Quotient Lie Group

Atonu Roy Chowdhury

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Let G be a Lie group and H a Lie subgroup, i.e. a subgroup which is also a (regular) submanifold. Then it follows that H is closed. The natural question to ask is: when H is normal, can G/H be given a Lie group structure? We shall explore the answer to this question in this note, and we will find out that the answer is affirmative!

§1 Quotient Manifold

Let's first recall the concepts of quotient topology:

> A surjective map $p : X \to Y$ between topological spaces is a **quotient map** if the following is satisfied: $V \subseteq Y$ is open **if and only if** $p^{-1}(V) \subseteq X$ is open.

Then we have the result: (we can call it the "universal property of quotient map")

> Suppose $p : X \to Y$ is a quotient map. Let Z be a topological space, and $f : X \to Z$ a map such that f is constant on each set $p^{-1}(\{y\})$ for $y \in Y$. Then there exist a unique map $\widetilde{f} : Y \to Z$ such that $\widetilde{f} \circ p = f$, i.e. the following diagram commutes:
>
> $$\begin{tikzcd} X \arrow[r, "p"] \arrow[dr, "f"'] & Y \arrow[d, dashed, "\widetilde{f}"] \\ & Z \end{tikzcd} \tag{1}$$
>
> Furthermore, $\widetilde{f}$ is continuous if and only if f is continuous.

Equivalently,

> Suppose $p : X \to Y$ is a quotient map. Let Z be a topological space, and consider the map $f : Z \to Y$.
>
> $$\begin{tikzcd} X \arrow[r, "p"] \arrow[dr, "f\circ\pi"'] & Y \arrow[d, "f"] \\ & Z \end{tikzcd} \tag{2}$$
>
> Then f is continuous if and only if $f \circ \pi$ is continuous.

Then given any equivalence relation $\sim$ on a topological space X, we give the set $X/\sim$ the **unique** topology such that

$$\begin{aligned} \pi : X &\to X/\sim, \\ x &\mapsto [x] \end{aligned} \tag{3}$$

is a quotient map. In other words, $X/\sim$ is equipped with the **finest** (i.e. "largest") topology such that $\pi : X \to X/\sim$ is continuous.

We may be prompted to do a similar construction for manifolds as well. But the truth is, manifolds are much more intricate structures, so they need a little bit more care. For example, if we define the equivalence relation on $\mathbb{R}$

$$0 \sim 1, \text{ and } a \sim a \text{ for } a \in \mathbb{R},$$

then the quotient space is

Clearly this doesn't have a manifold structure. So we can't just get away with giving a manifold **any** equivalence relation. The equivalence relation has to be "nice enough". The exact meaning of "nice enough" is given by *Godement's Criterion*:

> **Theorem 1.1** (Godement's Criterion)
>
> Let M be a manifold and $R \subseteq M \times M$ an equivalence relation. Then there is **at most one** manifold structure on M/R such that $p : M \to M/R$ (maps $m \in M$ to its equivalence class $[m]$) is a submersion. Furthermore, M/R can be given a manifold structure **if and only if**
>
> (a) R is a closed embedded submanifold of $M \times M$; and
>
> (b) the map $R \hookrightarrow M \times M \xrightarrow{\pi_1} M$ is a submersion.

We won't go into the proof of this result. The interested readers are encouraged to go through Chapter 9 of [3], or the French-speaking audience may refer to Section 5.9.5 of [4]. In the following section, we shall work with the special case where $M = G$ is a Lie group, and the equivalence relation is $g_1 \sim g_2$ if and only if $g_1 H = g_2 H$ for a Lie subgroup H.

Before that, it's important to note a few parallels between quotient map and submersions. We know that submersions are open maps. Hence, a surjective smooth submersion is a quotient map. So submersions are much more stronger than quotient maps. But they are the parallel of quotient maps in the world of manifolds.

In topology, the set of equivalence class is given the **unique** topology such that $\pi : X \to X/\sim$ is a quotient map. In this case, we are giving M/R a manifold structure, and in doing so, we are making sure that $\pi : M \to M/R$ becomes a submersion. This is because submersions have a similar "universal property" like quotient map.

Proposition 1.2

Let $\pi : M \to N$ be a surjective smooth submersion, and P a manifold. Consider the map $F : N \to P$.

$$\begin{tikzcd} M \arrow[r, "\pi"] \arrow[dr, "F\circ\pi"'] & N \arrow[d, "F"] \\ & P \end{tikzcd} \tag{4}$$

Then F is smooth if and only if $F \circ \pi$ is smooth.

Proof. If F is smooth, then clearly $F \circ \pi$ is smooth as the composition of two smooth maps. Now suppose $F \circ \pi$ is smooth. Take $q \in N$.

Since π is surjective, we can find $p \in M$ such that $\pi(p) = q$. Since π is a submersion, we claim that there is a neighborhood $V \ni q$ and a smooth map $\sigma : V \to M$ such that $\sigma(q) = p$ and $\pi \circ \sigma = \mathrm{id}_V$. This is also known as the local section lemma.

Lemma 1.3 (Local section lemma)

Let $\pi : M \to N$ be a smooth map. Then π is a submersion if and only if for any $p \in M$ and $q = \pi(p)$, there is a neighborhood $V \ni q$ and a smooth map $\sigma : V \to M$ such that $\sigma(q) = p$ and $\pi \circ \sigma = \mathrm{id}_V$.

Proof. **($\Rightarrow$)** Suppose π is a submersion. By the submersion theorem, there are charts (U, φ) centered at p and (V, ψ) centered at $\pi(p) = q$ such that

$$\left(\psi \circ \pi \circ \varphi^{-1}\right)\left(r^1, \ldots, r^k, r^{k+1}, \ldots, r^n\right) = \left(r^1, \ldots, r^k\right). \tag{5}$$

(Here, $n = \dim M$ and $k = \dim N$) Now we define, $\sigma : V \to M$ as follows:

$$\sigma(n) = \varphi^{-1}\left[\psi(n), 0, 0, 0, \ldots, 0\right]. \tag{6}$$

In other words, if $\psi(n) = \left(r^1, \ldots, r^k\right)$, then $\varphi\left(\sigma(n)\right)$ has the coordinates $\left(r^1, \ldots, r^k, 0, \ldots, 0\right)$. Clearly, σ is smooth, and $\sigma(q) = p$. Furthermore,

$$\left(\psi \circ \pi \circ \sigma\right)(n) = \left(\psi \circ \pi\right)\varphi^{-1}\left(\psi(n), 0, 0, \ldots, 0\right) = \psi(n). \tag{7}$$

So $\psi \circ \pi \circ \sigma = \psi$, proving that $\pi \circ \sigma = \mathrm{id}_V$. ■

Now, on V, F is

$$F|_V = F \circ \mathrm{id}_V = F \circ (\pi \circ \sigma) = (F \circ \pi) \circ \sigma, \tag{8}$$

the composition of two smooth maps. Hence, F is smooth on V. Since $q \in N$ is arbitrary, F is smooth on the whole N.

(⇐) Conversely, suppose that given any $p \in M$ there is a neighborhood $V \ni q = \pi(p)$ and a smooth map $\sigma : V \to M$ such that $\sigma(q) = p$ and $\pi \circ \sigma = \mathrm{id}_V$. Then at the level of tangent spaces,

$$(d\pi)_p \circ (d\sigma)_q = \mathbb{1}_{T_q V}, \tag{9}$$

proving that $(d\pi)_p$ is surjective, i.e. π is a submersion. ■

Proposition 1.4

Suppose $\pi : M \to N$ is a surjective smooth submersion. Let P be a manifold, and $F : M \to P$ a map such that F is constant on each set $\pi^{-1}(\{q\})$ for $q \in N$. Then there exists a unique map $\widetilde{F} : N \to P$ such that $\widetilde{F} \circ \pi = F$, i.e. the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "\pi"] \arrow[dr, "F"'] & N \arrow[d, dashed, "\widetilde{F}"] \\ & P \end{tikzcd} \tag{10}$$

Furthermore, $\widetilde{F}$ is smooth if and only if F is smooth.

Proof. Since surjective smooth surjection is a quotient map, there exists a unique map $\widetilde{F} : N \to P$ such that $\widetilde{F} \circ \pi = F$, and $\widetilde{F}$ is continuous if and only if F is continuous. Furthermore, $\widetilde{F}$ is smooth if and only if F is smooth follows from Proposition 1.2. ■

These results should be enough to convince the reader that a surjective smooth submersion is the differential geometric analogue of quotient map in topology. That's why when we are giving M/R a manifold structure, we also require that it's "compatible" with the map $\pi : M \to M/R$, i.e. the manifold structure on M/R makes π a surjective smooth submersion.

Now, let's take a moment to show that if such a manifold structure on M/R exists, then it's unique. Suppose M/R can be given two manifold structures, namely $(M/R)_1$ and $(M/R)_2$ such that both

$$\pi_1 : M \to (M/R)_1 \text{ and } \pi_2 : M \to (M/R)_2$$

are smoth surjective submersions. Now consider the following commutative diagram:

$$\begin{tikzcd} M \arrow[r, "\pi_1"] \arrow[dr, "\pi_2"'] & (M/R)_1 \arrow[d, "\mathrm{id}_{M/R}"] \\ & (M/R)_2 \end{tikzcd} \tag{11}$$

Since π_1 is a surjective smooth submersion and π_2 is smooth, $\mathrm{id}_{M/R} : (M/R)_1 \to (M/R)_2$ is smooth, by Proposition 1.2. Similarly, $\mathrm{id}_{M/R} : (M/R)_2 \to (M/R)_1$ is also smooth. Therefore, the identity map

$$\mathrm{id}_{M/R} : (M/R)_1 \to (M/R)_2$$

is a diffeomorphism. Hence, the manifold structures are the same!

§2 Lie Group Setting

Let G be a Lie group, and H a Lie subgroup, i.e. a subgroup (not necessarily normal) and a (regular) submanifold. Consider the set G/H of all left-cosets, and the map $\pi : G \to G/H$ that takes $g \in G$ to its coset gH. We can also define the equivalence relation

$$g_1 \sim g_2 \iff g_1 H = g_2 H.$$

Then $g_1 H$ is the equivalence class containing g_2. We can give G/H the quotient topology with respect to π. In this section, we shall prove that G/H can be given a (unique) manifold structure.

Lemma 2.1

$\sim$ is an open equivalence relation, i.e. $\pi : G \to G/H$ is an open map.

Proof. Let $U \subseteq G$ be open. Then

$$\begin{aligned} \pi(U) \text{ is open} &\iff \pi^{-1}(\pi(U)) \subseteq G \text{ is open} \\ &\iff \pi^{-1}(\pi(U)) = UH \text{ is open.} \end{aligned}$$

Now, clearly UH is open, since

$$UH = \bigcup_{h \in H} Uh, \tag{12}$$

and right-multiplication by $h \in H$ is a diffeomorphism. Therefore, π is an open map. ■

Corollary 2.2

G/H is second coutable.

Proof. Since π is an open map, if $\{B_i\}_{i=1}^{\infty}$ is a countable basis for G, $\{\pi(B_i)\}_{i=1}^{\infty}$ is a countable basis for G/H. ∎

It's well known that a Lie subgroup is closed. It's important to note that this is not generally true if H is an immersed submanifold instead of regular (embedded) submanifold. For example, consider the map $\phi : \mathbb{R} \to S^1 \times S^1$ given by

$$\phi(x) = \left(e^{ix}, e^{ix\sqrt{2}}\right). \tag{13}$$

Here, ϕ is an immersion, but not an embedding. Hence, $\phi(\mathbb{R})$ is an immersed (but not embdeed) submanifold of $S^1 \times S^1$. Furthermore, since $\phi(\mathbb{R})$ is dense in $S^1 \times S^1$, it's not closed.

Note that, if H wasn't closed, G/H can't be Hausdorff. If G/H is Hausdorff, the singleton $\{\bar{e}\}$ is closed. Since $\pi : G \to G/H$ is continuous,

$$\pi^{-1}(\{\bar{e}\}) = H$$

has to be closed in G. Therefore, H being closed is a necessary condition for G/H being Hausdorff. The following result tells us that it's also sufficient.

Corollary 2.3

G/H is Hausdorff.

Proof. We know that

> If $\sim$ is an open equivalence relation on X, then $X/\sim$ is Hausdorff if and only if the graph $R = \{(x, y) \mid x \sim y\}$ is closed in $X \times X$.

In this case, $g_1 \sim g_2 \iff g_1 H = g_2 H \iff g_1^{-1} g_2 \in H$. Now consider the map

$$\begin{aligned} f : G \times G &\to G \\ (g_1, g_2) &\mapsto g_1^{-1} g_2. \end{aligned} \tag{14}$$

This is a smooth map, since it's composed of smooth maps. In particular, f is continuous. H is closed in G, hence $R = f^{-1}(H)$ is closed in $G \times G$. Therefore, G/H is Hausdorff. ∎

Now all that is left to prove is that G/H is locally Euclidean, and the charts are compatible.

Proposition 2.4

G/H is locally Euclidean at $\bar{e}$.

Proof. Take a chart (U, φ) of G centered at e. Since H is a regular submanifold, we can choose U to be a "adapted chart", i.e. such that $U \cap H$ is defined by the vanishing of $n-k$ coordinate functions. Now we define the slice

$$S = \left\{ g \in U \mid \varphi(g) = \left(0, 0, \ldots, 0, r^{k+1}, \ldots, r^n\right) \right\}. \tag{15}$$

Then S is also an embedded submanifold of G, but of dimension $n-k$. Consider the map

$$\begin{aligned} \psi : S \times H &\to G \\ (s, h) &\mapsto sh. \end{aligned} \tag{16}$$

Consider the differential

$$(d\psi)_{(e,e)} : T_{(e,e)} \left(S \times H\right) \cong T_e S \times T_e H \to T_e G. \tag{17}$$

The dimensions of the domain space and the codomain space are the same. This is also surjective, since any basis element $\frac{\partial}{\partial x^i}\big|_e$ is the image of either $\left(\frac{\partial}{\partial x^i}\big|_e, \mathbf{0}\right)$ (if $i \leq k$) or $\left(\mathbf{0}, \frac{\partial}{\partial x^i}\big|_e\right)$ (if $i > k$). Therefore, $(d\psi)_{(e,e)}$ is an isomorphism of vector spaces. (S is called the submanifold transversal to H. They intersect at e only, and at that intersection point, $T_e S \times T_e H \cong T_e G$.) By the inverse function theorem, there are open neighborhoods

$$e \in S_0 \subseteq S, \qquad e \in H_0 \subseteq H, \qquad e \in U_0 \subseteq U$$

such that $\psi : S_0 \times H_0 \to U_0$ is a diffeomorphism. Therefore, each point of U_0 can be written **uniquely** as sh for some $s \in S_0$ and $h \in H_0$.

Now, given any two points $s_1 h_1, s_2 h_2 \in U_0$, they belong in the same coset if and only if $s_1^{-1} s_2 \in H$. We can take S_0 to be as small as we want, so that $s_1^{-1} s_2 \in H$ is only possible when $s_1 = s_2$. For instance, we can replace S_0 by $S_0 \cap S_0^{-1} S_0 = S_0 \cap m\left(i(S_0), S_0\right)$ (we can do this because $S_0 \cap S_0^{-1} S_0$ is a nonempty open set containing e), if required. That would mean $s_1^{-1} s_2 \in S_0$. Since $S \cap H = \{e\}$, Therefore, each coset intersects S_0 at most once. So we have the map

$$\begin{aligned} f : \pi(U_0) &\to S_0 \\ [sh] &\mapsto s. \end{aligned} \tag{18}$$

Clearly, it's well-defined and bijective. Now, consider the commutative diagram:

$$\begin{tikzcd} U_0 \arrow[r, "\pi"] \arrow[dr, "sh \mapsto s"'] & \pi(U_0) \arrow[d, "f"] \\ & S_0 \end{tikzcd} \tag{19}$$

Since π is a quotient map, and the map $U_0 \to S_0, sh \mapsto s$ is continuous, we have that f is continuous. In the other direction, f^{-1} is the composition:

$$S_0 \xrightarrow{s \mapsto (s, h_0)} S_0 \times H_0 \xrightarrow{\psi} U_0 \xrightarrow{\pi} \pi(U_0) \tag{20}$$

Hence, f^{-1} is also continuous, i.e. $f : \pi(U_0) \to S_0$ is a homeomorphism. Furthermore, S_0 is homeomorphic to some open set of $\mathbb{R}^{n-k}$. Therefore, $\overline{e} \in G/H$ has a neighborhood, namely $\pi(U_0)$, which is homeomorphic to some open set of $\mathbb{R}^{n-k}$, proving that G/H is locally Euclidean at $\overline{e}$. ■

Now, in order to show that G/H is locally Euclidean at every $\overline{g} \in G/H$, we just left multiply everything by g.

Proposition 2.5

G/H is locally Euclidean at $\overline{g}$.

Proof. The left-multiplication-by-g map $L_g : G \to G$ is a diffeomorphism. We define

$$U_g = gU_0, \qquad S_g := gS_0. \tag{21}$$

Then, $\pi(U_g) = \pi(gU_0) = g\pi(U_0)$ is homeomorphic to gS_0 via left multiplying the homeomorphism $f : \pi(U_0) \to S_0$ by g. Furthermore, gS_0 is homeomorphic to S_0, which is also homeomorphic to some open subset of $\mathbb{R}^{n-k}$. Hence, G/H is locally Euclidean at every $\overline{g} \in G/H$. ■

The homeomorphism $f_g : \pi(U_g) \to S_g$ acts like this:

$$f_g([gu]) = gf([u]). \tag{22}$$

Then the total chain of going from an open set $\pi(U_g)$ containing $\overline{g}$ to an open set of $\mathbb{R}^{n-k}$ is:

$$\pi(U_g) \xrightarrow{f_g} S_g = gS_0 \xrightarrow{L_{g^{-1}}} S_0 \xrightarrow{\xi} V_0 \subset \mathbb{R}^{n-k} \tag{23}$$

Then the collection

$$\left\{\left(\pi(U_g), \xi \circ L_{g^{-1}} \circ f_g\right)\right\}_{g \in G}$$

forms an atlas for G/H. Now we need to show that this atlast is C^∞-compatible. Before that, let's focus on one single chart $\left(\pi(U_g), \xi \circ L_{g^{-1}} \circ f_g\right)$. Clearly, $\pi(U_g)$ is a smooth manifold, since $\left\{\left(\pi(U_g), \xi \circ L_{g^{-1}} \circ f_g\right)\right\}$ is an atlas.

Lemma 2.6

$f_g : \pi(U_g) \to S_g$ is a diffeomorphism.

Proof. Take the atlas $(U, \varphi) := \left(\pi(U_g), \xi \circ L_{g^{-1}} \circ f_g\right)$ on the domain and the atlas $(V, \psi) := \left(S_g, \xi \circ L_{g^{-1}}\right)$ on the codomain. Then

$$\begin{aligned}\psi \circ f_g \circ \varphi^{-1} &= \xi \circ L_{g^{-1}} \circ f_g \circ \left(\xi \circ L_{g^{-1}} \circ f_g\right)^{-1} \\ &= \xi \circ L_{g^{-1}} \circ f_g \circ f_g^{-1} \circ L_g \circ \xi^{-1} \\ &= \xi \circ L_{g^{-1}} \circ L_g \circ \xi^{-1},\end{aligned} \tag{24}$$

which is smooth. Therefore, f_g is smooth. In the other direction, one can show that

$$\varphi \circ f_g^{-1} \circ \psi = \xi \circ L_{g^{-1}} \circ L_g \circ \xi^{-1}, \tag{25}$$

which is also smooth. Therefore, f_g^{-1} is a diffeomorphism. ■

> **Lemma 2.7**
>
> $\pi|_{U_g} : U_g \to \pi(U_g)$ is smooth.

Proof. Choose the atlas $\left(U_g, \varphi|_{U_0} \circ L_{g_{-1}}\right)$ on the domain and the atlas $\left(\pi(U_g), \xi \circ L_{g^{-1}} \circ f_g\right)$ on the codomain, and consider:

$$\xi \circ L_{g^{-1}} \circ f_g \circ \pi|_{U_g} \circ L_g \circ \varphi|_{U_0}^{-1}. \tag{26}$$

First note that, $f_g \circ \pi|_{U_g}$ takes an element $gsh \in gU_0 = U_g$ to $gs \in gS_0 = S_g$. Therefore, $L_{g^{-1}} \circ f_g \circ \pi|_{U_g} \circ L_g$ will take $sh \in U_0$ to $s \in S_0$. In other words, it's nothing but the composition

$$U_0 \xrightarrow{\psi^{-1}} S_0 \times H_0 \xrightarrow{(s,h)\mapsto s} S_0 \tag{27}$$

Therefore, this is definitely smooth. Composing by diffeomorphisms to the left or right also preserves smoothness, so the map on (26) is smooth, proving that $\pi|_{U_g}$ is smooth. ■

> **Lemma 2.8**
>
> $\pi|_{U_g} : U_g \to \pi(U_g)$ is a smooth submersion.

Proof. Consider the map $f_g : \pi(U_g) \to S_g \subseteq U_g$. Both $\pi|_{U_g}$ and f_g are smooth. Furthermore.

$$\left(\pi|_{U_g} \circ f_g\right)[gsh] = \pi|_{U_g}(gs) = [gs]\,. \tag{28}$$

$[gsh]$ and $[gs]$ are the same since they differ by multiplication by $h \in H_0 \subseteq H$. Hence, $\pi|_{U_g} \circ f_g = \mathrm{id}_{\pi(U_g)}$. Therefore, by Local section lemma, $\pi|_{U_g} : U_g \to \pi(U_g)$ is a smooth submersion. ■

Proposition 2.9

The atlas $\left\{\left(\pi\left(U_g\right), \xi \circ L_{g^{-1}} \circ f_g\right)\right\}_{g \in G}$ is C^∞-compatible.

Proof. Take two intersecting charts

$$\left(\pi\left(U_{g_1}\right), \xi \circ L_{g_1^{-1}} \circ f_{g_1}\right) \text{ and } \left(\pi\left(U_{g_2}\right), \xi \circ L_{g_2^{-1}} \circ f_{g_2}\right).$$

Then the transition map is

$$\left(\xi \circ L_{g_1^{-1}} \circ f_{g_1}\right) \circ \left(f_{g_2}^{-1} \circ L_{g_2} \circ \xi^{-1}\right) \tag{29}$$

ξ, L_g are diffeomorphisms. Now, consider $f_{g_1} \circ f_{g_2}^{-1}$, we need to show that this is smooth. Note the domain and codomain of this map:

$$f_{g_1} \circ f_{g_2}^{-1} : f_{g_2}\left(\pi\left(U_{g_1}\right) \cap \pi\left(U_{g_2}\right)\right) \subseteq S_{g_2} \to f_{g_1}\left(\pi\left(U_{g_1}\right) \cap \pi\left(U_{g_2}\right)\right) \subseteq S_{g_1}.$$

Both f_{g_1} and $f_{g_2}^{-1}$ are smooth as we have seen in Lemma 2.6. Therefore, their composition $f_{g_1} \circ f_{g_2}^{-1}$ is smooth, and we are done! ■

Thus we have given G/H a manifold structure! All that's left is to show that this actually makes $\pi : G \to G/H$ a smooth submersion.

Theorem 2.10

$\pi : G \to G/H$ is a smooth submersion.

Proof. For every $g \in G$, $\pi|_{U_g}$ is smooth. Therefore, π is smooth. Now, given any $g \in G$ and $[g] \in G/H$, one can choose the neighborhood $\pi\left(U_g\right)$ of $[g]$, and the map

$$f_g : \pi\left(U_g\right) \to U_g \subseteq G.$$

f_g is smooth, as seen earlier. Furthermore,

$$\pi|_{U_g} \circ f_g = \mathrm{id}_{\pi(U_g)}. \tag{30}$$

Therefore, by Local section lemma, $\pi : G \to G/H$ is a smooth submersion. ■

Proposition 2.11

$T_{\bar{e}}(G/H) \cong T_e G / T_e H.$

Proof. Since $\pi : G \to G/H$ is a smooth submersion, $(d\pi)_e : T_e G \to T_e(G/H)$ is surjective. Hence,

$$T_e(G/H) \cong \frac{T_e G}{\operatorname{Ker}\,(d\pi)_e}. \tag{31}$$

Now, consider the maps

$$S_0 \times H_0 \xrightarrow{\psi} U_0 \xrightarrow{\pi} \pi(U_0) \xrightarrow{f} S_0 \tag{32}$$

At the level of tangent spaces,

$$T_e S_0 \times T_e H_0 \xrightarrow[\cong]{(d\psi)_{e,e}} T_e U_0 \xrightarrow{(d\pi)_e} T_{\overline{e}}(\pi(U_0)) \xrightarrow[\cong]{(df)_{\overline{e}}} T_e S_0 \tag{33}$$

Since $S_0 \times H_0 \xrightarrow{f\circ\pi\circ\psi} S_0$ is the inclusion, so is $(df)_{\overline{e}} \circ (d\pi)_e \circ (d\psi)_{e,e}$ at the level of tangent spaces. Hence, it's kernel is

$$\operatorname{Ker}\left[(df)_{\overline{e}} \circ (d\pi)_e \circ (d\psi)_{e,e}\right] = \{\mathbf{0}\} \times T_e H_0. \tag{34}$$

It gets mapped to $T_e H_0 \subseteq T_e U_0$ under $(d\psi)_{e,e}$. Therefore,

$$\operatorname{Ker}(d\pi)_e = T_e H_0 \subseteq T_e U_0. \tag{35}$$

Hence,

$$T_e(G/H) \cong \frac{T_e G}{\operatorname{Ker}(d\pi)_e} = \frac{T_e G}{T_e H_0} = \frac{T_e G}{T_e H}. \tag{36}$$

■

§3 Fiber bundles and lifting

We have shown earlier that $\pi : G \to G/H$ is a smooth surjective submersion. Furthermore, it has an extra structure of a fiber bundle, i.e. locally G looks like $G/H \times H$. Let's first recall the definition of a fiber bundle.

> **Definition 3.1.** Let M and F be smooth manifolds. A **smooth fiber bundle over** M **with model fiber** F is a manifold E together with a smooth surjection $\pi : E \to M$ with the property that for each $x \in M$, there exists a neighborhood U of x in M and a diffeomorphism
>
> $$\Phi : \pi^{-1}(U) \to U \times F,$$
>
> called a **local trivialization of** E **over** U, such that the following diagram commutes:
>
> $$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\Phi"] \arrow[d, "\pi"'] & U \times F \arrow[dl, "\pi_1"] \\ U & \end{tikzcd} \tag{37}$$
>
> The space E is called the **total space** of the bundle, M is its **base**, and π is its **projection**.

Proposition 3.1

$\pi : G \to G/H$ is a fiber bundle with G being the total space, G/H being the base, and H being the model fiber.

Proof. Essentially, we have to show that given any $\overline{g} \in G/H$, there is a neighborhood U of $\overline{g}$ in G/H and a diffeomorphism $\Phi : \pi^{-1}(U) \to U \times H$, such that the following diagram commutes:

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[r, "\Phi"] \arrow[d, "\pi"'] & U \times H \arrow[dl, "\pi_1"] \\
U &
\end{tikzcd} \tag{38}$$

We are gonna choose $U = \pi\left(U_g\right)$. So we need to show that

$$\Phi : \pi^{-1}\left(\pi\left(U_g\right)\right) \to \pi\left(U_g\right) \times H$$

is a diffeomorphism, and the following diagram commutes:

$$\begin{tikzcd}
\pi^{-1}\left(\pi\left(U_g\right)\right) \arrow[r, "\Phi"] \arrow[d, "\pi"'] & \pi\left(U_g\right) \times H \arrow[dl, "\pi_1"] \\
\pi\left(U_g\right) &
\end{tikzcd} \tag{39}$$

Note that $\pi^{-1}\left(\pi\left(U_g\right)\right) = U_g H = gS_0H_0H = gS_0H$. An arbitrary element from $\pi^{-1}\left(\pi\left(U_g\right)\right)$ will then look like gs_0h for some $s_0 \in S, h \in H$. We then define

$$\Phi\left(gs_0h\right) = \left(\left[gs_0\right], h\right). \tag{40}$$

Note that this decomposition of an element of $\pi^{-1}\left(\pi\left(U_g\right)\right)$ into gs_0h is unique. Indeed, if $s_0h = s_0'h'$, then $[s_0] = [s_0']$. But since each coset intersects S_0 at most once, we must have $s_0 = s_0'$. Hence, $h = h'$ as well. Note that, Φ can be expressed as the composition

$$\pi^{-1}\left(\pi\left(U_g\right)\right) = gS_0H \xrightarrow{\sigma} gS_0 \times H \xrightarrow{f_g^{-1} \times \mathrm{id}_H} \pi(U_g) \times H \tag{41}$$

Note that σ is locally a diffeomorphism, i.e. $U_g \to gS_0 \times H_0$ is a diffeomorphism. Furthermore, it is "globally invertible". Therefore, it's a diffeomorphism. Hence, Φ is a diffeomorphism, as a composition of two diffeomorphisms.

Now we are only left to show the commutativity of (39), which is clearly true because

$$\pi_1 \circ \Phi\left(gs_0h\right) = \pi_1\left(\left[gs_0\right], h\right) = \left[gs_0\right] = \pi\left(gs_0h\right). \tag{42}$$

Therefore, $\pi = \pi_1 \circ \Phi$, and hence $\pi : G \to G/H$ is a fiber bundle. ∎

It's important to note some similarities. A fiber bundle is almost like a covering map from algebraic topology.

Definition 3.2 (Covering map). A surjective continuous map $p : E \to B$ is called a covering map if every $b \in B$ has a neighborhood U such that $p^{-1}(U)$ is a disjoint union of V_α's, i.e. $p^{-1}(U) = \bigsqcup_{\alpha \in J} V_\alpha$ such that

$$p|_{V_\alpha} : V_\alpha \to U$$

is a homeomorphism for each α.

Note that this essentially means that $p^{-1}(U)$ is homeomorphic to $U \times J$, for some discrete space J. Therefore, a covering map is a fiber bundle with a discrete fiber. This allows **unique** path lifting and **unique** path-homotopy lifting (See [1], §54). Similar properties also hold for fiber bundles as well. However, we lose uniqueness.

Theorem 3.2 (Path lifting of fiber bundle)

Let $\pi : E \to M$ be a fiber bundle, and $\gamma : [0,1] \to M$ is a path. Take $e_0 \in E$ such that $\pi(e_0) = \gamma(0)$. Then there exists a path (not necessarily unique) $\widetilde{\gamma} : [0,1] \to E$ such that $\widetilde{\gamma}(0) = e_0$, and $\gamma = \pi \circ \widetilde{\gamma}$, i.e. the following diagram commutes:

$$\begin{tikzcd} & {[0,1]} \arrow[dl, dashed, "\widetilde{\gamma}"'] \arrow[d, "\gamma"] \\ E \arrow[r, "\pi"'] & M \end{tikzcd} \tag{43}$$

Theorem 3.3 (Path-homotopy lifting of fiber bundle)

Let $\pi : E \to M$ be a fiber bundle. Let $H : [0,1]^2 \to M$ be a continuous homotopy of paths in M. Suppose there exists a continuous lift $\tilde{\alpha}_0 : [0,1] \to E$ of the initial path $\alpha_0(t) = H(0,t)$ such that $\pi \circ \tilde{\alpha}_0 = \alpha_0$. Then there exists a continuous map (not necessarily unique) $\tilde{H} : [0,1] \times [0,1] \to E$ such that

$$\pi \circ \tilde{H} = H, \qquad \tilde{H}(0,t) = \tilde{\alpha}_0(t) \quad \text{for all } t \in [0,1].$$

In particular, for each $s \in [0,1]$, the map $\tilde{\alpha}_s(t) := \tilde{H}(s,t)$ is a lift of the path $\alpha_s(t) := H(s,t)$.

We will not give the proofs here. An interested reader may follow the proof almost verbatim from [1], §54. Although those proofs are given for covering maps, they hold for fiber bundles as well. We conclude this section with the following result:

Theorem 3.4

Let G be a Lie group, and H its Lie subgroup. Suppose G, H are connected. Then we have the following exact sequence:

$$\pi_1(H) \xrightarrow{\iota_*} \pi_1(G) \xrightarrow{\pi_*} \pi_1(G/H) \longrightarrow 1 \tag{44}$$

where $\iota : H \to G$ is the inclusion map and $\pi : G \to G/H$ is the projection.

Proof. Note that G/H is also connected, since it's the continuous image of a connected space. So we can take the basepoints to be identities. We have to show that π_* is surjective and $\operatorname{Ker} \pi_* = \operatorname{im} \iota_*$.

Note that $\pi \circ \iota$ maps everything to the identity of G/H. Hence, $\pi_* \circ \iota_*$ maps everything to the identity of $\pi_1 (G/H)$. Therefore, $\operatorname{im} \iota_* \subseteq \operatorname{Ker} \pi_*$.

Now, suppose $[\gamma] \in \operatorname{Ker} \pi_*$. Then $\pi \circ \gamma$ is null-homotopic. So there is a homotopy

$$F : [0, 1]^2 \to G/H$$

such that $F(-, 0) = \pi \circ \gamma$ and $F(-, 1) = \overline{e}$. Since $\pi : G \to G/H$ is a fiber bundle, by Theorem 3.3, there is a homotopy lifting

$$\widetilde{F} : [0, 1]^2 \to G$$

satisfying $\widetilde{F}(0, -) = \gamma$ and $\alpha := \widetilde{F}(1, -)$ is contained in H. Furthermore, each $\widetilde{F}(s, -)$ is a lift of $F(s, -)$. Now, $\alpha(0) = \widetilde{F}(1, 0) = \widetilde{F}(1, 1) = \alpha(1) = e$ since $\widetilde{F}$ is a path homotopy. Therefore, γ is path-homotopic to α, which is, in fact, a loop in H. In other words,

$$[\gamma] = [\alpha] = [\iota \circ \alpha] = \iota_* [\alpha], \tag{45}$$

proving that $\operatorname{Ker} \pi_* \subseteq \operatorname{im} \iota_*$. Therefore, $\operatorname{Ker} \pi_* = \operatorname{im} \iota_*$.

Now, we need to show that π_* is surjective. Let $\alpha : [0, 1] \to G/H$ be a loop based at $\overline{e}$. Then α has a lift $\widetilde{\alpha} : [0, 1] \to G$ such that $\pi \circ \widetilde{\alpha} = \alpha$, and $\widetilde{\alpha}(0) = e$ and $\widetilde{\alpha}(1) \in H$. Take any path β from $\widetilde{\alpha}(1)$ to e in H (this is possible since H is connected, also connected and path-connected are equivalent for manifolds).

Then $\widetilde{\alpha} * \beta$ is a loop based at e in G. Furthermore,

$$\begin{aligned} \pi_* [\widetilde{\alpha} * \beta] &= [\pi \circ (\widetilde{\alpha} * \beta)] \\ &= [\pi \circ \widetilde{\alpha} * \pi \circ \beta] \\ &= [\pi \circ \widetilde{\alpha}] * [\pi \circ \beta] \\ &= [\alpha] * [\mathrm{const}_{\overline{e}}] = [\alpha]. \end{aligned} \tag{46}$$

Therefore, π_* is surjective, and hence (44) is exact, ■

§4 Normal subgroup H

In case H is normal, G/H can be given a group structure. We have given G/H a manifold structure in the previous section. We will now see that G/H is a Lie group. For that purpose, we need to show that the multuplication map, and the inverse map

$$\widetilde{m} : G/H \times G/H \to G/H \text{ and } \widetilde{i} : G/H \to G/H$$

are both smooth.

> **Lemma 4.1**
>
> $\widetilde{m}$ is smooth.

Proof. Since $\pi : G \to G/H$ is a smooth submersion, so is $\pi \times \pi : G \times G \to G/H \times G/H$. Now, consider the following diagram:

$$\begin{tikzcd} G \times G \arrow[r, "\pi\times\pi"] \arrow[dr, "\pi\circ m"'] & G/H \times G/H \arrow[d, "\widetilde{m}"] \\ & G/H \end{tikzcd} \tag{47}$$

This diagram commutes, because given $g_1, g_2 \in G$,

$$\begin{aligned} \widetilde{m} \circ (\pi \times \pi)(g_1, g_2) &= \widetilde{m}([g_1], [g_2]) = [g_1][g_2] = [g_1 g_2] \\ &= \pi(g_1 g_2) = \pi \circ m(g_1, g_2). \end{aligned} \tag{48}$$

Furthermore, $\pi \circ m$ is smooth as a composition of two smooth maps. Hence, by Proposition 1.2, $\widetilde{m}$ is smooth. ■

> **Lemma 4.2**
>
> $\widetilde{i}$ is smooth.

Proof. $\pi : G \to G/H$ is a smooth submersion. Now, consider the following diagram:

$$\begin{tikzcd} G \arrow[r, "\pi"] \arrow[dr, "\pi\circ i"'] & G/H \arrow[d, "\widetilde{i}"] \\ & G/H \end{tikzcd} \tag{49}$$

This diagram commutes, because given $g \in G$,

$$\begin{aligned}\widetilde{i} \circ \pi (g) = \widetilde{i} ([g]) = [g]^{-1} = \left[g^{-1}\right] \\ = \pi \left(g^{-1}\right) = \pi \circ i (g) .\end{aligned} \tag{50}$$

Furthermore, $\pi \circ i$ is smooth as a composition of two smooth maps. Hence, by Proposition 1.2, $\widetilde{i}$ is smooth. ■

Combining everything, we have the following result:

> **Theorem 4.3**
>
> If G is a Lie group and H is a normal Lie subgroup, then G/H has the structure of a Lie group.

Now we would've loved to prove the first isomorphism theorem for Lie groups.

> **Theorem 4.4** (First isomorphism theorem for Lie groups)
>
> Suppose $f : G \to H$ is a Lie group homomomorphism. Then $\operatorname{Ker} f$ is a closed normal Lie subgroup of G. Furthermore, f gives rise to an injective Lie group homomorphism
>
> $$\widetilde{f} : G/\operatorname{Ker} f \to H.$$
>
> $\operatorname{im} f$ has a unique smooth manifold structure such that it's an immersed submanifold of H, and $\widetilde{f} : G/\operatorname{Ker} f \to \operatorname{im} f$ is a Lie group isomorphism.

To prove this, we will require the following result, which is also known as Cartan's closed subgroup theorem:

> Let H be a closed subgroup of a Lie group G. Then H is a Lie subgroup.

The proof of Cartan's closed subgroup theorem requires some Lie algebra machinaries, so we will skip it for now. Assuming this result, let's present a proof for First isomorphism theorem for Lie groups.

Proof of First isomorphism theorem for Lie groups. $\operatorname{Ker} f = f^{-1}(\{e\})$. Since H is a manifold, it's Hausdorff, and hence, singletons are closed. Therefore, $\operatorname{Ker} f$ is a closed subgroup of G. By Cartan's closed subgroup theorem, $\operatorname{Ker} f$ is a Lie subgroup, which is also normal. Therefore, $G/\operatorname{Ker} f$ is a Lie group, and the map $\pi : G \to G/\operatorname{Ker} f$ is a smooth submersion.

$$\begin{tikzcd} G \arrow[r, "\pi"] \arrow[dr, "f"'] & G/\operatorname{Ker} f \arrow[d, "\widetilde{f}"] \\ & H \end{tikzcd} \tag{51}$$

Since f is smooth, so is $\widetilde{f}$, by Proposition 1.2. $\widetilde{f}$ being injective is also well known.

Furthermore, since $\widetilde{f} : G/\operatorname{Ker} f \to H$ is injective, $\operatorname{im} f = \operatorname{im} \widetilde{f}$ has a unique smooth manifold structure such that it's an immersed submanifold of H, and $\widetilde{f}$ is a diffeomorphism onto its image. Therefore, $\widetilde{f} : G/\operatorname{Ker} f \to \operatorname{im} f$ is a group isomorphism, which is also a diffeomorphism, i.e. a Lie group isomorphism. ■

Note that here we didn't use the full generality of Cartan's closed subgroup theorem. We used a slightly weaker version:

> If H is a closed subgroup of a Lie group G, then there is a unique manifold structure on G/H such that $\pi : G \to G/H$ is a submersion.

This weaker version can be proved after developing some tools about orbits and stabilizers.

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