$\boxed{\text{Lie Groups \#3}}$

$$\begin{tikzcd}
\tilde{G} \arrow[r, "\pi"] & G \\
M \arrow[u, dashed, "\exists!\,\tilde{f}"] \arrow[ur, "f"'] &
\end{tikzcd}$$

$\tilde{f}(m_0)$

$f_1(m_0) = f_2(m_0)$

$\Rightarrow \boxed{f_1 = f_2}$

$$\boxed{\begin{tikzcd}
\tilde{G} \arrow[r, "\pi"] & G \\
M \arrow[u, shift left, "f_1"] \arrow[u, shift right, "f_2"'] \arrow[ur, "f"'] \arrow[ur, phantom, bend left, "\checkmark"] &
\end{tikzcd}}$$

$\boxed{\text{Lie Subgroups}}$

$f : G_1 \to G_2$

"substructures"

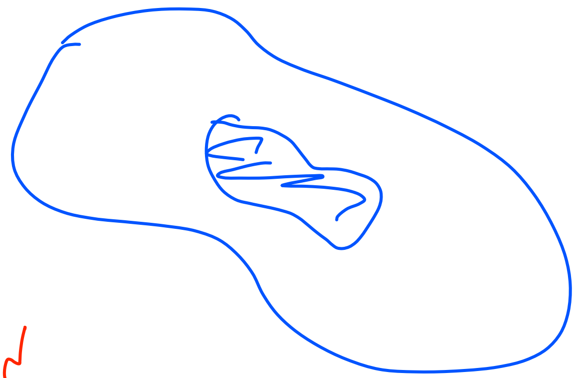
G is a Lie group.

$H \subseteq G$ is a Lie subgroup if

(i) H is a subgroup

(ii) $\boxed{\text{submanifold}}$

$\boxed{\text{regular submanifold (embedded)}}$ | immersed submanifold



M is a manifold, $S \subseteq M$.

$$S \xhookrightarrow{i} M$$

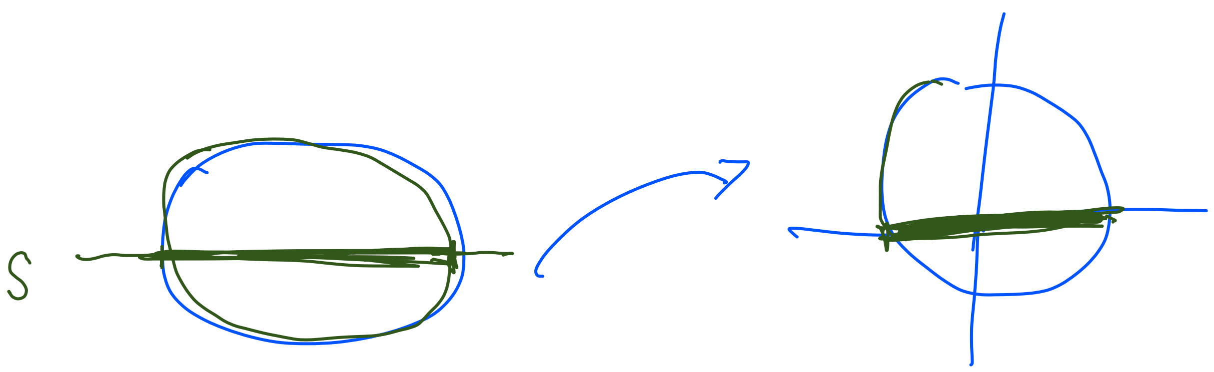
. injective

. embedding $\quad S \xhookrightarrow{\cong} i(S)$

a chart on S $\quad \varphi_s$ $\quad$ is defined by vanishing of coordinates in a chart of M.

$(\mathcal{U}, \varphi)$ chart of M

$(\mathcal{U} \cap S, \varphi_s)$ $\qquad \varphi_s(\mathcal{U} \cap S) = \varphi(\mathcal{U}) \cap \mathbb{R}^k \times \{\bar{0}\}$

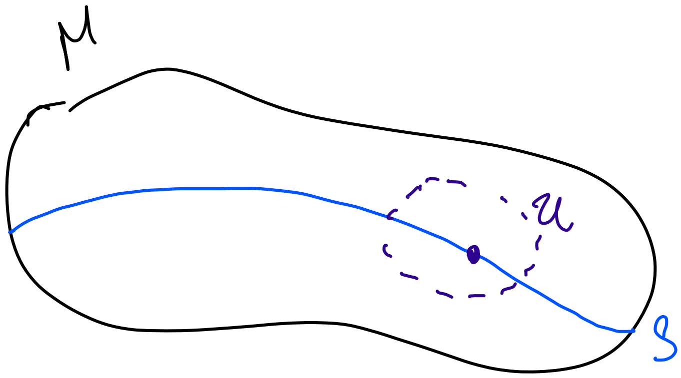


Q: is S closed in M?

$S = \mathcal{U}$ open in M

S is $\boxed{\text{locally closed}}$ in M.

$\hookrightarrow$ given any $x \in S$,
there exists a nbhd $V \ni x$ in M.
s.t. $S \cap V$ is closed in V.



$$\mathcal{U} \longrightarrow \varphi(\mathcal{U}) \subseteq \mathbb{R}^n$$

$$\mathcal{U} \cap S \longrightarrow \varphi(\mathcal{U} \cap S) = \boxed{\varphi(\mathcal{U}) \cap (\mathbb{R}^k \times \{\bar{0}\})}$$

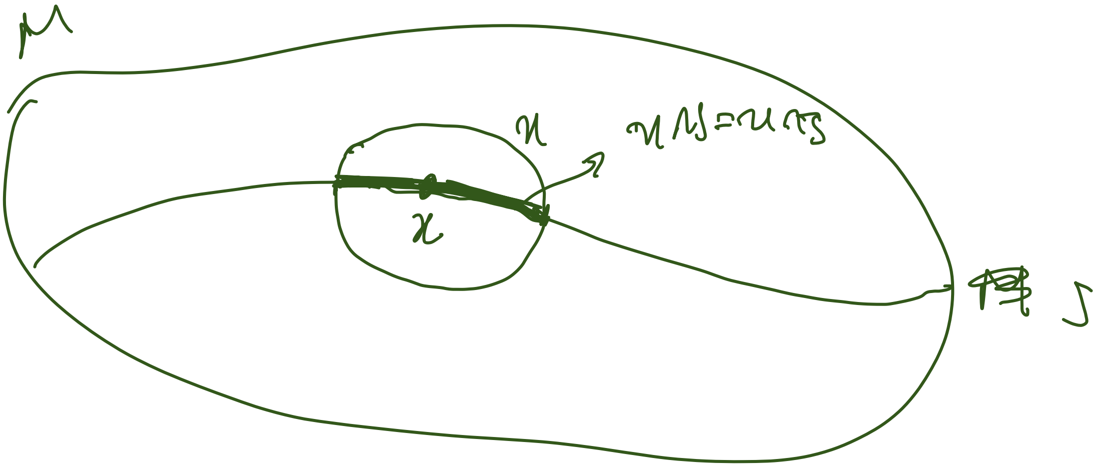
$\Rightarrow \varphi(\mathcal{U} \cap S)$ is closed in $\varphi(\mathcal{U})$

$\Rightarrow \mathcal{U} \cap S$ - - - $\mathcal{U}$.

Exercise: ① S is open is $\bar{S}$.

② $\forall x \in S$, there is a nbhd $\mathcal{U} \ni x$ in M such that $\mathcal{U} \cap S = \mathcal{U} \cap \bar{S}$.

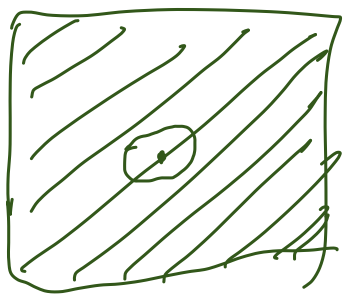
② $\Rightarrow$ ①



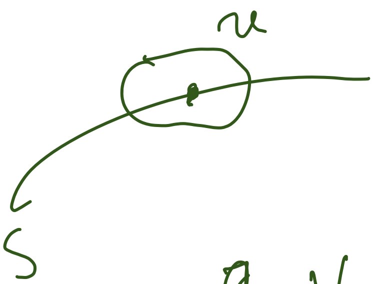
$$f : \mathbb{R} \to S^1 \times S^1 = T^2$$

$$f(t) = \left(e^{2\pi i t}, e^{2\pi i \sqrt{2} t}\right)$$

$f(\mathbb{R})$ is an immersed submanifold of T^2



S is open in $\bar{S}$.



$\forall x \in S$

$U \cap S = U \cap \bar{S}$ is open in $\bar{S}$

$S = \bigcup (U \cap S)$ open in $\bar{S}$.

Theorem: H is a lie subgroup, and $\bar{H}$ is closure.

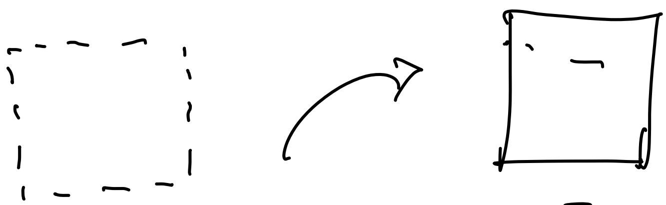
① $\bar{H}$ is a group.

purpose: to show that H is closed.

$\boxed{\text{we'll show } \bar{H} = H}$

Proof: $\bar{H}$ is a subgroup of G.

Take $a, b \in \bar{H}$. Show $ab^{-1} \in \bar{H}$.



X $\qquad\qquad$ $\bar{X}$

Suppose A is a first countable top. space, $X \subseteq A$. Then $x \in \bar{X}$ iff there is a sequence $x_n \in X$
$$x_n \longrightarrow x.$$

$$\left.\begin{array}{l} x_n \longrightarrow a \\ y_n \longrightarrow b \end{array}\right\} \quad x_n, y_n \in H.$$

$$x_n y_n^{-1} \in H \longrightarrow ab^{-1} \in \bar{H}.$$

$m(x_n, i(y_n)) \longrightarrow m(a, i(b))$

(ii) H is open in $\bar{H}$

(Trivial)

$\Rightarrow \forall x \in \bar{H}$, the coset $\underline{Hx}$ is also open in $\bar{H}$.

$R_x : \bar{H} \longrightarrow \bar{H}$

since H is open in $\bar{H}$

$\Rightarrow Hx$ is open in $\bar{H}$.

(iii) $\boxed{H = \bar{H}.}$

Compare H vs Hx, $x \in \underline{\bar{H} - H}$

they are disjoint.

$x = \lim x_n$, $x_n \in H$.

Given any nbhd $V \ni x$,
there is some N s.t.
$x_n \in V \quad \forall n \geq N$.

choose $V := Hx$. $\exists N$:

then $x_n \in Hx \quad \forall n \geq \underline{N}$.

$x_n \in H$.

$\Rightarrow Hx$ and H intersects.

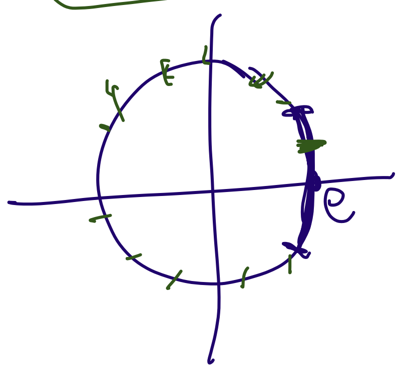
$\Rightarrow Hx = H$.

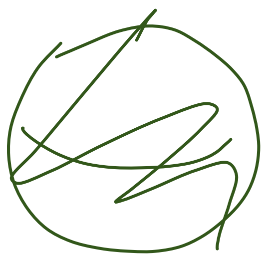
$\Rightarrow H = \bar{H}$.

$\therefore$ Lie Subgroups are closed. ∎

Consequence/Corollary:

Suppose G is a CLG.
And $\mathcal{U}$ is a nbhd of e.
Then $\mathcal{U}$ generates G.





SO(3)

Take $H = \langle \mathcal{U} \rangle$ is a subgroup of G.

H is open: $\boxed{\text{Take } h \in H. \quad h \in V \subseteq H \quad V = \mathcal{U}h \subseteq H \quad \checkmark}$

$\Rightarrow H$ is a regular submanifold of G.

$\Rightarrow H$ is a Lie subgroup

$\Rightarrow H$ is closed

$\Rightarrow H = G$. $\blacksquare$

$f : N \longrightarrow M$

$(df)_p : T_pN \longrightarrow T_{f(p)}M$

<u>Corollary</u>: $f : G_1 \longrightarrow G_2$ is a ~~Lie group~~ smooth <u>homomorphism</u>.

G_2 is connected.

Suppose $(df)_e : T_eG_1 \longrightarrow T_{e'}G_2$ is surjective

then: f is surjective.

f is a $\boxed{\text{submersion}}$ at e.

$F: N \to M$ is smooth.

F is a submersion at $p \in N$ if

$(dF)_p : T_p N \longrightarrow T_{F(p)} M$ is surjective.

F is a submersion if it's a submersion at every $p \in N$.

Claim: f is a submersion.

$(df)_g : T_g G_1 \longrightarrow T_{f(g)} G_2$ is surjective.

because f is a homomorphism:

$$f(g g') = f(g) f(g')$$

$$\Leftrightarrow f(L_g(g')) = L_{f(g)}(f(g'))$$

$$\Leftrightarrow f \circ L_g = L_{f(g)} \circ f.$$

$$\Leftrightarrow \boxed{f = L_{f(g)} \circ f \circ L_{g^{-1}}}$$

$$f(g') = (L_{f(g)} \circ f \circ L_{g^{-1}})(g')$$

$$= f(g) f(g^{-1} g')$$

$= f(g')$

level of tangent spaces

$$(df)_g = \boxed{(dL_{f(g)})_e} \circ \underline{(df)_e} \circ \boxed{(dL_{g^{-1}})_g}$$

vect space isomorphisms

$\Rightarrow (df)_g$ is surjective

$\Rightarrow f$ is a submersion $\Rightarrow$ open map.

$\Rightarrow$ $f: G_1 \longrightarrow G_2$

$\Rightarrow$ $f(G_1)$ is open in G_2.

$f(G_1)$ is a subgroup

$\Rightarrow$ $f(G_1) = \langle f(G_1) \rangle = G_2$.

$\Rightarrow f$ is surjective.

$\blacksquare$

Goal: G/H is a manifold,

where G is a Lie group

H is a Lie subgroup.

· Hausdorff:

~~$G \to H$~~ $\quad$ true because H is closed.

$H \longrightarrow G$

Chapter 21 of Lee's Smooth Manifolds

M is a manifold, $R \subseteq M \times M$ is an equiv. relation.

M/R has a manifold structure if

(i) $R \hookrightarrow M \times M \xrightarrow{\pi_1} M$ is an ~~im~~submersion.

(ii) R is closed in $M \times M$. (submanifold)

(iii) TBD

if H were not closed.

$\pi : G \longrightarrow \overline{G/H}$.

$\{\bar{e}\}$ is closed

$\pi^{-1}\{\bar{e}\} = H$ is closed.

Second ctable:

G/H second countable

$\boxed{\pi \text{ is an open map.}}$

$\{B_i\}$ ctble basis of G

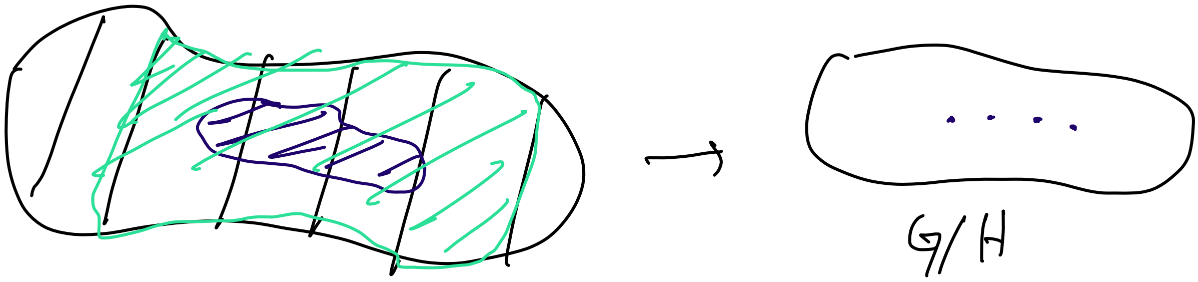
$\downarrow$

$\{\pi(B_i)\}_i$ . . . G/H.

$\pi(\mathcal{U})$ is open in G/H

$\Leftrightarrow \boxed{\pi^{-1}(\pi(\mathcal{U}))}$ is open in G

$= \mathcal{U}H$.


G/H