeomorphism,  $f^{-1} : S \rightarrow U$  is smooth with  $S$  being an arbitrary subset. By Lemma 7.1, there exist an open set  $V \subseteq \mathbb{R}^n$  containing  $S$  and a  $C^\infty$  function  $g : V \rightarrow \mathbb{R}^n$  such that  $g|_S = f^{-1}$ .

$$U \xrightarrow{f} V \xrightarrow{g} \mathbb{R}^n$$

$g$  agrees  $f^{-1}$  on  $f(U) = S$ . Therefore,  $g \circ f = \mathbb{1}_U : U \rightarrow U \subseteq \mathbb{R}^n$ . Given  $p \in U$ , by the chain rule, one has

$$g_{*,f(p)} \circ f_{*,p} = \mathbb{1}_{T_p U} : T_p U \rightarrow T_p U, \quad (7.1)$$

the identity map on the tangent space  $T_p U$ . So  $g_{*,f(p)}$  is the left inverse of  $f_{*,p}$ . The existence of left inverse implies injectivity, so  $f_{*,p} : T_p U \rightarrow T_{f(p)} V$  is injective. Since  $U$  and  $V$  are open subsets of the same Euclidean space, they have the same dimension as manifolds. So  $T_p U$  and  $T_{f(p)} V$  have the same dimension as vector spaces. Now,  $f_{*,p} : T_p U \rightarrow T_{f(p)} V$  is an injective linear map between vector spaces of same dimension. So  $f_{*,p} : T_p U \rightarrow T_{f(p)} V$  is invertible.

Now we recall *inverse function theorem*:

A  $C^\infty$  map  $F : N \rightarrow M$  between two manifolds of same dimension is locally invertible at  $p \in N$  (i.e.  $p$  has a neighborhood  $U$  on which  $F|_U : U \rightarrow F(U)$  is a diffeomorphism) if and only if the differential  $F_{*,p} : T_p N \rightarrow T_{F(p)} M$  is an isomorphism of vector spaces.

We have that  $f_{*,p} : T_p U \rightarrow T_{f(p)} V$  is an isomorphism of vector spaces. Hence, by *inverse function theorem*,  $f : U \rightarrow V$  is locally invertible at  $p$ . This means that there are open neighborhoods  $U_p$  of  $p \in U$  and  $V_{f(p)}$  of  $f(p) \in V$  such that

$$f|_{U_p} : U_p \rightarrow V_{f(p)}$$

is a diffeomorphism. Then it follows that

$$f(p) \in V_{f(p)} = f(U_p) \subseteq f(U) = S. \quad (7.2)$$

For every  $f(p) \in S$ , we can find an neighborhood  $V_{f(p)} \ni f(p)$  open in  $\mathbb{R}^n$  ( $V_{f(p)}$  is open in  $V$ ,  $V$  is open in  $\mathbb{R}^n$ ; hence  $V_{f(p)}$  is open in  $\mathbb{R}^n$ ) that is contained in  $S$ . Therefore,  $S$  is open in  $\mathbb{R}^n$ . ■

### Proposition 7.3

Let  $U$  and  $V$  be open subsets of the upper half space  $\mathbb{H}^n$ , and  $f : U \rightarrow V$  be a diffeomorphism. Then  $f$  maps interior points to interior points, and boundary points to boundary points.

*Proof.* Let  $p \in U$  be an interior point. Then there is an open ball  $B$  in  $\mathbb{R}^n$  containing  $p$ , which is contained in  $U$ . Restriction of a diffeomorphism to an open subset is still a diffeomorphism. Hence,

$$f|_B : B \rightarrow f(B)$$

is a diffeomorphism, with  $B$  being open in  $\mathbb{R}^n$ . By Theorem 7.2,  $f(B)$  is open in  $\mathbb{R}^n$ . Hence,

$$f(p) \in f(B) \subseteq f(U) = V \subseteq \mathbb{H}^n.$$



$f(B)$  is open in  $\mathbb{R}^n$ , and it is contained in  $\mathbb{H}^n$ . Therefore,  $f(B) \subseteq (\mathbb{H}^n)^\circ$ . In other words,  $f(p) \in (\mathbb{H}^n)^\circ$ , since  $f(p) \in f(B)$ . So  $f$  maps interior points to interior points.

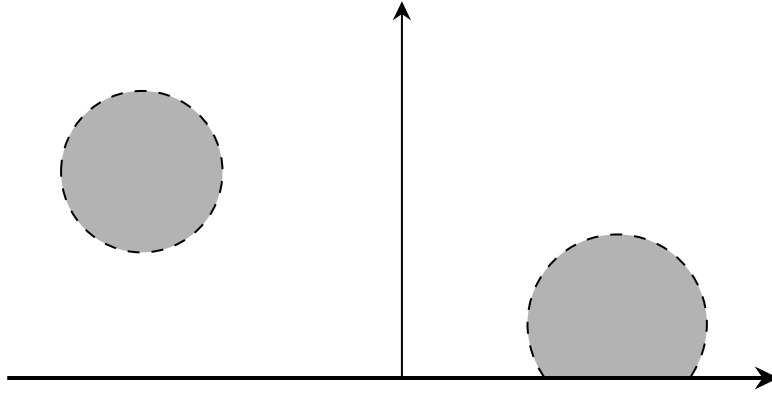
If  $p$  is a boundary point in  $U \cap \partial\mathbb{H}^n$ , then  $f^{-1}(f(p)) = p$  is a boundary point. Since  $f^{-1} : V \rightarrow U$  is a diffeomorphism, by the previous argument,  $f^{-1}$  takes interior points to interior points. If  $f(p)$  were an interior point, then  $f^{-1}$  would have mapped it to an interior point. But  $f^{-1}$  maps  $f(p)$  to a boundary point. So  $f(p)$  cannot be an interior point. Therefore,  $f(p)$  is a boundary point. ■

**Remark 7.2.** Replacing Euclidean spaces by manifolds throughout, one can prove in exactly the same way the **smooth invariance of domain for manifolds**:

Suppose  $N$  and  $M$  are  $n$ -dimensional manifolds. Let  $U \subseteq N$  be open, and  $S \subset M$  be any arbitrary subset. If there is a diffeomorphism  $F : U \rightarrow S$ , then  $S$  is open in  $M$ .

## §7.2 Manifolds with Boundary

In the upper half space  $\mathbb{H}^n$ , there are 2 types of open sets as seen in the following diagram:



In the left one, the set is disjoint from the boundary of  $\mathbb{H}^n$ , while in the right one the open set has nontrivial intersection with  $\partial\mathbb{H}^n$ . We say that a topological space  $M$  is locally  $\mathbb{H}^n$  if every point  $p \in M$  has a neighborhood  $U$  homeomorphic to an open subset of  $\mathbb{H}^n$ .

**Definition 7.2.** A **topological  $n$ -manifold with boundary** is a second countable, Hausdorff topological space that is locally  $\mathbb{H}^n$ .

Let  $M$  be a topological  $n$ -manifold with boundary. For  $n \geq 2$ , a chart on  $M$  is defined to be a pair  $(U, \varphi)$  consisting of an open set  $U \subseteq M$  and a homeomorphism

$$\varphi : U \rightarrow \varphi(U) \subseteq \mathbb{H}^n$$

of  $U$  with an open subset  $\varphi(U)$  of  $\mathbb{H}^n$ . A slight modification is necessary for the definition of a chart in the case  $n = 1$ .

Note that  $\mathbb{H}^1 = [0, \infty)$  is the right half-line. We also need the left half line  $\mathbb{L}^1 = (-\infty, 0]$  to model a 1-manifold with boundary locally. A chart  $(U, \varphi)$  in dimension 1 consists of an open set  $U \subseteq M$  and a homeomorphism  $\varphi$  of  $U$  with an open subset of  $\mathbb{H}^1$  or  $\mathbb{L}^1$ . With this slight modification of definition of chart in dimension 1, it can be seen that if  $(U, x^1, x^2, \dots, x^n)$  is a chart of an  $n$ -dimensional manifold with boundary, then so is  $(U, -x^1, x^2, \dots, x^n)$  for  $n \geq 1$ . A manifold with boundary has dimension at least 1, since a manifold of dimension 0, being a discrete set of points, necessarily has empty boundary.

**Definition 7.3.** A collection  $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$  of charts is a  $C^\infty$  **atlas** for the topological manifold  $M$  with boundary if

$$\bigcup_{\alpha \in A} U_\alpha = M,$$

and for any two charts  $(U_\alpha, \varphi_\alpha)$  and  $(U_\beta, \varphi_\beta)$ , the transition map

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \rightarrow \varphi_\beta(U_\alpha \cap U_\beta)$$

is a diffeomorphism. A  $C^\infty$  **manifold with boundary** is a topological manifold with boundary together with a maximal  $C^\infty$  atlas.

A point  $p \in M$  is called an **interior point** in some chart  $(U, \varphi)$  if the point  $\varphi(p)$  is an interior point of  $\mathbb{H}^n$ , i.e.  $\varphi(p) \in (\mathbb{H}^n)^\circ$ . Similarly,  $p \in M$  is a **boundary point** if  $\varphi(p)$  is a boundary point of  $\mathbb{H}^n$ , i.e.  $\varphi(p) \in \partial\mathbb{H}^n$ . These concepts are independent of charts. Suppose  $(V, \psi)$  is another chart about  $p$ . Then the diffeomorphism  $\psi \circ \varphi^{-1}$  maps  $\varphi(p)$  to  $\psi(p)$ . By [Proposition 7.3](#),  $\varphi(p)$  and  $\psi(p)$  are both interior points, or both boundary points.

The set of all boundary points of  $M$  is denoted by  $\partial M$ . On the contrary, the set of all interior points of  $M$  is denoted by  $M^\circ$ .

In contrast to the geometric notion of the interior and boundary of a manifold, there is the topological notion of the interior and boundary of a subset  $A$  of a topological space  $S$ . A point  $p \in S$  is said to be an interior point of  $A$  if there exists an open subset  $U \subseteq S$  such that

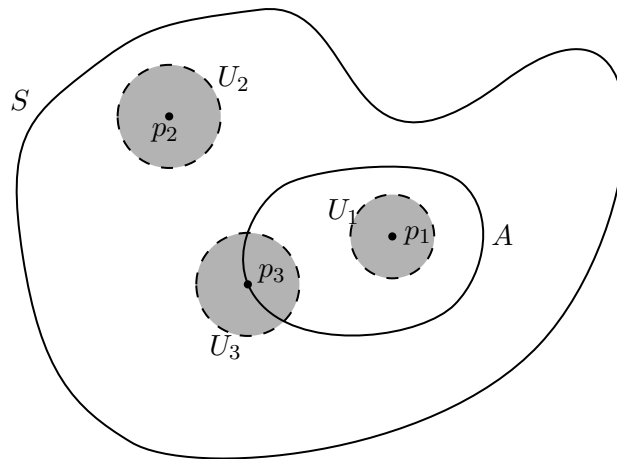
$$p \in U \subseteq A.$$

The point  $p \in S$  is an exterior point of  $A$  if there exists an open set  $U$  of  $S$  such that

$$p \in U \subseteq A.$$

Finally  $p \in S$  is a boundary point of  $A$  if every neighborhood of  $p$  contains both a point of  $A$  and a point not in  $A$ . One denotes by  $\text{int}_S(A)$ ,  $\text{ext}_S(A)$ ,  $\text{bd}_S(A)$  the sets of interior, exterior and boundary points of  $A$  in  $S$ , respectively. Clearly, the topological space  $S$  is the disjoint union

$$S = \text{int}_S(A) \sqcup \text{ext}_S(A) \sqcup \text{bd}_S(A). \quad (7.3)$$



In the above diagram,  $p_1 \in \text{int}_S(A)$ ,  $p_2 \in \text{ext}_S(A)$ ,  $p_3 \in \text{bd}_S(A)$ .

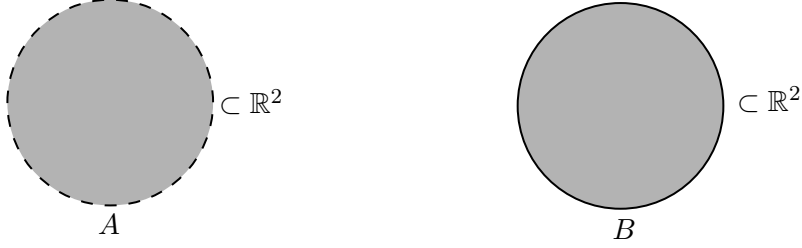
In case the subset  $A \subseteq S$  is a manifold with boundary, we call  $\text{int}_S(A)$  the topological interior and  $\text{bd}_S(A)$  the topological boundary of  $A$ , to distinguish them from the manifold interior  $A^\circ$  and the manifold boundary  $\partial A$ .

Note that the topological interior and the topological boundary of a set depend on an ambient space, while the manifold interior and the manifold boundary are both intrinsic.

**Example 7.1** (Topological boundary vs. manifold boundary). Let  $A$  be the open unit disk in  $\mathbb{R}^2$ :

$$A = \{\mathbf{x} \in \mathbb{R}^2 \mid \|\mathbf{x}\| < 1\}.$$

Then its topological boundary  $\text{bd}_{\mathbb{R}^2} A$  in  $\mathbb{R}^2$  is the unit circle, while  $A$  being a 2-dimensional manifold (without boundary) has its manifold boundary to be the empty set  $\emptyset$ .



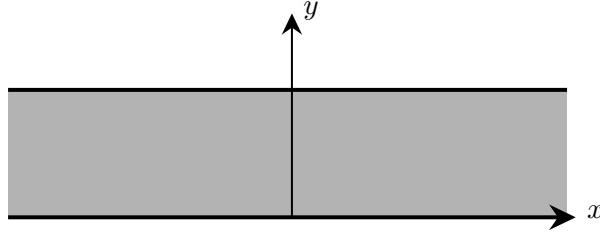
Now, consider  $B$  to be the closed unit disk in  $\mathbb{R}^2$ :

$$B = \{\mathbf{x} \in \mathbb{R}^2 \mid \|\mathbf{x}\| \leq 1\}.$$

It is a 2-dimensional manifold with boundary, with its manifold boundary  $\partial B$  being the unit circle. The topological boundary  $\text{bd}_{\mathbb{R}^2}(B)$  of  $B$  in  $\mathbb{R}^2$  is also the unit circle and hence  $\partial B$  and  $\text{bd } B$  coincide with each other.

**Example 7.2** (Topological interior vs. manifold interior). Let  $S$  be the upper half-plane  $\mathbb{H}^2$ , and  $A$  be the subset

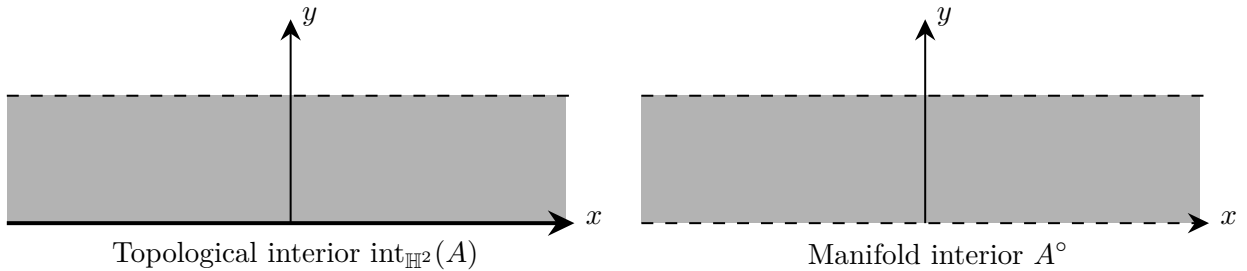
$$A = \{(x, y) \in \mathbb{H}^2 \mid y \leq 1\}.$$



The topological interior  $\text{int}_{\mathbb{H}^2}(A)$  of  $A$  in  $\mathbb{H}^2$  is the set

$$\text{int}_{\mathbb{H}^2}(A) = \{(x, y) \in \mathbb{H}^2 \mid 0 \leq y < 1\},$$

containing the  $x$ -axis.



On the other haand, the manifold interior  $A^\circ$  of the 2-dimensional manifold with boundary  $A$  is the set

$$A^\circ = \{(x, y) \in \mathbb{H}^2 \mid 0 < y < 1\},$$

not containing the  $x$ -axis.

Let us now consider the same set  $A$ , but now as a subset of  $\mathbb{R}^2$  instead of  $\mathbb{H}^2$ :

$$A = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq 1\}.$$

Then the topological interior  $\text{int}_{\mathbb{R}^2}(A)$  of  $A$  in  $\mathbb{R}^2$  is the set

$$\text{int}_{\mathbb{R}^2}(A) = \{(x, y) \in \mathbb{H}^2 \mid 0 < y < 1\},$$

which coincides with  $A^\circ$ .

### The boundary of a manifold with boundary

Let  $M$  be a manifold of dimension  $n$  with boundary  $\partial M$ . If  $(U, \varphi)$  is a chart of  $M$ , we denote by

$$\varphi' = \varphi|_{U \cap \partial M},$$

the restriction of the coordinate map  $\varphi$  to the boundary  $\partial M$ . Since  $\varphi$  maps boundary points to boundary points,

$$\varphi' = \varphi|_{U \cap \partial M} : U \cap \partial M \rightarrow \partial \mathbb{H}^n = \mathbb{R}^{n-1}.$$

Additionally, if  $(U, \varphi)$  and  $(V, \psi)$  are two charts on  $M$ , then

$$\psi' \circ (\varphi')^{-1} : \varphi(U \cap V \cap \partial M) \rightarrow \psi(U \cap V \cap \partial M)$$

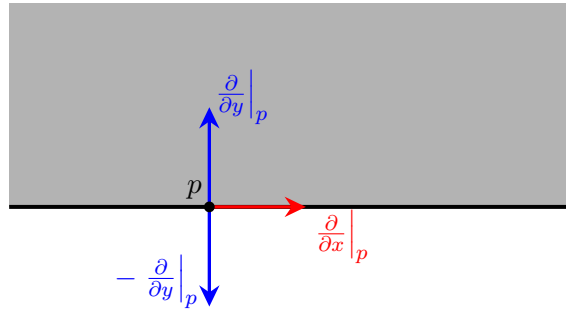
is  $C^\infty$ . Thus, an atlas  $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$  induces an atlas  $\{(U_\alpha \cap \partial M, \varphi_\alpha|_{U_\alpha \cap \partial M})\}_{\alpha \in A}$  for  $\partial M$ , making  $\partial M$  into a  $C^\infty$  manifold of dimension  $n - 1$  (without boundary).

### §7.3 Tangent Vectors, Differential Forms, and Orientations

Let  $M$  be a manifold with boundary and  $p \in \partial M$ . Let us first understand what  $C_p^\infty(M)$  is. Two  $C^\infty$  functions  $f : U \rightarrow \mathbb{R}$  and  $g : V \rightarrow \mathbb{R}$  defined on neighborhoods  $U$  and  $V$  of  $p \in M$  and are said to be equivalent if  $f$  and  $g$  agree on some neighborhood  $W$  of  $p$  contained in  $U \cap V$ . It can easily be verified that the relation defined above is an equivalence relation. Under this equivalence relation, a germ of  $C^\infty$  functions at  $p$  is an equivalence class of such functions. With the usual pointwise addition, multiplication and scalar multiplication, the set  $C_p^\infty(M)$  of germs of  $C^\infty$  functions at  $p$  is an  $\mathbb{R}$ -algebra.

The **tangent space**  $T_p M$  at  $p$  is then defined to be the vector space of all point derivations on the algebra  $C_p^\infty(M)$  [Recall that a point derivation of  $C_p^\infty(M)$  is a linear map  $D_p : C_p^\infty(M) \rightarrow \mathbb{R}$  obeying Leibniz condition].

Let us now take the example where  $p \in \partial \mathbb{H}^2$ .



$\frac{\partial}{\partial x}|_p$  and  $\frac{\partial}{\partial y}|_p$  are both point derivations on  $C_p^\infty(\mathbb{H}^2)$ . The tangent space  $T_p \mathbb{H}^2$  is represented by a 2-dimensional vector space with origin at  $p$  and spanned by the tangent vectors  $\frac{\partial}{\partial x}|_p$  and  $\frac{\partial}{\partial y}|_p$ . Since  $T_p \mathbb{H}^2$  is a vector space and  $\frac{\partial}{\partial y}|_p \in T_p \mathbb{H}^2$ , we have  $-\frac{\partial}{\partial y}|_p \in \mathbb{H}^2$ .

The **cotangent space**  $T_p^* M$  to the point  $p \in \partial M$  of the manifold  $M$  with boundary  $\partial M$  is defined to be the dual of the tangent space  $T_p M$ :

$$T_p^* M = \text{Hom}(T_p M, \mathbb{R}). \quad (7.4)$$

By taking the disjoint union of the cotangent spaces  $T_p^* M$  for all points  $p \in M$ , i.e. over all interior and boundary points of  $M$ , one arrives at the cotangent bundle  $T^* M$  of the manifold with boundary. Define the vector bundle.

$$\Lambda^k(T^* M) = \bigsqcup_{p \in M} \Lambda^k(T_p^* M). \quad (7.5)$$

Then a **differential  $k$ -form** on  $M$  is a section of the vector bundle  $\Lambda^k(T^*M)$ . A differential  $k$ -form is  $C^\infty$  if it is  $C^\infty$  as a section of the vector bundle  $\Lambda^k(T^*M)$ . For example,  $dx \wedge dy$  is a  $C^\infty$  2-form on  $\mathbb{H}^2$ .

An **orientation** on an  $n$ -dimensional manifold  $M$  with boundary is again a  $C^\infty$  nowhere vanishing  $n$ -form on  $M$ . We've seen in Proposition 6.4 that the orientability of a manifold without boundary (or equivalently the existence of a  $C^\infty$  nowhere-vanishing top degree form by the definition of orientability of a manifold) is equivalent to the existence of an oriented atlas. The same goes for manifold with boundary.

In the proof of Proposition 6.4 for establishing the necessary and sufficient condition for the orientability of a manifold without boundary, it was necessary to replace the chart  $(U, x^1, \dots, x^n)$  by  $(U, -x^1, \dots, x^n)$ . This would not be possible to carry out in the case  $n = 1$  for manifold with boundary if we had not allowed the left-half line  $\mathbb{L}^1$  as a local model in the definition of a chart on a 1-dimensional manifold with boundary. It would be better understood if we look at the following example.

**Example 7.3.** The closed interval  $[0, 1]$  is a  $C^\infty$  manifold with boundary. It has an atlas with 2 charts  $(U_1, \phi_1)$  and  $(U_2, \phi_2)$ , where  $U_1 = [0, 1]$ ,  $\phi_1(x) = x$ , and  $U_2 = (0, 1]$ ,  $\phi_2(x) = 1 - x$ .

With  $dx$  as the orientation form,  $[0, 1]$  should be an oriented manifold with boundary. However,  $\{(U_1, \phi_1), (U_2, \phi_2)\}$  is not an oriented atlas as the transition map

$$(\phi_2 \cdot \phi_1^{-1})(x) = 1 - x$$

has negative Jacobian determinant on  $\phi_1(0, 1) = (0, 1)$ . If one flips the sign of  $\phi_2$ , then  $\{(U_1, \phi_1), (U_2, -\phi_2)\}$  becomes an oriented atlas as

$$(\phi_2 \cdot \phi_1^{-1})(x) = x - 1$$

has positive Jacobian determinant on  $\phi_1(0, 1) = (0, 1)$ . It's important to note that  $-\phi_2(x) = x - 1$  maps  $U_2 = (0, 1]$  to  $(-1, 0]$  being an open set of the left half-line  $\mathbb{L}^1 = (-\infty, 0]$ . If we had allowed only  $\mathbb{H}^1$  as a local model for a 1-dimensional manifold with boundary, the closed interval  $[0, 1]$  wouldn't have an oriented atlas.

## §7.4 Outward-Pointing Vector Fields

**Definition 7.4.** Let  $M$  be a manifold with boundary  $\partial M$ , and  $p \in \partial M$ . We say that a tangent vector  $X_p \in T_p M$  is **inward-pointing** if  $X_p \notin T_p(\partial M)$ , and there are a positive real number  $\varepsilon$  and a curve  $c: [0, \varepsilon) \rightarrow M$  such that  $c(0) = p$ ,  $c(0, \varepsilon) \in M^\circ$ , and  $c'(0) = X_p$ . A vector  $X_p \in T_p M$  is **outward-pointing** if  $-X_p$  is inward-pointing.

For example, on the upper half-plane  $\mathbb{H}^2$ , the tangent vector  $\frac{\partial}{\partial y}\Big|_p$  is inward-pointing, and  $-\frac{\partial}{\partial y}\Big|_p$  is outward-pointing at  $p \in \partial\mathbb{H}^2$ .

A vector field along  $\partial M$  is a map that assigns to each point  $p \in \partial M$ , a tangent vector  $X_p \in T_p M$  (as opposed to  $T_p(\partial M)$ ). We say that a vector field  $X$  along the boundary  $\partial M$  is **outward-pointing** if for all  $p \in \partial M$ ,  $X_p \in T_p M$  is outward pointing.

In a coordinate neighborhood  $(U, x^1, \dots, x^n)$  of  $p$  in  $M$ , a vector field  $X$  along  $\partial M$  can be written as a linear combination

$$X_q = \sum_i a^i(q) \frac{\partial}{\partial x^i} \Big|_q, \quad (7.6)$$

for  $q \in \partial M$ . The vector field  $X$  along  $\partial M$  is said to be smooth at  $p \in M$  if there exists a coordinate neighborhood of  $p$  for which the functions  $a^i$  on  $\partial M$  are  $C^\infty$  at  $p$ ; it is said to be smooth if it is smooth at every point  $p$ .

**Lemma 7.4**

Let  $M$  be a manifold with boundary and let  $p \in \partial M$ . Suppose  $X_p \in T_p M$  is expressed as a linear combination of basis vectors as follows:

$$X_p = \sum_i a^i(p) \frac{\partial}{\partial x^i} \Big|_p.$$

Then  $X_p$  is outward pointing if and only if  $a^n(p) < 0$ .

**Proposition 7.5**

On a manifold  $M$  with boundary  $\partial M$ , there is a smooth outwardpointing vector field along  $\partial M$ .

**§7.5 Interior Multiplication**

If  $\beta$  is a  $k$ -covector on a vector space  $V$ , and  $\mathbf{v} \in V$ , for  $k \geq 2$ , the **interior multiplication** or **contraction** of  $\beta$  with  $\mathbf{v}$  is the  $(k-1)$ -covector  $\iota_{\mathbf{v}}\beta$  defined by

$$(\iota_{\mathbf{v}}\beta)(\mathbf{v}_2, \dots, \mathbf{v}_k) = \beta(\mathbf{v}, \mathbf{v}_2, \dots, \mathbf{v}_k), \quad (7.7)$$

with  $\mathbf{v}_2, \dots, \mathbf{v}_k \in V$ . When  $\beta$  is a 1-covector, then  $\iota_{\mathbf{v}}\beta$  is supposed to be a constant real number (i.e. a 0-covector). This is defined by

$$\iota_{\mathbf{v}}\beta = \beta(\mathbf{v}) \in \mathbb{R}. \quad (7.8)$$

Finally, when  $\beta$  is a 0-covector on  $V$  (i.e. a constant real number), we define

$$\iota_{\mathbf{v}}\beta = 0. \quad (7.9)$$

Interior multiplication on a manifold is defined pointwise. If  $X$  is a smooth vector field on  $M$  and  $\omega \in \Omega^k(M)$ , then  $\iota_X\omega \in \Omega^{k-1}(M)$ , defined by

$$(\iota_X\omega)_p = \iota_{X_p}\omega_p. \quad (7.10)$$

The right side of (7.10) makes sense according to (7.7), (7.8), and (7.9). Indeed, for  $(k-1)$  many tangent vectors  $X_p^2, \dots, X_p^k$  with  $k \geq 2$ , (7.10) can be recast into the following using (7.7) as

$$(\iota_X\omega)_p(X_p^2, \dots, X_p^k) = (\iota_{X_p}\omega_p)(X_p^2, \dots, X_p^k) = \omega_p(X_p, X_p^2, \dots, X_p^k). \quad (7.11)$$

If  $X, X^2, \dots, X^k$  are  $k$ -many smooth vector fields on  $M$ , then the RHS of (7.11) is  $[\omega(X, X^2, \dots, X^k)](p)$  while the LHS of (7.11) is  $[(\iota_X\omega)(X^2, \dots, X^k)](p)$ . Therefore, for  $k \geq 2$ , one has

$$(\iota_X\omega)(X^2, \dots, X^k) = \omega(X, X^2, \dots, X^k), \quad (7.12)$$

for  $(k-1)$  many  $C^\infty$  vector fields  $X^2, \dots, X^k$  on  $M$ . Now, since  $\omega$  is a smooth  $k$ -form, for any smooth vector fields  $X, X^2, \dots, X^k$  on  $M$ ,  $\omega(X, X^2, \dots, X^k)$  is a smooth function on  $M$ . As a result, for any  $C^\infty$  vector fields  $X^2, \dots, X^k$ ,  $(\iota_X\omega)(X^2, \dots, X^k)$  is a smooth function on  $M$ . Therefore,  $\iota_X\omega$  is a smooth  $(k-1)$ -form on  $M$ .

**Proposition 7.6**

For 1-covectors  $\alpha^1, \dots, \alpha^k$  on a vector space  $V$  and  $\mathbf{v} \in V$ ,

$$\iota_{\mathbf{v}}(\alpha^1 \wedge \dots \wedge \alpha^k) = \sum_{i=1}^k (-1)^{i-1} \alpha^i(\mathbf{v}) \alpha^1 \wedge \dots \wedge \widehat{\alpha^i} \wedge \dots \wedge \alpha^k,$$

where the caret  $\widehat{\phantom{x}}$  on the  $i$ -th covector  $\alpha^i$  means that  $\alpha^i$  is omitted from the wedge product.

*Proof.* Consider  $k \geq 2$ . For any  $\mathbf{v}_2, \dots, \mathbf{v}_k \in V$ ,

$$\begin{aligned}
 \left[ \iota_{\mathbf{v}} (\alpha^1 \wedge \dots \wedge \alpha^k) \right] (\mathbf{v}_2, \dots, \mathbf{v}_k) &= (\alpha^1 \wedge \dots \wedge \alpha^k) (\mathbf{v}, \mathbf{v}_2, \dots, \mathbf{v}_k) \\
 &= \det \begin{bmatrix} \alpha^1(\mathbf{v}) & \alpha^1(\mathbf{v}_2) & \dots & \alpha^1(\mathbf{v}_k) \\ \alpha^2(\mathbf{v}) & \alpha^2(\mathbf{v}_2) & \dots & \alpha^2(\mathbf{v}_k) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha^k(\mathbf{v}) & \alpha^k(\mathbf{v}_2) & \dots & \alpha^k(\mathbf{v}_k) \end{bmatrix} \\
 &= \sum_{i=1}^k (-1)^{i+1} \alpha^i(\mathbf{v}) \det [\alpha^l(\mathbf{v}_j)]_{\substack{1 \leq l \leq k, l \neq i \\ 2 \leq j \leq k}} \\
 &= \sum_{i=1}^k (-1)^{i-1} \alpha^i(\mathbf{v}) (\alpha^1 \wedge \dots \wedge \widehat{\alpha^i} \wedge \dots \wedge \alpha^k) (\mathbf{v}_2, \dots, \mathbf{v}_k). \quad (7.13)
 \end{aligned}$$

Therefore,

$$\iota_{\mathbf{v}} (\alpha^1 \wedge \dots \wedge \alpha^k) = \sum_{i=1}^k (-1)^{i-1} \alpha^i(\mathbf{v}) \alpha^1 \wedge \dots \wedge \widehat{\alpha^i} \wedge \dots \wedge \alpha^k. \quad (7.14)$$

If  $k = 1$ ,

$$\iota_{\mathbf{v}} (\alpha^1) = \alpha^1(\mathbf{v}) = (-1)^{1-1} \alpha^1(\mathbf{v}). \quad (7.15)$$

So, the equality holds for  $k = 1$  as well.  $\blacksquare$

### Lemma 7.7

The interior multiplication  $\iota_X \omega$  of a smooth  $k$ -form  $\omega$  on  $M$  with a smooth vector field  $X$  on  $M$  has the following properties:

- (i)  $\iota_{fX} \omega = f \iota_X \omega$ ;
- (ii)  $\iota_X (f\omega) = f \iota_X \omega$ ;

for  $f \in C^\infty(M)$ .

*Proof.* (i) Suppose  $k \geq 2$ . For any  $p \in M$ , and any  $X_p^2, \dots, X_p^k$ ,

$$\begin{aligned}
 (\iota_{fX} \omega)_p (X_p^2, \dots, X_p^k) &= (\iota_{f(p)X_p} \omega_p) (X_p^2, \dots, X_p^k) \\
 &= \omega_p (f(p)X_p, X_p^2, \dots, X_p^k) \\
 &= f(p) \omega_p (X_p, X_p^2, \dots, X_p^k) \\
 &= f(p) (\iota_X \omega)_p (X_p, X_p^2, \dots, X_p^k). \quad (7.16)
 \end{aligned}$$

Since  $p \in M$  is arbitrary, we have

$$\iota_{fX} \omega = f \iota_X \omega. \quad (7.17)$$

Now consider the case  $k = 1$ .

$$(\iota_{fX} \omega)_p = \omega_p (f(p) X_p) = f(p) [\omega(X)]_p = (f \iota_X \omega)_p. \quad (7.18)$$

Therefore, for  $k = 1$  as well, since  $p \in M$  is arbitrary,

$$\iota_{fX} \omega = f \iota_X \omega. \quad (7.19)$$

(ii) Again, let us first consider the case  $k \geq 2$ . For any  $p \in M$ , and any  $X_p^2, \dots, X_p^k$ ,

$$\begin{aligned} (\iota_X(f\omega))_p(X_p^2, \dots, X_p^k) &= (\iota_{X_p}(f(p)\omega_p))(X_p^2, \dots, X_p^k) \\ &= f(p)\omega_p(X_p, X_p^2, \dots, X_p^k) \\ &= f(p)(\iota_X\omega)_p(X_p, X_p^2, \dots, X_p^k). \end{aligned} \quad (7.20)$$

Since  $p \in M$  is arbitrary, we have

$$\iota_X(f\omega) = f\iota_X\omega. \quad (7.21)$$

Now consider the case  $k = 1$ .

$$(\iota_X(f\omega))_p = \iota_{X_p}(f(p)\omega_p) = f(p)\omega_p(X_p) = f(p)[\omega(X)]_p = (f\iota_X\omega)_p. \quad (7.22)$$

Therefore, for  $k = 1$  as well, since  $p \in M$  is arbitrary,

$$\iota_X(f\omega) = f\iota_X\omega. \quad (7.23)$$

■

## §7.6 Boundary Orientation

Now, we show that the boundary of an orientable manifold  $M$  with boundary is an orientable manifold without boundary.

### Proposition 7.8

Let  $M$  be an orientable  $n$ -manifold with boundary  $\partial M$ . If  $\omega$  is an orientation form on  $M$  and  $X$  is a smooth outward-pointing vector field along  $\partial M$ , then  $\iota_X\omega$  is a smooth nowhere vanishing  $(n-1)$  form on  $\partial M$ . Hence,  $\partial M$  is orientable.

*Proof.* Since  $\omega$  is smooth on  $M$ ,  $\omega$  is also smooth on  $\partial M$ . By hypothesis,  $X$  is smooth on  $\partial M$ . Hence, the contraction  $\iota_X\omega$  is also smooth on  $\partial M$ . We will now prove by contradiction that  $\iota_X\omega$  is nowhere-vanishing on  $\partial M$ . Suppose.  $\iota_X\omega$  vanishes at some  $p \in \partial M$ . It means that

$$(\iota_X\omega)_p(X_p^1, \dots, X_p^{n-1}) = 0, \quad (7.24)$$

for any  $X_p^1, \dots, X_p^{n-1} \in T_p(\partial M)$ . Let  $Y_p^1, \dots, Y_p^{n-1}$  be a basis for  $T_p(\partial M)$ . Since  $X$  is a smooth outward-pointing vector field along  $\partial M$ ,  $X_p \notin T_p(\partial M)$ . Now,

$$\dim T_p M = \dim T_p(\partial M) + 1,$$

since  $\partial M$  is a manifold of dimension  $n-1$  (without boundary). Since  $X_p \notin T_p(\partial M)$  and  $Y_p^1, \dots, Y_p^{n-1}$  is a basis for  $T_p(\partial M)$ , one finds that  $X_p, Y_p^1, \dots, Y_p^{n-1}$  is a basis for  $T_p M$ . Hence,

$$\omega_p(X_p, Y_p^1, \dots, Y_p^{n-1}) = (\iota_X\omega)_p(X_p^1, \dots, X_p^{n-1}) = 0. \quad (7.25)$$

Now, by Lemma 6.1, since  $X_p, Y_p^1, \dots, Y_p^{n-1}$  forms a basis for  $T_p M$ ,

$$\omega_p(Z_p^1, \dots, Z_p^n) = 0, \quad (7.26)$$

for all  $Z_p^1, \dots, Z_p^n \in T_p M$ . In other words,  $\omega_p \equiv 0$  on  $T_p M$ , a contradiction. ■

**Example 7.4** (The boundary orientation on  $\partial\mathbb{H}^n$ ). An orientation form for the standard orientation on the upper half-space  $\mathbb{H}^n$  is  $\omega = dx^1 \wedge \dots \wedge dx^n$ . A smooth outward pointing vector field on  $\partial\mathbb{H}^n$  is  $-\frac{\partial}{\partial x^n}$ . By Proposition 7.8, an orientation form on  $\partial\mathbb{H}^n$  is given by the contraction

$$\iota_{-\frac{\partial}{\partial x^n}}(\omega) = \iota_{-\frac{\partial}{\partial x^n}}(dx^1 \wedge \dots \wedge dx^n). \quad (7.27)$$



Using [Proposition 7.6](#) and [Lemma 7.7\(i\)](#), we get

$$\begin{aligned}
 \iota_{-\frac{\partial}{\partial x^n}}(\omega) &= - \sum_{i=1}^n (-1)^{i-1} \left[ dx^i \left( \frac{\partial}{\partial x^n} \right) \right] dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \\
 &= \sum_{i=1}^n (-1)^i \delta_n^i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^{n-1} \wedge dx^n \\
 &= (-1)^n dx^1 \wedge \cdots \wedge dx^{n-1}.
 \end{aligned} \tag{7.28}$$

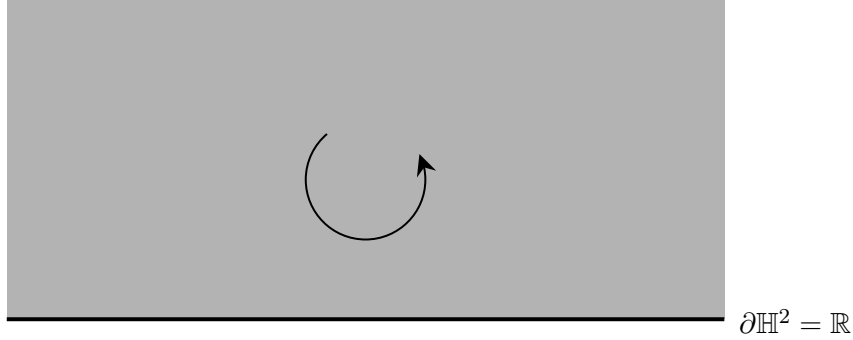
The  $n = 1$  case needs a separate treatment. Recall that for a 1-covector  $\beta$  on a vector space  $V$ , and any vector  $\mathbf{v} \in V$ , the interior multiplication of  $\beta$  with  $\mathbf{v}$  is defined as

$$\iota_{\mathbf{v}}\beta = \beta(\mathbf{v}) \in \mathbb{R}.$$

Now, for  $\mathbb{H}^1 = [0, \infty)$ ,  $\partial\mathbb{H}^1 = \{0\}$ . Fix the orientation 1-form  $dx$  corresponding to the orientation directed from left to right. An outward pointing vector field on  $\partial\mathbb{H}^1 = \{0\}$  is given by  $-\frac{\partial}{\partial x}$ . By [Proposition 7.8](#), an orientation form on  $\partial\mathbb{H}^1$  is given by the contraction

$$\iota_{-\frac{\partial}{\partial x}}(dx) = -1. \tag{7.29}$$

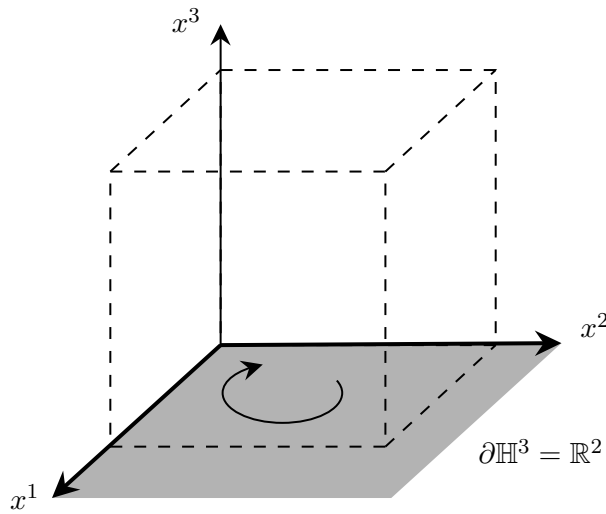
Let us now consider the case  $n = 2$ .



By [\(7.28\)](#), the boundary orientation on  $\partial\mathbb{H}^2 = \mathbb{R}$  is given by the contraction

$$\iota_{-\frac{\partial}{\partial x^2}}(dx^1 \wedge dx^2) = dx^1. \tag{7.30}$$

Now consider  $n = 3$ .

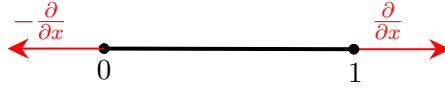


Here the manifold is  $\mathbb{H}^3$  with boundary  $\partial\mathbb{H}^3 = \mathbb{R}^2$ , the  $x^1$ - $x^2$  plane (corresponding to  $x^3 = 0$ ). According to [\(7.28\)](#), the boundary orientation on  $\partial\mathbb{H}^3 = \mathbb{R}^2$  is given by the 2-form

$$(-1)^3 dx^1 \wedge dx^2 = -dx^1 \wedge dx^2. \tag{7.31}$$

This yields the anti-clockwise orientation on the  $x^1$ - $x^2$  plane.

**Example 7.5.** Consider the closed interval  $[0, 1]$  in  $\mathbb{R}$ . One has  $\partial[0, 1] = \{0, 1\}$ . Also consider the orientation 1-form  $dx$  on  $[0, 1]$  corresponding to the standard orientation directed from left to right.



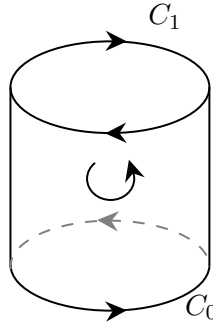
At the right boundary point 1, an outward pointing vector field reads  $\frac{\partial}{\partial x}$ . Hence, by [Proposition 7.8](#), the boundary orientation at 1 is given by

$$\iota_{\frac{\partial}{\partial x}}(dx) = 1. \quad (7.32)$$

On the other hand, at the left boundary point 0, an outward pointing vector field reads  $-\frac{\partial}{\partial x}$ . Hence, by [Proposition 7.8](#), the boundary orientation at 0 is given by the contraction

$$\iota_{-\frac{\partial}{\partial x}}(dx) = -1. \quad (7.33)$$

**Example 7.6.** Let  $M$  be the cylinder  $S^1 \times [0, 1]$  with the counterclockwise orientation when viewed from the exterior. Let us determine the boundary orientation on  $C_0 = S^1 \times \{0\}$  and  $C_1 = S^1 \times \{1\}$ .



The counterclockwise orientation on  $M$  is given by the orientation form  $\omega = d\theta \wedge dt$ . An outward-pointing vector field on  $C_0$  is given by  $-\frac{\partial}{\partial t}$ , so that the relevant contraction of  $\omega$  with  $-\frac{\partial}{\partial t}$  reads

$$\iota_{-\frac{\partial}{\partial t}}(d\theta \wedge dt) = -\frac{\partial}{\partial t}(dt)(-1)^{2-1}d\theta = d\theta. \quad (7.34)$$

Hence, the boundary orientation on  $C_0$  is given by the 1-form  $d\theta$ .

Now, to determine the boundary orientation on  $C_1 = S^1 \times \{1\}$ , let us compute the contraction of  $\omega$  on with an outward-pointing vector field  $\frac{\partial}{\partial t}$  on  $C_1$ :

$$\iota_{\frac{\partial}{\partial t}}(d\theta \wedge dt) = \frac{\partial}{\partial t}(dt)(-1)^{2-1}d\theta = -d\theta. \quad (7.35)$$

So the boundary orientation on  $C_1$  is given by the 1-form  $-d\theta$ . Therefore, on  $C_0$ , the orientation is given by the counterclockwise orientation and on  $C_1$ , the orientation is given by the clockwise orientation viewed from the top.

**Remark 7.3.** Recall the nowhere-vanishing 1 form on  $S^1$  from [Example 5.2](#):

$$\omega = \begin{cases} \frac{dy}{x} & \text{on } U_x = \{(x, y) \in S^1 \mid x \neq 0\}, \\ -\frac{dx}{y} & \text{on } U_y = \{(x, y) \in S^1 \mid y \neq 0\}. \end{cases} \quad (7.36)$$

If we go back to polar coordinates, i.e.  $x = \cos \theta$  and  $y = \sin \theta$ , then

$$\frac{dy}{x} = \frac{d(\sin \theta)}{\cos \theta} = \frac{\cos \theta d\theta}{\cos \theta} = d\theta, \quad (7.37)$$

$$-\frac{dx}{y} = -\frac{d(\cos \theta)}{\sin \theta} = -\frac{-\sin \theta d\theta}{\sin \theta} = d\theta. \quad (7.38)$$

Therefore, this nowhere vanishing 1-form  $\omega$  is, in fact,  $d\theta$ .

# 8 Integration on Manifolds

## §8.1 Riemann Integral Review

Let us first recall the subject of Riemann integration over a closed rectangle in Euclidean space  $\mathbb{R}^n$ . A closed rectangle in  $\mathbb{R}^n$  is a cartesian product

$$R = [a^1, b^1] \times \dots \times [a^n, b^n]$$

of closed intervals in  $\mathbb{R}$  with  $a^i, b^i \in \mathbb{R}$ . Let  $f$  be a bounded function  $f : R \rightarrow \mathbb{R}$  defined on a closed rectangle  $R$ . The volume  $\text{vol}(R)$  of the closed rectangle is defined to be

$$\text{vol}(R) = \prod_{i=1}^n (b_i - a_i). \quad (8.1)$$

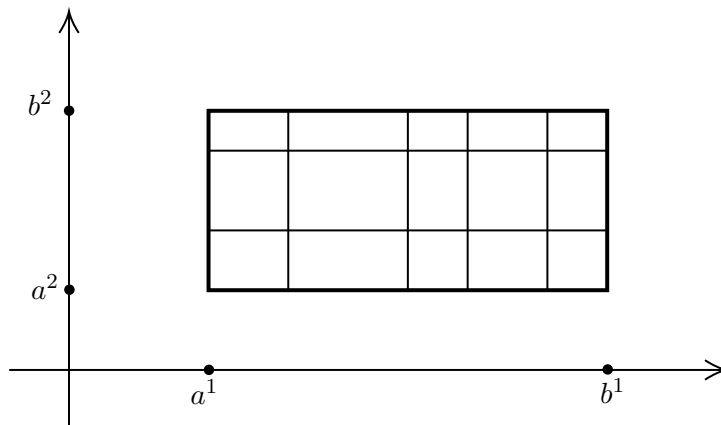
A partition of the closed interval  $[a, b]$  is a set of real numbers  $\{p_0, \dots, p_n\}$  arranged in ascending order, i.e.

$$a = p_0 < p_1 < \dots < p_n = b \quad (8.2)$$

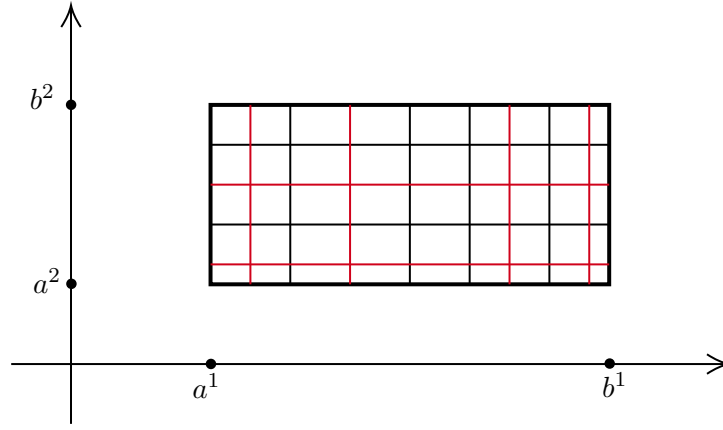
A partition of the rectangle  $R$  is a collection  $P = \{P_1, \dots, P_n\}$  such that each  $P_i$  is a partition of  $[a^i, b^i]$ . In other words, each  $P_i$  is an increasing sequence of real numbers

$$a_i^i = p_0^i < p_1^i < \dots < p_{k_i}^i = b^i \quad (8.3)$$

for  $i = 1, 2, \dots, n$ . This way, the partition  $P$  of the closed rectangle  $R$  divides it into closed subrectangles, which we denote by  $R_j$ . One possible partition of a closed rectangle  $[a^1, b^1] \times [a^2, b^2]$  is pictured below in  $\mathbb{R}^2$ :



A partition  $P' = \{P'_1, \dots, P'_n\}$  of the same rectangle  $R$  is called a **refinement** of the partition  $P = \{P_1, \dots, P_n\}$  of  $R$  if  $P_i \subset P'_i$  for each  $i = 1, 2, \dots, n$ . For example, the following partition of  $[a^1, b^1] \times [a^2, b^2]$  is a refinement of the partition shown above. (The original partition  $P$  is drawn in black, while the new lines arising in the refined partition are drawn in red.)



We immediately see that each subrectangle  $R_j$  of  $P$  is subdivided into subrectangles  $R'_{jk}$  of the refinement  $P'$ . It's now time to define the lower and upper sum of the bounded function  $f : R \rightarrow \mathbb{R}$  with respect to the partition  $P$ :

$$L(f, P) := \sum_{R_j} \left( \inf_{R_j} f \right) \text{vol}(R_j), \quad (8.4)$$

$$U(f, P) := \sum_{R_j} \left( \sup_{R_j} f \right) \text{vol}(R_j). \quad (8.5)$$

It's clear that for any partition  $P$ ,

$$L(f, P) \leq U(f, P). \quad (8.6)$$

Now, suppose  $P'$  is a refinement of the partition  $P$ . Then each subrectangle  $R_j$  of  $P$  is subdivided into subrectangles  $R'_{jk}$  of the refinement  $P'$ . Furthermore,  $\text{vol}(R_j) = \sum_k \text{vol}(R'_{jk})$ . Now, since  $R'_{jk} \subset R_j$ , one has

$$\inf_{R_j} f \leq \inf_{R'_{jk}} f \text{ and } \sup_{R_j} f \geq \sup_{R'_{jk}} f. \quad (8.7)$$

Now, from (8.4),

$$\begin{aligned} L(f, P) &= \sum_{R_j} \left( \inf_{R_j} f \right) \text{vol}(R_j) \\ &\leq \sum_{R_j} \left( \inf_{R'_{jk}} f \right) \sum_k \text{vol}(R'_{jk}) \\ &\leq \sum_{R'_{jk}} \left( \inf_{R'_{jk}} f \right) \text{vol}(R'_{jk}) = L(f, P'). \end{aligned}$$

In other words,

$$L(f, P) \leq L(f, P'). \quad (8.8)$$

Similarly,

$$U(f, P) \geq U(f, P'). \quad (8.9)$$

Any two partitions  $P$  and  $P'$  of the rectangle  $R$  have a common refinement  $Q = \{Q_1, \dots, Q_n\}$  with  $Q_i = P_i \cup P'_i$ . Then by (8.8) and (8.9),

$$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P').$$

Therefore, for any two partitions  $P$  and  $P'$ ,

$$L(f, P) \leq U(f, P'). \quad (8.10)$$

From (8.10), one sees that  $U(f, P')$  is an upper bound of  $L(f, P)$  for any partition  $P$  of the rectangle  $R$ . As a result,

$$\sup_P L(f, P) \leq U(f, P'). \quad (8.11)$$

Again, from (8.11), one sees that  $\sup_P L(f, P)$  is a lower bound of  $U(f, P')$  for any partition  $P'$  of the rectangle  $R$ . Hence,

$$\sup_P L(f, P) \leq \inf_{P'} U(f, P'). \quad (8.12)$$

The supremum of the lower-sum  $L(f, P)$  as  $P$  varies over all partitions of the rectangle is called the **lower integral**, and is denoted by

$$\int_R f := \sup_P L(f, P) \quad (8.13)$$

On the contrary, the infimum of the upper-sum  $U(f, P)$  as  $P$  varies over all partitions of the rectangle is called the **upper integral**, and is denoted by

$$\bar{\int}_R f := \inf_P U(f, P). \quad (8.14)$$

Using these notations (8.12) reads

$$\int_R f \leq \bar{\int}_R f. \quad (8.15)$$

**Definition 8.1** (Riemann integrability). Let  $R$  be a closed rectangle in  $\mathbb{R}^n$ . A bounded function  $f : R \rightarrow \mathbb{R}$  is said to be **Riemann integrable** if  $\int_R f = \bar{\int}_R f$ . In this case, the **Riemann integral** of  $f$  is this common value, usually denoted by

$$\int_R f(\mathbf{x}) \, dx^1 dx^2 \cdots dx^n,$$

where  $x^1, \dots, x^n$  are the standard coordinates on  $\mathbb{R}^n$ .

**Definition 8.2.** If  $f : A \subset \mathbb{R}^n \rightarrow \mathbb{R}$ , then the **extension of  $f$  by zero** is the function  $\bar{f} : \mathbb{R}^n \rightarrow \mathbb{R}$  such that

$$\bar{f}(x) = \begin{cases} f(x) & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

If  $f$  is a bounded function on a bounded set  $A$ , then one encloses  $A$  in a closed rectangle  $R \subset \mathbb{R}^n$ . The Riemann integral of  $f : A \rightarrow \mathbb{R}$  over  $A$  is defined to be

$$\int_A f(\mathbf{x}) \, dx^1 dx^2 \cdots dx^n := \int_R \bar{f}(\mathbf{x}) \, dx^1 dx^2 \cdots dx^n, \quad (8.16)$$

provided the RHS exists, i.e. the extension  $\bar{f}$  of  $f$  by zero is Riemann integrable over the closed rectangle  $R$  enclosing  $A$ . The **volume**  $\text{vol } A$  of a subset  $A \subset \mathbb{R}^n$  is defined to be the integral  $\int_A 1 \, dx^1 dx^2 \cdots dx^n$  if the integral exists.

### Integrability conditions

**Definition 8.3.** A set  $A \subset \mathbb{R}^n$  is said to have **measure zero** if for every  $\varepsilon > 0$ , there is a countable collection of closed rectangles  $\{R_i\}_{i=1}^\infty$  such that  $A \subset \bigcup_{i=1}^\infty R_i$  and

$$\sum_{i=1}^\infty \text{vol}(R_i) < \varepsilon.$$

**Theorem 8.1** (Lebesgue's theorem)

A bounded function  $f : A \rightarrow \mathbb{R}$  on a bounded subset  $A \subset \mathbb{R}^n$  is Riemann integrable if and only if the set  $\text{Disc}(\bar{f})$  of discontinuities of the extended function  $\bar{f}$  has measure zero.

**Proposition 8.2**

If a continuous function  $f : U \rightarrow \mathbb{R}$  defined on an open subset  $U$  of  $\mathbb{R}^n$  has compact support, then  $f$  is Riemann integrable on  $U$ .

*Proof.*  $f : U \rightarrow \mathbb{R}$  is continuous. On  $U \setminus \text{supp } f$ ,  $f$  is zero. Since  $f$  is continuous on  $\text{supp } f \subset U$ , and  $\text{supp } f$  is compact (by hypothesis),  $f$  is bounded on  $\text{supp } f$ . Also,  $f$  is zero on  $U \setminus \text{supp } f$ . Hence,  $f : U \rightarrow \mathbb{R}$  is a bounded continuous function on the open subset  $U \subset \mathbb{R}^n$ . We claim that the extension  $\bar{f} : \mathbb{R}^n \rightarrow \mathbb{R}$  is continuous.

By the definition of extension of a function by zero,  $\bar{f}$  agrees with  $f$  on  $U$ , and hence  $\bar{f}$  is continuous on  $U$ . It remains to show that  $\bar{f}$  is continuous on the complement  $\mathbb{R}^n \setminus U$  of  $U$ . Since  $\text{supp } f \subset U$ , if  $p \notin U$ , then  $p \notin \text{supp } f$ .

$\text{supp } f$  being a compact subset of  $\mathbb{R}^n$  is closed and bounded. Hence,  $\mathbb{R}^n \setminus \text{supp } f$  is open and  $p \in \mathbb{R}^n \setminus \text{supp } f$ . Therefore, there exist an open ball  $B$  such that

$$p \in B \subset \mathbb{R}^n \setminus \text{supp } f,$$

i.e. an open ball containing  $p$  and disjoint from  $\text{supp } f$ . On this open ball  $B$  containing  $p$ ,  $\bar{f} \equiv 0$ . Therefore,  $\bar{f}$  is

$$\bar{f}(x) = \begin{cases} f(x) & \text{if } x \in U, \\ 0 & \text{if } x \notin U. \end{cases} \quad (8.17)$$

On  $B \setminus U$ ,  $\bar{f}$  is clearly 0. Since  $B \cap U \cap \text{supp } f = \emptyset$ , and on  $B \cap U$ ,  $f$  and  $\bar{f}$  agree with other, one must have,  $\bar{f} = 0$  on  $B \cap U$ . Hence, on  $B$ ,  $\bar{f} \equiv 0$ . This implies that  $\bar{f}$  is continuous at  $p \in U$ . We, therefore, have,  $\bar{f}$  to be continuous on the whole of  $\mathbb{R}^n$ .

Note that  $\bar{f}$ , defined by (8.17) is also the zero extension of  $f|_{\text{supp } f} : \text{supp } f \rightarrow \mathbb{R}$  with  $\text{supp } f$  being a bounded subset of  $\mathbb{R}^n$ . Now, we are all good to apply [Lebesgue's theorem](#) by which  $f|_{\text{supp } f} : \text{supp } f \rightarrow \mathbb{R}$  is Riemann integrable. Since  $f$  is zero on  $U \setminus \text{supp } f$ ,  $f : U \rightarrow \mathbb{R}$  is also Riemann integrable. ■

**Definition 8.4** (Domain of integration). A subset  $A \subset \mathbb{R}^n$  is called a **domain of integration** if it is bounded and its topological boundary  $\text{bd } A$  is a set of measure zero.

Familiar plane figures, such as triangles, rectangles, disks are all domains of integration in  $\mathbb{R}^2$ .

**Proposition 8.3**

Every bounded continuous function  $f$  defined on a domain of integration  $A$  in  $\mathbb{R}^n$  is Riemann integrable over  $A$ .

*Proof.* Let  $\bar{f} : \mathbb{R}^n \rightarrow \mathbb{R}$  be the extension of  $f$  by zero, i.e.,

$$\bar{f}(x) = \begin{cases} f(x) & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases} \quad (8.18)$$

Since  $f$  is continuous on  $A$  by hypothesis,  $\bar{A}$  is necessarily continuous in the open set  $\text{int}(A)$ . One also observes that if  $p$  is an exterior point of  $A$ , i.e., if  $p \in \mathbb{R}^n \setminus \bar{A}$ , being an open set, there is an open set  $U \subset \mathbb{R}^n$  such that

$$p \in U \subseteq \mathbb{R}^n \setminus \bar{A}.$$

Since  $U \cap A = \emptyset$ ,  $\bar{f} \equiv 0$  on  $U$ . Hence,  $\bar{f}$  is continuous at  $p$ . One, thus, verifies that  $\bar{f}$  is continuous at all interior and exterior points of  $A$ . Therefore, the set  $\text{Disc}(\bar{f})$  of discontinuities of  $\bar{f}$  is a subset of  $\text{bd}(A)$ , a set of measure zero. By Lebergues' theorem,  $f$  is Riemann integrable on  $A$ . ■

## §8.2 Integral of an $n$ -form on $\mathbb{R}^n$

**Definition 8.5.** Let  $\omega = f(\mathbf{x}) \, dx^1 \wedge \cdots \wedge dx^n$  be a  $C^\infty$   $n$ -form on an open subset  $U \subset \mathbb{R}^n$ , with standard coordinates  $x^1, \dots, x^n$ . Its integral over a subset  $A \subset U$  is defined to be the Riemann integral of  $f(\mathbf{x})$

$$\int_A \omega = \int_A f(\mathbf{x}) \, dx^1 \wedge \cdots \wedge dx^n := \int_A f(\mathbf{x}) \, dx^1 \cdots dx^n, \quad (8.19)$$

if the Riemann integral exists.

If  $f$  is a bounded continuous function on a domain of integration  $A$  in  $\mathbb{R}^n$ , then the integral  $\int_A f \, dx^1 \wedge \cdots \wedge dx^n$  exists by [Proposition 8.3](#).