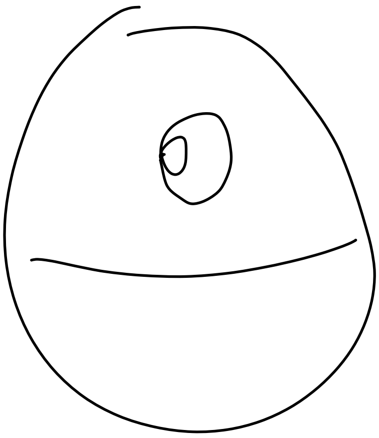
Lie Group #2

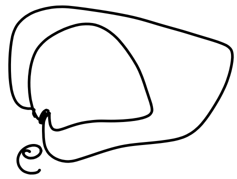
Simply connected $\Rightarrow$ $\pi_1(G) = \cancel{0}\ \{i\}$

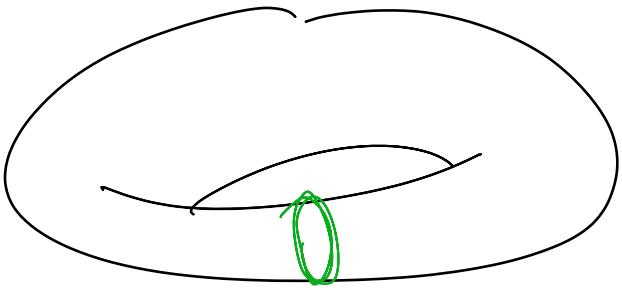
$\cancel{\pi_1(G, g)}$

Homotopy class of loops based at e.



S^2





$\pi_1(T^2) \neq 0$

$\mathbb{Z}^2$

$\tilde{X} \xrightarrow{\pi} X$ path connected

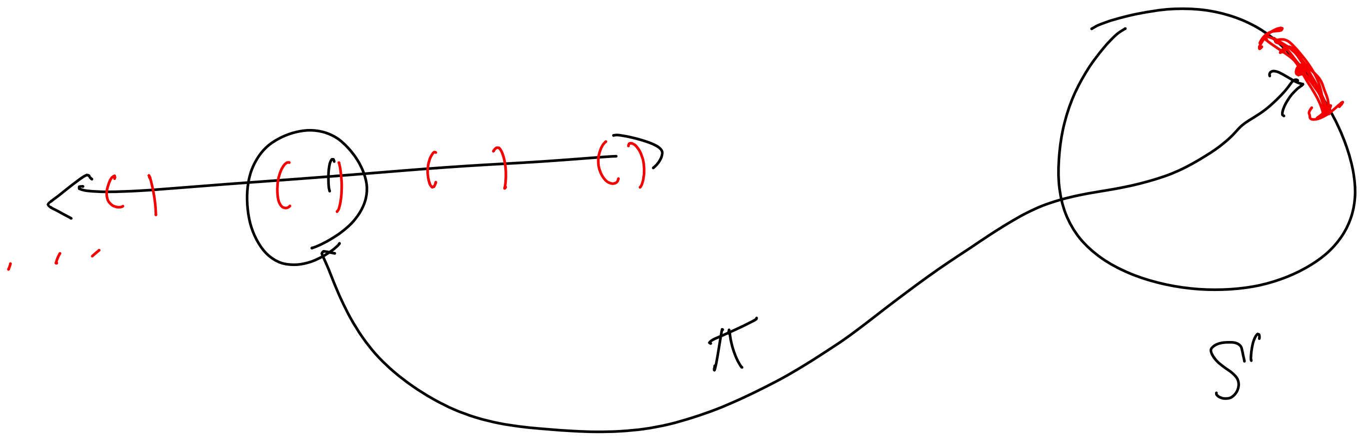
universal cover / simply connected cover

$\tilde{X}$

$$\boxed{\begin{array}{c} \mathbb{R} \longrightarrow S^1 \\ \pi(t) = e^{2\pi i t} \end{array}}$$

$\operatorname{Ker}\pi \cong \mathbb{Z}$

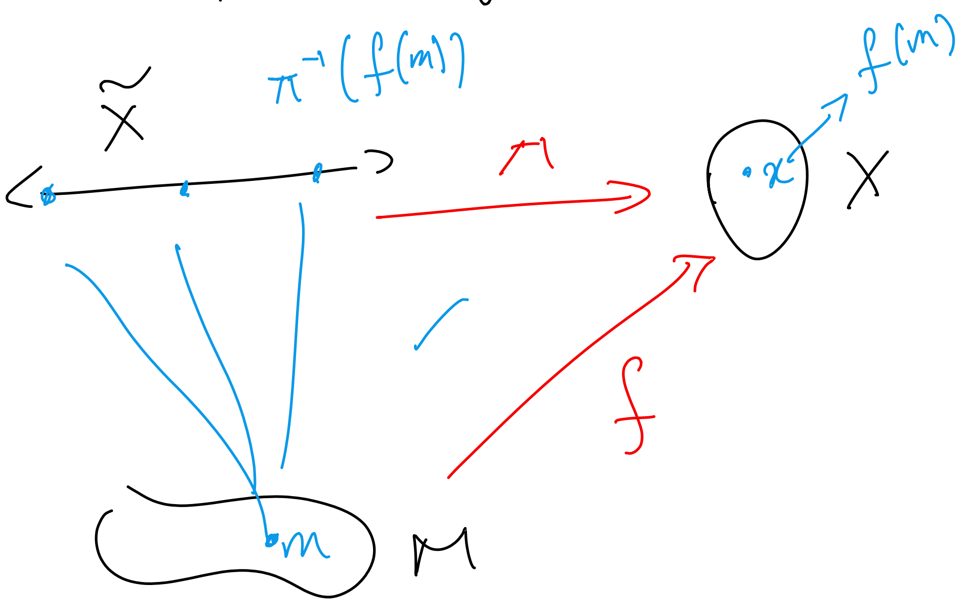
$\pi_1(S^1) \cong \mathbb{Z}$

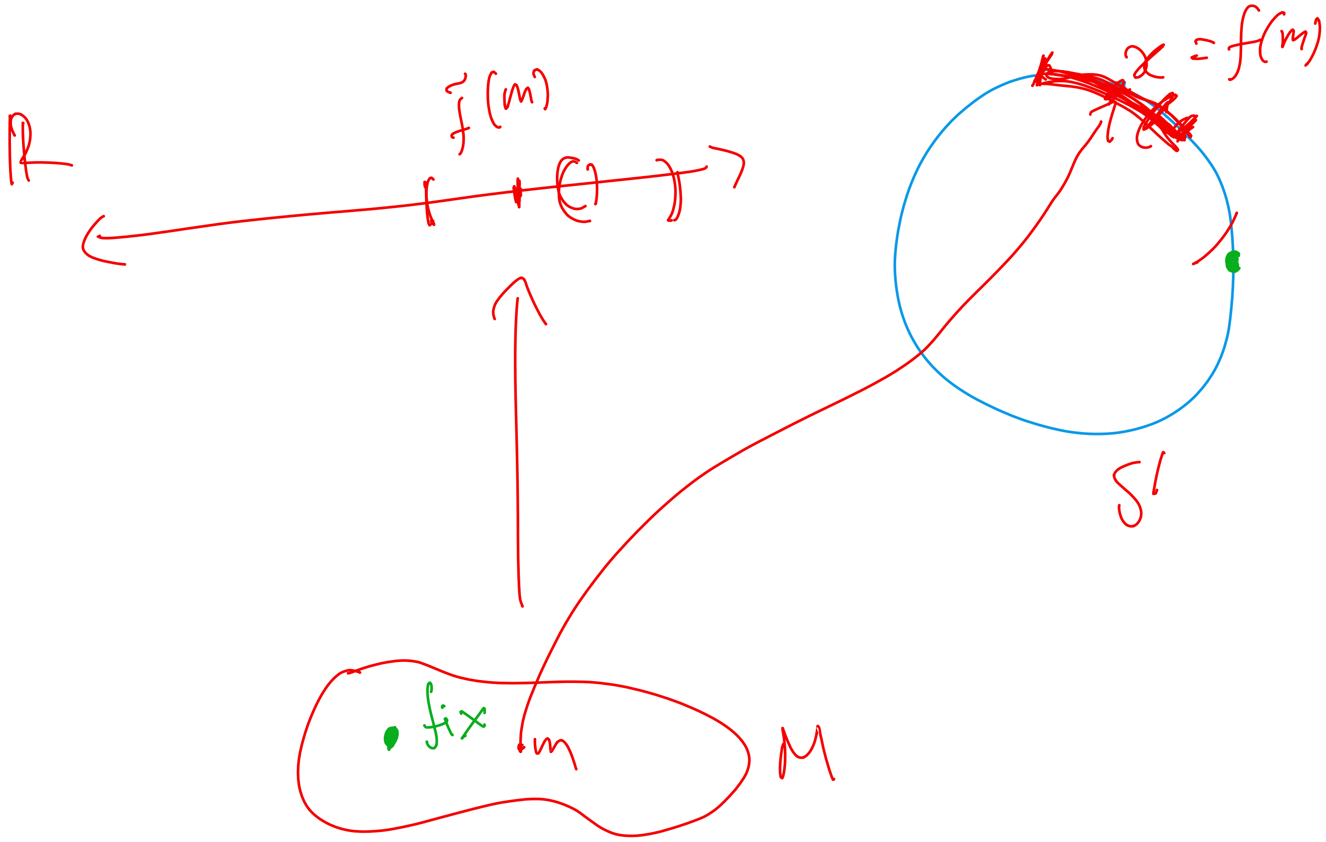


$\pi : \tilde{X} \longrightarrow X$ has a universal property

$$\begin{tikzcd} \tilde{X} \arrow[r, "\pi"] & X \\ M \arrow[u, dashed, "\exists!\, \tilde{f}"] \arrow[ur, "\forall f"'] \arrow[ur, phantom, "\checkmark"] & \end{tikzcd}$$

M simply connected





Goal: G is a (connected) Lie group.

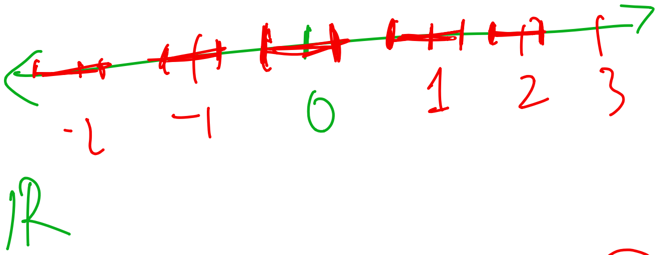
$\tilde G$ is also a Lie group.

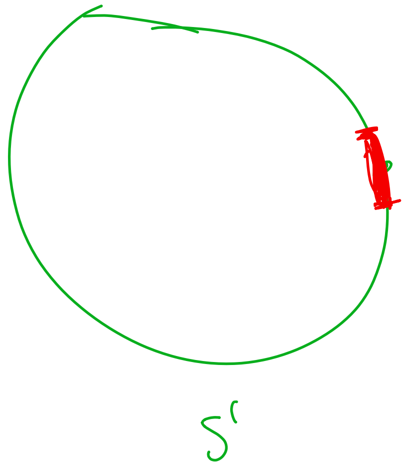
$\underbrace{\ker(\pi : \tilde G \to G)}_{\text{abelian.}} \cong \pi_1(G)$

Lemma: Suppose H is a discrete normal subgroup of a CLG G. Then $H \subseteq Z(G)$.

Relevant because it shows this [illegible] $\ker(\pi : \tilde G \to G)$.

$$\boxed{\ker \pi \subseteq Z(\tilde G)}$$





$\pi^{-1}(U) = \bigsqcup \ldots V_\alpha$

V_α is open $\tilde{G}$

$V_\alpha \cap \operatorname{Ker} \pi$ is singleton.

$\Rightarrow \operatorname{Ker} \pi$ is ~~disjoint~~ discrete.

G is connected.

H is discrete normal $\Rightarrow H \subseteq Z(G)$

$\{g' : gg' = g'g \ \forall g\}$

$\Rightarrow g' = gg'g^{-1}$

Proof: WTS

$\boxed{\text{Given any } h \in H, \ ghg^{-1} = h \ \forall g.}$

$C_h = \{ghg^{-1} : g \in G\} \overset{?}{=} \{h\}.$

$f : G \to C_h$ continuous

$g \mapsto ghg^{-1}$

C_n is connected.

$\boxed{H \text{ is normal}} \Rightarrow C_n \subset H.$

C_n has to be singleton.

$gHg^{-1} \subseteq H$

∎

$\tilde{G}$ is a lie group.

$$\begin{tikzcd}
 & \tilde{G} \arrow[r, "\pi"] & G \\
 & \tilde{G}\times\tilde{G} \arrow[u, "\exists!\, \tilde{s}"] \arrow[ur, "s"'] &
\end{tikzcd}$$

smooth

$\pi_1(\tilde{G}\times\tilde{G}) = \pi_1(\tilde{G})\times\pi_1(\tilde{G}) = 0.$

$$\boxed{s(\bar{g}, \bar{h}) := \pi(\bar{g})\,\pi(\bar{h})^{-1}}$$

happening in G.

$\therefore \exists!\ \tilde{s} : \tilde{G}\times\tilde{G} \longrightarrow \tilde{G}$ s.t. $\boxed{\pi\circ\tilde{s} = s.}$

Define: $\bar{h}^{-1} := \tilde{s}(\bar{e}, \bar{h}).$

a fixed element from $\pi^{-1}(e)$.

$\bar{g}\cdot\bar{h} := \tilde{s}(\bar{g}, \bar{h}^{-1})$

$$\bar g_1 \cdot (\bar g_2 \cdot \bar g_3) = \tilde s\left(\bar g_1, (\bar g_2 \cdot \bar g_3)^{-1}\right)$$

$$= \tilde s\left(\bar g_1, \tilde s(e, \bar g_2 \bar g_3)\right)$$

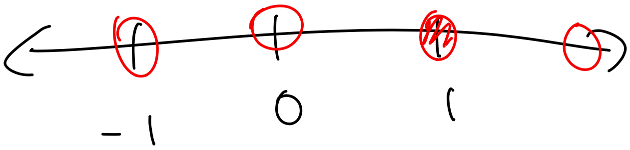
$$(\bar g_1 \cdot \bar g_2) \cdot \bar g_3 = \tilde s\left(\bar g_1 \cdot \bar g_2, \bar g_3^{-1}\right)$$

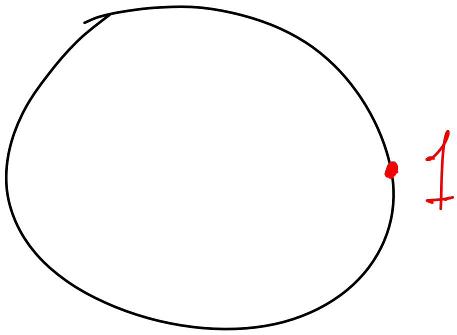
$$= \tilde s\left(\tilde s(\bar g_1, \bar g_2^{-1}), \bar g_3^{-1}\right)$$

$$\begin{tikzcd} \tilde G \arrow[r, "\pi"] & G \\ \tilde G \times \tilde G \arrow[u, "\exists ! \tilde s"] \arrow[ur, "s"'] & \end{tikzcd}$$

$\boxed{\pi \circ \tilde s = s}$ Hint: apply π.

$$\bar g \cdot \bar g^{-1} = \tilde s(\bar g, \bar g) \xrightarrow{\pi} e \in G.$$





S^1

inversion $\bar{h}^{-1} := \tilde{s}(\bar{e}, \bar{h})$

smooth

$\bar{g}\cdot\bar{h} := \tilde{s}(\bar{g}, \tilde{s}(\bar{e}, h))$

Smooth.

$\tilde{G}$ is a lie group.

$\boxed{\text{Ker } \pi \text{ is a Discrete central subgroup of } \tilde{G}.}$

Final Part: $\pi_1(G) \cong \text{Ker } \pi_1$.

change of notation $p: \tilde{G} \to G$.

$\boxed{\pi_1(G) = \text{Ker } p.}$

construct an isomorphism

$f: \pi_1(G) \to \text{Ker } p \subseteq \tilde{G}$

basepoint is e

$f([\alpha]) = ?$

α starts and ends at $e \in G$.

One way to define f

$$f([\alpha]) = \bar{e}$$

$$f : \pi_1(G) \to \mathrm{Ker}\, p$$

$$\begin{tikzcd}
\tilde{G} \arrow[r, "p"] & G \\
{[0,1]} \arrow[u, "\exists!\, \tilde{\alpha}"] \arrow[ur, "\alpha"'] &
\end{tikzcd}$$

$\tilde{\alpha}(0) = \bar{e}$

$\tilde{\alpha} : [0,1] \to \tilde{G}$

$$\boxed{f([\alpha]) := \tilde{\alpha}(1)}$$

Ker p?

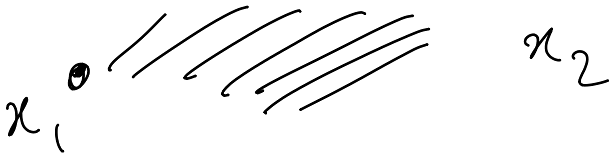
$p(\tilde{\alpha}(1)) = \alpha(1) = e$

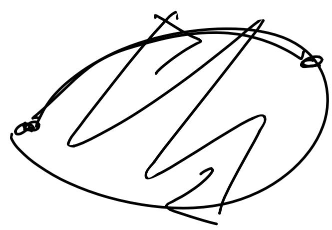
Well defined!

$$\boxed{[\alpha] = [\beta]} \Rightarrow \underline{\tilde{\alpha}(1)} = \underline{\tilde{\beta}(1)}$$

Ker p

Mystery: $\tilde{G}$ is simply connected.





$$\boxed{\alpha \sim_p \beta} \quad \Rightarrow \quad \boxed{\begin{gathered}\tilde\alpha \sim_p \tilde\beta \\ \tilde\alpha(1) = \tilde\beta(1)\end{gathered}}$$

$G \qquad\qquad\qquad\qquad \tilde G$

$$H : [0,1]^2 \longrightarrow G$$

$$H(-,0) = \alpha \qquad H(0,-) = e$$

$$H(-,1) = \beta \qquad H(1,-) = e$$

$\forall M$ simp. conn.

$$\begin{tikzcd} \tilde G \arrow[r] & G \\ M \arrow[u, "\exists!\,\tilde f"] \arrow[ur, "\forall f"'] \arrow[ur, phantom, "\checkmark"] & \end{tikzcd}$$

$$\begin{tikzcd} \tilde G \arrow[r, "p"] & G \\ {[0,1]^2} \arrow[u, "\exists!\,\tilde H"] \arrow[ur, "H"'] & \end{tikzcd}$$

simply connected

$\tilde H(-,0) =$? should be $\tilde\alpha$

$$\begin{tikzcd} \tilde G \arrow[r, "p"] & G \\ {[0,1]} \arrow[u, shift right=2, "\tilde H(-,0)"] \arrow[u, shift left=2, "\tilde\alpha"'] \arrow[ur, "\alpha = H(-,0)"'] \arrow[ur, phantom, "\checkmark"] & \end{tikzcd}$$

$\Rightarrow \boxed{\tilde H(-,0) = \tilde\alpha}$

Analogously

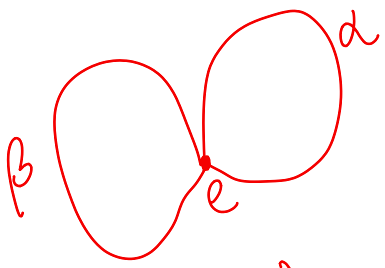
$\tilde H(-,1) = \tilde\beta$.

$\tilde{H}$ is a path-homotopy between $\tilde{\alpha}$ & $\tilde{\beta}$.

$\Rightarrow \tilde{\alpha}(1) = \tilde{\beta}(1)$.

$\Rightarrow f$ is well-defined.

f is group homomorphism:



$$f([\alpha+\beta]) = \widetilde{\alpha+\beta}(1) = \underbrace{\tilde{\alpha}(1)\cdot\tilde{\beta}(1)}_{\text{mult in } \tilde{G}}$$

$\cancel{= f([\alpha])} +$

$$f([\alpha]+[\beta]) = f([\alpha]) * f([\beta])$$

$$\Leftrightarrow f([\alpha+\beta]) = \tilde{\alpha}(1) * \tilde{\beta}(1)$$

$$\boxed{\Leftrightarrow \widetilde{\alpha+\beta}(1) = \tilde{\alpha}(1) * \tilde{\beta}(1)}$$ have to show.

$$\begin{tikzcd} \tilde{G} \arrow[r, "p"] & G \\ \cancel{[0,1]}\ [0,2] \arrow[u, "\widetilde{\alpha+\beta}"] \arrow[ur, "\alpha+\beta"'] & \end{tikzcd}$$

$$(\alpha+\beta)(t) = \begin{cases} \alpha(t) & t \le 1 \\ \cancel{\beta(t)}\ \beta(t-1) & t \ge 1 \end{cases}$$

$$(\alpha+\beta)(1+t) = \beta(t)$$

$$\boxed{\begin{tikzcd} \tilde{G} \arrow[r, "p"] & G \\ {[0,1]} \arrow[u, "\tilde{\alpha}"] \arrow[ur, "\alpha"'] & \end{tikzcd} \qquad \begin{tikzcd} \tilde{G} \arrow[r, "p"] & G \\ {[1,2]} \arrow[u, "\tilde{\beta}"] \arrow[ur, "\beta"'] & \end{tikzcd}}$$

add

$$\begin{tikzcd} \tilde{G} \arrow[r, "p"] & G \\ {[0,2]} \arrow[u] \arrow[ur, "\alpha+\beta"'] & \end{tikzcd}$$

✓

$\cancel{(\widetilde{\alpha+\beta})(t)}$ $\leftarrow$ $\tilde{\alpha}\cdot\tilde{\beta}$

$\cancel{t\le 1}$

$\tilde{\alpha}(t) \quad t \le 1$

$\tilde{\alpha}(1)\cdot\tilde{\beta}(t) \quad t \ge 1$

$$\Rightarrow \boxed{\widetilde{\alpha+\beta} = \tilde{\alpha}\cdot\tilde{\beta}}$$

$$(\tilde{\alpha}\cdot\tilde{\beta})(t) = \begin{cases} \tilde{\alpha}(t) & t\leq 1 \\ \tilde{\alpha}(1)\cdot \cancel{\tilde{\beta}(t-1)} \; \tilde{\beta}(t) & t > 1 \end{cases}$$

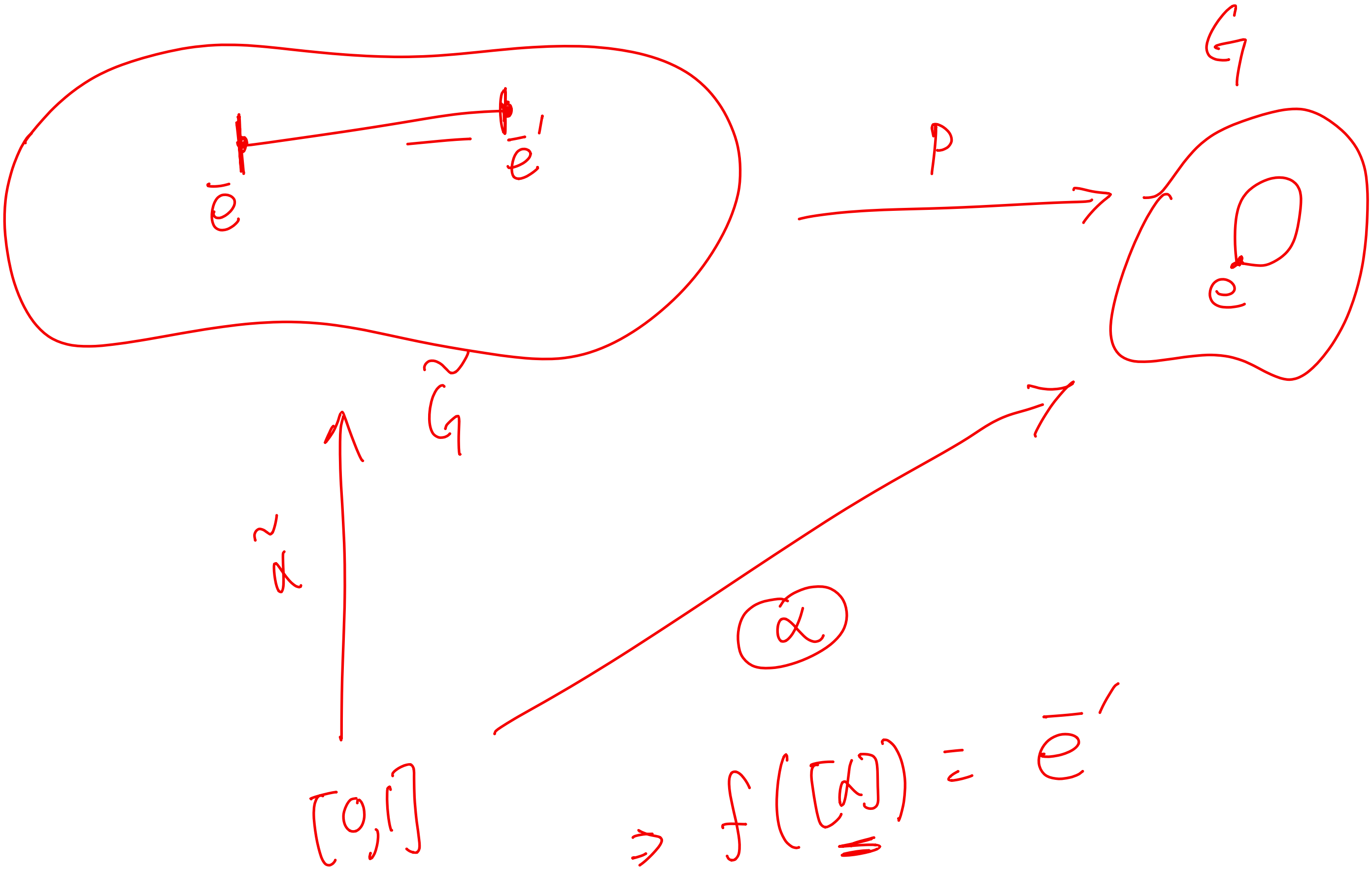
$$(p\circ\tilde{\alpha}\cdot\tilde{\beta})(t) = \begin{cases} p(\tilde{\alpha}(t)) \longrightarrow \alpha(t) \\ p(\tilde{\alpha}(1)\cdot\tilde{\beta}(t)) \end{cases}$$

$$= \cancel{\beta t}\;\beta(t)$$

endpoint of $\widetilde{\alpha+\beta} = \tilde{\alpha}\,\cdot$ endpoint of $\tilde{\alpha}\,\cdot$ endp. $\tilde{\beta}$.

<u>f is surjective</u>

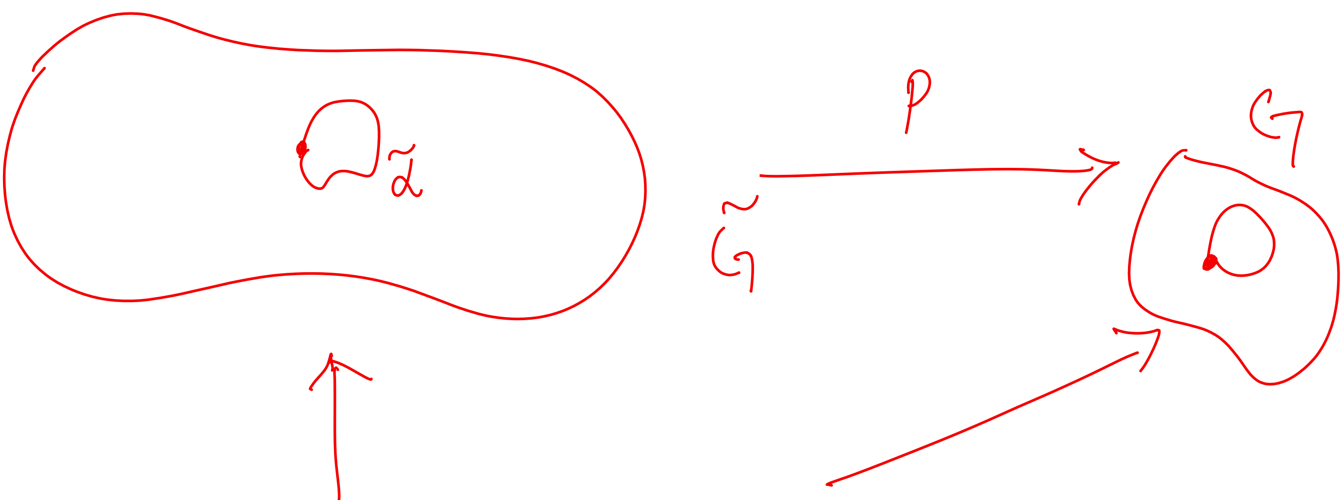
$$f : \pi_1(G) \to \operatorname{Ker} p$$



$$\Rightarrow f([\alpha]) = \bar{e}'$$

<u>f is injective</u> :

$$\operatorname{Ker} f = \{ [\alpha] : \tilde{\alpha}(1) = \bar{e} \}$$



H $\qquad\qquad$ $\tilde{H} = p \circ H$.

$[0,1]^2$

$\tilde{\alpha}(1) = \bar{e} \Rightarrow \tilde{\alpha}$ and $\cdot$ are path homo

$\Rightarrow \alpha$ and $\cdot$ are path hom

$$\Rightarrow [\alpha] = [c_e]$$

∎

Correct version of universal property