



Inspiring Excellence

## **Differential Geometry II (MAT401)**

**Lecture Notes**

# Preface

This series of lecture notes has been prepared for aiding students who took the BRAC University course **Differential Geometry II (MAT401)** in Fall 2023 semester. These notes were typeset under the supervision of mathematician **Dr. Syed Hasibul Hassan Chowdhury**. The main goal of this typeset is to have an organized digital version of the notes, which is easier to share and handle. If you see any mistakes or typos, please send me an email at [atonuroychowdhury@gmail.com](mailto:atonuroychowdhury@gmail.com)

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## References:

- *An Introduction to Manifolds*, by **Loring W. Tu**
- *An Introduction to Differentiable Manifolds and Riemannian Geometry*, by **William Boothby**
- *Introduction to Smooth Manifolds*, by **John M. Lee**
- *Lectures on Differential geometry*, by **S.S Chern, W.H. Chen and K.S. Lam**
- *Geometry of Differential Forms*, by **Shigeyuki Morita**
- *From Calculus to Cohomology: De Rham Cohomology and Characteristic Classes*, by **Ib Madsen and Jxrgen Tornehave**.

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# 1 Review of Multilinear Algebra

## §1.1 Dual Space

Let  $V$  and  $W$  be real vector spaces. We denote by  $\text{Hom}(V, W)$  the vector space of all linear maps  $f : V \rightarrow W$ . In particular, if we choose  $W = \mathbb{R}$ , we get the **dual space**  $V^*$ .

$$V^* = \text{Hom}(V, \mathbb{R}).$$

The elements of  $V^*$  are called covectors on  $V$ . In the rest of the lecture, we will assume  $V$  to be a finite dimensional vector space. Let  $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$  be a basis for  $V$ . Then every  $\mathbf{v} \in V$  is a unique linear combination

$$\mathbf{v} = \sum_{i=1}^n v^i \mathbf{e}_i, \quad (1.1)$$

with  $v^i \in \mathbb{R}$ .  $v^i$ 's are called the coordinates of  $\mathbf{v}$  relative to the basis  $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ . Let  $\hat{\alpha}^i$  be the linear function on  $V$  that picks up the  $i$ -th coordinate of the vector, i.e.

$$\hat{\alpha}^i(\mathbf{v}) = \hat{\alpha}^i\left(\sum_{i=1}^n v^i \mathbf{e}_i\right) = v^i. \quad (1.2)$$

When  $\mathbf{v}$  is one of the basis vectors,

$$\hat{\alpha}^i(\mathbf{e}_j) = \delta^i_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \quad (1.3)$$

### Proposition 1.1

The functions  $\hat{\alpha}^1, \dots, \hat{\alpha}^n$  form a basis for  $V^*$ .

*Proof.* Suppose  $f \in V^*$ . Then for any  $\mathbf{v} = \sum_{i=1}^n v^i \mathbf{e}_i \in V$ ,

$$f(\mathbf{v}) = f\left(\sum_{i=1}^n v^i \mathbf{e}_i\right) = \sum_{i=1}^n v^i f(\mathbf{e}_i) = \sum_{i=1}^n f(\mathbf{e}_i) \hat{\alpha}^i(\mathbf{v}).$$

Since this holds for any  $\mathbf{v} \in V$ ,

$$f = \sum_{i=1}^n f(\mathbf{e}_i) \hat{\alpha}^i. \quad (1.4)$$

Therefore,  $\hat{\alpha}^1, \dots, \hat{\alpha}^n$  span  $V^*$ . As for linear independence, suppose

$$\sum_{i=1}^n c_i \hat{\alpha}^i = \mathbf{0}, \quad (1.5)$$

where  $\mathbf{0}$  is the function that takes all of  $V$  to 0 in  $\mathbb{R}$ . If we evaluate (1.5) at  $\mathbf{e}_j$ , we get

$$0 = \sum_{i=1}^n c_i \hat{\alpha}^i(\mathbf{e}_j) = \sum_{i=1}^n c_i \delta^i_j = c_j. \quad (1.6)$$

So  $c_j = 0$ , and this holds for each  $j = 1, 2, \dots, n$ . Therefore,  $\{\hat{\alpha}^1, \dots, \hat{\alpha}^n\}$  is a linearly independent set that spans  $V^*$ , i.e. a basis. ■

### Corollary 1.2

The dual space  $V^*$  of a finite dimensional vector space has the same dimension as  $V$ .

The basis  $\{\hat{\alpha}^1, \dots, \hat{\alpha}^n\}$  for  $V^*$  is said to be dual to the basis  $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$  for  $V$ .

## §1.2 Permutations

Fix a positive integer  $k$ . A permutation of the set  $A = \{1, 2, \dots, k\}$  is a bijection  $\sigma : A \rightarrow A$ . The product of two permutations  $\tau$  and  $\sigma$  is the composition  $\tau \circ \sigma : A \rightarrow A$ . The **cyclic permutation**  $(a_1 a_2 \cdots a_r)$  is the permutation  $\sigma$  such that

$$\sigma(a_1) = a_2, \sigma(a_2) = a_3, \dots, \sigma(a_{r-1}) = a_r, \text{ and } \sigma(a_r) = 1,$$

leaving all other elements unchanged, i.e.  $\sigma(j) = j$  if  $j$  is not one of the  $a_i$ 's. A cyclic permutation  $(a_1 a_2 \cdots a_r)$  is also called a **cycle** of length  $r$  or an  $r$ -cycle. A **transposition** is a permutation of the form  $(a b)$  that interchanges  $a$  and  $b$ , leaving all other elements of  $A$  fixed.

A permutation  $\sigma : A \rightarrow A$  can be described by

$$\begin{bmatrix} 1 & 2 & \cdots & k \\ \sigma(1) & \sigma(2) & \cdots & \sigma(k) \end{bmatrix}.$$

We also write it as

$$[1 \ 2 \ \cdots \ k] \xrightarrow{\sigma} [\sigma(1) \ \sigma(2) \ \cdots \ \sigma(k)].$$

**Example 1.1.** Suppose  $\sigma : \{1, 2, 3, 4, 5\} \rightarrow \{1, 2, 3, 4, 5\}$  is the permutation given by

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 5 & 1 & 3 \end{bmatrix}.$$

In other words,  $\sigma(1) = 2$ ,  $\sigma(2) = 4$ ,  $\sigma(3) = 5$ ,  $\sigma(4) = 1$ , and  $\sigma(5) = 3$ .

$$[1 \ 2 \ 3 \ 4 \ 5] \xrightarrow{\sigma} [2 \ 4 \ 5 \ 1 \ 3].$$

Observe that the cyclic permutation  $\sigma' = (1 \ 2 \ 4)$  acts as  $\sigma'(1) = 2$ ,  $\sigma'(2) = 4$  and  $\sigma'(4) = 1$ , keeping 3 and 5 unchanged, i.e.  $\sigma'(3) = 3$  and  $\sigma'(5) = 5$ .

$$[1 \ 2 \ 3 \ 4 \ 5] \xrightarrow{(1 \ 2 \ 4)} [2 \ 4 \ 3 \ 1 \ 5].$$

Now the transposition  $\sigma'' = (3 \ 5)$  acts as  $\sigma''(3) = 5$  and  $\sigma''(5) = 3$ , keeping 1, 2, 4 unchanged. Therefore,

$$\begin{array}{ccccc} [1 \ 2 \ 3 \ 4 \ 5] & \xrightarrow{(1 \ 2 \ 4)} & [2 \ 4 \ 3 \ 1 \ 5] & \xrightarrow{(3 \ 5)} & [2 \ 4 \ 5 \ 1 \ 3] \\ & \searrow & & \nearrow & \\ & & (3 \ 5)(1 \ 2 \ 4) & & \end{array}$$

so that  $\sigma = (3 \ 5)(1 \ 2 \ 4)$ .

Let  $S_k$  be the group of permutations of the set  $\{1, 2, \dots, k\}$ . The order of this group is  $k!$ . A permutation is *even* or *odd* depending on whether it is the product of an even or an odd number of transpositions. The sign of a permutation  $\sigma$  is 1 if the permutation is even, and  $-1$  otherwise. It is denoted by  $\text{sgn } \sigma$ . For example, in [Example 1.1](#),  $\sigma = (3 \ 5)(1 \ 2 \ 4)$ . Note that we can write  $(1 \ 2 \ 4)$  as a product of two transpositions:

$$\begin{array}{ccccc} [1 \ 2 \ 3 \ 4 \ 5] & \xrightarrow{(1 \ 2)} & [2 \ 1 \ 3 \ 4 \ 5] & \xrightarrow{(1 \ 4)} & [2 \ 4 \ 3 \ 1 \ 5] \\ & \searrow & & \nearrow & \\ & & (1 \ 4)(1 \ 2) = (1 \ 2 \ 4) & & \end{array}$$

In other words,  $\sigma = (3 \ 5)(1 \ 4)(1 \ 2)$ . Hence,  $\text{sgn } \sigma = -1$ . One can easily check that

$$\text{sgn } (\sigma\tau) = (\text{sgn } \sigma)(\text{sgn } \tau). \quad (1.7)$$

So  $\text{sgn} : S_k \rightarrow \{1, -1\}$  is a group homomorphism.

**Example 1.2.** Observe that the 5-cycle  $(1\ 2\ 3\ 4\ 5)$  can be written as

$$(1\ 2\ 3\ 4\ 5) = (1\ 5)(1\ 4)(1\ 3)(1\ 2).$$

Indeed,

$$[1\ 2\ 3\ 4\ 5] \xrightarrow{(1\ 2)} [2\ 1\ 3\ 4\ 5] \xrightarrow{(1\ 3)} [2\ 3\ 1\ 4\ 5] \xrightarrow{(1\ 4)} [2\ 3\ 4\ 1\ 5] \xrightarrow{(1\ 5)} [2\ 3\ 4\ 5\ 1]$$

$(1\ 2\ 3\ 4\ 5) = (1\ 5)(1\ 4)(1\ 3)(1\ 2)$

Therefore,  $\text{sgn}(1\ 2\ 3\ 4\ 5) = 1$ .

An **inversion** in a permutation  $\sigma$  is an ordered pair  $(\sigma(i), \sigma(j))$  such that  $i < j$  but  $\sigma(i) > \sigma(j)$ . In [Example 1.1](#),  $\sigma(1) = 2$ ,  $\sigma(2) = 4$ ,  $\sigma(3) = 5$ ,  $\sigma(4) = 1$ , and  $\sigma(5) = 3$ . So, the inversions in this permutation are

$$(\sigma(1), \sigma(4)), (\sigma(2), \sigma(4)), (\sigma(2), \sigma(5)), (\sigma(3), \sigma(4)), (\sigma(3), \sigma(5)).$$

Hence, there are 5 inversions associated with the permutation  $\sigma$ . There is an efficient way of determining the sign of a permutation.

### Proposition 1.3

A permutation is even if and only if it has an even number of inversions.

*Proof.* Let  $\sigma \in S_k$  with  $n$  inversions. We shall prove that we can multiply  $\sigma$  by  $n$  transpositions and get the identity permutation. This will prove that  $\text{sgn } \sigma = (-1)^n$ .

Suppose  $\sigma(j_1) = 1$ . Then for each  $i < j_1$ ,  $(\sigma(i), \sigma(j_1))$  is an inversion, and there are  $j_1 - 1$  many of them. These are all the inversions with 1 in the second slot of the ordered pair of inversion. If we now multiply  $\sigma$  by the  $j_1$ -cycle

$$(\sigma(1)\ 1)(\sigma(2)\ 1) \cdots (\sigma(j_1 - 1)\ 1)$$

to the left of  $\sigma$ , the resulting permutation  $\sigma_1$  would be

$$\begin{bmatrix} 1 & 2 & 3 & \cdots & j_1 & j_1 + 1 & \cdots & k \\ 1 & \sigma(1) & \sigma(2) & \cdots & \sigma(j_1 - 1) & \sigma(j_1 + 1) & \cdots & \sigma(k) \end{bmatrix}$$

This permutation has no inversion with 1 in the second slot of the ordered pair of inversion. Suppose now that  $\sigma(j_2) = 2$ . Now observe that if  $(\sigma_1(i), 2)$  is an inversion in  $\sigma_1$ , then either  $(\sigma(i), 2)$  (if  $i \geq j_1 + 1$ ) or  $\sigma(i - 1), 2$  (if  $i \leq j_1 - 1$ ) is an inversion in  $\sigma$ . Therefore, the number of inversions in  $\sigma_1$  ending in 2 is precisely the same as the number of inversions in  $\sigma$  ending in 2. So following a similar procedure as above, we can multiply  $\sigma_1$  by  $i_2$ -many transpositions to the left ( $i_2$  is the number of transpositions ending in 2) and get

$$\begin{bmatrix} 1 & 2 & 3 & \cdots & k \\ 1 & 2 & \sigma(1) & \cdots & \sigma(k) \end{bmatrix}$$

We can continue these steps for each  $j = 1, 2, \dots, k$ , and the number of transpositions required to move  $j$  to its natural position is the same as the number of inversions ending in  $j$ . In the end we achieve the identity permutation. Therefore,  $\text{sgn } \sigma = (-1)^n$ , where  $n$  is the number of inversions. ■

### §1.3 Multilinear Functions

**Definition 1.1.** Let  $V^k$  be the cartesian product of  $k$ -copies of a real vector space  $V$ .

$$V^k = \underbrace{V \times V \times \cdots \times V}_{k\text{-copies}}$$

A function  $f : V^k \rightarrow \mathbb{R}$  is called  $k$ -linear if it is linear in each of its  $k$  arguments:

$$f(\dots, a\mathbf{v} + b\mathbf{w}, \dots) = a f(\dots, \mathbf{v}, \dots) + b f(\dots, \mathbf{w}, \dots), \quad (1.8)$$

for  $a, b \in \mathbb{R}$  and  $\mathbf{v}, \mathbf{w} \in V$ .

Instead of 2-linear and 3-linear, it's customary to call “bilinear” and “trilinear”, respectively. A  $k$ -linear function on  $V$  is called a  **$k$ -tensor** on  $V$ . We will denote the vector space of all  $k$ -tensors on  $V$  by  $L_k(V)$ . The vector addition and scalar multiplication of the real vector space  $L_k(V)$  is the straightforward pointwise operation.

**Example 1.3.** The dot product  $f(\mathbf{v}, \mathbf{w}) = \mathbf{v} \cdot \mathbf{w}$  on  $\mathbb{R}^n$  is bilinear: if  $\mathbf{v} = \sum_{i=1}^n v^i \mathbf{e}_i$  and  $\mathbf{w} = \sum_{i=1}^n w^i \mathbf{e}_i$ , then

$$f(\mathbf{v}, \mathbf{w}) = \mathbf{v} \cdot \mathbf{w} = \sum_{i=1}^n v^i w^i.$$

**Example 1.4.** The determinant

$$f(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n) = \det \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}$$

viewed as a function of the  $n$  column vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  is  $n$ -linear.

**Definition 1.2** (Symmetric and alternating function). A  $k$ -linear function  $f : V^k \rightarrow \mathbb{R}$  is **symmetric** if

$$f(\mathbf{v}_{\sigma(1)}, \mathbf{v}_{\sigma(2)}, \dots, \mathbf{v}_{\sigma(k)}) = f(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k), \quad (1.9)$$

for all permutations  $\sigma \in S_k$ . It is **alternating** if

$$f(\mathbf{v}_{\sigma(1)}, \mathbf{v}_{\sigma(2)}, \dots, \mathbf{v}_{\sigma(k)}) = (\text{sgn } \sigma) f(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k), \quad (1.10)$$

for all permutations  $\sigma \in S_k$ .

The dot product function on  $\mathbb{R}^n$  in [Theorem 1.3](#) is symmetric, and the determinant function on  $\mathbb{R}^n$  in [Theorem 1.4](#) is alternating.

We are especially interested in the vector space  $A_k(V)$  of all alternating  $k$ -linear functions on a vector space  $V$ , for  $k > 0$ . The elements of  $A_k(V)$  are called alternating  $k$ -tensors (also known as  $k$ -covectors). We define  $A_0(V)$  to be  $\mathbb{R}$ . The elements of  $A_0(V)$  are simply constants, which we call 0-covectors. The elements of  $A_1(V)$  are simply covectors, i.e. the elements of  $V^*$ .

#### Permutation action on $k$ -linear functions

If  $f \in L_k(V)$  and  $\sigma \in S_k$ , define  $\sigma f \in L_k(V)$  as follows:

$$(\sigma f)(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k) = f(\mathbf{v}_{\sigma(1)}, \mathbf{v}_{\sigma(2)}, \dots, \mathbf{v}_{\sigma(k)}). \quad (1.11)$$

Thus,  $f$  is symmetric if and only if  $f = \sigma f$  for all  $\sigma \in S_k$ ; and  $f$  is alternating if and only if  $\sigma f = (\text{sgn } \sigma) f$  for all  $\sigma \in S_k$ . When  $k = 1$ ,  $S_k$  only has the identity permutation. In that case, a 1-linear function or simply linear function on  $V$  is both symmetric and alternating. In particular,

$$A_1(V) = L_1(V) = V^*.$$

**Lemma 1.4**

If  $\sigma, \tau \in S_k$  and  $f \in L_k(V)$ , then  $\tau(\sigma f) = (\tau\sigma)f$ .

*Proof.* For any  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in V$ ,

$$\begin{aligned}
 (\tau(\sigma f))(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k) &= (\sigma f)(\mathbf{v}_{\tau(1)}, \mathbf{v}_{\tau(2)}, \dots, \mathbf{v}_{\tau(k)}) \\
 &= (\sigma f)(\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_k) && [\mathbf{w}_i = \mathbf{v}_{\tau(i)}] \\
 &= f(\mathbf{w}_{\sigma(1)}, \mathbf{w}_{\sigma(2)}, \dots, \mathbf{w}_{\sigma(k)}) \\
 &= f(\mathbf{v}_{\tau(\sigma(1))}, \mathbf{v}_{\tau(\sigma(2))}, \dots, \mathbf{v}_{\tau(\sigma(k))}) \\
 &= ((\tau\sigma)f)(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k).
 \end{aligned}$$

Therefore,  $\tau(\sigma f) = (\tau\sigma)f$ . ■

**Definition 1.3.** If  $G$  is a group and  $X$  is a set, a map

$$\begin{aligned}
 G \times X &\rightarrow X \\
 (g, x) &\mapsto g \cdot x
 \end{aligned}$$

is called a **left action** of  $G$  on  $X$  if

- (i)  $e \cdot x = x$ , where  $e$  is the identity element in  $G$  and  $x$  is any element in  $X$ ; and
- (ii)  $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$ , for all  $g_1, g_2 \in G$  and  $x \in X$ .

Similarly, a **right action** of  $G$  on  $X$  is a map

$$\begin{aligned}
 X \times G &\rightarrow X \\
 (x, g) &\mapsto x \cdot g
 \end{aligned}$$

such that

- (i)  $x \cdot e = x$ , for all  $x \in X$ ; and
- (ii)  $(x \cdot g_1) \cdot g_2 = x \cdot (g_1 g_2)$ , for all  $g_1, g_2 \in G$  and  $x \in X$ .

**Symmetrizing and alternating operators**

Given  $f \in L_k(V)$ , there is a way to make it a symmetric  $k$ -linear function  $\mathcal{S}f$  from it:

$$(\mathcal{S}f)(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k) = \sum_{\sigma \in S_k} f(\mathbf{v}_{\sigma(1)}, \mathbf{v}_{\sigma(2)}, \dots, \mathbf{v}_{\sigma(k)}). \quad (1.12)$$

In other words,

$$\mathcal{S}f = \sum_{\sigma \in S_k} \sigma f. \quad (1.13)$$

Similarly, there is a way to make an alternating  $k$ -linear function from  $f$ :

$$\mathcal{A}f = \sum_{\sigma \in S_k} (\text{sgn } \sigma) \sigma f. \quad (1.14)$$

**Proposition 1.5** (i) The  $k$ -linear function  $\mathcal{S}f$  is symmetric.

(ii) The  $k$ -linear function  $\mathcal{A}f$  is alternating.



*Proof.* (i) Let  $\tau \in S_k$ . Then

$$\tau(\mathcal{S}f) = \tau\left(\sum_{\sigma \in S_k} \sigma f\right). \quad (1.15)$$

The group action of  $S_k$  on  $L_k(V)$  is distributive over the vector space addition. Therefore,

$$\tau(\mathcal{S}f) = \sum_{\sigma \in S_k} \tau(\sigma f) = \sum_{\sigma \in S_k} (\tau\sigma)f. \quad (1.16)$$

As  $\sigma$  varies over  $S_k$ ,  $\tau\sigma$  also varies over  $S_k$ . Therefore,  $\sum_{\sigma \in S_k} (\tau\sigma)f = \mathcal{S}f$ . In other words,

$$\tau(\mathcal{S}f) = \mathcal{S}f, \quad (1.17)$$

i.e.  $\mathcal{S}f$  is symmetric.

(ii) Let  $\tau \in S_k$ . Then

$$\tau(\mathcal{A}f) = \tau\left(\sum_{\sigma \in S_k} (\text{sgn } \sigma) \sigma f\right) = \sum_{\sigma \in S_k} (\text{sgn } \sigma) \tau(\sigma f) = \sum_{\sigma \in S_k} (\text{sgn } \sigma) (\tau\sigma)f. \quad (1.18)$$

Since  $(\text{sgn } \tau)^2 = 1$ ,

$$\begin{aligned} \tau(\mathcal{A}f) &= \sum_{\sigma \in S_k} (\text{sgn } \tau)^2 (\text{sgn } \sigma) (\tau\sigma)f \\ &= (\text{sgn } \tau) \sum_{\sigma \in S_k} (\text{sgn } \tau) (\text{sgn } \sigma) (\tau\sigma)f \\ &= (\text{sgn } \tau) \sum_{\sigma \in S_k} (\text{sgn } (\tau\sigma)) (\tau\sigma)f. \end{aligned} \quad (1.19)$$

As  $\sigma$  varies over  $S_k$ ,  $\tau\sigma$  also varies over  $S_k$ . Therefore,  $\sum_{\sigma \in S_k} (\text{sgn } (\tau\sigma)) (\tau\sigma)f = \mathcal{A}f$ . In other words,

$$\tau(\mathcal{A}f) = \mathcal{A}f, \quad (1.20)$$

i.e.  $\mathcal{A}f$  is alternating. ■

### Lemma 1.6

If  $f \in A_k(V)$ , then  $\mathcal{A}f = (k!)f$ .

*Proof.* Since  $f$  is alternating,

$$\mathcal{A}f = \sum_{\sigma \in S_k} (\text{sgn } \sigma) \sigma f = \sum_{\sigma \in S_k} (\text{sgn } \sigma)^2 f = \sum_{\sigma \in S_k} f = (k!)f, \quad (1.21)$$

because the order of  $S_k$  is  $k!$ . ■

## §1.4 Tensor Product and Wedge Product

**Definition 1.4** (Tensor Product). Let  $f$  be a  $k$ -linear function and  $g$  an  $l$ -linear function on a vector space  $V$ . Their tensor product  $f \otimes g$  is the  $(k+l)$ -linear function defined by

$$(f \otimes g)(\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{v}_{k+1}, \dots, \mathbf{v}_{k+l}) = f(\mathbf{v}_1, \dots, \mathbf{v}_k) g(\mathbf{v}_{k+1}, \dots, \mathbf{v}_{k+l}). \quad (1.22)$$

$(k+l)$ -linearity of  $f \otimes g$  follows from  $k$ -linearity of  $f$  and  $l$ -linearity of  $g$ .

**Lemma 1.7** (Associativity of Tensor Product)

Let  $f \in L_k(V)$ ,  $g \in L_l(V)$  and  $h \in L_m(V)$ . Then

$$(f \otimes g) \otimes h = f \otimes (g \otimes h).$$

*Proof.* For  $\mathbf{v}_1, \dots, \mathbf{v}_{k+l+m}$ ,

$$\begin{aligned} [(f \otimes g) \otimes h](\mathbf{v}_1, \dots, \mathbf{v}_{k+l+m}) &= (f \otimes g)(\mathbf{v}_1, \dots, \mathbf{v}_{k+l}) h(\mathbf{v}_{k+l+1}, \dots, \mathbf{v}_{k+l+m}) \\ &= f(\mathbf{v}_1, \dots, \mathbf{v}_k) g(\mathbf{v}_{k+1}, \dots, \mathbf{v}_{k+l}) h(\mathbf{v}_{k+l+1}, \dots, \mathbf{v}_{k+l+m}). \end{aligned} \quad (1.23)$$

$$\begin{aligned} [f \otimes (g \otimes h)](\mathbf{v}_1, \dots, \mathbf{v}_{k+l+m}) &= f(\mathbf{v}_1, \dots, \mathbf{v}_k) (g \otimes h)(\mathbf{v}_{k+1}, \dots, \mathbf{v}_{k+l+m}) \\ &= f(\mathbf{v}_1, \dots, \mathbf{v}_k) g(\mathbf{v}_{k+1}, \dots, \mathbf{v}_{k+l}) h(\mathbf{v}_{k+l+1}, \dots, \mathbf{v}_{k+l+m}). \end{aligned} \quad (1.24)$$

Therefore,  $(f \otimes g) \otimes h = f \otimes (g \otimes h)$ , i.e. tensor product is associative.  $\blacksquare$

**Example 1.5.** Let  $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$  be the standard basis for  $\mathbb{R}^n$ , and  $\{\hat{\alpha}^1, \dots, \hat{\alpha}^n\}$  its dual basis. The Euclidean inner product on  $\mathbb{R}^n$  is the bilinear function

$$\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$$

defined by

$$\langle \cdot, \cdot \rangle(\mathbf{v}, \mathbf{w}) = \langle \mathbf{v}, \mathbf{w} \rangle = \sum_{i=1}^n v^i w^i,$$

for  $\mathbf{v} = \sum_{i=1}^n v^i \mathbf{e}_i$  and  $\mathbf{w} = \sum_{i=1}^n w^i \mathbf{e}_i$ . We can express  $\langle \cdot, \cdot \rangle$  in terms of tensor product as follows:

$$\langle \cdot, \cdot \rangle(\mathbf{v}, \mathbf{w}) = \sum_{i=1}^n v^i w^i = \sum_{i=1}^n \hat{\alpha}^i(\mathbf{v}) \hat{\alpha}^i(\mathbf{w}) = \sum_{i=1}^n (\hat{\alpha}^i \otimes \hat{\alpha}^i)(\mathbf{v}, \mathbf{w}).$$

Since  $\mathbf{v}, \mathbf{w}$  are arbitrary,

$$\langle \cdot, \cdot \rangle = \sum_{i=1}^n (\hat{\alpha}^i \otimes \hat{\alpha}^i). \quad (1.25)$$

If  $f \in A_k(V)$  and  $g \in A_l(V)$ , then it's not true that  $f \otimes g \in A_{k+l}(V)$ , in general. We need to construct a product that is also alternating.

**Definition 1.5** (Wedge Product). For  $f \in A_k(V)$  and  $g \in A_l(V)$ , the wedge product of  $f$  and  $g$  is defined as follows:

$$f \wedge g = \frac{1}{k!l!} \mathcal{A}(f \otimes g). \quad (1.26)$$

Explicitly,

$$\begin{aligned} (f \wedge g)(\mathbf{v}_1, \dots, \mathbf{v}_{k+l}) &= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (\text{sgn } \sigma) \sigma(f \otimes g)(\mathbf{v}_1, \dots, \mathbf{v}_{k+l}) \\ &= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (\text{sgn } \sigma) (f \otimes g)(\mathbf{v}_{\sigma(1)}, \dots, \mathbf{v}_{\sigma(k+l)}) \\ &= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (\text{sgn } \sigma) f(\mathbf{v}_{\sigma(1)}, \dots, \mathbf{v}_{\sigma(k)}) g(\mathbf{v}_{\sigma(k+1)}, \dots, \mathbf{v}_{\sigma(k+l)}). \end{aligned} \quad (1.27)$$

When  $k = 0$ , the element  $f \in A_0(V)$  is simply a constant  $c \in \mathbb{R}$  as discussed earlier. In this case, the wedge product  $c \wedge g$  is just scalar multiplication as is evident from (1.27).

$$\begin{aligned}
 (c \wedge g)(\mathbf{v}_1, \dots, \mathbf{v}_l) &= \frac{1}{l!} \sum_{\sigma \in S_l} (\text{sgn } \sigma) c g(\mathbf{v}_{\sigma(1)}, \dots, \mathbf{v}_{\sigma(l)}) \\
 &= \frac{1}{l!} \sum_{\sigma \in S_l} (\text{sgn } \sigma) c (\text{sgn } \sigma) g(\mathbf{v}_1, \dots, \mathbf{v}_l) \\
 &= \frac{1}{l!} \sum_{\sigma \in S_l} c g(\mathbf{v}_1, \dots, \mathbf{v}_l) \\
 &= \frac{1}{l!} l! c g(\mathbf{v}_1, \dots, \mathbf{v}_l) \\
 &= c g(\mathbf{v}_1, \dots, \mathbf{v}_l).
 \end{aligned}$$

Thus  $c \wedge g = cg$ , for  $c \in \mathbb{R}$  and  $g \in A_l(V)$ .

**Example 1.6.** For  $f \in A_2(V)$  and  $g \in A_1(V)$ ,

$$\begin{aligned}
 \mathcal{A}(f \otimes g)(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) &= f(\mathbf{v}_1, \mathbf{v}_2)g(\mathbf{v}_3) - f(\mathbf{v}_1, \mathbf{v}_3)g(\mathbf{v}_2) - f(\mathbf{v}_2, \mathbf{v}_1)g(\mathbf{v}_3) \\
 &\quad - f(\mathbf{v}_3, \mathbf{v}_2)g(\mathbf{v}_1) + f(\mathbf{v}_2, \mathbf{v}_3)g(\mathbf{v}_1) + f(\mathbf{v}_3, \mathbf{v}_1)g(\mathbf{v}_2).
 \end{aligned}$$

Among these 6 terms, there are 3 pairs of equal terms due to the alternating nature of  $f$ .

$$\begin{aligned}
 f(\mathbf{v}_1, \mathbf{v}_2)g(\mathbf{v}_3) &= -f(\mathbf{v}_2, \mathbf{v}_1)g(\mathbf{v}_3), \\
 f(\mathbf{v}_3, \mathbf{v}_1)g(\mathbf{v}_2) &= -f(\mathbf{v}_1, \mathbf{v}_3)g(\mathbf{v}_2), \\
 f(\mathbf{v}_2, \mathbf{v}_3)g(\mathbf{v}_1) &= -f(\mathbf{v}_3, \mathbf{v}_2)g(\mathbf{v}_1).
 \end{aligned}$$

Therefore,

$$\mathcal{A}(f \otimes g)(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = 2f(\mathbf{v}_1, \mathbf{v}_2)g(\mathbf{v}_3) + 2f(\mathbf{v}_3, \mathbf{v}_1)g(\mathbf{v}_2) + 2f(\mathbf{v}_2, \mathbf{v}_3)g(\mathbf{v}_1). \quad (1.28)$$

Hence,

$$\begin{aligned}
 (f \wedge g)(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) &= \frac{1}{2!1!} \mathcal{A}(f \otimes g)(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \\
 &= f(\mathbf{v}_1, \mathbf{v}_2)g(\mathbf{v}_3) + f(\mathbf{v}_3, \mathbf{v}_1)g(\mathbf{v}_2) + f(\mathbf{v}_2, \mathbf{v}_3)g(\mathbf{v}_1).
 \end{aligned} \quad (1.29)$$

**Example 1.7** (Wedge product of 2 covectors). If  $f, g \in A_1(V)$ , and  $\mathbf{v}_1, \mathbf{v}_2 \in V$ , then

$$(f \wedge g)(\mathbf{v}_1, \mathbf{v}_2) = \frac{1}{1!1!} \mathcal{A}(f \otimes g)(\mathbf{v}_1, \mathbf{v}_2).$$

$S_2$  has 2 elements: the identity element  $e$  and  $(1\ 2)$ . Therefore,

$$(f \wedge g)(\mathbf{v}_1, \mathbf{v}_2) = \mathcal{A}(f \otimes g)(\mathbf{v}_1, \mathbf{v}_2) = f(\mathbf{v}_1)g(\mathbf{v}_2) - f(\mathbf{v}_2)g(\mathbf{v}_1).$$

### Proposition 1.8 (Anticommutativity of wedge product)

The wedge product is anticommutative: if  $f \in A_k(V)$  and  $g \in A_l(V)$ , then

$$f \wedge g = (-1)^{kl} g \wedge f.$$

*Proof.* Define  $\tau \in S_{k+l}$  to be the following permutation:

$$\begin{bmatrix} 1 & 2 & \cdots & l & l+1 & l+2 & \cdots & l+k \\ k+1 & k+2 & \cdots & k+l & 1 & 2 & \cdots & k \end{bmatrix}.$$

In other words,

$$\tau(i) = \begin{cases} k+i & \text{if } 1 \leq i \leq l, \\ i-l & \text{if } l+1 \leq i \leq l+k. \end{cases}$$

Then for any  $\sigma \in S_{k+l}$ ,

$$\sigma(j) = \begin{cases} \sigma(\tau(l+j)) & \text{if } 1 \leq j \leq k, \\ \sigma(\tau(j-k)) & \text{if } k+1 \leq j \leq k+l. \end{cases} \quad (1.30)$$

Now, for any  $\mathbf{v}_1, \dots, \mathbf{v}_{k+l} \in V$ ,

$$\begin{aligned} \mathcal{A}(f \otimes g)(\mathbf{v}_1, \dots, \mathbf{v}_{k+l}) &= \sum_{\sigma \in S_{k+l}} (\text{sgn } \sigma) f(\mathbf{v}_{\sigma(1)}, \dots, \mathbf{v}_{\sigma(k)}) g(\mathbf{v}_{\sigma(k+1)}, \dots, \mathbf{v}_{\sigma(k+l)}) \\ &= \sum_{\sigma \in S_{k+l}} (\text{sgn } \sigma) f(\mathbf{v}_{\sigma(\tau(l+1))}, \dots, \mathbf{v}_{\sigma(\tau(l+k))}) g(\mathbf{v}_{\sigma(\tau(1))}, \dots, \mathbf{v}_{\sigma(\tau(l))}) \\ &= (\text{sgn } \tau) \sum_{\sigma \in S_{k+l}} (\text{sgn } \sigma) (\text{sgn } \tau) g(\mathbf{v}_{\sigma(\tau(1))}, \dots, \mathbf{v}_{\sigma(\tau(l))}) f(\mathbf{v}_{\sigma(\tau(l+1))}, \dots, \mathbf{v}_{\sigma(\tau(l+k))}) \\ &= (\text{sgn } \tau) \sum_{\sigma \in S_{k+l}} (\text{sgn } \sigma \tau) g(\mathbf{v}_{\sigma(\tau(1))}, \dots, \mathbf{v}_{\sigma(\tau(l))}) f(\mathbf{v}_{\sigma(\tau(l+1))}, \dots, \mathbf{v}_{\sigma(\tau(l+k))}). \end{aligned}$$

Again, as  $\sigma$  varies over  $S_{k+l}$ ,  $\sigma\tau$  also varies over  $S_{k+l}$ . Therefore,

$$\mathcal{A}(f \otimes g)(\mathbf{v}_1, \dots, \mathbf{v}_{k+l}) = (\text{sgn } \tau) \mathcal{A}(g \otimes f)(\mathbf{v}_1, \dots, \mathbf{v}_{k+l}). \quad (1.31)$$

Now, let us evaluate the sign of the permutation  $\tau$ . Let  $(\tau(i), \tau(j))$  be an inversion of  $\tau$ . Then it's not possible that  $1 \leq i < j \leq l$ , or  $l+1 \leq i < j \leq l+k$ ; because if we have  $1 \leq i < j \leq l$  or  $l+1 \leq i < j \leq l+k$ , then  $\tau(i) < \tau(j)$ . Therefore,  $i$  must be in between 1 and  $l$  (inclusive), and  $j$  must be in between  $l+1$  and  $l+k$  (inclusive). So there are  $l$  options for  $i$ , and  $k$  options for  $j$ . Therefore,  $\tau$  has  $kl$  many inversions. So  $\text{sgn } \tau = (-1)^{kl}$ . Using (1.31),

$$\mathcal{A}(f \otimes g) = (-1)^{kl} \mathcal{A}(g \otimes f). \quad (1.32)$$

Dividing by  $k!l!$ , we obtain

$$f \wedge g = (-1)^{kl} g \wedge f. \quad (1.33)$$

■

### Corollary 1.9

If  $f$  is a  $k$ -covector on  $V$ , i.e.  $f \in A_k(V)$ , and  $k$  is odd, then  $f \wedge f = 0$ .

*Proof.* By anticommutativity of wedge product,

$$f \wedge f = (-1)^{k^2} f \wedge f = -f \wedge f.$$

Therefore,  $f \wedge f = 0$ . ■

If  $f$  is a  $k$ -covector and  $g$  is an  $l$ -covector, i.e.  $f \in A_k(V)$  and  $g \in A_l(V)$ , then we have defined their wedge product to be the  $(k+l)$ -covector

$$f \wedge g = \frac{1}{k!l!} \mathcal{A}(f \otimes g). \quad (1.34)$$

We have the following lemmas associated with the alternating operator  $\mathcal{A}$ .

**Lemma 1.10**

Suppose  $f \in L_k(V)$  and  $g \in L_l(V)$ . Then

- (i)  $\mathcal{A}(\mathcal{A}(f) \otimes g) = k! \mathcal{A}(f \otimes g)$ .
- (ii)  $\mathcal{A}(f \otimes \mathcal{A}(g)) = l! \mathcal{A}(f \otimes g)$ .

*Proof.* (i) By definition,

$$\begin{aligned} \mathcal{A}(\mathcal{A}(f) \otimes g) &= \sum_{\sigma \in S_{k+l}} (\text{sgn } \sigma) \sigma(\mathcal{A}(f) \otimes g) \\ &= \sum_{\sigma \in S_{k+l}} (\text{sgn } \sigma) \sigma \left[ \sum_{\tau \in S_k} (\text{sgn } \tau) (\tau f) \otimes g \right]. \end{aligned} \quad (1.35)$$

We can view  $\tau \in S_k$  as a permutation in the following way: define  $\tau' \in S_{k+l}$  as follows

$$\tau'(i) = \begin{cases} \tau(i) & \text{if } i \leq k, \\ i & \text{if } i > k. \end{cases} \quad (1.36)$$

Then for  $\mathbf{v}_1, \dots, \mathbf{v}_{k+l}$ , we have

$$\begin{aligned} [(\tau f) \otimes g](\mathbf{v}_1, \dots, \mathbf{v}_{k+l}) &= (\tau f)(\mathbf{v}_1, \dots, \mathbf{v}_k) g(\mathbf{v}_{k+1}, \dots, \mathbf{v}_{k+l}) \\ &= f(\mathbf{v}_{\tau(1)}, \dots, \mathbf{v}_{\tau(k)}) g(\mathbf{v}_{k+1}, \dots, \mathbf{v}_{k+l}) \\ &= f(\mathbf{v}_{\tau'(1)}, \dots, \mathbf{v}_{\tau'(k)}) g(\mathbf{v}_{\tau'(k+1)}, \dots, \mathbf{v}_{\tau'(k+l)}) \\ &= [\tau'(f \otimes g)](\mathbf{v}_1, \dots, \mathbf{v}_{k+l}). \end{aligned}$$

Therefore,  $(\tau f) \otimes g = \tau'(f \otimes g)$ . Furthermore,  $\text{sgn } \tau = \text{sgn } \tau'$  since the inversions  $(\tau'(i), \tau'(j))$  occur only when  $1 \leq i < j \leq k$ , so that the  $\tau$  and  $\tau'$  has the same number of inversions.

Let us abuse notation a bit and denote by  $S_k$  the subgroup of permutations in  $S_{k+l}$  by keeping the last  $l$  arguments fixed. This subgroup of  $S_{k+l}$  is indeed isomorphic to  $S_k$ , so we will denote both these groups by  $S_k$ . Therefore, from (1.35),

$$\begin{aligned} \mathcal{A}(\mathcal{A}(f) \otimes g) &= \sum_{\sigma \in S_{k+l}} (\text{sgn } \sigma) \sigma \left[ \sum_{\tau' \in S_k \subseteq S_{k+l}} (\text{sgn } \tau') \tau'(f \otimes g) \right] \\ &= \sum_{\sigma \in S_{k+l}} \sum_{\tau' \in S_k \subseteq S_{k+l}} (\text{sgn } \sigma) (\text{sgn } \tau') \sigma \tau'(f \otimes g) \\ &= \sum_{\tau' \in S_k \subseteq S_{k+l}} \sum_{\sigma \in S_{k+l}} (\text{sgn } \sigma \text{sgn } \tau') ((\sigma \tau')(f \otimes g)). \end{aligned}$$

For a fixed  $\tau'$ , as  $\sigma$  varies over  $S_{k+l}$ ,  $\sigma \tau'$  also varies over  $S_{k+l}$ . Therefore,

$$\mathcal{A}(\mathcal{A}(f) \otimes g) = \sum_{\tau' \in S_k \subseteq S_{k+l}} \mathcal{A}(f \otimes g) = k! \mathcal{A}(f \otimes g). \quad (1.37)$$

(ii) By (1.32),

$$\begin{aligned} \mathcal{A}(f \otimes \mathcal{A}(g)) &= \mathcal{A}((-1)^{kl} \mathcal{A}(g) \otimes f) \\ &= (-1)^{kl} \mathcal{A}(\mathcal{A}(g) \otimes f) \\ &= (-1)^{kl} l! \mathcal{A}(g \otimes f) \\ &= l! \mathcal{A}((-1)^{kl} g \otimes f) \\ &= l! \mathcal{A}(f \otimes g). \end{aligned} \quad (1.38)$$

■

**Proposition 1.11** (Associativity of wedge product)

Let  $V$  be a real vector space and  $f, g, h$  be alternating multilinear functions on  $V$  of degree  $k, l, m$ , respectively. Then

$$(f \wedge g) \wedge h = f \wedge (g \wedge h).$$

*Proof.* Using the definition of wedge product,

$$\begin{aligned} (f \wedge g) \wedge h &= \frac{1}{(k+l)!m!} \mathcal{A}[(f \wedge g) \otimes h] \\ &= \frac{1}{(k+l)!m!} \mathcal{A}\left[\frac{1}{k!l!} \mathcal{A}(f \otimes g) \otimes h\right] \\ &= \frac{1}{(k+l)!k!l!m!} \mathcal{A}[\mathcal{A}(f \otimes g) \otimes h] \\ &= \frac{(k+l)!}{(k+l)!k!l!m!} \mathcal{A}[(f \otimes g) \otimes h] \\ &= \frac{1}{k!l!m!} \mathcal{A}[(f \otimes g) \otimes h]. \end{aligned}$$

On the other hand,

$$\begin{aligned} f \wedge (g \wedge h) &= \frac{1}{k!(l+m)!} \mathcal{A}[f \otimes (g \wedge h)] \\ &= \frac{1}{k!(l+m)!} \mathcal{A}\left[f \otimes \left(\frac{1}{l!m!} \mathcal{A}(g \otimes h)\right)\right] \\ &= \frac{1}{k!(l+m)!l!m!} \mathcal{A}[f \otimes \mathcal{A}(g \otimes h)] \\ &= \frac{(l+m)!}{k!(l+m)!l!m!} \mathcal{A}[f \otimes (g \otimes h)] \\ &= \frac{1}{k!l!m!} \mathcal{A}[f \otimes (g \otimes h)]. \end{aligned}$$

Since tensor product is associative (by [Lemma 1.7](#)), we conclude that

$$(f \wedge g) \wedge h = f \wedge (g \wedge h). \quad (1.39)$$

■

By associativity, we can omit the parenthesis and write univocally  $f \wedge g \wedge h$  instead of  $(f \wedge g) \wedge h$  or  $f \wedge (g \wedge h)$ .

**Corollary 1.12**

Under the hypothesis of [Proposition 1.11](#),

$$f \wedge g \wedge h = \frac{1}{k!l!m!} \mathcal{A}[f \otimes g \otimes h]. \quad (1.40)$$

This easily generalizes to an arbitrary number of factors: if  $f_i \in A_{d_i}(V)$  for  $i = 1, 2, \dots, r$ , i.e.  $f_i$  is an alternating  $d_i$ -linear function on  $V$ , then

$$f_1 \wedge \dots \wedge f_r = \frac{1}{d_1! \dots d_r!} \mathcal{A}(f_1 \otimes \dots \otimes f_r). \quad (1.41)$$

**Proposition 1.13**

Let  $\hat{\alpha}^1, \hat{\alpha}^2, \dots, \hat{\alpha}^k$  be linear functions on a real vector space  $V$  (i.e.  $\hat{\alpha}^i : V \rightarrow \mathbb{R}$ ) and  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in V$ . Then

$$\begin{aligned} (\hat{\alpha}^1 \wedge \dots \wedge \hat{\alpha}^k)(\mathbf{v}_1, \dots, \mathbf{v}_k) &= \det [\hat{\alpha}^i(\mathbf{v}_j)] \\ &= \det \begin{bmatrix} \hat{\alpha}^1(\mathbf{v}_1) & \hat{\alpha}^1(\mathbf{v}_2) & \dots & \hat{\alpha}^1(\mathbf{v}_k) \\ \hat{\alpha}^2(\mathbf{v}_1) & \hat{\alpha}^2(\mathbf{v}_2) & \dots & \hat{\alpha}^2(\mathbf{v}_k) \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\alpha}^k(\mathbf{v}_1) & \hat{\alpha}^k(\mathbf{v}_2) & \dots & \hat{\alpha}^k(\mathbf{v}_k) \end{bmatrix}. \end{aligned}$$

*Proof.* By 1.41,

$$(\hat{\alpha}^1 \wedge \dots \wedge \hat{\alpha}^k)(\mathbf{v}_1, \dots, \mathbf{v}_k) = \mathcal{A}(\hat{\alpha}^1 \otimes \dots \otimes \hat{\alpha}^k)(\mathbf{v}_1, \dots, \mathbf{v}_k).$$

By the definition of the action of alternating operator,

$$\mathcal{A}(\hat{\alpha}^1 \otimes \dots \otimes \hat{\alpha}^k)(\mathbf{v}_1, \dots, \mathbf{v}_k) = \sum_{\sigma \in S_k} (\text{sgn } \sigma) \hat{\alpha}^1(\mathbf{v}_{\sigma(1)}) \dots \hat{\alpha}^k(\mathbf{v}_{\sigma(k)}). \quad (1.42)$$

By the definition of determinant of a  $k \times k$  matrix  $A = [a_{ij}]$ ,

$$\det A = \sum_{\sigma \in S_k} (\text{sgn } \sigma) a_{1\sigma(1)} a_{2\sigma(2)} \dots a_{k\sigma(k)}. \quad (1.43)$$

Using (1.43) in (1.42), we get

$$\mathcal{A}(\hat{\alpha}^1 \otimes \dots \otimes \hat{\alpha}^k)(\mathbf{v}_1, \dots, \mathbf{v}_k) = \det [\hat{\alpha}^i(\mathbf{v}_j)]. \quad (1.44)$$

■

**§1.5 A Basis for  $A_k(V)$** 

Let  $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$  be a basis for a real vector space  $V$ , and let  $\{\hat{\alpha}^1, \dots, \hat{\alpha}^n\}$  be the dual basis for  $V^*$ . Introduce the multi-index notation

$$I = (i_1, i_2, \dots, i_k)$$

and write  $\mathbf{e}_I$  for  $(\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \dots, \mathbf{e}_{i_k})$  and  $\hat{\alpha}^I$  for  $\hat{\alpha}^{i_1} \wedge \hat{\alpha}^{i_2} \wedge \dots \wedge \hat{\alpha}^{i_k}$ .

A  $k$ -linear function  $f$  on  $V$  is completely determined by its values on all  $k$ -tuples  $(\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \dots, \mathbf{e}_{i_k})$ . If  $f$  is alternating, then  $f$  is completely determined by its values on all  $k$ -tuples  $(\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \dots, \mathbf{e}_{i_k})$  with

$$1 \leq i_1 < i_2 < \dots < i_k \leq n.$$

In other words, it's sufficient to consider  $\mathbf{e}_I$  with  $I$  in ascending order.

**Lemma 1.14**

Suppose  $I$  and  $J$  are ascending multi-indices of length  $k$ . Then

$$\hat{\alpha}^I(\mathbf{e}_J) = \delta^I_J := \begin{cases} 1 & \text{if } I = J, \\ 0 & \text{if } I \neq J. \end{cases}$$

*Proof.* Suppose  $I = (i_1, \dots, i_k)$  and  $J = (j_1, \dots, j_k)$ . Using (1.42), we get

$$\begin{aligned} \hat{\alpha}^I(\mathbf{e}_J) &= (\hat{\alpha}^{i_1} \wedge \hat{\alpha}^{i_2} \wedge \dots \wedge \hat{\alpha}^{i_k})(\mathbf{e}_{j_1}, \mathbf{e}_{j_2}, \dots, \mathbf{e}_{j_k}) \\ &= \sum_{\sigma \in S_k} (\text{sgn } \sigma) \hat{\alpha}^{i_1}(\mathbf{e}_{j_{\sigma(1)}}) \dots \hat{\alpha}^{i_k}(\mathbf{e}_{j_{\sigma(k)}}) \\ &= \sum_{\sigma \in S_k} (\text{sgn } \sigma) \delta^{i_1}_{j_{\sigma(1)}} \dots \delta^{i_k}_{j_{\sigma(k)}}. \end{aligned} \quad (1.45)$$

The terms in the sum (1.45) contribute  $\text{sgn } \sigma$  if and only if

$$(i_1, i_2, \dots, i_k) = (j_{\sigma(1)}, j_{\sigma(2)}, \dots, j_{\sigma(k)});$$

otherwise they contribute 0 to the sum. Both  $I$  and  $J$  are ascending multi-indices. Permuting the elements of  $J$  no longer gives an ascending multi-index (unless the permutation  $\sigma$  is the identity permutation). Therefore, in (1.45), all the summands corresponding to  $\sigma$  being a non-identity permutation contribute 0.

$$\hat{\alpha}^I(\mathbf{e}_J) = \sum_{\sigma \in S_k} (\text{sgn } \sigma) \delta^{i_1}_{j_{\sigma(1)}} \cdots \delta^{i_k}_{j_{\sigma(k)}} = \delta^{i_1}_{j_1} \cdots \delta^{i_k}_{j_k} = \begin{cases} 1 & \text{if } I = J, \\ 0 & \text{if } I \neq J. \end{cases} \quad (1.46)$$

■

### Proposition 1.15

The alternating  $k$ -linear functions  $\hat{\alpha}^I$ ,  $I = (i_1, \dots, i_k)$ , with  $1 \leq i_1 < \dots < i_k \leq n$  form a basis for the space  $A_k(V)$  of alternating  $k$ -linear functions on  $V$ .

*Proof.* Let us first show linear independence. Suppose

$$\sum_I c_I \hat{\alpha}^I = \mathbf{0}, \quad (1.47)$$

$c_I \in \mathbb{R}$  with  $I$  running over ascending multi-indices of length  $k$ . Applying  $\mathbf{e}_J$  to both sides, we get

$$0 = \sum_I c_I \hat{\alpha}^I(\mathbf{e}_J) = \sum_I c_I \delta^I_J = c_J. \quad (1.48)$$

Therefore,  $\{\hat{\alpha}^I \mid I \text{ is ascending multi-index of length } k\}$  is a linearly independent set. Now let us prove that this set spans  $A_k(V)$ . Let  $f \in A_k(V)$ . We claim that

$$f = \sum_I f(\mathbf{e}_I) \hat{\alpha}^I. \quad (1.49)$$

Let  $g = \sum_I f(\mathbf{e}_I) \hat{\alpha}^I$ . We need to prove that  $f = g$ . By  $k$ -linearity and alternating property, if two  $k$ -covectors agree on all  $\mathbf{e}_J$  where  $J$  is an ascending multi-index, then they are equal. Now,

$$g(\mathbf{e}_J) = \sum_I f(\mathbf{e}_I) \hat{\alpha}^I(\mathbf{e}_J) = \sum_I f(\mathbf{e}_I) \delta^I_J = f(\mathbf{e}_J). \quad (1.50)$$

Therefore,  $f = g = \sum_I f(\mathbf{e}_I) \hat{\alpha}^I$ . ■

### Corollary 1.16

If the vector space  $V$  has dimension  $n$ , then the vector space  $A_k(V)$  of  $k$ -covectors on  $V$  has dimension  $\binom{n}{k}$ .

*Proof.* An ascending multi-index  $I = (i_1, i_2, \dots, i_k)$ ,  $1 \leq i_1 < i_2 < \dots < i_k \leq n$  is obtained by choosing a  $k$ -element subset of  $\{1, 2, \dots, n\}$ . This can be done in  $\binom{n}{k}$  ways. ■

### Corollary 1.17

If  $k > \dim V$ , then  $A_k(V) = 0$ .

*Proof.* If  $k > \dim V = n$ , then in the expression

$$\hat{\alpha}^{i_1} \wedge \hat{\alpha}^{i_2} \wedge \dots \wedge \hat{\alpha}^{i_k}$$

with each  $i \in \{1, 2, \dots, n\}$ , there must be a repeated  $i_j$ 's, say  $\hat{\alpha}^r$ . Then  $\hat{\alpha}^r \wedge \hat{\alpha}^r$  arises in the expression  $\hat{\alpha}^{i_1} \wedge \hat{\alpha}^{i_2} \wedge \dots \wedge \hat{\alpha}^{i_k}$ . But  $\hat{\alpha}^r \wedge \hat{\alpha}^r = 0$  by Corollary 1.9. Hence,  $\hat{\alpha}^{i_1} \wedge \hat{\alpha}^{i_2} \wedge \dots \wedge \hat{\alpha}^{i_k} = 0$ . Therefore, the basis set of  $A_k(V)$  is empty, meaning  $A_k(V) = 0$ . ■