# On the Category Theoretic Implications on Induced Representation of Finite Groups

## Part 2: Induced Representation Theory

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I've now discovered a lifehack for learning math, which is to sign up to give a talk about something I don't know super well
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# Representations

Groups are abstract objects, and hard to deal with.

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# Representations

$\mathbb{Z} \to$ $n \longrightarrow \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$

Groups are abstract objects, and hard to deal with. What if group elements were matrices?

Voila! We have Representation Theory!!!

$$\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & m+n \\ 0 & 1 \end{bmatrix}$$

$$\rho : \mathbb{Z} \longrightarrow GL_2(\mathbb{K})$$

# Representations

## Definition 1

A **representation** of a group $G$ on a $\mathbb{K}$-vector space $V$ is a homomorphism

$$\boxed{\rho : G \to \mathrm{GL}(V).}$$

$$\left\{ f : V \to V \mid f \text{ is invertible} \right\}$$

# Representations

**Definition 1**

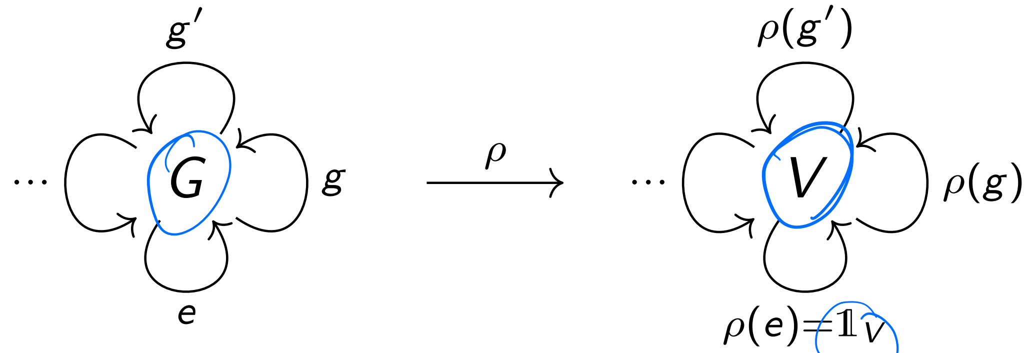
A **representation** of a group $G$ on a $\mathbb{K}$-vector space $V$ is a homomorphism

$$\rho : G \to \mathrm{GL}\,(V)\,.$$

Sometimes we call $V$ is the representation of $G$.

# Representations

Categorically speaking, a representation of a group $G$ is a functor $\mathcal{C}(G) \to \boxed{\mathbf{Vect}_{\mathbb{K}}}$.




$$\rho(g_1 \circ g_2) = \rho(g_1) \circ \cancel{gf}\ \rho(g_2)$$

$$\boxed{\rho(g_1 g_2) = \rho(g_1) \circ \rho(g_2)}$$

# Representations

$f : V \to W$

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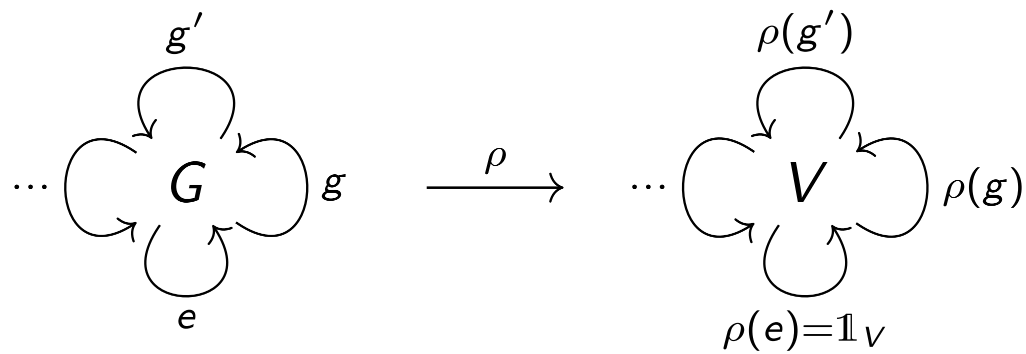


So we can form the functor category $\mathrm{Fun}\left(\mathcal{C}\left(G\right), \mathbf{Vect}_{\mathbb{K}}\right)$. This is the category of all $\mathbb{K}$-representations of the group $G$.

$$\mathrm{Fun}\left(\mathcal{C}(G), \mathrm{Vect}_{\mathbb{K}}\right) =: \mathrm{Rep}_{\mathbb{K}}(G)$$

# Representations

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$$\begin{tikzcd}
G \arrow[loop above, "g'"] \arrow[loop right, "g"] \arrow[loop below, "e"] \arrow[loop left, "\cdots"] \arrow[rr, "\rho"] & & V \arrow[loop above, "\rho(g')"] \arrow[loop right, "\rho(g)"] \arrow[loop below, "\rho(e)=\mathbb{1}_V"] \arrow[loop left, "\cdots"]
\end{tikzcd}$$

So we can form the functor category $\mathrm{Fun}\left(\mathcal{C}\left(G\right), \mathbf{Vect}_{\mathbb{K}}\right)$. This is the category of all $\mathbb{K}$-representations of the group $G$. We also call this category $\mathrm{Rep}_{\mathbb{K}}\left(G\right)$ or $\mathrm{Rep}(G)$.

# Representations

What are the arrows in $\mathrm{Fun}\left(\mathcal{C}\left(G\right), \mathbf{Vect}_{\mathbb{K}}\right)$?

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Natural transformations $\eta : \rho \Rightarrow \sigma$!!!

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Natural transformations $\eta : \rho \Rightarrow \sigma$!!!

Recall the definition of natural transformations: Let $F, G : \mathcal{C} \to \mathcal{D}$ be two functors. Then a **natural transformation** $\eta : F \Rightarrow G$ is a family of arrows

$$\boxed{\left\{\eta_X : F\left(X\right) \to G\left(X\right)\right\}_{X \in \mathcal{C}_0}}$$

in $\mathcal{D}$ such that for every arrow $f : X \to Y$ in $\mathcal{C}$, the following diagram commutes:

$$\begin{tikzcd}
F(X) \arrow[r, "F(f)"] \arrow[d, "\eta_X"'] & F(Y) \arrow[d, "\eta_Y"] \\
G(X) \arrow[r, "G(f)"'] & G(Y)
\end{tikzcd}$$

$f : X \to Y$

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G(X) \arrow[r, "G(f)"'] & G(Y)
\end{tikzcd}$$

In other words, $\eta_Y \circ F\left(f\right) = G\left(f\right) \circ \eta_X$.

# Representations

In our case, $\mathcal{C} = \mathcal{C}(G)$, $\mathcal{D} = \textbf{Vect}_{\mathbb{K}}$.

$$\{\eta_x : F(x) \to G(x)\}_{x \in \mathcal{C}_0}$$

In our case, $\mathcal{C} = \mathcal{C}(G)$, $\mathcal{D} = \mathbf{Vect}_{\mathbb{K}}$. There is only one object in $\mathcal{C}$, i.e. $G$. So $\eta$ has only one component. So we write $\eta_G : \rho(G) =: V \to \sigma(G) =: W$.

$$\rho, \sigma : \mathcal{C}(G) \longrightarrow \mathrm{Vect}_{\mathbb{K}}$$

$$\eta_G$$

In our case, $\mathcal{C} = \mathcal{C}(G)$, $\mathcal{D} = \mathbf{Vect}_{\mathbb{K}}$. There is only one object in $\mathcal{C}$, i.e. $G$. So $\eta$ has only one component. So we write $\eta_G : \rho(G) =: V \to \sigma(G) =: W$.

The arrows in the domain category are $g \in G$. We have to make the following diagram commute for every $g \in G$:

$$\begin{tikzcd} V \arrow[r, "\rho(g)"] \arrow[d, "\eta_G"'] & V \arrow[d, "\eta_G"] \\ W \arrow[r, "\sigma(g)"'] & W \end{tikzcd}$$

→ in $\mathcal{C}(G)$

$g : G \to G$

$$\begin{tikzcd} \rho(G) \arrow[r, "\rho(g)"] \arrow[d, "\eta_G"'] & \rho(G) \arrow[d, "\eta_G"] \\ \sigma(G) \arrow[r, "\sigma(g)"'] & \sigma(G) \end{tikzcd}$$

$\rho, \sigma$ :

In our case, $\mathcal{C} = \mathcal{C}(G)$, $\mathcal{D} = \textbf{Vect}_{\mathbb{K}}$. There is only one object in $\mathcal{C}$, i.e. $G$. So $\eta$ has only one component. So we write $\eta_G : \rho(G) =: V \to \sigma(G) =: W$.

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When a linear map $\boxed{V \to W}$ satisfies this commutativity, we call it a homomorphism of representations.

# Representations

**Definition 2**

Let $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$ be two representations of a group $G$. A **homomorphism of representations** $\varphi$ between two representations $V$ and $W$ of $G$ is a linear map $\boxed{\varphi : V \to W}$ such that the following diagram commutes for every $g \in G$:

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "\rho(g)"'] \arrow[dr, phantom, "\checkmark"] & W \arrow[d, "\sigma(g)"] \\ V \arrow[r, "\varphi"'] & W \end{tikzcd}$$

$\eta_G$

$\varphi \circ \rho(g) = \sigma(g) \circ \varphi$

$\boxed{\varphi(\rho(g)\bar{v}) = \sigma(g)\varphi(\bar{v})}$

In other words, $\varphi \circ \rho(g) = \sigma(g) \circ \varphi$.

$\varphi : V \longrightarrow W$

$\cancel{\rho(g)\varphi(\bar{v})}$ $\sigma(g)\varphi(\bar{v})$

# Representations

## Definition 2

Let $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$ be two representations of a group $G$. A **homomorphism of representations** $\varphi$ between two representations $V$ and $W$ of $G$ is a linear map $\varphi : V \to W$ such that the following diagram commutes for every $g \in G$:

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "\rho(g)"'] & W \arrow[d, "\sigma(g)"] \\ V \arrow[r, "\varphi"'] & W \end{tikzcd}$$

$\varphi(\rho(g)v)$

$= \sigma(g)\varphi(v)$

In other words, $\varphi \circ \rho(g) = \sigma(g) \circ \varphi$.

We also call it a $G$-linear map, or intertwining operator.

$\rho, \sigma$

Let $V$ and $W$ be two representations of $G$. Then the space of all $G$-linear maps from $V$ to $W$ is

$$\boxed{\mathrm{Hom}_{\mathrm{Rep}(G)}(V, W)\,.}$$

$\mathrm{Hom}_G(V, W)$

$\mathrm{Hom}_{\mathrm{Rep}(G)}(\rho, \sigma)$

# The category Rep($G$)

Let $V$ and $W$ be two representations of $G$. Then the space of all $G$-linear maps from $V$ to $W$ is

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We also write it as $\mathrm{Hom}_G(V, W)$.

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We also write it as $\mathrm{Hom}_G(V, W)$.

It has a vector space structure!!

Suppose $\varphi, \psi \in \boxed{\mathrm{Hom}_G(V, W)}$ and $a \in \mathbb{K}$. Then we have the following commutative squares:

$(a\varphi + \psi)$

$$
\begin{tikzcd}
V \arrow[r, "\varphi"] \arrow[d, "\rho(g)"'] \arrow[dr, phantom, "\checkmark"] & W \arrow[d, "\sigma(g)"] \\
V \arrow[r, "\varphi"'] & W
\end{tikzcd}
\qquad
\begin{tikzcd}
V \arrow[r, "\psi"] \arrow[d, "\rho(g)"'] \arrow[dr, phantom, "\checkmark"] & W \arrow[d, "\sigma(g)"] \\
V \arrow[r, "\psi"'] & W
\end{tikzcd}
$$

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$a\varphi + \psi \in \mathrm{Hom}_G(V, W)$

$$\sigma(g) \circ (a\varphi + \psi)(\mathbf{v}) = a\sigma(g)(\varphi(\mathbf{v})) + \sigma(g)(\psi(\mathbf{v}))$$

$$\begin{tikzcd} V \arrow[r, "a\varphi+\psi"] \arrow[d, "\rho(g)"'] & W \arrow[d, "\sigma(g)"] \\ V \arrow[r, "a\varphi+\psi"'] & W \end{tikzcd}$$

$\sigma(g) \circ (a\varphi + \psi)(\bar{v})$

$= \sigma(g)\big(a\ \varphi(\bar{v}) + \psi(\bar{v})\big)$

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Suppose $\varphi, \psi \in \mathrm{Hom}_G(V, W)$ and $a \in \mathbb{K}$. Then we have the following commutative squares:

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$$\begin{aligned} \sigma(g) \circ (a\varphi + \psi)(\mathbf{v}) &= a\sigma(g)(\varphi(\mathbf{v})) + \sigma(g)(\psi(\mathbf{v})) \\ &= a\varphi(\rho(g)(\mathbf{v})) + \psi(\rho(g)(\mathbf{v})) \end{aligned}$$

$= \cancel{\varphi}(a\varphi + \psi)(\rho(g)\bar{v})$

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$$\begin{aligned} \sigma(g) \circ (a\varphi + \psi)(\mathbf{v}) &= a\sigma(g)(\varphi(\mathbf{v})) + \sigma(g)(\psi(\mathbf{v})) \\ &= a\varphi(\rho(g)(\mathbf{v})) + \psi(\rho(g)(\mathbf{v})) \\ &= (a\varphi + \psi)(\rho(g)\mathbf{v}). \end{aligned}$$

This proves the commutativity of the following square:

$$\begin{tikzcd} V \arrow[r, "a\varphi+\psi"] \arrow[d, "\rho(g)"'] & W \arrow[d, "\sigma(g)"] \\ V \arrow[r, "a\varphi+\psi"'] & W \end{tikzcd}$$

Therefore, $a\varphi + \psi \in \mathrm{Hom}_G(V, W)$, i.e. $\mathrm{Hom}_G(V, W)$ is a $\mathbb{K}$-vector space.

$V \oplus W = \left\{ (\bar{v}, \bar{w}) \,\middle|\, \begin{array}{l} \bar{v} \in V \\ \bar{w} \in W \end{array} \right\}$

Let $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$ be representations. Then there is a representation of $G$ on the vector space $V \oplus W$.

$$\boxed{\rho \oplus \sigma : G \to \mathrm{GL}\,(V \oplus W)\,;}$$

$$(\rho \oplus \sigma)(g)(\mathbf{v}, \mathbf{w}) = (\rho(g)\mathbf{v}, \sigma(g)\mathbf{w})\,, \tag{1}$$

for $g \in G$.

$\rho \oplus \sigma : G \to GL(V \oplus W)$

$(\rho \oplus \sigma)(g)(\bar{v}, \bar{w}) = (\rho(g)\bar{v}, \sigma(g)\bar{w})$

# New representations from old ones

Let $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$ be representations. Then there is a representation of $G$ on the tensor product vector space $V \otimes W$.

$$\rho \otimes \sigma : G \to \mathrm{GL}\left(V \otimes W\right);$$

$$(\rho \otimes \sigma)(g)(\mathbf{v} \otimes \mathbf{w}) = \rho(g)\mathbf{v} \otimes \sigma(g)\mathbf{w}, \tag{2}$$

for $g \in G$.

# New representations from old ones

$F : V \to W$, $F^T : W^* \to V^*$, $(F^T)(\alpha)(\bar{v}) = \alpha(F(\bar{v}))$

Let $\rho : G \to \mathrm{GL}(V)$ be a representation of $G$. Then there is a representation of $G$ on the dual vector space $V^*$.

$$\rho^* : G \to \mathrm{GL}\left(V^*\right);$$

$V^* = \{f : V \to \mathbb{K}\}$

$= \mathrm{Hom}_{\mathrm{vect}_{\mathbb{K}}}(V, \mathbb{K})$

$\rho^*(g) : V^* \to V^*$

$$\boxed{\rho^*(g)(f)(\bar{v}) = f(\rho(g)\bar{v})}$$

$\rho^*(g) = \rho(g)^T$

$\rho^*(gh) = \rho^*(g)\rho^*(h)$

$\rho(gh)^T = \rho(g)^T \rho(h)^T$

$(\rho(g)\rho(h))^T = \rho(g)^T \rho(h)^T$

# New representations from old ones

Let $\rho : G \to \mathrm{GL}(V)$ be a representation of $G$. Then there is a representation of $G$ on the dual vector space $V^*$.

$$\rho^* : G \to \mathrm{GL}\left(V^*\right);$$

$$V \xrightarrow{\rho(g)} V$$

$$V^* \xleftarrow[\rho(g)^T]{} V^*$$

$$V^* \longrightarrow V^*$$

# New representations from old ones

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$$\rho^* : G \to \mathrm{GL}\left(V^*\right);$$

$$V \xrightarrow{\rho(g)} V$$

$$V^* \xleftarrow[\rho(g)^T]{} V^*$$

So we define

$$\rho^*(g) = \left[\rho\left(g^{-1}\right)\right]^T. \tag{3}$$

# New representations from old ones

Let $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$ be representations. Then we can define a representation on the vector space $\boxed{\mathrm{Hom}(V, W)}$. $\to \mathrm{Vect}_k$

$$\gamma : G \to \mathrm{GL}\left(\mathrm{Hom}\left(V, W\right)\right).$$

$$\gamma(g) : \mathrm{Hom}(V,W) \to \mathrm{Hom}(V,W)$$

$$\begin{tikzcd}
V \arrow[r, "f"] \arrow[d, "\rho(g)"'] \arrow[dr, phantom, "\checkmark"] & W \arrow[d, "\sigma(g)"] \\
V \arrow[r, "\gamma(g)f"'] & W
\end{tikzcd}$$

$$\gamma(g)f = \sigma(g) \circ f \circ \rho(g)^{-1}$$

# New representations from old ones

Let $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$ be representations. Then we can define a representation on the vector space $\mathrm{Hom}\,(V, W)$.

$$\gamma : G \to \mathrm{GL}\,(\mathrm{Hom}\,(V, W))\,.$$

Given $f \in \mathrm{Hom}\,(V, W)$, $\gamma(g)f$ is supposed to be another element of $\mathrm{Hom}\,(V, W)$.

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$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\rho(g)"'] & W \arrow[d, "\sigma(g)"] \\ V & W \end{tikzcd}$$

# New representations from old ones

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$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\rho(g)"'] & W \arrow[d, "\sigma(g)"] \\ V & W \end{tikzcd}$$

So we can define $\gamma(g)f$ to be the missing arrow in the above diagram!

$$\gamma(g)f = \sigma\,(g) \circ f \circ \rho\,(g)^{-1}\,.$$

$\mathrm{Vect}_{\mathbb{k}}$ $\mathrm{Vect}_{\mathbb{k}}^{fin}$

From now on, for the rest of the talk, all groups are finite groups. Also, all the vector spaces are finite dimensional.

# Subgroup Representation

Let $\rho : G \to \mathrm{GL}(V)$ be a representation of $G$, and $H \subseteq G$ be a subgroup. Then $\rho$ defines a representation of $H$ as well!

$$\boxed{\rho\big|_H : H \to \mathrm{GL}(V).}$$

# Subgroup Representation

Let $\rho : G \to \mathrm{GL}(V)$ be a representation of $G$, and $H \subseteq G$ be a subgroup. Then $\rho$ defines a representation of $H$ as well!

$$\rho\big|_H : H \to \mathrm{GL}(V).$$

Since we are restricting the domain of the representation $\rho : G \to \mathrm{GL}(V)$, we call this the **restriction** of the representation $\rho$ of $G$.

$\rho\big|_H : H \longrightarrow GL(V)$

$\rho\big|_H (h) = \rho(h)$ .

# Subgroup Representation

This gives rise to a functor

$$\mathrm{Res}_H^G : \mathrm{Rep}(G) \to \mathrm{Rep}(H),$$

called the restriction functor. It takes a representation of $G$ and restricts it to a representation of $H$.

$\mathrm{Res}^{G}_{H}$

$h \mapsto h$

$\iota : H \to G$ is an inclusion, and it gives a functor between the category of representations, but in the reverse direction $\mathrm{Res}_H^G : \mathrm{Rep}(G) \to \mathrm{Rep}(H)$.

$f : H \to G$

$\mathrm{Res} : \mathrm{Rep}(G) \to \mathrm{Rep}(H)$

# Subgroup Representation

$$(\rho \circ i)(h) = \rho(h).$$

$$i(h) = h.$$

$\iota : H \to G$ is an inclusion, and it gives a functor between the category of representations, but in the reverse direction $\mathrm{Res}_H^G : \mathrm{Rep}(G) \to \mathrm{Rep}(H)$.

Can we do the same for any group homomorphism $f : G_1 \to G_2$? Does it give us a restriction functor

$$\mathrm{Res} : \mathrm{Rep}\left(G_2\right) \to \mathrm{Rep}\left(G_1\right)?$$

$$\rho|_H(h) = \rho(h)$$

$$\boxed{\rho|_H = \rho \circ \iota}$$

$$\begin{tikzcd} H \arrow[r, "i"] \arrow[rr, bend right, "\rho\circ i"'] & G \arrow[r, "\rho"] & GL(V) \end{tikzcd}$$

# Subgroup Representation

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How did we define $\mathrm{Res}_H^G$? We took a representation $\rho : G \to \mathrm{GL}(V)$, and we restricted it to $H$. Now notice that

$$\rho\big|_H = \rho \circ \iota. \tag{4}$$

# Subgroup Representation

$$\mathrm{Res} : \mathrm{Rep}(G_2) \longrightarrow \mathrm{Rep}(G_1)$$

$$G_2 \xleftarrow{f} G_1$$

How did we define $\mathrm{Res}_H^G$? We took a representation $\rho : G \to \mathrm{GL}(V)$, and we restricted it to $H$. Now notice that

$$\rho\big|_H = \rho \circ \iota. \tag{4}$$

We can do the same for any group homomorphism $\boxed{f : G_1 \to G_2}$. If $\rho : G_2 \to \mathrm{GL}(V)$ is a representation of $G_2$, we get a representation of $G_1$:

$$\rho \circ f : G_1 \to \mathrm{GL}\,(V)\,. \tag{5}$$

$$\begin{tikzcd} G_1 \arrow[r, "f"] \arrow[rr, bend right, "\rho \circ f"'] & G_2 \arrow[r, "\rho"] & GL(V) \end{tikzcd}$$

# Induced Representation

Now suppose we have a representation $W$ of a subgroup $H$. Can we generate a representation of $G$ from this subgroup representation?

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- When we restrict this new representation to $H$, we get $W$ back. In other words, $\mathrm{Res}_H^G V = W$.

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- Also, we wish to construct the "most free" representation of $G$ so as to not lose any information.

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- Also, we wish to construct the "most free" representation of $G$ so as to not lose any information.

Information are encoded in Hom-sets, i.e. arrows (Yoneda lemma)!

## Definition 3

Let $W$ be a representation of $H$ and $V$ be a representation of $G$. We say that $W \xrightarrow{\alpha} V$ is an **induction** if it satisfies the following universal property:

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1. $\alpha$ is $H$-linear;

# Induced Representation

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1. $\alpha$ is $H$-linear;
2. if $Z$ is another representation of $G$, and $\beta : W \to Z$ is a $H$-linear map, then there exists a unique $G$-linear map $\overline{\beta} : V \to Z$ such that the following diagram commutes:

$$\begin{tikzcd} W \arrow[r, "\alpha"] \arrow[dr, "\forall \beta"'] & V \arrow[d, dashed, "\exists!\ \overline{\beta}"] \\ & Z \end{tikzcd}$$

H-rep

G-rep

$$\begin{tikzcd} W \arrow[r, "\alpha"] \arrow[dr] & V \arrow[d, dashed, "\exists!\ \bar{\beta}"] \\ & Z \end{tikzcd}$$

$\bar{\beta} \circ \alpha = \beta$

$\mathrm{Hom}_H(W, Z) \longleftrightarrow \mathrm{Hom}_G(V, Z)$

How does this definition satisfy our requirements?

# Induced Representation

$$\beta = \bar{\beta} \circ \alpha$$

$$\begin{tikzcd} W \arrow[r, "\alpha"] \arrow[dr, "\forall \beta"'] & V \arrow[d, "\exists!\, \bar{\beta}"] \\ & Z \end{tikzcd}$$

How does this definition satisfy our requirements?

All $H$-linear maps $\beta : W \to Z$ gets uniquely factored through $\alpha : W \to V$. Therefore, $V$ preserves the "information" of $\mathrm{Hom}_H(W, -)$, where the black $-$ is replaced by a representation of $G$.

$$\begin{array}{ccc} \mathrm{Hom}_H(W, Z) & & \boxed{\mathrm{Hom}_G(V, Z)} \\ \beta & \longrightarrow & \bar{\beta} \\ \theta \circ \alpha & \longleftarrow & \theta \end{array}$$

$$\mathrm{Hom}_H(W, -) \longleftrightarrow \mathrm{Hom}_G(V, -)$$

## Proposition 1

If induction of a $H$-representation $W$ exists, then it's unique (up to isomorphism).

$$\mathrm{Hom}_H(W, Z) \longleftrightarrow \mathrm{Hom}_G(V, Z)$$

$V = \mathrm{Ind}\, W$

$$\underline{\mathrm{Hom}_H(W, \mathrm{Res}(\_))} \longleftrightarrow \underline{\mathrm{Hom}_G(\mathrm{Ind}\, W, \_)}$$

$$\mathrm{Hom}_G(\underline{\mathrm{Ind}}(\_), \_) \longleftrightarrow \mathrm{Hom}_H(\_, \underline{\mathrm{Res}}(\_))$$

$W \longrightarrow V_1, V_2$

$V_1 = \mathrm{Ind}\, W, \quad V_2 = \mathrm{Ind}\, W$

# Induced Representation

## Proposition 1

If induction of a $H$-representation $W$ exists, then it's unique (up to isomorphism).

## Proof.

Suppose $W \xrightarrow{\alpha_1} V_1$ and $W \xrightarrow{\alpha_2} V_2$ are two inductions of $W$.

$$\begin{tikzcd} W \arrow[r, "\alpha_1"] \arrow[dr, "\forall \beta"'] & V_1 \arrow[d, dashed, "\exists \bar{\beta}"] \\ & Z \end{tikzcd}$$

$$\begin{tikzcd} V_1 \arrow[r, bend left, "f"] & V_2 \arrow[l, bend left, "g"] \end{tikzcd}$$

$$\begin{tikzcd} W \arrow[r, "\alpha_2"] \arrow[dr, "\forall \beta'"'] & V_2 \arrow[d, dashed, "\exists ! \bar{\beta}'"] \\ & Z \end{tikzcd}$$

$f \circ g = \mathbb{1}_{V_2}, \quad g \circ f = \mathbb{1}_{V_1}$

# Induced Representation

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$$\begin{tikzcd} W \arrow[r, "\alpha_1"] \arrow[dr, "\alpha_2"'] & V_1 \arrow[d, dashed, "\exists!\ f"] \\ & V_2 \end{tikzcd}$$

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# Induced Representation

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$$\begin{tikzcd} W \arrow[r, "\alpha_1"] \arrow[dr, "\alpha_2"'] & V_1 \arrow[d, dashed, "\exists!\ f"] \\ & V_2 \end{tikzcd} \qquad \begin{tikzcd} W \arrow[r, "\alpha_2"] \arrow[dr, "\alpha_1"'] & V_2 \arrow[d, dashed, "\exists!\ g"] \\ & V_1 \end{tikzcd}$$

$\alpha_1 : W \to V_1$

$$\begin{tikzcd} W \arrow[r, "\alpha_1"] \arrow[dr, "\forall \beta"'] & V_1 \arrow[d, dashed, "\exists!\ \bar{\beta}"] \\ & Z \end{tikzcd}$$

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 & V_2
\end{tikzcd}
\qquad
\begin{tikzcd}
W \arrow[r, "\alpha_2"] \arrow[dr, "\alpha_1"'] & V_2 \arrow[d, dashed, "\exists!\ g"] \\
 & V_1
\end{tikzcd}
\implies
\begin{tikzcd}
 & V_2 \arrow[d, "g"] \arrow[dd, bend left=60, "f \circ g"] \\
W \arrow[ur, "\alpha_2"] \arrow[r, "\alpha_1"] \arrow[dr, "\alpha_2"'] & V_1 \arrow[d, "f"] \\
 & V_2
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & V_2 \arrow[dd, bend left, "f \circ g"] \\
W \arrow[ur, "\alpha_2"] \arrow[dr, "\alpha_2"'] & \\
 & V_2
\end{tikzcd}
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& V_2 \arrow[d, "g"] \\
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& V_2
\end{tikzcd}
$$

$$
\begin{tikzcd}
W \arrow[r, "\alpha_2"] \arrow[dr, "\alpha_2"'] & V_2 \arrow[d, "f \circ g"] \\
& V_2
\end{tikzcd}
\qquad
\begin{tikzcd}
W \arrow[r, "\alpha_2"] \arrow[dr, "\forall \beta = \alpha_2"'] & V_2 \arrow[d, dashed, "\exists!\ \cancel{f}\ \bar{\alpha}_2"] \\
& \cancel{X}\ V_2
\end{tikzcd}
$$

# Induced Representation

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$$\begin{tikzcd} W \arrow[r, "\alpha_2"] \arrow[dr, "\alpha_2"'] & V_2 \arrow[d, "f \circ g"] \\ & V_2 \end{tikzcd} \implies f \circ g = \mathbb{1}_{V_2}.$$

# Induced Representation

## Proposition 1

If induction of a $H$-representation $W$ exists, then it's unique (up to isomorphism).

## Proof.

Suppose $W \xrightarrow{\alpha_1} V_1$ and $W \xrightarrow{\alpha_2} V_2$ are two inductions of $W$.

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& V_2
\end{tikzcd}
\qquad
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& V_1
\end{tikzcd}
\implies
\begin{tikzcd}
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W \arrow[ur, "\alpha_2"] \arrow[r, "\alpha_1"] \arrow[dr, "\alpha_2"'] & V_1 \arrow[d, "f"] \\
& V_2
\end{tikzcd}
$$

$$
\begin{tikzcd}
W \arrow[r, "\alpha_2"] \arrow[dr, "\alpha_2"'] & V_2 \arrow[d, "f \circ g"] \\
& V_2
\end{tikzcd}
\implies f \circ g = \mathbb{1}_{V_2}. \text{ Similarly, } g \circ f = \mathbb{1}_{V_1}.
$$

∎

# Induced Representation

But does it exist tho?

$$Hg_i = \{hg_i \mid h \in H\}$$

But does it exist tho?

$$\boxed{G = Hg_1 \cup Hg_2 \cup \cdots \cup Hg_n.} \tag{6}$$

$G$ is made out of $n$ "copies" of $H$, so it's expected that a representation of $G$ that doesn't lose any information will be made out of $n$ copies of a representation of $H$.

$$Hg_i \cap Hg_j = \phi$$

$$|Hg_i| = |H|$$

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$$V = W \oplus W \oplus \cdots \oplus W = W^n. \tag{7}$$

H-rep

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$$V = W \oplus W \oplus \cdots \oplus W = W^n. \tag{7}$$

We label them with the cosets, so

$$V = W_{Hg_1} \oplus W_{Hg_2} \oplus \cdots \oplus W_{Hg_n}. \tag{8}$$

Also, we take $\boxed{g_1 = e}$.

$Hg_1 = H$

# Induced Representation

Suppose $\rho_H : H \to \mathrm{GL}(W)$ be the representation. We define $\rho : G \to GL(V)$.

# Induced Representation

$\rho(g) : W_\sigma \to W_{g\sigma}$

Suppose $\rho_H : H \to \mathrm{GL}(W)$ be the representation. We define $\boxed{\rho : G \to GL(V)}$.

For $g \in G$, we are going to define $\boxed{\rho(g)}(\mathbf{v})$ for $\mathbf{v} \in W_{Hg_j}$.

$\boxed{\rho(g)} : V \to V$

$= \bigoplus_{i=1}^{n} W_{Hg_i}$

$V = \bigoplus W_\sigma$

$\rho(g) : W_\sigma \to W_{g\sigma}$

$V = W_{Hg_1} \oplus \cdots \oplus W_{Hg_n}$

$W_{Hg_j}$

# Induced Representation

$\bar{v} \in W_{Hg_j}$ $\rho(g)(\bar{v})$ $\{g_1, \dots, g_n\} \longleftrightarrow \{g_1, \dots, g_n\}$

Suppose $\rho_H : H \to \mathrm{GL}(W)$ be the representation. We define $\rho : G \to GL(V)$.

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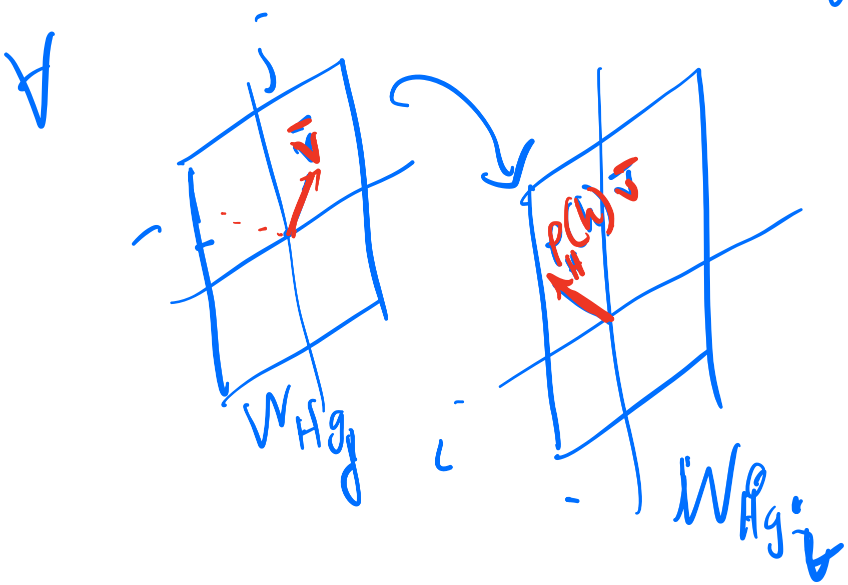
$$\boxed{\rho(g)\mathbf{v}} := \text{ the copy of } \boxed{\rho_H(h)\mathbf{v}} \text{ in } W_{Hg_i}. \tag{9}$$

$\to W_{Hg_j}$




$\bar{v} \in W_{Hg_j}$

$\rho_H : H \to GL(W)$

$\rho_H(h)\bar{v} \in W$

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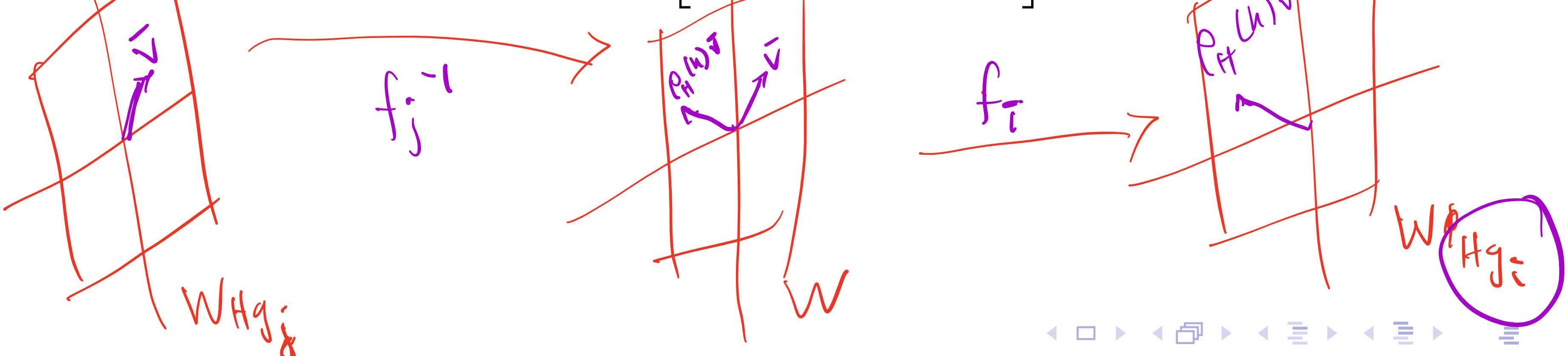
More formally speaking, suppose $f_i : W = W_{He} \to W_{Hg_i}$ be the identification. Then for $\mathbf{v} \in W_{Hg_j}$ and $gg_j^{-1} = g_i^{-1}h$,

$$\rho(g)\mathbf{v} = \left[f_i \circ \rho_H(h) \circ f_j^{-1}\right]\mathbf{v}. \tag{10}$$



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**Exercise:** Show that $\boxed{\rho : G \to \mathrm{GL}(V)}$ is a homomorphism.

# Induced Representation

$$W \xrightarrow{\alpha} V = \bigoplus W_{Hg_i}$$

Now we are going to show that this construction satisfies the universal property of induction.

Suppose $Z$ is another representation of $G$, and $\beta : W \to Z$ is a $H$-linear map.

$$\begin{tikzcd}
W \arrow[r, "\alpha"] \arrow[drr, "\beta"'] & V = \bigoplus_{i=1}^{n} W_{Hg_i} \arrow[dr, dashed, "\overline{\beta}"] & \\
& & Z
\end{tikzcd}$$

We need to show the existence and uniqueness of a $G$-linear map $\overline{\beta} : V \to Z$ such that the diagram commutes.

Suppose the corresponding group homomorphism of the representation $Z$ is $\sigma : G \to \mathrm{GL}(Z)$.

$g_1 = e$  $g = g_j$

First we show the uniqueness of $\overline{\beta}$.
Given $\mathbf{v} \in W_{Hg_j}$, $\boxed{\rho(g_j)}\mathbf{v} \in W_{He}$.

$\rho(g)\bar{v}$  $\bar{v} \in W_{Hg_i}$

$g g_j^{-1} = g_i^{-1} h$

$\rho(g)\bar{v}$ = copy of $\rho_H(h)\bar{v}$ in $W_{Hg_i}$

$$\begin{tikzcd} W \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & \bigoplus_{i=1}^{n} W_{Hg_i} \arrow[d, dashed, "\overline{\beta}"] \\ & Z \end{tikzcd}$$

# Induced Representation

First we show the uniqueness of $\overline{\beta}$.
Given $\mathbf{v} \in W_{Hg_j}$, $\rho(g_j)\mathbf{v} \in W_{He}$. Since $\alpha$ is the inclusion of $W$ into $W_{He}$,

$$\alpha(\rho(g_j)\mathbf{v}) = \rho(g_j)\mathbf{v}.$$

$$W \xhookrightarrow{\alpha} W_{He}$$

$$\begin{tikzcd} W \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & \bigoplus_{i=1}^{n} W_{Hg_i} \arrow[d, dashed, "\overline{\beta}"] \\ & Z \end{tikzcd}$$

# Induced Representation

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$\overline{\beta}$ is $G$-linear, so

$$\boxed{\beta(\rho(g_j)\mathbf{v}) = \sigma(g_j)\overline{\beta}(\mathbf{v}).}$$

$$\bar{\beta}(\bar{v}) = \boxed{\sigma(g_j)^{-1}\beta\left(\rho(g_j)\bar{v}\right)}$$

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$\overline{\beta}$ is $G$-linear, so

$$\beta(\rho(g_j)\mathbf{v}) = \sigma(g_j)\overline{\beta}(\mathbf{v}).$$

$$\therefore \overline{\beta}(\mathbf{v}) = \sigma(g_j)^{-1}\beta(\rho(g_j)\mathbf{v}).$$

# Induced Representation

Now we show the existence of $\overline{\beta}$, i.e. the formula we got earlier satisfies the required properties of $\overline{\beta}$.

$$\underbrace{\overline{\beta}(\bar{v}) = \sigma(g_j)^{-1}\beta(\rho(g_j)\bar{v})} \quad \text{for } \bar{v} \in W_{Hg_j}$$

$$\begin{tikzcd} W \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & \bigoplus_{i=1}^{n} W_{Hg_i} \arrow[d, dashed, "\overline{\beta}"] \\ & Z \end{tikzcd}$$

# Induced Representation

Now we show the existence of $\overline{\beta}$, i.e. the formula we got earlier satisfies the required properties of $\overline{\beta}$.

It makes the diagram commute. We just need to check that $\overline{\beta}$ is $G$-linear.

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W \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & \bigoplus_{i=1}^{n} W_{Hg_i} \arrow[d, dashed, "\overline{\beta}"] \\
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Now we show the existence of $\overline{\beta}$, i.e. the formula we got earlier satisfies the required properties of $\overline{\beta}$.

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$$\overline{\beta}\left(\mathbf{v}\right) = \sigma\left(g_j\right)^{-1} \beta\left(\rho\left(g_j\right)\mathbf{v}\right) \text{ for } \mathbf{v} \in W_{Hg_j}.$$

Let $gg_j^{-1} = g_i^{-1}h$ for some $h \in H$.

$$\begin{tikzcd}
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Now we show the existence of $\overline{\beta}$, i.e. the formula we got earlier satisfies the required properties of $\overline{\beta}$.

$$\bar{\beta}(\rho(g)\bar{v}) = \sigma(g)\bar{\beta}(\bar{v})$$

It makes the diagram commute. We just need to check that $\overline{\beta}$ is $G$-linear.

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$$\overline{\beta}\left(\boxed{\rho(g)\mathbf{v}}\right) = \sigma(g_i)^{-1}\beta(\rho(g_i)\rho(g)\mathbf{v})$$

$$\rho(g)\bar{v} \in W_{Hg_i}$$

$$gg_j^{-1} = g_i^{-1}h$$

$$g_i g = hg_j$$

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It makes the diagram commute. We just need to check that $\overline{\beta}$ is $G$-linear.

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Let $gg_j^{-1} = g_i^{-1}h$ for some $h \in H$.

$$\begin{aligned} \overline{\beta}(\rho(g)\mathbf{v}) &= \sigma(g_i)^{-1}\beta(\rho(g_i)\rho(g)\mathbf{v}) \\ &= \sigma(g_i)^{-1}\beta(\rho(h)\rho(g_j)\mathbf{v}) \end{aligned}$$

$\sigma(h)\,\boxed{\beta(\rho(g_j)\bar{v})}$

# Induced Representation

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Let $gg_j^{-1} = g_i^{-1} h$ for some $h \in H$.

$$\begin{aligned} \overline{\beta}\left(\rho\left(g\right)\mathbf{v}\right) &= \sigma\left(g_i\right)^{-1} \beta\left(\rho\left(g_i\right)\rho\left(g\right)\mathbf{v}\right) \\ &= \sigma\left(g_i\right)^{-1} \beta\left(\rho\left(h\right)\rho\left(g_j\right)\mathbf{v}\right) \\ &= \sigma\left(g_i\right)^{-1} \sigma\left(h\right) \beta\left(\rho\left(g_j\right)\mathbf{v}\right) \end{aligned}$$

# Induced Representation

Now we show the existence of $\overline{\beta}$, i.e. the formula we got earlier satisfies the required properties of $\overline{\beta}$.

It makes the diagram commute. We just need to check that $\overline{\beta}$ is $G$-linear.

$$\overline{\beta}(\mathbf{v}) = \sigma(g_j)^{-1}\beta(\rho(g_j)\mathbf{v}) \text{ for } \mathbf{v} \in W_{Hg_j}.$$

$$\begin{tikzcd}
W \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & \bigoplus_{i=1}^{n} W_{Hg_i} \arrow[d, dashed, "\overline{\beta}"] \\
& Z
\end{tikzcd}$$

Let $gg_j^{-1} = g_i^{-1}h$ for some $h \in H$.

$$\begin{aligned}
\overline{\beta}(\rho(g)\mathbf{v}) &= \sigma(g_i)^{-1}\beta(\rho(g_i)\rho(g)\mathbf{v}) \\
&= \sigma(g_i)^{-1}\beta(\rho(h)\rho(g_j)\mathbf{v}) \\
&= \sigma(g_i)^{-1}\sigma(h)\beta(\rho(g_j)\mathbf{v}) \\
&= \sigma(g)\sigma(g_j)^{-1}\beta(\rho(g_j)\mathbf{v})
\end{aligned}$$

$\overline{\beta}(\bar{v})$

# Induced Representation

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W \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & \bigoplus_{i=1}^{n} W_{Hg_i} \arrow[d, dashed, "\overline{\beta}"] \\
& Z
\end{tikzcd}$$

Let $gg_j^{-1} = g_i^{-1}h$ for some $h \in H$.

$$\begin{aligned}
\overline{\beta}\left(\rho\left(g\right)\mathbf{v}\right) &= \sigma\left(g_i\right)^{-1}\beta\left(\rho\left(g_i\right)\rho\left(g\right)\mathbf{v}\right) \\
&= \sigma\left(g_i\right)^{-1}\beta\left(\rho\left(h\right)\rho\left(g_j\right)\mathbf{v}\right) \\
&= \sigma\left(g_i\right)^{-1}\sigma\left(h\right)\beta\left(\rho\left(g_j\right)\mathbf{v}\right) \\
&= \sigma\left(g\right)\sigma\left(g_j\right)^{-1}\beta\left(\rho\left(g_j\right)\mathbf{v}\right) \\
&= \sigma\left(g\right)\overline{\beta}\left(\mathbf{v}\right).
\end{aligned}$$

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$$\mathrm{Ind}_H^G : \mathrm{Rep}(H) \to \mathrm{Rep}(G).$$

Does this work for group homomorphisms $f : G_1 \to G_2$?

# Induced Representation

Let $f : G_1 \to G_2$ be a group homomorphism. If $\rho : G_1 \to \mathrm{GL}(W)$ is a representation of $G_1$, how can we get a representation of $G_2$ using $f$?

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We are going to use the **first isomorphism theorem**: $\operatorname{im} f \cong \frac{G_1}{\operatorname{Ker} f}$. We factorize $f$ as follows:

$$G_1 \longrightarrow \frac{G_1}{\operatorname{Ker} f} \cong \operatorname{im} f \longrightarrow G_2$$

Now, given a representation $W$ of $G_1$, we just have to find a representation of $\operatorname{im} f$. Then we can induce it to a representation of $G_2$ as earlier.

Let $\rho' : \operatorname{im} f \to \mathrm{GL}(W)$ be defined by

$$\rho'(f(g_1)) = \rho(g_1) \in \mathrm{GL}(W). \tag{11}$$

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But for $f(g_1) = f(g_2)$, we know $g_1 g_2^{-1} \in \operatorname{Ker} f$. This motivates us to define

$$W' = \frac{W}{\langle \mathbf{w} - \rho(k)\,\mathbf{w} \mid \mathbf{w} \in W, k \in \operatorname{Ker} f \rangle}. \tag{13}$$

Now $W'$ is a representation of $\operatorname{im} f$, and using this, we can induce a representation of $G_2$.

# Adjunction!

Recall the universal property of induced representation: $\alpha : W \to \operatorname{Ind} W$ is universal in the sense that if $Z$ is another representation of $G$, and $\beta : W \to Z$ is a $H$-linear map, then there exists a unique $G$-linear map $\overline{\beta} : \operatorname{Ind} W \to Z$ such that the following diagram commutes:

$$\begin{tikzcd} W \arrow[r, "\alpha"] \arrow[dr, "\forall\, \beta"'] & \operatorname{Ind} W \arrow[d, dashed, "\exists!\ \overline{\beta}"] \\ & Z \end{tikzcd}$$

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Or in other words,

$$\boxed{\operatorname{Hom}_{\mathsf{Rep}(H)}(W, Z) \cong \operatorname{Hom}_{\mathsf{Rep}(G)}(\operatorname{Ind} W, Z).} \tag{15}$$

# Adjunction!

So we have

$$\mathrm{Hom}_{\mathrm{Rep}(G)}\left(\mathrm{Ind}\, W, Z\right) \cong \mathrm{Hom}_{\mathrm{Rep}(H)}\left(W, \mathrm{Res}\, Z\right). \tag{16}$$

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So we have

$$\mathrm{Hom}_{\mathrm{Rep}(G)}(\mathrm{Ind}\, W, Z) \cong \mathrm{Hom}_{\mathrm{Rep}(H)}(W, \mathrm{Res}\, Z)\,. \tag{16}$$

This also gives a natural isomorphism of functors

$$\begin{tikzcd}
\mathrm{Rep}(H)^{\mathrm{op}} \times \mathrm{Rep}(G) \arrow[rr, bend left, "\mathrm{Hom}_{\mathrm{Rep}(G)}(\mathrm{Ind}(-),-)"] \arrow[rr, bend right, "\mathrm{Hom}_{\mathrm{Rep}(H)}(-,\mathrm{Res}(-))"'] \arrow[rr, phantom, "\Downarrow\eta"] & & \textbf{Sets}
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Therefore, $\mathrm{Ind}_H^G$ is the left-adjoint functor of $\mathrm{Res}_H^G$.

# Adjunction!

## Proposition 2

Let $H \leq K \leq G$. Then $\mathrm{Ind}_H^G = \mathrm{Ind}_K^G \mathrm{Ind}_H^K$.

$$\mathrm{Ind}_H^G W = \mathrm{Ind}_K^G \mathrm{Ind}_H^K W \to \mathrm{Ind}_K^G \bigoplus W_{Hk_i}$$

$$= \bigoplus \left( \bigoplus W_{Hk_i} \right)_{Kg_i'}$$

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$$\mathrm{Ind}_H^G \dashv \mathrm{Res}_H^G = \underbrace{\mathrm{Res}_H^K}_{F_1} \circ \underbrace{\mathrm{Res}_K^G}_{F_2}$$

$$\underbrace{\mathrm{Ind}_K^G}_{G_2} \circ \underbrace{\mathrm{Ind}_H^K}_{G_1}$$

$$G_2 \circ G_1 \dashv F_1 \circ F_2$$

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## Theorem 3

Let $F_1 : \mathcal{C} \to \mathcal{D}$ and $F_2 : \mathcal{D} \to \mathcal{E}$ be left adjoints of the functors $G_1 : \mathcal{D} \to \mathcal{C}$ and $G_2 : \mathcal{E} \to \mathcal{D}$, respectively. Then $F_2 \circ F_1$ is the left adjoint of $G_1 \circ G_2$.

$F_2 \circ F_1 : \mathcal{C} \longrightarrow \mathcal{E}$

$G_1 \circ G_2 : \mathcal{E} \longrightarrow \mathcal{C}$

$\mathrm{Hom}_{\mathcal{E}}((F_2 \circ F_1)(-), -) \cong \mathrm{Hom}_{\mathcal{C}}(-, G_1 \circ G_2(-))$

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## Proof.

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## Proof.

$$\mathrm{Hom}_{\mathcal{E}}\left(F_2\left(F_1(-)\right), -\right) \cong \mathrm{Hom}_{\mathcal{D}}\left(F_1(-), G_2(-)\right)$$

$$\mathrm{Hom}_{\mathcal{E}}(F_2(\underset{F(-)}{\underline{x}}), -) \cong \mathrm{Hom}_{\mathcal{D}}(\underset{F(-)}{\underline{x}}, G_2(-))$$

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## Proof.

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$F_2 \circ F_1 \vdash G_1 \circ G_2$

■

## Theorem 4

Suppose $H \leq G$. Let $U$ be a representation of $G$ and $W$ be a representation of $H$. Then

$$\boxed{U \otimes \mathrm{Ind}_H^G W \cong \mathrm{Ind}_H^G \left( \mathrm{Res}_H^G U \otimes W \right).}$$

$$U \otimes \mathrm{Ind}(W) \cong \mathrm{Ind}\left(\mathrm{Res}\, U \otimes W\right)$$

$$U \otimes \bigoplus W_{Hg_i} \longleftrightarrow \bigoplus \left(\mathrm{Res}\, U \otimes W\right)_{Hg_i}$$

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$$\cong \mathrm{Hom}_{\mathrm{Rep}(G)} \left( \mathrm{Ind} \left( \mathrm{Res}\, U \otimes W \right), - \right)$$

$$- \otimes V \dashv \mathrm{Hom}(V, -)$$

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$$\mathrm{Hom}\left(U \otimes V, \underline{W}\right) \cong \mathrm{Hom}\left(V, \mathrm{Hom}(U, \underline{W})\right)$$

$$\mathrm{Hom}(U, V) \cong U^* \otimes V$$

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\end{aligned}$$

$\rho : G \to GL(V)$

$\rho^* : G \to GL(V^*)$

$\rho^*(g) = \rho(g^{-1})^T$

$\rho|_H : H \to GL(V)$

$(\rho|_H)^* : H \to GL(V^*)$

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&\cong \operatorname{Hom}_{\operatorname{Rep}(H)} (\operatorname{Res}(U) \otimes W, \operatorname{Res}(-)) \\
&\cong \operatorname{Hom}_{\operatorname{Rep}(G)} (\operatorname{Ind}(\operatorname{Res}(U) \otimes W), -)
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&\cong \operatorname{Hom}_{\operatorname{Rep}(G)}\left(\operatorname{Ind}\left(\operatorname{Res}\left(U\right) \otimes W\right), -\right)
\end{aligned}$$

All of these isomorphisms are natural isomorphisms. Therefore, by Yoneda lemma, we are done! ∎

# (Co)-induction?

Category theorists love to reverse the arrows and add the prefix "co". If we reverse the arrows in the universal property of induction, we get coinduction.

$$\begin{tikzcd} W \arrow[r] \arrow[dr, bend right, "\forall \beta"'] & \mathrm{Ind}\, W \arrow[d, dashed, "\exists !\, \bar{\beta}"] \\ & Z \end{tikzcd}$$

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Let $H \subset G$. Given a representation $W$ of $H$, and a representation $V$ of $G$, we say that $V \xrightarrow{\alpha} W$ is a **coinduction** if it satisfies the following universal property:

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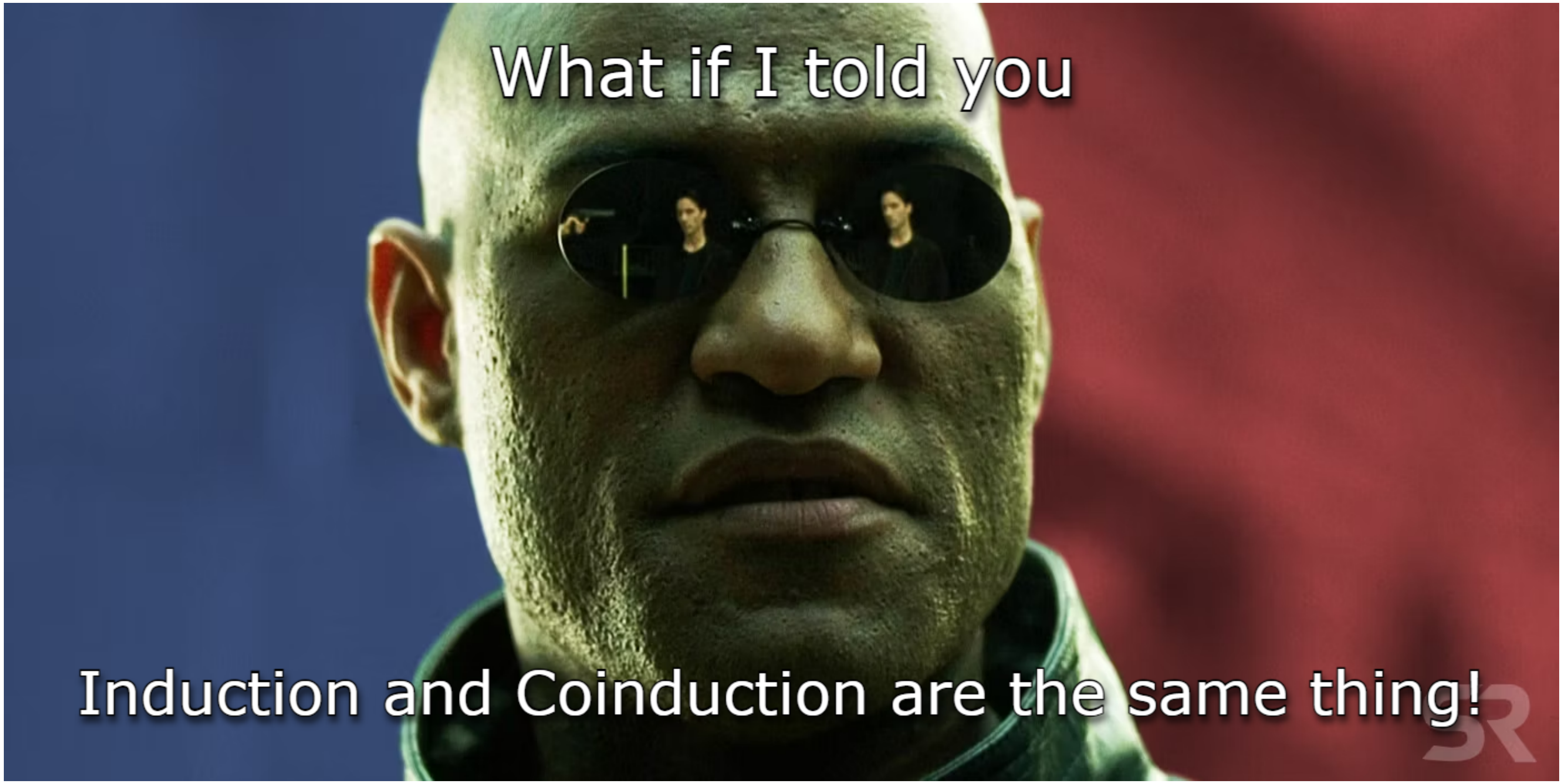
$$\mathrm{Hom}_{\mathrm{Rep}(H)}(\mathrm{Res}(Z), W) \cong \mathrm{Hom}_{\mathrm{Rep}(G)}(Z, \text{Coind } W) \tag{18}$$

So Coind is the **right adjoint** of Res!


What if I told you
Induction and Coinduction are the same thing!

# Induction and Coinduction

Our construction of induction also satisfies the universal property of coinduction!

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Suppose $V$ is another representation of $G$, and $\beta : V \to W$ is a $H$-linear map.

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We need to show the existence and uniqueness of a $G$-linear map $\overline{\beta} : V \to \bigoplus_{i=1}^{n} W_{Hg_i}$ such that the diagram commutes.

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Suppose the corresponding group homomorphism of the representation $V$ is $\sigma : G \to \mathrm{GL}(V)$.

# Induction and Coinduction

First we show the uniqueness of $\overline{\beta}$.

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$\alpha$ is the projection onto the first component
$W_{Hg_1} = W$. Therefore, the first component of $\overline{\beta}(\mathbf{v})$ is
$\alpha \circ \overline{\beta}(\mathbf{v}) = \beta(\mathbf{v})$. In other words, $\overline{\beta}(\mathbf{v}) = (\beta(\mathbf{v}), \ldots)$.

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. So we get

$$\overline{\beta}(\mathbf{v}) = (\beta(\mathbf{v}), \beta(\sigma(g_2)\mathbf{v}), \cdots, \beta(\sigma(g_n)\mathbf{v})).$$

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Now we show the existence of $\overline{\beta}$. Clearly, $\overline{\beta} : V \to \bigoplus_{i=1}^{n} W_{Hg_i}$ defined by

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makes the diagram commute! So we only need to show that it is $G$-linear.

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\end{aligned}$$

$g_i g = h_i g_{j(i)}$ is equivalent to $gg_{j(i)}^{-1} = g_i^{-1} h_i$. Then $\rho(g)$ takes $W_{Hg_{j(i)}}$ on to $W_{Hg_i}$.

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So $\overline{\beta}\left(\sigma(g)\,\mathbf{v}\right) = \rho(g)\,\beta(\mathbf{v})$.

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$$\text{Coind}\, W = \{f : G \to W \mid f(hg) = \rho(h) f(g) \ \forall h \in H\} \tag{19}$$

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This is not isomorhic to $\bigoplus W_{Hg_i}$ when the index of the subgroup is infinite.

$\mathbb{K}[G] = \text{span}\, G$

$\sum a_g g$.

$\bigoplus W_{g_i H}$

$\mathbb{K}[G] \otimes_{\mathbb{K}[H]} W$

# References


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Thank you for joining!

The slides are available in my webpage
`https://atonurc.github.io/assets/catrep_talk_2.pdf`