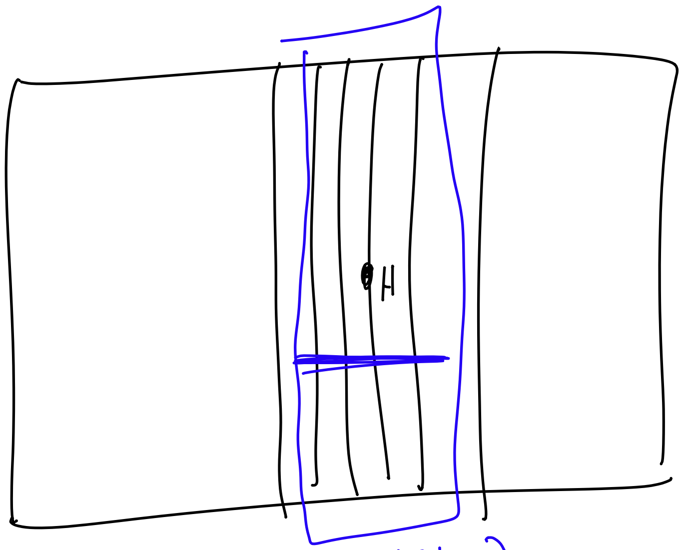
Lie Grps #5

$G/H \longrightarrow$ subgroup
regular submanifold

$\pi : G \longrightarrow \boxed{G/H}$ submersion.



$\pi(\mathcal{U}_g) \longrightarrow S_g \longrightarrow \mathbb{R}^{n-k}$

$$\begin{tikzcd} M \arrow[r, "\pi"] \arrow[dr, "F\circ\pi"'] & N \arrow[d, "F"] \\ & P \end{tikzcd}$$

F smooth $\Leftrightarrow$ $F\circ\pi$ smooth

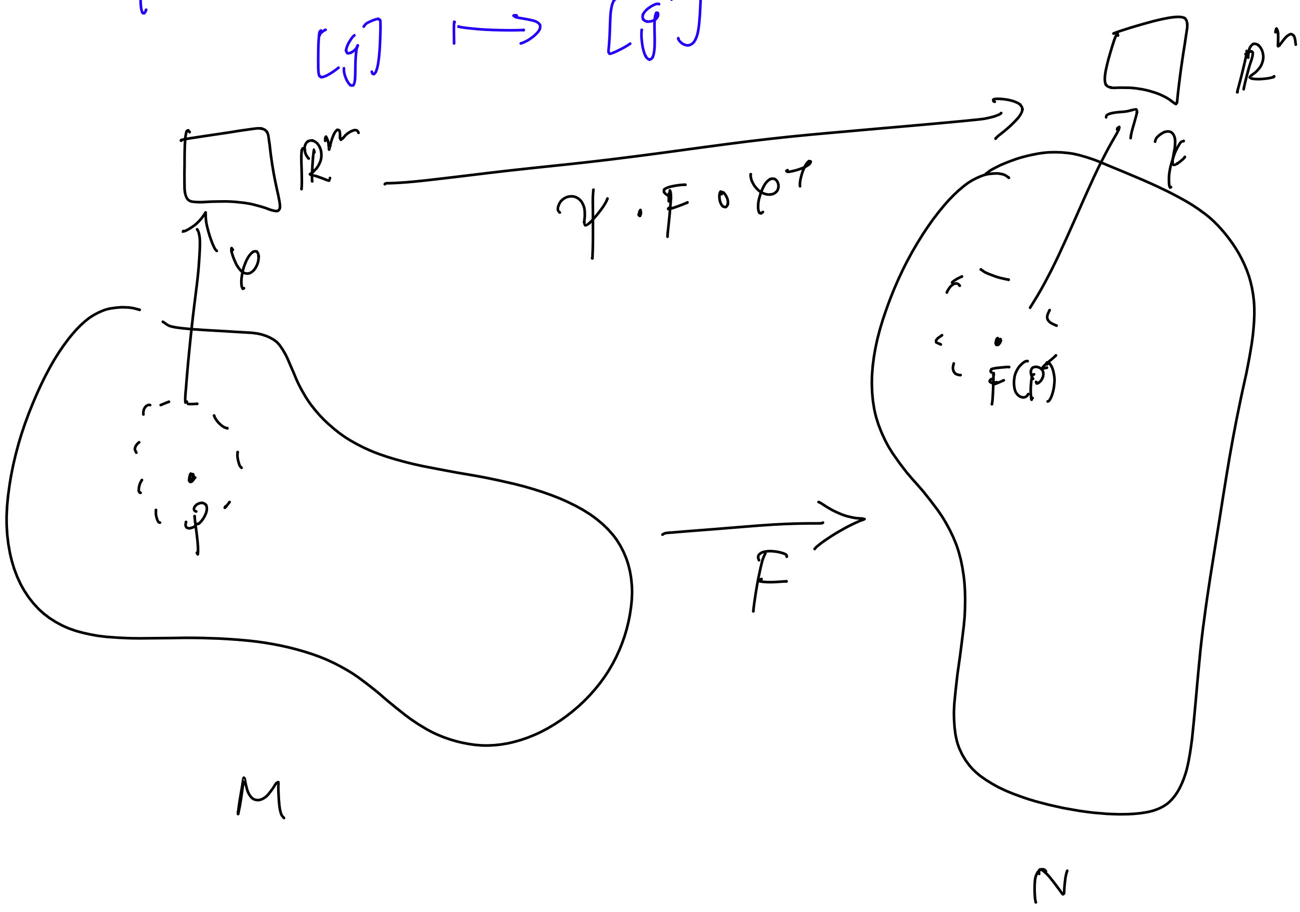
What if H normal Lie subgroup.

$$\left.\begin{array}{ll} G/H & \text{group} \\ G/H & \text{manifold} \end{array}\right\} \quad G/H \text{ Lie grp?}$$

$$\boxed{\text{exercise: } \quad m : G \times G \to G \text{ smooth} \Rightarrow i : G \to G \text{ smooth.}}$$

$$\tilde{m} : G/H \times G/H \to G/H$$
$$([g], [h]) \mapsto [gh]$$

$$\tilde{i} : G/H \to G/H$$
$$[g] \mapsto [g^{-1}]$$



$\pi : G \longrightarrow G/H$ submersion

$\pi \times \pi : G \times G \longrightarrow G/H \times G/H$.

$$
\begin{tikzcd}
G \times G \arrow[r, "\pi \times \pi"] \arrow[dr, "m' = \boxed{\pi \circ m}"'] & G/H \times G/H \arrow[d, "\tilde{m}"] \\
 & \boxed{G/H}
\end{tikzcd}
$$

✓

$m : G \times G \longrightarrow G \longrightarrow G/H$

$(\tilde{m} \circ (\pi \times \pi))(g,h) = \tilde{m}([g],[h])$

$= [gh]$

$(\pi \circ m)(g,h) = \pi(gh) = [gh]$

$$
\begin{tikzcd}
g \arrow[ddrr, bend right] & G \arrow[r, "\pi"] \arrow[dr, "\pi \circ i"'] & G/H \arrow[d, "\tilde{i}"] & {[g]} \arrow[dd, bend left] \\
 & & G/H & \\
 & & \boxed{[g^{-1}] = [g]^{-1}} &
\end{tikzcd}
$$

$\therefore$ G/H is a Lie group. ∎

First isomorphism theorem

$f: G \to H$ Lie group homomorphism

$$\boxed{\operatorname{im} f \cong G/\operatorname{Ker} f.}$$

does it hold?

$\operatorname{Ker} f = f^{-1}(\{e\}) \Rightarrow \boxed{\text{closed}}$ subgroup is a Lie subgroup

Cartan's closed subgroup theorem

Let H be a closed subgroup of a Lie group G.
Then $\underbrace{H \text{ is a Lie subgroup}}$.

H is a regular/embedded submanifold.

Proof: (later).

First isom theorem: ~~#2~~:

$f: G \to H$ is a smooth homomorphism. $\boxed{\text{as Lie groups}}$

Then $\operatorname{im} f \cong G/\operatorname{Ker} f$

Proof:

$$\begin{tikzcd} G \arrow[r, "\pi"] \arrow[dr, "f"'] & \boxed{G/\operatorname{Ker} f} \arrow[d, "\tilde{f}"] \\ & H \end{tikzcd} \qquad \begin{array}{c} [g] \\ \downarrow \\ f(g) \end{array}$$

$\Rightarrow \tilde{f}$ is smooth.

$\tilde{f}: \underline{G/\operatorname{Ker} f} \longrightarrow H$ injective, smooth.

$\operatorname{im} \tilde{f} = \operatorname{im} f$

$F: M \to N$ injective smooth map

$F(M)$ has a unique manifold structure

s.t. F is a diffeo $M \to F(M)$.

Lee's Intro to Smooth Manifold.

H is a closed subgroup, then there is a unique manifold structure on G/H s.t. $\pi : G \longrightarrow G/H$ is a submersion

Proof: (after Lie alg & orbit/stabilizer)

Some algebraic topology

$$\tilde{G} \xrightarrow{\pi} G$$

$\pi_1(G) \cong \text{Ker}\,\pi$ normal discrete contained in $Z(\tilde{G})$

abelian.

Goal: Understand relation between $\pi_1(G)$, $\pi_1(H)$, $\pi_1(G/H)$.

Question: Is it true $(G/H) \times H \cong G$?

$\boxed{\text{NO}}$

$$0 \to A \xrightarrow{i} B \xrightarrow{\pi} C \to 0 \quad \text{SES}$$

this SES splits if $B = A \oplus C$.

$$0 \to H \xrightarrow{i} G \xrightarrow{\pi} G/H \to 0$$

$G \cong H \oplus G/H$ X

$G = \mathbb{Z}$, $H = 2\mathbb{Z}$

$G/H = \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2$

$(G/H) \times H = \mathbb{Z}_2 \times 2\mathbb{Z}$ ~ $\mathbb{Z}$ ~

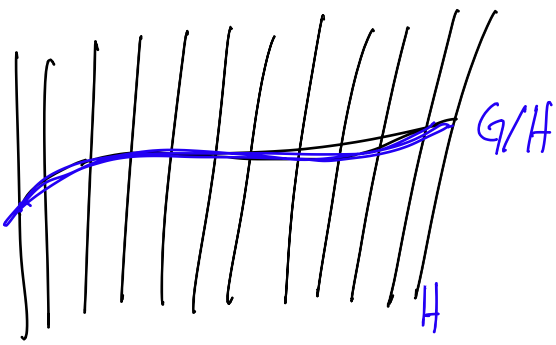
$(0, 2n)$

$(1, 2n)$

$(1, 0)$

For lie groups,
G "locally looks like" $G/H \times H$.

$\boxed{\text{Fiber bundle}}$



Fiber bundle

E, M, F manifolds.

A smooth fiber bundle over M with model fiber F is a smooth surjection $\pi: E \to M$, s.t. $\forall p \in M$, $\exists$ nbhd $U \ni p$ and a diffeo $\Phi: \pi^{-1}(U) \to U \times F$. and

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[r, "\Phi"] \arrow[d, "\pi"'] & U \times F \arrow[dl, "p_1"] \\
U &
\end{tikzcd}$$

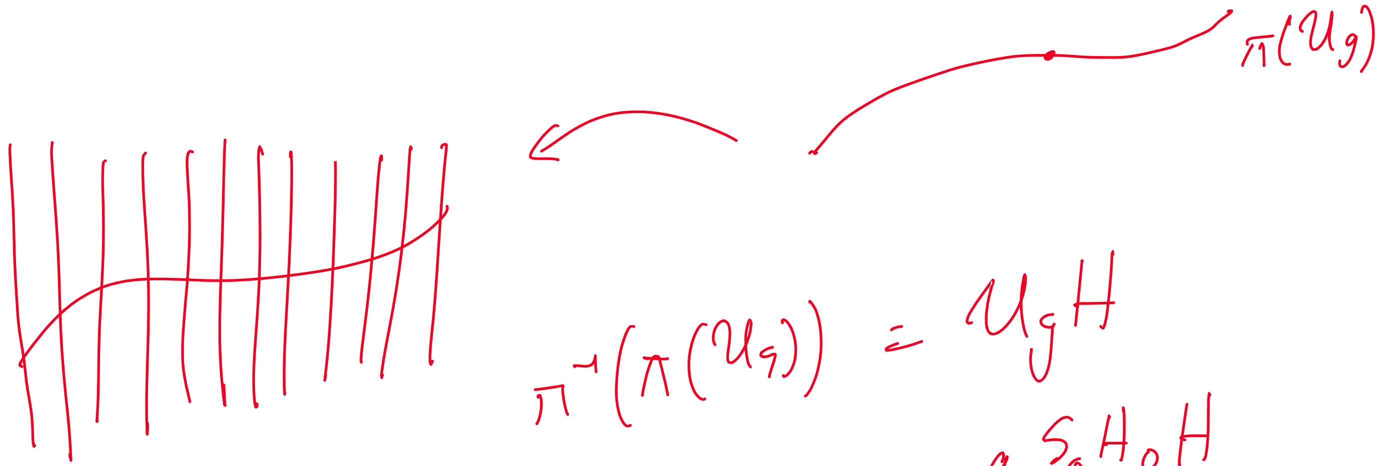
For covering maps, $\boxed{F \text{ is discrete.}}$

Proposition: $\pi: G \to G/H$ fiber bundle.

Pf: Take $\bar{g} \in G/H$, and nbhd $\pi(U_g)$.

Diffeo: $\boxed{\Phi: \pi^{-1}(\pi(U_g)) \to \pi(U_g) \times H.}$

$$\begin{tikzcd}
\pi^{-1}(\pi(\mathcal{U}_g)) \arrow[r, "\Phi"] \arrow[d, "\pi"'] & \pi(\mathcal{U}_g)\times H \arrow[dl, "p_1"] \\
\pi(\mathcal{U}_g) &
\end{tikzcd}$$



$$\pi^{-1}(\pi(\mathcal{U}_g)) = \mathcal{U}_g H$$
$$= g\, S_0 H_0 H$$
$$= g\, S_0 H$$

$$\begin{tikzcd}
g S_0 H \arrow[r, "\Phi"] \arrow[d] & \cancel{\pi(\mathcal{U}_g)}\ \pi(gS_0) \times H \arrow[dl] \\
\cancel{\pi(\mathcal{U}_g)}\ \pi(gS_0) &
\end{tikzcd}$$

$$\Phi(gsh) = ([gs], h) \qquad \text{diffeo.}$$

$$g S_0 H \xrightarrow{\sigma} g S_0 \times H \xrightarrow{f_g^{-1} \times 1_H} \pi(\mathcal{U}_g) \times H$$

$\pi(\mathcal{U}_g)$

$= g S_0 H_0 H$

$\psi : g S_0 \times H_0 \longrightarrow g \mathcal{U}_0$

Lee

σ is a local diffeo, and it's globally invertible.

$\Rightarrow \sigma$ is globally diffeo.

G, H connected, $G \longrightarrow G/H$ is a fiber bundle.

$$\pi_1(H) \xrightarrow{\iota_*} \pi_1(G) \xrightarrow{\pi_*} \pi_1(G/H) \rightarrow \{1\}$$

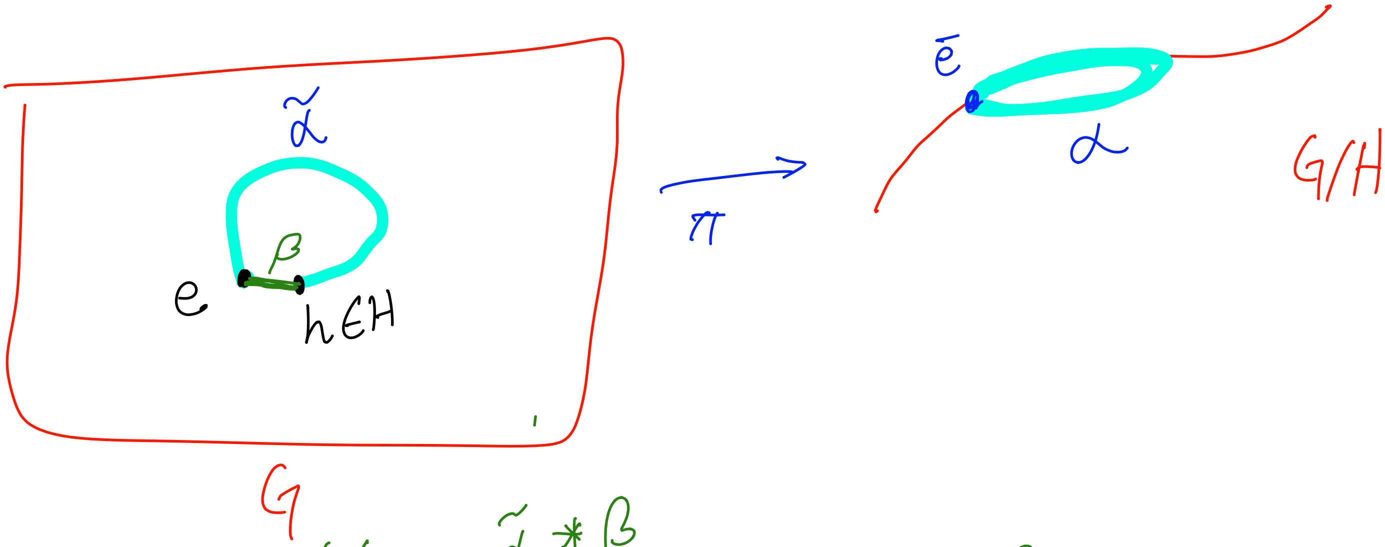
is exact.

$$\pi_* : \pi_1(G) \longrightarrow \pi_1(G/H)$$

surjective.

Take $[\alpha] \in \pi_1(G/H)$
$\alpha : [0,1] \longrightarrow G/H$, s.t. $\alpha(0) = \alpha(1) = \bar{e}$.

Have to show that $[\alpha] = \pi_*[\gamma] = [\pi \circ \gamma]$.



Take $\gamma = \tilde{\alpha} * \beta$

$$\pi_*[\gamma] = [\pi \circ \gamma] = [\pi \circ (\tilde{\alpha} * \beta)]$$
$$= [\pi \circ \tilde{\alpha}] * [\pi \circ \beta]$$
$$= [\alpha] * [\mathrm{cost}_{\bar{e}}] = [\alpha].$$

$\Rightarrow \pi_*$ is surjective.

$$\pi_1(H) \xrightarrow{i_*} \boxed{\pi_1(G) \xrightarrow{\pi_*} \pi_1(G/H)}$$

$$\operatorname{im} i_* = \ker \pi_* .$$

What is $\pi \circ i$

$$\pi_* \circ i_* = \text{const}_e$$

$$\Rightarrow \boxed{\operatorname{im} i_* \subseteq \ker \pi_*}$$

Suppose $[\gamma] \in \ker \pi_*$.

$\Rightarrow$ $\cancel{\gamma}$ $\pi \circ \gamma$ is nullhomotopic

Homotopy $F : [0,1]^2 \to G/H$

$$F(-,0) = \pi \circ \gamma$$

$$F(-,1) = \text{const}_e$$

Homotopy lifting property.

$$\begin{tikzcd}
G \arrow[r, "\pi"] & G/H \\
{[0,1]^2} \arrow[u, dotted, "\tilde{F}"] \arrow[ur, "F"'] &
\end{tikzcd}$$

$$\tilde{F}(-,0) = \gamma \qquad \tilde{F}(-,1) = \alpha$$

$\pi \circ \tilde{F}(-,1) = \pi \circ \alpha$

$\tilde{e} = F(-,1) = \pi \circ \alpha$

α is a loop in H

$[\gamma] = [\alpha] = [\iota \circ \alpha] = i_*[\alpha]$

$\Rightarrow [\gamma] \in \operatorname{im} i_*$

$\Rightarrow \operatorname{im} i_* = \operatorname{Ker} \pi_*$

$\blacksquare$

Next week: Left-invariant vect. field

M, $\mathfrak{X}(M)$ $\longrightarrow$ $X : C^\infty(M) \to C^\infty(M)$

$X(fg) = (Xf)g + f(Xg)$

$\boxed{L(G) \cong T_e G}$

$\boxed{\exp : T_e G \longrightarrow G}$