Lie Group #6

12/11/2025

Lie Algebra

Vect space ($\mathbb{R}$) V, $[-,-] : V \times V \longrightarrow V$

① Bilinear

② antisymm $[v_1, v_2] = -[v_2, v_1]$

③ Jacobi

$$[[u,v],w] + [[w,u],v] + [[v,w],u] = 0$$

$\mathrm{End}(\mathbb{R}^n)$

$[A, B] = AB - BA$

V is an associative alg

$[v_1, v_2] = v_1 v_2 - v_2 v_1$

M manifold:

$\mathfrak{X}(M) \ni X$ $\quad$ $\mathbb{R}$ lin

$$\boxed{\begin{gathered} X : \underbrace{C^\infty(M)} \to \underbrace{C^\infty(M)} \\ X(fg) = f \cdot Xg + g \cdot Xf \end{gathered}}$$

$X_p \in T_p M$

$[X \cdot Y - Y \cdot X]$

$$\boxed{\begin{aligned} [X, Y] &= X \cdot Y - Y \cdot X \\ &= XY - YX \end{aligned}}$$

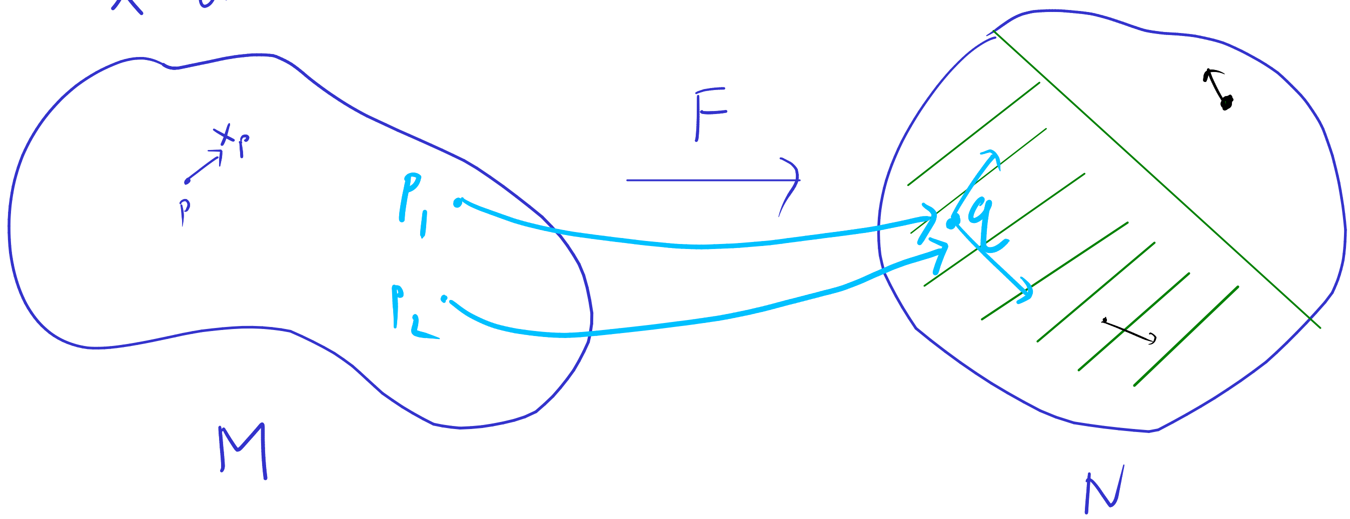
$\underbrace{\mathfrak{X}(M)}_{\dim ?}, [-,-]$ Lie alg

$M = \mathbb{R}$

$\frac{d}{dx}, x \frac{d}{dx}, x^2 \frac{d}{dx}, \dots$

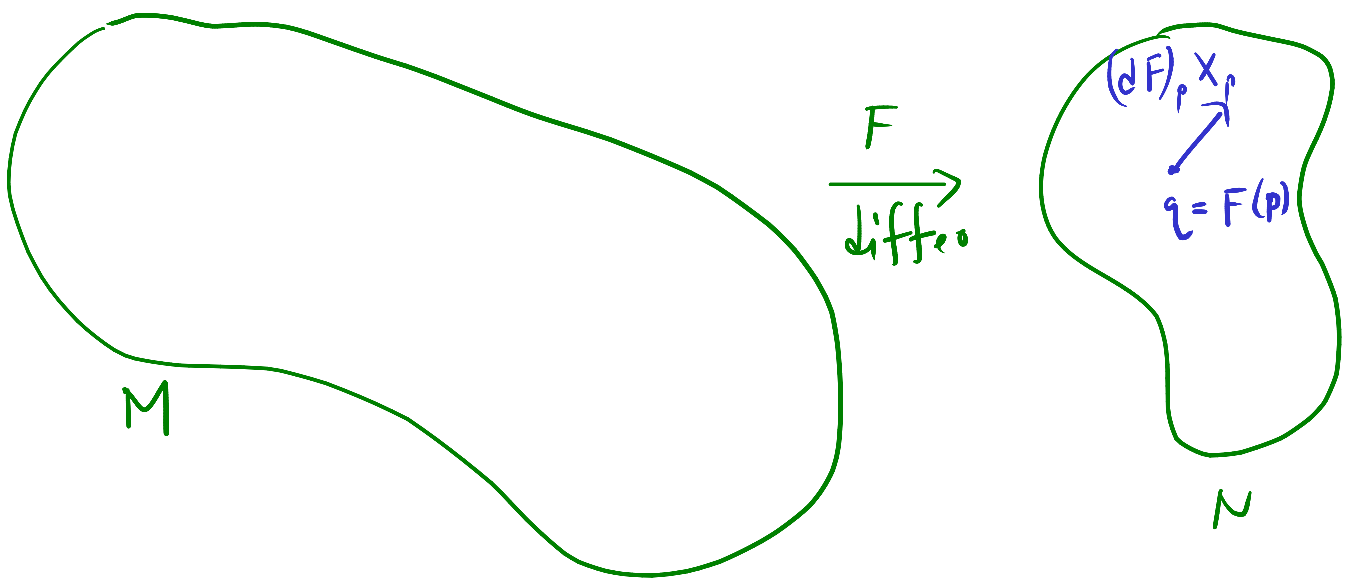
Lie Groups

X on M



$(dF)_p X_p \in T_{F(p)} N$

$(dF)_{p_1} X_{p_1}$ $(dF)_{p_2} X_{p_2}$



$$\boxed{F_*X \in \mathfrak{X}(N)}$$

$$(F_*X)_{F(p)} = (dF)_p X_p \in T_{F(p)} N.$$

Suppose $h : N \to \mathbb{R}$

$$[(F_*X)h](q) = (dF)_p X_p\, h$$

$$= X_p(h \circ F) \qquad p = F(q)$$

$$\Leftrightarrow [(F_*X)h](F(p)) = X_p(h \circ F).$$

<u>Def:</u> $F : M \to M$ is a diffeo.

$X \in \mathfrak{X}(M)$ is F-invariant if $F_*X = X$.

$$\boxed{\mathfrak{X}_F(M) = \text{all F-inv vect field}}$$

Lie subalgebra?

$$\boxed{X, Y \text{ F-inv} \Rightarrow [X, Y] \text{ F-inv}}$$

$$F_*X = X \Leftrightarrow (F_*X)_{F(p)} = X_{F(p)}$$

$$\Leftrightarrow (F_*X)_{F(p)} h = X_{F(p)} h \qquad \forall h \in C^\infty(M)$$

$$\Leftrightarrow X_p(h \circ F) = (Xh)\, F(p)$$

$$\Leftrightarrow [X(h\circ F)](p) = (Xh\circ F)(p)$$

$$\boxed{\Leftrightarrow X(h\circ F) = (Xh)\circ F \quad \forall h \in C^\infty(M)}$$

X, Y are F-inv $\Rightarrow [X,Y]$ is F-inv

$$\Leftrightarrow [X,Y](h\circ F) = ([X,Y]h)\circ F \quad \forall h$$

$$= XY(h\circ F) - YX(h\circ F)$$

$$= X[Yh\circ F] - Y[Xh\circ F]$$

$$= XYh\circ F - YXh\circ F$$

$$= [(XY-YX)h]\circ F$$

$\mathfrak{X}_F(M)$ is a Lie subalg. of $\mathfrak{X}(M)$.

Lie Group

$$L_g : G \longrightarrow G \qquad \text{diffeo.}$$
$$h \longmapsto gh.$$

$$L(G) = \left\{ X \in \mathfrak{X}(G) \;\middle|\; X \text{ is } L_g\text{-inv } \forall g \right\}.$$

left-invariant

$$\boxed{(L_g)_* X = X \quad \forall g}.$$

$$\Leftrightarrow X(h \circ L_g) = Xh \circ L_g \quad \forall h \in C^\infty(M) \;\; \forall g \in G.$$

Claim: $L(G)$ is a fin-dim Lie alg.
$\cong T_e G.$

$$\left.\begin{aligned} \Phi : L(G) &\longrightarrow T_e G \\ X &\longmapsto X_e \end{aligned}\right\}$$

$$\Psi : T_e G \longrightarrow L(G)$$
$$X_e \longrightarrow \Psi(X_e)_g = \boxed{(dL_g)_e X_e}.$$

Exercise: $\Psi(X_e)$ is left-invariant.

$$T_eG \xrightarrow{\Psi} L(G) \xrightarrow{\Phi} T_eG$$
$$X_e \longmapsto \Psi(X_e) \longmapsto \Psi(X_e)_e = (dL_e)_e X_e$$
$$= X_e .$$

$$L(G) \xrightarrow{\Phi} T_eG \xrightarrow{\Psi} L(G)$$
$$X \longmapsto X_e \longmapsto \Psi(X_e) =$$

$$X_g = \Psi(X_e)_g$$

$$(dL_g)_{g'} X_{g'} = X_{gg'} \; ; \quad \boxed{X_g = (dL_g)_e X_e}$$

$$\therefore \dim L(G) = \dim \underline{T_eG} = \dim G .$$

Lie alg associated w/ a Lie group.

Bracket on T_eG:

$$[X_e, Y_e] = \Phi\left(\left[\Psi(X_e), \Psi(Y_e)\right]\right) .$$

$$\Phi : L(G) \longrightarrow T_e G. \qquad X \longmapsto X_e$$

"Natural"

$$\begin{tikzcd}[column sep=huge]
\text{LieGrps} \arrow[r, bend left=50, "G \mapsto L(G)"{name=U}] \arrow[r, bend right=50, "G \to T_e G"'{name=D}] & \mathrm{Vect}_{\mathbb{R}} \arrow[Rightarrow, from=U, to=D, "\Phi"]
\end{tikzcd}$$

$$\begin{tikzcd}
T_e G \arrow[r, "(df)_e"'] & T_{e'} H \arrow[d, "\Psi_H"] \\
L(G) \arrow[u, "\Phi_G"] \arrow[r, "\hat{f}"'] & L(H)
\end{tikzcd}$$

<u>Theorem:</u> $\hat{f} = \Psi_H \circ (df)_e \circ \Phi_G$ is a Lie alg homo.

$$\Leftrightarrow \quad \hat{f}[X,Y] = [\hat{f}X, \hat{f}Y]$$

$$\Leftrightarrow \quad \Psi_H \circ (df)_e \circ \Phi_G [X,Y] = [\hat{f}X, \hat{f}Y]$$

$$\Leftrightarrow \quad (df)_e \circ \Phi_G [X,Y] = \Phi_H [\hat{f}X, \hat{f}Y]$$

$\Leftrightarrow (df)_e [X,Y]_e = [\hat{f}X, \hat{f}Y]_{e'}$; these are in $T_{e'}H$.

$\Leftrightarrow (df)_e [X,Y]_e h = [\hat{f}X, \hat{f}Y]_{e'} h$; $\forall h \in C^\infty(H)$.

$\Leftrightarrow (XY - YX)(h \cdot f)(e)$

$$\boxed{= X_e Y(h \cdot f) - Y_e X(h \cdot f)}.$$

$[\hat{f}X, \hat{f}Y]_{e'} h$

$= (\hat{f}X)_{e'} (\hat{f}Y) h - (\hat{f}Y)_e (\hat{f}X) h$

$= (df)_e X_e (\hat{f}Y) h - (df)_e Y_e (\hat{f}X) h$

$= X_e ((\hat{f}Y) h \cdot f) - Y_e ((\hat{f}X) h \cdot f)$

Sufficient $Y(h \cdot f) = (\hat{f}Y) h \cdot f$ functions in G

$LHS = Y(h \cdot f)(g)$

$= Y_g (h \cdot f)$

$= (dL_g)_e Y_e (h \cdot f)$

$= \boxed{Y_e (h \cdot f \cdot L_g)}$

$RHS = ((\hat{f}Y) h \cdot f)(g)$

$= (\hat{f}Y) h (f(g))$

$= (\hat{f}Y)_{f(g)} h$

$= (dL_{f(g)})_{e'} (df)_e Y_e h$

$= \boxed{Y_e (h \circ L_{f(g)} \cdot f)}$

$$f(gg') = f(g)\,f(g')$$

$$\Leftrightarrow\ f(L_g\, g') = L_{f(g)}\, f(g')$$

$$\Leftrightarrow\ f \circ L_g = L_{f(g)} \circ f$$

$\Rightarrow \hat{f}$ is a Lie alg hom . . . $\boxed{\S}$

$\exp : T_e G \longrightarrow G$

$V \cong T_e G$ Lie's 3rd

$T_e G = T_e G^\circ$