On the Category Theoretic Implications on Induced Representation of Finite Groups

Part 1: Categories

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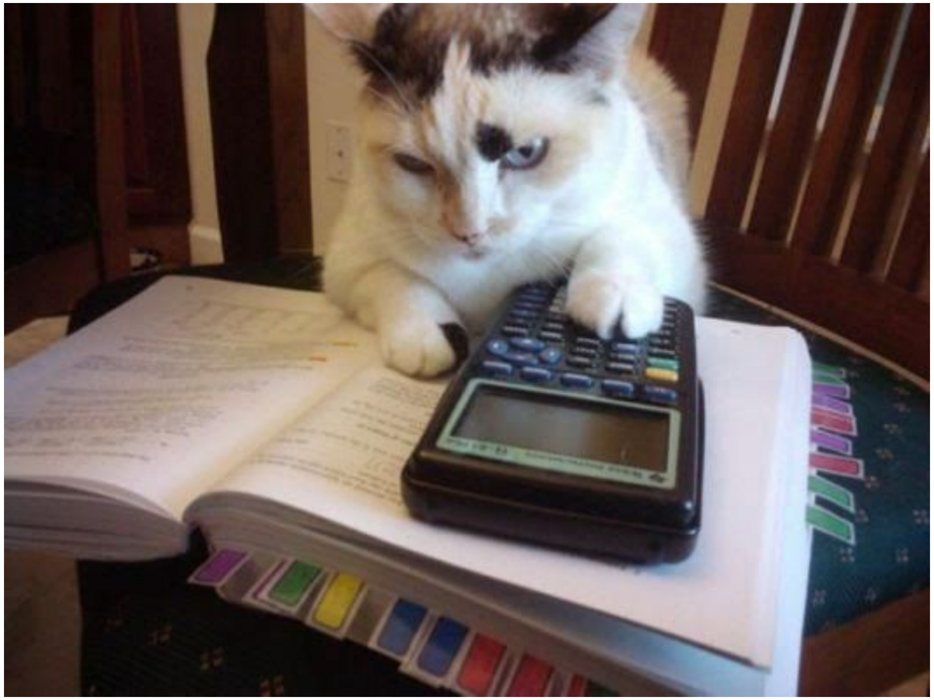

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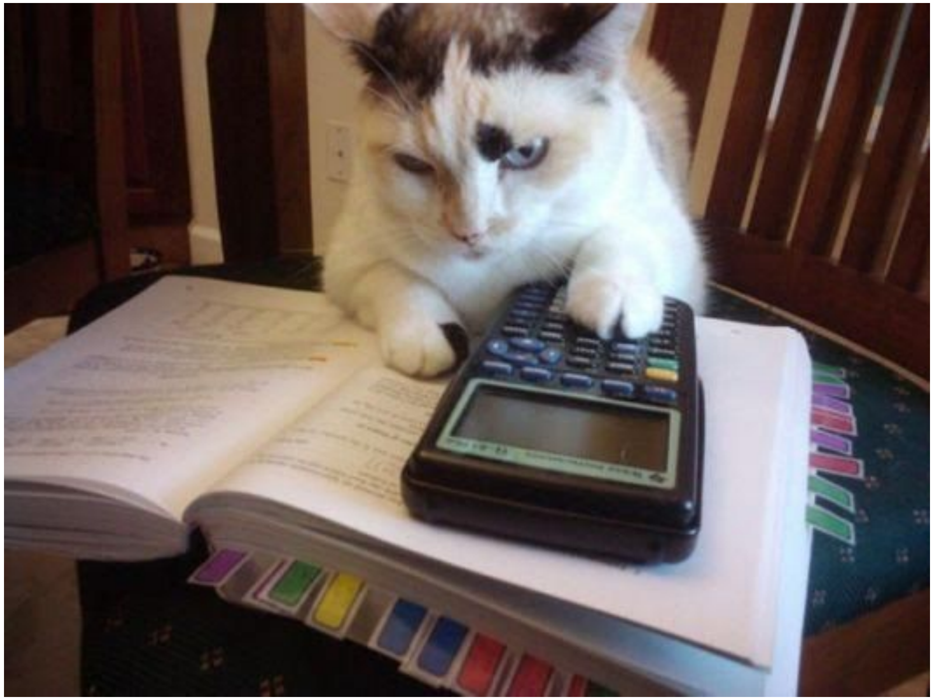

Sadly, no! :(

What do we do in math?

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In Linear Algebra

we study vector spaces and linear transformations between them

$$
\begin{tikzcd}
U \arrow[d, "g"'] \arrow[dr, "h"'] \arrow[dr, shift left=2, "j"] & \\
V \arrow[r, "f"'] & W \arrow[loop below, "i"]
\end{tikzcd}
$$

In Group Theory

we study groups and group homomorphisms between them

$$\begin{tikzcd} K \arrow[d, "g"'] \arrow[dr, "h"'] \arrow[dr, shift left=2, "j"] & \\ G \arrow[r, "f"'] & H \arrow[loop below, "i"] \end{tikzcd}$$

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In Topology

we study topological spaces and conotinuous between them

$$
\begin{tikzcd}
Z \arrow[d, "g"'] \arrow[dr, shift right, "h"'] \arrow[dr, shift left, "j"] & \\
X \arrow[r, "f"'] & Y \arrow[loop below, "i"]
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- We can compose them:
    - Composition of group homomorphisms is a group homomorphism:

$$\begin{tikzcd} G_1 \arrow[r, "f_1"] \arrow[rr, bend right, "f_2 \circ f_1"'] & G_2 \arrow[r, "f_2"] & G_3 \end{tikzcd}$$

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    - Composition of linear maps is a linear map:

$$\begin{tikzcd} U \arrow[r, "f_1"] \arrow[rr, bend right, "f_2 \circ f_1"'] & V \arrow[r, "f_2"] & W \end{tikzcd}$$

    - and so on . . .

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- There is an identity of the composition!!
  - Every group has an identity group homomorphism: id_G

$$\mathbb{1}_G : G \to G$$

$$id_G(g) = g$$

$$\begin{tikzcd} G \arrow[loop left, "\mathrm{id}_G"] \arrow[r, "f"] & H \arrow[loop right, "\mathrm{id}_H"] \end{tikzcd}$$

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  - Every vector space has an identity transformation: $\mathbb{1}_V$

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  - and so on ...

$$
\begin{tikzcd}
G \arrow[loop below] \arrow[r] & H \arrow[loop above] \arrow[r] & K
\end{tikzcd}
$$

Now we strip down all the details, and we have Category Theory!

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> *Category theory is the bird's eye view of mathematics.*
>
> — *Tom Leinster*

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A **category** $\mathcal{C}$ consists of

$A \xrightarrow{f} B$

- a collection of **objects**, often denoted as $\mathcal{C}_0$;
- a collection **arrows** from one object to another, often denoted as $\mathcal{C}_1$; if f is an arrow from A to B, we write $f : A \to B$ and call $A = \operatorname{dom} f$ and $B = \operatorname{cod} f$; $\boxed{\operatorname{Hom}_{\mathcal{C}}(A, B)}$ is the collection of all arrows from the object A to the object B.

$$\boxed{\mathcal{C}_1 = \bigcup_{A,B \in \mathcal{C}_0} \operatorname{Hom}_{\mathcal{C}}(A, B)}$$

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- given any two arrows f, g with $\operatorname{cod} f = \operatorname{dom} g$, there is a composition $g \circ f : \operatorname{dom} f \to \operatorname{cod} g$;

$$\left.\begin{array}{l} f : A \to B \\ g : B \to C \end{array}\right\} g \circ f : A \to C$$

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These data have the following properties:

- composition is associative: given $f : A \to B, g : B \to C, h : C \to D$, $\boxed{h \circ (g \circ f)} = \cancel{(h \circ g)} \circ f$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & B \arrow[r, "g"] \arrow[rr, bend left] & C \arrow[r, "h"] & D \end{tikzcd}$$

Definition

$\mathbb{Z} \to \mathbb{Z}$ $f(n)=n$
$f(n)=2n$

Definition 1

$f(n)=pn$

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- composition is associative: given $f : A \to B, g : B \to C, h : C \to D$, $h \circ (g \circ f) = (h \circ g) \circ f$.
- identity arrow is the identity of composition: for any $f : A \to B$, $\mathbb{1}_B \circ f = f = f \circ \mathbb{1}_A$.

$$\begin{tikzcd} A \arrow[loop left, "\mathbb{1}_A"] \arrow[r, "f"] & B \arrow[loop right, "\mathbb{1}_B"] \end{tikzcd}$$

$f \circ \mathbb{1}_A = \mathbb{1}_B \circ f = f$

Caution!

Objects need not be sets, and arrows need not be functions!

Examples of Categories

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2. $\boxed{\textbf{Vect}_{\boxed{\mathbb{C}}}}$ is the category of all vector spaces over the field $\mathbb{C}$. $(\textbf{Vect}_{\mathbb{C}})_0$ is the collection of all vector spaces over $\mathbb{C}$, $(\textbf{Vect}_{\mathbb{C}})_1$ is the collection of all linear maps between them.

$\mathrm{Vect}_{\mathbb{K}}$

$V \longrightarrow W$

$\mathbb{K}_1$ $\quad$ $\mathbb{K}_2$

$f(a\bar{v}+\bar{w}) = a f(\bar{v}) + f(\bar{w})$

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3. **Sets** is the category of all sets. $\mathbf{Sets}_0$ is the collection of all sets, $\mathbf{Sets}_1$ is the collection of all functions between sets.

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4. and so on ...

Top

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$G = \{e, g_1, g_2, \ldots\}$

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$\mathcal{C}(G)_0 = \{G\}$, $\mathcal{C}(G)_1 = G$. In other words, there is only one object, G itself. The arrows are the elements of G.

$\mathrm{Hom}_{\mathcal{C}(G)}(G, G) = G$

$= e, g_1$

$g_1 \circ g_2 = g_1 g_2$

$G \xrightarrow{g_1} G$

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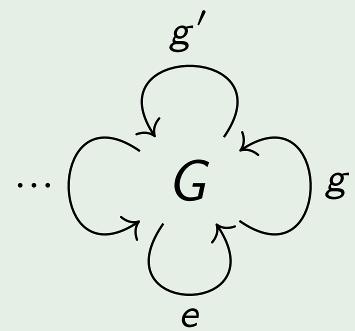

$\boxed{\begin{array}{l} g : G \to G \\ g(g') = gg' \end{array}}$

$Hom_{\mathcal{C}}(A, A)$

Examples of Categories

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$\mathcal{C}(G)_0 = \{G\}$, $\mathcal{C}(G) = G$. In other words, there is only one object, G itself. The arrows are the elements of G.

$$\begin{tikzcd}
\cdots & G \arrow[loop above, "g'"] \arrow[loop right, "g"] \arrow[loop below, "e"] \arrow[loop left]
\end{tikzcd}$$

The identity arrow $\mathbb{1}_G$ is the identity element e of G. The composition of arrows is given by the group operation: $g_1 \circ g_2 = g_1 g_2$.

Category of Categories

So, a group is a category. Then **Groups** is a category of some categories.

Is there a category of all categories? What are the arrows in that category?

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Suppose $\mathcal{C}$ and $\mathcal{D}$ are two categories. Then an arrow $F : \mathcal{C} \to \mathcal{D}$ is supposed to preserve the "structure" of the categories.

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The only structure on the catrgories we have are composition of arrows and the identity arrow.

Definition 2

A **functor** $F : \mathcal{C} \to \mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$ is two mappings $F_0 : \mathcal{C}_0 \to \mathcal{D}_0$, and $F_1 : \mathcal{C}_1 \to \mathcal{D}_1$ such that

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- if $f : A \to B$ is an arrow in $\mathcal{C}$, then $F_1(f) : F_0\,(A) \to F_0\,(B)$ in $\mathcal{D}$, i.e. F preserves domains and codomains;

$\mathcal{C}$ $\qquad\qquad$ $\mathcal{D}$

$A \xrightarrow{f} B$ $\qquad\qquad$ $F_0(A) \xrightarrow{F_1(f)} F_0(B)$

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- for every $A \in \mathcal{C}_0$, $F_1(\mathbb{1}_A) = \mathbb{1}_{F_0(A)}$;

$\mathcal{C}$

$A \xrightarrow{\mathbb{1}_A} A$

$\mathcal{D}$

$$\begin{tikzcd} F_0(A) \arrow[r, "F_1(\mathbb{1}_A)"] \arrow[r, bend right, "\mathbb{1}_{F_0(A)}"'] & F_0(A) \end{tikzcd}$$

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- given two composable arrows $f : A \to B$ and $g : B \to C$ in $\mathcal{C}$,

f

$$F_1(g \circ f) = F_1(g) \circ F_1(f).$$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & B \arrow[r, "g"] & C \end{tikzcd}
\qquad
\begin{tikzcd} F_0(A) \arrow[r, "F_1(f)"] \arrow[rr, bend left, "F_1(g)\circ F_1(f)"] \arrow[rr, bend right, "F_1(g \circ f)"'] & F_0(B) \arrow[r, "F_1(g)"] & F_0(C) \end{tikzcd}$$

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- given two composable arrows $f : A \to B$ and $g : B \to C$ in $\mathcal{C}$,

$$F_1(g \circ f) = F_1(g) \circ F_1(f).$$

We shall often abuse the notation by writing $F(f) : F(A) \to F(B)$.

$F_1(f) : F_0(A) \to F_0(B)$

$$\begin{tikzcd} \mathcal{C} \arrow[dd, "F"'] & A \arrow[r, "f"] \arrow[dr, "g\circ f"'] & B \arrow[d, "g"] \\ & & C \\ \mathcal{D} & F_0(A) \arrow[r, "F_1(f)"] \arrow[dr] & F_0(B) \arrow[d, "F_1(g)"] \\ & & F_0(C) \end{tikzcd}$$

$F_1(g\circ f) = F_1(g)\circ F_1(f)$

$\mathcal{C}$

$\mathcal{D}$

Category of Categories

In order to make a category of categories, we need identity functors $\mathbb{1}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$. This is easy, we can just map everything to itself.

$$\mathbb{1}_{\mathcal{C}}(A) = A$$

$$\mathbb{1}_{\mathcal{C}}(f) = f$$

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What about composition?

$$\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}$$

Given $A \in \mathcal{C}_0$, we can just define $(G \circ F)_0 (A) = G_0 (F_0 (A))$. For an arrow $f : A \to B$ in $\mathcal{C}$, we can define $(G \circ F)_1 (f) = G_1 (F_1 (f))$.

$$\mathcal{C} \xrightarrow{G \circ F} \mathcal{E}$$

$$A \longmapsto G(F(A))$$

$$f \longmapsto G(F(f))$$

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Then we have the category of all categories, **Cat**.

Making New categories From Old Ones

Example 3

The opposite category $\mathcal{C}^{\mathrm{op}}$ has the same objects as $\mathcal{C}$, but the arrows are reversed.

$f : A \to B$

$\mathcal{C}$

$f^{op} : B \to A$

$\mathcal{C}^{op}$

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$$\mathcal{C}$$

$$\begin{tikzcd}
C \arrow[loop left, "\mathbb{1}_C"] \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & D \arrow[r, "g"] & E
\end{tikzcd}$$

$$\mathcal{C}^{\mathrm{op}}$$

$$\begin{tikzcd}
C \arrow[loop above, "(\mathbb{1}_C)^{\mathrm{op}}"] & D \arrow[l, "f^{\mathrm{op}}"'] & E \arrow[l, "g^{\mathrm{op}}"'] \arrow[ll, bend left, "f^{\mathrm{op}}\circ g^{\mathrm{op}}=(g\circ f)^{\mathrm{op}}"]
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$$\mathcal{C}^{\mathrm{op}} \qquad \begin{tikzcd} C \arrow[loop above, "(\mathbb{1}_C)^{\mathrm{op}}"] & D \arrow[l, "f^{\mathrm{op}}"'] & E \arrow[l, "g^{\mathrm{op}}"'] \arrow[ll, bend left, "f^{\mathrm{op}} \circ g^{\mathrm{op}} = (g \circ f)^{\mathrm{op}}"] \end{tikzcd}$$

Often we use this interpretation that an arrow $f^{op} : X \to Y$ in $\mathcal{C}^{\mathrm{op}}$ is really an arrow $f : Y \to X$ in $\mathcal{C}$. This is an abuse of notation since we are dropping the superscript op from the arrows in $\mathcal{C}^{\mathrm{op}}$.

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The product of 2 categories $\mathcal{C}$ and $\mathcal{D}$, written as $\mathcal{C} \times \mathcal{D}$.

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The product of 2 categories $\mathcal{C}$ and $\mathcal{D}$, written as $\mathcal{C} \times \mathcal{D}$.

- the objects are (C, D) for $C \in \mathcal{C}_0$ and $D \in \mathcal{D}_0$,
- and the arrows are

$$(f, g) : (C, D) \to (C', D'),$$

where $f : C \to C'$ and $g : D \to D'$ are arrows in $\mathcal{C}$ and $\mathcal{D}$, respectively.

Example 4

The product of 2 categories $\mathcal{C}$ and $\mathcal{D}$, written as $\mathcal{C} \times \mathcal{D}$.

- the objects are (C, D) for $C \in \mathcal{C}_0$ and $D \in \mathcal{D}_0$,
- and the arrows are
$$(f, g) : (C, D) \to (C', D'),$$
where $f : C \to C'$ and $g : D \to D'$ are arrows in $\mathcal{C}$ and $\mathcal{D}$, respectively.

$\mathcal{C}$

$$\begin{tikzcd} C \arrow[loop left, "\mathbb{1}_C"] \arrow[r, "f"] \arrow[rr, bend right, "f' \circ f"'] & C' \arrow[r, "f'"] & C'' \end{tikzcd}$$

$$\begin{tikzcd} D \arrow[loop left, "\mathbb{1}_D"] \arrow[r, "g"] \arrow[rr, bend right, "g' \circ g"'] & D' \arrow[r, "g'"] & D'' \end{tikzcd}$$

$\mathcal{D}$

$\mathcal{C} \times \mathcal{D}$

$$\begin{tikzcd} (C, D) \arrow[loop left, "\mathbb{1}_{(C,D)} = (\mathbb{1}_C, \mathbb{1}_D)"] \arrow[r, "(f,g)"] \arrow[rr, bend right, "(f',g') \circ (f,g) = (f' \circ f, g' \circ g)"'] & (C', D') \arrow[r, "(f',g')"] & (C'', D'') \end{tikzcd}$$

Example 5

We can also form a category with the arrows of a category $\mathcal{C}$. It is known as the **arrow category** of $\mathcal{C}$, and is denoted as $\mathrm{Arr}(\mathcal{C})$. The objects are arrows of $\mathcal{C}$. What are the arrows of $\mathrm{Arr}(\mathcal{C})$?

$$\begin{array}{ccc} f : A & \longrightarrow & B \\ {\scriptstyle x_1}\big\downarrow & & \big\downarrow{\scriptstyle x_2} \\ g : C & \longrightarrow & D \end{array}$$

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Suppose $f : A \to B, g : C \to D \in \mathrm{Arr}(\mathcal{C})_0$. An arrow $x : f \to g$ in the arrow category is a pair of arrows $x_1 : A \to C$ and $x_2 : B \to D$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "x_1"] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "x_2"'] & D \end{tikzcd} \text{, i.e. } \boxed{x_2 \circ f = g \circ x_1.}$$

f

$x_2 \circ f \neq g \circ x_1$

$$\begin{tikzcd} A \arrow[r, "x'"] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "x_2"'] & D \end{tikzcd}$$

Making New categories From Old Ones

Example 5

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$$\begin{tikzcd} A \arrow[r, "x_1"] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "x_2"'] & D \end{tikzcd} \quad \text{, i.e. } x_2 \circ f = g \circ x_1.$$

$$\begin{tikzcd} A \arrow[r, "\mathbb{1}_A"] \arrow[d, "f"'] & A \arrow[d, "f"] \\ B \arrow[r, "\mathbb{1}_B"'] & B \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, "x_1"] \arrow[d, "f"'] \arrow[dr, phantom, "\checkmark"] & C \arrow[r, "y_1"] \arrow[d, "g"] \arrow[dr, phantom, "\checkmark"] & E \arrow[d, "h"] \\ B \arrow[r, "x_2"'] & D \arrow[r, "y_2"'] & F \end{tikzcd}$$

Identity and composition are defined in the obvious way.

$$\begin{tikzcd} A \arrow[r, "y_1 \circ x_1"] \arrow[d, "f"'] \arrow[dr, phantom, "\checkmark"] & E \arrow[d, "h"] \\ B \arrow[r, "y_2 \circ x_2"'] & F \end{tikzcd}$$

Hom-sets

The categories we see in our everyday life are **locally small** categories, i.e. $\mathrm{Hom}_{\mathcal{C}}(A, B)$ are sets. We shall not worry about it anymore and assume that all categories are locally small.

Hom-sets

For a locally small category $\mathcal{C}$, and $X \in \mathcal{C}_0$, we have a functor

$$\mathrm{Hom}_{\mathcal{C}}(X, -) : \mathcal{C} \to \mathbf{Sets},$$

called the Hom functor.

$$Y \longmapsto \mathrm{Hom}_{\mathcal{C}}(X, Y)$$

Hom-sets

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Given an arrow $\boxed{f : A \to B \text{ in}}$ $\mathcal{C}$, the Hom functor is supposed to take it to a set function $f^* : \underline{\mathrm{Hom}_{\mathcal{C}}(X, A)} \to \underline{\mathrm{Hom}_{\mathcal{C}}(X, B)}$. How does f^* work?

$$(g : X \to A) \mapsto (f^*(g) : X \to B)$$

$$\begin{tikzcd} X \arrow[d, "g"'] \arrow[dr, "f^*(g)", "f \circ g"'] & \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

Hom-sets

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$$\begin{tikzcd} X \arrow[d, "g"'] \arrow[dr, "f^*(g) = f \circ g"] & \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

Similarly can we define a functor

$$\mathrm{Hom}_{\mathcal{C}}(-, X) : \mathcal{C} \to \mathbf{Sets}?$$

$$A \longrightarrow \mathrm{Hom}_{\mathcal{C}}(A, X)$$

Hom-sets

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$$\mathrm{Hom}_{\mathcal{C}}\left(-, X\right) : \mathcal{C} \to \mathbf{Sets}?$$

It will take objects $A \in \mathcal{C}_0$ to the set $\mathrm{Hom}_{\mathcal{C}}\left(A, \right)$.

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$$\left(g : A \to X\right) \longmapsto \left(f_*(g) : B \to X\right)$$

$$\begin{tikzcd} & X \\ A \arrow[ur, "g"] \arrow[r, "f"'] & B \arrow[u, "f_*(g)"'] \end{tikzcd}$$

Hom-sets

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$$\mathrm{Hom}_{\mathcal{C}}\,(-, X) : \mathcal{C} \to \textbf{Sets}?$$

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$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[l, bend right, "f"'] \arrow[dl, "f_*(g)=???"] \\
X &
\end{tikzcd}$$

Hom-sets

Given an arrow $f : A \to B$, seemingly there's an issue with defining $f_* : \mathrm{Hom}_{\mathcal{C}}(A, X) \to \mathrm{Hom}_{\mathcal{C}}(B, X)$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[dl, "f_*(g)=???"] \\ X & \end{tikzcd}$$

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But what we can do is, we can define $f_* : \mathrm{Hom}_{\mathcal{C}}(B, X) \to \mathrm{Hom}_{\mathcal{C}}(A, X)$.

$$
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So $\mathrm{Hom}_{\mathcal{C}}(-, X) : \mathcal{C} \to$ **Sets** reverses the direction of arrows?

Hom-sets

The solution is tweaking the domain category to be the opposite category of $\mathcal{C}$.

$$\mathrm{Hom}_{\mathcal{C}}\left(-, X\right) : \mathcal{C}^{\mathrm{op}} \to \textbf{Sets}.$$

Hom-sets

The solution is tweaking the domain category to be the opposite category of $\mathcal{C}$.

$$\mathrm{Hom}_{\mathcal{C}}(-, X) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Sets}.$$

The objects of $\mathcal{C}^{\mathrm{op}}$ are the same as $\mathcal{C}$.

$f : A \to B$ in $\mathcal{C}$

$f^{op} : B \to A$

Hom-sets

The solution is tweaking the domain category to be the opposite category of $\mathcal{C}$.

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The objects of $\mathcal{C}^{\mathrm{op}}$ are the same as $\mathcal{C}$. The arrows of $\mathcal{C}^{\mathrm{op}}$ are $f^{\mathrm{op}} : B \to A$, corresponding to an arrow $f : A \to B$ in $\mathcal{C}$.

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The objects of $\mathcal{C}^{\mathrm{op}}$ are the same as $\mathcal{C}$. The arrows of $\mathcal{C}^{\mathrm{op}}$ are $f^{\mathrm{op}} : B \to A$, corresponding to an arrow $f : A \to B$ in $\mathcal{C}$.

The functor $\mathrm{Hom}_{\mathcal{C}}(-, X)$ takes the arrow $\boxed{f^{\mathrm{op}} : B \to A}$ to a set function $f_* : \mathrm{Hom}_{\mathcal{C}}(B, X) \to \mathrm{Hom}_{\mathcal{C}}(A, X)$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "f_*(g) = g \circ f"'] & B \arrow[dl, "g"] \\ X & \end{tikzcd}$$

$\left(\mathrm{Vect}_{\mathbb{K}}\right)^{op}$

$f : V \longrightarrow W$

$f^T : W^* \longrightarrow V^*$

Generalizing further, we define the functor

$$\mathrm{Hom}_{\mathcal{C}}(-,-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \textbf{Sets}.$$

$$(A, B) \longmapsto \mathrm{Hom}_{\mathcal{C}}(A, B)$$

Hom-sets

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In the object level, it takes (A, B) to the set $\mathrm{Hom}_{\mathcal{C}}(A, B)$.

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In the arrow level, let (f^{op}, g) be an arrow in $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$. Then $f : A \to B$ and $g : X \to Y$ are arrows in $\mathcal{C}$.

$$(f^{op}, g) : \cancel{B}\,(B, X) \longmapsto (A, Y)$$

$$f^{op} : B \longrightarrow A, \quad f : A \longrightarrow B \ (\mathcal{C})$$

$$g : X \longrightarrow Y \ (\mathcal{C})$$

Hom-sets

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In the arrow level, let (f^{op}, g) be an arrow in $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$. Then $f : A \to B$ and $g : X \to Y$ are arrows in $\mathcal{C}$.
$\mathrm{Hom}_{\mathcal{C}}(f^{\mathrm{op}}, g)$ will be a set function $\mathrm{Hom}_{\mathcal{C}}(B, X) \to \mathrm{Hom}_{\mathcal{C}}(A, Y)$.

$$\begin{tikzcd} B \arrow[d, "a"] & A \arrow[l, "f"'] \arrow[d] \\ X \arrow[r, "g"] & Y \end{tikzcd}$$

$g \circ a \circ f$

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In the arrow level, let (f^{op}, g) be an arrow in $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$. Then $f : A \to B$ and $g : X \to Y$ are arrows in $\mathcal{C}$.
$\mathrm{Hom}_{\mathcal{C}}(f^{\mathrm{op}}, g)$ will be a set function $\mathrm{Hom}_{\mathcal{C}}(B, X) \to \mathrm{Hom}_{\mathcal{C}}(A, Y)$.

$$\begin{tikzcd} A \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[ur, "x"] & Y \end{tikzcd}$$

Given $x \in \mathrm{Hom}_{\mathcal{C}}(B, X)$, we define $\mathrm{Hom}_{\mathcal{C}}(f^{\mathrm{op}}, g) = g \circ x \circ f$.

Isomorphisms

$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \end{tikzcd}$$

$$f \circ g = 1\!\!1_Y$$

$$g \circ f = 1\!\!1_X$$

We define

- isomorphism between groups,
- isomorphism between vector spaces,
- homeomorphism between topological spaces,
- diffeomorphisms between smooth manifolds,
- and so on . . .

Category theory captures this pattern of “sameness” as well.

Isomorphisms

Definition 3

An arrow $f : A \to B$ in a category $\mathcal{C}$ is called an **isomorphism** if there exists another arrow $g : B \to A$ such that $f \circ g = \mathbb{1}_B$ and $g \circ f = \mathbb{1}_A$.
If there is an isomorphism from A to B, we call A and B **isomorphic objects**.

Isomorphisms

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If there is an isomorphism from A to B, we call A and B **isomorphic objects**.

This definition aligns with our definition of isomorphisms in other categories.

Isomorphism and Functors

Suppose A and B are isomorphic objects in a category $\mathcal{C}$. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then are $F(A)$ and $F(B)$ isomorphic objects in the category $\mathcal{D}$?

$\mathcal{C}$

$$\begin{tikzcd} A \arrow[r, bend left, "f"] & B \arrow[l, bend left, "g"] \end{tikzcd}$$

$\mathcal{D}$

$F(A)$ $F(B)$

Isomorphism and Functors

Suppose A and B are isomorphic objects in a category $\mathcal{C}$. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then are $F(A)$ and $F(B)$ isomorphic objects in the category $\mathcal{D}$?

$$\mathcal{C} \qquad\qquad\qquad \mathcal{D}$$

$$A \underset{g}{\overset{f}{\rightleftarrows}} B \qquad\qquad F(A) \underset{F(g)}{\overset{F(f)}{\rightleftarrows}} F(B)$$

$$F(f) \circ F(g) = F(f \circ g)$$
$$= F(1_B) = 1_{F(B)}$$

$$F(g) \circ F(f) = F(g \circ f) = F(1_A)$$
$$= 1_{F(A)}$$

Isomorphism and Functors

Suppose A and B are isomorphic objects in a category $\mathcal{C}$. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then are $F(A)$ and $F(B)$ isomorphic objects in the category $\mathcal{D}$?

$\mathcal{C}$ $\qquad\qquad\qquad$ $\mathcal{D}$

$$A \underset{g}{\overset{f}{\rightleftarrows}} B \qquad\qquad F(A) \underset{F(g)}{\overset{F(f)}{\rightleftarrows}} F(B)$$

$$F(f) \circ F(g) = F(f \circ g) = F(\mathbb{1}_A) = \mathbb{1}_{F(A)}.$$

$$F(g) \circ F(f) = F(g \circ f) = F(\mathbb{1}_B) = \mathbb{1}_{F(B)}.$$

Isomorphism and Functors

Suppose A and B are isomorphic objects in a category $\mathcal{C}$. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then are $F(A)$ and $F(B)$ isomorphic objects in the category $\mathcal{D}$?

$\mathcal{C}$ $\qquad\qquad\qquad$ $\mathcal{D}$

$$A \underset{g}{\overset{f}{\rightleftarrows}} B \qquad\qquad F(A) \underset{F(g)}{\overset{F(f)}{\rightleftarrows}} F(B)$$

$$F(f) \circ F(g) = F(f \circ g) = F(\mathbb{1}_A) = \mathbb{1}_{F(A)}.$$

$$F(g) \circ F(f) = F(g \circ f) = F(\mathbb{1}_B) = \mathbb{1}_{F(B)}.$$

Therefore, functors preserve isomorphisms.

Isomorphism and Hom-sets

$\mathrm{Hom}(X, -) : \mathcal{C} \longrightarrow \mathrm{Sets}$

Suppose A and B are isomorphic objects in a category. Then for any other object X, since functors preserve isomorphisms, $\mathrm{Hom}_{\mathcal{C}}(X, A)$ and $\mathrm{Hom}_{\mathcal{C}}(X, B)$ are isomorphic objects in **Sets**.

$A \cong B \Rightarrow \underline{\mathrm{Hom}_{\mathcal{C}}(X, A)} \cong \underline{\mathrm{Hom}_{\mathcal{C}}(X, B)}$

Isomorphism and Hom-sets

Suppose A and B are isomorphic objects in a category. Then for any other object X, since functors preserve isomorphisms, $\mathrm{Hom}_{\mathcal{C}}(X, A)$ and $\mathrm{Hom}_{\mathcal{C}}(X, B)$ are isomorphic objects in **Sets**.

Similarly, $\mathrm{Hom}_{\mathcal{C}}(A, X)$ and $\mathrm{Hom}_{\mathcal{C}}(B, X)$ are also isomorphic objects in **Sets**.

Isomorphism and Hom-sets

$\mathrm{Hom}_{\mathcal{C}}(-, X)$

Suppose A and B are isomorphic objects in a category. Then for any other object X, since functors preserve isomorphisms, $\mathrm{Hom}_{\mathcal{C}}(X, A)$ and $\mathrm{Hom}_{\mathcal{C}}(X, B)$ are isomorphic objects in **Sets**.

Similarly, $\mathrm{Hom}_{\mathcal{C}}(A, X)$ and $\mathrm{Hom}_{\mathcal{C}}(B, X)$ are also isomorphic objects in **Sets**.

Does the converse hold? If $\mathrm{Hom}_{\mathcal{C}}(A, X)$ and $\mathrm{Hom}_{\mathcal{C}}(B, X)$ are also isomorphic objects in **Sets**, then can we say that A and B are isomorphic objects in $\mathcal{C}$?

Isomorphisms

Natural Transformation

Given two categories $\mathcal{C}$ and $\mathcal{D}$, we can form the functor category $\boxed{\mathrm{Fun}\left(\mathcal{C}, \mathcal{D}\right)}$ whose objects are all the functors from $\mathcal{C}$ to $\mathcal{D}$. What should be the arrows in this category?

This is similar to the construction of arrow category.

Natural Transformation

Definition 4

Let $F, G : \mathcal{C} \to \mathcal{D}$ be two functors. Then a **natural transformation** $\eta : F \Rightarrow G$ is a family of arrows

$$\boxed{\{\eta_X : F(X) \to G(X)\}_{X \in \mathcal{C}_0}}$$

in $\mathcal{D}$ such that for every arrow $f : X \to Y$ in $\mathcal{C}$, the following diagram commutes:

$$\begin{tikzcd} X \arrow[d, "f"] \\ Y \end{tikzcd}$$

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\eta_X"'] & F(Y) \arrow[d, "\eta_Y"] \\ G(X) \arrow[r, "G(f)"'] & G(Y) \end{tikzcd}$$

$f : X \to Y$ in $\mathcal{C}$

$\boxed{\eta_X : F(X) \to G(X)}$ in $\mathcal{D}$

$$\begin{tikzcd} F(X) \arrow[r, "\eta_X"'] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\eta_Y"'] & G(Y) \end{tikzcd}$$

In other words, $\eta_Y \circ F(f) = G(f) \circ \eta_X$.

Given such a natural transformation $\eta : F \Rightarrow G$, the arrow η_X is called the component of η at X.

Natural Transformation

$\eta : F \Rightarrow G$

If $F, G : \mathcal{C} \to \mathcal{D}$ are functors, and $\eta : F \Rightarrow G$ is a natural transformation, it is denoted as follows

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "F"{name=U}] \arrow[r, bend right=50, "G"'{name=D}] & \mathcal{D} \arrow[Rightarrow, from=U, to=D, "\eta"] \end{tikzcd}$$

Natural Transformation

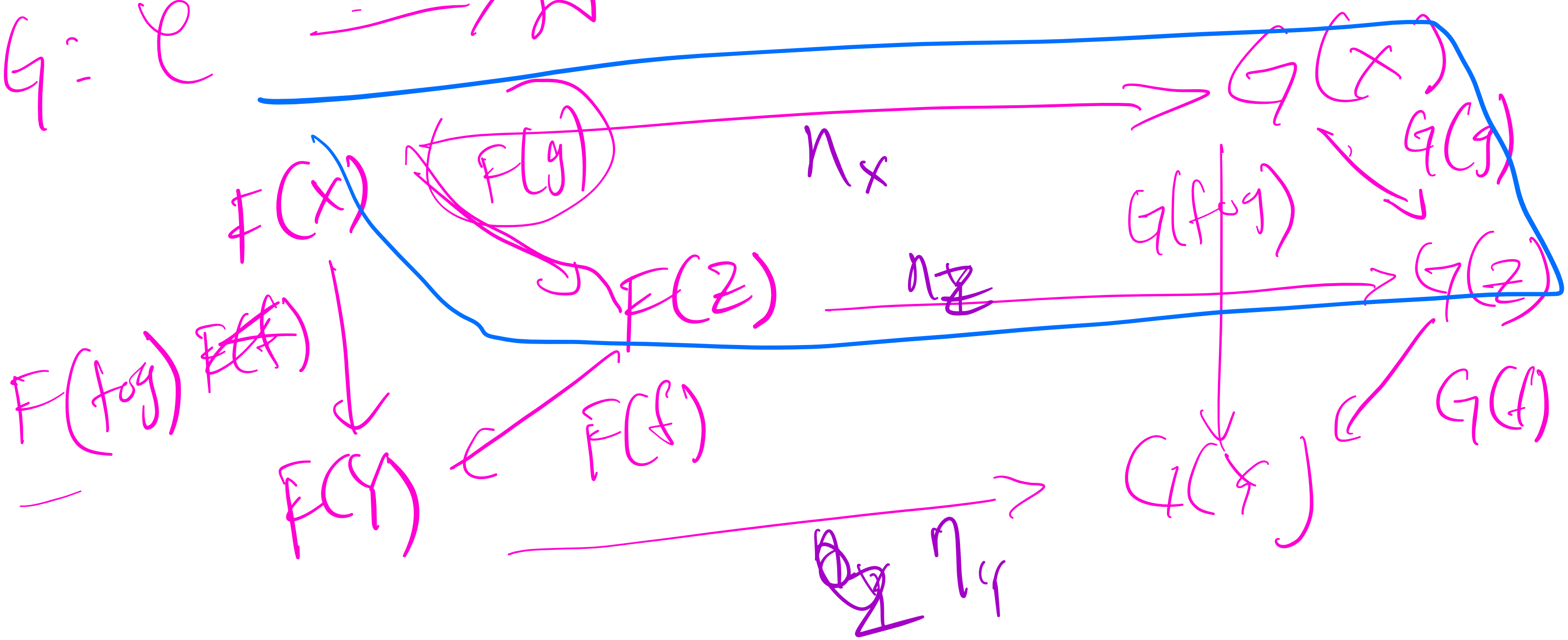
Why do we care about natural transformations?

Why do we care about natural transformations?

Because they take commutative diagrams to commutative diagrams!!!

$F : \mathcal{C} \longrightarrow \mathcal{D}$

$G : \mathcal{C} \longrightarrow \mathcal{D}$



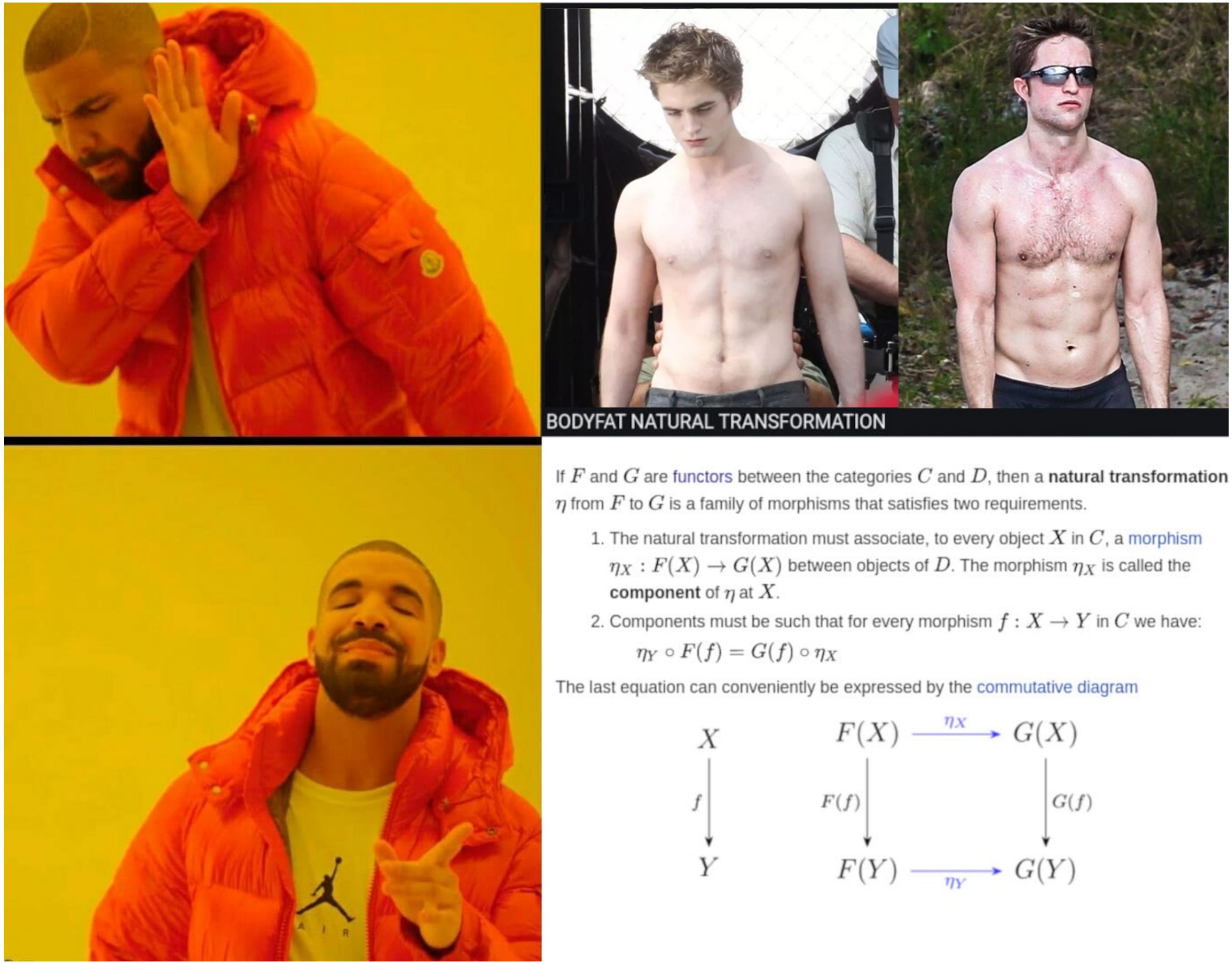
Why do we care about natural transformations?

Because they take commutative diagrams to commutative diagrams!!!

$$
\begin{tikzcd}
F(X) \arrow[rr, "\eta_X"] \arrow[dr, "F(f)"] \arrow[dd, "F(g\circ f)"'] & & G(X) \arrow[dr, "G(f)"] \arrow[dd, "G(g\circ f)"'] & \\
& F(Y) \arrow[rr, "\eta_Y"'] \arrow[dl, "F(g)"] & & G(Y) \arrow[dl, "G(g)"] \\
F(Z) \arrow[rr, "\eta_Z"'] & & G(Z) &
\end{tikzcd}
$$

$$
\begin{tikzcd}
G(X) \arrow[loop above] \arrow[d, bend left] \\
G(Z) \arrow[u, bend left]
\end{tikzcd}
$$

Natural Transformation



Natural Isomorphism

$$\begin{tikzcd} V \arrow[r] \arrow[d, "f"'] & V^* \arrow[d] \\ W \arrow[r] & W^* \end{tikzcd}$$

$F, G : \mathrm{Vect}_{\mathbb{K}}^{fin} \to \mathrm{Vect}_{\mathbb{K}}^{fin}$

$G : V \longmapsto V^*$

$V \longrightarrow V^*$

basis of V $= \{\bar{v}_1, \dots, \bar{v}_n\}$

$\{\hat{\alpha}^1, \hat{\alpha}^2, \dots, \hat{\alpha}^n\}$

$\hat{\alpha}^i(\bar{v}_j) = \delta^i_j$

Definition 5

Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A natural transformation $\eta : F \Rightarrow G$ is called a **natural isomorphism** if all its components

$$\{\eta_X : F(X) \to G(X)\}_{X \in \mathcal{C}_0}$$

are isomorphisms in $\mathcal{D}$.

$f : X \to Y$

$$\begin{tikzcd} F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f)"'] \arrow[dr, phantom, "\checkmark"] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\eta_Y"'] & G(Y) \end{tikzcd}$$

Natural Isomorphism

Definition 5

Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A natural transformation $\eta : F \Rightarrow G$ is called a **natural isomorphism** if all its components

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are isomorphisms in $\mathcal{D}$.

Since natural transformations are arrows in the functor category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, natural isomorphisms are just isomorphisms in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

$\pi_X = \eta_X^{-1} : G(X) \longrightarrow F(X)$

$\eta : F \Rightarrow G$

$\pi : G \Rightarrow F$

$\pi \circ \eta : F \to F$

$(\pi \circ \eta)_X = \pi_X \circ \eta_X$

$= 1_{F(X)}$

Natural Isomorphism

What do we mean when we say V and V^{**} are "naturally isomorphic" (when V is a finite dimensional vector space)? Where is the natural isomorphism here?

$$\begin{tikzcd} V \arrow[r, "\eta_V"] \arrow[d, "f"] & V^{**} \arrow[d, "(f^T)^T"] \\ W \arrow[r, "\eta_W"] & W^{**} \end{tikzcd}$$

Natural Isomorphism

$f: V \to W$

$f^T: W^* \to V^*$, $(f^T)^T: V^{**} \to W^{**}$

What do we mean when we say V and V^{**} are "naturally isomorphic" (when V is a finite dimensional vector space)? Where is the natural isomorphism here?

Let $\mathbf{Vect}_{\mathbb{K}}^{\text{fin}}$ be the category of all finite dimensional vector spaces over the field $\mathbb{K}$. Consider the functors $\mathbb{1}_{\mathbf{Vect}_{\mathbb{K}}^{\text{fin}}}, (-)^{**} : \mathbf{Vect}_{\mathbb{K}}^{\text{fin}} \to \mathbf{Vect}_{\mathbb{K}}^{\text{fin}}$.

$$V \longmapsto V^{**}$$

$$(f: V \to W) \longmapsto \left(V^{**} \longrightarrow W^{**} \right)$$

Natural Isomorphism

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The second functor sends a vector space V to its double dual V^{**}, and a linear map $f : V \to W$ to $\left(f^T\right)^T : V^{**} \to W^{**}$.

Natural Isomorphism

Then we can define an isomorphism $\eta_V : V \to V^{**}$ such that

$$\eta_V(\mathbf{v})(\varphi) = \varphi(\mathbf{v}),$$

for $\mathbf{v} \in V$ and $\varphi \in V^*$.

Groups

$V^* = \{ f : V \to \mathbb{C} \}$

$V' = \{ f : V \to \mathbb{C} \}$

$\langle \varphi | \psi \rangle$

$V \longleftrightarrow V'$

Riesz Representation theorem

$\longleftrightarrow \langle \varphi, - \rangle : V \to \mathbb{C}$

Natural Isomorphism

$\eta_v(\bar{v})(\varphi)$

$\varphi(\bar{v})$

Then we can define an isomorphism $\eta_V : V \to V^{**}$ such that

$$\eta_V(\mathbf{v})(\varphi) = \varphi(\mathbf{v}),$$

for $\mathbf{v} \in V$ and $\varphi \in V^*$.

v

Then $\{\eta_V : V \to V^{**}\}_{V \in \left(\mathbf{Vect}_{\mathbb{K}}^{\text{fin}}\right)_0}$ is a natural isomorphism, because the following diagram commutes:

$$\begin{tikzcd}
V \arrow[r, "f"] \arrow[d, "\eta_V"'] & W \arrow[d, "\eta_W"] \\
V^{**} \arrow[r, "\left(f^T\right)^T"'] & W^{**}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{Vect}_{\mathbb{K}}^{\text{fin}} \arrow[r, bend left=50, "\mathbb{1}_{\mathbf{Vect}_{\mathbb{K}}^{\text{fin}}}", ""{name=U, below}] \arrow[r, bend right=50, "(-)^{**}"', ""{name=D, above}] & \mathbf{Vect}_{\mathbb{K}}^{\text{fin}} \arrow[Rightarrow, from=U, to=D, "\eta"]
\end{tikzcd}$$

Adjoints

$\langle \bar{v}, \bar{w} \rangle$

$\langle f(\bar{v}), \bar{w} \rangle = \langle v, g(\bar{w}) \rangle$

Cat

Isomorphism	Equivalence	Adjoints
$F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$, s.t. $F \circ G = \mathbb{1}_{\mathcal{D}}$ and $G \circ F = \mathbb{1}_{\mathcal{C}}$	$F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$, s.t. **natural isomorphisms** $F \circ G \Rightarrow \mathbb{1}_{\mathcal{D}}$ and $G \circ F \Rightarrow \mathbb{1}_{\mathcal{C}}$	$F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$, s.t. **natural transformations** $F \circ G \Rightarrow \mathbb{1}_{\mathcal{D}}$ and $G \circ F \Leftarrow \mathbb{1}_{\mathcal{C}}$

$\mathcal{C}$ and $\mathcal{D}$ are isomorphic categories

$\mathcal{C}$ and $\mathcal{D}$ are equivalent categories

Definition 6

Two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are called **adjoints** to each other if

$$\mathrm{Hom}_{\mathcal{D}}\left(F\left(C\right), D\right) \cong \mathrm{Hom}_{\mathcal{C}}\left(C, G\left(D\right)\right) \quad \boxed{\textit{naturally.}}$$

$C \in \mathcal{C}_0 \quad D \in \mathcal{D}_0$

$\mathrm{Hom}_{\mathcal{C}}(F(C), D) \cong \mathrm{Hom}_{\mathcal{C}}(C, G(D))$

$\mathrm{Hom}_{\mathcal{D}}(F(-), -)$ ~~(C,D)~~

Definition 6

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$$\mathrm{Hom}_{\mathcal{D}}(F(C), D) \cong \mathrm{Hom}_{\mathcal{C}}(C, G(D)) \quad \textit{naturally.}$$

$\mathrm{Hom}_{\mathcal{C}}(-, G(-))$

In other words, there exists a natural isomorphism η:

$$\begin{tikzcd} \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \arrow[rr, bend left, "\mathrm{Hom}_{\mathcal{D}}(F(-),-)"] \arrow[rr, bend right, "\mathrm{Hom}_{\mathcal{C}}(-,G(-))"'] \arrow[rr, phantom, "\Downarrow \eta"] & & \mathbf{Sets} \end{tikzcd}$$

$$\eta : \mathrm{Hom}_{\mathcal{D}}(F(-), -) \Longrightarrow \mathrm{Hom}_{\mathcal{C}}(-, G(-))$$

$$\mathcal{C}^{op} \times \mathcal{D} \longrightarrow \mathrm{Sets}$$

Adjoints

Definition 6

Two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are called **adjoints** to each other if

$$\mathrm{Hom}_{\mathcal{D}}(F(C), D) \cong \mathrm{Hom}_{\mathcal{C}}(C, G(D)) \ \textit{naturally}.$$

In other words, there exists a natural isomorphism η:

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If this happens, we call F a **left adjoint** of G; and we call G a **right adjoint** of F. We write this as $\boxed{F \dashv G}$.

$F \dashv G$

Adjoints Example

Tensor product of vector spaces

The **tensor product** of two $\mathbb{K}$-vector spaces U and V is another $\mathbb{K}$-vector space $U \otimes V$ equipped with a bilinear map $\theta : U \times V \to U \otimes V$ that is *universal*:

Adjoints Example

$U \times V = \{(\bar{u}, \bar{v}) \}$

Tensor product of vector spaces

The **tensor product** of two $\mathbb{K}$-vector spaces U and V is another $\mathbb{K}$-vector space $U \otimes V$ equipped with a bilinear map $\theta : U \times V \to U \otimes V$ that is *universal*: for any bilinear map $\beta : U \times V \to W$, there exists a unique linear map $\alpha : U \otimes V \to W$ such that the following diagram commutes:

$$\begin{tikzcd} U \times V \arrow[r, "\theta"] \arrow[d, "\forall \text{ bilinear } \beta"'] & U \otimes V \arrow[dl, dashed, "\exists!\,\alpha"] \\ W & \end{tikzcd}$$

$\{\bar{e} \otimes \bar{f}\}$

In other words, $\beta = \alpha \circ \theta$.

So we have a 1-1 correspondence

$$\{\text{linear maps } U \otimes V \to W\} \leftrightarrow \{\text{bilinear maps } U \times V \to W\}.$$

$f : U \times V \longrightarrow W$

$f(\bar{u}, -) : V \longrightarrow W$

$f : U \times V \longrightarrow W \longrightarrow f(\bar{u}, -) : V \longrightarrow W$

$\mathrm{Hom}(V, W)$

Adjoints Example

So we have a 1-1 correspondence

$$\{\text{linear maps } U \otimes V \to W\} \leftrightarrow \{\text{bilinear maps } U \times V \to W\}\,.$$

We also have another 1-1 correspondence

$$\{\text{bilinear maps } U \times V \to W\} \leftrightarrow \{\text{linear maps } U \to \operatorname{Hom}(V, W)\}\,;$$

Adjoints Example

$\mathcal{U} \to V^{x}$

$\mathcal{U} \times V \longrightarrow W$

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$$\{\text{bilinear maps } U \times V \to W\} \leftrightarrow \{\text{linear maps } U \to \operatorname{Hom}(V, W)\}\,;$$
$$(f : U \times V \to W) \mapsto (\mathbf{u} \mapsto f(\mathbf{u}, -))\,,$$

bilinear $U \times V \longrightarrow W$

$f(\alpha \bar{u}, \alpha \bar{v}) = \alpha f(\bar{u}, \bar{v})$

$\hookrightarrow = \alpha^2 f(\bar{u}, \bar{v})$

$f(\alpha(\bar{u}_1, \bar{v}_1) + (\bar{u}_2, \bar{v}_2))$

$= \alpha f(\bar{u}_1, \bar{v}_1) + f(\bar{u}_2, \bar{v}_2)$

Adjoints Example

So we have a 1-1 correspondence

$$\{\text{linear maps } U \otimes V \to W\} \leftrightarrow \{\text{bilinear maps } U \times V \to W\}\,.$$

We also have another 1-1 correspondence

$$\{\text{bilinear maps } U \times V \to W\} \leftrightarrow \{\text{linear maps } U \to \text{Hom}\,(V, W)\}\,;$$

$$(f : U \times V \to W) \mapsto (\mathbf{u} \mapsto f(\mathbf{u}, -))\,,$$

$$((\mathbf{u}, \mathbf{v}) \mapsto g\,(\mathbf{u})\,(\mathbf{v})) \leftarrow\!\shortmid (g : U \to \text{Hom}(V, W))\,.$$

ϵ_W

$$\left\{\text{linear } U \otimes V \to W\right\} \leftrightarrow \left\{\text{linear } U \to \text{Hom}(V, W)\right\}$$

Adjoints Example

So we have a 1-1 correspondence

$$\{\text{linear maps } U \otimes V \to W\} \leftrightarrow \{\text{bilinear maps } U \times V \to W\}.$$

We also have another 1-1 correspondence

$$\begin{gathered}
\{\text{bilinear maps } U \times V \to W\} \leftrightarrow \{\text{linear maps } U \to \mathrm{Hom}\,(V, W)\}; \\
(f : U \times V \to W) \mapsto (\mathbf{u} \mapsto f(\mathbf{u}, -)), \\
((\mathbf{u}, \mathbf{v}) \mapsto g\,(\mathbf{u})\,(\mathbf{v})) \leftarrow\!\shortmid (g : U \to \mathrm{Hom}(V, W)).
\end{gathered}$$

Therefore, we have the 1-1 correspondence

$$\{\text{linear maps } U \otimes V \to W\} \leftrightarrow \{\text{linear maps } U \to \mathrm{Hom}\,(V, W)\}.$$

Adjoints Example

$\mathrm{Hom}_{\mathbf{Vect}_{\mathbb{K}}}$

In other words,

$$\boxed{\mathrm{Hom}\,(U \otimes V, W) \cong \mathrm{Hom}\,(U, \mathrm{Hom}\,(V, W))\,.}$$

Does that mean $- \otimes V$ and $\mathrm{Hom}(V, -)$: $\mathbf{Vect}_{\mathbb{K}} \to \mathbf{Vect}_{\mathbb{K}}$ are adjoint functors?

$\{U \otimes V \to W \text{ linear}\}$

$\longleftrightarrow \{U \to \mathrm{Hom}(V, W)\}$

$- \otimes V$

$\mathrm{Hom}(V, -) \to \mathrm{Vect}_{\mathbb{K}}$

Adjoints Example

In other words,

$$\mathrm{Hom}\,(U \otimes V, W) \cong \mathrm{Hom}\,(U, \mathrm{Hom}\,(V, W))\,.$$

Does that mean $- \otimes V$ and $\mathrm{Hom}(V, -)$: $\mathbf{Vect}_{\mathbb{K}} \to \mathbf{Vect}_{\mathbb{K}}$ are adjoint functors? Well, not yet. We need to show the naturality of this isomorphism.

$$\begin{tikzcd}
\mathbf{Vect}_{\mathbb{K}}^{\mathrm{op}} \times \mathbf{Vect}_{\mathbb{K}} \arrow[rr, bend left, "\mathrm{Hom}_{\mathbf{Vect}_{\mathbb{K}}}(-\otimes V,-)"] \arrow[rr, bend right, "\mathrm{Hom}_{\mathbf{Vect}_{\mathbb{K}}}(-,\mathrm{Hom}(V,-))"'] \arrow[rr, phantom, "\Downarrow\eta"] & & \mathbf{Sets}
\end{tikzcd}$$

$F : \mathcal{C} \to \mathcal{D}$
$G : \mathcal{D} \to \mathcal{C}$

$$\begin{tikzcd}
\mathcal{C}^{op} \times \mathcal{D} \arrow[rr, bend left, "\mathrm{Hom}_{\mathcal{D}}(F(-),-)"] \arrow[rr, bend right, "\mathrm{Hom}_{\mathcal{C}}(-,G(-))"'] \arrow[rr, phantom, "\Downarrow\eta"] & & \mathrm{Sets}
\end{tikzcd}$$

Adjoints Example

Given an arrow $(\alpha_1^{op}, \alpha_2) : (U, W) \to (U', W')$ in $\mathbf{Vect}_{\mathbb{K}}^{op} \times \mathbf{Vect}_{\mathbb{K}}$, we need to show the commutativity of the following diagram in the category **Sets**:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{Vect}_{\mathbb{K}}}(U \otimes V, W) \arrow[r, "\eta_{(U,W)}"] \arrow[d, "F(\alpha_1^{op}, \alpha_2)"'] & \mathrm{Hom}_{\mathbf{Vect}_{\mathbb{K}}}(U, \mathrm{Hom}(V, W)) \arrow[d, "G(\alpha_1^{op}, \alpha_2)"] \\
\mathrm{Hom}_{\mathbf{Vect}_{\mathbb{K}}}(U' \otimes V, W') \arrow[r, "\eta_{(U',W')}"'] & \mathrm{Hom}_{\mathbf{Vect}_{\mathbb{K}}}(U', \mathrm{Hom}(V, W'))
\end{tikzcd}$$

where $F = \mathrm{Hom}_{\mathbf{Vect}_{\mathbb{K}}}(- \otimes V, -)$ and $G = \mathrm{Hom}_{\mathbf{Vect}_{\mathbb{K}}}(-, \mathrm{Hom}(V, -))$.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathcal{D}
\end{tikzcd}$$

$f : X \to Y$

$$\begin{tikzcd}
F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\
F(Y) \arrow[r, "\eta_Y"] & G(Y)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Vect}_{\mathbb{K}}^{op} \times \mathrm{Vect}_{\mathbb{K}} \arrow[r, bend left, "\mathrm{Hom}(-\otimes V, -)"] \arrow[r, bend right, "\mathrm{Hom}(-, \mathrm{Hom}(V, -))"'] & \mathrm{Sets}
\end{tikzcd}$$

Adjoints Example

Given an arrow $(\alpha_1^{\mathrm{op}}, \alpha_2) : (U, W) \to (U', W)$ in $\mathbf{Vect}_{\mathbb{K}}^{\mathrm{op}} \times \mathbf{Vect}_{\mathbb{K}}$, we need to show the commutativity of the following diagram in the category **Sets**:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{Vect}_{\mathbb{K}}}(U \otimes V, W) \arrow[r, "\eta_{(U,W)}"] \arrow[d, "F(\alpha_1^{\mathrm{op}}, \alpha_2)"'] & \mathrm{Hom}_{\mathbf{Vect}_{\mathbb{K}}}(U, \mathrm{Hom}(V, W)) \arrow[d, "G(\alpha_1^{\mathrm{op}}, \alpha_2)"] \\
\mathrm{Hom}_{\mathbf{Vect}_{\mathbb{K}}}(U' \otimes V, W') \arrow[r, "\eta_{(U',W')}"'] & \mathrm{Hom}_{\mathbf{Vect}_{\mathbb{K}}}(U', \mathrm{Hom}(V, W'))
\end{tikzcd}$$

where $F = \mathrm{Hom}_{\mathbf{Vect}_{\mathbb{K}}}(- \otimes V, -)$ and $G = \mathrm{Hom}_{\mathbf{Vect}_{\mathbb{K}}}(-, \mathrm{Hom}(V, -))$.
And this diagram indeed commutes! So $- \otimes V$ is the left adjoint of $\mathrm{Hom}(V, -)$.

Exercise

Adjoints

$F \dashv G_1$ $\qquad$ $F \dashv G_2$

Are adjoints unique? Can a functor $F : \mathcal{C} \to \mathcal{D}$ have two right adjoints $G_1, G_2 : \mathcal{D} \to \mathcal{C}$?

Adjoints

Are adjoints unique? Can a functor $F : \mathcal{C} \to \mathcal{D}$ have two right adjoints $G_1, G_2 : \mathcal{D} \to \mathcal{C}$? If this happens, then for any $C \in \mathcal{C}_0$ and $D \in \mathcal{D}_0$,

$$\mathrm{Hom}_{\mathcal{D}}(F(-), -) \cong \mathrm{Hom}_{\mathcal{C}}(-, G_1(-))$$

$$\cong \mathrm{Hom}_{\mathcal{C}}(-, G_2(-))$$

For any $C \in \mathcal{C}_0$, $\boxed{\mathrm{Hom}_{\mathcal{C}}(C, G_1(-)) \cong \mathrm{Hom}_{\mathcal{C}}(C, G_2(-))}$

Adjoints

Are adjoints unique? Can a functor $F : \mathcal{C} \to \mathcal{D}$ have two right adjoints $G_1, G_2 : \mathcal{D} \to \mathcal{C}$? If this happens, then for any $C \in \mathcal{C}_0$ and $D \in \mathcal{D}_0$,

$$\mathrm{Hom}_{\mathcal{C}}\left(C, G_1\left(D\right)\right) \cong \mathrm{Hom}_{\mathcal{D}}\left(F\left(C\right), D\right)$$

Adjoints

Are adjoints unique? Can a functor $F : \mathcal{C} \to \mathcal{D}$ have two right adjoints $G_1, G_2 : \mathcal{D} \to \mathcal{C}$? If this happens, then for any $C \in \mathcal{C}_0$ and $D \in \mathcal{D}_0$,

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}\left(C, G_1\left(D\right)\right) &\cong \mathrm{Hom}_{\mathcal{D}}\left(F\left(C\right), D\right) \\ &\cong \mathrm{Hom}_{\mathcal{C}}\left(C, G_2\left(D\right)\right). \end{aligned}$$

Adjoints

Are adjoints unique? Can a functor $F : \mathcal{C} \to \mathcal{D}$ have two right adjoints $G_1, G_2 : \mathcal{D} \to \mathcal{C}$? If this happens, then for any $C \in \mathcal{C}_0$ and $D \in \mathcal{D}_0$,

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(C, G_1(D)) &\cong \mathrm{Hom}_{\mathcal{D}}(F(C), D) \\ &\cong \mathrm{Hom}_{\mathcal{C}}(C, G_2(D)). \end{aligned}$$

Therefore,

$$\mathrm{Hom}_{\mathcal{C}}(C, G_1(D)) \cong \mathrm{Hom}_{\mathcal{C}}(C, G_2(D)).$$

Does this mean G_1 and G_2 are isomorphic functors?

Yoneda Lemma

Yoneda Lemma

Theorem 6 (Yoneda Lemma)

For any functor $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Sets}$ and any $X \in \mathcal{C}_0$, the natural transformations $\mathrm{Hom}_{\mathcal{C}}(-, X) \Rightarrow F$ are in bijection with the elements of the set $F(X)$. In other words,

$$\mathrm{Hom}_{\boxed{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Sets})}}(\mathrm{Hom}_{\mathcal{C}}(-, X), F) \cong \boxed{F(X)},$$

and this isomorphism is natural in both F and X.

$F : \mathcal{C}^{op} \longrightarrow Sets \longleftarrow \mathrm{Hom}_{\mathcal{C}}(-, X) : \mathcal{C}^{op} \longrightarrow Sets$

$\mathrm{Fun}(\mathcal{C}^{op}, Sets)$

$\mathrm{Hom}_{\mathcal{D}}(D_1, D_2)$

Yoneda Lemma

Theorem 6 (Yoneda Lemma)

For any functor $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Sets}$ and any $X \in \mathcal{C}_0$, the natural transformations $\mathrm{Hom}_{\mathcal{C}}(-, X) \Rightarrow F$ are in bijection with the elements of the set $F(X)$. In other words,

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathbf{Sets})}\left(\mathrm{Hom}_{\mathcal{C}}(-,X), F\right) \cong F(X),$$

and this isomorphism is natural in both F and X.

The last line means that we have a natural isomorphism

$$\begin{tikzcd}
\mathcal{C}^{\mathrm{op}} \times \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Sets}) \arrow[rr, bend left=60, "{(X,F) \mapsto \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathbf{Sets})}(\mathrm{Hom}_{\mathcal{C}}(-,X),F)}"] \arrow[rr, bend right=60, "{(X,F) \mapsto F(X)}"'] \arrow[rr, phantom, "\Downarrow \eta"] & & \mathbf{Sets}
\end{tikzcd}$$

Yoneda Lemma

What does it even mean?

What does it even mean?

Definition 7

A functor $F : \mathcal{C} \to \mathcal{D}$ is *fully faithful* if the set-functions

$$F_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$$

are bijective for all $X, Y \in \mathcal{C}_0$.

$$X \xrightarrow{f} Y \qquad\qquad F(X) \xrightarrow{F(f)} F(Y)$$

$$F_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X, Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$$

Yoneda Lemma

$$\begin{tikzcd} F(X) \arrow[r, bend left, "F(f)"] & F(Y) \arrow[l, bend left, "F(g)"] \end{tikzcd}$$

$F(f) \circ F(g) = 1_{F(Y)}$

$F(f \circ g) = \cancel{1}\, F(1_Y)$

$f \circ g = 1_Y$.

What does it even mean?

Definition 7

A functor $F : \mathcal{C} \to \mathcal{D}$ is *fully faithful* if the set-functions

$$F_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$$

are bijective for all $X, Y \in \mathcal{C}_0$.

Theorem 7

Let $F : \mathcal{C} \to \mathcal{D}$ be a fully faithful functor. Then $X \cong Y$ in $\mathcal{C}$ if and only if $F(X) \cong F(Y)$ in $\mathcal{D}$.

$h = F(f)$

$$\begin{tikzcd} F(X) \arrow[r, shift left, "F(f)"] & F(Y) \arrow[l, shift left, "F(g)"] \end{tikzcd}$$

$\mathrm{Hom}_{\mathcal{D}}(F(X), F(Y)) \longleftrightarrow \mathrm{Hom}_{\mathcal{C}}(X, Y)$

Yoneda Lemma

$F : \mathcal{C}^{op} \to Sets$ $\quad \mathrm{Hom}_{Fun(\mathcal{C}^{op}, Sets)}(\mathrm{Hom}_{\mathcal{C}}(-,X), F) \cong F(X)$

Corollary 8

The **Yoneda embedding**

$$\begin{aligned} \mathscr{Y} : \mathcal{C} &\to \boxed{\mathrm{Fun}\left(\mathcal{C}^{\mathrm{op}}, \mathbf{Sets}\right)} \\ X &\mapsto \mathrm{Hom}_{\mathcal{C}}(-, X) \\ (f : X \to Y) &\mapsto \left(f_* : \mathrm{Hom}_{\mathcal{C}}(-, X) \to \mathrm{Hom}_{\mathcal{C}}(-, Y)\right) \end{aligned}$$

is *fully faithful*.

$\mathrm{Hom}_{\mathcal{C}}(X,Y) \cong \mathrm{Hom}_{Fun(\mathcal{C}^{op}, Sets)}(\mathscr{Y}(X), \mathscr{Y}(Y))$

$\cong \mathrm{Hom}_{Fun(\mathcal{C}^{op}, Sets)}(\mathrm{Hom}_{\mathcal{C}}(-,X), \mathrm{Hom}_{\mathcal{C}}(-,Y))$

$\mathrm{Hom}_{\mathcal{C}}(-,Y)(X)$

$F = \mathrm{Hom}_{\mathcal{C}}(-,Y)$

Corollary 8

The **Yoneda embedding**

$$\begin{aligned} \mathscr{Y} : \mathcal{C} &\to \boxed{\mathrm{Fun}\left(\mathcal{C}^{\mathrm{op}}, \mathbf{Sets}\right)} \\ X &\mapsto \mathrm{Hom}\left(-, X\right) \\ (f : X \to Y) &\mapsto \left(f_* : \mathrm{Hom}_{\mathcal{C}}\left(-, X\right) \to \mathrm{Hom}_{\mathcal{C}}\left(-, Y\right)\right) \end{aligned}$$

is *fully faithful*.

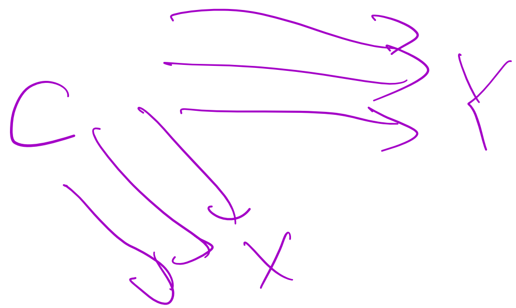
Proof.

Take $F = \mathrm{Hom}_{\mathcal{C}}\left(-, Y\right)$ in Yoneda Lemma. This gives us

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Sets})}\left(\mathrm{Hom}_{\mathcal{C}}(-, X), \mathrm{Hom}_{\mathcal{C}}(-, Y)\right) \cong \mathrm{Hom}_{\mathcal{C}}\left(X, Y\right).$$

Isomorphism in sets is bijection!! ∎

Yoneda Lemma



Corollary 9

In a category $\mathcal{C}$, $X \cong Y$ if and only if $\mathrm{Hom}_{\mathcal{C}}(-, X)$ and $\mathrm{Hom}_{\mathcal{C}}(-, Y)$ are naturally isomorhic functors.

$$X \cong Y \iff y(X) \cong y(Y)$$

$$\iff \mathrm{Hom}_{\mathcal{C}}(-, X) \cong \mathrm{Hom}_{\mathcal{C}}(-, Y) \text{ in } \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Sets})$$

$$\iff \mathrm{Hom}_{\mathcal{C}}(-, X) \text{ is nat isom } \mathrm{Hom}_{\mathcal{C}}(-, Y)$$

Yoneda Lemma

Corollary 9

In a category $\mathcal{C}$, $X \cong Y$ if and only if $\mathrm{Hom}_{\mathcal{C}}(-, X)$ and $\mathrm{Hom}_{\mathcal{C}}(-, Y)$ are naturally isomorhic functors.

Proof.

Because Yoneda embedding $X \mapsto \mathrm{Hom}_{\mathcal{C}}(-, X)$ is fully faithful!!! ■

Yoneda Lemma

Corollary 9

In a category $\mathcal{C}$, $X \cong Y$ if and only if $\mathrm{Hom}_{\mathcal{C}}(-, X)$ and $\mathrm{Hom}_{\mathcal{C}}(-, Y)$ are naturally isomorhic functors.

Proof.

Because Yoneda embedding $X \mapsto \mathrm{Hom}_{\mathcal{C}}(-, X)$ is fully faithful!!! ■

We usually use this variant of Yoneda lemma to prove isomorphisms.

Yoneda Lemma Summarized

Interchanging $\mathcal{C}$ and $\mathcal{C}^{\text{op}}$ throughout, we get the following result:

Corollary 10

In a category $\mathcal{C}$, $X \cong Y$ if and only if $\text{Hom}_{\mathcal{C}}(X, -)$ and $\text{Hom}_{\mathcal{C}}(Y, -)$ are naturally isomorhic functors.

Yoneda Lemma Summarized

Interchanging $\mathcal{C}$ and $\mathcal{C}^{\text{op}}$ throughout, we get the following result:

Corollary 10

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∎

Interchanging $\mathcal{C}$ and $\mathcal{C}^{\mathrm{op}}$ throughout, we get the following result:

Corollary 10

In a category $\mathcal{C}$, $X \cong Y$ if and only if $\mathrm{Hom}_{\mathcal{C}}(X, -)$ and $\mathrm{Hom}_{\mathcal{C}}(Y, -)$ are naturally isomorhic functors.

Proof.

$$X \cong Y \text{ in } \mathcal{C} \iff X \cong Y \text{ in } \mathcal{C}^{\mathrm{op}}$$

$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r, bend left, leftarrow, "f^{op}"] & Y \arrow[l, bend left, leftarrow, "g^{op}"] \end{tikzcd}$$

■

Yoneda Lemma Summarized

Interchanging $\mathcal{C}$ and $\mathcal{C}^{\text{op}}$ throughout, we get the following result:

Corollary 10

In a category $\mathcal{C}$, $X \cong Y$ if and only if $\text{Hom}_{\mathcal{C}}(X, -)$ and $\text{Hom}_{\mathcal{C}}(Y, -)$ are naturally isomorhic functors.

Proof.

$$\begin{aligned} X \cong Y \text{ in } \mathcal{C} &\iff X \cong Y \text{ in } \mathcal{C}^{\text{op}} \\ &\iff \underline{\text{Hom}_{\mathcal{C}^{\text{op}}}(-, X)} \cong \underline{\text{Hom}_{\mathcal{C}^{\text{op}}}(-, Y)} \end{aligned}$$

■

Yoneda Lemma Summarized

Interchanging $\mathcal{C}$ and $\mathcal{C}^{\text{op}}$ throughout, we get the following result:

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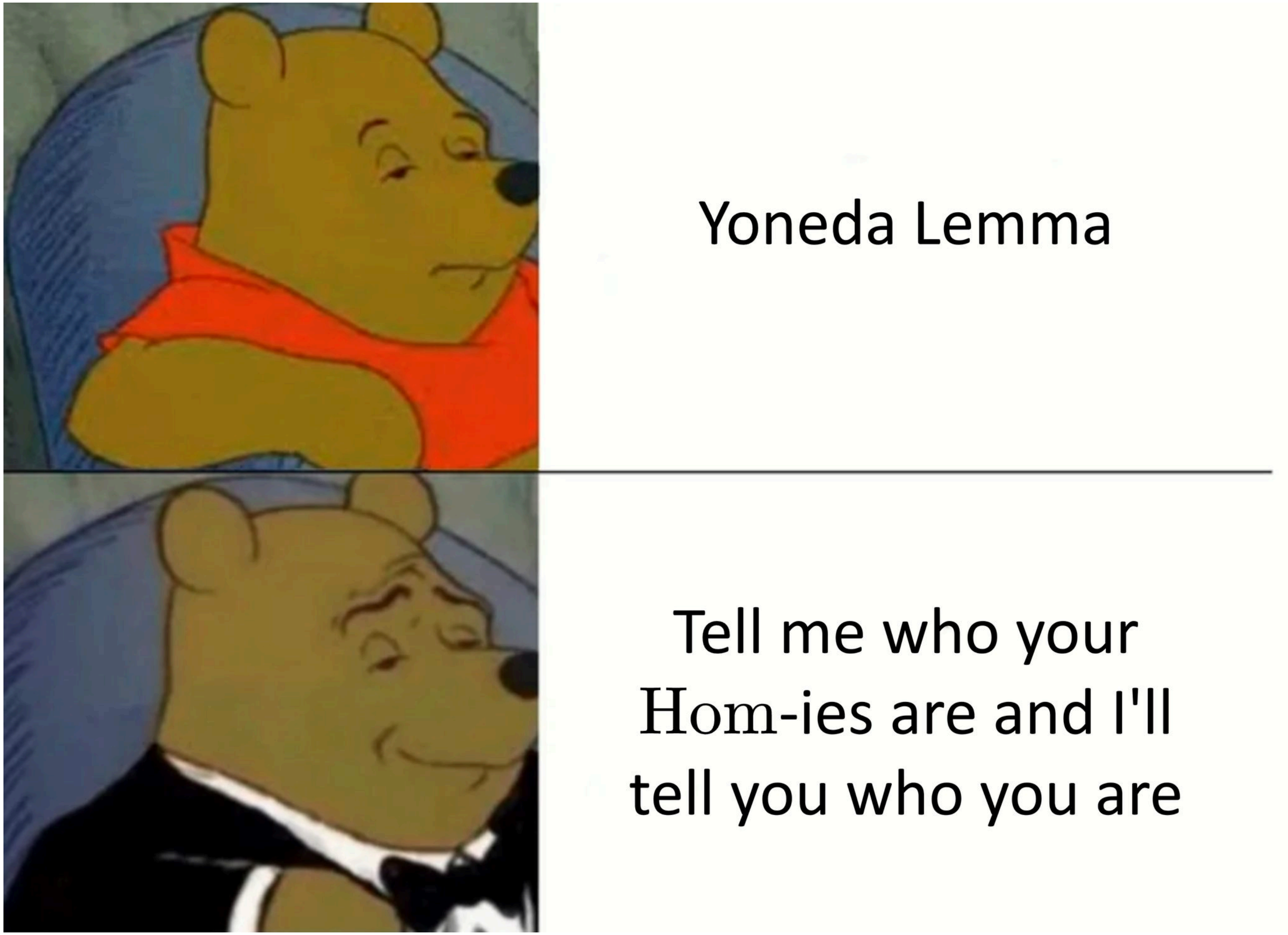
$\text{Hom}_{\mathcal{C}}(-, X)$ $\text{Hom}_{\mathcal{C}}(-, Y)$

Proof.

$$\begin{aligned} X \cong Y \text{ in } \mathcal{C} &\iff X \cong Y \text{ in } \mathcal{C}^{\text{op}} \\ &\iff \text{Hom}_{\mathcal{C}^{\text{op}}}(-, X) \cong \text{Hom}_{\mathcal{C}^{\text{op}}}(-, Y) \\ &\iff \text{Hom}_{\mathcal{C}}(X, -) \cong \text{Hom}_{\mathcal{C}}(Y, -). \end{aligned}$$

■

Yoneda Lemma Summarized



References

1. *Category Theory*, by Steve Awodey
2. *Category Theory in Context*, by Emily Riehl
3. *Basic Category Theory*, by Tom Leinster
4. *Categories for the Working Mathematician*, by Saunders Mac Lane
5. Math3ma blog: `https://www.math3ma.com/blog/the-yoneda-lemma`
6. Ncatlab: `https://ncatlab.org/nlab/show/Yoneda+lemma`

Thank you for joining!

The slides are available in my webpage
https://atonurc.github.io/assets/catrep_talk_1.pdf