

# Adjoint Action and Lie Algebra

ATONU ROY CHOWDHURY

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This is a continuation of [15], where we explored the Lie algebra  $\mathfrak{g}$  associated with a Lie group  $G$ , and we constructed the exponential map  $\exp : \mathfrak{g} \rightarrow G$ . We have seen that  $\exp$  is a local diffeomorphism, i.e. there is a neighborhood  $U \ni e$  and a neighborhood  $\mathfrak{u} \ni \mathbf{0}$  such that  $\exp$  is a diffeomorphism of  $\mathfrak{u}$  and  $U$ . Let's write  $\log = \exp^{-1}$ .

$$\begin{array}{ccc} \mathfrak{u} \subseteq \mathfrak{g} & \xrightarrow{\exp} & U \subseteq G \\ & \xleftarrow{\log} & \end{array}$$

Now, we know that a connected Lie group is generated by any neighborhood of the identity. So if  $\exp$  was a group homomorphism, i.e. if  $\exp(X_e + Y_e) = \exp(X_e) \exp(Y_e)$ , then  $\exp$  would've been surjective, since it's surjective on  $U$  and that generates the whole group  $G$ . In fact, that's indeed the case for the connected abelian Lie groups  $\mathbb{R}$ ,  $S^1$ . In this note we'll explore to what extent this holds.

## §1 Adjoint Representation

Previously we gave a Lie bracket on  $\mathfrak{L}(G)$ , which was the usual Lie bracket of (left-invariant) vector fields,

$$[X, Y] = XY - YX. \tag{1}$$

We also saw that  $\mathfrak{L}(G) \cong T_e G$  naturally as vector spaces. So this natural isomorphism gives a Lie bracket on  $T_e G$ . But this doesn't really reflect how the Lie bracket on  $T_e G$  works. Furthermore, it doesn't give any insight on the group structure of the Lie group, since the bracket  $[X, Y] = XY - YX$  can be defined on  $\mathfrak{X}(M)$ , for any manifold  $M$ . Now we shall give a description of the Lie bracket on  $\mathfrak{g} = T_e G$ , that we'll prove to be equivalent to the Lie bracket obtained from  $\mathfrak{L}(G)$ .

We start with the conjugation map

$$\begin{aligned} \text{conj} : G \times G &\rightarrow G \\ (g, h) &\mapsto ghg^{-1}. \end{aligned} \tag{2}$$

This is a smooth map since the multiplication map  $m : G \times G \rightarrow G$  and the inversion map  $i : G \rightarrow G$  are smooth. We denote

$$\text{conj}_g = \text{conj}(g, -) : G \rightarrow G, \quad (3)$$

which is a diffeomorphism. Therefore, its differential at identity,

$$(d \text{conj}_g)_e : T_e G \rightarrow T_e G,$$

is an invertible map between vector spaces, i.e.  $(d \text{conj}_g)_e \in \text{GL}(T_e G)$ . So we have the map

$$\begin{aligned} \text{Ad} : G &\rightarrow \text{GL}(T_e G) = \text{GL}(\mathfrak{g}) \\ g &\mapsto (d \text{conj}_g)_e. \end{aligned} \quad (4)$$

This is a *representation* of the Lie group  $G$  on its Lie algebra  $\mathfrak{g} = T_e G$ . This representation is called the **adjoint representation**.

**Definition 1.1** (Representation). A **representation** of a Lie group  $G$  on a finite dimensional  $\mathbb{R}$ -vector space  $V$  is a homomorphism of Lie groups  $\rho : G \rightarrow \text{GL}(V)$ .

Here, since  $V$  is a finite dimensional vector space, so is  $\text{End}(V)$ , because its dimension is  $(\dim V)^2$ . Therefore,  $\text{End}(V)$  has a manifold structure. Then  $\text{GL}(V)$  is easily seen to be an open subspace of  $\text{End}(V)$ , as

$$\text{GL}(V) = \det^{-1}(\mathbb{R} \setminus \{0\}), \quad (5)$$

and the determinant function is continuous. Therefore,  $\text{GL}(V)$  has a manifold structure. It's also a group. Furthermore, the multiplication map and inverse maps are easily seen to be smooth maps, upon choosing a basis for  $V$ . Therefore,  $\text{GL}(V)$  is a Lie group. Let's now show that for  $V = T_e G$ ,  $\text{Ad} : G \rightarrow \text{GL}(T_e G)$  is indeed a representation of the Lie group  $G$ .

### Proposition 1.1

$\text{Ad} : G \rightarrow \text{GL}(T_e G)$  is a representation of the Lie group  $G$  on  $T_e G$ .

*Proof.* First,  $\text{Ad} : G \rightarrow \text{GL}(T_e G)$  is a group homomorphism, since

$$\text{conj}_g \circ \text{conj}_{g'}(h) = gg'h(g')^{-1}g^{-1} = (gg')h(gg')^{-1} = \text{conj}_{gg'}(h), \quad (6)$$

so that  $\text{conj}_g \circ \text{conj}_{g'} = \text{conj}_{gg'}$ . Therefore,

$$\begin{aligned} \text{Ad}(g) \circ \text{Ad}(g') &= (d \text{conj}_g)_e \circ (d \text{conj}_{g'})_e \\ &= (d(\text{conj}_g \circ \text{conj}_{g'}))_e \\ &= (d \text{conj}_{gg'})_e = \text{Ad}(gg'). \end{aligned} \quad (7)$$

For showing the smoothness of  $\text{Ad}$ , we shall use the following result: (Lee [1], Proposition 3.21)

If  $F : M \rightarrow N$  is smooth, then its global differential  $dF : TM \rightarrow TN$  is smooth.

Now, since the conjugation map  $\text{conj} : G \times G \rightarrow G$  is smooth, then so is

$$d\text{conj} : T(G \times G) \cong TG \times TG \rightarrow TG.$$

Now, consider the composition:

$$\begin{array}{ccccc} G \times T_e G & \hookrightarrow & TG \times TG & \xrightarrow{d\text{conj}} & TG \\ (g, X_e) & \mapsto & (\mathbf{0}_g, X_e) & \mapsto & (d\text{conj})(\mathbf{0}_g, X_e) \end{array} \quad (8)$$

where  $\mathbf{0}_g \in T_g G$  is the zero vector of the vector space  $T_g G$ . We now claim that

$$(d\text{conj})(\mathbf{0}_g, X_e) = (d\text{conj}_g)_e X_e. \quad (9)$$

The LHS is actually  $(\text{conj} \circ c)'(0)$ , where  $c : (-\varepsilon, \varepsilon) \rightarrow G \times G$  is a curve satisfying  $c(0) = (g, e)$  and  $c'(0) = (\mathbf{0}_g, X_e)$ . We can simply take  $c(t) = (g, \exp(tX_e))$ .

$$\begin{aligned} (\text{conj} \circ c)'(0) &= \left. \frac{d}{dt} \right|_{t=0} (\text{conj}(g, \exp(tX_e))) \\ &= \left. \frac{d}{dt} \right|_{t=0} (\text{conj}_g(\exp(tX_e))) \\ &= (\text{conj}_g \circ \alpha)'(0), \end{aligned} \quad (10)$$

where  $\alpha(t) = \exp(tX_e)$  is the curve that satisfies  $\alpha(0) = e$  and  $\alpha'(0) = X_e$ . Therefore,  $(\text{conj}_g \circ \alpha)'(0) = (d\text{conj}_g)_e X_e$ . Let's denote the composition (8) by  $\mathcal{F}$ .

$$\begin{aligned} \mathcal{F} : G \times T_e G &\rightarrow T_e G \\ (g, X_e) &\mapsto (d\text{conj}_g)_e X_e = \text{Ad}(g)(X_e). \end{aligned} \quad (11)$$

$\mathcal{F}$  is smooth since it's the composition of two smooth maps. Now, we need to show that  $\mathcal{F} : G \times T_e G \rightarrow T_e G$  being smooth implies the smoothness of its “currying”  $\text{Ad} : G \rightarrow \text{Hom}(T_e G, T_e G) = \text{End}(T_e G)$ . However, this is not immediately obvious, since the category of manifolds don't have exponential objects. But we have the following result:

### Lemma 1.2

Let  $V, W$  be finite dimensional vector spaces, and let  $M$  be a manifold. If  $F : M \times V \rightarrow W$  is a smooth map, then so is its “currying”  $G : M \rightarrow \text{Hom}(V, W)$  given by  $G(p) = F(p, -)$ .

*Proof.* Upon choosing a basis  $\{e_1, \dots, e_n\}$  of  $V$ , the space  $\text{Hom}(V, W)$  is diffeomorphic to  $W^n$ .

Each  $f_i = F(-, e_i)$  is smooth. Therefore,

$$G(p) = (f_1(p), \dots, f_n(p)). \quad (12)$$

Since each component of  $G$  is smooth,  $G$  is also smooth. ■

Now, by Lemma 1.2,  $\text{Ad} : G \rightarrow \text{Hom}(T_e G, T_e G) = \text{End}(T_e G)$  is smooth. But the image of  $\text{Ad}$  is contained in  $\text{GL}(T_e G)$ , which is an open subspace of  $\text{End}(T_e G)$ . Therefore,  $\text{Ad} : G \rightarrow \text{GL}(T_e G)$  is smooth, and hence it's a Lie group representation. ■

## §2 Lie Bracket on $T_e G$

In the previous section, we have seen that  $\text{Ad} : G \rightarrow \text{GL}(T_e G)$  is smooth. Its differential at identity gives a linear map at the level of tangent spaces:

$$(d\text{Ad})_e : T_e G \rightarrow T_I \text{GL}(T_e G).$$

Now, since  $\text{GL}(T_e G)$  is an open subspace of  $\text{End}(T_e G)$ , which is a finite dimensional vector space, we have

$$T_I \text{GL}(T_e G) = T_I \text{End}(T_e G) \cong \text{End}(T_e G). \quad (13)$$

(This isomorphism and its naturality is explored in [15], Lemma 6.1 and Proposition 6.2) Therefore,  $(d\text{Ad})_e$  gives a map

$$\text{ad} : T_e G \rightarrow \text{End}(T_e G).$$

So, given any  $X_e \in T_e G$ ,  $\text{ad } X_e$  is a linear map from  $T_e G$  to itself. As a result, for  $Y_e \in T_e G$ ,  $(\text{ad } X_e)(Y_e)$  will give us another element of  $T_e G$ . We may be prompted to define

$$[X_e, Y_e] = (\text{ad } X_e)(Y_e). \quad (14)$$

This is clearly bilinear: it's linear in the first component since  $\text{ad} : T_e G \rightarrow \text{End}(T_e G)$  is linear; and it's linear in the second component since  $\text{ad } X_e \in \text{End}(T_e G)$  is linear. One can show that it has all the required properties of a Lie bracket.

Now we have two Lie brackets on  $T_e G$ : one inherited from  $\mathfrak{L}(G)$ , and the other is (14). Turns out they are, in fact, the same! We shall prove it in this section.

### Theorem 2.1

Given  $X_e, Y_e \in T_e G$ , let  $X, Y$  be their corresponding left-invariant vector fields. Then

$$(\text{ad } X_e)(Y_e) = [X, Y]_e, \quad (15)$$

where  $[X, Y]$  is the Lie bracket of vector fields given by  $[X, Y] = XY - YX$ .

$$\begin{array}{ccc}
& & \text{End}(T_e G) \\
& \nearrow \text{ad} & \uparrow \Phi^{-1} \downarrow \Phi \\
T_e G & \xrightarrow{(d \text{Ad})_e} & T_I \text{GL}(T_e G) = T_I \text{End}(T_e G)
\end{array} \tag{16}$$

So we have  $\text{ad} = \Phi^{-1} \circ (d \text{Ad})_e$ . Now,

$$(\text{ad } X_e)(Y_e) = (\text{ev}_{Y_e} \circ \text{ad})(X_e), \tag{17}$$

where  $\text{ev}_{Y_e} : \text{End}(T_e G) \rightarrow T_e G$  is the “evaluating at  $Y_e$ ” map:

$$\begin{aligned}
\text{ev}_{Y_e} : \text{End}(T_e G) &\rightarrow T_e G \\
F &\mapsto F(Y_e).
\end{aligned} \tag{18}$$

This is a linear map, so we have the following commutative diagram, by the naturality of the isomorphism  $T_{\mathbf{v}} V \cong V$ .

$$\begin{array}{ccc}
T_I \text{End}(T_e G) & \xrightarrow{(d \text{ev}_{Y_e})_I} & T_{Y_e}(T_e G) \\
\uparrow \Phi & & \downarrow \Phi_t^{-1} \\
\text{End}(T_e G) & \xrightarrow{\text{ev}_{Y_e}} & T_e G
\end{array} \tag{19}$$

where  $\Phi_t : T_e G \rightarrow T_{Y_e}(T_e G)$  is the isomorphism.

### Lemma 2.2

Let  $c : \mathbb{R} \rightarrow T_e G$  be defined as  $c(t) = \text{Ad}(\exp(tX_e))(Y_e)$ . Then

$$(\text{ad } X_e)(Y_e) = \Phi_t^{-1} [c'(0)]. \tag{20}$$

*Proof.* We can combine the diagrams (16) and (19):

$$\begin{array}{ccccc}
& & T_I \text{End}(T_e G) & \xrightarrow{(d \text{ev}_{Y_e})_I} & T_{Y_e}(T_e G) \\
& \nearrow (d \text{Ad})_e & \uparrow \Phi \downarrow \Phi^{-1} & & \downarrow \Phi_t^{-1} \\
T_e G & \xrightarrow{\text{ad}} & \text{End}(T_e G) & \xrightarrow{\text{ev}_{Y_e}} & T_e G \\
& \nearrow \text{Ad} & & & \\
G & & & & 
\end{array} \tag{21}$$

Therefore, by the commutativity of this diagram,

$$\begin{aligned}
(\operatorname{ad} X_e)(Y_e) &= (\operatorname{ev}_{Y_e} \circ \operatorname{ad})(X_e) \\
&= (\Phi_t^{-1} \circ (d \operatorname{ev}_{Y_e})_I \circ (d \operatorname{Ad})_e)(X_e) \\
&= (\Phi_t^{-1} \circ (d(\operatorname{ev}_{Y_e} \circ \operatorname{Ad}))_e)(X_e)
\end{aligned} \tag{22}$$

Now,  $(d(\operatorname{ev}_{Y_e} \circ \operatorname{Ad}))_e X_e$  is equal to  $(\operatorname{ev}_{Y_e} \circ \operatorname{Ad} \circ \gamma)'(0)$  where  $\gamma : (-\varepsilon, \varepsilon) \rightarrow G$  is a curve such that  $\gamma(0) = e$  and  $\gamma'(0) = X_e$ . We can simply take  $\gamma(t) = \exp(tX_e)$ . Therefore,

$$\begin{aligned}
(d(\operatorname{ev}_{Y_e} \circ \operatorname{Ad}))_e X_e &= \left. \frac{d}{dt} \right|_{t=0} \left( \operatorname{ev}_{Y_e} \circ \operatorname{Ad} \exp(tX_e) \right) \\
&= \left. \frac{d}{dt} \right|_{t=0} \left( \operatorname{Ad}(\exp(tX_e))(Y_e) \right) \\
&= c'(0).
\end{aligned} \tag{23}$$

Therefore, combining (22) and (23),

$$(\operatorname{ad} X_e)(Y_e) = \Phi_t^{-1} [c'(0)]. \tag{24}$$

■

### Lemma 2.3

For a fixed  $t \in \mathbb{R}$ , define  $\alpha_t : \mathbb{R} \rightarrow G$  as

$$\alpha_t(s) = \exp(tX_e) \exp(sY_e) \exp(-tX_e) \tag{25}$$

Then  $c(t) = \alpha'_t(0)$ .

*Proof.*

$$\begin{aligned}
c(t) &= \operatorname{Ad}(\exp(tX_e))(Y_e) = \left( d \operatorname{conj}_{\exp(tX_e)} \right)_e Y_e \\
&= \left( \operatorname{conj}_{\exp(tX_e)} \circ \gamma \right)'(0),
\end{aligned} \tag{26}$$

where  $\gamma : (-\varepsilon, \varepsilon) \rightarrow G$  is a smooth curve such that  $\gamma(0) = e$  and  $\gamma'(0) = Y_e$ . We can simply take  $\gamma(s) = \exp(sY_e)$ . Then

$$\begin{aligned}
\operatorname{conj}_{\exp(tX_e)} \circ \gamma(s) &= \operatorname{conj}_{\exp(tX_e)}(\exp(sY_e)) \\
&= \exp(tX_e) \exp(sY_e) \exp(-tX_e) \\
&= \alpha_t(s).
\end{aligned} \tag{27}$$

Therefore,  $c(t) = \alpha'_t(0)$ .

■

So we have

$$\begin{aligned}
(\operatorname{ad} X_e) Y_e &= \Phi_t^{-1} c'(0) \\
&= \Phi_t^{-1} \left. \frac{d}{dt} \right|_{t=0} c(t) \\
&= \Phi_t^{-1} \left. \frac{d}{dt} \right|_{t=0} \left. \frac{d}{ds} \right|_{s=0} \left[ \exp(tX_e) \exp(sY_e) \exp(-tX_e) \right]
\end{aligned} \tag{28}$$

Now, observe where each terms are in:

$$\begin{aligned}
&\underbrace{\underbrace{\underbrace{\Phi_t^{-1} \left. \frac{d}{dt} \right|_{t=0} \left. \frac{d}{ds} \right|_{s=0}}_{\in T_e G} \left[ \underbrace{\exp(tX_e) \exp(sY_e) \exp(-tX_e)}_{\in G} \right]}_{\in T_{Y_e}(T_e G)}}_{\in T_e G}
\end{aligned}$$

Thus, the overall term is indeed in  $T_e G$ . So it will act on a smooth function  $f : G \rightarrow \mathbb{R}$  and will yield a real number. From (28), it's natural to expect that

$$[(\operatorname{ad} X_e) Y_e] f = \left. \frac{\partial^2}{\partial t \partial s} \right|_{t=0, s=0} f(\exp(tX_e) \exp(sY_e) \exp(-tX_e)). \tag{29}$$

We will now prove (29). We need to prove a preliminary lemma first.

#### Lemma 2.4

Let  $f : M \rightarrow \mathbb{R}$  be smooth. Then  $(df)_p : T_p M \rightarrow T_{f(p)} \mathbb{R} \cong \mathbb{R}$  can be considered as a covector, i.e.  $(df)_p \in (T_p M)^*$ . Then for any  $X_p \in T_p M$ ,

$$X_p f = (df)_p(X_p). \tag{30}$$

More concretely, define  $\Psi : T_{f(p)} \mathbb{R} \rightarrow \mathbb{R}$  as  $\beta \left. \frac{d}{dt} \right|_{t=f(p)} \mapsto \beta$ , so that  $\Psi \circ (df)_p \in (T_p M)^*$ . Then

$$X_p f = [\Psi \circ (df)_p] X_p. \tag{31}$$

*Proof.* Let  $\gamma : (-\varepsilon, \varepsilon) \rightarrow M$  be a smooth curve such that  $\gamma(0) = p$  and  $\gamma'(0) = X_p$ . Then

$$\begin{aligned}
X_p f &= \left. \frac{d}{dt} \right|_{t=0} (f \circ \gamma) = \Psi \left[ d(f \circ \gamma)_0 \left. \frac{d}{dt} \right|_{t=0} \right] \\
&= \Psi \left[ (df)_p (d\gamma)_0 \left. \frac{d}{dt} \right|_{t=0} \right] \\
&= \Psi \left[ (df)_p \gamma'(0) \right] = [\Psi \circ (df)_p] X_p.
\end{aligned} \tag{32}$$

■

Before going into the proof of (29), let's first recall how  $\Phi_t^{-1}$  works. Let  $\Phi : V \rightarrow T_{\mathbf{v}}V$  be the natural isomorphism. We defined it as follows: given  $\mathbf{u} \in V$ ,  $\Phi(\mathbf{u}) =: D_{\mathbf{u}} \in T_{\mathbf{v}}V$  is the point-derivation that acts like

$$D_{\mathbf{u}}f = \left. \frac{d}{dt} \right|_{t=0} f(t\mathbf{u} + \mathbf{v}). \quad (33)$$

The inverse  $\Phi^{-1}$  has the following defining property: suppose  $D \in T_{\mathbf{v}}V$ . Take any  $\lambda \in V^*$ . If  $D = D_{\mathbf{u}}$  for some  $\mathbf{u} \in V$ , then

$$D\lambda = D_{\mathbf{u}}\lambda = \left. \frac{d}{dt} \right|_{t=0} \lambda(t\mathbf{u} + \mathbf{v}) = \left. \frac{d}{dt} \right|_{t=0} [t\lambda(\mathbf{u}) + \lambda(\mathbf{v})] = \lambda(\mathbf{u}). \quad (34)$$

Therefore,

$$D\lambda = \lambda(\Phi^{-1}D), \quad (35)$$

for every  $\lambda \in V^*$ . Now let's prove (29).

### Proposition 2.5

Given  $X_e, Y_e \in T_eG$  and  $f : G \rightarrow \mathbb{R}$  smooth,

$$[(\text{ad } X_e) Y_e] f = \left. \frac{\partial^2}{\partial t \partial s} \right|_{t=0, s=0} f(\exp(tX_e) \exp(sY_e) \exp(-tX_e)). \quad (36)$$

*Proof.* Let  $\Psi : T_{f(e)}\mathbb{R} \rightarrow \mathbb{R}$  be defined as above. By Lemma 2.4,

$$\begin{aligned} \text{LHS} &= [(\text{ad } X_e) Y_e] f \\ &= \Psi \circ (df)_e \left( (\text{ad } X_e) Y_e \right) \\ &= \Psi \circ (df)_e \left( \Phi_t^{-1} c'(0) \right). \end{aligned} \quad (37)$$

Now we're gonna use (35). Since  $\Psi \circ (df)_e \in (T_eG)^*$ ,

$$\begin{aligned} \text{LHS} &= c'(0) [\Psi \circ (df)_e] \\ &= \left. \frac{d}{dt} \right|_{t=0} \left( \Psi \circ (df)_e \circ c \right). \end{aligned} \quad (38)$$

Note the compositions

$$\mathbb{R} \xrightarrow{c} T_eG \xrightarrow{(df)_e} T_{f(e)}\mathbb{R} \xrightarrow{\Psi} \mathbb{R} \quad (39)$$

Now, since  $c(t) = \alpha'(0)$  by Lemma 2.3,

$$(df)_e \circ c(t) = (df)_e [\alpha'_t(0)] = (df)_e (d\alpha_t)_0 \left. \frac{d}{ds} \right|_{s=0} = (d(f \circ \alpha_t))_0 \left. \frac{d}{ds} \right|_{s=0}. \quad (40)$$

Composing it with  $\Psi$ , we get

$$\begin{aligned}\Psi \circ (df)_e \circ c(t) &= \Psi \left[ (d(f \circ \alpha_t))_0 \frac{d}{ds} \Big|_{s=0} \right] \\ &= \frac{d}{ds} \Big|_{s=0} (f \circ \alpha_t).\end{aligned}\tag{41}$$

Therefore,

$$\begin{aligned}\text{LHS} &= \frac{d}{dt} \Big|_{t=0} \left( (\Psi \circ (df)_e \circ c)(t) \right) \\ &= \frac{d}{dt} \Big|_{t=0} \frac{d}{ds} \Big|_{s=0} (f \circ \alpha_t(s)) \\ &= \frac{\partial^2}{\partial t \partial s} \Big|_{t=0, s=0} f(\exp(tX_e) \exp(sY_e) \exp(-tX_e)).\end{aligned}\tag{42}$$

■

### Lemma 2.6

$$\begin{aligned}& \frac{\partial^2}{\partial t \partial s} \Big|_{t=0, s=0} f(\exp(tX_e) \exp(sY_e) \exp(-tX_e)) \\ &= \frac{\partial^2}{\partial t \partial s} \Big|_{t=0, s=0} f(\exp(tX_e) \exp(sY_e)) - \frac{\partial^2}{\partial t \partial s} \Big|_{t=0, s=0} f(\exp(sY_e) \exp(tX_e))\end{aligned}\tag{43}$$

*Proof.* For simplicity of notations, let's write  $e^{tX_e} := \exp(tX_e)$ . The function on the LHS,  $(t, s) \mapsto f(e^{tX_e} e^{sY_e} e^{-tX_e})$  is, in fact, the following composition:

$$\begin{array}{ccccc} \mathbb{R}^2 & \xrightarrow[\substack{(t,s) \mapsto (t,s,-t)}]{a} & \mathbb{R}^3 & \xrightarrow[\substack{(x,y,z) \mapsto e^{xX_e} e^{yY_e} e^{zX_e}}]{b} & G \xrightarrow{f} \mathbb{R} \\ & & & \text{F} = f \circ b & \\ & & & \text{F} \circ a & \\ & & & (t,s) \mapsto f(e^{tX_e} e^{sY_e} e^{-tX_e}) & \end{array}\tag{44}$$

We are now gonna apply the multivariable chain rule on  $F \circ a$ .

$$\frac{\partial a}{\partial t} = (1, 0, -1), \quad \frac{\partial a}{\partial s} = (0, 1, 0), \quad \frac{\partial^2 a}{\partial t \partial s} = (0, 0, 0).\tag{45}$$

Therefore, we get

$$\frac{\partial^2}{\partial t \partial s} \Big|_{t=0, s=0} (F \circ a) = \sum_{i,j=1}^3 \frac{\partial^2 F}{\partial x^i \partial x^j} \Big|_0 \left( \frac{\partial a}{\partial t} \right)^i \left( \frac{\partial a}{\partial s} \right)^j = F_{xy}|_0 - F_{yz}|_0.\tag{46}$$

Now, consider

$$\begin{array}{c}
 \mathbb{R}^2 \xrightarrow[\substack{(t,s) \mapsto (t,s,0)}]{a'} \mathbb{R}^3 \xrightarrow[\substack{(x,y,z) \mapsto e^{xX_e} e^{yY_e} e^{zX_e}}]{b} G \xrightarrow{f} \mathbb{R} \\
 \text{---} \xrightarrow[\substack{(t,s) \mapsto f(e^{tX_e} e^{sY_e})}]{F \circ a'} \text{---}
 \end{array}
 \quad (47)$$

$F = f \circ b$

We are now gonna apply the multivariable chain rule on  $F \circ a'$ .

$$\frac{\partial a'}{\partial t} = (1, 0, 0), \quad \frac{\partial a'}{\partial s} = (0, 1, 0), \quad \frac{\partial^2 a'}{\partial t \partial s} = (0, 0, 0). \quad (48)$$

Therefore, we get

$$\left. \frac{\partial^2}{\partial t \partial s} \right|_{t=0, s=0} (F \circ a') = \sum_{i,j=1}^3 \left. \frac{\partial^2 F}{\partial x^i \partial x^j} \right|_0 \left( \frac{\partial a'}{\partial t} \right)^i \left( \frac{\partial a'}{\partial s} \right)^j = F_{xy}|_0. \quad (49)$$

Similarly,

$$\begin{array}{c}
 \mathbb{R}^2 \xrightarrow[\substack{(t,s) \mapsto (0,s,t)}]{a''} \mathbb{R}^3 \xrightarrow[\substack{(x,y,z) \mapsto e^{xX_e} e^{yY_e} e^{zX_e}}]{b} G \xrightarrow{f} \mathbb{R} \\
 \text{---} \xrightarrow[\substack{(t,s) \mapsto f(e^{sY_e} e^{tX_e})}]{F \circ a''} \text{---}
 \end{array}
 \quad (50)$$

$F = f \circ b$

We are now gonna apply the multivariable chain rule on  $F \circ a''$ .

$$\frac{\partial a''}{\partial t} = (0, 0, 1), \quad \frac{\partial a''}{\partial s} = (0, 1, 0), \quad \frac{\partial^2 a''}{\partial t \partial s} = (0, 0, 0). \quad (51)$$

Therefore, we get

$$\left. \frac{\partial^2}{\partial t \partial s} \right|_{t=0, s=0} (F \circ a'') = \sum_{i,j=1}^3 \left. \frac{\partial^2 F}{\partial x^i \partial x^j} \right|_0 \left( \frac{\partial a''}{\partial t} \right)^i \left( \frac{\partial a''}{\partial s} \right)^j = F_{yz}|_0. \quad (52)$$

Now, combining (46), (49), (52), we have

$$\left. \frac{\partial^2}{\partial t \partial s} \right|_{t=0, s=0} (F \circ a) = \left. \frac{\partial^2}{\partial t \partial s} \right|_{t=0, s=0} (F \circ a') - \left. \frac{\partial^2}{\partial t \partial s} \right|_{t=0, s=0} (F \circ a''), \quad (53)$$

as required. ■

### Lemma 2.7

Let  $X$  be a left-invariant vector field on  $G$ , and  $h \in C^\infty(G)$ . Then

$$(Xh)(g) = \left. \frac{d}{dt} \right|_{t=0} h(g \exp(tX_e)). \quad (54)$$

*Proof.* By Proposition 5.2 from [15], the integral curve for  $X$  starting at  $g$  is  $\alpha(t) = g \exp(tX_e)$ . Therefore,  $\alpha(0) = g$  and  $\alpha'(0) = X_g$ . Hence,

$$(Xh)(g) = X_g h = \left. \frac{d}{dt} \right|_{t=0} (h \circ \alpha) = \left. \frac{d}{dt} \right|_{t=0} h(g \exp(tX_e)). \quad (55)$$

■

Now we're ready to finally prove Theorem 2.1.

*Proof of Theorem 2.1.* Combining Proposition 2.5 and Lemma 2.6, we have

$$[(\text{ad } X_e) Y_e] f = \left. \frac{\partial^2}{\partial t \partial s} \right|_{t=0, s=0} f(e^{tX_e} e^{sY_e}) - \left. \frac{\partial^2}{\partial t \partial s} \right|_{t=0, s=0} f(e^{sY_e} e^{tX_e}). \quad (56)$$

Now, applying Lemma 2.7 on  $h = f$ ,  $g = \exp(tX_e)$ , we get

$$\begin{aligned} \left. \frac{\partial^2}{\partial t \partial s} \right|_{t=0, s=0} f(e^{tX_e} e^{sY_e}) &= \left. \frac{d}{dt} \right|_{t=0} \left. \frac{d}{ds} \right|_{s=0} f(e^{tX_e} e^{sY_e}) \\ &= \left. \frac{d}{dt} \right|_{t=0} (Yf)(e^{tX_e}). \end{aligned} \quad (57)$$

Then, applying Lemma 2.7 on  $h = Yf$ ,  $g = e$ , we get

$$\left. \frac{d}{dt} \right|_{t=0} (Yf)(e^{tX_e}) = X(Yf)(e). \quad (58)$$

Therefore,

$$\left. \frac{\partial^2}{\partial t \partial s} \right|_{t=0, s=0} f(e^{tX_e} e^{sY_e}) = X(Yf)(e). \quad (59)$$

Similarly, since partial derivatives commute,

$$\left. \frac{\partial^2}{\partial t \partial s} \right|_{t=0, s=0} f(e^{sY_e} e^{tX_e}) = Y(Xf)(e). \quad (60)$$

Hence,

$$[(\text{ad } X_e) Y_e] f = X(Yf)(e) - Y(Xf)(e) = [X, Y]_e f. \quad (61)$$

■

Although we haven't proved that  $[X_e, Y_e] = (\text{ad } X_e)Y_e$  is a Lie bracket on  $T_e G$ , Theorem 2.1 proves that it is, indeed, a Lie bracket. Not only that, it's exactly the same Lie bracket as the one inherited from  $\mathfrak{L}(G)$ .

### Proposition 2.8

Given  $X_e, Y_e \in T_e G$ ,

$$\text{ad } [X_e, Y_e] = [\text{ad } X_e, \text{ad } Y_e], \quad (62)$$

i.e.  $\text{ad} : T_e G \rightarrow \text{End}(T_e G)$  is a Lie algebra homomorphism. (Here,  $[X_e, Y_e]$  is the Lie bracket on  $T_e G$ , and  $[\text{ad } X_e, \text{ad } Y_e]$  is the Lie bracket on  $\text{End}(T_e G)$ .)

*Proof.* Since  $[X_e, Y_e] = (\text{ad } X_e)Y_e$  is a Lie bracket on  $T_e G$ , it satisfies the Jacobi identity, i.e. for any  $Z_e \in T_e G$ ,

$$\begin{aligned} & [[X_e, Y_e], Z_e] + [[Z_e, X_e], Y_e] + [[Y_e, Z_e], X_e] = 0 \\ \implies & [[X_e, Y_e], Z_e] = [X_e, [Y_e, Z_e]] - [Y_e, [X_e, Z_e]] \\ \implies & (\text{ad } [X_e, Y_e]) Z_e = (\text{ad } X_e)((\text{ad } Y_e)Z_e) - (\text{ad } Y_e)((\text{ad } X_e)Z_e) \\ \implies & (\text{ad } [X_e, Y_e]) Z_e = \left( (\text{ad } X_e) \circ (\text{ad } Y_e) - (\text{ad } Y_e) \circ (\text{ad } X_e) \right) Z_e \\ \implies & \text{ad } [X_e, Y_e] = [\text{ad } X_e, \text{ad } Y_e]. \end{aligned} \quad (63)$$

■

One thing that's fascinating about the fact that the Lie brackets on  $\mathfrak{L}(G)$  and  $T_e G$  are the same is that they connect a global statement with a local statement. The Lie bracket on  $\mathfrak{L}(G)$  is a “global” statement, because the Lie bracket of left-invariant vector fields,

$$[X, Y] = XY - YX,$$

deals with the global phenomena of the vector fields and how they act on functions. On the other hand, the Lie bracket on  $T_e G$  is given by  $\text{ad}$ ,

$$[X_e, Y_e]_{T_e G} = (\text{ad } X_e)Y_e,$$

which is essentially the differential of  $\text{Ad}$  at  $e$ , which is a “local” statement as it only deals with the behavior of  $\text{Ad}$  near  $e$ .

## §3 A Brief Detour on $\text{GL}(V)$

In Example 6.1 of [15], we remarked that the exponential map  $T_I \text{GL}(n, \mathbb{R}) \cong \text{End}(\mathbb{R}^n) \rightarrow \text{GL}(n, \mathbb{R})$  is actually the matrix exponential  $A \mapsto e^A := \sum_{n=0}^{\infty} \frac{1}{n!} A^n$ . In this section, we shall

prove that the matrix exponential is indeed the exponential map for  $\mathrm{GL}(V)$ , where  $V$  is a finite-dimensional real vector space.

First,  $\mathrm{GL}(V)$  is an open subspace of  $\mathrm{End}(V)$ , and hence we have that  $T_I \mathrm{GL}(V) = T_I \mathrm{End}(V)$ .  $\mathrm{End}(V)$  is a finite dimensional vectpr space, so we have the natural isomorphism  $\Phi : \mathrm{End}(V) \rightarrow T_I \mathrm{End}(V)$ . The Lie group exponential is the map

$$\exp : T_I \mathrm{GL}(V) \rightarrow \mathrm{GL}(V),$$

and let's denote the matrix exponential with  $\mathrm{Exp}$ :

$$\begin{aligned} \mathrm{Exp} : \mathrm{End}(V) &\rightarrow \mathrm{GL}(V) \\ A &\mapsto \sum_{n=0}^{\infty} \frac{1}{n!} A^n. \end{aligned} \tag{64}$$

We know that this infinite sum converges absolutely and uniformly on every bounded subset of  $\mathrm{End}(V)$ . When we say that the matrix exponential exactly coincides with the Lie group exponential, we mean that the following diagram commutes:

$$\begin{array}{ccc} & T_I \mathrm{GL}(V) = T_I \mathrm{End}(V) & \\ & \nearrow \Phi & \downarrow \exp \\ \mathrm{End}(V) & \xleftarrow{\Phi^{-1}} & \\ & \searrow \mathrm{Exp} & \\ & \mathrm{GL}(V) & \end{array} \tag{65}$$

### Theorem 3.1

(65) commutes, i.e.  $\mathrm{Exp} \circ \Phi^{-1} = \exp$ .

*Proof.* Given  $X_I \in T_I \mathrm{GL}(V)$ , we define a function

$$\begin{aligned} \gamma : \mathbb{R} &\rightarrow \mathrm{GL}(V) \\ t &\mapsto \mathrm{Exp}(t \Phi^{-1}(X_I)) \end{aligned} \tag{66}$$

We claim that  $\gamma(t) = \exp(tX_I)$ . Proving this claim is sufficient, since that would imply  $\mathrm{Exp} \circ \Phi^{-1} = \exp$  (upon taking  $t = 1$ ). Clearly  $\gamma$  is a smooth map, and it's a group homomorphism. Therefore, it's a one-parameter subgroup. Now, in order to show that  $\gamma(t) = \exp(tX_I)$ ,

we just need to show that  $\gamma'(0) = X_I$ . Suppose  $X_I = \Phi(A)$  for some  $A \in \text{End}(V)$ .

$$\begin{aligned}\gamma'(0) = X_I &\iff \Phi^{-1}\gamma'(0) = \Phi^{-1}(X_I) = A \\ &\iff \lambda[\Phi^{-1}\gamma'(0)] = \lambda(A) \text{ for every } \lambda \in \text{End}(V)^* \\ &\iff \gamma'(0)[\lambda] = \lambda(A) \text{ for every } \lambda \in \text{End}(V)^*,\end{aligned}$$

since  $\lambda(\Phi^{-1}D) = D\lambda$  by (35). Now,

$$\begin{aligned}\gamma'(0)[\lambda] &= \left. \frac{d}{dt} \right|_{t=0} \lambda(\gamma(t)) \\ &= \left. \frac{d}{dt} \right|_{t=0} \lambda(\text{Exp}(tA)) \\ &= \left. \frac{d}{dt} \right|_{t=0} \lambda\left(\sum_{n=0}^{\infty} \frac{t^n}{n!} A^n\right).\end{aligned}\tag{67}$$

Now,  $\lambda$  is a covector on a finite-dimensional vector space, i.e.  $\lambda \in \text{End}(V)^*$ . Therefore,  $\lambda$  being linear implies continuity. So we can take the sum out of  $\lambda$ . Also since the converges uniformly, we can swap the sum and differentials:

$$\begin{aligned}\gamma'(0)[\lambda] &= \left. \frac{d}{dt} \right|_{t=0} \sum_{n=0}^{\infty} \frac{t^n}{n!} \lambda(A^n) \\ &= \sum_{n=0}^{\infty} \left. \frac{d}{dt} \right|_{t=0} \frac{t^n}{n!} \lambda(A^n) \\ &= \lambda(A).\end{aligned}\tag{68}$$

Here,  $n = 0$  term vanished because it's a constant term (i.e. doesn't depend on  $t$ ).  $n \geq 2$  terms also vanished because after completing the differentiation, there are  $ts$  in the expression and upon setting  $t = 0$ , those terms vanish. Therefore, we are done proving  $\gamma'(0) = X_I$ .  $\blacksquare$

### Corollary 3.2

The following diagram commutes:

$$\begin{array}{ccc} T_e G & \xrightarrow{\text{ad}} & \text{End}(T_e G) \\ \exp_G \downarrow & & \downarrow \text{Exp} \\ G & \xrightarrow{\text{Ad}} & \text{GL}(T_e G) \end{array}\tag{69}$$

i.e.  $\text{Exp} \circ \text{ad} = \text{Ad} \circ \exp_G$ .

*Proof.* We know that the following diagram commutes:

$$\begin{array}{ccc}
 & \text{End}(T_e G) & \\
 \text{ad} \nearrow & \uparrow \Phi^{-1} \downarrow \Phi & \\
 T_e G & \xrightarrow{(d \text{Ad})_e} & T_l \text{GL}(T_e G) \\
 \exp_G \downarrow & & \downarrow \exp_{\text{GL}(T_e G)} \\
 G & \xrightarrow{\text{Ad}} & \text{GL}(T_e G)
 \end{array}
 \quad \begin{array}{l} \text{Exp} = \exp_{\text{GL}(T_e G)} \circ \Phi \end{array}
 \quad (70)$$

The square at the bottom commutes because of the naturality of  $\exp$ , the triangle at the top commutes by the definition of  $\text{ad}$ . Furthermore,  $\text{Exp} = \exp_{\text{GL}(T_e G)} \circ \Phi$  by Theorem 3.1. Therefore, this square commutes:

$$\begin{array}{ccc}
 T_e G & \xrightarrow{\text{ad}} & \text{End}(T_e G) \\
 \exp_G \downarrow & & \downarrow \exp_{\text{GL}(T_e G)} \circ \Phi = \text{Exp} \\
 G & \xrightarrow{\text{Ad}} & \text{GL}(T_e G)
 \end{array}
 \quad (71)$$

which is exactly what we needed to show! ■

## §4 Relationship Between Lie Bracket and Lie Group Commutator

We now have a Lie bracket on the Lie algebra  $\mathfrak{g} = T_e G$  of our Lie group  $G$ :

$$[X_e, Y_e]_{T_e G} = (\text{ad } X_e)Y_e. \quad (72)$$

Furthermore, we have a “group commutator” on  $G$ :

$$[g, h]_G = ghg^{-1}h^{-1}. \quad (73)$$

Two group elements  $g$  and  $h$  commute if and only if  $[g, h]_G = e$ .  $G$  is abelian if and only if  $[-, -]_G = e$ . We shall now establish a relationship between the Lie bracket and the Lie group commutator.

### Proposition 4.1

If  $G$  is abelian,  $[-, -]_{\mathfrak{g}} = 0$ .

*Proof.* If  $G$  is abelian,  $\text{conj}_g = \text{id}_G$ . Therefore,  $(d \text{conj}_g)_e = \mathbb{1}_{T_e G}$ . In other words,

$$\begin{aligned} \text{Ad} : G &\rightarrow \text{GL}(T_e G) \\ g &\mapsto (d \text{conj}_g)_e, \end{aligned} \quad (74)$$

maps every  $g$  to  $\mathbb{1}_{T_e G}$ , so  $\text{Ad}$  is the constant map. Therefore,  $(d \text{Ad})_e : T_e G \rightarrow T_I \text{GL}(T_e G)$  is the zero map. Hence,

$$\begin{array}{ccc} & & \text{End}(T_e G) \\ & \nearrow \text{ad} & \uparrow \Phi^{-1} \downarrow \Phi \\ T_e G & \xrightarrow{(d \text{Ad})_e} & T_I \text{GL}(T_e G) = T_I \text{End}(T_e G) \end{array} \quad (75)$$

$$\text{ad} = \Phi^{-1} \circ (d \text{Ad})_e = 0. \quad (76)$$

As a result, given any  $X_e, Y_e \in T_e G$ ,

$$[X_e, Y_e]_{T_e G} = (\text{ad } X_e)Y_e = 0. \quad (77)$$

■

#### Corollary 4.2

If  $G$  is connected, and  $[-, -]_{T_e G} = 0$ , then  $G$  is abelian.

*Proof.* Since  $[-, -]_{T_e G} = 0$ , we must have  $\text{ad} = 0$ .

$$\begin{array}{ccc} & & \text{End}(T_e G) \\ & \nearrow \text{ad} & \uparrow \Phi^{-1} \downarrow \Phi \\ T_e G & \xrightarrow{(d \text{Ad})_e} & T_I \text{GL}(T_e G) = T_I \text{End}(T_e G) \end{array} \quad (78)$$

Now,  $\text{ad} = \Phi^{-1} \circ (d \text{Ad})_e$  and  $\Phi$  being an isomorphism gives us that  $(d \text{Ad})_e = 0$ .

At the end of [15], we have seen that

If  $G$  is connected, any Lie group homomorphism  $\varphi : G \rightarrow H$  is completely determined by the Lie algebra homomorphism  $\widehat{\varphi} = (d\varphi)_e : \mathfrak{g} \rightarrow \mathfrak{h}$

Since  $(d \text{Ad})_e = 0$ , the composition  $\exp_{\text{GL}(T_e G)} \circ (d \text{Ad})_e$  takes everything to the identity element

of  $\text{GL}(T_e G)$ . The following diagram commutes by the naturality of  $\exp$  ([15] Proposition 6.5):

$$\begin{array}{ccc}
T_e G & \xrightarrow{(d \text{Ad})_e} & T_e \text{GL}(T_e G) \\
\exp_G \downarrow & & \downarrow \exp_{\text{GL}(T_e G)} \\
G & \xrightarrow{\text{Ad}} & \text{GL}(T_e G)
\end{array} \tag{79}$$

Since  $\exp_G$  is a local diffeomorphism, and

$$\text{Ad} \circ \exp_G = \exp_{\text{GL}(T_e G)} \circ (d \text{Ad})_e \tag{80}$$

is a constant map, we can conclude that  $\text{Ad}$  is constant on a neighborhood of  $e$ . But since any neighborhood of  $e$  generates  $G$  (because  $G$  is connected), we can conclude that  $\text{Ad}$  is constant on the whole  $G$ , i.e. for any  $g \in G$ ,

$$\text{Ad}(g) = (d \text{conj}_g)_e = \text{id}_{T_e G}. \tag{81}$$

Now, the following diagram commutes by the naturality of  $\exp$  ([15] Proposition 6.5):

$$\begin{array}{ccc}
T_e G & \xrightarrow{(d \text{conj}_g)_e} & T_e G \\
\exp_G \downarrow & & \downarrow \exp_G \\
G & \xrightarrow{\text{conj}_g} & G
\end{array} \tag{82}$$

Since  $(d \text{conj}_g)_e = \text{id}_{T_e G}$ , we have  $\exp_G = \text{conj}_g \circ \exp_G$ . Furthermore,  $\exp$  is a local diffeomorphism, so we have  $\text{conj}_g$  is identity at a neighborhood of  $e$ . But since any neighborhood of  $e$  generates  $G$  (because  $G$  is connected), we can conclude that  $\text{conj}_g$  is identity on the whole  $G$ , i.e.  $\text{conj}_g = \text{id}_G$  for any  $g \in G$ , proving that  $G$  is abelian.  $\blacksquare$

**Example 4.1.** The connected assumption cannot be dropped from Corollary 4.2. Consider  $G = \text{O}(2)$ . It's also non-connected: it has two connected components, namely the elements with determinant  $+1$  and  $-1$ . However, its connected component of identity is  $G^\circ = \text{SO}(2)$ , which is abelian. Therefore, the Lie algebra is

$$T_e G = T_e G^\circ = \mathfrak{so}(2), \tag{83}$$

which is the Lie algebra of an abelian Lie group. Hence, the Lie bracket is trivial. However, despite having a trivial Lie bracket,  $G = \text{O}(2)$  is non-abelian, since a rotation doesn't commute with a reflection. So we need the assumption of connectedness in Corollary 4.2.

### Proposition 4.3

If  $[X_e, Y_e]_{T_e G} = 0$ , then  $[\exp X_e, \exp Y_e]_G = e$ , i.e.  $\exp X_e$  and  $\exp Y_e$  commute.

*Proof.* We'll first prove a lemma:

### Lemma 4.4

If  $[X_e, Y_e]_{T_e G} = 0$ , then

$$[\text{Ad}(\exp Y_e)] X_e = X_e. \quad (84)$$

*Proof.* Consider this commutative diagram:

$$\begin{array}{ccc}
 & \text{End}(T_e G) & \\
 \text{ad} \nearrow & \uparrow \Phi^{-1} \downarrow \Phi & \\
 T_e G & \xrightarrow{(d \text{Ad})_e} & T_l \text{GL}(T_e G) \\
 \exp_G \downarrow & \text{exp}_{\text{GL}(T_e G)} \downarrow & \\
 G & \xrightarrow{\text{Ad}} & \text{GL}(T_e G)
 \end{array}
 \quad \text{Exp} = \text{exp}_{\text{GL}(T_e G)} \circ \Phi \quad (85)$$

$$\begin{aligned}
 [\text{Ad}(\exp Y_e)] X_e &= [\text{Exp}(\text{ad } Y_e)] X_e \\
 &= \sum_{n=0}^{\infty} \frac{1}{n!} (\text{ad } Y_e)^n (X_e).
 \end{aligned} \quad (86)$$

But since we have  $(\text{ad } Y_e)(X_e) = [Y_e, X_e] = -[X_e, Y_e] = 0$ , only the  $n = 0$  term will have nonzero contribution to the sum. Therefore,

$$[\text{Ad}(\exp Y_e)] X_e = X_e. \quad (87)$$

■

Now we get back to our proof. We define the following curve:

$$\begin{aligned}
 \gamma : \mathbb{R} &\rightarrow G \\
 t &\mapsto \exp(Y_e) \exp(tX_e) \exp(-Y_e) = \text{conj}_{\exp(Y_e)}(\exp(tX_e)).
 \end{aligned} \quad (88)$$

Then  $\gamma$  is a smooth group homomorphism, and

$$\gamma'(0) = \left( d \text{conj}_{\exp(Y_e)} \right)_e X_e = [\text{Ad}(\exp Y_e)] X_e = X_e, \quad (89)$$

because  $\exp(tX_e)$ 's velocity at  $t = 0$  is  $X_e$ . Therefore,  $\gamma(t) = \exp(tX_e)$ . At  $t = 1$ , we have

$$\begin{aligned} & \exp(Y_e) \exp(X_e) \exp(-Y_e) = \exp(X_e) \\ \implies & \exp(Y_e) \exp(X_e) = \exp(X_e) \exp(Y_e). \end{aligned} \quad (90)$$

■

#### Theorem 4.5

If  $[X_e, Y_e]_{T_e G} = 0$ , then

$$\exp(X_e + Y_e) = \exp(X_e) \exp(Y_e). \quad (91)$$

*Proof.* We define a curve as follows:

$$\begin{aligned} \gamma : \mathbb{R} &\rightarrow G \\ t &\mapsto \exp(tX_e) \exp(tY_e). \end{aligned} \quad (92)$$

We shall prove that this is a Lie group homomorphism with velocity vector at  $t = 0$  being equal to  $X_e + Y_e$ .

$$\begin{aligned} \gamma(s+t) &= \exp((s+t)X_e) \exp((s+t)Y_e) \\ &= \exp(sX_e) \exp(tX_e) \exp(sY_e) \exp(tY_e) \\ &= \exp(sX_e) \exp(sY_e) \exp(tX_e) \exp(tY_e) \\ &= \gamma(s) \gamma(t). \end{aligned} \quad (93)$$

Here we used the fact that  $\exp(tX_e)$  and  $\exp(sY_e)$  commute, which follows from Proposition 4.3, since  $[tX_e, sY_e] = st[X_e, Y_e] = 0$ .

Now, we can write  $\gamma = m \circ (\alpha_1, \alpha_2)$ , where  $m$  is the multiplication map of the Lie group  $G$ ,  $\alpha_1(t) = \exp(tX_e)$ ,  $\alpha_2(t) = \exp(tY_e)$ . Therefore,

$$\begin{aligned} \gamma'(0) &= (dm)_{(e,e)}(\alpha'_1(0), \alpha'_2(0)) \\ &= (dm)_{(e,e)}(X_e + Y_e) \\ &= X_e + Y_e, \end{aligned} \quad (94)$$

since  $(dm)_{(e,e)} : T_e G \times T_e G \rightarrow T_e G$  is the addition map. Therefore,  $\gamma$  is the smooth group homomorphism with initial velocity vector  $X_e + Y_e$ . Hence,  $\gamma(t) = \exp(t(X_e + Y_e))$ . Taking  $t = 1$ , we have

$$\exp(X_e + Y_e) = \gamma(1) = \exp(X_e) \exp(Y_e). \quad (95)$$

■

Therefore, for an abelian Lie group,  $\exp : T_e G \rightarrow G$  is a group homomorphism. If, furthermore,

$G$  is connected, then we can conclude that  $\exp$  is also surjective. Because  $\exp$  is surjective on a neighborhood of the identity (since it's a local diffeomorphism). If  $G$  is connected, it's generated by any neighborhood of the identity. So if  $\exp$  happens to be a group homomorphism, its image is a subgroup, which contains a neighborhood of the identity, and hence it has to be the whole  $G$ . This gives rise to two natural questions.

1. What are all the connected abelian Lie groups? Can we classify them?
2. In the non-abelian case, how exactly does  $\exp$  fail to be a homomorphism?

We shall end this note with the answer to these questions.

## §5 Abelian Lie Groups and Their Classification

In this section, we shall classify all the connected abelian Lie groups (up to isomorphism). Throughout this section,  $G$  will denote a connected abelian Lie group. We have already shown that  $\exp : T_e G \rightarrow G$  is a **surjective** group homomorphism. So we have

$$G \cong \frac{T_e G}{\text{Ker } \exp}. \quad (96)$$

### Proposition 5.1

$\text{Ker } \exp$  is discrete.

*Proof.* We know that  $\exp$  is a local diffeomorphism near the identity. So there is a neighborhood  $U \ni e$  and a neighborhood  $\mathfrak{u} \ni 0$  such that  $\exp$  is a diffeomorphism of  $\mathfrak{u}$  and  $U$ .

$$\begin{array}{ccc} \mathfrak{u} \subseteq \mathfrak{g} & \xrightarrow{\exp} & U \subseteq G \\ & \xleftarrow{\log} & \end{array}$$

Given  $X_e \in \text{Ker } \exp$ , define  $\mathfrak{u}' = \mathfrak{u} + X_e$ . We claim that  $\mathfrak{u}'$  and  $U$  are diffeomorphic through the exponential map. Given  $Y_e \in \mathfrak{u}$ ,

$$\exp(Y_e + X_e) = \exp(Y_e) \exp(X_e) = \exp(Y_e). \quad (97)$$

Therefore, the following diagram commutes:

$$\begin{array}{ccccc} \mathfrak{u}' & \xrightarrow{Y_e + X_e \mapsto Y_e} & \mathfrak{u} & \xrightarrow{\exp|_{\mathfrak{u}}} & U \\ & \searrow \exp|_{\mathfrak{u}'} & & & \end{array} \quad (98)$$

$\exp|_{\mathfrak{u}}$  is a diffeomorphism, so is the translation map  $\mathfrak{u}' \rightarrow \mathfrak{u}$ . Therefore,  $\exp|_{\mathfrak{u}'}$  is a diffeomorphism. In particular,  $\mathfrak{u}'$  doesn't contain any other element of  $\text{Ker exp}$  than  $X_e$ .

Now, given any  $X_e \in \text{Ker exp}$ , it has a neighborhood, namely  $\mathfrak{u} + X_e$  that does not contain any other element from the kernel. Therefore,  $\text{Ker exp}$  is discrete.  $\blacksquare$

There is a classification of the discrete subgroups of a finite-dimensional vector space  $V$ .

### Lemma 5.2

If  $\Lambda$  is a discrete subgroup of a finite-dimensional real vector space  $V$ , then it is generated by linearly independent vectors  $v_1, \dots, v_k$ .

We won't go into the proof here. Interested readers can go through [4], Lemma 3.8.

### Theorem 5.3

If  $G$  is a connected abelian Lie group, then

$$G \cong (S^1)^k \times \mathbb{R}^m, \quad (99)$$

for some non-negative integers  $k, m$ .

*Proof.* Since  $\exp$  is a surjective group homomorphism,

$$G \cong \frac{T_e G}{\text{Ker exp}} \cong \frac{\mathbb{R}^n}{\text{Ker exp}}. \quad (100)$$

$\text{Ker exp}$  is discrete, so it is generated by a set of linearly independent vectors  $\mathbf{v}_1, \dots, \mathbf{v}_k$ . We can extend them to a basis of  $\mathbb{R}^n$ :  $\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{v}_{k+1}, \dots, \mathbf{v}_n$ . Therefore,

$$\begin{aligned} G &\cong \frac{\mathbb{R}^n}{\langle \mathbf{v}_1, \dots, \mathbf{v}_k \rangle_{\mathbb{Z}}} \\ &= \frac{\mathbb{R}\mathbf{v}_1 \oplus \dots \oplus \mathbb{R}\mathbf{v}_k \oplus \mathbb{R}\mathbf{v}_{k+1} \oplus \dots \oplus \mathbb{R}\mathbf{v}_n}{\mathbb{Z}\mathbf{v}_1 \oplus \dots \oplus \mathbb{Z}\mathbf{v}_k} \\ &\cong (\mathbb{R}/\mathbb{Z})^k \oplus \mathbb{R}^{n-k} \\ &\cong (S^1)^k \times \mathbb{R}^{n-k}. \end{aligned} \quad (101)$$

$\blacksquare$

## §6 BCH Formula

In general, when  $[X_e, Y_e] \neq 0$ , Theorem 4.5 doesn't hold. So we can expect that

$$\exp(X_e) \exp(Y_e) = \exp(X_e + Y_e + \text{some "error term" of order } [X_e, Y_e]). \quad (102)$$

Furthermore, this “error terms” vanish when  $[X_e, Y_e] = 0$ . But there are some subtleties in writing (102). Suppose  $\exp$  is a diffeomorphism on  $\mathfrak{u} \subseteq \mathfrak{g}$  onto  $U \subseteq G$ .

$$\begin{array}{ccc} & \xrightarrow{\exp} & \\ \mathfrak{u} \subseteq \mathfrak{g} & & U \subseteq G \\ & \xleftarrow{\log} & \end{array}$$

We have to pick  $X_e, Y_e$  in a (possibly smaller than  $\mathfrak{u}$ ) neighborhood of  $\mathbf{0} \in \mathfrak{g}$  such that  $\exp(X_e)\exp(Y_e) \in \mathfrak{u}$ , so that  $\log[\exp(X_e)\exp(Y_e)]$  exists. With all the necessary assumptions, we can write the series expansion of  $\log[\exp(X_e)\exp(Y_e)]$ . This is known as the Baker–Campbell–Hausdorff formula, or BCH formula, in short.

**Theorem 6.1** (Baker–Campbell–Hausdorff formula)

For small enough  $X_e, Y_e \in \mathfrak{g}$  one has

$$\exp(X_e)\exp(Y_e) = \exp(\mu(X_e, Y_e)) \quad (103)$$

for some  $\mathfrak{g}$ -valued function  $\mu(X_e, Y_e)$  which is given by the following series convergent in some neighborhood of  $(0, 0)$ :

$$\mu(X_e, Y_e) = X_e + Y_e + \sum_{n \geq 2} \mu_n(X_e, Y_e), \quad (104)$$

where  $\mu_n(X_e, Y_e)$  is a Lie polynomial in  $X_e, Y_e$  of degree  $n$ , i.e. an expression consisting of commutators of  $X_e, Y_e$ , their commutators, etc., of total degree  $n$  in  $X_e, Y_e$ . This expression is universal: it does not depend on the Lie algebra  $\mathfrak{g}$  or on the choice of  $X_e, Y_e$ .

It is possible to write the expression for  $\mu$  explicitly. However, this is rarely useful, so we only write the first few terms:

$$\mu(X_e, Y_e) = X_e + Y_e + \frac{1}{2}[X_e, Y_e] + \frac{1}{12}\left([X_e, [X_e, Y_e]] + [Y_e, [Y_e, X_e]]\right) + \cdots \quad (105)$$

We won’t go into the full proof of Theorem 6.1 here right now, as this note is getting pretty long. Interested readers are encouraged to go through [6], Section 1.6 and 1.7. Let’s end this note with a few consequences of the BCH formula.

For a connected Lie group, it’s generated by a neighborhood of the identity, and near identity, the group multiplication is given by

$$\exp(X_e)\exp(Y_e) = \exp\left(X_e + Y_e + \frac{1}{2}[X_e, Y_e] + \frac{1}{12}\left([X_e, [X_e, Y_e]] + [Y_e, [Y_e, X_e]]\right) + \cdots\right)$$

Therefore, using the BCH formula, one can recover the group structure of a Lie group just using the Lie algebra structure  $\mathfrak{g}$ . However, this does not mean that given isomorphic Lie algebras  $\mathfrak{g}$  and  $\mathfrak{h}$ , they come from isomorphic Lie groups  $G$  and  $H$ . A counterexample of this would be

$G = S^1$  and  $H = \mathbb{R}$ . Both of them have isomorphic Lie algebras,  $\mathfrak{g} = \mathfrak{h} = \mathbb{R}$  with the trivial Lie bracket  $[s, t] = 0$ . But the groups are not isomorphic. Therefore, from the Lie algebra alone, we cannot determine the Lie group. If, along with the Lie algebra structure, we have the manifold structure, then we can determine the Lie group structure.

Notice, however, that  $\mathbb{R}$  is the simply connected cover of  $S^1$ . Indeed, a connected Lie group  $G$  and its simply connected cover  $\tilde{G}$  has the same Lie algebra, because the covering map

$$p : \tilde{G} \rightarrow G$$

is a local diffeomorphism near the identity, making the differential

$$(dp)_{\bar{e}} : T_{\bar{e}}\tilde{G} = \tilde{\mathfrak{g}} \rightarrow T_e G = \mathfrak{g}$$

an isomorphism. Using the BCH formula, we can actually answer the following question:

Let  $G$  and  $H$  be Lie groups. Given a Lie algebra homomorphism  $\Phi : \mathfrak{g} \rightarrow \mathfrak{h}$ , is there a Lie group homomorphism  $\phi : G \rightarrow H$  such that the following diagram commutes?

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\Phi} & \mathfrak{h} \\ \exp_G \downarrow & & \downarrow \exp_H \\ G & \xrightarrow{\phi} & H \end{array} \quad (106)$$

The answer is affirmative if  $G$  is simply connected (this result is also known as Lie's Third Theorem). Then we first construct  $\phi$  on a neighborhood of identity, by using the BCH formula. Then we extend it to the whole  $G$ . The details can be found in [12].

Note that the hypothesis that  $G$  is simply connected is necessary. Consider  $G = S^1$  and  $H = \mathbb{R}$ . Both of them have isomorphic Lie algebras,  $\mathfrak{g} = \mathfrak{h} = \mathbb{R}$ . So there is a Lie algebra homomorphism  $\Phi : \mathfrak{g} \rightarrow \mathfrak{h}$ . But there are no corresponding nontrivial Lie group homomorphism. Assume there is one, say  $\phi$ . Suppose

$$\phi(e^{i\theta}) = x \neq 0.$$

Then

$$\phi(e^{in\theta}) = nx, \text{ for any } n \in \mathbb{Z}.$$

But since  $S^1$  is compact,  $\phi(S^1)$  needs to be compact, i.e. closed and bounded. But it's not bounded if  $x \neq 0$ . Therefore, the only Lie group homomorphism  $\phi : S^1 \rightarrow \mathbb{R}$  is the trivial one. But its corresponding Lie algebra homomorphism is also trivial, NOT an isomorphism.

We will return to the proof of the Baker–Campbell–Hausdorff formula, and to the ideas that follow from it, in a future note, where these threads can be developed more fully.

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