Lie groups #4

H is a Lie subgroup

$\Rightarrow H$ is closed.

$\boxed{G/H}$

Quotient topology

$p : X \to Y$ is a q map

$V \subseteq Y$ open $\Leftrightarrow p^{-1}(V) \subseteq X$ is open.

then give set A, and surjective map

$p : X \to A$,

$X, \sim,$

$\pi : X \to X/\sim$ is a q map.

unique topolgy of $X/\sim$

s.t. π is q

suppose p is q map

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[dr, "f \circ p"'] & Y \arrow[d, "f"] \\ & Z \end{tikzcd}$$

f is continuous $\iff$ $f \circ p$ is continuous.

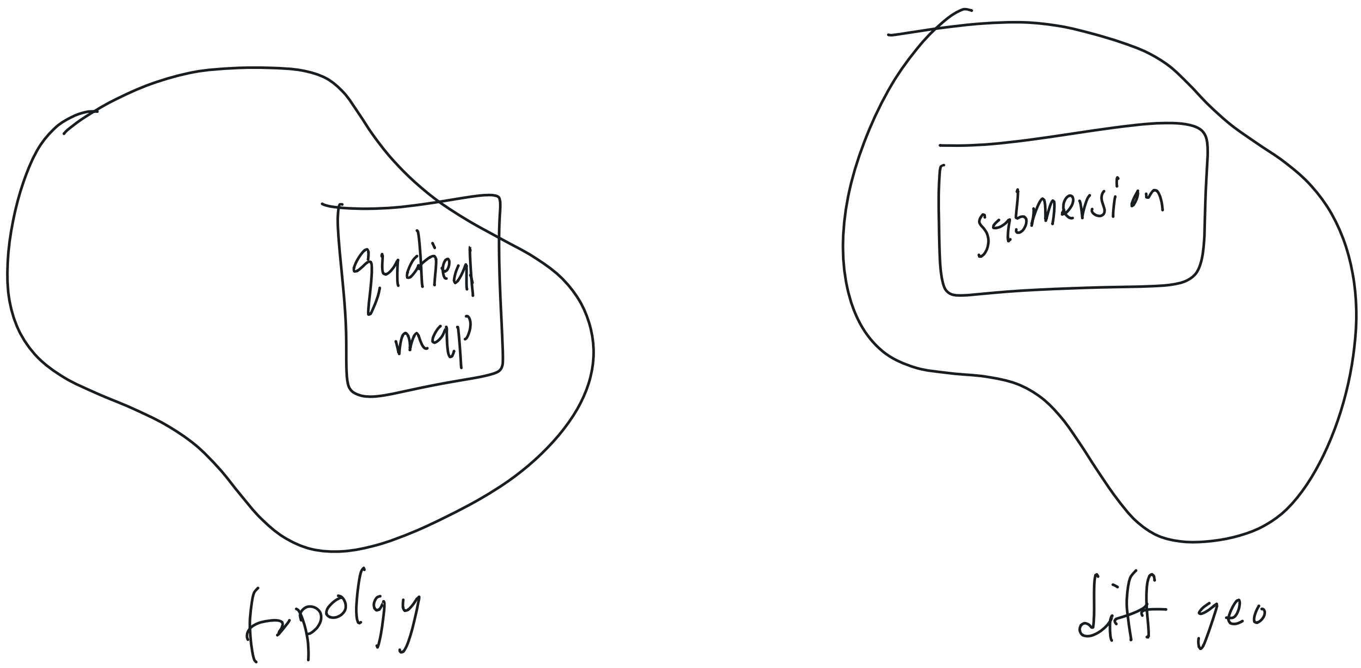
$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[dr, "f \circ \pi"'] & X/\sim \arrow[d, "f"] \\ & Y \end{tikzcd}$$

Manifolds

$0 \sim 1$ on $\mathbb{R}$

suppose $R \subseteq M \times M$ is a eq relation.
then there is at most one manifold structure on $\underline{M/R}$ s.t. $\pi : M \to M/R$ is a submersion.

"submersions are manifold-equivalent of quotient map."



$\pi : M \longrightarrow N$ is a smooth surjective $\boxed{\text{submersion}}$.

$(d\pi)_p : \underline{T_pM} \to \underline{T_pN}$ is surjective $\forall p$.

$\dim M \geq \dim N$.

Given any $p \in M$, $\boxed{\text{there exists charts}}$ (U, φ) centered at p

(V, ψ) ″ ″ $\pi(p)$

s.t. $\boxed{\psi \circ \pi \circ \varphi^{-1}}$ is a projection.

$(x^1, \dots, x^n) = (x^1, \dots, x^m)$.

$n \geq m$

dim M dim N.

dom $\varphi(\mathcal{U}) \subseteq \mathbb{R}^n$

cod: $\psi(\pi(\mathcal{U}) \cap V) \subseteq \mathbb{R}^m$

<u>Cool Property</u>: π is smooth surj. submersion

so

$$\begin{tikzcd} M \arrow[r, "\pi"] \arrow[dr, "F \circ \pi"'] & N \arrow[d, "F"] \\ & P \end{tikzcd}$$

F is smooth $\iff$ $F \circ \pi$ is smooth.

Proof: Lee or $\underbrace{\text{my notes}}_{}$.

atonurc.github.io/liegrp/quotient_lie.pdf

Why is there exactly one manifold structure on M/R?

$\cancel{\mathbb{R}, x}$ $\qquad$ $\cancel{\mathbb{R}, x^3}$

AFTSOC, $\quad (M/R)_1 \quad (M/R)_2$

$\pi_1 : M \to (M/R)_1$, $\quad \pi_2 : M \to (M/R)_2$

submersions.

$$\begin{tikzcd}
M \arrow[r, "\pi_1"] \arrow[dr, "\pi_2"'] \arrow[dr, phantom, "\checkmark", shift left=3ex] & (M/R)_1 \arrow[d, "id"] \\
& (M/R)_2
\end{tikzcd}$$

diffeo $\left\{ \begin{array}{l} \Rightarrow id : (M/R)_1 \longrightarrow (M/R)_2 \quad \text{smooth} \\ \text{similarly } id : (M/R)_2 \longrightarrow (M/R)_1 \quad \sim \end{array} \right.$

Familiar Setting

G a lie grp

H " " subgrp

$\boxed{\text{goal: } G/H \text{ is a manifold}}$

= all left ~~coest~~ cosets gH

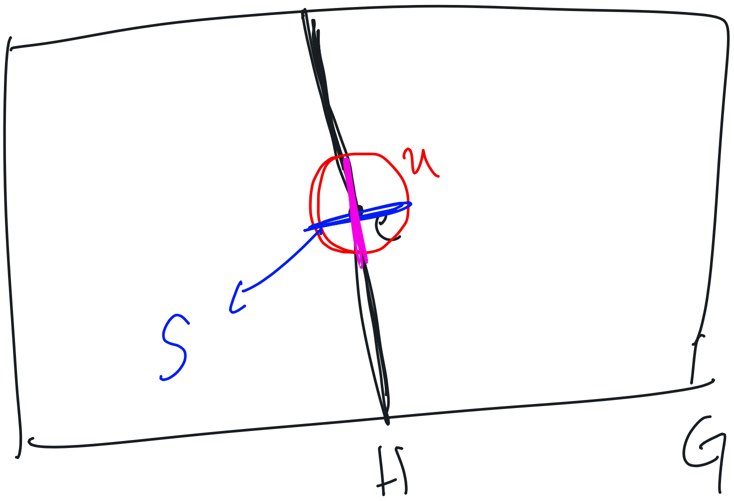
Step 1: G/H is locally euclidean at $\bar{e}$.

$\boxed{\mathcal{U} \ni e \text{ then } \mathcal{U} \text{ generates } G}$

L_g is a diffeo

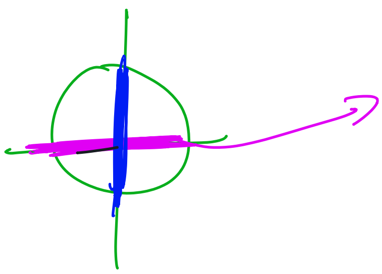
$(\mathcal{U}, \varphi)$ is the chart

$\varphi(\mathcal{U} \cap H) = \varphi(\mathcal{U}) \cap (\mathbb{R}^k \times \{\bar{0}\})$



$\dim H = k$

$\dim G = n$



$\varphi(\mathcal{U} \cap H)$

$S = \{ g \in \mathcal{U} \mid \varphi(g) = (0, 0, \dots, 0, r^{k+1}, \dots, r^n) \}$

$\to$ manifold transversal to H.

$(S) \cap (\mathcal{U} \cap H)$

Define $\psi : S \times H \to G$
$(s, h) \mapsto sh$

$(d\psi)_{(e,e)} : \boxed{T_e S \times T_e H} \to \underline{T_e G}$ $\quad \to T_{(e,e)}(S \times H)$

$\left.\frac{\partial}{\partial x^i}\right|_e$

$i \leq k$

$i \geq k$

exercise:

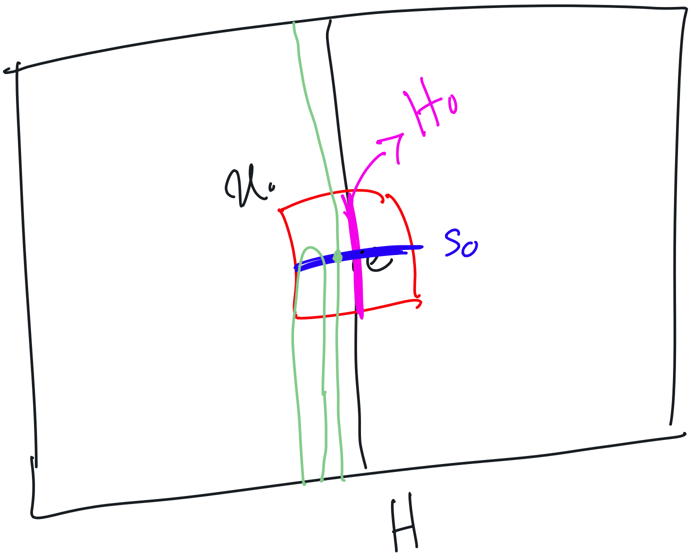
$T_{(m,n)}(M \times N) \cong T_m M \times T_n N$ (natural isomorphism)

invoke inverse function theorem

$e \in S_0 \subseteq S$
$e \in H_0 \subseteq H$
$e \in \mathcal{U}_0 \subseteq G$

s.t. $\psi|_{S_0 \times H_0} : S_0 \times H_0 \to \mathcal{U}_0$ is a diffeomorphism

$\boxed{\text{write } \psi \text{ to mean } \psi|_{S_0 \times H_0}}$



$$S_0 \times H_0 \longrightarrow \mathcal{U}_0$$

$$\Downarrow$$

$$s_1 h_1 \sim s_2 h_2$$

$$\Leftrightarrow s_1 = s_2$$

$$\Leftrightarrow \boxed{s_1^{-1} s_2 \in H}$$

$$\Downarrow$$

$$s_1^{-1} s_2 = e.$$

$$\Uparrow$$

$$\boxed{s_1^{-1} s_2 \in S_0}$$

We can choose S_0 to be as small as we want.

replace S_0 by $S_0 \cap (S_0^{-1} S_0)$

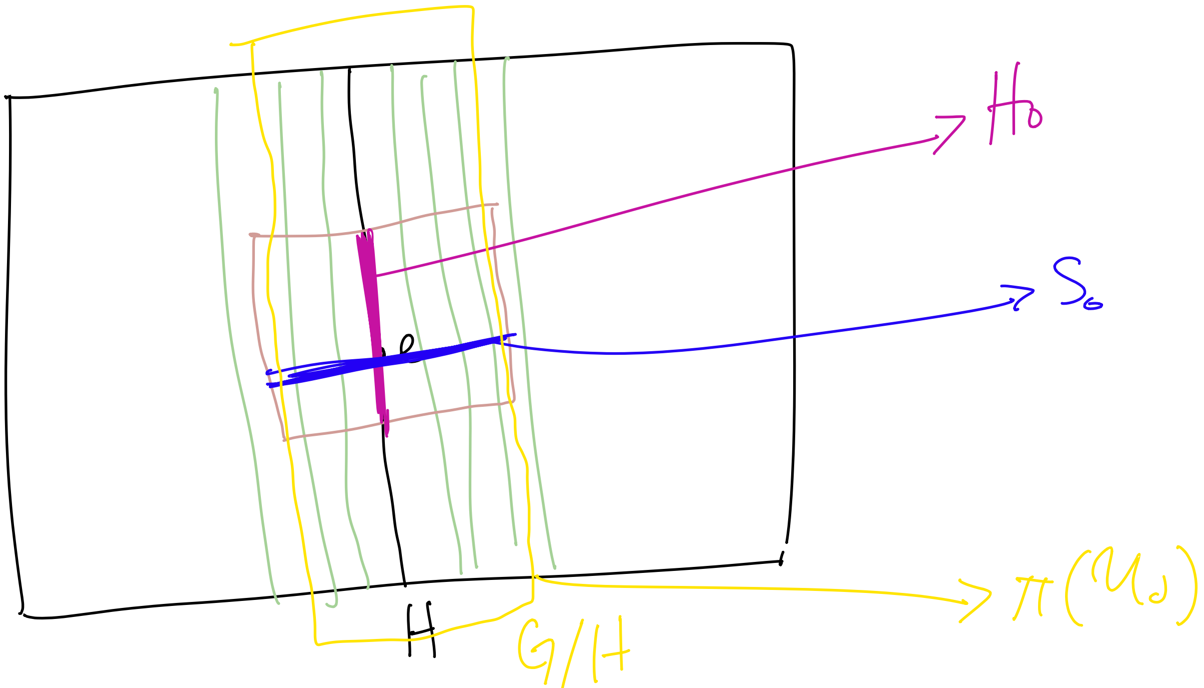
$$\psi : S_0 \times H_0 \longrightarrow \mathcal{U}_0 \quad \text{diffeo.}$$

<u>Claim:</u> $\pi(\mathcal{U}_0) \subseteq G/H$ and S_0 are homeomorphic.

$$f : \pi(\mathcal{U}_0) \longrightarrow S_0$$
$$[sh] \longmapsto s$$

exercise: f is homeo.

$$\pi(\mathcal{U}_0) \xrightarrow[\cong]{f} S_0 \xrightarrow[\cong]{} \text{some open set } \mathbb{R}^{n-k}.$$



$$\bar{g} \longrightarrow g\,\mathcal{U}_0$$

$$g\,S_0$$

$$\pi(\mathcal{U}_g) \xrightarrow[\cong]{f_g} g\,S_0 \xrightarrow{L_{g^{-1}}} S_0 \xrightarrow{\xi} V \subseteq \mathbb{R}^{n-k}$$

compatibility: left as an exercise.

(NOT) trivial.

Furthermore $\pi : G \longrightarrow G/H$ is a submersion.

$$\begin{array}{ccc} G & \longrightarrow & G/H \\ \downarrow & & \downarrow \\ \mathcal{U}_g & & \pi(\mathcal{U}_g) \end{array}$$

Local section lemma:

let $\pi : M \to N$ be smooth.

π is submersion

$\Leftrightarrow$ $\forall p \in M$, $q = \pi(p)$, there is a nbhd $V \ni q$ and smooth $\sigma : V \to M$ $\sigma(q) = p$, and $\pi \circ \sigma = id_V$.

$$\pi_1(H) \longrightarrow \pi_1(G) \longrightarrow \pi_1(G/H) \longrightarrow \{1\}$$

exact seq.