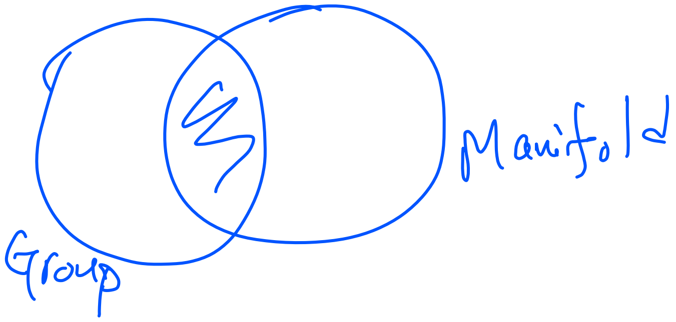
Lie Groups

S^1 $\quad SO(2)$

manifold

$\boxed{\text{Group + Manifold}}$



multiplication map

$$\left.\begin{aligned} m : G\times G &\longrightarrow G \\ (g_1, g_2) &\longmapsto g_1 g_2 \end{aligned}\right\} \text{ smooth } C^\infty$$

inverse map

$$\begin{aligned} i : G &\longrightarrow G \\ g &\longmapsto g^{-1} \end{aligned} \qquad \text{smooth}$$

Man category of smooth manifolds

LieGrps is $\underbrace{\text{cat of groups}}_{\substack{G\times G \to G \\ i : G \to G}}$ in Man

$m : G \times G \to G$

$i : G \to G$

open map?

sufficient

$$\underbrace{m(\mathcal{U} \times V) \text{ is open}}$$

$$m\left(\bigcup \mathcal{U}_a \times V_a\right) = \bigcup m(\mathcal{U}_a \times V_a)$$

$$l_g : G \to G \qquad \text{left translation map}$$
$$h \mapsto gh$$

smooth?

$$\begin{tikzcd} G \arrow[r, "j_g"] \arrow[rr, bend right, "l_g"'] & G \times G \arrow[r, "m"] & G \end{tikzcd}$$

$j_g(h) = (g, h)$

l_g ✓

l_g smooth

$\cancel{l_{g^{-1}}}$

$l_{g^{-1}} \circ l_g = id$

$l_g \circ l_{g^{-1}} = id$

l_g is a diffeomorphism.

in particular, l_g is an open map.

$l_g(V)$ is open $\forall$ open $V \subseteq G$
$\forall g \in G$.

$$m(\mathcal{U}\times V) = \underbrace{\boxed{\bigcup_{g\in\mathcal{U}} l_g(V)}}_{\text{open}} = \bigcup_{g\in V} r_g(\mathcal{U})$$

$l_g(V) := gV$

$$\boxed{\begin{array}{l}\text{Given any open } \mathcal{U}, V \subseteq G, \\ \quad \mathcal{U}V \text{ is open} \longrightarrow \{gh : g\in\mathcal{U},\ h\in V\}\end{array}}$$

$\hookrightarrow$ if either of $\mathcal{U}, V$ are open

$\mathcal{U}V$ is open.

if V open $\quad \mathcal{U}V = \bigcup l_g(V) = \bigcup gV$

$\mathcal{U}$ open $\quad \mathcal{U}V = \bigcup r_g(\mathcal{U}) = \bigcup \mathcal{U}g$

$m|_{\mathcal{U}\times V} : \mathcal{U}\times V \longrightarrow \mathcal{U}V$ open

$\mathbb{R}$, $\mathbb{R}^n$, $(\mathbb{R}^{>0}, \times)$, $S^1 =$

$GL(n,\mathbb{R})$, $SL(n,\mathbb{R})$, $O(n,\mathbb{R})$, $U(n)$, $SO(n,\mathbb{R})$, $SU(n)$

complex Lie groups

$O(n) = \{A : AA^T = \mathbb{I}\}$

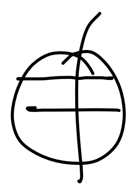
det $+1$

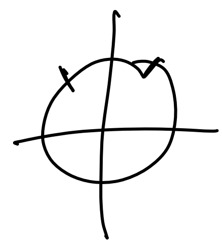
det -1

<u>Theorem</u>: If X is a connected manifold, X is path connected.

$\mathbb{R}^n$ is locally connected

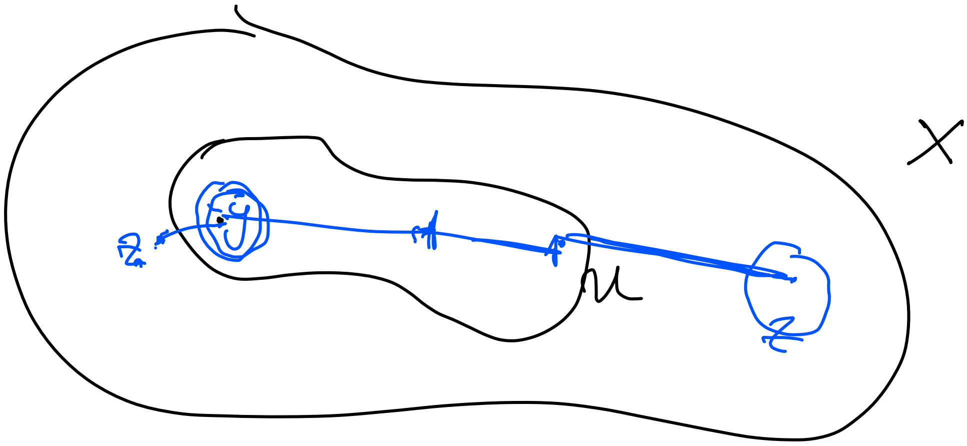
" path connected





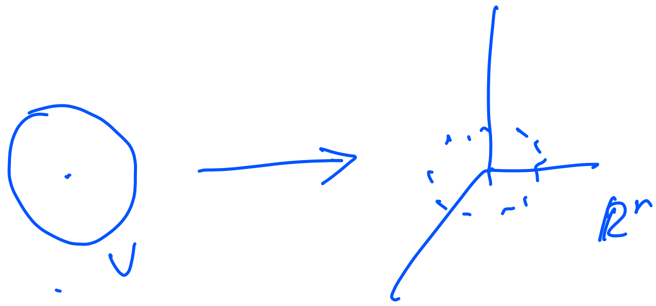
If a top space X is locally path conn, connected $\iff$ path connected.

Proof:



$$\mathcal{U} = \{ y : \boxed{y \sim x} \}$$

$\mathcal{U}$ open



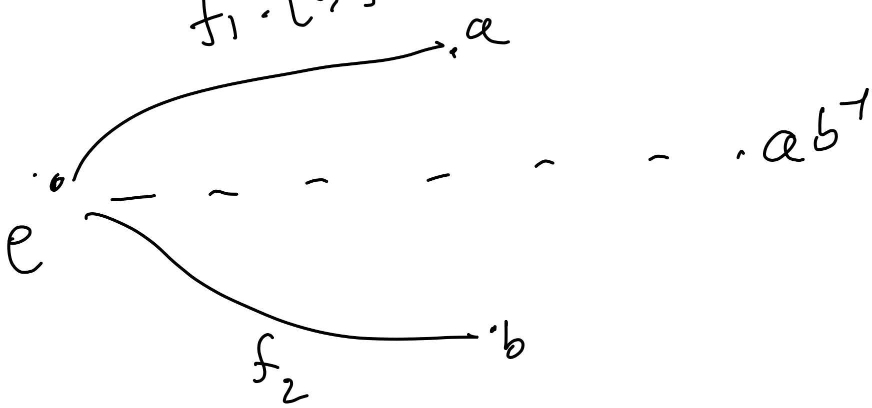
$\mathcal{U}^c$ open

$$X = \mathcal{U} \cup \mathcal{U}^c$$

Theorem: G° is the connected component of $e \in G$.

1. $\underbrace{G^\circ \text{ is a subgroup of } G.}_{a,b \in G^\circ \Rightarrow ab^{-1} \in G^\circ.}$

$f_1 : [0,1] \to G$



e ——— ab

$e \longrightarrow a^{\gamma}$

$f : [0,1] \longrightarrow G$

$f(0) = e$

$f(1) = ab$

$$\begin{tikzcd}
& b \arrow[dl, bend right, "\overline{f_2}"'] \\
e \arrow[ur, "f_2"'] \arrow[dr, bend right, "f_1"'] & \\
& a
\end{tikzcd}$$

$$\begin{tikzcd}
e \arrow[r, "f_1"] \arrow[d, phantom, "\Downarrow r_b"] & a \\
b \arrow[r, "\cancel{f_1 \circ r_b}"'] & ab
\end{tikzcd}
\qquad\Bigg|\qquad
\begin{tikzcd}
e \arrow[r, "f_1"] \arrow[d, phantom, "\Downarrow i"] & a \\
e \arrow[r, "i \cdot f_1"'] & a^{\gamma}
\end{tikzcd}$$

$r_b \circ f_1$

alt proof: $f : [0,1] \to G$

$$f(t) = m\left(f_1(t),\ i(f_2(t))\right)$$

$$f(0) = m(e,e) = e$$

$$f(1) = m(a, b^{-1}) = ab^{-1}$$

$$f := m \circ (f_1,\ i \circ f_2)$$

11 G^0 is a normal subgroup.

$$\boxed{\forall g \in G,\ a \in G^0,\ gag^{-1} \in G^0.}$$ WTS.

$$e \xrightarrow{f} a$$

have to construct

$$e \xrightarrow{f'} gag^{-1}$$

$$f'(t) = g \cdot f(t) \cdot g^{-1}$$

$$= m\left(m(g, \underline{f(t)}),\ \underline{i(g)}\right)$$

$O(1,3)$

$\Downarrow$

$SO(1,3)^{+}$

connected component of identity

$\Rightarrow G^0$ is normal subgroup.

(iii) G/G^0 is a countable $\underline{\text{discrete group.}}$

quotient topology

G is ~~...~~

$\overset{?}{\cong} \underline{G^0} \times \underline{G/G^0}$

$\cancel{G}$

0-dim Lie groups: H

$|H| \le \cancel{\aleph_0} |\mathbb{N}|$

G/G^0 discrete?

$\Leftrightarrow$ any singleton $\{\bar{g}\} \subseteq G/G^0$ is open

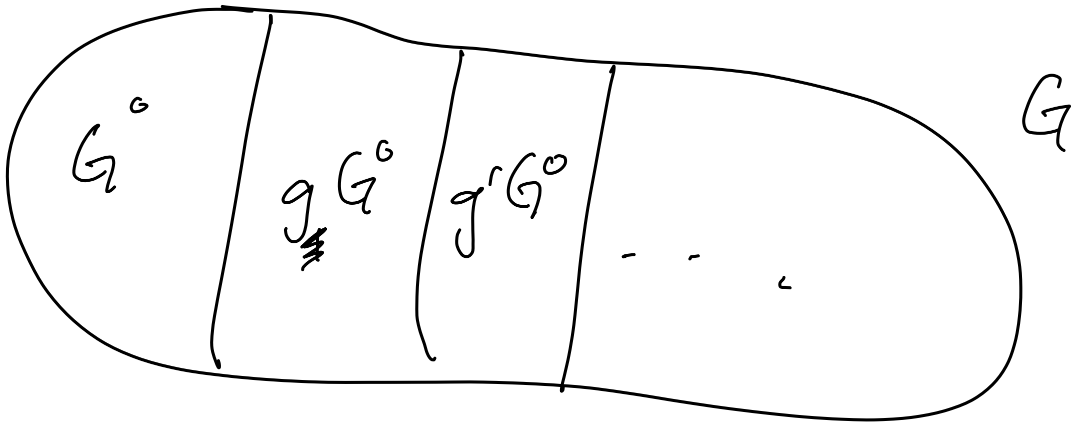
$\pi : X \longrightarrow X/\sim$

$U \subseteq X/\sim$ is open

$\Leftrightarrow \pi^{-1}(U) \subseteq X$ is open

$\Leftrightarrow \pi^{-1}(\bar{g}) \subseteq G$ is open ✓

$= g G^0$



gG^0 is clopen, connected.

$\Rightarrow$ $gG^0, \ldots$ all the connected components of G.

2nd countable

Claim: a manifold can only have countably many components.

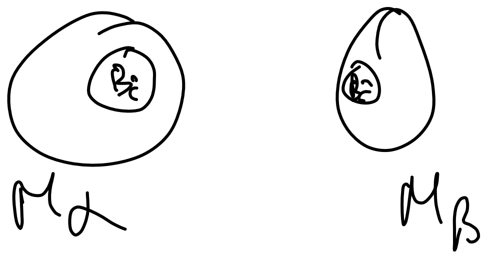
M has $\{M_\alpha\}$, $\alpha \in J$, $|J| > \aleph_0$

$\downarrow$

$\mathcal{B}_d$ $\{B_i\}_{i=0}^{\infty}$ $\quad$ $\forall M_\alpha$, $\exists B_i$ s.t. $B_i \subseteq M_\alpha$

$J \longrightarrow \mathbb{N}$

$\alpha \longmapsto i$ s.t. $B_i \subseteq M_\alpha$



injective $\Rightarrow |J| \leq |\mathbb{N}|$.

$\Rightarrow$ G has countably many components.

$\{gG^0_{\alpha}, \ldots\} = G/G^0$ countable discrete group.

$\blacksquare$

Theorem: G is a Lie group, H is a subgroup. if H is closed, then H is a Lie subgroup.

subgroup + embedded submanifold

(i) Alexander Kirillov Jr. lecture notes, STONY Brook

(ii) Lecture Notes by Etingof, MIT

(iii) Humphrey's for Lie Algebra.