Group Structure on $\tilde{G}$

Recall: universal property of cover

$\tilde{G} \xrightarrow{\pi} G$ is universal, in the sense that

$\forall$ simply connected manifold M and smooth map $f: M \to G$, with $m_0 \in M$ and $g_0 \in \pi^{-1}(f(m_0))$,

$\exists! \; \tilde{f}: M \to \tilde{G}$ smooth s.t. $f(m_0) = g_0$ &

$$\begin{tikzcd}
\tilde{G} \arrow[r, "\pi"] & G \\
M \arrow[u, dotted, "\exists!\tilde{f}"] \arrow[ur, "\forall f"'] \arrow[ur, phantom, shift left=3ex, "\checkmark"] &
\end{tikzcd}$$

.

Now, consider $s : \tilde{G} \times \tilde{G} \longrightarrow G$

$$(\bar{g}, \bar{h}) \longmapsto \pi(\bar{g})\,\pi(\bar{h})^{-1}$$

$\tilde{G} \times \tilde{G}$ is simply connected, so

$$\begin{tikzcd}
\tilde{G} \arrow[r, "\pi"] & G \\
\tilde{G} \times \tilde{G} \arrow[u, "\exists!\, \tilde{s}"] \arrow[ur, "s"'] \arrow[ur, phantom, bend left, "\checkmark"] &
\end{tikzcd}$$

Fix $\bar{e} \in \pi^{-1}(e)$ and $\tilde{s}(\bar{e}, \bar{e}) = \bar{e}$.

Define $\bar{h}^{-1} := \tilde{s}(\bar{e}, \bar{h})$

$\bar{g} \cdot \bar{h} := \tilde{s}(\bar{g}, \bar{h}^{-1})$.

<u>Theorem</u>: This defines a group structure.

Claim 1: $\cancel{\tilde{s}(\bar{g}, \bar{h}}$ $\pi(\bar{g} \cdot \bar{h}) = \pi(\bar{g}) \cdot \pi(\bar{h})$

Proof: $\pi(\bar{g} \cdot \bar{h}) = \pi(\tilde{s}(\bar{g}, \bar{h}^{-1}))$

$= s(\bar{g}, \bar{h}^{-1}) = \pi(\bar{g})\,\pi(\bar{h}^{-1})^{-1}$

$\pi(\bar{h}^{-1})^{-1} = \pi(\tilde{s}(\bar{e}, \bar{h}))^{-1} = s(\bar{e}, \bar{h})^{-1} = (\pi(\bar{e})\,\pi(\bar{h})^{-1})^{-1} = \pi(\bar{h})$.

Claim 2: $\bar{e}$ is two-sided identity.

consider $\cancel{l_{\bar{x}} = \tilde{s}(\bar{e}, \bar{x}^{-1}) : \tilde{G} \to \tilde{G}}$

$\cancel{\mathrm{id}}$

$$l(\bar{x}) = \bar{e}\cdot\bar{x} = \tilde{s}(\bar{e}, \bar{x}^{-1}) : \tilde{G} \to \tilde{G}$$

$$\mathrm{id}(\bar{x}) = \bar{x} : \tilde{G} \to \tilde{G}$$

$$\begin{tikzcd}
\tilde{G} \arrow[r, "\pi"] & G \\
\tilde{G} \arrow[u, shift left, "l"'] \arrow[u, shift right, "\mathrm{id}"'] \arrow[ur, "\pi"'] \arrow[u, phantom, "\checkmark", shift left=8ex]
\end{tikzcd}$$

$$\begin{aligned}
&\pi(l(\bar{x})) \\
&= \pi(\tilde{s}(\bar{e}, \bar{x}^{-1})) \\
&= s(\bar{e}, \bar{x}^{-1}) \\
&= \pi(\bar{e})\, \pi(\bar{x}^{-1})^{-1} \\
&= \pi(\bar{x})
\end{aligned}$$

$\Rightarrow$ By uniqueness, $l = \mathrm{id}$, i.e. $\bar{e}\cdot\bar{x} = \bar{x}$ $\forall \bar{x} \in \tilde{G}$.

Similarly,

$$\begin{tikzcd}
\tilde{G} \arrow[r, "\pi"] & G \\
\tilde{G} \arrow[u, shift left, "\gamma"'] \arrow[u, shift right, "\mathrm{id}"'] \arrow[ur, "\pi"'] \arrow[u, phantom, "\checkmark", shift left=8ex]
\end{tikzcd}$$

$$\gamma(\bar{x}) = \bar{x}\cdot\bar{e}$$

$$\begin{aligned}
\pi(\gamma(\bar{x})) &= \pi(\tilde{s}(\bar{x}, \bar{e}^{-1})) \\
&= s(\bar{x}, \bar{e}) \\
&= \pi(x)
\end{aligned}$$

$\Rightarrow \mathrm{id} = \gamma$. $\square$

Claim 3: Inverses are two-sided.

$$\begin{tikzcd} \tilde{G} \arrow[r, "\pi"] & G \\ \tilde{G} \arrow[u, shift left, "i_1"] \arrow[u, shift right, "i_2"'] \arrow[ur, "\text{const map to } e \in G"'] & \end{tikzcd}$$

$i_1(\bar{x}) = \bar{x} \cdot \bar{x}^{-1}$

$i_2(\bar{x}) = \bar{e}$.

$$\pi(i_1(\bar{x})) = \pi(\tilde{s}(\bar{x}, (\bar{x}^{-1})^{-1}))$$
$$= \pi(\tilde{s}(\bar{x}, \bar{x}))$$
$$= s(\bar{x}, \bar{x}) = e$$

$\Rightarrow i_1 = i_2$, since they agree at $\bar{e}$.

similarly,

$$\begin{tikzcd} \tilde{G} \arrow[r, "\pi"] & G \\ \tilde{G} \arrow[u, shift left, "j_1"] \arrow[u, shift right, "j_2"'] \arrow[ur, "\text{const map to } e \in G"'] & \end{tikzcd}$$

$j_1(\bar{x}) = \bar{x}^{-1} \cdot x$

$j_2(x) = \bar{e}$

$$\pi(j_1(\bar{x})) = \pi(\tilde{s}(\bar{x}^{-1}, \bar{x}^{-1})) = e$$

$\Rightarrow j_1 = j_2$ $\square$

~~A~~ Claim 4: Associativity:

$$\begin{tikzcd}
\tilde{G} \arrow[r, "\pi"] & G \\
\tilde{G}\times\tilde{G}\times\tilde{G} \arrow[u, shift left, "m_1"] \arrow[u, shift right, "m_2"'] \arrow[ur, "(\bar{g},\bar{h},\bar{k}) \longmapsto \pi(\bar{g})\pi(\bar{h})\pi(\bar{k})"'] &
\end{tikzcd}$$

$$m_1(\bar{g},\bar{h},\bar{k}) = (\bar{g}\cdot\bar{h})\cdot\bar{k}$$
$$m_2(\bar{g},\bar{h},\bar{k}) = \bar{g}\cdot(\bar{h}\cdot\bar{k})$$

$$\pi(m_1(\bar{g},\bar{h},\bar{k})) = \pi(\bar{g}\cdot\bar{h})\pi(\bar{k})$$
$$= \pi(\bar{g})\pi(\bar{h})\pi(\bar{k})$$
(claim 1)

$$\pi(m_2(\bar{g},\bar{h},\bar{k})) = \pi(\bar{g})\,\pi(\bar{h}\cdot\bar{k})$$
$$= \pi(\bar{g})\pi(\bar{h})\pi(\bar{k})$$
(1)

$$\Rightarrow m_1 = m_2$$

$\therefore$ $\tilde{G}$ is a group. $\blacksquare$