

Lie Algebra Associated with a Lie Group

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Given a Lie group G , its tangent space at the identity, $T_e G$, is isomorphic to the vector space $\mathfrak{L}(G)$ of all left-invariant vector fields on G . $\mathfrak{L}(G)$ has the structure of a Lie algebra under the operation $[X, Y] = XY - YX$. In fact, it's a Lie subalgebra of the space of all vector fields $\mathfrak{X}(G)$. We then establish a correspondence between the Lie algebra and the Lie group, namely the exponential map.

§1 Lie algebra

Definition 1.1 (Lie algebra). A **Lie algebra** over a field \mathbb{K} is a \mathbb{K} -vector space V equipped with a bilinear operation $[-, -] : V \times V \rightarrow V$ (called the **Lie bracket**) such that

- (a) $[-, -]$ is antisymmetric, i.e. $[v_1, v_2] = -[v_2, v_1]$ for all $v_1, v_2 \in V$; and
- (b) for any $u, v, w \in V$,

$$[[u, v], w] + [[w, u], v] + [[v, w], u] = 0. \quad (1)$$

(1) is also known as the Jacobi identity.

Example 1.1. If V is an associative \mathbb{K} -algebra with respect to the product \cdot , then we can define the Lie bracket as

$$[v_1, v_2] = v_1 \cdot v_2 - v_2 \cdot v_1. \quad (2)$$

Then V is a Lie algebra with respect to $[-, -]$.

Example 1.2. Let M be a manifold, and $\mathfrak{X}(M)$ is the vector space of all C^∞ vector fields on M . If $X \in \mathfrak{X}(M)$, then X assigns to each point $p \in M$ a tangent vector $X_p \in T_p M$, and this assignment is smooth. Alternatively, $X : C^\infty(M) \rightarrow C^\infty(M)$ is a linear map such that

$$X(fg) = fX(g) + X(f)g. \quad (3)$$

(3) is the Leibniz rule. If $X, Y \in \mathfrak{X}(M)$, then $X \circ Y$ is not, in general, a vector field. However,

$X \circ Y - Y \circ X$ is a vector field, since it obeys the Leibniz rule.

$$\begin{aligned}
(X \circ Y - Y \circ X)(fg) &= XY(fg) - YX(fg) \\
&= X(Y(f)g + fY(g)) - Y(X(f)g + fX(g)) \\
&= XY(f)g + Y(f)X(g) + X(f)Y(g) + fXY(g) \\
&\quad - YX(f)g - X(f)Y(g) - Y(f)X(g) - fYX(g) \\
&= (X \circ Y - Y \circ X)(f)g + f(X \circ Y - Y \circ X)(g).
\end{aligned}$$

Now we define the Lie bracket on $\mathfrak{X}(M)$ as

$$[X, Y] = X \circ Y - Y \circ X, \tag{4}$$

and then $\mathfrak{X}(M)$ becomes a Lie algebra with respect to $[-, -]$.

Note that $\mathfrak{X}(M)$ is, in general, an infinite dimensional Lie algebra. For example, if we take $M = \mathbb{R}$, then

$$\frac{d}{dx}, x \frac{d}{dx}, x^2 \frac{d}{dx}, x^3 \frac{d}{dx}, \dots$$

are all linearly independent vector fields on \mathbb{R} . However, if our manifold is a Lie group, we can actually get a finite dimensional Lie subalgebra of $\mathfrak{X}(M)$.

§2 Invariant vector fields

Let $f : M \rightarrow N$ be a smooth map of manifolds. If X is a vector field on M , we can't always push it forward to a vector field f_*X on N . Sure, we can try to define

$$(f_*X)_{f(p)} = (df)_p X_p \in T_{f(p)}N. \tag{5}$$

But the issue with this definition is that not all the points in N can be written as $f(p)$ if f is not surjective. Furthermore, even if f is surjective, if we take $q \in N$, there can be multiple choices for p , i.e. we may have $q = f(p_1) = f(p_2)$ if f is not injective. But we can pushforward a vector field when f is a diffeomorphism.

Definition 2.1 (Pushforward of vector field). Let $f : M \rightarrow N$ be a diffeomorphism. If $X \in \mathfrak{X}(M)$, the pushforward vector field $f_*X \in \mathfrak{X}(N)$ is defined as

$$(f_*X)_{f(p)} = (df)_p X_p \in T_{f(p)}N. \tag{6}$$

Alternatively, f_*X acting on a smooth function h on N is defined as

$$(f_*X)(h)(q) = X_{f^{-1}(q)}(h \circ f), \text{ or alternatively, } (f_*X)(h)(f(p)) = X_p(h \circ f). \tag{7}$$

Definition 2.2 (Invariant vector field). Let $f : M \rightarrow M$ be a diffeomorphism. A vector field $X \in \mathfrak{X}(M)$ is called **f -invariant** if $f_*X = X$. Let $\mathfrak{X}_f(M)$ denote the space of all f -invariant vector fields on M .

Note that

$$\begin{aligned}
f_*X = X &\iff (f_*X)_{f(p)} = X_{f(p)} \text{ for all } p \in M \\
&\iff (f_*X)_{f(p)} h = X_{f(p)} h \text{ for all } p \in M \text{ and } h \in C^\infty(M) \\
&\iff X_p(h \circ f) = (Xh)(f(p)) \text{ for all } p \in M \text{ and } h \in C^\infty(M) \\
&\iff (X(h \circ f))(p) = (Xh) \circ f(p) \text{ for all } p \in M \text{ and } h \in C^\infty(M) \\
&\iff X(h \circ f) = (Xh) \circ f \text{ for all } h \in C^\infty(M).
\end{aligned} \tag{8}$$

In other words, in the expression $X(h \circ f)$, we can take the composition with f out of the parenthesis. We shall often use (8) as the definition of an invariant vector field.

Proposition 2.1

If $X, Y \in \mathfrak{X}(M)$ are f -invariant, so is $[X, Y]$.

Proof. Take $h \in C^\infty(M)$.

$$\begin{aligned}
[X, Y](h \circ f) &= XY(h \circ f) - YX(h \circ f) \\
&= X((Yh) \circ f) - Y((Xh) \circ f) \\
&= X((Yh)) \circ f - Y((Xh)) \circ f \\
&= ((XY - YX)h) \circ f \\
&= ([X, Y]h) \circ f.
\end{aligned} \tag{9}$$

Hence, $[X, Y]$ is also f -invariant. ■

Therefore, $\mathfrak{X}_f(M)$ is a Lie subalgebra of $\mathfrak{X}(M)$. Now we come back to the context of Lie group, i.e. the manifold $M = G$ is a Lie group. Let $L_g : G \rightarrow G$ denote the “left multiplication by g ” map, i.e.

$$L_g(h) = gh. \tag{10}$$

We know L_g is a diffeomorphism.

Definition 2.3 (Left-invariant vector field). A vector field $X \in \mathfrak{X}(G)$ is called **left-invariant** if it is L_g -invariant for every $g \in G$. Then $\mathfrak{L}(G)$ denotes the space of all left-invariant vector fields.

In other words, $X \in \mathfrak{X}(G)$ is left-invariant if $(dL_g)_{g'} X_{g'} = X_{gg'}$ for every $g, g' \in G$. Alternatively, if $X(h \circ L_g) = (Xh) \circ L_g$ for every $h \in C^\infty(G)$ and $g \in G$.

By Proposition 2.1, $\mathfrak{L}(G)$ is a Lie subalgebra of $\mathfrak{X}(G)$. We shall now prove that a left-invariant vector field is completely determined by its value at the identity.

Theorem 2.2

The map $\Phi : \mathfrak{L}(G) \rightarrow T_e G$ given by $X \mapsto X_e$ is an isomorphism of vector spaces.

Proof. We define $\Psi : T_e G \rightarrow \mathfrak{L}(G)$ as follows: given $X_e \in T_e(G)$,

$$\Psi(X_e)_g = (dL_g)_e X_e \in T_g G. \quad (11)$$

Let's first verify that $X := \Psi(X_e)$ is left-invariant.

$$\begin{aligned} (dL_g)_{g'} X_{g'} &= (dL_g)_{g'} (dL_{g'})_e X_e \\ &= (d(L_g \circ L_{g'}))_e X_e \\ &= (dL_{gg'})_e X_e \\ &= X_{gg'}. \end{aligned} \quad (12)$$

Therefore, Ψ is indeed a linear map from $T_e G$ to $\mathfrak{L}(G)$.

$$\begin{array}{ccccc} T_e G & \xrightarrow{\Psi} & \mathfrak{L}(G) & \xrightarrow{\Phi} & T_e(G) \\ X_e & \longmapsto & \Psi(X_e) & \longmapsto & \Psi(X_e)_e = (dL_e)_e X_e = X_e, \end{array} \quad (13)$$

since L_e is the identity map. Therefore, $\Phi \circ \Psi = \text{id}_{T_e G}$. On the other hand, if $X \in \mathfrak{L}(G)$, then $(dL_g)_{g'} X_{g'} = X_{gg'}$ for every $g, g' \in G$. Taking $g' = e$, we have $(dL_g)_e X_e = X_g$.

$$\begin{array}{ccccc} \mathfrak{L}(G) & \xrightarrow{\Phi} & T_e(G) & \xrightarrow{\Psi} & \mathfrak{L}(G) \\ X & \longmapsto & X_e & \longmapsto & \Psi(X_e) \end{array} \quad (14)$$

Now,

$$(\Psi(X_e))_g = (dL_g)_e X_e = X_g. \quad (15)$$

Therefore, $\Psi \circ \Phi = \text{id}_{\mathfrak{L}(G)}$. Hence, Φ is a vector space isomorphism. ■

Since $T_e G$ is a finite-dimensional vector space, so is $\mathfrak{L}(G)$. Since $\mathfrak{L}(G)$ and $T_e G$ are isomorphic as vector spaces, the Lie algebra structure on $\mathfrak{L}(G)$ can be translated to $T_e G$, by defining the Lie bracket $[-, -]_{T_e G}$ on $T_e G$

$$[X_e, Y_e]_{T_e G} = \Phi([\Psi(X_e), \Psi(Y_e)]). \quad (16)$$

§3 Lie algebra of a Lie group

Definition 3.1 (Lie algebra associated with a Lie group). Let G be a Lie group. Then $\mathfrak{g} := \mathfrak{L}(G)$ is called the **Lie algebra associated with the Lie group G** .

Note that in general, Lie algebras don't come with a Lie group. However, every Lie group G gives rise to a finite-dimensional Lie algebra \mathfrak{g} . It turns out that every finite dimensional Lie algebra is isomorphic to the Lie algebra of a Lie group! This is known as Lie's third theorem.

Now, suppose $f : G \rightarrow H$ is a Lie group homomorphism. Then at the tangent space level, we have the linear map $(df)_e : T_e G \rightarrow T_{e'} H$. Under the identification $T_e G \cong \mathfrak{g} = \mathfrak{L}(G)$, we get an induced map $\mathfrak{g} \rightarrow \mathfrak{h}$.

$$\begin{array}{ccc} T_e G & \xrightarrow{(df)_e} & T_{e'} H \\ \Phi_G \uparrow & & \downarrow \Psi_H \\ \mathfrak{g} & \xrightarrow{\Psi_H \circ (df)_e \circ \Phi_G} & \mathfrak{h} \end{array} \quad (17)$$

Theorem 3.1

$\widehat{f} := \Psi_H \circ (df)_e \circ \Phi_G : \mathfrak{g} \rightarrow \mathfrak{h}$ is a Lie algebra homomorphism.

Proof. We have to verify that \widehat{f} preserves Lie bracket. In other words, we need to show that $\widehat{f}([X, Y]) = [\widehat{f}(X), \widehat{f}(Y)]$.

$$\begin{aligned} \widehat{f}([X, Y]) &= [\widehat{f}(X), \widehat{f}(Y)] \\ \iff (df)_e \circ \Phi_G([X, Y]) &= \Phi_H [\widehat{f}(X), \widehat{f}(Y)] \\ \iff (df)_e [X, Y]_e &= [\widehat{f}(X), \widehat{f}(Y)]_{e'} \\ \iff (df)_e [X, Y]_e(h) &= [\widehat{f}(X), \widehat{f}(Y)]_{e'}(h) \text{ for any } h \in C^\infty(H) \\ \iff X_e(Y(h \circ f)) - Y_e(X(h \circ f)) &= [\widehat{f}(X), \widehat{f}(Y)]_{e'}(h) \text{ for any } h \in C^\infty(H). \end{aligned} \quad (18)$$

Now,

$$\begin{aligned} [\widehat{f}(X), \widehat{f}(Y)]_{e'}(h) &= \widehat{f}(X)_e(\widehat{f}(Y)h) - \widehat{f}(Y)_e(\widehat{f}(X)h) \\ &= (df)_e X_e(\widehat{f}(Y)h) - (df)_e Y_e(\widehat{f}(X)h) \\ &= X_e(\widehat{f}(Y)h \circ f) - Y_e(\widehat{f}(X)h \circ f). \end{aligned} \quad (19)$$

Therefore, it suffices to verify that $Y(h \circ f) = \widehat{f}(Y)h \circ f$ as functions on G . If we manage to

prove it, $X(h \circ f) = \widehat{f}(X)h \circ f$ follows similarly. Now, let's evaluate both sides at $g \in G$.

$$\begin{aligned} \text{LHS} &= Y(h \circ f)(g) = Y_g(h \circ f) \\ &= (dL_g)_e Y_e(h \circ f) = Y_e(h \circ f \circ L_g) \end{aligned} \tag{20}$$

$$\begin{aligned} \text{RHS} &= [\widehat{f}(Y)h \circ f](g) = [\widehat{f}(Y)h](f(g)) \\ &= (\widehat{f}(Y))_{f(g)} h = (dL_{f(g)})_{e'} (df)_e Y_e(h) \\ &= (d(L_{f(g)} \circ f))_e Y_e(h) \\ &= Y_e(h \circ L_{f(g)} \circ f). \end{aligned} \tag{21}$$

Now, $f \circ L_g = L_{f(g)} \circ f$, since f is a group homomorphism. Therefore, the LHS and RHS are equal, and hence we are done! \blacksquare

One can easily verify that if $f_1 : G \rightarrow H, f_2 : H \rightarrow K$ are Lie group homomorphisms, then

$$\widehat{f_2 \circ f_1} = \widehat{f}_2 \circ \widehat{f}_1, \tag{22}$$

$$G \xrightarrow{f_1} H \xrightarrow{f_2} K \quad \mathfrak{g} \xrightarrow{\widehat{f}_1} \mathfrak{h} \xrightarrow{\widehat{f}_2} \mathfrak{k} \tag{23}$$

$\widehat{f_2 \circ f_1} = \widehat{f}_2 \circ \widehat{f}_1$

Furthermore, $\widehat{1}_G = 1_{\mathfrak{g}}$. Therefore, this defines a functor from the category of Lie groups to the category of Lie algebras:

$$\begin{aligned} \text{Lie} : \mathbf{LieGrp} &\rightarrow \mathbf{LieAlg} \\ G &\mapsto \mathfrak{g} \\ (f : G \rightarrow H) &\mapsto (\widehat{f} : \mathfrak{g} \rightarrow \mathfrak{h}) \end{aligned} \tag{24}$$

§4 Exponential map

Tangent spaces give us a local linearization of the manifold. This often helps us translate tangent space properties to local properties of manifolds, e.g. inverse function theorem, constant rank theorem, immersion/submersion theorems etc. This “tangent space to manifold” correspondence is formally defined using something called exponential map. In Riemannian geometry, we define the exponential map as follows:

$\exp_p : T_p M \rightarrow M$ is defined in the following way: given $X_p \in T_p M$, let γ be the **unique** geodesic such that $\gamma(0) = p$ and $\gamma'(0) = X_p$. Then we define $\exp_p(X_p) = \gamma(1)$.

(Well, technically \exp_p is defined on an open subset of $T_p M$, which might not be the whole M .) Here, we need the notion of an affine connection for existence and uniqueness of geodesics. Furthermore, one needs to verify that the domain of the geodesic γ contains 1.

Note that we need to impose some further restriction on γ . Because the curve satisfying $\gamma(0) = p$ and $\gamma'(0) = X_p$ only is not unique. The uniqueness is guaranteed by requiring the curve to be a geodesic.

Now suppose $M = G$ is a Lie group. Suppose we want to define $\exp : T_e G \rightarrow G$. Given $X_e \in T_e G$, our idea is to take a “unique” curve γ satisfying $\gamma(0) = p$ and $\gamma'(0) = X_p$ along with some other condition. Then we’re gonna define $\exp(X_e) = \gamma(1)$.

By Theorem 2.2, every X_e uniquely extends to a left-invariant vector field X . Then we can take γ to be the integral curve of the vector field X , satisfying $\gamma(0) = p$ and $\gamma'(0) = X_p$. By the existence and uniqueness theorem of ODEs, such a γ exists in a neighborhood of 0, and it’s unique.

Now, we just know that γ is defined on an interval $(-\varepsilon, \varepsilon)$ containing 0. We don’t know how long the interval is, or if it even contains 1. It turns out that the maximal domain of γ is the whole \mathbb{R} .

Proposition 4.1

Let X be a left-invariant vector field on G , and γ the maximal integral curve of the vector field X starting at e . Then the domain of γ is \mathbb{R} .

Proof. Let α be any integral curve of the vector field X (not necessarily starting at e). Then given $g \in G$, we take the curve

$$\alpha_1(t) = g \cdot \alpha(t) = (L_g \circ \alpha)(t). \quad (25)$$

Then we have

$$\begin{aligned} \alpha'_1(t) &= (d\alpha_1)_t \left. \frac{d}{dx} \right|_t \\ &= (dL_g)_{\alpha(t)} (d\alpha)_t \left. \frac{d}{dx} \right|_t \\ &= (dL_g)_{\alpha(t)} X_{\alpha(t)} = X_{g\alpha(t)} \\ &= X_{\alpha_1(t)}. \end{aligned} \quad (26)$$

Therefore, α_1 is also an integral curve. Now let I be the domain of the maximal integral curve γ starting at e . We need to show that $I = \mathbb{R}$.

Fix $t_1 \in I$ with $t_1 \neq 0$, and take $g_1 = \gamma(t_1)$. Now, the curve

$$\alpha_1(t) = g_1 \cdot \gamma(t) \quad (27)$$

is an integral curve for X (as above), starting at g_1 . It has domain I . On the other hand, the

maximal integral curve of X starting at g_1 is given by

$$\alpha_2(t) = \gamma(t_1 + t). \quad (28)$$

This is maximal because otherwise we'd be able to extend γ to a larger interval. Now, the domain of α_2 is $I - t_1 := (a - t_1, b - t_1)$ if $I = (a, b)$. Since α_2 is maximal, it contains the domain of α_1 , i.e.

$$(a, b) \subseteq (a - t_1, b - t_1), \quad \text{or equivalently } (a + t_1, b + t_1) \subseteq (a, b). \quad (29)$$

In particular, given any $s \in I$, $s + t_1 \in (a + t_1, b + t_1) \subseteq (a, b)$, i.e. $s + t_1 \in I$. This forces I to be equal to \mathbb{R} . ■

Corollary 4.2

$\alpha(t) = g \cdot \gamma(t)$ is the integral curve for X starting at g .

Proof. We have already shown it in the proof for Proposition 4.1. ■

Proposition 4.3

$\gamma : \mathbb{R} \rightarrow G$ is a group homomorphism.

Proof. Fix $s \in \mathbb{R}$. Consider the following curves:

$$\alpha_1(t) = \gamma(s)\gamma(t), \quad (30)$$

$$\alpha_2(t) = \gamma(s + t). \quad (31)$$

Both of them are integral curves of the vector field X , and both starting at $\gamma(s)$. Furthermore, both are defined on the whole \mathbb{R} . Therefore, by the uniqueness of the maximal integral curve, we have $\alpha_1 = \alpha_2$, i.e.

$$\gamma(s)\gamma(t) = \gamma(s + t). \quad (32)$$

Therefore, $\gamma : \mathbb{R} \rightarrow G$ is a group homomorphism. ■

Now we can define the exponential map.

Definition 4.1 (Exponential map). Let G be a Lie group. The **exponential map**, $\exp : T_e G \rightarrow G$ is defined by

$$\exp(X_e) = \gamma(1), \quad (33)$$

where $\gamma : \mathbb{R} \rightarrow G$ is the maximal integral curve of the left-invariant vector field X , starting at e . $\gamma : \mathbb{R} \rightarrow G$ is also called the **one-parameter subgroup** corresponding to $X_e \in T_e G$.

Lemma 4.4

Under the above notation, $\gamma(s) = \exp(sX_e)$ for any $s \in \mathbb{R}$.

Proof. Fix $s \in \mathbb{R}$. We then define the curve $\alpha : \mathbb{R} \rightarrow G$ as follows:

$$\alpha(t) = \gamma(st). \quad (34)$$

Then $\alpha(0) = 0$, and

$$\begin{aligned} \alpha'(t) &= (d\alpha)_t \frac{d}{dx}\Big|_t \\ &= (d\gamma)_{st} s \frac{d}{dx}\Big|_t \\ &= sX_{\gamma(st)} = sX_{\alpha(t)}. \end{aligned} \quad (35)$$

Therefore, α is the integral curve of the left-invariant vector field sX , starting at e . Therefore,

$$\exp(sX_e) = \alpha(1) = \gamma(s). \quad (36)$$

■

Lemma 4.5

For $s, t \in \mathbb{R}$, $\exp((s+t)X_e) = \exp(sX_e)\exp(tX_e)$.

Proof. Let $\gamma : \mathbb{R} \rightarrow G$ be the one-parameter subgroup corresponding to $X_e \in T_e G$. Then

$$\exp((s+t)X_e) = \gamma(s+t) = \gamma(s)\gamma(t) = \exp(sX_e)\exp(tX_e). \quad (37)$$

■

Remark 4.1. In many literature, \exp is defined in a slightly different, albeit equivalent, way. Given $X_e \in T_e G$, they define γ to be the unique Lie group homomorphism $\gamma : \mathbb{R} \rightarrow G$ such that $\gamma'(0) = X_e$. We prove the existence and uniqueness of the Lie group homomorphism $\gamma : \mathbb{R} \rightarrow G$ in the following proposition.

Proposition 4.6

There is a unique Lie group homomorphism $\gamma : \mathbb{R} \rightarrow G$ such that $\gamma'(0) = X_e$.

Proof. Such a γ satisfies

$$\gamma(t+s) = \gamma(t)\gamma(s) = L_{\gamma(t)} \circ \gamma(s). \quad (38)$$

Differentiating with respect to s , we get

$$(d\gamma)_{t+s} = (dL_{\gamma(t)})_{\gamma(s)} (d\gamma)_s. \quad (39)$$

This gives us

$$\gamma' (t + s) = (dL_{\gamma(t)})_{\gamma(s)} \gamma'(s). \quad (40)$$

At $s = 0$,

$$\gamma' (t) = (dL_{\gamma(t)})_e \gamma'(0) = (dL_{\gamma(t)})_e X_e. \quad (41)$$

Therefore, γ is the integral curve to the left-invariant vector field X defined as $X_g = (dL_g)_e X_e$. Although we saw a proof of extendability to \mathbb{R} earlier, let's present a different proof now.

We know that γ exists and is unique on $(-\varepsilon, \varepsilon)$. Furthermore, if $|s| + |t| < \varepsilon$, both $\alpha_1(t) := \gamma(s + t)$ and $\alpha_2(t) := \gamma(s)\gamma(t)$ are integral curves to the vector field X with initial point $\gamma(s)$. Therefore,

$$\gamma(s + t) = \gamma(s)\gamma(t), \quad \text{for } |s| + |t| < \varepsilon. \quad (42)$$

Differentiating with respect to s , we get

$$(d\gamma)_{s+t} = (dR_{\gamma(t)})_{\gamma(s)} (d\gamma)_s \quad (43)$$

This gives us

$$\gamma' (t + s) = (dR_{\gamma(t)})_{\gamma(s)} \gamma'(s). \quad (44)$$

At $s = 0$,

$$\gamma' (t) = (dR_{\gamma(t)})_e \gamma'(0) = (dR_{\gamma(t)})_e X_e, \quad (45)$$

proving that $\gamma'(t) = (dR_{\gamma(t)})_e X_e = (dL_{\gamma(t)})_e X_e$. Now, one can extend γ to $(-2\varepsilon, 2\varepsilon)$ by

$$\gamma(2t) = \gamma(t)^2. \quad (46)$$

Then one can verify that this satisfies $\gamma'(2t) = (dL_{\gamma(2t)})_e X_e$. This can be easily done by an application of Leibniz-like formula (see here, [10] for details)

$$\begin{aligned} 2\gamma'(2t) &= (dL_{\gamma(t)})_{\gamma(t)} \gamma'(t) + (dR_{\gamma(t)})_{\gamma(t)} \gamma'(t) \\ &= (dL_{\gamma(t)})_{\gamma(t)} (dL_{\gamma(t)})_e X_e + (dR_{\gamma(t)})_{\gamma(t)} (dR_{\gamma(t)})_e X_e \\ &= (dL_{\gamma(t)^2})_e X_e + (dR_{\gamma(t)^2})_e X_e \\ &= 2(dL_{\gamma(t)^2})_e X_e \\ &= 2(dL_{\gamma(2t)})_e X_e. \end{aligned} \quad (47)$$

So one can extend γ on $(-2\varepsilon, 2\varepsilon)$. By induction, it can be extended to $(-2^n\varepsilon, 2^n\varepsilon)$, and hence on the whole \mathbb{R} . ■

§5 Flow

Definition 5.1 (The flow of a vector field). Let M be a smooth m -manifold and $X \in \mathfrak{X}(M)$ be a smooth vector field on M . For $p_0 \in M$, the **maximal existence interval** of p_0 is the maximal open interval $I(p_0)$ in which $\gamma : I(p_0) \rightarrow M$ is the integral curve of the vector field X , starting at p_0 . The **flow** of X is the map $\phi : \mathcal{D} \rightarrow M$ defined by

$$\mathcal{D} := \{(t, p_0) \mid p_0 \in M, t \in I(p_0)\}, \quad \text{and } \phi(t, p_0) := \gamma(t),$$

where $\gamma : I(p_0) \rightarrow M$ is the integral curve of the vector field X , starting at p_0 . If $\mathcal{D} = \mathbb{R} \times M$, the vector field X is called **complete**.

Theorem 5.1

\mathcal{D} is an open subset of $\mathbb{R} \times M$, and ϕ is smooth.

For a proof, interested readers can consult §2.4.2 from the preprint linked in [7]. We have seen in Proposition 4.1 and Corollary 4.2 that if X is a left-invariant vector field on a Lie group G , then its integral curve starting at g has domain \mathbb{R} . Therefore, X is complete.

Proposition 5.2

Let X be a left-invariant vector field on a Lie group G . Then X is complete, and its flow is given by

$$\begin{aligned} \phi : \mathbb{R} \times G &\rightarrow G \\ (t, g) &\mapsto g \exp(tX_e). \end{aligned} \tag{48}$$

Proof. By Corollary 4.2, the integral curve for X , starting at g is given by

$$\alpha(t) = g\gamma(t) = g \exp(tX_e), \tag{49}$$

where γ is the integral curve for X , starting at e . Hence, $\phi(t, g) = \alpha(t) = g \exp(tX_e)$. ■

Now, $T_e G$ is a finite-dimensional vector space. We can give it an inner product or a norm and that induces a topology on it. We know that on a finite dimensional vector space, **all** norms are equivalent, i.e. they all generate the same topology. Under this topology, it's homeomorphic to \mathbb{R}^n . Therefore, $T_e G$ canonically has a manifold structure, that doesn't depend on the choice of the norm.

Theorem 5.3

$\exp : T_e G \rightarrow G$ is smooth.

Proof. Consider the map

$$\begin{aligned}\Phi : \mathbb{R} \times G \times T_e G &\rightarrow G \times T_e G \\ (t, g, X_e) &\mapsto (g \exp(tX_e), X_e).\end{aligned}\tag{50}$$

Let X be the left-invariant vector field $X_g = (dL_g)_e X_e$. Consider the vector field $(X, 0)$ on $G \times T_e G$. Then Φ is the flow of $(X, 0)$ on $G \times T_e G$, since $\Phi(-, g, X_e)$ is the integral curve for $(X, 0)$, starting at the point (g, X_e) . Therefore, Φ is smooth. It follows that

$$\exp = \pi_1 \circ \Phi|_{\{1\} \times \{e\} \times T_e G},\tag{51}$$

$$T_e G \xhookrightarrow{\quad} \{1\} \times \{e\} \times T_e G \xhookrightarrow{\quad} \mathbb{R} \times G \times T_e G \xrightarrow{\Phi} G \times T_e G \xrightarrow{\pi_1} G$$

$\exp = \pi_1 \circ \Phi|_{\{1\} \times \{e\} \times T_e G}$

Therefore, as a composition of smooth maps, $\exp : T_e G \rightarrow G$ is smooth. ■

§6 Further properties of \exp

Since we have proved that $\exp : T_e G \rightarrow G$ is smooth, the next natural question that arises is whether \exp is injective/surjective or diffeomorphism. We shall attempt to answer this question by first computing the differential $(d\exp)_0$ at $0 \in T_e G$.

On $(\mathbb{R}_{>0}, \times)$, under the identification $T_1 \mathbb{R}_{>0} \cong \mathbb{R}$, the exponential map is $\exp(x) = e^x$. The differential of this map $\exp : \mathbb{R} \rightarrow \mathbb{R}_{>0}$ is actually

$$(d\exp)_0 : T_0 \mathbb{R} \cong \mathbb{R} \rightarrow T_1 \mathbb{R}_{>0} \cong \mathbb{R},$$

which can easily be seen to be the identity map $\text{id}_{\mathbb{R}}$ under the aforementioned identifications. So it's natural to expect that for a general Lie group G , the differential of the exponential map

$$(d\exp)_0 : T_0(T_e G) \rightarrow T_e G,\tag{53}$$

is gonna be the identity map under some canonical identification. The identification is actually $T_0(T_e G) \cong T_e G$.

Lemma 6.1

If V is a finite dimensional \mathbb{R} -vector space, $T_0 V \cong V$.

Proof. We define $\Phi : V \rightarrow T_0 V$ as follows: given $\mathbf{v} \in V$, $\Phi(\mathbf{v}) =: D_{\mathbf{v}} \in T_0 V$ is the point-derivation that acts like

$$D_{\mathbf{v}} f = \frac{d}{dt} \Big|_{t=0} f(t\mathbf{v}),\tag{54}$$

for $f \in C_0^\infty(V)$. Clearly $D_{\mathbf{v}}$ is linear, and it's a derivation. In fact, it works exactly like point-derivations, since $D_{\mathbf{v}}f = \frac{d}{dt}\Big|_{t=0}(f \circ c)$, where $c : (-\varepsilon, \varepsilon) \rightarrow V, t \mapsto t\mathbf{v}$ is the curve such that $c(0) = \mathbf{0}$ and $c'(0) = D_{\mathbf{v}}$. (Check out Proposition 6.3.3 from [9])

Furthermore, Φ is linear. Also notice that the domain and codomain of Φ has the same dimension. Therefore, we only need to check for injectivity. Suppose $\mathbf{v} \in \text{Ker } \Phi$, i.e. $D_{\mathbf{v}} = 0$. In other words, $D_{\mathbf{v}}f = 0$ for every $f \in C_0^\infty(V)$. Now take $f \in V^*$.

$$0 = D_{\mathbf{v}}f = \frac{d}{dt}\Big|_{t=0} f(t\mathbf{v}) = \frac{d}{dt}\Big|_{t=0} tf(\mathbf{v}) = f(\mathbf{v}). \quad (55)$$

Therefore, $f(\mathbf{v}) = 0$ for every $f \in V^*$. Hence, $\mathbf{v} = 0$, and thus Φ is injective, and we are done! ■

Theorem 6.2

If G is a Lie group, then $(d\exp)_0 \circ \Phi = \text{id}_{T_e G}$.

$$\begin{array}{ccc} T_e G & \xrightarrow{\Phi} & T_{\mathbf{0}}(T_e G) \xrightarrow{(d\exp)_0} T_e G \\ & \searrow & \swarrow \\ & (d\exp)_0 \circ \Phi = \text{id}_{T_e G} & \end{array} \quad (56)$$

Proof. Given $X_e \in T_e G$, we have to show that $(d\exp)_0 D_{X_e} = X_e$. Let's evaluate the LHS on an arbitrary $f \in C_e^\infty(G)$.

$$(d\exp)_0 D_{X_e} f = D_{X_e}(f \circ \exp) = \frac{d}{dt}\Big|_{t=0} f(\exp(tX_e)). \quad (57)$$

Now, the curve $\gamma : \mathbb{R} \rightarrow G$ given by $\gamma(t) = \exp(tX_e)$ has the property that $\gamma(0) = e, \gamma'(0) = X_e$. Therefore,

$$\frac{d}{dt}\Big|_{t=0} f(\exp(tX_e)) = \frac{d}{dt}\Big|_{t=0} (f \circ \gamma) = X_e f. \quad (58)$$

(The last equality follows from Proposition 6.3.3 from [9]) Therefore, $(d\exp)_0 D_{X_e} = X_e$. In other words, $(d\exp)_0 \circ \Phi = \text{id}_{T_e G}$. ■

Corollary 6.3

\exp is a local diffeomorphism near $\mathbf{0} \in T_e G$.

Proof. Since $(d\exp)_0$ is invertible, by the inverse function theorem, \exp is a diffeomorphism in a neighborhood of $\mathbf{0} \in T_e G$. ■

Therefore, there is a neighborhood $U \ni e$ and a neighborhood $\mathfrak{u} \ni \mathbf{0}$ such that \exp is a diffeo-

morphism of \mathfrak{u} and U . Let's call the inverse log.

$$U \subseteq G \xrightleftharpoons[\text{exp}]{\log} \mathfrak{u} \subseteq \mathfrak{g} \quad (59)$$

(Here, we are using the same symbol \mathfrak{g} for the Lie algebra $\mathfrak{L}(G)$ and $T_e G$. This abuse of notation will be used often, and the meaning will be clear from the context.) Then the next question arises: is \exp globally a diffeomorphism? If so, when?

Example 6.1. For the matrix Lie groups, say $\mathrm{GL}(n, \mathbb{R})$ or $\mathrm{SL}(n, \mathbb{R})$, the \exp map is the matrix exponential

$$\exp(A) = e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n. \quad (60)$$

The Lie algebra of $\mathrm{GL}(n, \mathbb{R})$ is $\mathfrak{gl}(n, \mathbb{R}) = \mathbb{R}^{n^2}$. Note that $\mathrm{GL}(n, \mathbb{R})$ is not connected, since we can write it as the union of two disjoint open sets:

$$\mathrm{GL}(n, \mathbb{R}) = \det^{-1}((0, \infty)) \cup \det^{-1}((-∞, 0)). \quad (61)$$

Now, given $A \in \mathfrak{gl}(n, \mathbb{R}) = \mathbb{R}^{n^2}$, $\exp(A)$ is in the $\det^{-1}((0, \infty))$ component, since

$$\det e^A = e^{\mathrm{Tr} A} > 0. \quad (62)$$

Therefore, $\exp : \mathfrak{gl}(n, \mathbb{R}) \rightarrow \mathrm{GL}(n, \mathbb{R})$ is not surjective.

In fact, for any Lie group, $\exp : \mathfrak{g} \rightarrow G$ is not surjective, if G is not connected. Indeed, if G° is the connected component of the identity element, then G° is a clopen subset of G , so $T_e G = T_e G^\circ$. Therefore,

$$\exp : T_e G^\circ \rightarrow G^\circ,$$

the \exp map takes $T_e G = T_e G^\circ$ to G° , not the whole G . So \exp can't be surjective if G is not connected. However, this is not a sufficient condition. Even if G is connected, \exp can still fail to be surjective.

Example 6.2. Consider $G = \mathrm{SL}(n, \mathbb{R})$, it's a connected Lie group. Its Lie algebra is $\mathfrak{sl}(n, \mathbb{R})$, all real traceless $n \times n$ matrices. For $A \in \mathfrak{sl}(n, \mathbb{R})$, if $\lambda_1, \dots, \lambda_n$ are (complex) eigenvalues of A , the eigenvalues of e^A are $e^{\lambda_1}, \dots, e^{\lambda_n}$. If $n = 2$, either

- (i) both λ_1, λ_2 are real, in which case $e^{\lambda_1}, e^{\lambda_2} > 0$, and hence $\mathrm{Tr} e^A > 0$; or
- (ii) λ_1, λ_2 are complex conjugates, in which case, we must have λ_1, λ_2 to be purely imaginary, because $\det e^A = 1$. Then,

$$\mathrm{Tr} e^A = e^{\lambda_1} + e^{\bar{\lambda}_1} = e^{ix} + e^{-ix} = 2 \cos x \in [-2, 2]. \quad (63)$$

In any case, $\text{Tr } e^A \geq -2$. Now consider the matrix

$$X = \begin{bmatrix} -2 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \in \text{SL}(2, \mathbb{R}). \quad (64)$$

Its trace is $-\frac{5}{2}$, which is less than -2 . Therefore, $X \notin \text{im exp}$, i.e. \exp is not surjective.

Example 6.3. Consider $G = S^1$, which is a compact connected Lie group. Its Lie algebra is $T_1 S^1 \cong \mathbb{R}$. The exp map is given by

$$\begin{aligned} \exp : T_1 S^1 &\cong \mathbb{R} \rightarrow S^1 \\ x &\mapsto e^{ix}. \end{aligned} \quad (65)$$

Then \exp is a surjective map, although it's not injective.

It's true that for a compact connected Lie group, \exp is surjective. The proof amounts to showing that a compact connected Lie group can be given a Riemannian metric, and the two exponentials (the Riemannian one and the Lie group one) agree. Then the fact that \exp is surjective is a consequence of a theorem from Riemannian geometry, called the Hopf-Rinow theorem. The proof is outside the scope of this note, so we'll skip it. The interested readers can go through Terence Tao's blog [13] and CUHK's blog [14] for a detailed account.

The answer to the question of when \exp is injective is much more nontrivial. Two StackExchange questions [11], [12] give us some classification that G needs to be solvable, simply connected, and \mathfrak{g} does not admit \mathfrak{e} as subalgebra of a quotient. Interested readers are encouraged to read the papers linked in the StackExchange answers.

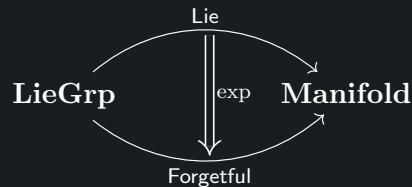
Proposition 6.4 (\exp is natural)

Given any Lie group homomorphism $\varphi : G \rightarrow H$, the following diagram commutes:

$$\begin{array}{ccc} T_e G & \xrightarrow{(d\varphi)_e} & T_{\varphi(e)} H \\ \exp_G \downarrow & & \downarrow \exp_H \\ G & \xrightarrow{\varphi} & H \end{array} \quad (66)$$

i.e. $\varphi \circ \exp_G = \exp_H \circ (d\varphi)_e$.

From a categorical standpoint, this result basically means that \exp is a natural transformation between two functors:



The **Forgetful** functor essentially forgets the group structure, so it takes a Lie group G to its underlying manifold, and a smooth homomorphism map to its underlying smooth map. The functor **Lie** takes a Lie group G to its Lie algebra $\mathfrak{g} = T_e G$'s underlying manifold, and a Lie group homomorphism φ to its differential map $(d\varphi)_e$.

Proof of Proposition 6.4. Given $X_e \in T_e G$, consider the curve

$$\begin{aligned}\alpha : \mathbb{R} &\rightarrow H \\ t &\mapsto \varphi(\exp_G(tX_e)),\end{aligned}\tag{67}$$

i.e. $\alpha = \varphi \circ \gamma$, where $\gamma : \mathbb{R} \rightarrow G$ is defined by $\gamma(t) = \exp_G(tX_e)$. Then $\alpha(0) = e' \in H$, and

$$\alpha'(0) = (d\varphi)_e \gamma'(0) = (d\varphi)_e X_e.\tag{68}$$

Therefore, by the definition of \exp_H ,

$$\alpha(1) = \exp_H((d\varphi)_e X_e),\tag{69}$$

and we are done! ■

If G is connected, any Lie group homomorphism $\varphi : G \rightarrow H$ is completely determined by the Lie algabra homomorphism $\widehat{\varphi} = (d\varphi)_e : \mathfrak{g} \rightarrow \mathfrak{h}$. Because \exp is a diffeomorphism on a neighborhood of the identity $U \ni e$. So $(d\varphi)_e$ completely determines φ on a neighborhood U . But since G is connected, it's generated by any neighborhood of the identity. Hence, $(d\varphi)_e$ completely determines φ on the whole G .

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