

# Leibniz-like Rule in Lie Groups

ATONU ROY CHOWDHURY

October 24, 2025

## §1 Background

Let  $M$  be a manifold, and  $p \in M$ . Then we have the notion of the tangent space  $T_p M$  at the point  $p$ . This is the space of all point-derivations at  $p$ , i.e. all linear maps  $X_p : C_p^\infty(M) \rightarrow \mathbb{R}$  such that

$$X_p(fg) = (X_p f)g(p) + f(p)(X_p g). \quad (1)$$

Here  $C_p^\infty(M)$  is the germs of  $C^\infty$  functions  $p \in M$ , i.e. the set of equivalence classes of smooth functions that agree on some neighborhood of  $p$ . (1) can be thought of as the analogue of the Leibniz (product) rule for single-variable derivatives:

$$(fg)' = f'g + fg'. \quad (2)$$

Locally speaking, on a coordinate chart  $(U, x^1, \dots, x^n)$  around  $p$ , the tangent vector  $X_p$  is a linear combination of the partial derivatives

$$\left. \frac{\partial}{\partial x^1} \right|_p, \left. \frac{\partial}{\partial x^2} \right|_p, \dots, \left. \frac{\partial}{\partial x^n} \right|_p.$$

Now, given a smooth map  $F : N \rightarrow M$  and  $p \in N$ , one can define the differential of  $F$ ,

$$\begin{aligned} (dF)_p : T_p N &\rightarrow T_{F(p)} M \\ [(dF)_p X_p] f &= X_p(f \circ F), \end{aligned} \quad (3)$$

for  $f \in C_{F(p)}^\infty(M)$ . Locally,  $(dF)_p$  is the Jacobian matrix

$$\left[ \frac{\partial F^i}{\partial x^j}(p) \right]_{1 \leq i \leq m, 1 \leq j \leq n}.$$

The alternate notation for  $(dF)_p$  is  $f_{*,p}$ , which is also used in some literatures. But we will stick to the notation  $(dF)_p$ , since it manifests the fact that it's the differential of the smooth map  $F$ .

$(dF)_p$  takes us from a smooth map of  $F : N \rightarrow M$  manifolds to a linear map  $(dF)_p : T_p N \rightarrow T_{F(p)} M$  between vector spaces. So we can expect that it defines a functor from the category of

pointed manifolds to the category of vector spaces, given by

$$\begin{aligned} (M, p) &\mapsto T_p M \\ F : (N, p) \rightarrow (M, q) &\mapsto (dF)_p : T_p N \rightarrow T_{F(p)} M = T_q M. \end{aligned} \quad (4)$$

This, indeed, defines a functor since the differential has the following functorial property:

### Theorem 1.1

Let  $F : N \rightarrow M$  and  $G : M \rightarrow P$  be smooth maps of manifolds, and  $p \in N$ . Then,

$$d(G \circ F)_p = (dG)_{F(p)} \circ (dF)_p. \quad (5)$$

In other words, the following diagram commutes:

$$\begin{array}{ccccc} T_p N & \xrightarrow{(dF)_p} & T_{F(p)} M & \xrightarrow{(dG)_{F(p)}} & T_{G(F(p))} P \\ & \searrow & & \nearrow & \\ & & d(G \circ F)_p = (dG)_{F(p)} \circ (dF)_p & & \end{array}$$

Furthermore, if  $\mathbb{1}_M : M \rightarrow M$  is the identity map, then

$$(d\mathbb{1}_M)_p = \mathbb{1}_{T_p M} : T_p M \rightarrow T_p M. \quad (6)$$

We won't be proving it here, but the proof can be found in any elementary differential geometry textbook/course notes, for instance [here](#).

## §2 Lie group setting

Now suppose the codomain of the smooth map is a Lie group  $G$ . Now, using the smooth multiplication in  $G$ , we can take the product of two smooth maps into  $G$ . In other words, given smooth maps  $F_1, F_2 : N \rightarrow G$ , we can consider the map

$$\begin{aligned} F_1 F_2 : N &\rightarrow G \\ p &\mapsto F_1(p) F_2(p). \end{aligned} \quad (7)$$

But then, naively, you may be prompted to write the differential  $d(F_1 F_2)_p$  like this:

$$d(F_1 F_2)_p \stackrel{?}{=} (dF_1)_p F_2(p) + F_1(p) (dF_2)_p. \quad (8)$$

But if you look at it carefully, (8) makes no sense.  $(dF_1)_p$  is a linear map  $T_p N \rightarrow T_{F_1(p)} G$ , and  $F_2(p)$  is a Lie group element. How are we actually multiplying/composing them? Is there some sort of group action going on here? The same can be asked for  $F_1(p)$  and  $(dF_2)_p$  as well. But although (8) looks complete nonsense, there's a way to make sense of it. We can define the

left/right actions in such a way that (8) actually makes sense. Let

$$\begin{aligned} m : G \times G &\rightarrow G \\ (g_1, g_2) &\mapsto g_1 g_2 \end{aligned} \tag{9}$$

be the multiplication map on  $G$ , since  $G$  is a Lie group. Let  $L_g$  and  $R_g$  be the left and the multiplication by  $g$ , respectively. In other words,

$$\begin{aligned} L_g = m(g, -) : G &\rightarrow G \\ h &\mapsto gh; \end{aligned} \tag{10}$$

$$\begin{aligned} R_g = m(-, g) : G &\rightarrow G \\ h &\mapsto hg. \end{aligned} \tag{11}$$

The goal of this note is to prove the following:

### Theorem 2.1

If  $G$  is a Lie group and  $F_1, F_2 : N \rightarrow M$  are smooth maps, then

$$d(F_1 F_2)_p = (dR_{F_2(p)})_{F_1(p)} \circ (dF_1)_p + (dL_{F_1(p)})_{F_2(p)} \circ (dF_2)_p. \tag{12}$$

Note that the compositions make sense, because

$$T_p N \xrightarrow{(dF_1)_p} T_{F_1(p)} G \xrightarrow{(dR_{F_2(p)})_{F_1(p)}} T_{F_1(p)F_2(p)} G \tag{13}$$

$$T_p N \xrightarrow{(dF_2)_p} T_{F_2(p)} G \xrightarrow{(dL_{F_1(p)})_{F_2(p)}} T_{F_1(p)F_2(p)} G \tag{14}$$

Furthermore, the addition is also understood since we are adding two linear maps  $T_p N \rightarrow T_{F_1(p)F_2(p)} G$ .

## §3 Proof

Throughout this section, we shall write  $g = F_1(p)$  and  $h = F_2(p)$ .

### Lemma 3.1

$$T_{(g,h)}(G \times G) \cong T_g G \times T_h G.$$

*Proof.* Consider the maps

$$G \xleftarrow{\pi_1} G \times G \xrightarrow{\pi_2} G \tag{15}$$

where  $\pi_1(g_1, g_2) = g_1$  and  $\pi_2(g_1, g_2) = g_2$ . These are smooth maps. So we consider

$$T_g G \xleftarrow{(d\pi_1)_{(g,h)}} T_{(g,h)}(G \times G) \xrightarrow{(d\pi_2)_{(g,h)}} T_h G \quad (16)$$

Now we define

$$\begin{aligned} \Psi : T_{(g,h)}(G \times G) &\rightarrow T_g G \times T_h G \\ Z_{(g,h)} &\mapsto ((d\pi_1)_{(g,h)} Z_{(g,h)}, (d\pi_2)_{(g,h)} Z_{(g,h)}). \end{aligned} \quad (17)$$

In the other direction, we consider

$$G \xleftarrow{i_1} G \times G \xleftarrow{i_2} G \quad (18)$$

where  $i_1(g_1) = (g_1, h)$  and  $i_2(g_1) = (g, g_1)$ . These are clearly smooth maps, since they are inclusions. Now,

$$T_g G \xrightarrow{(di_1)_g} T_{(g,h)}(G \times G) \xleftarrow{(di_2)_h} T_h G \quad (19)$$

Then we define

$$\begin{aligned} \Phi : T_g G \times T_h G &\rightarrow T_{(g,h)}(G \times G) \\ (X_g, Y_h) &\mapsto (di_1)_g X_g + (di_2)_h Y_h. \end{aligned} \quad (20)$$

Then one can check that both  $\Psi \circ \Phi$  and  $\Phi \circ \Psi$  are identity maps. The first one follows from the fact that  $\pi_1 \circ i_1 = \pi_2 \circ i_2 = \text{id}_G$  so their differentials are also identity maps. The second one follows using a computation in local coordinates.  $\blacksquare$

### Proposition 3.2

Consider the smooth map  $F = (F_1, F_2) : N \rightarrow G \times G$  defined by  $p \mapsto (F_1(p), F_2(p))$ . Then

$$(dF)_p = \left( (dF_1)_p, (dF_2)_p \right), \quad (21)$$

under the identification  $T_{(g,h)}(G \times G) \cong T_g G \times T_h G$ . More concretely,

$$\Psi \circ (dF)_p = \left( (dF_1)_p, (dF_2)_p \right). \quad (22)$$

*Proof.* Consider this commutative diagram in the category of smooth manifolds and smooth maps:

$$\begin{array}{ccccc} & & N & & \\ & \swarrow F_1 & \downarrow F & \searrow F_2 & \\ G & \xleftarrow{\pi_1} & G \times G & \xrightarrow{\pi_2} & G \end{array} \quad (23)$$

Then we have the following commutative diagram in the category of real vector spaces, by the functorial property of differential (Theorem 1.1):

$$\begin{array}{ccccc}
 & & T_p N & & \\
 & \swarrow (dF_1)_p & \downarrow (dF)_p & \searrow (dF_2)_p & \\
 T_g G & \xleftarrow{(d\pi_1)_{(g,h)}} & T_{(g,h)}(G \times G) & \xrightarrow{(d\pi_2)_{(g,h)}} & T_h G
 \end{array} \tag{24}$$

Now, given any  $X_p \in T_p N$ ,

$$\begin{aligned}
 [\Psi \circ (dF)_p] X_p &= ((d\pi_1)_{(g,h)}(dF)_p X_p, (d\pi_2)_{(g,h)}(dF)_p X_p) \\
 &= ((dF_1)_p X_p, (dF_2)_p X_p),
 \end{aligned} \tag{25}$$

by the commutativity of (24). Therefore,  $\Psi \circ (dF)_p = ((dF_1)_p, (dF_2)_p)$ . ■

### Proposition 3.3

For any  $X_g \in T_g G$  and  $Y_h \in T_h G$ ,

$$(dm)_{(g,h)} \circ \Phi(X_g, Y_h) = (dR_h)_g X_g + (dL_g)_h Y_h. \tag{26}$$

*Proof.* Consider this commutative diagram in the category of smooth manifolds and smooth maps:

$$\begin{array}{ccccc}
 G & \xrightarrow{i_1} & G \times G & \xleftarrow{i_2} & G \\
 & \searrow R_h = m \circ i_1 & \downarrow m & \swarrow L_g = m \circ i_2 & \\
 & & G & & 
 \end{array} \tag{27}$$

Then we have the following commutative diagram in the category of real vector spaces, by the functorial property of differential (Theorem 1.1):

$$\begin{array}{ccccc}
 T_g G & \xrightarrow{(di_1)_g} & T_{(g,h)}(G \times G) & \xleftarrow{(di_2)_h} & T_h G \\
 & \searrow (dR_h)_g & \downarrow (dm)_{(g,h)} & \swarrow (dL_g)_h & \\
 & & T_{gh} G & & 
 \end{array} \tag{28}$$

Now,

$$\begin{aligned}
(dm)_{(g,h)} \circ \Phi(X_g, Y_h) &= (dm)_{(g,h)} [(di_1)_g X_g + (di_2)_h Y_h] \\
&= (dm)_{(g,h)} (di_1)_g X_g + (dm)_{(g,h)} (di_2)_h Y_h \\
&= (dR_h)_g X_g + (dL_g)_h Y_h,
\end{aligned} \tag{29}$$

by the commutativity of (28). Therefore,  $(dm)_{(g,h)} \circ \Phi(X_g, Y_h) = (dR_h)_g X_g + (dL_g)_h Y_h$ .  $\blacksquare$

Now we are ready to finally conclude the proof of Theorem 2.1.

*Proof of Theorem 2.1.* Note that

$$F_1 F_2 = m \circ (F_1, F_2) = m \circ F. \tag{30}$$

Therefore, by Theorem 1.1,

$$d(F_1 F_2)_p = (dm)_{gh} \circ (dF)_p. \tag{31}$$

$$\begin{array}{ccccc}
& & T_{(g,h)}(G \times G) & & \\
& \nearrow (dF)_p & \uparrow \Psi & \nwarrow (dm)_{(g,h)} & \\
T_p N & & & & T_{gh} G \\
& \downarrow \Phi & & & \\
& & T_g G \times T_h G & & 
\end{array} \tag{32}$$

Now, in light of (32), since  $\Phi \circ \Psi$  is identity, we can conclude that

$$\begin{aligned}
d(F_1 F_2)_p &= (dm)_{gh} \circ (dF)_p = (dm)_{gh} \circ (\Phi \circ \Psi) \circ (dF)_p \\
&= (dm)_{gh} \circ \Phi \circ \left( (dF_1)_p, (dF_2)_p \right)
\end{aligned} \tag{33}$$

$$= (dR_h)_g (dF_1)_p + (dL_g)_h (dF_2)_p. \tag{34}$$

Here (33) follows from Proposition 3.2 and (34) follows from Proposition 3.3. Hence, (12) is verified!  $\blacksquare$