The Baker–Campbell–Hausdorff Formula and Lie's Third Theorem

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Lie Groups

Definition 1

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A **Lie group** is a group G, that also has a manifold structure; and these two structures are compatible, i.e. the multiplication map

$$\begin{aligned} m : G \times G &\to G \\ (g, h) &\mapsto gh \end{aligned}$$

and the inverse map

$$\begin{aligned} i : G &\to G \\ g &\mapsto g^{-1} \end{aligned}$$

are smooth.

Examples of Lie Groups

Example 1

- $(\mathbb{R}, +)$

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Lie Algebra

Definition 2 (Lie algebra)

A **Lie algebra** over a field $\mathbb{K}$ is a $\mathbb{K}$-vector space V equipped with a bilinear operation $[-,-] : V \times V \to V$ (called the **Lie bracket**) such that

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$$[[u,v], w] + [[w,u], v] + [[v,w], u] = 0. \tag{1}$$

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$$[[u, v], w] + [[w, u], v] + [[v, w], u] = 0. \tag{1}$$

(1) is also known as the Jacobi identity.

Examples of Lie Algebra

Example 2

If V is an associative $\mathbb{K}$-algebra with respect to the product $\cdot$, then we can define the Lie bracket as

$$[v_1, v_2] = v_1 \cdot v_2 - v_2 \cdot v_1. \tag{2}$$

Then V is a Lie algebra with respect to $[-, -]$.

Example 3

Let M be a manifold, and $\mathfrak{X}(M)$ be the space of smooth vector fields on M. Given two vector fields X and Y, XY is not a vector field, in general. However, $XY - YX$ is a vector field.

Examples of Lie Algebra

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$$[X, Y] = XY - YX, \tag{3}$$

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and then $\mathfrak{X}(M)$ becomes a Lie algebra with respect to $[-, -]$.

Note that $\mathfrak{X}(M)$ is, in general, an infinite dimensional Lie algebra. For example, if we take $M = \mathbb{R}$, then

$$\frac{\mathrm{d}}{\mathrm{d}x},\ x\frac{\mathrm{d}}{\mathrm{d}x},\ x^2\frac{\mathrm{d}}{\mathrm{d}x},\ x^3\frac{\mathrm{d}}{\mathrm{d}x}, \ldots$$

are all linearly independent vector fields on $\mathbb{R}$.

Examples of Lie Algebra

For a Lie group, we can get a finite dimensional subalgebra of $\mathfrak{X}(G)$. It's the space of all left-invariant vector fields on G.

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Definition 3

A vector field $X \in \mathfrak{X}(G)$ is called **left-invariant** if it is L_g-invariant for every $g \in G$, i.e. $(L_g)_* X = X$. Then $\mathfrak{L}(G)$ denotes the space of all left-invariant vector fields.

Left-invariant Vector Field

Theorem 4

$\mathfrak{L}(G)$ is a Lie algebra with respect to the Lie bracket $[X, Y] = XY - YX$.

Proof.

See any introductory Lie groups textbook. (Don't ask a toddler on the street please.) ■

Left-invariant Vector Field

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$\mathfrak{L}(G)$ is a Lie algebra with respect to the Lie bracket $[X, Y] = XY - YX$.

Proof.

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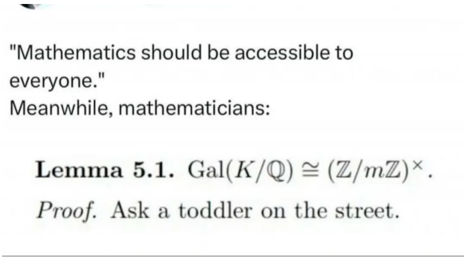

Theorem 5

$\mathfrak{L}(G)$ is finite dimensional. Moreover, $\mathfrak{L}(G) \cong T_e G$.

Proof.

Again, see any introductory Lie groups textbook. ∎

Lie Algebra Associated to a Lie Group

If G is a Lie group, then $\mathfrak{g} = \mathfrak{L}(G)$ (or equivalently, $\mathfrak{g} = T_e G$) is called the Lie algebra associated to G.

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Exponential map

The exponential map is a way of going from the Lie algebra $\mathfrak{g}$ to the Lie group G. So, it's a map

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Given $X \in \mathfrak{g} = \mathfrak{L}(G) \cong T_e G$, let $\alpha : \mathbb{R} \to G$ be the integral curve of X starting at the identity, i.e. $\alpha(0) = e$. Then we define

$$\exp(X) = \alpha(1). \tag{4}$$

Examples of Exponential Map

Example 6

- For $G = \mathbb{R}_{>0}$, $\mathfrak{g} = T_e G = T_1 \mathbb{R}_{>0} \cong \mathbb{R}$. Then the exponential map $\exp : \mathfrak{g} \to G$ is the usual exponential map

$$\begin{aligned} \exp : \mathbb{R} &\to \mathbb{R}_{>0} \\ t &\mapsto e^t. \end{aligned} \tag{5}$$

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- For $G = \mathrm{GL}(n, \mathbb{R})$, $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{R}) = \mathrm{End}(\mathbb{R}^n) = \mathbb{R}^{n \times n}$. Then the exponential map $\exp : \mathfrak{g} \to G$ is the matrix exponential

$$\begin{aligned} \exp : \mathbb{R}^{n \times n} &\to \mathrm{GL}(n, \mathbb{R}) \\ A &\mapsto e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n. \end{aligned} \tag{6}$$

Example 7

- For $G = S^1$, $\mathfrak{g} = T_1 S^1 \cong \mathbb{R}$. Then the exponential map $\exp : \mathfrak{g} \to G$ is

$$\begin{aligned} \exp : \mathbb{R} &\to S^1 \\ t &\mapsto e^{it}. \end{aligned} \tag{7}$$

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- For $G = \mathrm{SL}(n, \mathbb{R})$, $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$ = all traceless $n \times n$ matrices. Then the exponential map $\exp : \mathfrak{g} \to G$ is the matrix exponential

$$\begin{aligned} \exp : \mathfrak{sl}(n, \mathbb{R}) &\to \mathrm{SL}(n, \mathbb{R}) \\ A &\mapsto e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n. \end{aligned} \tag{8}$$

Properties of Exponential Map

Proposition 8

If $f : G \to H$ is a Lie group homomorphism, then $(df)_e : T_eG \to T_eH$ is a Lie algebra homomorphism. Furthermore, the following diagram commutes:

$$\begin{tikzcd} T_eG \arrow[r, "(df)_e"] \arrow[d, "\exp_G"'] & T_{e'}H \arrow[d, "\exp_H"] \\ G \arrow[r, "f"'] & H \end{tikzcd} \tag{9}$$

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Proposition 9

$(d\exp)_{\mathbf{0}} : T_{\mathbf{0}}(T_e G) \cong T_e G \to T_e G$ is the identity map.

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$(d\exp)_{\mathbf{0}} : T_{\mathbf{0}}(T_eG) \cong T_eG \to T_eG$ is the identity map.

Therefore, exp is a local diffeomorphism near $\mathbf{0} \in T_eG$, i.e. there is a neighborhood $U \ni e$ and a neighborhood $\mathfrak{u} \ni \mathbf{0}$ such that exp is a diffeomorphism of $\mathfrak{u}$ and U. Let's write $\log = \exp^{-1}$.

$$\begin{tikzcd} \mathfrak{u} \subseteq \mathfrak{g} \arrow[r, bend left, "\exp"] & U \subseteq G \arrow[l, bend left, "\log"] \end{tikzcd}$$

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But it's not true for non-abelian Lie groups, for example $\mathrm{GL}(n,\mathbb{R})$ or $\mathrm{SL}(n,\mathbb{R})$.

To what extent does $\exp : \mathfrak{g} \to G$ fail to be a group homomorphism for nonabelian G?

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BCH Formula

Proposition 10

If $X, Y \in \mathfrak{g}$ with $[X, Y] = 0$,

$$\exp(X)\exp(Y) = \exp\left(X + Y\right).$$

BCH Formula

Theorem 11 (BCH Formula)

For "small" $X, Y \in \mathfrak{g}$,

$$\exp(X)\exp(Y) = \exp\left(X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}([X, [X, Y]] + [Y, [Y, X]]) + \cdots\right)$$

The latter terms are nested commutators. Here, "small" $X, Y \in \mathfrak{g}$ means that there exists a neighborhood $\mathfrak{u}$ of $\mathbf{0} \in \mathfrak{g}$ such that for $X, Y \in \mathfrak{u}$, the RHS converges, and the equation above holds.

BCH Formula

The exact coefficients, or the exact terms of the BCH formula have been computed. The following is taken from [3].

Dynkin's formula:

$$\mu(X,Y) = Y + X + \sum_{k=1}^{\infty} \frac{(-1)^k}{k+1} \sum_{\substack{l_1,\dots,l_k \geq 0,\\ m_1,\dots,m_k \geq 0,\\ l_j+m_j>0}} \frac{1}{l_1+\ldots+l_k+1} \tag{1.7.3}$$

$$\times \left(\frac{(\operatorname{ad} X)^{l_1}}{l_1!} \circ \frac{(\operatorname{ad} Y)^{m_1}}{m_1!} \circ \cdots \circ \frac{(\operatorname{ad} X)^{l_k}}{l_k!} \circ \frac{(\operatorname{ad} Y)^{m_k}}{m_k!} \right)(X).$$

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But the exact formula is rarely useful. What's useful is the mere fact that this formula exists! Because it gives a way of multiplying Lie group elements (near the identity) just from the Lie bracket.

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Lie Algebra to Lie Group

We already know that if $f : G \to H$ is a Lie group homomorphism, then $(df)_e : T_e G \to T_e H$ is a Lie algebra homomorphism, and the following diagram commutes:

$$\begin{tikzcd} T_e G \arrow[r, "(df)_e"] \arrow[d, "\exp_G"'] & T_{e'} H \arrow[d, "\exp_H"] \\ G \arrow[r, "f"'] & H \end{tikzcd} . \tag{10}$$

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Now we want to know whether the converse holds, i.e. given any Lie algebra homomorphism $\Phi : \mathfrak{g} \to \mathfrak{h}$, is there a corresponding Lie group homomorphism $f : G \to H$ such that $\Phi = (df)_e$?

Lie Algebra to Lie Group

$$
\begin{tikzcd}
T_e G \arrow[r, "\Phi"] \arrow[d, "\exp_G"'] & T_{e'} H \arrow[d, "\exp_H"] \\
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We want to define something like $f(\exp(X)) = \exp(\Phi(X))$. But it has several issues:

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- Can every element $g \in G$ be written as $\exp X$ for some $X \in \mathfrak{g}$?
- If $g = \exp(X)$, is this unique? If not, why is $\exp(\Phi(X))$ independent of the choice of X?

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- If $g = \exp(X)$, is this unique? If not, why is $\exp(\Phi(X))$ independent of the choice of X?
- Is f a group homomorphism?

Lie Algebra to Lie Group

Consider $G = S^1$ and $H = \mathbb{R}$. Both of them have isomorphic Lie algebras, $\mathfrak{g} = \mathfrak{h} = \mathbb{R}$. So there is a Lie algebra isomorphism $\Phi : \mathfrak{g} \to \mathfrak{h}$. But there is no corresponding nontrivial Lie group homomorphism. Assume there is one, say ϕ.

$$\begin{tikzcd}
\mathfrak{g} = \mathbb{R} \arrow[r, "\Phi"] \arrow[d, "\exp_{S^1}"'] & \mathfrak{h} = \mathbb{R} \arrow[d, "\exp_{\mathbb{R}}"] \\
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$$\varphi\left(e^{in\theta}\right) = nx, \text{ for any } n \in \mathbb{Z}.$$

But since S^1 is compact, $\varphi\left(S^1\right)$ needs to be compact, i.e. closed and bounded. But it's not bounded if $x \neq 0$. Therefore, the only Lie group homomorphism $\varphi : S^1 \to \mathbb{R}$ is the trivial one. But its corresponding Lie algebra homomorphism is also trivial, NOT an isomorphism.

Lie Algebra to Lie Group

Therefore, the group homomorphism $f : G \to H$ cannot always be defined. We need some extra condition(s).

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So we can guess that for simply connected Lie group G, such a group homomorphism $f : G \to H$ exists for every Lie group homomorphism $\Phi : \mathfrak{g} \to \mathfrak{h}$.

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Turns out, this is actually correct!!

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1. We define $f(\exp(X)) = \exp(\Phi(X))$ on a neighborhood of $e \in G$.
2. We extend this definition by taking **ANY** path from $e \in G$ to $g \in G$, and then define $f(g)$.
3. Then we show that this is independent of the path we choose, since any two path are homotopic, because G is simply connected.

Step 1

We take a neighborhood $\mathfrak{u} \subseteq \mathfrak{g}$ and $\mathfrak{v} \subseteq \mathfrak{h}$ such that $\Phi(\mathfrak{u}) \subseteq \mathfrak{v}$, and BCH formula holds on $\mathfrak{u}$ and $\mathfrak{v}$. Since exp is a local diffeomorphism, it's a diffeomorphism of $\mathfrak{u}$ and $\exp(\mathfrak{u}) =: U$.

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So, on $U \subseteq G$, we can uniquely define $f(\exp(X)) = \exp(\Phi(X))$. Then why is f a (locally) group homomorphism?

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The answer is BCH formula!!

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$$\begin{aligned} f(\exp(X)\exp(Y)) &= f\left(\exp\left(X + Y + \frac{1}{2}[X, Y] + \cdots\right)\right) \\ &= \exp\left(\Phi\left(X + Y + \frac{1}{2}[X, Y] + \cdots\right)\right) \end{aligned}$$

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&= \exp\left(\Phi(X) + \Phi(Y) + \frac{1}{2}[\Phi(X), \Phi(Y)] + \cdots\right) \\
&= \exp(\Phi(X))\exp(\Phi(Y))
\end{aligned}
$$

Step 1

We want to show that $f(\exp(X)\exp(Y)) = f(\exp X)f(\exp Y)$.

$$\begin{aligned}
f(\exp(X)\exp(Y)) &= f\left(\exp\left(X + Y + \frac{1}{2}[X, Y] + \cdots\right)\right) \\
&= \exp\left(\Phi\left(X + Y + \frac{1}{2}[X, Y] + \cdots\right)\right) \\
&= \exp\left(\Phi(X) + \Phi(Y) + \frac{1}{2}[\Phi(X), \Phi(Y)] + \cdots\right) \\
&= \exp(\Phi(X))\exp(\Phi(Y)) \\
&= f(\exp X)f(\exp Y).
\end{aligned}$$

So f acts like a group homomorphism on U.

Step 2: Define $f(g)$

Now take any $g \in G$. Consider any path $\gamma : [0, 1] \to G$ from e to g.

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Since $[0,1]$ is compact, there is a partition $0 = t_0 < t_1 < t_2 < \cdots < t_m = 1$ such that for all $s, t \in [t_i, t_{i+1}]$ and for each i,

$$\gamma(t)\gamma(s)^{-1} \in \exp(\mathfrak{u}) = U. \tag{14}$$

We call such a partition a *"good partition" for* γ.

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We call such a partition a *"good partition" for* γ. Then we write

$$g = \left(\gamma(t_m)\gamma(t_{m-1})^{-1}\right)\left(\gamma(t_{m-1})\gamma(t_{m-2})^{-1}\right)\cdots\left(\gamma(t_2)\gamma(t_1)^{-1}\right)\left(\gamma(t_1)\gamma(t_0)^{-1}\right).$$

Then we define

$$\begin{aligned} f(g) = f\left(\gamma(t_m)\gamma(t_{m-1})^{-1}\right) f\left(\gamma(t_{m-1})\gamma(t_{m-2})^{-1}\right)\cdots \\ \cdots f\left(\gamma(t_2)\gamma(t_1)^{-1}\right) f\left(\gamma(t_1)\gamma(t_0)^{-1}\right). \end{aligned} \tag{15}$$

Step 2: Independence of Partition

There can be another partition of the interval $[0,1]$ such that for each i, for all $s, t \in \left[t_i', t_{i+1}'\right]$, $\gamma(t)\gamma(s)^{-1} \in \exp(\mathfrak{u}) = U$. But $f(g)$ stays invariant!

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we add a point s in between t_i and t_{i+1}. Then the value of $f(g)$ with respect to this refined partition is

$$f(g) = \cdots f\left(\gamma(t_{i+1})\gamma(s)^{-1}\right) f\left(\gamma(s)\gamma(t_i)^{-1}\right) \cdots$$

Step 2: Independence of Partition

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which is the same as the value obtained from the previous partition! Therefore, $f(g)$ is invariant under refinement of partition.

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we add a point s in between t_i and t_{i+1}. Then the value of $f(g)$ with respect to this refined partition is

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which is the same as the value obtained from the previous partition! Therefore, $f(g)$ is invariant under refinement of partition. Given two partitions, one may take their union, which is refinement of both partitions, and thus we achieve the independence of partition!

Step 3: Independence of Path

Simply-connectedness comes into play here!

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$$H(0,-) = \beta, \quad H(1,-) = \gamma; \tag{16}$$

$$H(-,0) = e, \quad H(-,1) = g. \tag{17}$$

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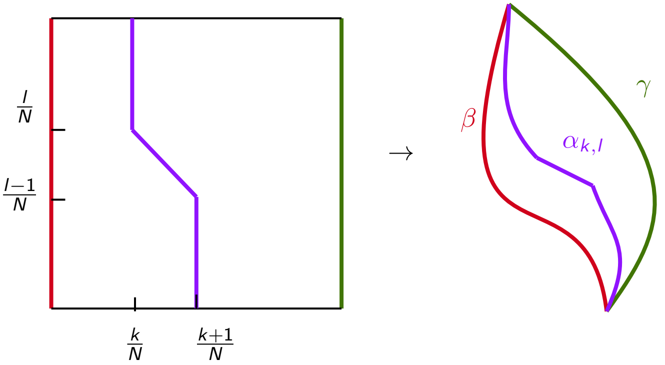
Since $[0,1]^2$ is compact, there exists a positive integer N such that for every $(s,t), (s',t') \in [0,1]^2$ with $|s - s'| < \frac{2}{N}, |t - t'| < \frac{2}{N}$,

$$H(s,t)\, H(s',t')^{-1} \in \exp(\mathfrak{u}) = U. \tag{18}$$

Now we define a sequence of paths $\alpha_{k,l}$ for $k = 0, 1, 2, \ldots, N-1$ and $l = 0, 1, 2, \ldots, N$:

$$\alpha_{k,l}(t) = \begin{cases} H\left(\frac{k+1}{N}, t\right) & \text{if } 0 \leq t \leq \frac{l-1}{N}, \\ H\left(\frac{k}{N}, t\right) & \text{if } \frac{l}{N} \leq t \leq 1, \\ H\,(\text{connecting diagonal}) & \text{if } \frac{l-1}{N} \leq t \leq \frac{l}{N}. \end{cases} \tag{19}$$



Step 3: Independence of Path

We will show that the value of $f(g)$ is the same across all $\alpha_{k,l}$'s. Then we're done, since $\alpha_{0,0} = \beta$ and $\alpha_{N,0} = \gamma$.

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Note that $\alpha_{k,l}$ and $\alpha_{k,l+1}$ agree everywhere except on $\left[\frac{l-1}{N}, \frac{l+1}{N}\right]$. We consider the partition

$$0, \frac{1}{N}, \ldots, \frac{l-1}{N}, \frac{l+1}{N}, \frac{l+2}{N}, \ldots, 1.$$

This is a good partition for both $\alpha_{k,l}$ and $\alpha_{k,l+1}$.

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Since the value of $f(g)$ across a path depends only on the points of the partition, the value of $f(g)$ is the same across $\alpha_{k,l}$ and $\alpha_{k,l+1}$.

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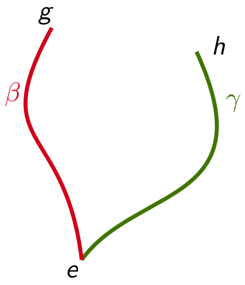
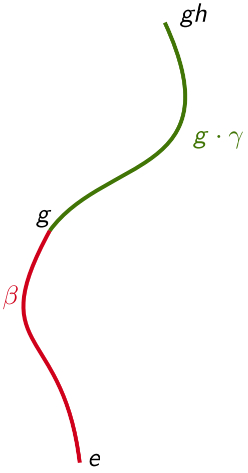
Therefore, $f(g)$ is well-defined.

Step 4: f is a group homomorphism

Let β and γ be paths from e to g and h, respectively. Then from that, we can construct a path from e to gh in the following way:



$\Longrightarrow$



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$$\alpha(t) = \begin{cases} \beta(2t) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ g \cdot \gamma(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases} \tag{20}$$

We then combine the good partitions for γ and β to find a good partition for α.

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$$\frac{t_0}{2} < \frac{t_1}{2} < \cdots < \frac{t_m}{2} = \frac{s_0+1}{2} < \frac{s_1+1}{2} < \cdots < \frac{s_n+1}{2}.$$

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$$f\left(gh\right)=\prod_{i=n}^{1}f\left(\alpha\left(\frac{s_i+1}{2}\right)\alpha\left(\frac{s_{i-1}+1}{2}\right)^{-1}\right)\prod_{j=m}^{1}f\left(\alpha\left(\frac{t_j}{2}\right)\alpha\left(\frac{t_{j-1}}{2}\right)^{-1}\right)$$

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$$\begin{aligned}f\left(gh\right)&=\prod_{i=n}^{1}f\left(\alpha\left(\frac{s_i+1}{2}\right)\alpha\left(\frac{s_{i-1}+1}{2}\right)^{-1}\right)\prod_{j=m}^{1}f\left(\alpha\left(\frac{t_j}{2}\right)\alpha\left(\frac{t_{j-1}}{2}\right)^{-1}\right)\\&=\prod_{i=n}^{1}f\left(g\gamma(s_i)\gamma(s_{i-1})^{-1}g^{-1}\right)\prod_{j=m}^{1}f\left(\beta(t_i)\beta(t_{i-1})^{-1}\right)\end{aligned}$$

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$$\begin{aligned} f(gh) &= \prod_{i=n}^{1} f\left(\alpha\left(\frac{s_i+1}{2}\right)\alpha\left(\frac{s_{i-1}+1}{2}\right)^{-1}\right) \prod_{j=m}^{1} f\left(\alpha\left(\frac{t_j}{2}\right)\alpha\left(\frac{t_{j-1}}{2}\right)^{-1}\right) \\ &= \prod_{i=n}^{1} f\left(g\gamma(s_i)\gamma(s_{i-1})^{-1}g^{-1}\right) \prod_{j=m}^{1} f\left(\beta(t_i)\beta(t_{i-1})^{-1}\right) \\ &= f\left(ghg^{-1}\right) f(g). \end{aligned}$$

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So f is a group homomorphism.

Step 5: $(df)_e = \Phi$

$$\begin{tikzcd} T_eG \arrow[r, "\Phi"] \arrow[d, "\exp_G"'] & T_{e'}H \arrow[d, "\exp_H"] \\ G \arrow[r, "f"'] & H \end{tikzcd} \tag{23}$$

For $X_e \in T_eG$,

$$(df)_e X_e = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f\left(\exp(tX_e)\right)$$

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So $\Phi = (df)_e$.

Summary so far

So, we have proved the following:

Theorem 12 (Lie's second theorem)

If G is a simply connected Lie group, and $\Phi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism, there is a Lie group homomorphism $f : G \to H$ such that $\Phi = (df)_e$, and the following diagram commutes:

$$\begin{tikzcd} T_e G \arrow[r, "\Phi"] \arrow[d, "\exp_G"'] & T_{e'} H \arrow[d, "\exp_H"] \\ G \arrow[r, "f"'] & H \end{tikzcd} \tag{24}$$

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Furthermore, the Lie group is **unique** if we require it to be simply connected!

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Theorem 14 (Ado's theorem)

Every finite dimensional real Lie algabra is isomorphic to a Lie subalgebra of $\mathfrak{gl}(n, \mathbb{R})$ for some n.

Proof

By Ado's theorem, $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}(n, \mathbb{R})$.

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Theorem 15 (Chevalley's theorem, also known as Lie's first theorem)

Let G be a Lie group, and $\mathfrak{g}$ its Lie algebra. Then there is a one-to-one correspondence

$$\{\text{connected Lie subgroups } H \subseteq G\} \leftrightarrow \{\text{Lie subalgebras } \mathfrak{h} \subseteq \mathfrak{g}\}.$$

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Therefore, there is a connected Lie subgroup of $G' \subseteq \mathrm{GL}(n, \mathbb{R})$ whose Lie algebra is $\mathfrak{g}$.

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Therefore, there is a connected Lie subgroup of $G' \subseteq \mathrm{GL}(n, \mathbb{R})$ whose Lie algebra is $\mathfrak{g}$.

Take G to be the simply connected cover of G'. It has a group structure, and its Lie algebra is the same as the Lie algebra of G (details are in [7]).

Proof

Hence, given **any** finite dimensional Lie algebra $\mathfrak{g}$, there is a simply connected Lie group G whose Lie algebra is $\mathfrak{g}$. As for the uniqueness, we'll use the following:

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If G and H are simply connected Lie groups with $\mathfrak{g} \cong \mathfrak{h}$, then $G \cong H$.

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$$\begin{tikzcd}
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G \arrow[r, "f"'] & H \arrow[r, "f'"'] & G
\end{tikzcd}$$

Therefore, $(d(f' \circ f))_e = \mathbb{1}_{\mathfrak{g}}$.

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Hence, given **any** finite dimensional Lie algebra $\mathfrak{g}$, there is a simply connected Lie group G whose Lie algebra is $\mathfrak{g}$. As for the uniqueness, we'll use the following:

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G \arrow[r, "f"'] & H \arrow[r, "f'"'] & G
\end{tikzcd}$$

Therefore, $(d(f' \circ f))_e = \mathbb{1}_{\mathfrak{g}}$. Since any connected Lie group is generated by any neighborhood of identity and exp is a local diffeomorphism, we have $f' \circ f = \mathrm{id}_G$. Similarly, $f \circ f' = \mathrm{id}_H$. ■

Proof

Therefore, for any finite dimensional Lie algebra $\mathfrak{g}$, there is a unique simply connected Lie group G whose Lie algebra is $\mathfrak{g}$.

References

[1] A. Kirillov Jr., *An Introduction to Lie Groups and Lie Algebras*, Cambridge Studies in Advanced Mathematics, vol. 113, Cambridge University Press, Cambridge, 2008.

[2] B. C. Hall, *Lie Groups, Lie Algebras, and Representations*, 2nd ed., Graduate Texts in Mathematics, vol. 222, Springer, New York, 2015.

[3] J. A. C. Kolk and J. J. Duistermaat, *Lie Groups*, Universitext, Springer, Berlin, 2000.

[4] T. Bröcker and T. tom Dieck, *Representations of Compact Lie Groups*, Springer–Verlag, New York, 1985.

[5] P. Etingof, *Lie Groups and Lie Algebras*, Lecture notes, MIT, 2020-2021; arXiv:2201.09397 [math.RT], 2022. URL: `arxiv.org/pdf/2201.09397.pdf`

[6] C. Daly, *Baker-Campbell-Hausdorff Formula*, available at `https://www.math.brown.edu/cdaly2/Notes/BCH_Formula.pdf`.

[7] Expository notes I've written: `https://atonurc.github.io/liegrp`.

Thank you for joining!

The slides are available in my webpage

https://atonurc.github.io/liegrp/BCH_Lie3_Talk.pdf