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Algebraic Topology I (MAT431)

Lecture Notes

Preface

This series of lecture notes has been prepared for aiding students who took the BRAC University course **Algebraic Topology I (MAT431)** in Summer 2021 semester. These notes were typeset under the supervision of mathematician **Dr. Syed Hasibul Hassan Chowdhury**. The video lectures can be found here. The main goal of this typeset is to have an organized digital version of the notes, which is easier to share and handle. If you see any mistakes or typos, please send me an email at atonuroychowdhury@gmail.com

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References:

- *Topology*, by James R. Munkres
- *Elements of Algebraic Topology*, by James R. Munkres
- *A Basic Course in Algebraic Topology*, by William S. Massey

Contents

0 Topology Review

§0.1 Euclidean Space $\mathbb{R}^n$

Before embarking on the concept of general topological space, let us look at the Euclidean space $\mathbb{R}^n$. $\mathbb{R}^n$ is equipped with the notion of distance between 2 points p and q.

Definition 0.1.1 (Distance)**.** Let the coordinates of p and q be $(p^1, p^2, ...p^n)$ and $(q^1, q^2, ...q^n)$, respectively. The distance between p and q is given by

$$d(p,q) = \left[\sum_{i=1}^{n}(p^i - q^i)^2\right]^{\frac{1}{2}}$$

Definition 0.1.2 (Open ball)**.** An open ball $B(p,r)$ in $\mathbb{R}^n$ with center $p \in \mathbb{R}^n$ and radius $r > 0$ is defined as the set

$$B(p,r) = \{x \in \mathbb{R}^n \ : \ d(x,p) < r\}$$

A set equipped with the notion of distance between its elements is called a metric space[1]. Thus the Euclidean space $\mathbb{R}^n$ is a metric space. And we can talk about open balls in $\mathbb{R}^n$ using this metric. We can define open sets in $\mathbb{R}^n$ using open balls $B(p,r)$ defined above.

Definition 0.1.3 (Open Set in $\mathbb{R}^n$)**.** A set U in $\mathbb{R}^n$ is said to be open if for every p in U, there is an open ball $B(p,r)$ such that $B(p,r) \subseteq U$.

Proposition 0.1.1

The union of an arbitrary collection of $\{U_\alpha\}$ of open sets is open. The intersection of finite collection of open sets is open.

Proof. Trivial. ■

Example 0.1.1

The intervals $\left(-\frac{1}{n}, \frac{1}{n}\right)$, $n = 1, 2, 3, ...$ are all open in $\mathbb{R}$ but their intersection

$$\bigcap_{n\in\mathbb{N}} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$$

is not open.

The metric d in $\mathbb{R}^n$ allows us to define open sets in $\mathbb{R}^n$. In other words, given a subset of $\mathbb{R}^n$, we can tell if it is open or not. This situation is a special case called **metric topology in** $\mathbb{R}^n$.

§0.2 Topology

[1]There are some properties that a metric (distance) function should have. We won't go into much details

Definition 0.2.1 (Topology). A topology on a set S is a collection $\mathcal{T}$ of subsets of S containing both the empty set $\varnothing$ and the S such that $\mathcal{T}$ is closed under arbitrary union and finite intersection. In other words,

- If $U_\alpha \in \mathcal{T}$ for all α in an index set A, then $\bigcup\limits_{\alpha \in A} U_\alpha \in \mathcal{T}$
- If $U_i \in \mathcal{T}$ for $i \in \{1, 2, ..., n\}$, then $\bigcap\limits_{i=1}^{n} U_i \in \mathcal{T}$

The elements of $\mathcal{T}$ are called open sets.

Definition 0.2.2 (Topological Space). The pair $(S, \mathcal{T})$ consisting of a set S together with a topology $\mathcal{T}$ on S is called a **topological space**.

Abuse of Notation. We shall often say "S is a topological space" in short. But there is always a topology $\mathcal{T}$ on S, which we recall when necessary.

Definition 0.2.3 (Neighborhood). A **neighbourhood** of a point $p \in S$ is called an open set U containing p.

Definition 0.2.4 (Closed Set). The complement of an open set is called a **closed set**.

Proposition 0.2.1

The union of a finite collection of closed sets is closed. The intersection of an arbitrary collection of closed sets is closed.

Proof. Let $\{F_i\}_{i=1}^{n}$ be a finite collection of closed sets. Then, $\{S \setminus F_i\}_{i=1}^{n}$ is a finite collection of open sets. The intersection of a finite collection of open sets is open, therefore $\bigcap\limits_{i=1}^{n} (S \setminus F_i)$ is open. By De Morgan's law,

$$\bigcap_{i=1}^{n} (S \setminus F_i) = S \setminus \left(\bigcup_{i=1}^{n} F_i \right) \quad \text{is open} \implies \bigcup_{i=1}^{n} F_i \quad \text{is closed}$$

Therefore, the union of a finite collection of closed sets is closed.
Now, let $\{F_\alpha\}_{\alpha \in A}$ be an arbitrary collection of closed sets with A being an index set. Then $\{S \setminus F_\alpha\}_{\alpha \in A}$ is an arbitrary collection of open sets. We know that the union of an arbitrary collection of open sets is open, therefore $\bigcup_{\alpha \in A} (S \setminus F_\alpha)$ is open. By De Morgan's law,

$$\bigcup_{\alpha \in A} (S \setminus F_\alpha) = S \setminus \left(\bigcap_{\alpha \in A} F_\alpha \right) \quad \text{is open} \implies \bigcap_{\alpha \in A} F_\alpha \quad \text{is closed}$$

Therefore, the intersection of an arbitrary collection of closed sets is closed. ■

Definition 0.2.5 (Subspace Topology). Let $(S, \mathcal{T})$ be a topological space and A a subset of S. Define $\mathcal{T}_A$ to be the collection of subsets

$$\mathcal{T}_A = \{U \cap A \mid U \in \mathcal{T}\}$$

$\mathcal{T}_A$ is called the **subspace topology** of A in S.

It is not hard to see that $\mathcal{T}_A$ satisfies the conditions of a Topology. Firstly, $\mathcal{T}_A$ contains both $\varnothing$ and A. For these, taking $U = \varnothing$ and $U = S$, respecitvely, suffices. By the distributive property of union and intersection

$$\bigcup_{\alpha}(U_\alpha \cap A) = \left(\bigcup_{\alpha} U_\alpha\right) \cap A \text{ and } \bigcap_{i=1}^{n}(U_i \cap A) = \left(\bigcap_{i=1}^{n} U_i\right) \cap A$$

which shows that $\mathcal{T}_A$ is closed under arbitrary union and finite intersection. So $\mathcal{T}$ is a Topology indeed.

Example 0.2.1

Consider the subset $A = [0,1]$ of $\mathbb{R}$. In the subspace topology, the half-open interval $\left[0, \frac{1}{2}\right)$ is an open subset of A, because $\left[0, \frac{1}{2}\right) = \left(-\frac{1}{2}, \frac{1}{2}\right) \cap [0,1]$

Lemma 0.2.2

Let Y be a subspace of X (that is Y has the subspace topology inherited from X). If U is open in Y and Y is open in X, then U is open in X.

Proof. Since U is open in Y, $U = Y \cap V$ for some V open in X. Both Y and V are open in X, hence $Y \cap V = U$ is also open in X. ■

§0.3 Bases and Countability

Definition 0.3.1 (Basis and Basic Open Sets). A subcollection $\mathcal{B}$ of a topology $\mathcal{T}$ is a **basis** for $\mathcal{T}$ if given an open set U and a point p in U, there is an open set $\mathrm{B} \in \mathcal{B}$ such that $p \in \mathrm{B} \subseteq u$. An element of $\mathcal{B}$ is called a **basic open set**.

Example 0.3.1

The collection of all open balls $B(p,r)$ in $\mathbb{R}^n$ with $p \in \mathbb{R}^n$ and $r > 0$ is a basis for the standard topology (metric topology) on $\mathbb{R}^n$.

Proposition 0.3.1

A collection $\mathcal{B}$ of open sets of S is a basis if and only if every open set in S is a union of sets in $\mathcal{B}$.

Proof. ($\Rightarrow$) We are given a collection of $\mathcal{B}$ of open sets of S that is a basis. U is any open set in S. Also, let $p \in U$. Therefore, there is a basic open set $B_p \in \mathcal{B}$ such that $p \in B_p \subseteq U$. Hence, one can show that $U = \bigcup\limits_{p \in U} B_p$.

($\Leftarrow$) Suppose, every open set in S is a union of open sets in $\mathcal{B}$. Now, given an open set U and a point $p \in U$, since $U = \bigcup\limits_{B_\alpha \in \mathcal{B}} B_\alpha$, there is a $B_\alpha \in \mathcal{B}$, such that $p \in B_\alpha \subseteq U$. Hence $\mathcal{B}$ is a basis. ■

We say that a point in $\mathbb{R}^n$ is rational if all of its coordinates are rational numbers. Let $\mathbb{Q}$ be the set of rational numbers and $\mathbb{Q}^+$ the set of positive rational numbers.

Lemma 0.3.2

Every open set in $\mathbb{R}^n$ contains a rational point.

Proof. An open set U in $\mathbb{R}^n$ contains an open ball $B(p,r)$ which, in turn, contains an open cube $\prod\limits_{i=1}^{n} I_i$ where I_i is the open interval $\left(p^i - \frac{r}{\sqrt{n}}, p^i + \frac{r}{\sqrt{n}}\right)$. Here is a visual example for $n = 2$.

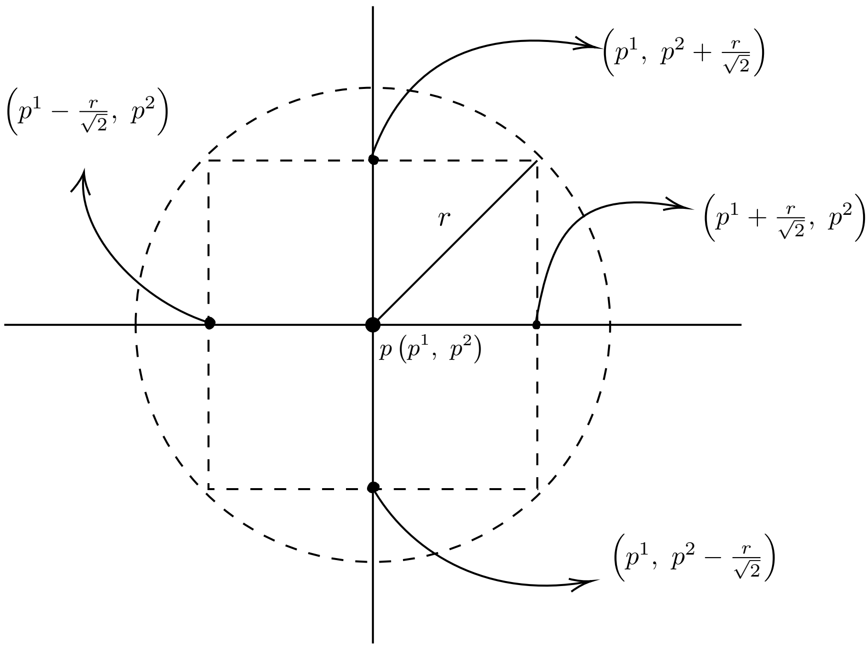


Figure 1: $B(p, r)$ contains $\left(p^1 - \frac{r}{\sqrt{n}}, p^1 + \frac{r}{\sqrt{n}}\right) \times \left(p^2 - \frac{r}{\sqrt{n}}, p^2 + \frac{r}{\sqrt{n}}\right)$

Now back to general n. For each i, let q^i be a rational number in I_i. Then $(q^1, q^2, ..., q^n)$ is a rational point in $\prod_{i=1}^{n} I_i \subseteq B(p, r)$. Therefore, every open set contains a rational point. ■

Proposition 0.3.3

The collection $\mathcal{B}_{\mathbb{Q}}$ of all open balls in $\mathbb{R}^n$ with rational centers and rational radii is a basis for $\mathbb{R}^n$.

Proof. Given an open set U in $\mathbb{R}^n$ and $p \in U$, there is an open ball $B(p, r')$ with positive real radius r' such that $p \in B(p, r') \subseteq U$. Take a rational number $r \in (0, r')$. Then we have

$$p \in B(p, r) \subseteq B(p, r') \subseteq U$$

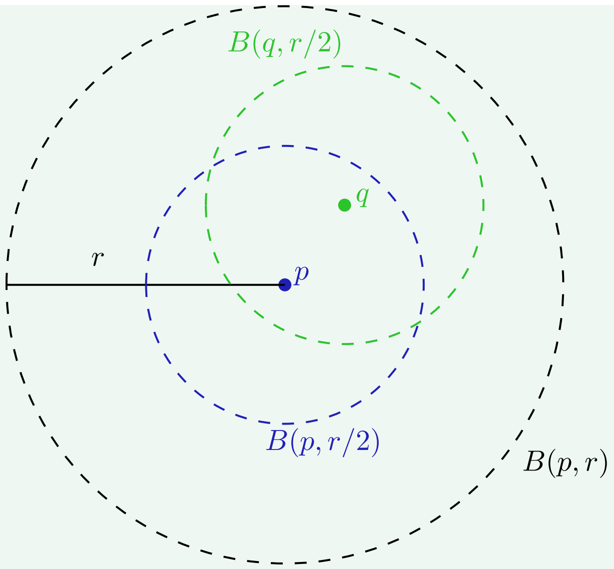
By Lemma 0.3.2 , there is a rational point in the smaller ball $B\left(p, \frac{r}{2}\right)$.

Claim — $p \in B\left(q, \frac{r}{2}\right) \subseteq B(p, r)$

Proof. Since $d(p, q) < \frac{r}{2}$, we have $p \in B\left(q, \frac{r}{2}\right)$. Next, if $x \in B\left(q, \frac{r}{2}\right)$, then by triangle inequality

$$d(x, p) \leq d(x, q) + d(q, p) < \frac{r}{2} + \frac{r}{2} = r$$

Therefore, $x \in B(p, r)$.



So, $p \in B\left(q, \frac{r}{2}\right)$ and $B\left(q, \frac{r}{2}\right) \subseteq B(p, r)$. □

As a result, $p \in B\left(q, \frac{r}{2}\right) \subseteq B(p, r) \subseteq B\left(p, r'\right) \subseteq U$. Hence we proved,

$$p \in B\left(q, \frac{r}{2}\right) \subseteq U$$

In other words, the collection $\mathcal{B}_{\mathbb{Q}}$ of open balls with rational centers and rational radii is a basis for $\mathbb{R}^n$. ■

Both the sets $\mathbb{Q}$ and $\mathbb{Q}^+$ are countable. Since the centers of the open balls in $\mathcal{B}_{\mathbb{Q}}$ are indexed by $\mathbb{Q}^n$, a countable set, and the radii are indexed by $\mathbb{Q}^+$, also a countable set, the collection $\mathcal{B}_{\mathbb{Q}}$ is countable.

Definition 0.3.2 (Second Countable). A topological space is said to be second countable if it has a countable basis.

Proposition 0.3.3 shows that $\mathbb{R}^n$ with its standard topology is second countable.

Definition 0.3.3 (Neighborhood Basis). Let S be a topological space and p be a point in S. A **basis of neighbourhoods** or a **neighbourhood basis** at p is a collection $\mathcal{B} = \{B_\alpha\}$ of neighbourhoods of p such that for any neighbourhood U of p there is a $B_\alpha \in \mathcal{B}$ such that $p \in B_\alpha \subseteq U$.

Definition 0.3.4 (First Countable). A topological space S is first countable if it has a countable basis of neighbourhoods at every point $p \in S$.

Example 0.3.2

For $p \in \mathbb{R}^n$, let $B\left(p, \frac{1}{n}\right)$ be the open ball of center p and radius $\frac{1}{n}$ in $\mathbb{R}^n$. Then $\left\{B\left(p, \frac{1}{n}\right)\right\}_{n=1}^{\infty}$ is a neighbourhood basis at p. Thus $\mathbb{R}^n$ is first countable.

An important note: An uncountable discrete topological space is first countable but not second countable. A second countable topological space is always first countable.

§0.4 Hausdorff Space

Definition 0.4.1 (Hausdorff Space). A topological space S is Hausdorff if given any 2 distinct points x, y in S there exist disjoint open sets U, V such that $x \in U$ and $y \in V$.

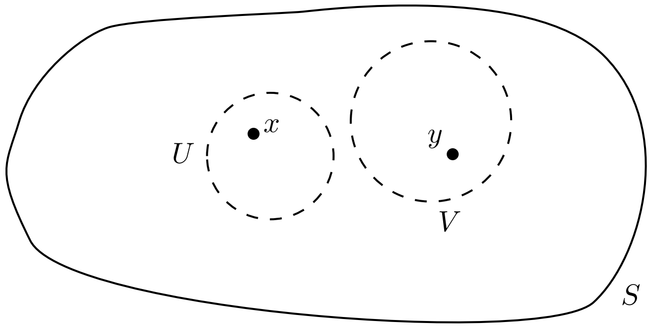


Figure 2: Here S is a Hausdorff space, U and V are disjoint open sets containing x and y respectively.

Proposition 0.4.1

Every singleton set (a one-point set) in a Hausdorff space S is closed.

Proof. Let $x \in S$. We want to prove that $\{x\}$ is closed, i.e. $S \setminus \{x\}$ is open.

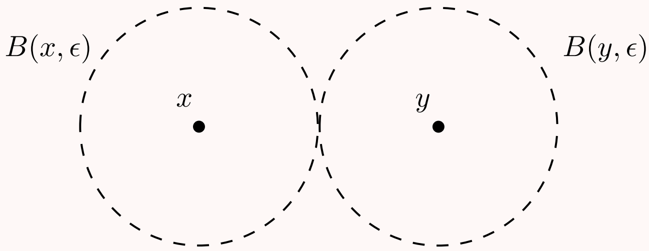
Let $y \in S \setminus \{x\}$. Since S is Hausdorff, we can find disjoint open sets U_y and V_y such that $x \in U_y$ and $y \in V_y$. No such V_y contains x. Therefore

$$S \setminus \{x\} = \bigcup_{y \in S \setminus \{x\}} V_y$$

So $S \setminus \{x\}$ is union of open sets, hence open. So $\{x\}$ is closed. ■

Example 0.4.1

The Euclidean space $\mathbb{R}^n$ (equipped with standard/ metric topology) is Hausdorff, for given distinct points x, y in $\mathbb{R}^n$, if $\epsilon = \frac{1}{2}d(x, y)$, then the open balls $B(x, \epsilon)$ and $B(y, \epsilon)$ will be disjoint.



In a similar manner, one can show that every metric space is Hausdorff.

Lemma 0.4.2

Let A be a subspace of X. If X is a Hausdorff space, then so is A.

Proof. Take $x, y \in A \subseteq X$ with $x \neq y$. As X is Hausdorff, we can find disjoint open sets U and V in X, such that $U \ni x$ and $V \ni y$. $x \in A$ and $x \in U$, so $x \in A \cap U$. Similarly, $y \in A \cap V$.

Now, both $A \cap U$ and $A \cap V$ are open in A, with respect to the subspace topology. Furthermore, $(A \cap U) \cap (A \cap V) = A \cap (U \cap V) = \varnothing$. Therefore, for $x, y \in A$ we've found disjoint open sets $A \cap U$ and $A \cap V$, containing x and y respectively. So A is Hausdorff. ■

§0.5 Continuity and Homeomorphism

Definition 0.5.1 (Continuous Maps)**.** Let $f : X \to Y$ be a map of topological spaces. f is said to be continuous if for each open subset V of Y, the set $f^{-1}(V)$ is an open subset of X.

Proposition 0.5.1

$f : X \to Y$ is continuous if and only if for every closed subset B of Y, the set $f^{-1}(B)$ will be closed in X.

Proof. ($\Rightarrow$) Suppose f is continuous. B is closed, so $Y \setminus B$ is open in Y. Therefore, by the continuity of f, $f^{-1}(Y \setminus B) = X \setminus f^{-1}(B)$ is open in X, so $f^{-1}(B)$ is closed.

($\Leftarrow$) Suppose $f^{-1}(B)$ is closed in X for any closed $B \subseteq Y$. Take any open set U in Y. Choose $B = Y \setminus U$. Then by the assumption $f^{-1}(Y \setminus U) = X \setminus f^{-1}(U)$ is closed in X. This gives us $f^{-1}(U)$ is open. So f is continuous. ∎

Definition 0.5.2 (Homeomorphism)**.** Let X and Y be topological spaces; let $f : X \to Y$ be a bijection. If both f and the inverse function $f^{-1} : Y \to X$ are continuous, then f is called a **homeomorphism**.

Example 0.5.1

The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 3x + 1$ is a homeomorphism. We define $g : \mathbb{R} \to \mathbb{R}$ by $g(y) = \frac{1}{3}(y - 1)$. Then we have

$$f(g(y)) = y \quad \text{and} \quad g(f(x)) = x \qquad \forall\ x, y \in \mathbb{R}$$

This proves $g = f^{-1}$. It is easy to see that both f and g are continuous functions. Therefore f is a homeomorphism.

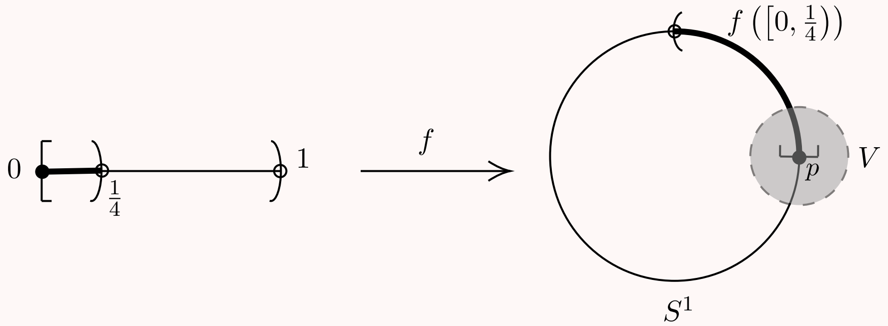
However, a bijective function can be continuous without being a homeomorphism.

Example 0.5.2

Let S^1 denote the unit circle in $\mathbb{R}^2$; that is $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$, considered as a **subspace**[a] of the space $\mathbb{R}^2$. Let $f : [0, 1) \to S^1$ be the

$$f(t) = (\cos 2\pi t, \sin 2\pi t)$$

It is left as an exercise for the reader to show that f is a continuous bijective function. But the function f^{-1} is not continuous.



$U = \left[0, \frac{1}{4}\right)$ is an open set in $[0, 1)$ according to the subspace topology. We want to show that $f(U)$ is not open in S^1. That would prove the discontinuity of f^{-1}.

Let p be the point $f(0)$. And $p \in f(U)$. We need to find an open set of S^1 in subspace topology containing $p = f(0)$ and contained in $f(U)$ to show that $f(U)$ is open in S^1, i.e we have to find an open set in V of $\mathbb{R}^2$ such that $f(0) = p \in V \cap S^1 \subseteq f(U)$. But it is impossible as is evident from the figure above. No matter what V we choose, some part of $V \cap S^1$ would lie outside $f(U)$.

[a] Subset of $\mathbb{R}^2$ equipped with subspace topology.

Lemma 0.5.2 (Pasting Lemma)

Let $X = A \cup B$, where A and B are closed in X. Let $f : A \to Y$ and $g : B \to Y$ be continuous. If $f(x) = g(x)$ for every $x \in A \cap B$, then f and g combine to give a continuous function $h : X \to Y$ defined by

$$h(x) = \begin{cases} f(x) & \text{if } \ x \in A \\ g(x) & \text{if } \ x \in B \end{cases}$$

Proof. Let C be a closed subset of Y. Now,

$$h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$$

Since f is continuous, $f^{-1}(C)$ is closed in A, hence closed in X. Similarly, $g^{-1}(C)$ is closed in X. So $h^{-1}(C)$ is the union of two closed sets in X, hence it is closed in X. Therefore, h is continuous. ■

Lemma 0.5.3

Let X, Y be topological spaces and $f : X \to Y$. Then f is continuous if and only if for every $x \in X$ and each neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subseteq V$.

Proof. ($\Rightarrow$) Suppose f is continuous. Let $x \in X$ and $V \ni f(x)$ is open in Y. We take $U = f^{-1}(V)$. Since f is open and U is preimage of open set, so U is open. Also,

$$f(x) \in V \implies x \in f^{-1}(V) = U \text{ and } f(U) = f\left(f^{-1}(V)\right) \subseteq V$$

($\Leftarrow$) Let $V \subseteq Y$ be open. We need to show that $f^{-1}(V)$ is open. Take $x \in f^{-1}(V)$. Then $f(x) \in V$, so V is a neighborhood of $f(x)$. By assumption, there exists open $U \ni x$ such that

$$f(U) \subseteq V \implies U \subseteq f^{-1}(V)$$

So for every $x \in f^{-1}(V)$, there exists a neighborhood of x that is contained in $f^{-1}(V)$. So $f^{-1}(V)$ is open, and hence f is continuous. ■

§0.6 Quotient Topology

Quotient topology is defined using an equivalence relation. An equivalence relation is a binary relation on a set that has some properties.

Definition 0.6.1 (Equivalence Relation and Equivalence Class). An equivalence relation $\sim$ on a set S is a binary relation which is reflexive, symmetric and transitive. That is

(i) $a \sim a$ for every $a \in S$

(ii) $a \sim b \implies b \sim a$

(iii) $a \sim b$, $b \sim c \implies a \sim c$

The equivalence class $[x]$, if $x \in S$, is the set of all elements in S equivalent to x.

An equivalence relation on S partitions S into disjoint equivalence classes. We denote the set of all equivalence classes with $S/{\sim}$ and call this the quotient of S by the equivalence relation $\sim$. There is a natural projection map $\pi : S \to S/{\sim}$ which projects $x \in S$ to its own equivalence class $[x] \in S/{\sim}$.

Abuse of Notation. Ideally $[x]$ denotes a point in $S/{\sim}$. But we will use the same notation $[x]$ to identify a set in S whose elements are all equivalent to each other under the given equivalence relation.

> **Definition 0.6.2** (Quotient Topology)**.** Let S be a topological space. We define a topology called **quotient topology** on $S/{\sim}$ by declaring a set U in $S/{\sim}$ to be open if and only if $\pi^{-1}(U)$ is open in S.

It's not hard to see that quotient topology is a well defined topology. Note that $\pi^{-1}(\varnothing) = \varnothing$ and $\pi^{-1}(S/{\sim}) = S$ and hence $\varnothing$ and $S/{\sim}$ are both open sets in quotient topology. Now let $\{U_\alpha\}_{\alpha\in A}$ be an arbitrary collection of open sets in $S/{\sim}$. Then $\left\{\pi^{-1}(U_\alpha)\right\}_{\alpha\in A}$ is an an arbitrary collection of open sets in S. So,

$$\bigcup_{\alpha\in A} \pi^{-1}(U_\alpha) = \pi^{-1}\left(\bigcup_{\alpha\in A} U_\alpha\right) \text{ is open in } S \implies \bigcup_{\alpha\in A} U_\alpha \text{ open in } S/{\sim}$$

So arbitrary union of open sets is open in $S/{\sim}$. Now for a finite collection of open sets $\{U_i\}_{i=1}^n$ in $S/{\sim}$, $\left\{\pi^{-1}(U_i)\right\}_{i=1}^n$ is a finite collection of open sets in S. So,

$$\bigcap_{i=1}^{n} \pi^{-1}(U_i) = \pi^{-1}\left(\bigcap_{i=1}^{n} U_\alpha\right) \text{ is open in } S \implies \bigcap_{i=1}^{n} U_\alpha \text{ open in } S/{\sim}$$

So finite intersection of open sets is open in $S/{\sim}$. Therefore, we've verified that the open sets defined on $S/{\sim}$ indeed form a topology.

Continuity on Quotient Topology

Let $\sim$ be a equivalence relation on the topological space S and give $S/{\sim}$ the quotient topology. Suppose that the function $f : S \to Y$ is continuous from S to another topological space Y. Further assume that f is constant on each equivalence class. Then f induces a map

$$\bar{f} : S/{\sim} \to Y \;;\; \bar{f}([p]) = f(p) \quad \forall\, p \in S$$

Note that this latter function $\bar{f}$ wouldn't be well-defined had we not assumed f to be constant on each equivalence class in $S/{\sim}$.

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "\pi"'] & Y \\ S/{\sim} \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

$$f = \bar{f} \circ \pi$$
$$f(p) = \bar{f}(\pi(p)) = f([p])$$

> **Proposition 0.6.1**
>
> The induced map $\bar{f} : S/{\sim} \to Y$ is continuous if and only if the map $f : S \to Y$ is continuous.

Proof. ($\Rightarrow$). Suppose $f : S \to Y$ is continuous. Let V be open in Y. Then $f^{-1}(V) = \pi^{-1}\left(\bar{f}^{-1}(V)\right)$ is open in S. Therefore, by the definition of quotient topology, then $\bar{f}^{-1}(V)$ is open in $S/{\sim}$. Hence, we've shown that for a given open set V in Y, $\bar{f}^{-1}(V)$ is open in $S/{\sim}$. So, $\bar{f} : S/{\sim} \to Y$ is continuous.

($\Leftarrow$). That the continuity of $\bar{f} : S/{\sim} \to Y$ implies the continuity of $f : S \to Y$ is easy to see using the equality $f = \bar{f} \circ \pi$ and the fact that $\pi : S \to S/{\sim}$ is continuous. ■

Identification of a subset to a point

If A is a subspace of a topological space S, we can define a relation $\sim$ on S by declaring

$$x \sim x \;,\; \forall\, x \in S \qquad \text{and} \qquad x \sim y \;,\; \forall\, x, y \in A$$

It is immediate that $\sim$ is an equivalence relation. We say that the quotient space $S/\sim$ is obtained from S by identifying A to a point.

§0.7 Compactness

Definition 0.7.1 (Open Cover). Let S be a topological space. A collection $\{U_\alpha\}_{\alpha\in I}$ of open subsets of S is said to be an open cover of S if

$$S \subseteq \bigcup_{\alpha\in I} U_\alpha$$

Since the open sets are in the topology of S and consequently $U_\alpha \subseteq S$ for every $\alpha \in I$, one has $\bigcup_{\alpha\in I} U_\alpha \subseteq S$. Therefore, the open cover condition in this case reduces to $S = \bigcup_{\alpha\in I} U_\alpha$.

With the subspace topology, a subset A of a topological space S is a topological space by its own right. The subspace A can be covered by **open sets in A** or **by open sets in S**.

- An **open cover of A in S** is a collection $\{U_\alpha\}_\alpha$ of open sets in S that covers A. In other words, $A \subseteq \bigcup_\alpha U_\alpha$ (Note that in this case $A = \bigcup_\alpha U_\alpha$ might not hold in general).
- An **open cover of A in A** is a collection $\{U_\alpha\}_\alpha$ of open sets in A in subsapce topology that covers A. In other words, $A \subseteq \bigcup_\alpha U_\alpha$ (Here, in fact, $A = \bigcup_\alpha U_\alpha$ as each $U_\alpha \subseteq A$).

Definition 0.7.2 (Compact Set). Let S be a topological space and $A \subseteq S$. A is **compact** if and only if every open cover of A in A has finite subcover.

Proposition 0.7.1

A subspace A of a topological space S is **compact** if and only if every **open cover of A in S** has a finite subcover.

Proof. ($\Rightarrow$) Assume A is compact and let $\{U_\alpha\}$ be an open cover of A in S. This means that $A \subseteq \bigcup\limits_\alpha U_\alpha$. Hence,

$$A \subseteq \left(\bigcup_\alpha U_\alpha\right) \bigcap A = \bigcup_\alpha \left(U_\alpha \bigcap A\right)$$

Now, $\{U_\alpha \cap A\}_\alpha$ is an open cover of A in A. Since A is compact, every open cover of A in A has a finite subcover. Let the finite sub-cover be $\{U_{\alpha_i} \cap A\}_{i=1}^n$. Thus,

$$A \subseteq \bigcup_{i=1}^n U_{\alpha_i}$$

which means that $\{U_{\alpha_i}\}_{i=1}^n$ is a finite sub-cover of the open cover $\{U_\alpha\}_\alpha$ of A in S.

($\Leftarrow$) Suppose every open cover of A in S has a finite subcover, and let $\{V_\alpha\}_\alpha$ be an open cover of A in A. Then each V_α is an open set of A in subspace topology. According to the definition of subspace topology, there is an open set U_α in S such that $V_\alpha = U_\alpha \cap A$. Now,

$$A \subseteq \bigcup_\alpha V_\alpha = \bigcup_\alpha (U_\alpha \cap A) = \left(\bigcup_\alpha U_\alpha\right) \cap A \subseteq \bigcup_\alpha U_\alpha$$

Therefore, $\{U_\alpha\}_\alpha$ is an open cover of A in S. By hypothesis, there are finitely many sets $\{U_{\alpha_i}\}_{i=1}^n$ such that $A \subseteq \bigcup\limits_{i=1}^n U_{\alpha_i}$. Hence,

$$A \subseteq \left(\bigcup_{i=1}^n U_{\alpha_i}\right) \cap A = \bigcup_{i=1}^n (U_{\alpha_i} \cap A) = \bigcup_{i=1}^n V_{\alpha_i}$$

So $\{V_{\alpha_i}\}_{i=1}^n$ is a finite subcover of $\{V_\alpha\}$ that covers A in A. Therefore, A is compact. ■

Proposition 0.7.2

Every compact subset of K of a Hausdorff space S is closed.

Proof. We shall prove that $S \setminus K$ is open. Let's take $x \in S \setminus K$. We claim that there is a neighborhood U_x of x that is disjoint from K.

Since S is hausdorff, for each $y \in K$, we can choose disjoint open sets U_y and V_y such that $U_y \ni x$ and $V_y \ni y$. The collection $\{V_y \ : \ y \in K\}$ is an open cover of K in S. Since K is compact, there exists a finite subcover $\{V_{y_i}\}_{i=1}^n$. That is $K \subseteq \bigcup_{i=1}^n V_{y_i}$. Since $U_{y_i} \cap V_{y_i}$ for every i, we have

$$\left(\bigcap_{i=1}^n U_{y_i}\right) \cap \left(\bigcup_{i=1}^n V_{y_i}\right) = \varnothing \implies U_x \cap K = \varnothing \text{ where } U_x = \bigcap_{i=1}^n U_{y_i}$$

U_x is the finite intersection of open sets, hence open. Also, every U_{y_i} contains x, hence their intersection U_x also contains x. So U_x is the desired open set that is disjoint from K, in other words $x \in U_x \subseteq S \setminus K$. As a result,

$$S \setminus K \subseteq \bigcup_{x \in S \setminus K} U_x \subseteq S \setminus K \implies S \setminus K = \bigcup_{x \in S \setminus K}$$

$S \setminus K$ is the union of open sets, hence open. Therefore K is closed. ■

Proposition 0.7.3

The image of a compact set under a continuous map is compact.

Proof. Let $f : X \to Y$ be a continuous and K a compact subset of X. Suppose $\{U_\alpha\}$ is an open cover of $f(K)$ by open subsets of Y. Since, f is continuous, the inverse images of $f^{-1}(U_\alpha)$ are all open in X. Moreover,

$$K \subseteq f^{-1}(f(K)) \subseteq f^{-1}\left(\bigcup_\alpha U_\alpha\right) = \bigcup_\alpha f^{-1}(U_\alpha)$$

So $\left\{f^{-1}(U)_\alpha\right\}$ is an open cover of K in X. By Proposition 0.7.1, there is a finite sub-collection $\left\{f^{-1}(U_{\alpha_i})\right\}_{i=1}^n$ such that

$$K \subseteq \bigcup_{i=1}^n f^{-1}(U_{\alpha_i}) = f^{-1}\left(\bigcup_{i=1}^n U_{\alpha_i}\right) \implies f(K) \subseteq f\left(f^{-1}\left(\bigcup_{i=1}^n U_{\alpha_i}\right)\right) \subseteq \bigcup_{i=1}^n U_{\alpha_i}$$

Thus $f(K)$ is compact. ■

Lemma 0.7.4

A closed subset F of a compact topological space S is compact.

Proof. Let $\{U_\alpha\}_\alpha$ be an open cover of F in S. The collection $\{U_\alpha, S \setminus F\}$ is an open cover of S itself. By compactness of S, there is a finite sub-cover $\{U_{\alpha_i}, S \setminus F\}_{i=1}^n$ of S, that is,

$$F \subseteq S \subseteq \left(\bigcup_{i=1}^n U_{\alpha_i}\right) \cup (S \setminus F) \implies F \subseteq \bigcup_{i=1}^n U_{\alpha_i}$$

Therefore, $\{U_{\alpha_i}\}_{i=1}^n$ is a finite subcover of the open cover $\{U_\alpha\}$ of F in S. Hence, F is also compact. ■

Proposition 0.7.5

A continuous map $f : X \to Y$ form a compact space X to a Hausdorff space Y is a closed map (a map that takes closed sets to closed sets).

Proof. Let $F \subseteq X$ be closed. Then F is compact by Lemma 0.7.4. Since $f : X \to Y$ is a continuous map, by Proposition 0.7.3, $f(F)$ is compact in Y. Since Y is Hausdorff, by Proposition 0.7.2, $f(F)$ is closed in Y. Hence, f is a closed map. ■

Corollary 0.7.6

A continuous bijection $f : X \to Y$ from a compact space X to a Hausdorff space is a homeomorphism.

Proof. We want to show thaat $f^{-1} : Y \to X$ is continuous. And in order to that it suffices to show that for every closed set F in X, $\left(f^{-1}\right)^{-1}(F) = f(F)$ is closed in Y. In other words, it suffices to show that f is a closed map. The corollary then follows from Proposition 0.7.5. ■

Definition 0.7.3 (Bounded Set)**.** A subset A of $\mathbb{R}^n$ is said to be bounded if it is contained in some open ball $B(p, r)$. otherwise, it is unbounded.

Theorem 0.7.7 (Heine-Borel Theorem)

A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

Definition 0.7.4 (Diameter of Set)**.** Let $A \subseteq X$ be a bounded subset of a metric space (X, d). The diameter of A is defined by

$$\operatorname{diam}(A) := \sup \{d(a_1, a_2) \ : \ a_1, a_2 \in A\}$$

Lemma 0.7.8 (Lebesgue Number Lemma)

Let (X, d) be a compact metric space. Given an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in J}$ of X, there exists a number $\delta > 0$ — called the Lebesgue number associated with the cover — such that for a given $A \subseteq X$ with $\operatorname{diam}(A) < \delta$, one must have $A \subseteq U_\alpha$ for some $\alpha \in J$.

Proof. Take $x \in X$. As $\mathcal{U}$ covers X, we can find $U_\alpha \in \mathcal{U}$ such that $x \in U_\alpha$. Since U_α is open and $x \in U_\alpha$, there exists $r_x > 0$ such that

$$B(x, r_x) \subseteq U_\alpha$$

We do this for every $x \in X$. So we get an open cover of X

$$X = \bigcup_{x \in X} B\left(x, \frac{r_x}{2}\right)$$

Since X is compact, there exists a finite subcover of this open cover. So

$$X = \bigcup_{i=1}^{n} B\left(x_i, \frac{r_{x_i}}{2}\right)$$

We define $\delta > 0$ in the following way:

$$\delta = \min\left\{\frac{r_{x_i}}{2} \; : \; i = 1, 2, \ldots, n\right\}$$

We claim that this δ is our desired Lebesgue number of the open cover $\mathcal{U}$. Let $A \subseteq X$ with $\operatorname{diam}(A) < \delta$. Fix $a \in A$. Then there exists $j \in \{1, 2, \ldots, n\}$ such that

$$a \in B\left(x_j, \frac{r_{x_j}}{2}\right) \implies \boxed{d\left(x_j, a\right) < \frac{r_{x_j}}{2}}$$

By the construction of r_{x_j}, there exists $U_\beta \in \mathcal{U}$ such that $B\left(x_j, r_{x_j}\right) \subseteq U_\beta$. We claim that $A \subseteq U_\beta$. Take any $b \in A$.

$$d\left(a, b\right) \leq \operatorname{diam}(A) < \delta \leq \frac{r_{x_j}}{2} \implies \boxed{d\left(a, b\right) < \frac{r_{x_j}}{2}}$$

$$d\left(x_j, b\right) \leq d\left(x_j, a\right) + d\left(a, b\right) < \frac{r_{x_j}}{2} + \frac{r_{x_j}}{2} = r_{x_j} \implies b \in B\left(x_j, r_{x_j}\right)$$

For every $b \in A$, we have $b \in B\left(x_j, r_{x_j}\right)$. Therefore, $A \subseteq B\left(x_j, r_{x_j}\right) \subseteq U_\beta$. ■

§0.8 Quotient Topology Continued

Let I be the closed interval $[0, 1]$ in the standard topology of $\mathbb{R}^n$ and $I/\sim$ be the quotient space obtained from I by identifying the 2 points $\{0, 1\}$ to a point. Denote by S^1 the unit circle in the complex plane. Define f by $f(x) = e^{2\pi i x}$.

Now the function $f : I \to S^1$ defined above assumes the same value at 0 and 1 and based on the discussion prior to Proposition 0.6.1, f induces the map $\bar{f} : I/\sim \to S^1$.

Proposition 0.8.1

The function $\bar{f} : I/\sim \to S^1$ is a homeomorphism.

Proof. The function $f : I \to S^1$ defined by $f(x) = e^{2\pi i x}$ is continuous (check!). Therefore, by Proposition 0.6.1, $\bar{f} : I \to S^1$ is also continuous.

Note that $I = [0, 1]$ in $\mathbb{R}$ is closed and bounded and hence by Heine-Borel Theorem, I is compact. Since the projection $\pi : I \to I/\sim$ is continuous, by Proposition 0.7.3, the image of I under π, *i.e.*, $I/\sim$ is compact.

It should also be obvious that $\bar{f} : I/\sim \to S^1$ is a bijection. Since S^1 is a of the Hausdorff space $\mathbb{R}^2$, by Lemma 0.4.2, S^1 is also Hausdorff. Hence, $\bar{f}$ is a continuous bijection from the compact space $I/\sim$ to the Hausdorff topological space S^1. Therefore, by Corollary 0.7.6, $\bar{f} : I/\sim \to S^1$ is a homeomorphism. ■

Necessary Condition for a Hausdorff quotient

Even if S is a Hausdorff space, the quotient space $S/\sim$ may fail to be Hausdorff.

Proposition 0.8.2

If the quotient space $S/\sim$ is Hausdorff, then the equivalence class $[p]$ of any point p in S is closed in S.

Proof. By Proposition 0.4.1, every singleton set is closed in a Hausdorff topological space. Now, consider the canonical projection map $\pi : S \to S/\!\sim$. For a point $p \in S$, $\{\pi(p)\}$ is a singleton set in $S/\!\sim$.

Since, by hypothesis $S/\!\sim$ is Hausdorff, $\{\pi(p)\}$ must be closed in $S/\!\sim$ with respect to quotient topology. By continuity of π, $\pi^{-1}(\{\pi(p)\})$ is closed in S. But $\pi^{-1}(\{\pi(p)\}) = [p]$. Hence, $[p]$ is a closed set in S. ■

Remark. In order to prove that a quotient space $S/\!\sim$ is not Hausdorff it is sufficient to prove that the equivalence class $[p]$ of some point $p \in S$ is not closed in S. We have the following example to elucidate this remark.

Example 0.8.1

Define an equivalence relation $\sim$ on $\mathbb{R}$ by identifying the open interval $(0, \infty)$ to a point. The resulting quotient space $\mathbb{R}/\!\sim$ is not Hausdorff since the equivalence class $(0, \infty)$ is not a closed subset of $\mathbb{R}$.

§0.9 Open Equivalence Relations

Definition 0.9.1. An equivalence relation $\sim$ on a topological space S is said to be open if the underlying projection map $\pi : S \to S/\!\sim$ is open (maps open sets to open sets).

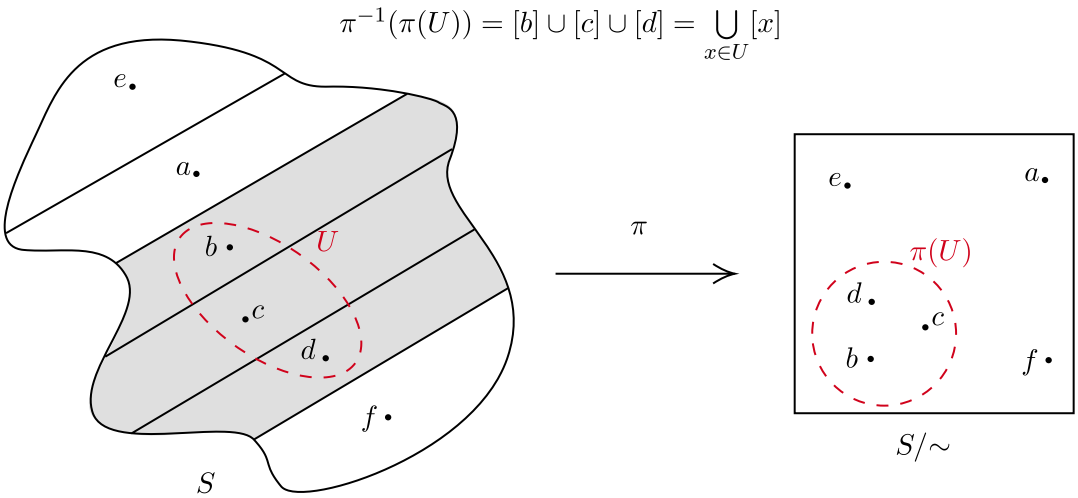


Figure 3: Indeed $\pi^{-1}(\pi(U)) = \bigcup_{x \in U} [x]$

In other words, the equivalence relation $\sim$ on S is open if and only if for every open set $U \in S$, the set $\pi(U) \in S/\!\sim$ is open. Or equivalently, by definition of quotient topology,

$$\pi^{-1}(\pi(U)) = \bigcup_{x \in U} [x] \quad \text{is open in } S$$

$\bigcup_{x \in U} [x]$ denotes all points equivalent to some point of U (shaded region in Figure 3).

Example 0.9.1

The projection map onto a quotient space is, in general, not open. For example, let $\sim$ be the equivalence relation on the real line $\mathbb{R}$ that identifies the two points 1 and -1, and $\pi : \mathbb{R} \to \mathbb{R}/\!\sim$ the projection map.

The map π is open if and only if for every open set V in $\mathbb{R}$, its image $\pi(V)$ is open in $\mathbb{R}/\!\sim$, or equivalently $\pi^{-1}(\pi(V))$ is open in $\mathbb{R}$. Let V be the open interval $(-2,0)$ in $\mathbb{R}$. Then,

$$\pi^{-1}(\pi(V)) = (-2,0) \cup \{1\}\ , \qquad [\text{Since } \pi(1) \in \pi(V)]$$

which is not open in $\mathbb{R}$ and hence π is not an open map. In other words, the equivalence relation $\sim$ is not open.

Definition 0.9.2 (Graph of Equivalence Relation)**.** Given an equivalence relation$\sim$ on S, let R be the subset of $S \times S$ that defines the relation $R = \{(x,y) \in S \times S \mid x \sim y\}$. We call R the **graph** of the equivalence relation $\sim$.

We have a necessary and sufficient condition for a quotient space to be Hausdorff if the underlying equivalence relation is an open equivalence relation. We state this condition by means of a theorem (the proof is omitted).

Theorem 0.9.1

Suppose $\sim$ is an open equivalence relation on a topological space S. Then the quotient space $S/\!\sim$ is Hausdorff if and only if the graph R of $\sim$ is closed in $S \times S$.

If the equivalence relation $\sim$ is equality, *i.e.*, $x \sim y$ iff $x = y$, then the quotient space $S/\!\sim$ is S itself and the graph R of $\sim$ is simply the diagonal $\Delta = \{(x,x) \in S \times S\}$.

Corollary 0.9.2

A topological space is Hausdorff if and only if the diagonal Δ is closed in $S \times S$.

Theorem 0.9.3

Let $\sim$ be an open equivalence relation on a topological space S with projection $\pi : S \to S/\!\sim$. If $\mathcal{B} = \{B_\alpha\}$ is a basis for S, then its image $\{\pi(B_\alpha)\}$ under π is a basis for $S/\!\sim$.

Proof. Since π is open, $\{\pi(B_\alpha)\}$ is a collection of open sets in $S/\!\sim$. Let W be an open set in $S/\!\sim$ and $[x] \in W$ with $x \in S$. So $\pi(x) \in W$, *i.e.*, $x \in \pi^{-1}(W)$. Since $\pi^{-1}(W)$ is open in S, there is a basic open set $B \in \mathcal{B}$ such that, $x \in B \subseteq \pi^{-1}(W)$. Hence

$$[x] = \pi(x) \in \pi(B) \subseteq \pi(\pi^{-1}(W)) \subseteq W$$

Now, we have seen that given W open in $S/\!\sim$ and $[x] \in W$, there exists an open set $\pi(B)$ in the collection $\{\pi(B_\alpha)\}$ such that $[x] \in \pi(B) \subseteq W$. This proves that $\{\pi(B_\alpha)\}$ is a basis for $S/\!\sim$. ■

Corollary 0.9.4

If $\sim$ is an open equivalence relation on a second-countable topological space, then the quotient space $S/\!\sim$ is second countable.

§0.10 An Alternate Approach to Quotient Topology

Instead of considering the projection map $\pi : X \to X/\!\sim$ under some equivalence relation $\sim$ defined on the given topological space X, we shall consider a surjective map $p : X \to Y$ between two topological spaces X and Y.

Definition 0.10.1 (Quotient Map)**.** Let X and Y be topological spaces, and $p : X \to Y$ be surjective. Then p is said to be a **quotient map** provided $U \subseteq Y$ is open if and only if $p^{-1}(U) \subseteq X$ is open.

Another equivalent way to define quotient map would be using closed sets instead of open sets. In other words, $F \subseteq Y$ is closed if and only if $p^{-1}(F) \subseteq X$ is closed. The equivalence follows directly from the following equation:

$$p^{-1}(Y \setminus F) = X \setminus p^{-1}(F)$$

Lemma 0.10.1

If $p : X \to Y$ is a surjective continuous map, and p is either open or closed, then p is a quotient map.

Proof. Suppose p is an open map. Continuity of p gives us $p^{-1}(U)$ is open in X, if U is open in Y. Now for the other direction, suppose $p^{-1}(U)$ is open in X. p is an open map, so $p\left(p^{-1}(U)\right)$ is open in Y. As p is surjective, $p\left(p^{-1}(U)\right) = U$. So U is open in Y.

Now suppose p is a closed map. Similar as before, continuity of p gives one direction. For the other direction, suppose $p^{-1}(U)$ is open in X. Then $X \setminus p^{-1}(U)$ is closed in X. As p is a closed map, the image of closed sets are closed. Therfore,

$$p\left(X \setminus p^{-1}(U)\right) = Y \setminus p\left(p^{-1}(U)\right) = Y \setminus U$$

is closed in Y. So U is open in Y.

Henceforth, p is a quotient map. ■

However, the converse is not necessarily true. There are quotient maps that are neither open nor closed.

Example 0.10.1

Let A be a subset of $\mathbb{R}^2$ defined by

$$A = \left\{(x, y) \in \mathbb{R}^2 \ : \ x \geq 0 \text{ or } y = 0 \text{ (or both)}\right\}$$

Let $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ be the projection on the first coordinate. Let $q : A \to \mathbb{R}$ be the restriction of π_1, *i.e.* $q = \pi_1\big|_A$. Then q is a quotient map. But it is neither open, nor closed.

Definition 0.10.2 (Quotient Topology)**.** Let X be a topological space, and A be a set. If $p : X \to A$ is a surjective map, then there exists exactly one topology τ on A, relative to which p is a quotient map. It is called the **quotient topology** induced by p.

τ consists of the subsets U of A such that $p^{-1}(U)$ is open in X. It's easy to check that τ is a topology. τ contains $\varnothing$ and A, because $p^{-1}(\varnothing) = \varnothing$ and $p^{-1}(A) = X$. The other two conditions are also satisfied, because

$$p^{-1}\left(\bigcup_{\alpha \in J} U_\alpha\right) = \bigcup_{\alpha \in J} p^{-1}(U_\alpha) \quad \text{and} \quad p^{-1}\left(\bigcap_{i=1}^{n} U_i\right) = \bigcap_{i=1}^{n} p^{-1}(U_i)$$

Definition 0.10.3 (Quotient Space)**.** Let X be a topological space and $X*$ be a partition of X into disjoint subsets. Let $p : X \to X*$ be the surjective map, that carries each point of X to the element of $X*$ containing it. In the quotient topology induced by p, $X*$ is called a **quotient space** of X.

It is often said that there is an equivalence relation $\sim$ defined on X, by means of which elements of $X*$ are said to be the equivalence classes. That is the reason we denoted the quotient space by $X/\sim$ previously.

This broader definition quotient map p is a surjective map betweem topologiacal spaces. This is not necessairly the canonical projection map $\pi : X \to X/\sim$. But essentially it turns our that if $p : X \to Y$ is a quotient map, then Y is homeomorphic to the quotient space $X/\sim$, provided that p is constant on each equivalence classes of $\sim$.

Lemma 0.10.2

Composition of quotient maps is also a quotient map.

Proof. Let $p : X \to Y$ and $q : Y \to Z$ be quotient maps. We need to show that $q \circ p : X \to Z$ is also a quotient map.

Let $U \subseteq Z$. As q is a quotient map, U is open if and only if $q^{-1}(U)$ is open in Y. Since p is a quotient map, $q^{-1}(U)$ is open in Y if and only if $p^{-1}\left(q^{-1}(U)\right)$ is open in X. Therefore, U is open in Z if and only if $p^{-1}\left(q^{-1}(U)\right) = (q \circ p)^{-1}(U)$ is open in X. Hence $q \circ p$ is a quotient map. ∎

Theorem 0.10.3

Let $p : X \to Y$ be a quotient map. Let Z be a topological space, and $g : X \to Z$ be a map that is constant on each set $p^{-1}(\{y\})$ for every $y \in Y$. Then g induces a map $f : Y \to Z$ such that $f \circ p = g$. The induced map f is continuous if and only if g is continuous; f is a quotient map if and only if g is a quotient map.

Proof. For each $y \in Y$, the set $g\left(p^{-1}(\{y\})\right)$ is a singleton. If we let $f(y)$ denote this point, then we have defined a map $f : Y \to Z$; and for every $x \in X$, $f(p(x)) = g(x)$, so $f \circ p = g$.

$$\begin{tikzcd}
X \arrow[d, "p"'] \arrow[dr, "g"] & \\
Y \arrow[r, "f"'] & Z
\end{tikzcd}$$

If f is continuous, then g is a composition of continuous map, hence continuous. Now suppose g is continuous. So for V open in Z, $g^{-1}(V)$ is open in X. As $f \circ p = g$, $g^{-1}(V) = p^{-1}\left(f^{-1}(V)\right)$ is open in X. Since p is a quotient map, $p^{-1}(U)$ open in X implies U is open in Y. Therefore, $f^{-1}(V)$ is open in Y. Hence f is continuous.

If f is a quotient map, then g is a composition of quotient map. So g is also a quotient map by Lemma 0.10.2.

Now suppose g is a quotient map. Then g is surjective. $g = f \circ p$ gives us f is also surjective. Since g is continuous (because it's a quotient map), f is also continuous. So V is open in Z gives us $f^{-1}(V)$ is open in Y. For the other direction, assume $f^{-1}(V)$ is open in Y. As p is a quotient map, $f^{-1}(V)$ is open in Y implies $p^{-1}\left(f^{-1}(V)\right)$ is open in X. This is the same as $g^{-1}(V)$. g is a quotient map, so $g^{-1}(V)$ is open in X gives us V is open in Z.

Assuming $f^{-1}(V)$ is open in Y we proved that V is open in Z. So we've proven that V is open in Z if and only if $f^{-1}(V)$ is open in Y. Henceforth, f is a quotient map. ∎

Proposition 0.10.4

Let $p : X \to Y$ be a continuous map. If p has a continuous right inverse, *i.e.* there exists a continuous map $f : Y \to X$ such that $p \circ f = \mathrm{id}_Y$, then p is a quotient map.

Proof. $p(f(y)) = y$ for every $y \in Y$. So p is a surjective map from X to Y. By the continuity of p, $p^{-1}(U)$ is open in X if U is open in Y.

For the converse, suppose $p^{-1}(U)$ is open in X. By the continuity of f, $f^{-1}\left(p^{-1}(U)\right)$ is open in Y. As $p \circ f = \mathrm{id}_Y$, $f^{-1} \circ p^{-1} = (p \circ f)^{-1} = \mathrm{id}_Y^{-1} = \mathrm{id}_Y$. So $f^{-1}\left(p^{-1}(U)\right) = \mathrm{id}_Y(U) = U$ is open in Y.

So we've proven that U is open in Y if and only if $p^{-1}(U)$ is open in X. Henceforth, p is a quotient map. ■

Lemma 0.10.5

The map $q : A \to \mathbb{R}$ in Example 0.10.1 is a quotient map, which is neither open nor closed.

Proof. Consider the map $f : \mathbb{R} \to \mathbb{R}^2$ defined by $f(x) = (x, 0)$. We claim that this is continuous. Consider a basis element $U \times V$ of $\mathbb{R}^2$, where both U and V are open in $\mathbb{R}$. Any open set in $\mathbb{R}^2$ is composed of such bases. So it's enough for us to prove that $f^{-1}(U \times V)$ is open in $\mathbb{R}$ for such U and V.

If $0 \notin V$, then $f^{-1}(U \times V) = \varnothing$ is open in $\mathbb{R}$. If $0 \in V$, $f^{-1}(U \times V) = U$ is open in $\mathbb{R}$. So f is continuous.

The range of f is a subset of A, so we can think of f as a map from $\mathbb{R}$ to A. The topology on A is the subspace topology inherited from $\mathbb{R}^2$, so $f : \mathbb{R} \to A$ is continuous.

π_1 is continuous. Because for open $W \subseteq R$, $\pi_1^{-1}(W) = W \times \mathbb{R}$, which is open in $\mathbb{R}^2$. As a restriction of a continuous map, q is also continuous.

Now, f is the right inverse of q. Because, for $x \in \mathbb{R}$, $q(f(x)) = q((x, 0)) = x = \mathrm{id}_{\mathbb{R}}(x)$. We've shown that both q and f are continuous, so by Proposition 0.10.4, q is a quotient map. Now we need to show that it's neither open nor closed.

$U = (-1, 1) \times (0, 1)$ is open in $\mathbb{R}^2$. Therefore, $V = A \cap U = [0, 1) \times (0, 1)$ is open in A. But $q(V) = [0, 1)$ is not open in $\mathbb{R}$. Because, no matter how small a neighborhood of 0 we choose, it will always contain some negative part. So the neighborhood will not be contained in $[0, 1)$. Hence, q is not an open map.

To show that q is not a closed map, take the set $X = \left\{\left(\frac{1}{n}, n\right) : n \in \mathbb{N}\right\}$. X is closed in $\mathbb{R}^2$, so $X \cap A = X$ is closed in A. Then $q(X) = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} = Y$ is not closed. Because $\mathbb{R} \setminus Y$ contains 0, but no neighborhood of 0 is contained in $\mathbb{R} \setminus Y$. No matter how small a neighborhood we choose, it will always contain some $\frac{1}{n}$. Therefore, $\mathbb{R} \setminus Y$ is not open, consequently Y is not closed, and thus q is not a closed map. ■

Proposition 0.10.6

Let $f : X \to Y$ be continuous surjective, where X is compact and Y is hausdorff. Then f is a quotient map.

Proof. f is a continuous map from a compact space to a Hausdorff space. Therefore, by Proposition 0.7.5, f is a closed map. f is a continuous surjective closed map. By Lemma 0.10.1, f is a quotient map. ■

1 Lecture 1

§1.1 Homotopy

Definition 1.1.1 (Homotopy). If f and f' are continuous maps of the space X into the space Y, we say that f is **homotopic** to f' if there is a continuous map $F : X \times I \to Y$ such that

$$F(x, 0) = f(x) \quad \text{and} \quad F(x, 1) = f'(x)$$

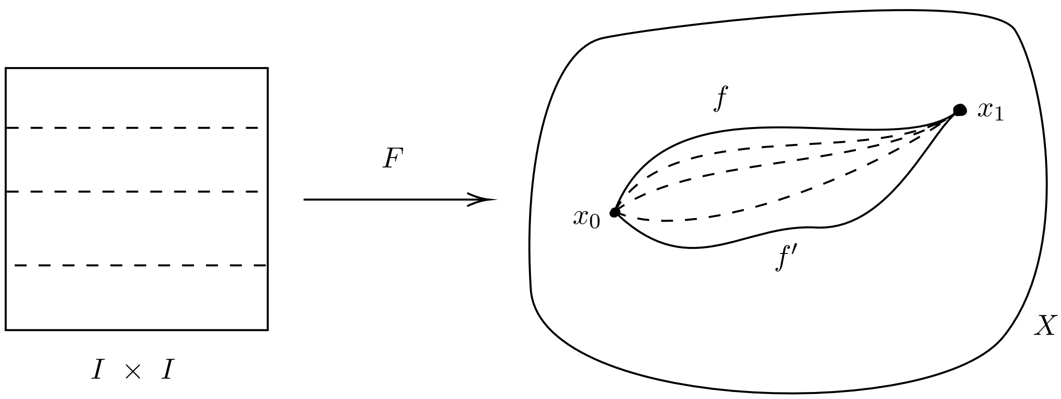
for each $x \in X$. (Here $I = [0, 1]$) The map F is called a **homotopy** between f and f'. If f is homotopic to f', we write $f \simeq f'$. If $f \simeq f'$ and f' is a constant map, we say that f is **nulhomotopic**.

We considered a continuous function $f : X \to Y$. Now we consider a special case where $f : [0, 1] \to X$ is a continuous map such that $f(0) = x_0 \in X$ and $f(1) = x_1 \in X$. We say that f is a path from x_0 (the initial point) to x_1 (the final point) in X.
If f and f' are 2 paths in X, there is a stronger relation between them than hometopy It is defined as follows:

Definition 1.1.2 (Path Homotopy). Two paths $f, f' : I \to X$, are said to be **path homotopic** if they have the same initial and final points, dented by x_0 and x_1 respectively, and if there is a continuous map $F : I \times I \to X$ such that

$$\begin{aligned} F(s, 0) &= f(s) \quad &\text{and} \quad F(s, 1) &= f'(s), \\ F(0, t) &= x_0 \quad &\text{and} \quad F(1, t) &= x_1, \end{aligned}$$

for each $s \in I$ and each $t \in I$. We call F a **path homotopy** between f and f'. If f is path homotopic to f', we write $f \simeq_p f'$.



The first condition simply says that F is a homotopy between f and f', and the second says that for each t, the path f_t defined by the equation $f_t(s) = F(s, t)$ is a path from x_0 to x_1, such that $f_0 = f$ and $f_1 = f'$.

Lemma 1.1.1
The relations $\simeq$ and $\simeq_p$ are equivalence relations.

Proof. We need to verify the three properties of an equivalence relation. Firstly, we need to check that

$f \simeq f$ for every f. That is, we need to find a homotopy $F : X \times I \to Y$ such that

$$F(x,0) = f(x) = F(x,1) \quad \forall x \in X$$

We choose $F(x,t) = f(x)$ for every $t \in I$. Continuity of F follows directly from continuity of f.
For the symmetry part, let $f \simeq f'$. We need to show that $f' \simeq f$. Let $F : X \times I \to Y$ be a homotopy between f and f'. That is,

$$F(x,0) = f(x) \quad \text{and} \quad F(x,1) = f'(x) \quad \forall x \in X$$

We need to find a homotopy G between f' and f. We choose $G : X \times I \to Y$ such that $G(x,t) = F(x, 1-t)$. Then it's immediate that

$$G(x,0) = F(x,1) = f'(x) \text{ and } G(x,1) = F(x,0) = f(x) \quad \forall x \in X$$

Continuity of G follows directly from continuity of F. Therefore, G is a homotopy between f' and f. In other words, $f' \simeq f$. The same construction of G works for path homotopy too.
Now for the transitivity part, let $f \simeq f'$ and $f' \simeq f''$. We need to show that $f \simeq f''$. Let F be a homotopy between f and f'; and F' be a homotopy between f' and f''. That is,

$$\begin{aligned} F(x,0) = f(x) \quad &\text{and} \quad F(x,1) = f'(x) \quad \forall x \in X \\ F'(x,0) = f'(x) \quad &\text{and} \quad F'(x,1) = f''(x) \quad \forall x \in X \end{aligned}$$

Now we define $G : X \times I \to Y$ as follows:

$$G(x,t) = \begin{cases} F(x,2t) & \text{for } t \in \left[0, \frac{1}{2}\right] \\ F'(x, 2t-1) & \text{for } t \in \left[\frac{1}{2}, 1\right] \end{cases}$$

Claim — G is a homotopy between f and f''.

Proof. Let $A = X \times \left[0, \frac{1}{2}\right]$ and $B = X \times \left[\frac{1}{2}, 1\right]$. So $A \cup B = X \times I$. X is both open and closed in X. The closed intervals $\left[0, \frac{1}{2}\right]$ and $\left[\frac{1}{2}, 1\right]$ are also closed. Therefore, A and B are both closed in $X \times I$. Here the underlying topology is product topology. $A \cap B = X \times \left\{\frac{1}{2}\right\}$. The functions indeed agree on $A \cap B$, because

$$F\left(x, 2 \cdot \frac{1}{2}\right) = F(x,1) = f'(x) = F'(x,0) = F'\left(x, 2 \cdot \frac{1}{2} - 1\right)$$

Therefore, by Pasting Lemma, G is continuous on $X \times I$. Now,

$$G(x,0) = F(x,0) = f(x) \text{ and } G(x,1) = F'(x,1) = f''(x) \quad \forall x \in X$$

Therefore, G is a homotopy between f and f''. □

Therefore, $\simeq$ is an equivalence relation. For path homotopy $\simeq_p$, transitivity can be checked in a similar way.

Let $f \simeq_p f'$ and F be a path homotopy between f and f'. That is,

$$\begin{aligned} F(s,0) = f(s) \text{ and } F(s,1) = f'(s)\ , \\ F(0,t) = x_0 \text{ and } F(1,t) = x_1\ \forall\ s,t \in I \end{aligned}$$

Let $f' \simeq_p f''$ and F' be a path homotopy between f' and f''. That is,

$$\begin{aligned} F'(s,0) = f'(s) \text{ and } F'(s,1) = f''(s)\ , \\ F'(0,t) = x_0 \text{ and } F'(1,t) = x_1\ \forall\ s,t \in I \end{aligned}$$

We construct $G : I \times I \to X$ similar as before.

$$G(s,t) = \begin{cases} F(s,2t) & \text{for } t \in \left[0, \frac{1}{2}\right] \\ F'(s, 2t-1) & \text{for } t \in \left[\frac{1}{2}, 1\right] \end{cases}$$

Now we want to show that G preserves the endpoints x_0 and x_1.

$$G(0,t) = \begin{cases} F(0,2t) & \text{for } t \in \left[0, \frac{1}{2}\right] \\ F'(0,2t-1) & \text{for } t \in \left[\frac{1}{2}, 1\right] \end{cases} = x_0 \quad [F(0,t) = F'(0,t) = x_0 \text{ for all } t]$$

$$G(1,t) = \begin{cases} F(1,2t) & \text{for } t \in \left[0, \frac{1}{2}\right] \\ F'(1,2t-1) & \text{for } t \in \left[\frac{1}{2}, 1\right] \end{cases} = x_1 \quad [F(1,t) = F'(1,t) = x_1 \text{ for all } t]$$

Therefore, $\simeq_p$ is also an equivalence relation. ∎

Example 1.1.1

Let f and g be any two continuous maps from a space X to $\mathbb{R}^2$. It's easy to see that $f \simeq g$. The map

$$F(x,t) = (1-t)f(x) + tg(x)$$

is a homotopy between them. It is called a **straight-line homotopy** because it moves the point $f(x)$ to the point $g(x)$ along the straight-line segment joining them.

If f is a path, we shall denote its path-homotopy equivalence class by $[f]$.

Abuse of Notation. When we say f is a path in X from x_0 to x_1, we generally mean that $f : I \to X$ is the path and $f(0) = x_0$, $f(1) = x_1$.

Definition 1.1.3 (Product of Path)**.** If f is a path in X from x_0 to x_1, and if g is a path in X from x_1 to x_2, we define the product $f * g$ of f and g to be the path h given by the equations

$$h(s) = \begin{cases} f(2s) & \text{for } s \in \left[0, \frac{1}{2}\right] \\ g(2s-1) & \text{for } s \in \left[\frac{1}{2}, 1\right] \end{cases}$$

The function h is well-defined and continuous, by Pasting Lemma. It is a path in X from x_0 to x_2, since $h(0) = f(0) = x_0$ and $h(1) = g(1) = x_2$. We think of h as the path whose first half is the path f and whose second half is the path g. The product operation on paths induces a well-defined operation on path-homotopy classes, defined by the equation $[f] * [g] = [f * g]$.

Definition 1.1.4. Let $[f]$ and $[g]$ be path classes such that $f(1) = g(0)$. Then we define the operation $*$ between path classes as follows:

$$[f] * [g] = [f * g]$$

Proposition 1.1.2

The operation $[f] * [g] = [f * g]$ is well defined.

Proof. For the operation to make sense, let f and f' be homotopic paths in X from x_0 to x_1; g and g' be homotopic paths in X from x_1 to x_2. Let F be a path homotopy between f and f' and let G be a path homotopy between g and g'. We need to prove that $f * g \simeq_p f' * g'$. Define $H : I \times I \to X$ by

$$H(s,t) = \begin{cases} F(2s,t) & \text{for } s \in \left[0, \frac{1}{2}\right] \\ G(2s-1,t) & \text{for } s \in \left[\frac{1}{2}, 1\right] \end{cases}$$

Claim — H is a path homotopy between $f * g$ and $f' * g'$.

Proof. F and G are path homotopies.

$f \simeq_p f'$ and F is a path homotopy between f and f'. That is,

$$F(s,0) = f(s) \text{ and } F(s,1) = f'(s) \ ,$$
$$F(0,t) = x_0 \text{ and } F(1,t) = x_1 \ \forall \ s,t \in I$$

$g \simeq_p g'$ and G be a path homotopy between g and g'. That is,

$$G(s,0) = g(s) \text{ and } G(s,1) = g'(s) \ ,$$
$$G(0,t) = x_1 \text{ and } G(1,t) = x_2 \ \forall \ s,t \in I$$

Now, from the definition of product,

$$\begin{aligned}
(f * g)(s) &= \begin{cases} f(2s) & \text{for } s \in \left[0, \frac{1}{2}\right] \\ g(2s-1) & \text{for } s \in \left[\frac{1}{2}, 1\right] \end{cases} \\
&= \begin{cases} F(2s,0) & \text{for } s \in \left[0, \frac{1}{2}\right] \\ G(2s-1,0) & \text{for } s \in \left[\frac{1}{2}, 1\right] \end{cases} = H(s,0) \\
(f' * g')(s) &= \begin{cases} f'(2s) & \text{for } s \in \left[0, \frac{1}{2}\right] \\ g'(2s-1) & \text{for } s \in \left[\frac{1}{2}, 1\right] \end{cases} \\
&= \begin{cases} F(2s,1) & \text{for } s \in \left[0, \frac{1}{2}\right] \\ G(2s-1,1) & \text{for } s \in \left[\frac{1}{2}, 1\right] \end{cases} = H(s,1)
\end{aligned}$$

So the first condition of path homotopy is satisfied. The second condition is also satisfied, since

$$H(0,t) = F(0,t) = x_0 \quad \text{and} \quad H(1,t) = G(1,t) = x_2$$

Hence H is a path homotopy between $f * g$ and $f' * g'$. □

Therefore, $f * g \simeq_p f' * g'$ and the product operation is well defined. ■

Theorem 1.1.3

The operation $*$ has the following properties:

(i) (Associativity) If $[f] * ([g] * [h])$ is defined, so is $([f] * [g]) * [h]$, and they are equal.

(ii) (Right and left identities) Given $x \in X$, let e_x denote the constant path $e_x : I \to X$ carrying all of I to the point x. If f is a path in X from x_0 to x_1, then

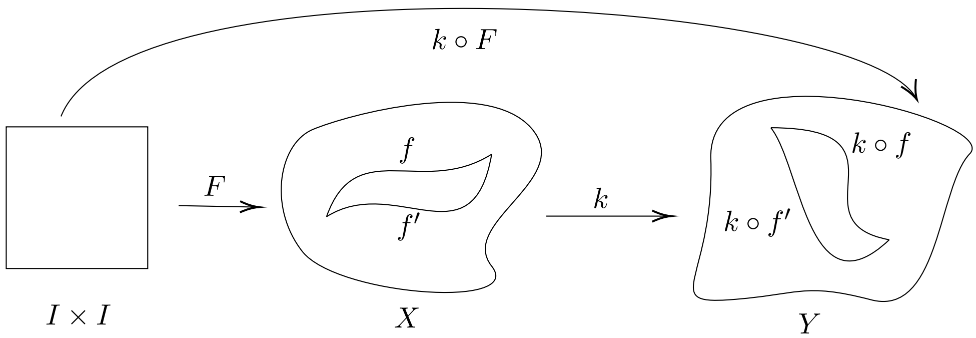
$$[f] * [e_{x_1}] = [f] \quad \text{and} \quad [e_{x_0}] * [f] = [f]$$

(iii) (Inverse) Given the path f in X from x_0 to x_1, let $\bar{f}$ be the path defined by $\bar{f}(s) = f(1-s)$. It is called the reverse of f. Then

$$[f] * [\bar{f}] = [e_{x_0}] \quad \text{and} \quad [\bar{f}] * [f] = [e_{x_1}]$$

Proof. The proof is based on two key facts.

1. If $k : X \to Y$ is a continuous map, and if F is a path homotopy in X between the paths $f, f' : I \to X$, then $k \circ F$ is a path homotopy in Y between the paths $k \circ f, k \circ f' : I \to Y$. See the figure below:



Given F is a path homotopy between f and f', we have

$$\begin{array}{lcl} F(s,0)=f(s) & \text{and} & F(s,1)=f'(s), \\ F(0,t)=x_0 & \text{and} & F(1,t)=x_1, \end{array}$$

Indeed we have

$$\begin{array}{lcl} (k\circ F)(s,0)=(k\circ f)(s) & \text{and} & (k\circ F)(s,1)=(k\circ f')(s), \\ (k\circ F)(0,t)=k(x_0) & \text{and} & (k\circ F)(1,t)=k(x_1) \end{array}$$

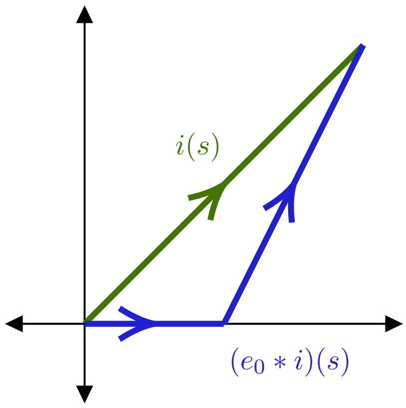
Both $k: X \to Y$ and $F: I\times I \to X$ are continuous, so is $k\circ F: I\times I\to Y$. Therefore, $k\circ F$ is a path homotopy between $k\circ f$ and $k\circ f'$.

2. The second is the fact that if $k: X\to Y$ is a continuous map, and if f and g are paths in X with $f(1)=g(0)$, then
$$k\circ(f*g)=(k\circ f)*(k\circ g)$$
Proving this fact is not hard. Indeed for continuous $f,g: I\to X$ and $x_0,x_1,x_2\in X$ with $f(0)=x_0$, $f(1)=x_1=g(0)$, $g(1)=x_2$ one finds $f*g$ making sense as a path in X with $(f*g)(0)=x_0$ and $(f*g)(1)=x_2$ defined by
$$(f*g)(s)=\begin{cases} f(2s) & \text{for } s\in\left[0,\frac{1}{2}\right] \\ g(2s-1) & \text{for } s\in\left[\frac{1}{2},1\right] \end{cases}$$
Hence, $k\circ(f*g): I\to Y$ is a path with $(k\circ(f*g))(0)=x_0$ and $(k\circ(f*g))(1)=x_2$ is given by
$$(k\circ(f*g))(s)=\begin{cases} (k\circ f)(2s) & \text{for } s\in\left[0,\frac{1}{2}\right] \\ (k\circ g)(2s-1) & \text{for } s\in\left[\frac{1}{2},1\right] \end{cases}$$
Since k is continuous, we have $k\circ f, k\circ g: I\to Y$ continuous. Also $(k\circ f)(1)=k(x_1)=(k\circ g)(0)$. So it makes sense to talk about the product of $k\circ f$ and $k\circ g$.
$$((k\circ f)*(k\circ g))(s)=\begin{cases} (k\circ f)(2s) & \text{for } s\in\left[0,\frac{1}{2}\right] \\ (k\circ g)(2s-1) & \text{for } s\in\left[\frac{1}{2},1\right] \end{cases}$$
So indeed we have $k\circ(f*g)=(k\circ f)*(k\circ g)$.

Now we proceed to prove the theorem. We shall prove the existence of identities and inverses first, then we will go on to prove associativity.
Let $e_0: I\to I$ be the constant path in I that maps all of I to 0; in other words, $e(t)=0$ for every $t\in I$. Also we denote by $i: I\to I$, the identity map on I; in other words, $i(s)=s$ for every $s\in I$. Note that, i is also a path on I from 0 to 1. Now, using the definition of product of paths,

$$(e_0*i)(s)=\begin{cases} e_0(2s) & \text{for } s\in\left[0,\frac{1}{2}\right] \\ i(2s-1) & \text{for } s\in\left[\frac{1}{2},1\right] \end{cases} = \begin{cases} 0 & \text{for } s\in\left[0,\frac{1}{2}\right] \\ 2s-1 & \text{for } s\in\left[\frac{1}{2},1\right] \end{cases}$$

We draw the graphs of the maps i and e_0*i below:



Claim — $i \simeq_p e_0 * i$

Proof. We construct a path homotopy $G : I \times I \to I$ between them as follows:

$$G(s,t) = (1-t)i(s) + t\,(e_0 * i)\,(s)$$

Indeed $G(s,0) = i(s)$ and $G(s,1) = (e_0 * i)\,(s)$. Also,

$$\begin{aligned} G(0,t) &= (1-t)i(0) + t\,(e_0 * i)\,(0) = 0 = i(0) = (e_0 * i)\,(0) \quad \text{and} \\ G(1,t) &= (1-t)i(1) + t\,(e_0 * i)\,(1) = 1 - t + t = 1 = i(1) = (e_0 * i)\,(1) \end{aligned}$$

Both i and $e_0 * i$ are continuous, therefore G is also continuous. Hence G is a path homotopy between i and $e_0 * i$. □

Now, let f be a path in X from x_0 to x_1. That is, $f : I \to X$ is a continuous map with $f(0) = x_0$ and $f(1) = x_1$. Since $i \simeq_p e_0 * i$ with G being a path homotopy between them, by the key facts stated above, $f \circ G$ is a path homotopy between $f \circ i$ and $f \circ (e_0 * i) = (f \circ e_0) * (f \circ i)$. Now, $(f \circ i)\,(s) = f\,(i(s)) = f(s)$, so $f \circ i = f$. Also,

$$(f \circ e_0)\,(s) = f\,(e_0(s)) = f(0) = x_0 \;\forall s \in I \implies f \circ e_0 = e_{x_0}$$

Therefore, $f \simeq_p e_{x_0} * f$. It makes sense to talk about $e_{x_0} * f$ because $e_{x_0}(1) = x_0 = f(0)$. So, by the well definedness of $*$,

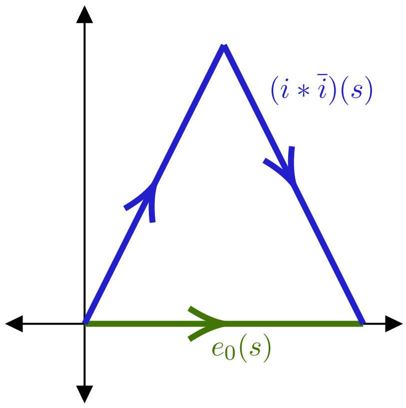
$$[e_{x_0}] * [f] = [e_{x_0} * f] = [f]$$

In a similar way, one can prove that $[f] * [e_{x_1}] = [f]$. For that we need $i \simeq_p i * e_1$, where e_1 is the constant path that sends all of I to 1.

Now we shall prove the existence of inverses. Firstly, we define the *reverse* path of i, $\bar{i} : I \to I$ by $\bar{i}(s) = 1 - s$ for every $t \in I$. Since i is a path in I from 0 to 1, $\bar{i}$ is a path in I from 1 to 0. So it makes sense to talk about $i * \bar{i}$. Using the definition of product of paths,

$$(i * \bar{i})\,(s) = \begin{cases} i(2s) & \text{for } s \in \left[0, \frac{1}{2}\right] \\ \bar{i}(2s-1) & \text{for } s \in \left[\frac{1}{2}, 1\right] \end{cases} = \begin{cases} 2s & \text{for } s \in \left[0, \frac{1}{2}\right] \\ 2 - 2s & \text{for } s \in \left[\frac{1}{2}, 1\right] \end{cases}$$

We draw the graphs of the maps e_0 and $i * \bar{i}$ below:



Claim — $i * \bar{i} \simeq_p e_0$

Proof. We construct a path homotopy $H : I \times I \to I$ between them as follows:

$$H(s,t) = (1-t)\left(i * \bar{i}\right)(s) + t\ e_0(s)$$

Indeed $H(s,0) = \left(i * \bar{i}\right)(s)$ and $H(s,1) = e_0(s)$. Also,

$$\begin{aligned} H(0,t) &= (1-t)\left(i * \bar{i}\right)(0) + t\ e_0(0) = 0 = \left(i * \bar{i}\right)(0) = e_0(0) \quad \text{and} \\ H(1,t) &= (1-t)\left(i * \bar{i}\right)(1) + t\ e_0(1) = 0 = \left(i * \bar{i}\right)(1) = e_0(1) \end{aligned}$$

Both $i * \bar{i}$ and e_0 are continuous, therefore H is also continuous. Hence H is a path homotopy between $i * \bar{i}$ and e_0. $\square$

Now, let f be a path in X from x_0 to x_1. That is, $f : I \to X$ is a continuous map with $f(0) = x_0$ and $f(1) = x_1$. Since $i * \bar{i} \simeq_p e_0$ with H being a path homotopy between them, by the key facts stated above, $f \circ H$ is a path homotopy between $f \circ e_0 = e_{x_0}$ and $f \circ (i * \bar{i}) = (f \circ i) * (f \circ \bar{i})$. Now, $(f \circ i)(s) = f(i(s)) = f(s)$, so $f \circ i = f$. Also,

$$\left(f \circ \bar{i}\right)(s) = f\left(\bar{i}(s)\right) = f(1-s) = \bar{f}(s) \implies f \circ \bar{i} = \bar{f}$$

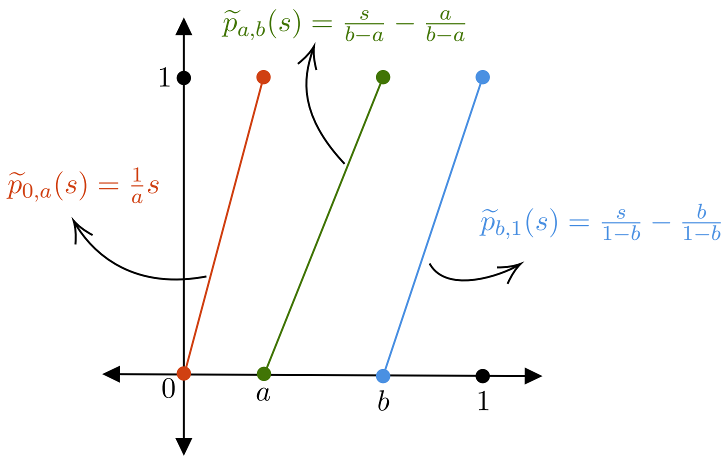
Therefore, $e_{x_0} \simeq_p f * \bar{f}$. It makes sense to talk about $f * \bar{f}$ because $f(1) = \bar{f}(0)$. So, by the well definedness of $*$,

$$[f] * \left[\bar{f}\right] = \left[f * \bar{f}\right] = [e_{x_0}]$$

In a similar way, one can prove that $[\bar{f}] * [f] = [e_{x_1}]$. For that we need $\bar{i} * i \simeq_p e_1$, where e_1 is the constant path that sends all of I to 1.

Now we shall move on to proving associativity. Let f, g, h be three paths in X such that $f(1) = g(0)$ and $g(1) = h(0)$, so that the products $(f * g) * h$ and $f * (g * h)$ are defined.

Now we define the *positive linear mapping* $\widetilde{p}_{m,n} : [m,n] \to [0,1]$ such that $\widetilde{p}_{m,n}(m) = 0$ and $\widetilde{p}_{m,n}(n) = 1$ and the map is linear, where $0 \leq m \leq n \leq 1$. Using this map, we will define the *"triple product"* of three paths f, g and h.



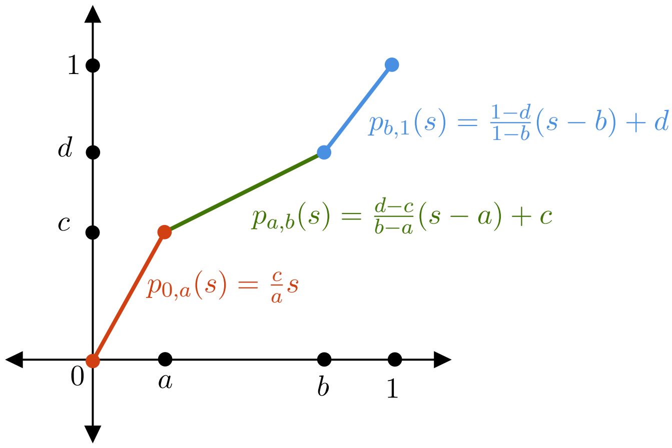
We choose a and b such that $0 < a < b < 1$. Then we define a path $K_{a,b} : I \to X$ (depending on the choice of a and b) in the following way:

$$K_{a,b}(s) = \begin{cases} (f \circ \widetilde{p}_{0,a})(s) & \text{if } s \in [0,a] \\ (g \circ \widetilde{p}_{a,b})(s) & \text{if } s \in [a,b] \\ (h \circ \widetilde{p}_{b,1})(s) & \text{if } s \in [b,1] \end{cases}$$

We wish to prove that, no matter what a, b we choose, the paths will belong to the same path homotopy class. That is $K_{a,b} \simeq_p K_{c,d}$ for all $0 < a < b < 1$ and $0 < c < d < 1$. For this, we need to define a new map $P : I \to I$ in the following way:

$$P(s) = \begin{cases} p_{0,a}(s) & \text{if } s \in [0,a] \\ p_{a,b}(s) & \text{if } s \in [a,b] \\ p_{b,1}(s) & \text{if } s \in [b,1] \end{cases}$$

where $p_{0,a} : [0,a] \to [0,c]$ is a positive linear map with $p_{0,a}(0) = 0$ and $p_{0,a}(a) = c$; $p_{a,b} : [a,b] \to [c,d]$ is a positive linear map with $p_{a,b}(a) = c$ and $p_{a,b}(b) = d$; $p_{b,1} : [b,1] \to [d,1]$ is a positive linear map with $p_{b,1}(b) = d$ and $p_{b,1}(1) = 1$.



Claim — $\widetilde{p}_{0,c} \circ p_{0,a} = \widetilde{p}_{0,a}$; $\widetilde{p}_{c,d} \circ p_{a,b} = \widetilde{p}_{a,b}$; $\widetilde{p}_{d,1} \circ p_{b,1} = \widetilde{p}_{b,1}$.

Proof. The proofs follow immediately from the equations shown in the figures.

$$\begin{aligned}
\left(\widetilde{p}_{0,c} \circ p_{0,a}\right)(s) &= \widetilde{p}_{0,c}\left(p_{0,a}(s)\right) = \frac{1}{c}\ p_{0,a}(s) = \frac{1}{c}\ \frac{c}{a}s \\
&= \frac{1}{a}s = \widetilde{p}_{0,a}(s) \\
\left(\widetilde{p}_{c,d} \circ p_{a,b}\right)(s) &= \widetilde{p}_{c,d}\left(p_{a,b}(s)\right) = \frac{1}{d-c}\ p_{a,b}(s) - \frac{c}{d-c} \\
&= \frac{1}{d-c}\left(\frac{d-c}{b-a}(s-a)+c\right) - \frac{c}{d-c} \\
&= \frac{s-a}{b-a} = \widetilde{p}_{a,b}(s) \\
\left(\widetilde{p}_{d,1} \circ p_{b,1}\right)(s) &= \widetilde{p}_{d,1}\left(p_{b,1}(s)\right) = \frac{1}{1-d}\ p_{b,1}(s) - \frac{d}{1-d} \\
&= \frac{1}{1-d}\left(\frac{1-d}{1-b}(s-b)+d\right) - \frac{d}{1-d} \\
&= \frac{s-b}{1-b} = \widetilde{p}_{b,1}(s)
\end{aligned}$$

So the claim is proved. □

Claim — $P \simeq_p i$, where $i : I \to I$ is the identity map.

Proof. We construct a path homotopy $F : I \times I \to I$ between them as follows:

$$F(s,t) = (1-t)P(s) + t\ i(s)$$

Indeed $F(s,0) = P(s)$ and $F(s,1) = i(s)$. Also,

$$\begin{aligned}
F(0,t) &= (1-t)P(0) + t\ i(0) = 0 = P(0) = i(0) \quad \text{and} \\
F(1,t) &= (1-t)P(1) + t\ i(1) = 1 = P(1) = i(1)
\end{aligned}$$

Both P and i are continuous, therefore F is also continuous. Hence F is a path homotopy between P and i. □

Now we wanna show that $K_{a,b} = K_{c,d} \circ P$. For this, let's recall the definition of $K_{c,d}$ once again.

$$K_{c,d}(s) = \begin{cases} \left(f \circ \widetilde{p}_{0,c}\right)(s) & \text{if } s \in [0,c] \\ \left(g \circ \widetilde{p}_{c,d}\right)(s) & \text{if } s \in [c,d] \\ \left(h \circ \widetilde{p}_{d,1}\right)(s) & \text{if } s \in [d,1] \end{cases}$$

Now we shall compose it with P. If $s \in [0,a]$, $P(s) = p_{0,a}(s) \in [0,c]$. That's why $P^{-1}\big([0,c]\big) = [a,b]$.

Similarly, $P^{-1}\big([c,d]\big) = [a,b]$ and $P^{-1}\big([d,1]\big) = [b,1]$. Now,

$$
\begin{aligned}
(K_{c,d} \circ P)(s) &= \begin{cases} (f \circ \widetilde{p}_{0,c} \circ P)(s) & \text{if } s \in P^{-1}\big([0,c]\big) \\ (g \circ \widetilde{p}_{c,d} \circ P)(s) & \text{if } s \in P^{-1}\big([c,d]\big) \\ (h \circ \widetilde{p}_{d,1} \circ P)(s) & \text{if } s \in P^{-1}\big([d,1]\big) \end{cases} \\
&= \begin{cases} (f \circ \widetilde{p}_{0,c} \circ P)(s) & \text{if } s \in [0,a] \\ (g \circ \widetilde{p}_{c,d} \circ P)(s) & \text{if } s \in [a,b] \\ (h \circ \widetilde{p}_{d,1} \circ P)(s) & \text{if } s \in [b,1] \end{cases} \\
&= \begin{cases} (f \circ \widetilde{p}_{0,c} \circ p_{0,a})(s) & \text{if } s \in [0,a] \\ (g \circ \widetilde{p}_{c,d} \circ p_{a,b})(s) & \text{if } s \in [a,b] \\ (h \circ \widetilde{p}_{d,1} \circ p_{b,1})(s) & \text{if } s \in [b,1] \end{cases} \\
&= \begin{cases} (f \circ \widetilde{p}_{0,a})(s) & \text{if } s \in [0,a] \\ (g \circ \widetilde{p}_{a,b})(s) & \text{if } s \in [a,b] \\ (h \circ \widetilde{p}_{b,1})(s) & \text{if } s \in [b,1] \end{cases} \\
&= K_{a,b}(s)
\end{aligned}
$$

We have shown that F is a path homotopy between P and i. So, by the first key fact stated at the beginning of the proof, $K_{c,d} \circ F$ is a path homotopy between $K_{c,d} \circ P$ and $K_{c,d} \circ i$. As a result,

$$
K_{c,d} \circ P \simeq_p K_{c,d} \circ i \implies \boxed{K_{a,b} \simeq_p K_{c,d}}
$$

Now we can go back to our original problem. The main idea is to write $f * (g * h)$ and $(f * g) * h$ as $K_{a,b}$ and $K_{c,d}$ for some $0 < a < b < 1$ and $0 < c < d < 1$.

$$
\begin{aligned}
(f * (g * h))(s) &= \begin{cases} f(2s) & \text{for } s \in \left[0, \frac{1}{2}\right] \\ (g * h)(2s-1) & \text{for } s \in \left[\frac{1}{2}, 1\right] \end{cases} \\
&= \begin{cases} f(2s) & \text{for } s \in \left[0, \frac{1}{2}\right] \\ g\left(2(2s-1)\right) & \text{for } 2s-1 \in \left[0, \frac{1}{2}\right] \\ h\left(2(2s-1)-1\right) & \text{for } 2s-1 \in \left[\frac{1}{2}, 1\right] \end{cases} \\
&= \begin{cases} f(2s) & \text{for } s \in \left[0, \frac{1}{2}\right] \\ g(4s-2) & \text{for } s \in \left[\frac{1}{2}, \frac{3}{4}\right] \\ h(4s-3) & \text{for } s \in \left[\frac{3}{4}, 1\right] \end{cases}
\end{aligned}
$$

From the equations of $\widetilde{p}_{a,b}$ maps,

$$
\begin{aligned}
\widetilde{p}_{0,\frac{1}{2}}(s) &= \frac{1}{\frac{1}{2}}(s) = 2s \\
\widetilde{p}_{\frac{1}{2},\frac{3}{4}}(s) &= \frac{s}{\frac{3}{4}-\frac{1}{2}} - \frac{\frac{1}{2}}{\frac{3}{4}-\frac{1}{2}} = \frac{s}{\frac{1}{4}} - \frac{\frac{1}{2}}{\frac{1}{4}} = 4s - 2 \\
\widetilde{p}_{\frac{3}{4},1}(s) &= \frac{s}{1-\frac{3}{4}} - \frac{\frac{3}{4}}{1-\frac{3}{4}} = \frac{s}{\frac{1}{4}} - \frac{\frac{3}{4}}{\frac{1}{4}} = 4s - 3
\end{aligned}
$$

As a result,

$$
\begin{aligned}
(f * (g * h))(s) &= \begin{cases} f\left(\widetilde{p}_{0,\frac{1}{2}}(s)\right) & \text{for } s \in \left[0, \frac{1}{2}\right] \\ g\left(\widetilde{p}_{\frac{1}{2},\frac{3}{4}}(s)\right) & \text{for } s \in \left[\frac{1}{2}, \frac{3}{4}\right] \\ h\left(\widetilde{p}_{\frac{3}{4},1}(s)\right) & \text{for } s \in \left[\frac{3}{4}, 1\right] \end{cases} \\
&= \begin{cases} \left(f \circ \widetilde{p}_{0,\frac{1}{2}}\right)(s) & \text{for } s \in \left[0, \frac{1}{2}\right] \\ \left(g \circ \widetilde{p}_{\frac{1}{2},\frac{3}{4}}\right)(s) & \text{for } s \in \left[\frac{1}{2}, \frac{3}{4}\right] \\ \left(h \circ \widetilde{p}_{\frac{3}{4},1}\right)(s) & \text{for } s \in \left[\frac{3}{4}, 1\right] \end{cases} \\
&= K_{\frac{1}{2},\frac{3}{4}}(s)
\end{aligned}
$$

So $f * (g * h) = K_{\frac{1}{2},\frac{3}{4}}$. In a similar manner, one can show that $(f * g) * h = K_{\frac{1}{4},\frac{1}{2}}$. Since $K_{a,b} \simeq_p K_{c,d}$, we get

$$
\begin{aligned}
K_{\frac{1}{2},\frac{3}{4}} \simeq_p K_{\frac{1}{4},\frac{1}{2}} &\implies f * (g * h) \simeq_p (f * g) * h \\
&\implies [f] * [g * h] = [f * g] * [h] \\
&\implies [f] * ([g] * [h]) = ([f] * [g]) * [h]
\end{aligned}
$$

Therefore, the operation $*$ on path homotopy classes is associative. ■

2 Lecture 2

§2.1 The Fundamental Group

The set of path-homotopy classes of paths in a space X does not form a group under the operation $*$ because the product of two path-homotopy classes is not always defined. But suppose we pick out a point (to be called "base point") $x_0 \in X$ and restrict ourselves to all the paths that begin and end at x_0. The set of these path-homotopy classes does form a group under $*$. It will be called the fundamental group of X relative to the base point x_0.

Definition 2.1.1 (Loop). Let X be a space and $x_0 \in X$. A path in X that begins and ends at x_0 is called a **loop** based at x_0.

A path f begins and ends at x_0 means $f : I \to X$ is a continuous map with $f(0) = f(1) = x_0$.

Definition 2.1.2 (The Fundamental Group). The set of path homotopy classes of loops based at x_0, equipped with the operation $*$ between any two path homotopy classes, is called the **fundamental group** of X **relative to the base point** x_0. It is denoted by $\pi_1(X, x_0)$.

Exercise 2.1. Convince yourself that $\pi_1(X, x_0)$ is indeed a group by using Theorem 1.1.3 from lecture 1.

Example 2.1.1

For the Euclidean space $\mathbb{R}^n$, the fundamental group $\pi_1(\mathbb{R}^n, x_0)$ is the trivial group (the group consisting of the identity alone). Recall that the identity element of $\pi_1(X, x_0)$ is $[e_{x_0}]$, the constant path in X beginning and ending at x_0. So in this example, $\pi_1(\mathbb{R}^n, x_0) = \{[e_{x_0}]\}$.

In fact, one can prove that if X is a convex subset of $\mathbb{R}^h$, then $\pi_1(X, x_0) = \{[e_{x_0}]\}$.

Lemma 2.1.1

If X is a convex subset of $\mathbb{R}^n$, then $\pi_1(X, x_0) = \{[e_{x_0}]\}$.

Proof. Let f be a loop in X based at x_0. That is, $f(0) = f(1) = x_0$. We need to show that $f \simeq_p e_{x_0}$. We define the continuous map $H : I \times I \to X$ as

$$H(s,t) = (1-t)f(s) + t\, e_{x_0}(s) = (1-t)f(t) + tx_0$$

It can be done because $X \subseteq \mathbb{R}^n$ is convex. That's why the straight line, given by the above equation, connecting $f(s) \in X$ and $x_0 \in X$ will lie in X. Here we have,

$$H(s,0) = f(s) \text{ and } H(s,1) = e_{x_0}(s)$$
$$H(0,t) = x_0 = f(0) = e_{x_0}(0) \text{ and } H(1,t) = x_0 = f(1) = e_{x_0}(1)$$

Also, H is a conntinuous map because both f and e_{x_0} are continuous. Hence H is a path homotopy between f and e_{x_0}. Therefore, for any loop f based at x_0, we have $f \in [e_{x_0}]$. So $\pi_1(X, x_0) = \{[e_{x_0}]\}$. ■

Question. To what extent the fundamental group depends on the base point?

§2.2 Path Connectedness and Simply Connectedness

Definition 2.2.1 (Path Connected Space)**.** A topological space X is **path connected** if for any $x_0, x_1 \in X$, there is a path from x_0 to x_1.

If X is a path connected space, then X is connected. But the converse is **not** true, a connected space may fail to be path connected.

Example 2.2.1

Consider "Topologist's sine curve"

$$A = \left\{(x, y) \in \mathbb{R}^2 \ : \ x > 0 \text{ and } y = \sin\left(\frac{1}{x}\right)\right\}$$

and it's closure $\overline{A} = A \cup \big(\{0\} \times [-1, 1]\big)$. Both A and $\overline{A}$ are connected, but $\overline{A}$ is not path connected.

Consider the relation $\sim$ on X by $x \sim y$ if there exists a path between x and y. Then $\sim$ is an equivalence relation and the equivalence classes are called the **path components** of X. If X is path connected, then there is only one equivalence class.

Definition 2.2.2. Let α be a path in X from x_0 to x_1. We define a map $\hat{\alpha} : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$ by the equation

$$\hat{\alpha}([f]) = [\bar{\alpha}] * [f] * [\alpha]$$

Here $\bar{\alpha}$ is the reverse of α.

Exercise 2.2. If f is a loop based at x_0, then check that $\bar{\alpha} * f * \alpha$ is a loop based at x_1.

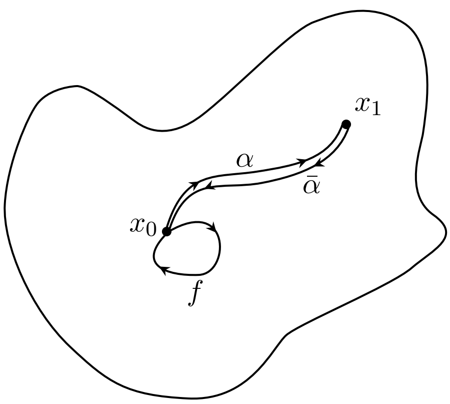


Figure 2.1: $\bar{\alpha} * f * \alpha$ is a loop based at x_1

The map $\hat{\alpha}$ is well-defined because the operation $*$ is a well defined product between path homotopy classes.

Theorem 2.2.1

The map $\hat{\alpha}$ is a group isomorphism.

Proof. Let us first show that $\hat{\alpha}$ is a group homomorphism.

$$\begin{aligned}
\hat{\alpha}([f]) * \hat{\alpha}([g]) &= ([\bar{\alpha}] * [f] * [\alpha]) * ([\bar{\alpha}] * [g] * [\alpha]) \\
&= [\bar{\alpha}] * [f] * ([\alpha] * [\bar{\alpha}]) * [g] * [\alpha] \\
&= [\bar{\alpha}] * [f] * [e_{x_0}] * [g] * [\alpha] \\
&= [\bar{\alpha}] * [f] * [g] * [\alpha] \\
&= [\bar{\alpha}] * ([f] * [g]) * [\alpha] = \hat{\alpha}([f] * [g])
\end{aligned}$$

Now we shall prove that $\hat{\alpha}$ is invertible. It would prove that $\hat{\alpha}$ is bijective. Let $\beta = \bar{\alpha}$ be the reverse of α. Consequently $\bar{\beta} = \alpha$. We claim that $\hat{\beta}$ is the inverse of $\hat{\alpha}$.
Let $[h] \in \pi_1(X, x_1)$. Then we have

$$\hat{\beta}([h]) = \left[\bar{\beta}\right] * [h] * [\beta] = [\alpha] * [h] * [\bar{\alpha}]$$

As a result,

$$\begin{aligned}\hat{\alpha}\left(\hat{\beta}([h])\right) &= [\bar{\alpha}] * \left(\hat{\beta}([h])\right) * [\alpha] \\ &= [\bar{\alpha}] * [\alpha] * [h] * [\bar{\alpha}] * [\alpha] \\ &= [e_{x_1}] * [h] * [e_{x_1}] \\ &= [h]\end{aligned}$$

Now let $[f] \in \pi_1(X, x_0)$. Then,

$$\begin{aligned}\hat{\beta}\big(\hat{\alpha}([f])\big) &= [\alpha] * \big(\hat{\alpha}([f])\big) * [\bar{\alpha}] \\ &= [\alpha] * [\bar{\alpha}] * [f] * [\alpha] * [\bar{\alpha}] \\ &= [e_{x_0}] * [f] * [e_{x_0}] \\ &= [f]\end{aligned}$$

So $\hat{\beta} = \hat{\alpha}^{-1}$. Therefore $\hat{\alpha}$ is invertible, and so $\hat{\alpha}$ is a bijective homomorphism. Hence $\hat{\alpha}$ is a group isomorphism. ■

Corollary 2.2.2
If X is path connected and x_0 and x_1 are two points of X then $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, x_1)$.

Let X be a topological space that is not path connected and $x_0 \in X$. Also let C be the path component of X that contains the point x_0. All the path homotopy classes of loops based at x_0 belong in the same **equivalence class** or **path component** of X containing x_0. So we have

$$\pi_1(X, x_0) = \pi_1(C, x_0)$$

Remark. If X is path connected, then all the groups $\pi_1(X, x)$ are isomorphic. In particular, $\pi_1(X, x_0) \cong \pi_1(X, x_1)$ for $x_0, x_1 \in X$. But the isomorphism between $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ depends on the choice of path from x_0 to x_1. However, the isomorphism will be independent of the chosen path between x_0 and x_1 if and only if the fundamental group is abelian (problem **??**).

Definition 2.2.3 (Simply Connected Space)**.** A space X is said to be **simply connected** if it is a path-connected space and if $\pi_1(X, x_0)$ is the trivial (one-element) group for some $x_0 \in X$, and hence for every $x_0 \in X$. In other words,

$$\pi_1(X, x_0) = \{[e_{x_0}]\} \quad \forall\, x_0 \in X$$

Abuse of Notation. We often express the fact that $\pi_1(X, x_0)$ is the trivial group by writing $\pi_1(X, x_0) = 0$.

Lemma 2.2.3

In a simply connected space X, any two paths having the same initial and final points are path homotopic.

Proof. Let α and β be two paths in X, both from x_0 to x_1. Then $\alpha * \bar{\beta}$ is a loop in X based at x_0. Since X is a simply connected space, we have $\left[\alpha * \bar{\beta}\right] = [e_{x_0}]$.

$$\begin{aligned} & \left[\alpha * \bar{\beta}\right] * [\beta] = [e_{x_0}] * [\beta] \\ \implies & [\alpha] * \left[\bar{\beta}\right] * [\beta] = [\beta] \\ \implies & [\alpha] * [e_{x_1}] = [\beta] \\ \implies & [\alpha] = [\beta] \end{aligned}$$

Therefore, $\alpha \simeq_p \beta$. ∎

§2.3 Induced Homomorphism and Its Properties

Let X be a topological space with a distinguished base point $x_0 \in X$. Such a topological space is called a **pointed topological space** and is denoted by (X, x_0).

Suppose $h_{x_0} : X \to Y$ is a continuous map that carries the point $x_0 \in X$ to the point $y_0 \in Y$; in other words, $h_{x_0}(x_0) = y_0$. We often write the map as a map between two pointed topological spaces (X, x_0) and (Y, y_0).

$$h_{x_0} : (X, x_0) \to (Y, y_0)$$

Now, if f is a loop in X based at x_0, then $h_{x_0} \circ f : I \to Y$ with

$$\begin{aligned} & (h_{x_0} \circ f)(0) = h_{x_0}(f(0)) = h_{x_0}(x_0) = y_0 \\ \text{and } & (h_{x_0} \circ f)(1) = h_{x_0}(f(1)) = h_{x_0}(x_0) = y_0 \end{aligned}$$

In other words, $h_{x_0} \circ f$ is a loop in Y based at y_0. The correspondence $f \to h_{x_0} \circ f$ gives rise to a map from $\pi_1(X, x_0)$ to $\pi_1(Y, y_0)$. We define it formally as follows:

Definition 2.3.1. Let $h_{x_0} : (X, x_0) \to (Y, y_0)$ be a continuous map. We define $(h_{x_0})_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ by the following equation:

$$(h_{x_0})_*([f]) = [h_{x_0} \circ f]$$

The map $(h_{x_0})_*$ is called the homomorphism induced by h_{x_0}, relative to the base point x_0 in X.

The map $(h_{x_0})_*$ is well-defined. To check the well-definedness, let $f \simeq_p f'$ and F is a path homotopy between them. Then by the key facts used in proving Theorem 1.1.3 in last lecture, $h_{x_0} \circ F$ is a path homotopy between $h_{x_0} \circ f$ and $h_{x_0} \circ f'$. So, for $[f] = [f']$, we have $[h_{x_0} \circ f] = [h_{x_0} \circ f']$. As a result, $(h_{x_0})_*$ is a well-defined map.

Proposition 2.3.1

$(h_{x_0})_*$ is a gruop homomorphism.

Proof. Let $[f], [g] \in \pi_1(X, x_0)$.

$$\begin{aligned} (h_{x_0})_*\big([f] * [g]\big) &= (h_{x_0})_*([f * g]) \\ &= [h_{x_0} \circ (f * g)] \\ &= [(h_{x_0} \circ f) * (h_{x_0} \circ g)] \\ &= [h_{x_0} \circ f] * [h_{x_0} \circ g] \\ &= (h_{x_0})_*\big([f]\big) * (h_{x_0})_*\big([g]\big) \end{aligned}$$

So $(h_{x_0})_*$ is indeed a group homomorphism. ∎

As the notation suggests, the induced homomorphism depends not only on the continuous map h_{x_0}, but also on the choice of the base point x_0.

Theorem 2.3.2

If $h_{x_0} : (X, x_0) \to (Y, y_0)$ and $k_{y_0} : (Y, y_0) \to (Z, z_0)$ are continuous maps between pointed topological spaces, then

$$(k_{y_0} \circ h_{x_0})_* = (k_{y_0})_* \circ (h_{x_0})_*$$

Furthermore, if $i_{x_0} : (X, x_0) \to (X, x_0)$ is the identity map, then $(i_{x_0})_* : \pi_1(X, x_0) \to \pi_1(X, x_0)$ is the identity homomorphism $(i_{x_0})_*([f]) = [f]$.

Proof. Note that, the composition map $k_{y_0} \circ h_{x_0}$ is continuous from (X, x_0) to (Z, z_0). Indeed $k_{y_0} \circ h_{x_0} : (X, x_0) \to (Z, z_0)$ as a map between pointed topological spaces, as $z_0 = k_{y_0}(y_0) = k_{y_0}(h_{x_0}(x_0)) = (k_{y_0} \circ h_{x_0})(x_0)$. Therefore,

$$(k_{y_0} \circ h_{x_0})_* : \pi_1(X, x_0) \to \pi_1(Z, z_0) \text{ with } \boxed{(k_{y_0} \circ h_{x_0})_*([f]) = [(k_{y_0} \circ h_{x_0}) \circ f]}$$

for a given $[f] \in \pi_1(X, x_0)$.

On the other hand, since $(h_{x_0})_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ and $(k_{y_0})_* : \pi_1(Y, y_0) \to \pi_1(Z, z_0)$, one has $(k_{y_0})_* \circ (h_{x_0})_* : \pi_1(X, x_0) \to \pi_1(Z, z_0)$ with

$$\begin{aligned} ((k_{y_0})_* \circ (h_{x_0})_*)([f]) &= (k_{y_0})_*((h_{x_0})_*([f])) \\ &= (k_{y_0})_*([h_{x_0} \circ f]) \\ &= \boxed{[k_{y_0} \circ (h_{x_0} \circ f)]} \end{aligned}$$

We know that composition of maps is associative. Therefore,

$$(k_{y_0} \circ h_{x_0})_* = (k_{y_0})_* \circ (h_{x_0})_*$$

Again for the identity map $i_{x_0} : (X, x_0) \to (X, x_0)$ we have $i_{x_0}(x) = x$ for every $x \in X$. So the induced homomorphism $(i_{x_0})_* : \pi_1(X, x_0) \to \pi_1(X, x_0)$ gives us

$$(i_{x_0})_*([f]) = [i_{x_0} \circ f] = [f] \quad \forall [f] \in \pi_1(X, x_0)$$

So $(i_{x_0})_*$ is indeed the identity group homomorphism. ■

Theorem 2.3.2 is often known as functorial properties[1] of induced homomorphism.

Corollary 2.3.3

If $h_{x_0} : (X, x_0) \to (Y, y_0)$ is a homeomorphism from X to Y, then $(h_{x_0})_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is a group isomorphism.

Proof. By Proposition 2.3.1, $(h_{x_0})_*$ is a group homomorphism. So we only need to prove that the inverse of $(h_{x_0})_*$ exists.

$h_{x_0} : (X, x_0) \to (Y, y_0)$ is continuous with continuous inverse. Let $k_{y_0} : (Y, y_0) \to (X, x_0)$ be the inverse of h_{x_0}. Hence $k_{y_0} \circ h_{x_0} : (X, x_0) \to (X, x_0)$ is the identity map i_{x_0} on (X, x_0). Similarly, $h_{x_0} \circ k_{x_0} : (Y, y_0) \to (Y, y_0)$ is the identity map j_{y_0} on (Y, y_0). That is

$$\boxed{k_{y_0} \circ h_{x_0} = i_{x_0}} \text{ and } \boxed{h_{x_0} \circ k_{x_0} = j_{y_0}}$$

Applying Theorem 2.3.2,

$$\begin{aligned} (k_{y_0} \circ h_{x_0})_* &= (k_{y_0})_* \circ (h_{x_0})_* && \text{and} && (h_{x_0} \circ k_{y_0})_* = (h_{x_0})_* \circ (k_{y_0})_* \\ \implies (k_{y_0})_* \circ (h_{x_0})_* &= (i_{x_0})_* && && \implies (h_{x_0})_* \circ (k_{y_0})_* = (j_{y_0})_* \end{aligned}$$

$(i_{x_0})_*$ and $(j_{y_0})_*$ are identity group homomorphism on $\pi_1(X, x_0)$ and $\pi_1(Y, y_0)$ respectively. Therefore, $(h_{x_0})_*$ and $(k_{y_0})_*$ are inverses of one another. It means the inverse of $(h_{x_0})_*$ exists, hence it's a group isomorphism. ■

[1] However, this naming is not random. See Appendix A for more details.

3 Lecture 3

Any convex subspace of the Euclidean space $\mathbb{R}^n$ has trivial fundamental group. We now need to compute fundamental groups of topological spaces that are not necessarily trivial. One useful technique to achieve this is to use the tool of *covering spaces*.

§3.1 Covering Space

Definition 3.1.1. Let $p : E \to B$ be a continuous surjective map. The open set U of B is said to be **evenly covered** by p if the inverse image $p^{-1}(U)$ can be written as the union of disjoint open sets V_α in E such that for each α, the restriction of p to V_α is a homeomorphism of V_α onto U. The collection $\{V_\alpha\}$ will be called a partition of $p^{-1}(U)$ into **slices**.

We often picture $p^{-1}(U)$ as a some copies of U; the map p maps them all down onto U (Firgure 3.1).

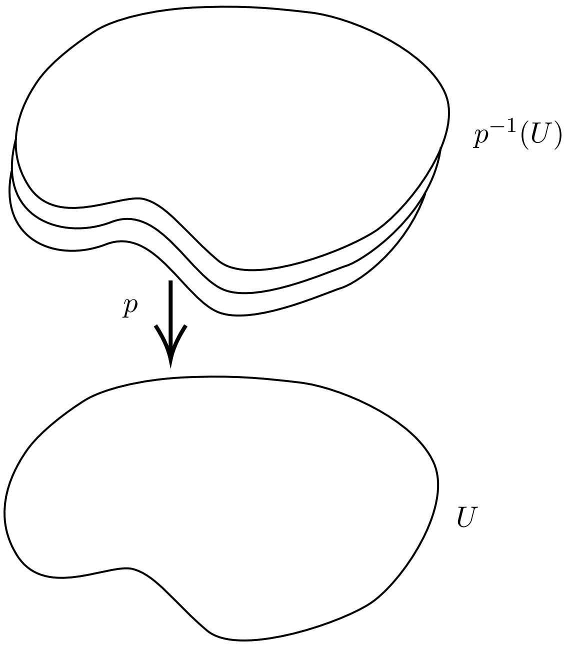


Figure 3.1: Even covering of $U \subseteq B$ by $p : E \to B$.

Definition 3.1.2. Let $p : E \to B$ be continuous and surjective. If every point $b \in B$ has a neighborhood U that is evenly covered by p, then p is called a **covering map**; and E is called a **covering space** of B.

There are a few properties of $p : E \to B$ that ensue from its definition.

Lemma 3.1.1

If $p : E \to B$ is a covering map, then for every $b \in B$ the subspace $p^{-1}(b)$, equipped with the subspace topology inherited from E, has the discrete topology.

Proof. We claim that $p^{-1}(b) \cap V_\alpha$ is a singleton set for each $\alpha \in J$. Assume for the sake of contradiction that, $c, d \in \left(p^{-1}(b) \cap V_\alpha\right)$ with $c \neq d$ for a given α. Then we would have that, for $c, d \in V_\alpha$,

$p(c) = p(d) = b$. As a result, $p|_{V_\alpha} : V_\alpha \to U$ is no longer injective, and hence not a homeomorphism. Contradiction! Also,

$$p^{-1}(b) \subseteq p^{-1}(U) = \bigsqcup_\alpha V_\alpha \implies p^{-1}(b) = p^{-1}(b) \cap \left(\bigsqcup_\alpha V_\alpha\right) = \bigsqcup_\alpha \left(p^{-1}(b) \cap V_\alpha\right)$$

Therefore, each element of $p^{-1}(b)$ is an element belonging to a singleton set of the mutually disjoint family $\{p^{-1}(b) \cap V_\alpha\}_\alpha$. Each V_α is open in E. According to the definition of Subspace topology, $p^{-1}(b) \cap V_\alpha$ is open in $p^{-1}(b)$. As a result, every singleton set is open in $p^{-1}(b)$. So $p^{-1}(b)$ has the discrete topology. ■

Lemma 3.1.2

If $p : E \to B$ is a covering map, then p is an open map.

Proof. Let $A \subseteq E$ be open. We need to show that $p(A)$ is open in B. Take $x \in p(A)$. We need to find a neighborhood of x that is contained inside $p(A)$.

Since p is a covering map of B, one can choose a neighborhood U of x that is evenly covered by p. Let $\{V_\alpha\}_{\alpha \in J}$ be a partition of $p^{-1}(U)$ into slices, *i.e.*, $p^{-1}(U) = \bigsqcup_{\alpha \in J} V_\alpha$.

Since $x \in p(A)$, there exists some $y \in A$ such that $x = p(y)$. Now $p(y) = x \in U \implies y \in p^{-1}(U) = \bigsqcup_{\alpha \in J} V_\alpha$. Let V_β be the slice containing y.

V_β is a subset of E, and A is open in E. Thus $V_\beta \cap A$ is open in V_β with respect to the subspace topology. p maps V_β homeomorphically onto U, so it maps open subsets of V_β to open subsets of U. Hence $p(V_\beta \cap A)$ is open in U. Since U is open in B, by Lemma 0.2.2, $p(V_\beta \cap A)$ is open in B.

We know that $y \in V_\beta$ and $y \in A$, so $y \in V_\beta \cap A \implies x = p(y) \in p(V_\beta \cap A) \subseteq p(A)$, in particular $x \in p(V_\beta \cap A) \subseteq p(A)$. So we have found our desired open neighborhood of x, contained in $p(A)$. Hence $p(A)$ is open. ■

Example 3.1.1

Let X be an arbitrary topological space. Set $E = X \times \{1, 2, \ldots, N\}$, with the latter set given the discrete topology. The the projection $p : E \to X$ given by $p(x, i) = x$ is a covering map.

Note that the whole set X, being open in X, is evenly covered by p. Indeed $p^{-1}(X) = \bigsqcup_{i=1}^N X \times \{i\}$, where each $X \times \{i\}$ is open in E. It's immediate that $p|_{X \times \{i\}} : X \times \{i\} \to X$ is a homeomorphism. So p is indeed a covering map.

Theorem 3.1.3

Define $p : \mathbb{R} \to S^1$ by $p(x) = (\cos 2\pi x,\ \sin 2\pi x)$. Then p is a covering map.

Proof. Clearly p is Surjective. Because every point y on S^1 has norm 1, so they can be expressed as $(\cos\theta, \sin\theta)$ for some $\theta \in \mathbb{R}$. So one can always find $x \in \mathbb{R}$ such that $p(x) = y$.
About the continuity, p can be expressed as follows:

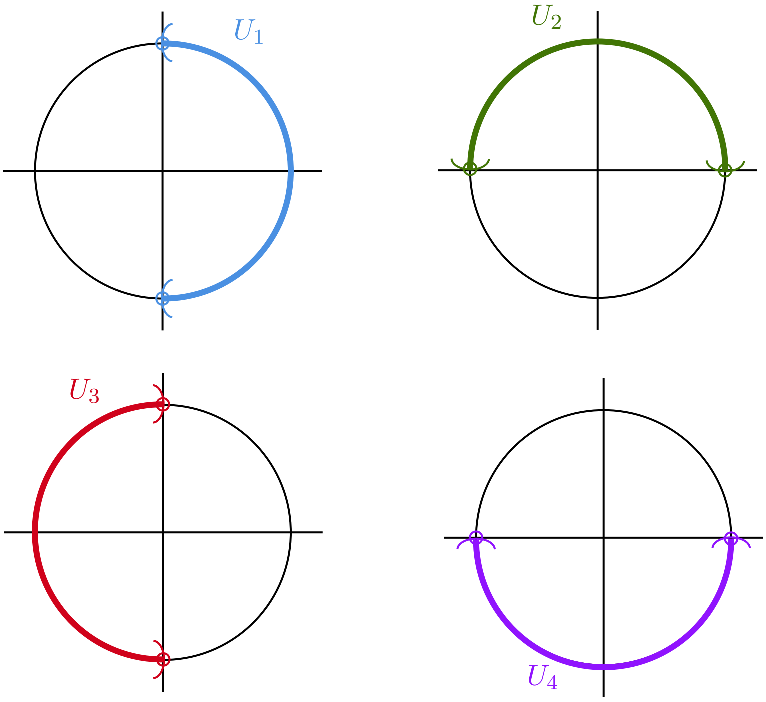
$$p : \mathbb{R} \to \mathbb{R} \times \mathbb{R}\ , \quad x \mapsto (p_1(x), p_2(x))$$

where $p_1, p_2 : \mathbb{R} \to \mathbb{R}$ are given by $p_1(x) = \cos 2\pi x$ and $p_2(x) = \sin 2\pi x$. Since both p_1 and p_2 are continuous, p is continuous (by Theorem 18.4 from Munkres).
Look at the point $(1, 0)$ in S^1. It's clear that $p^{-1}\big((1, 0)\big) = \mathbb{Z}$. Also, using the formula for p, one obtains easily that

$$p^{-1}\big((0, 1)\big) = \left\{n + \frac{1}{4} \mid n \in \mathbb{Z}\right\} \text{ and } p^{-1}\big((0, -1)\big) = \left\{n - \frac{1}{4} \mid n \in \mathbb{Z}\right\}$$

As a result, the open arc U_1 (colored blue in the figure below), has the preimage $p^{-1}(U_1) = \bigsqcup_{n \in \mathbb{Z}} \left(n - \frac{1}{4}, n + \frac{1}{4}\right)$. Similarly we find the preimage of other U_i's.

$p^{-1}(U_1) = \bigsqcup_{n\in\mathbb{Z}} \left(n - \frac{1}{4}, n + \frac{1}{4}\right)$ $\quad$ $p^{-1}(U_2) = \bigsqcup_{n\in\mathbb{Z}} \left(n, n + \frac{1}{2}\right)$



$p^{-1}(U_3) = \bigsqcup_{n\in\mathbb{Z}} \left(n + \frac{1}{4}, n + \frac{3}{4}\right)$ $\quad$ $p^{-1}(U_4) = \bigsqcup_{n\in\mathbb{Z}} \left(n + \frac{1}{2}, n + 1\right)$

Now, each point of S^1 belong to at least one of the U_i's. So our job is done if we can show that each U_i is evenly covered by p. Here we will only show that U_1 is evenly covered by p. The proofs for U_2, U_3, U_4 are similar.

We saw that $p^{-1}(U_1) = \bigsqcup V_n$, where $V_n = \left(n - \frac{1}{4}, n + \frac{1}{4}\right)$. It's immediate that the open sets V_n are disjoint. So we now need to show that the $p\big|_{V_n} : V_n \to U_1$ is a homeomorphism. But first we will show that $p\big|_{\overline{V_n}} : \overline{V_n} \to \overline{U_1}$ is a homeomorphism, where $\overline{X}$ stands for the closure of X.

Clearly $\overline{V_n} = \left[n - \frac{1}{4}, n + \frac{1}{4}\right]$. Firstly we need to check that $p\big|_{\overline{V_n}}$ is injective. Suppose we have $x_1, x_2 \in \overline{V_n}$ with $p\big|_{\overline{V_n}}(x_1) = p\big|_{\overline{V_n}}(x_2)$. That is, $(\cos 2\pi x_1,\ \sin 2\pi x_1) = (\cos 2\pi x_2,\ \sin 2\pi x_2)$. In other words,

$$\cos 2\pi x_1 = \cos 2\pi x_2 \quad \text{and} \quad \sin 2\pi x_1 = \sin 2\pi x_2$$

But $\sin\theta$ is a monotonically increasing continuous function for $\theta \in \left[2\pi n - \frac{\pi}{2}, 2\pi n + \frac{\pi}{2}\right]$. In this case $\sin 2\pi x$ is monotonically increasing continuous function for $x \in \overline{V_n}$. So, to acquire $\sin 2\pi x_1 = \sin 2\pi x_2$, we must have $x_1 = x_2$. Thus $p\big|_{\overline{V_n}}$ is injective.

Now we will prove the surjectivity of $p\big|_{\overline{V_n}}$. Observe that, $p\big|_{\overline{V_n}}$ maps the endpoints of $\overline{V_n}$ to the endpoints of $\overline{U}_1$, with he endpoints of $\overline{U}$ being $(0, 1)$ and $(0, -1)$.

$$\begin{aligned}
p\big|_{\overline{V_n}}\left(n - \frac{1}{4}\right) &= \left(\cos 2\pi\left(n - \frac{1}{4}\right),\ \sin 2\pi\left(n - \frac{1}{4}\right)\right) \\
&= \left(\cos\left(-\frac{\pi}{2}\right),\ \sin\left(-\frac{\pi}{2}\right)\right) = (0, -1) \\
p\big|_{\overline{V_n}}\left(n + \frac{1}{4}\right) &= \left(\cos 2\pi\left(n + \frac{1}{4}\right),\ \sin 2\pi\left(n + \frac{1}{4}\right)\right) \\
&= \left(\cos\left(\frac{\pi}{2}\right),\ \sin\left(\frac{\pi}{2}\right)\right) = (0, 1)
\end{aligned}$$

$p\big|_{\overline{V_n}}$ is continuous, $\overline{V_n}$ is connected, and $\overline{U_1}$ is ordered. So by *Intermediate value theorem*, for any $c \in U_1$ there exists $x \in \overline{V_n}$ such that $p\big|_{\overline{V_n}}(x) = c$. Hence $p\big|_{\overline{V_n}}$ is surjective.

We've proved that $p\big|_{\overline{V_n}}$ is a bijective continuous map from $\overline{V_n}$ to $\overline{U_1}$. $\overline{V_n}$ is a bounded closed subset of $\mathbb{R}$, hence compact (by Heine-Borel Theorem). $\overline{U_1}$ is a subset of the Hausdorff space $\mathbb{R}^2$, so it is also Hausdorff (by Lemma 0.4.2). So $p\big|_{\overline{V_n}}$ being a continuous bijective map from a compact set to a Hausdorff space, by Corollary 0.7.6, $p\big|_{\overline{V_n}}$ is a homeomorphism.

Restriction of a homeomorphism is again a homeomorphism. Therefore, $p\big|_{V_n} : V_n \to U_1$ is a homeomorphism for every $n \in \mathbb{Z}$. Hence U_1 is evenly covered by p. In a similar manner, one can prove that U_2, U_3, U_4 are also evenly covered by p. So $p : \mathbb{R} \to S^1$ is indeed a covering map, and $\mathbb{R}$ is a covering space of S^1. ■

Definition 3.1.3 (Local Homeomorphism). The map $p : E \to B$ is called a **local homeomorphism** if each point $e \in E$ has a neighborhood U which is mapped homeomorphically by p onto an open subset of B.

Let $p : E \to B$ be a covering map and $e \in E$. Also, let $p(e) = x \in B$. Since p is a covering map, there exists open $U \subseteq B$ containing x, that is evenly covered by p. In other words, $p^{-1}(U) = \bigsqcup\limits_{\alpha \in J} V_\alpha$ with each V_α being open in E.

Now, $p(e) = x \in U \implies e \in p^{-1}(U)$. Therefore, $e \in \bigsqcup_{\alpha \in J} V_\alpha$. So $e \in V_\alpha$ for some $\alpha \in J$. From the definition of covering map, $p\big|_{V_\alpha} : V_\alpha \to U$ is a homeomorphism. Therefore, for a given $e \in E$, we've found an open set $V_\alpha \subseteq E$ that contains e; and V_α gets homeomorphically mapped to an open subset U of B. Therefore, p is a local homeomorphism.

We've just seen that, if $p : E \to B$ is a covering map, then p is also a local homeomorphism. However, the converse is not true in general. The following example illustrates a counterexample to the converse.

Example 3.1.2

The map $\widetilde{p} : \mathbb{R}^+ \to S^1$ given by $\widetilde{p}(x) = (\cos 2\pi x,\ \sin 2\pi x)$ is surjective, and it's a local homeomorphism. We have seen that there is an open cover $\{U_i\}_{i=1}^4$ of S^1 with

$$p^{-1}(U_1) = \bigsqcup_{n \in \mathbb{Z}} \left(n - \frac{1}{4}, n + \frac{1}{4}\right) \quad , \quad p^{-1}(U_2) = \bigsqcup_{n \in \mathbb{Z}} \left(n, n + \frac{1}{2}\right)$$
$$p^{-1}(U_3) = \bigsqcup_{n \in \mathbb{Z}} \left(n + \frac{1}{4}, n + \frac{3}{4}\right) \quad , \quad p^{-1}(U_4) = \bigsqcup_{n \in \mathbb{Z}} \left(n + \frac{1}{2}, n + 1\right)$$

for the covering map $p : \mathbb{R} \to S^1$ given by $p(x) = (\cos 2\pi x,\ \sin 2\pi x)$. Now, for this map $\widetilde{p}$, we can calculate $\widetilde{p}^{-1}(U_i)$ in using $p^{-1}(U_i)$.

$$\widetilde{p}^{-1}(U_1) = p^{-1}(U_1) \cap \mathbb{R}^+ = \left(0, \frac{1}{4}\right) \sqcup \left(\bigsqcup_{n \in \mathbb{N}} \left(n - \frac{1}{4}, n + \frac{1}{4}\right)\right) = \bigsqcup_{n \in \mathbb{N} \cup \{0\}} \widetilde{V}_n$$
$$\widetilde{p}^{-1}(U_2) = p^{-1}(U_2) \cap \mathbb{R}^+ = \bigsqcup_{n \in \mathbb{N} \cup \{0\}} \left(n, n + \frac{1}{2}\right) = \bigsqcup_{n \in \mathbb{N} \cup \{0\}} \widetilde{V}'_n$$
$$\widetilde{p}^{-1}(U_3) = p^{-1}(U_3) \cap \mathbb{R}^+ = \bigsqcup_{n \in \mathbb{N} \cup \{0\}} \left(n + \frac{1}{4}, n + \frac{3}{4}\right) = \bigsqcup_{n \in \mathbb{N} \cup \{0\}} \widetilde{V}''_n$$
$$\widetilde{p}^{-1}(U_4) = p^{-1}(U_4) \cap \mathbb{R}^+ = \bigsqcup_{n \in \mathbb{N} \cup \{0\}} \left(n + \frac{1}{2}, n + 1\right) = \bigsqcup_{n \in \mathbb{N} \cup \{0\}} \widetilde{V}'''_n$$

The arc U_1, an open subset of S^1 containing $(1, 0)$, is not evenly covered by $\widetilde{p}$. Because, for the slice $\widetilde{V}_0 = \left(0, \frac{1}{4}\right)$, the map $\widetilde{p}\big|_{\widetilde{V}_0} : \widetilde{V}_0 \to U_1$ is not a homeomorphism, it's just a topological imbedding. So $\widetilde{p}$ is not a covering map.

For any point x in $\mathbb{R}^+$, some neighborhood of x will be contained in at least one of the open sets $\widetilde{V}_n$, $\widetilde{V}'_n$, $\widetilde{V}''_n$, $\widetilde{V}'''_n$ for some $n \in \mathbb{N} \cup \{0\}$. Let $W \ni x$ and $W \subseteq \widetilde{V}''_n$ (for example). Then one can easily show that the map $\widetilde{p}\big|_W : W \to \widetilde{p}(W) \subseteq U_3$ is a homeomorphism. Despite not being a covering map, $\widetilde{p}$ is a local homeomorphism.

In Example 3.1.2, we've seen that the restriction of a covering map, in general, is not a covering map. The following theorem tells us when the restriction of a covering map will be a covering map.

Theorem 3.1.4

Let $p : E \to B$ be a covering map. If B_0 is a subspace of B, and $E_0 = p^{-1}(B_0)$; then the map $p_0 : E_0 \to B_0$ obtained by restricting p is a covering map.

Proof. Given $b_0 \in B_0$, let U be an open set in B containing b_0, this is evenly covered by p. Let $\{V_\alpha\}_{\alpha \in J}$ be a partition of $p^{-1}(U)$ into slices. In other words, $p^{-1}(U) = \bigsqcup_{\alpha \in J} V_\alpha$.

Each V_α is open in E. $U \cap B_0$ is open in B_0, with respect to the subspace topology, that contains b_0. The sets $\{V_\alpha \cap E_0\}_{\alpha \in J}$ are disjoint open sets in E_0, because for distinct $\alpha, \beta \in J$

$$(V_\alpha \cap E_0) \cap (V_\beta \cap E_0) = (V_\alpha \cap V_\beta) \cap E_0 = \varnothing \cap E_0 = \varnothing$$

This disjoint collection of open sets cover $p^{-1}(U \cap B_0)$.

$$p^{-1}(U \cap B_0) = p^{-1}(U) \cap p^{-1}(B_0) = \left(\bigsqcup_{\alpha \in J} V_\alpha\right) \cap E_0 = \bigsqcup_{\alpha \in J} (V_\alpha \cap E_0)$$

In addition, since $p\big|_{V_\alpha} : V_\alpha \to U$ is a homeomorphism, $p\big|_{V_\alpha \cap E_0} : V_\alpha \cap E_0 \to U \cap B_0$ is also a homeomorphism. Hence, $p_0 : E_0 \to B_0$ is a covering map. ∎

4 Lecture 4

§4.1 Lifting Map

Definition 4.1.1 (Lifting). Let $p : E \to B$ be a map. If f is a continuous map from some topological space X to B, a **lifting of** f is a map $\bar{f} : X \to E$ such that $p \circ \bar{f} = f$.

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, "\bar{f}"] \arrow[r, "f"'] & B \end{tikzcd}$$

In this lecture we will be concerned with liftings when p is a covering map.

Example 4.1.1

Consider the covering map $p : \mathbb{R} \to S^1$ given by $p(x) = (\cos 2\pi x, \sin 2\pi x)$. The path $f : [0,1] \to S^1$ beginning at $(1,0) \in S^1$ given by $f(s) = (\cos \pi s, \sin \pi s)$ lifts to the path $\bar{f}(s) = \frac{s}{2}$ beginning at 0 and ending at $\frac{1}{2}$.

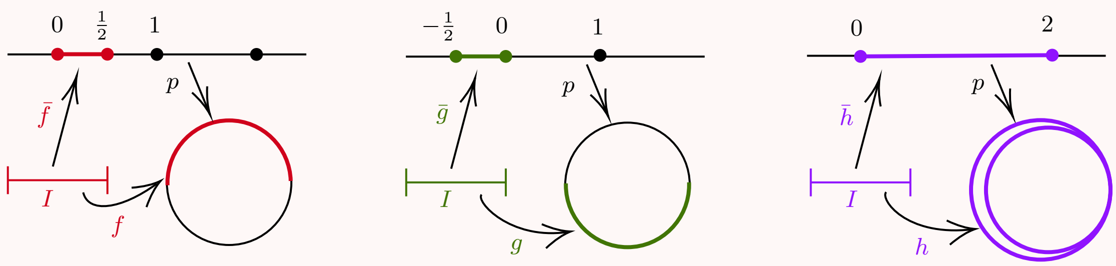
$$(\cos \pi s, \sin \pi s) = f(s) = p\left(\bar{f}(s)\right) = \left(\cos 2\pi \bar{f}(s), \sin 2\pi \bar{f}(s)\right) \implies \bar{f}(s) = \frac{s}{2}$$

The path $g : [0,1] \to S^1$ given by $g(s) = (\cos \pi s, -\sin \pi s)$ lifts to the path $\bar{g}(s) = -\frac{s}{2}$ beginning at 0 and ending at $-\frac{1}{2}$.

$$(\cos \pi s, -\sin \pi s) = g(s) = p\left(\bar{g}(s)\right) = \left(\cos 2\pi \bar{g}(s), \sin 2\pi \bar{g}(s)\right) \implies \bar{g}(s) = -\frac{s}{2}$$

The path $h : [0,1] \to S^1$ given by $f(s) = (\cos 4\pi s, \sin 4\pi s)$ lifts to the path $\bar{h}(s) = 2s$ beginning at 0 and ending at 2.

$$(\cos 4\pi s, \sin 4\pi s) = h(s) = p\left(\bar{h}(s)\right) = \left(\cos 2\pi \bar{h}(s), \sin 2\pi \bar{h}(s)\right) \implies \bar{h}(s) = 2s$$



Lemma 4.1.1

Let $p : E \to B$ be a covering map, and $p(e_0) = b_0$. Any path $f : I \to B$ beginning at b_0 has a unique lifting to a path $\bar{f}$ in E beginning at e_0.

Proof. Since $p : E \to B$ is a covering map, for every $x \in B$, there exists a neighborhood U_x that is evenly covered by p. U_x is open in B and it contains x, so the collection $\{U_x\}_{x \in B}$ is an open cover of B. $f : I \to B$ is a continuous map, so the preimages of open sets are open in I. Therefore, the collection $\left\{f^{-1}(U_x)\right\}_{x\in B}$ is an open cover of I.

I is bounded and closed subset of $\mathbb{R}$. By Heine-Borel Theorem, I is a compact metric space. Now we shall use Lebesgue Number Lemma. We have an open cover $\left\{f^{-1}(U_x)\right\}_{x\in B}$ of the compact metric space I. Let δ be the Lebesgue Number of this open cover. Now we are gonna partition $[0,1]$ into subintervals $[s_0, s_1], [s_1, s_2], \ldots, [s_{n-1}, s_n]$, with $s_0 = 0$ and $s_n = 1$, such that

$$\operatorname{diam}([s_i, s_{i+1}]) = s_i - s_{i-1} < \delta\ , \quad \forall\ i \in \{0, 1, 2, \ldots, n-1\}$$

Each $[s_i, s_{i+1}]$ has diameter less than δ. By Lebesgue Number Lemma,

$$[s_i, s_{i+1}] \subseteq f^{-1}(U_x) \text{ for some } x \in B \implies f([s_i, s_{i+1}]) \subseteq U_x \text{ for some } x \in B$$

Now we will inductively define the lift $\bar{f} : I \to E$ with $f(0) = e_0$. Note that, by the definition of lift, one must have $p \circ \bar{f} = f$. The base case of the inductive definition is

$$\bar{f}(0) = \bar{f}([0, s_0]) = e_0$$

Then we assume that $\bar{f}([0, s_i])$ is defined. We need to show that $\bar{f}([0, s_{i+1}])$ can be defined; in other words $\bar{f}([s_i, s_{i+1}])$ can be defined. Because if we can prove that $\bar{f}$ can be defined on $[s_i, s_{i+1}]$, we can conclude by pasting lemma that $\bar{f}$ can be defined on $[0, s_{i+1}]$. By inductive hypothesis, $\bar{f}([0, s_i])$ is defined. In particular, $\bar{f}(s_i)$ is defined. Since $p \circ \bar{f} = f$, we have

$$p\left(\bar{f}(s_i)\right) = f(s_i) \in U_x \implies \bar{f}(s_i) \in p^{-1}(U_x) = \bigsqcup_{\alpha\in J} V_\alpha \implies \bar{f}(s_i) \in V_{\alpha_0}$$

for a unique $\alpha_0 \in J$. For $s \in [s_i, s_{i+1}]$, we define $\bar{f}(s)$ as

$$\bar{f}(s) = \left(p\big|_{V_{\alpha_0}}^{-1} \circ f\right)(s)$$

$p\big|_{V_{\alpha_0}}$ is a homeomorphism, so $p\big|_{V_{\alpha_0}}^{-1}$ is continuous. Hence $\bar{f}$ is composition of continuous maps, thus continuous on $[s_i, s_{i+1}]$. This definition satisfies $p \circ \bar{f} = f$, because

$$\left(p \circ \bar{f}\right)(s) = \left(p \circ p\big|_{V_{\alpha_0}}^{-1} \circ f\right)(s) = f(s)$$

So $\bar{f}$ can be defined on $[s_i, s_{i+1}]$ with the required properties. $\bar{f}$ defined on the subintervals $[s_i, s_{i+1}]$ can be continuously extended to all of $[0,1]$ by pasting lemma, since $\bar{f}$ is defined on the closed intervals $[s_0, s_1], [s_1, s_2], \ldots, [s_{n-1}, s_n]$, with $s_0 = 0$ and $s_n = 1$ and $\bar{f}$ agrees on the pointwise overlaps.

We've shown that the lifting $\bar{f}$ can be defined on I with the desired properties. Now we need to show that, $\bar{f}$ is unique. The uniqueness can be proved inductively as well.

Suppose $\widetilde{f}$ is another lifting of f beginning at e_0, *i.e*, $\widetilde{f}(0) = e_0 = \bar{f}(0)$. So the base case of the induction is $\bar{f}(s) = \widetilde{f}(s)$ for every $s \in [0, s_0]$. Assume that $\bar{f}(s) = \widetilde{f}(s)$ for every $s \in [0, s_i]$; this is the inductive hypothesis. We want to show that $\bar{f}(s) = \widetilde{f}(s)$ for every $s \in [s_i, s_{i+1}]$. That will establish $\bar{f}(s) = \widetilde{f}(s)$ for every $s \in [0, s_{i+1}]$.

We've seen earlier from the inductive construction of $\bar{f}$ that $\bar{f}(s_i) \in V_{\alpha_0}$ for a unique $\alpha_0 \in J$. Also, we defined $\bar{f}$ on $[s_i, s_{i+1}]$ by

$$\bar{f}(s) = \left(p\big|_{V_{\alpha_0}}^{-1} \circ f\right)(s) = p\big|_{V_{\alpha_0}}^{-1}(f(s))$$

Since $\widetilde{f}$ is a lift of f, we must have $p \circ \widetilde{f} = f$. Now,

$$f([s_i, s_{i+1}]) \subseteq U_x \implies \left(p \circ \widetilde{f}\right)([s_i, s_{i+1}]) \subseteq U_x \implies \widetilde{f}([s_i, s_{i+1}]) \subseteq p^{-1}(U_x) = \bigsqcup_{\alpha\in J} V_\alpha$$

$[s_i, s_{i+1}]$ is an interval, thus connected. By *Theorem 23.5 from Munkres*, the image of a connected space under a continuous map is connected. So $\widetilde{f}([s_i, s_{i+1}])$ is connected. The slices V_α are open and disjoint. If $\widetilde{f}\left([s_i, s_{i+1}]\right)$ is distributed among the V_α's, connectedness of $\widetilde{f}\left([s_i, s_{i+1}]\right)$ is contradicted. So $\widetilde{f}\left([s_i, s_{i+1}]\right)$ must lie in a single slice, namely V_{α_1}.

But $\widetilde{f}\left(s_i\right) = \bar{f}\left(s_i\right) \in V_{\alpha_0}$, so $\alpha_1 = \alpha_0$. Hence $\widetilde{f}\left([s_i, s_{i+1}]\right) \subseteq V_{\alpha_0}$. Let $s \in [s_i, s_{i+1}]$ and $y_0 = \widetilde{f}(s) \in V_{\alpha_0}$. $p\big|_{V_{\alpha_0}}$ is a homeomorphism, so it's invertible.

$$f(s) = p\left(\widetilde{f}\left(s\right)\right) = p\big|_{V_{\alpha_0}}\left(y_0\right) \implies y_0 = p\big|_{V_{\alpha_0}}^{-1}\left(f(s)\right) = \bar{f}(s)$$

So $\widetilde{f}$ and $\bar{f}$ agrees on $[s_i, s_{i+1}]$ and we are done. ■

In fact, the uniqueness of lifting has a generalization.

Lemma 4.1.2

Let $p : E \to B$ be a covering map and X be a connected topological space. Given any two continuous maps $\widetilde{f}_0, \widetilde{f}_1 : X \to E$ such that $p \circ \widetilde{f}_0 = p \circ \widetilde{f}_1 =: f$ (in other words, both $\widetilde{f}_0$ and $\widetilde{f}_1$ are lifts of f), consider the set

$$A = \left\{x \in X \ : \ \widetilde{f}_0(x) = \widetilde{f}_1(x)\right\}$$

Then $A = \varnothing$ or $A = X$.

Proof. The only sets that are both open and closed in a connected space are the empty set and the whole set itself. So this lemma is equivalent to proving A is both open and closed.

Firstly we will show that A is closed. It is enough to show that $\overline{A} = A$. Let $y \in \overline{A}$, we nned to show that $y \in A$, *i.e.* $\widetilde{f}_0(y) = \widetilde{f}_1(y)$. Assume for the sake of contradiction that $\widetilde{f}_0(y) \neq \widetilde{f}_1(y)$.

$$p\left(\widetilde{f}_0(y)\right) = p\left(\widetilde{f}_1(y)\right) = f(y) =: x$$

Consider a neighborhood U_x of x that is evenly covered by p. Now,

$$p\left(\widetilde{f}_0(y)\right) = p\left(\widetilde{f}_1(y)\right) \in U_x \implies \widetilde{f}_0(y), \widetilde{f}_1(y) \in p^{-1}\left(U_x\right) = \bigsqcup_{\alpha \in J} V_\alpha$$

Let V_0 and V_1 be the disjoint open slices containg $\widetilde{f}_0(y)$ and $\widetilde{f}_1(y)$ respectively. $\widetilde{f}_0 :: X \to E$, V_0 is open in E, so V_0 is a neighborhood of $\widetilde{f}_0(y)$ in E. By Lemma 0.5.3, there exists a neighborhood W_0 of y in X such that $\widetilde{f}_0\left(W_0\right) \subseteq V_0$. Similarly, there exists a neighborhood W_1 of y in X such that $\widetilde{f}_1\left(W_1\right) \subseteq V_1$.

Now $W = W_0 \cap W_1$ is a neighborhood of y in X.

$$\widetilde{f}_0\left(W\right) \subseteq V_0 \ , \ \widetilde{f}_1\left(W\right) \subseteq V_1 \implies \widetilde{f}_0\left(W\right) \cap \widetilde{f}_1\left(W\right) = \varnothing$$

But since $y \in \overline{A}$ and W is a neighborhood of y, $W \cap A \neq \varnothing$. This contradicts with $\widetilde{f}_0\left(W\right) \cap \widetilde{f}_1\left(W\right) = \varnothing$. Therefore, $y \in A$. As a result $\overline{A} \subseteq A$.By the definition of closure, $A \subseteq \overline{A}$. Hence $A = \overline{A}$ and A is closed.

To prove that A is open, take any $y \in A$. So we have $\widetilde{f}_0(y) = \widetilde{f}_1(y)$.

$$p\left(\widetilde{f}_0(y)\right) = p\left(\widetilde{f}_1(y)\right) = f(y) =: x$$

Consider a neighborhood U_x of x that is evenly covered by p. Now,

$$p\left(\widetilde{f}_0(y)\right) = p\left(\widetilde{f}_1(y)\right) \in U_x \implies \widetilde{f}_0(y), \widetilde{f}_1(y) \in p^{-1}\left(U_x\right) = \bigsqcup_{\alpha \in J} V_\alpha$$

Let V_0 be the open slice containg $\widetilde{f}_0(y) = \widetilde{f}_1(y)$. $\widetilde{f}_0 :: X \to E$, V_0 is open in E, so V_0 is a neighborhood of $\widetilde{f}_0(y)$ in E. By Lemma 0.5.3, there exists a neighborhood W_0 of y in X such that $\widetilde{f}_0\left(W_0\right) \subseteq V_0$. Similarly, there exists a neighborhood W_1 of y in X such that $\widetilde{f}_1\left(W_1\right) \subseteq V_0$.

Now $W = W_0 \cap W_1$ is a neighborhood of y in X. $\widetilde{f}_0\left(W\right) \subseteq V_0$, $\widetilde{f}_1\left(W\right) \subseteq V_0$ Let $w \in W$, so $\widetilde{f}_0(w), \widetilde{f}_1(w) \in V_0$. $p_0 = p\big|_{V_0}$ is a homeomorphism, so it's injective.

$$f(w) = p_0\left(\widetilde{f}_0(w)\right) = p_0\left(\widetilde{f}_1(w)\right) \implies \widetilde{f}_0(w) = \widetilde{f}_1(w) \implies w \in A \implies W \subseteq A$$

W is a neighborhood of y that is fully contained in A. So every $y \in A$ is an interior point, hence A is open.

Therefore, A is both open and closed. Hence $A = \varnothing$ or $A = X$. ■

Lemma 4.1.3

Let $p : E \to B$ be a covering map; let $p(e_0) = b_0$. Let $F : I \times I \to B$ be continuous with $F(0,0) = b_0$. Then there is a unique lifting of F to a continuous map $\widetilde{F} : I \times I \to E$ such that $\widetilde{F}(0,0) = e_0$. Furthermore, if F is a path homotopy, so is $\widetilde{F}$.

Proof. Given F, we first define $F(0,0) = e_0$. Then we can extend F to the left hand edge $0 \times I$ and bottom edge $I \times 0$ of the square $I \times I$ using Lemma 4.1.1. Because F restricted on $0 \times I$ is basically a path, so $F\big|_{0 \times I} \equiv f : I \to B$. We can definitely find a unique lifting $\bar{f}$ of f that starts at b_0. We can then define $\widetilde{F}(0,t) = \bar{f}(t)$. In a similar manner, we can extend $\widetilde{F}$ on $I \times 0$ too. Now we will extend it to whole $I \times I$.

Since $p : E \to B$ is a covering map, for every $x \in B$, there exists a neighborhood U_x that is evenly covered by p. U_x is open in B and it contains x, so the collection $\{U_x\}_{x \in B}$ is an open cover of B. $F : I \times I \to B$ is a continuous map, so the preimages of open sets are open in $I \times I$. Therefore, the collection $\left\{F^{-1}(U_x)\right\}_{x \in B}$ is an open cover of I.

$I \times I$ is bounded and closed subset of $\mathbb{R}^2$. By Heine-Borel Theorem, I is a compact metric space. Let δ be the Lebesgue Number of this open cover. Now we are gonna choose subdivisions

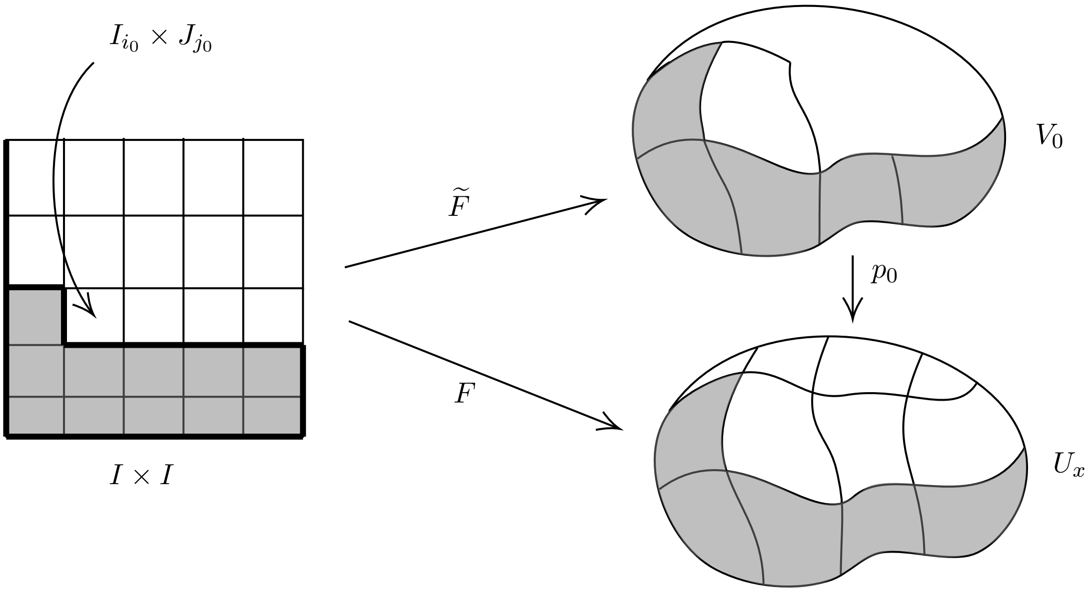
$$0 = s_0 < s_1 < s_2 \cdots < s_m = 1 \quad \text{and} \quad 0 = t_0 < t_1 < t_2 \cdots < t_n = 1$$

such that $\operatorname{diam}\left([s_{i-1}, s_i] \times [t_{j-1}, t_j]\right) < \delta$ for every $i \in \{1, 2, \ldots, m\}, j \in \{1, 2, \ldots, n\}$. Let $I_i = [s_{i-1}, s_i]$ and $J_j = [t_{j-1}, t_j]$. By Lebesgue Number Lemma,

$$I_i \times J_j \subseteq F^{-1}(U_x) \text{ for some } x \in B \implies F(I_i \times J_j) \subseteq U_x \text{ for some } x \in B$$

Now we shall define $F : I \times I \to E$ inductively beginning with $I_1 \times J_1$, then continuing with the other rectangles $I_i \times J_1$ of the "bottom row", and then with the rectangles $I_i \times J_2$ and so on. Now, fix $1 \leq i_0 \leq m$ and $1 \leq j_0 \leq n$. Denote by A the union of $0 \times I$ and $I \times 0$ and all the rectangles "previous" to $I_{i_0} \times J_{j_0}$. In other words,

$$A = (0 \times I) \cup (I \times 0) \cup \left(\bigcup_{j < j_0} (I_i \times J_j)\right) \cup \left(\bigcup_{i < i_0} (I_i \times J_{j_0})\right)$$



A is colored in grey in the figure above, and the edges of A are the thick black edges. For our inductive construction, the base case is $\widetilde{F}$ can be constructed on $I_1 \times J_1$. By the inductive hypothesis, the lifting $\widetilde{F}$ of $F\big|_A$ can be constructed. Now we will define $\widetilde{F}$ on $I_{i_0} \times J_{j_0}$ in the following way.

One can find U_x being open in B that is evenly covered by p such that $F\left(I_{i_0} \times J_{j_0}\right) \subseteq U_x$. Let $C = A \cap \left(I_{i_0} \times J_{j_0}\right)$ be the union of left and bottom edge of $I_{i_0} \times J_{j_0}$. According to the inductive hypothesis, $\widetilde{F}$ is defined on $C \subset A$.

$$p\left(\widetilde{F}\left(I_{i_0} \times J_{j_0}\right)\right) = F\left(I_{i_0} \times J_{j_0}\right) \subseteq U_x \implies \widetilde{F}\left(I_{i_0} \times J_{j_0}\right) \subseteq p^{-1}\left(U_x\right) = \bigsqcup_{\alpha \in J} V_\alpha$$
$$\implies \widetilde{F}(C) \subseteq \widetilde{F}\left(I_{i_0} \times J_{j_0}\right) \subseteq \bigsqcup_{\alpha \in J} V_\alpha$$

C is connected, $\widetilde{F}$ is continuous, so $\widetilde{F}(C)$ is connected. The slices V_α are open and disjoint. If $\widetilde{F}(C)$ is distributed among the V_α's, connectedness of $\widetilde{F}(C)$ is contradicted. So $\widetilde{F}(C)$ must lie in a single slice V_α. Let us denote the V_α entirely containing $\widetilde{F}(C)$ by V_0. Let $p_0 : V_0 \to U_x$ be the restriction of p on V_0. p_0 is a homeomorphism. We extend $\widetilde{F}$ from C to $I_{i_0} \times J_{j_0}$ by

$$\widetilde{F}(x) = \left(p_0^{-1} \circ F\right)(x) = p_0^{-1}\left(F(x)\right)$$

Now we are only left with the base case of $I_1 \times J_1$. We showed in the beginning that $\widetilde{F}$ can be constructed on $(0 \times I) \cup (I \times 0)$. And the left and bottom edge of $I_1 \times J_1$ is a subset of $(0 \times I) \cup (I \times 0)$. Using this, we can construct $\widetilde{F}$ on $I_1 \times J_1$ using a similar construction as we've just done for $I_{i_0} \times J_{j_0}$. Thus the base case is proved.

So we have proved that $\widetilde{F}$ can be constructed on each of the rectangles $I_i \times J_i$. The rectangles are closed, and $\widetilde{F}$ agrees on the boundaries of adjecent rectangles. Therefore, by pasting lemma, the extended map $\widetilde{F}$ on $I \times I$ is continuous. By construction, it satisfies $p \circ \widetilde{F} = F$. So $\widetilde{F}$ is indeed a lifting of F.

About the uniqueness issue, assume the contrary. If $\bar{F} : I \times I \to E$ is another lifting of $F : I \times I \to B$ with $\bar{F}(0,0) = e_0$, then the set

$$\left\{(x,y) \in I \times I \ : \ \widetilde{F}(x,y) = \bar{F}(x,y\right\}$$

contains $(0,0)$, so it is not empty. Hence, by Lemma 4.1.2, this set equal to $I \times I$. Therefore, $\widetilde{F}$ is unique.

Now, let g and h be two paths in B and F be a path homotopy between them. $F(0,0) = b_0$, so both g and h start at b_0. By Lemma 4.1.1, there exists unique lifts of g and h, namely $\widetilde{g}$ and $\widetilde{h}$, such that $\widetilde{g}(0) = \widetilde{h}(0) = e_0$. So $g = p \circ \widetilde{g}$ and $h = p \circ \widetilde{h}$. We shall prove that $\widetilde{F}$ is a path homotopy between $\widetilde{g}$ and $\widetilde{h}$ in E. Suppose they both end at b_1.

$$F(s,0) = (p \circ \widetilde{g})(s) \text{ and } F(s,1) = \left(p \circ \widetilde{h}\right)(s)$$
$$F(0,t) = (p \circ \widetilde{g})(0) = \left(p \circ \widetilde{h}\right)(0) = b_0 \text{ and } F(1,t) = (p \circ \widetilde{g})(1) = \left(p \circ \widetilde{h}\right)(1) = b_1$$

Recall that, we defined the unique map $\widetilde{F}$ as $\widetilde{F} = p\big|_{V_0}^{-1} \circ F$. Hence

$$\widetilde{F}(s,0) = p\big|_{V_0}^{-1}\left(F(s,0)\right) = p\big|_{V_0}^{-1}\left((p \circ \widetilde{g})(s)\right) = p\big|_{V_0}^{-1}\left(p\left(\widetilde{g}(s)\right)\right) = \widetilde{g}(s)$$
$$\widetilde{F}(s,1) = p\big|_{V_0}^{-1}\left(F(s,1)\right) = p\big|_{V_0}^{-1}\left(\left(p \circ \widetilde{h}\right)(s)\right) = p\big|_{V_0}^{-1}\left(p\left(\widetilde{h}(s)\right)\right) = \widetilde{h}(s)$$

We've seen before that $p\left(\widetilde{F}(0 \times I)\right) = F(0 \times I) = \{b_0\}$, so $\widetilde{F}(0 \times I) \subseteq p^{-1}(b_0)$. $0 \times I$ is connected, so is $\widetilde{F}(0 \times I)$. By Lemma 3.1.1, $p^{-1}(b_0)$ has the discrete topology; so every singleton set is open in $p^{-1}(b_0)$. If $\widetilde{F}(0 \times I)$ has more that one element, it can be splitted into two disjoint nonempty open sets, contradicting its connectedness. Hence $\widetilde{F}(0 \times I)$ must be a singleton set. Similarly $\widetilde{F}(1 \times I)$ is also a singleton set. In fact,

$$\begin{aligned}\widetilde{F}(0,t) = p|_{V_0}^{-1}(F(0,t)) &= p|_{V_0}^{-1}((p \circ \widetilde{g})(0)) = p|_{V_0}^{-1}(p(\widetilde{g}(0))) = \widetilde{g}(0)\\ &= p|_{V_0}^{-1}\left(\left(p \circ \widetilde{h}\right)(0)\right) = p|_{V_0}^{-1}\left(p\left(\widetilde{h}(0)\right)\right) = \widetilde{h}(0)\\ \widetilde{F}(1,t) = p|_{V_0}^{-1}(F(1,t)) &= p|_{V_0}^{-1}((p \circ \widetilde{g})(1)) = p|_{V_0}^{-1}(p(\widetilde{g}(1))) = \widetilde{g}(1)\\ &= p|_{V_0}^{-1}\left(\left(p \circ \widetilde{h}\right)(1)\right) = p|_{V_0}^{-1}\left(p\left(\widetilde{h}(1)\right)\right) = \widetilde{h}(1)\end{aligned}$$

Therefore, $\widetilde{F}$ is a path homotopy between $\widetilde{g}$ and $\widetilde{h}$. ■

Using Lemma 4.1.1 and Lemma 4.1.3, we can have the following theorem.

Theorem 4.1.4

Let $p : E \to B$ be a covering map and $p(e_0) = b_0$. Let f and g be two paths in B from b_0 to b_1; let $\bar{f}$ and $\bar{g}$ be their respective unique liftings to paths in E beginning at e_0. If $f \simeq_p g$, then $\bar{f}$ and $\bar{g}$ end at the same point and $\bar{f} \simeq_p \bar{g}$.

§4.2 Lifting Correspondence

Definition 4.2.1 (Lifting Correspondence). Let $p : E \to B$ be a covering map and $b_0 \in B$. Choose $e_0 \in E$ such that $p(e_0) = b_0$. Given an element $[f] \in \pi_1(B, b_0)$, let $\bar{f}$ be the unique lifting of f to a path in E such that $\bar{f}(0) = e_0$. Then we define a set map ϕ

$$\phi : \pi_1(B, b_0) \to p^{-1}(b_0)\ , \qquad \phi([f]) = \bar{f}(1)$$

We call ϕ the **lifting correspondence** derived from the covering map p. It depends on the choice of the point e_0.

ϕ is indeed a well-defined set map. Since $[f] \in \pi_1(B, b_0)$, f is a loop in B based at b_0.

$$b_0 = f(0) = f(1) = p\left(\bar{f}(1)\right) \implies \bar{f}(1) \in p^{-1}(b_0)$$

Now let $[f] = [g]$ for two loops f and g. So f and g are path homotopic. Let $\bar{f}$ and $\bar{g}$ be their respective unique liftings to paths in E beginning at e_0. Then by Theorem 4.1.4, $\bar{f}$ and $\bar{g}$ ends at the same point. So $\bar{f}(1) = \bar{g}(1)$. Therefore, $\phi([f]) = \phi([g])$. Hence ϕ is a well-defined set map.

Theorem 4.2.1

Let $p : E \to B$ be a covering map and $p(e_0) = b_0$. If E is path connected, then the lifting correspondece

$$\phi : \pi_1(B, b_0) \to p^{-1}(b_0)$$

is surjective. If E is simply connected, it is bijective.

Proof. Suppose E is path connected. $p(e_0) = b_0$, so $e_0 \in p^{-1}(b_0) \subseteq E$. Now, take any $e_1 \in p^{-1}(b_0)$. e_0 and e_1 are elements of E, so there exists a path $\bar{f}$ in E from e_0 to e_1. In other words, $\bar{f} : I \to E$ such that $\bar{f}(0) = e_0$ and $\bar{f}(1) = e_1$. Consider the map $f = p \circ \bar{f}$. It is a map from I to B. In fact, it is a loop based at b_0. Because

$$f(0) = p\left(\bar{f}(0)\right) = p(e_0) = b_0 \text{ and } f(1) = p\left(\bar{f}(1)\right) = p(e_1) = b_0$$

So $[f] \in \pi_1(B, b_0)$. It has a unique lifting in E that starts at e_0. $\bar{f}$ is one such lift. Therefore,

$$\phi([f]) = \bar{f}(1) = e_1$$

In other words, for every $e_1 \in p^{-1}(b_0)$, we can find $[f] \in \pi_1(B, b_0)$ such that $\phi([f]) = e_1$. Hence ϕ is surjective.

Now suppose E is simply connected. Then it is also path connected. As a result, ϕ is surjective. So we only need to prove that ϕ is injective. Let $[f], [g] \in \pi_1(B, b_0)$ such that $\phi([f]) = \phi([g])$. We need to show that $[f] = [g]$.

Let $\bar{f}$ and $\bar{g}$ be unique lifts of f and g, respectively, such that both of them start at e_0. In other words, $\bar{f}, \bar{g} : I \to E$ with $\bar{f}(0) = \bar{g}(0) = e_0$ and $p \circ \bar{f} = f, p \circ \bar{g} = g$. Since $\phi([f]) = \phi([g])$, we have $\bar{f}(1) = \bar{g}(1)$. Thus, $\bar{f}$ and $\bar{g}$ have the same initial and final points in a simply connected space E. Therefore, by Lemma 2.2.3, $\bar{f} \simeq \bar{g}$.

Let $\bar{F}$ be the path homotopy between $\bar{f}$ and $\bar{g}$. By the 1st key fact used in proving Theorem 1.1.3, $p \circ \bar{F}$ is a path homotopy between $p \circ \bar{f} = f$ and $p \circ \bar{g} = g$. So $f \simeq g$, and thus we have $[f] = [g]$. Hence ϕ is injective. ∎

Now the main theorem of this lecture.

Theorem 4.2.2

The fundamental group of S^1 is isomorphic to the additive group of integers.

Proof. Let $p : \mathbb{R} \to S^1$ be the covering map of Theorem 3.1.3, defined by $p(x) = (\cos 2\pi x, \sin 2\pi x)$. Let $e_0 = 0$, and $b_0 = p(e_0) = p(0) = (1, 0)$. We've seen earlier that $p^{-1}(b_0) = \mathbb{Z}$. Since $\mathbb{R}$ is simply connected, by Theorem 4.2.1, the lifting correspondence

$$\phi : \pi_1\left(S^1, b_0\right) \to \mathbb{Z}$$

is bijective. So we only need to show that ϕ is a group homomorphism.

Given $[f], [g] \in \pi_1\left(S^1, b_0\right)$, let $\bar{f}$ and $\bar{g}$ be their respective unique liftings on $\mathbb{R}$ both starting at $0 = e_0$. Let $\phi([f]) = \bar{f}(1) = n$ and $\phi([g]) = \bar{g}(1) = m$, where m and n are integers. Now, we define a new path $\widetilde{g}$ on $\mathbb{R}$ by

$$\widetilde{g} : I \to \mathbb{R}\ , \qquad \widetilde{g}(s) = n + \bar{g}(s)$$

From the definition of p, one immediately finds that

$$p(n + x) = (\cos(2\pi n + 2\pi x), \sin(2\pi n + 2\pi x)) = (\cos 2\pi x, \sin 2\pi x) = p(x)$$

$$(p \circ \widetilde{g})(s) = p(\widetilde{g}(s)) = p(n + \bar{g}(s)) = p(\bar{g}(s)) = g(s)$$

Also, $\widetilde{g}(0) = n + \bar{g}(0) = n$. So $\widetilde{g}$ is the unique lifting of g thet begins at n. Since $\bar{f}(1) = n = \widetilde{g}(0)$, we can form the product of paths $\bar{f} * \widetilde{g}$. We claim that $\bar{f} * \widetilde{g}$ is the lift of $f * g$ that begins at 0.

Using the 2nd key fact used in proving Theorem 1.1.3, we obtain that

$$p \circ \left(\bar{f} * \widetilde{g}\right) = \left(p \circ \bar{f}\right) * (p \circ \widetilde{g}) = f * g$$

Also, $\bar{f} * \widetilde{g}$ begins at where $\bar{f}$ begins, *i.e.* it begins at $\bar{f}(0) = 0$. So it is indeed the unique lift of $f * g$ that begins at 0. Now, the endpoint of $\bar{f} * \widetilde{g}$ is the endpoint of $\widetilde{g}$, *i.e.* it ends at $\widetilde{g}(1) = n + \bar{g}(1) = n + m$. Then using the definition of lifting correspondence, we get

$$\phi([f] * [g]) = \phi([f * g]) = \left(\bar{f} * \widetilde{g}\right)(1) = n + m = \phi([f]) + \phi([g])$$

So ϕ is indeed a homomorphism from $\pi_1\left(S^1, b_0\right)$ to the additive group of integers. We've seen earlier that ϕ is bijective. Therefore, $\pi_1\left(S^1, b_0\right)$ is isomorphic to $\mathbb{Z}$. ∎

5 Lecture 5

§5.1 Retraction

Definition 5.1.1 (Retraction). If $A \subset X$, a retraction of X onto A is a continuou map $r : X \to A$ such that $r\big|_A = \mathrm{id}_A$. If such r exists, we say that A is a retract of X.

Lemma 5.1.1

If A is a retract of X, then the homomorphism of fundamental groups induced by the inclusion map $j : A \to X$ is injective.

Proof. If $r : X \to A$ is a retraction, then $r \circ j = i_a$ is the identity map of the pointed topological space (A, a). By Theorem 2.3.2, $(r \circ j)_* = r_* \circ j_* = (i_a)_*$ is the identity homomorphism of the fundamental group $\pi_1(A, a)$. Therefore, j_* has a left inverse.

Claim — Let $f : X \to Y$ and $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$ (in other words, f has a left inverse). Then f is injective.

Proof. Suppose $f(x_1) = f(x_2)$, we need to show that $x_1 = x_2$.

$$f(x_1) = f(x_2) \implies g(f(x_1)) = g(f(x_2)) \implies \mathrm{id}_X(x_1) = \mathrm{id}_X(x_2) \implies x_1 = x_2$$

So f is injective. □

By the claim, j_* is injective. ■

Theorem 5.1.2 (No-retraction theorem)

There is no retraction of B^2 onto S^1.

Proof. Assume the contrary. By Lemma 5.1.1, the homomorphism induced by the inclusion map $j : S^1 \to B^2$ is injective. In other words, $j_* : \pi_1(S^1, b_0) \to \pi_1(B^2, b_0)$ is injective.

As B^2 is convex, the fundamental group $\pi_1(B^2, b_0)$ is trivial (contains only the identity element). But the fundamental group $\pi_1(S^1, b_0)$ is non-trivial (we know from Theorem 4.2.2 that this is isomorphic to $\mathbb{Z}$). So it's not possible to have an injective map from $\pi_1(S^1, b_0)$ to $\pi_1(B^2, b_0)$. Contradiction!

Hence S^1 is not a retract of B^2. ■

Lemma 5.1.3

Let $h : S^1 \to X$ be continuous. Then the following are equivalent:

(1) h is nulhomotopic.
(2) h extends to a continuos map $k : B^2 \to X$.
(3) h_* is the trivial homomorphism of fundamental groups.

Proof. **(1)⇒(2).** Let $H : S^1 \times I \to X$ be a path homotopy between h and a constant map. We define $\pi : S^1 \times I \to B^2$ by $\pi(x, t) = (1 - t)x$.

Claim — π is a quotient map.

Proof. Consider a map $\pi_1 : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $\pi_1(x_1, x_2, t) = ((1 - t)x_1, (1 - t)x_2)$. The map $(x_1, x_2, t) \mapsto 1 - t$ is continuous, so is $(x_1, x_2, t) \mapsto x_1$. Then as a product of continuous real values functions, $(x_1, x_2, t) \mapsto (1-t)\, x_1$ is continuous. Similarly, the second coordinate of π_1 is also continuous. Therefore, π_1 is continuous. Since π is the restriction of π_1 on $S^1 \times I$, π is also continuous.

If $y \in B^2$, then $\|y\| \leq 1$. If $\|y\| = 0$, $y = \mathbf{0}$. Then taking $t = 1$ gives us $\pi\,(x,t) = \mathbf{0}$. If $\|y\| \neq 0$, Then $\frac{y}{\|y\|}$ has unit norm, so

$$\pi\left(\frac{y}{\|y\|}, 1 - \|y\|\right) = \|y\| \frac{y}{\|y\|} = y$$

So π is surjective.

$S^1 \times I$ is a bounded closed subset of $\mathbb{R}^3$, so it's compact by Heine-Borel Theorem. B^2 is a subspace of $\mathbb{R}^2$, by Lemma 0.4.2, B^2 is hausdorff. π is a continuous map from a compact space to a hausdorff space. Therefore, by Proposition 0.7.5, π is a closed map.

π is continuous, surjective, closed. Lemma 0.10.1 gives us π is a quotient map. □

We want this following diagram to commute. In other words, we want $H = k \circ \pi$.

$$\begin{tikzcd} S^1 \times I \arrow[d, "\pi"'] \arrow[dr, "H"] & \\ B^2 \arrow[r, "k"'] & X \end{tikzcd}$$

Note that, the preimage of $\mathbf{0}$ under π is $S^1 \times \{1\}$. As H is a homotopy between h and a constant map,

$$H\,(x,1) = e_{x_0}\,(x) = x_0 \implies H\left(\pi^{-1}\left(\{0\}\right)\right) = H\left(S^1 \times \{1\}\right) = \{x_0\}$$

where e_{x_0} is a constant map that maps all of S^1 to $x_0 \in X$. So H is constant on $\pi^{-1}\left(\{0\}\right)$.

For $b \neq \mathbf{0}$, $\pi^{-1}\left(\{b\}\right)$ is a singleton set. Because,

$$(1-t)x = b \implies |1-t|\,\|x\| = \|b\| \implies 1 - t = \|b\| \implies t = 1 - \|b\| \text{ and } x = \frac{b}{\|b\|}$$

So H is trivially constant on $\pi^{-1}\left(\{b\}\right)$. Therefore, by Theorem 0.10.3, H induces a map $k : B^2 \to X$ such that $k \circ \pi = H$. Since H is continuous, by the same theorem, k is continuous.

Now we need to show that k is an extension of h. For $x \in S^1$, $H\,(x,0) = h(x)$ since H is a homotopy. As a result,

$$h(x) = H\,(x,0) = k\,(\pi\,(x,0)) = k\,(x) \implies k\big|_{S^1} = h$$

So h extends to a continuous map $k : B^2 \to X$.

(2)⇒(3). If $j : S^1 \to B^2$ is the inclusion map, then $h = k \circ j$.

$$\begin{tikzcd} S^1 \arrow[r, "j"] \arrow[rr, bend right, "h=k\circ j"'] & B^2 \arrow[r, "k"] & X \end{tikzcd}$$

Hence $h_* = (k \circ j)_* = k_* \circ j_*$. As B^2 is convex, the fundamental group $\pi_1\left(B^2, b_0\right)$ is trivial. So $j_* : \pi_1\left(S^1, b_0\right) \to \pi_1\left(B^2, b_0\right)$ must be the trivial homomorphism, a homomorphism that maps all of $\pi_1\left(S^1, b_0\right)$ to the identity element of the one-element group $\pi_1\left(B^2, b_0\right)$.

We take any $g \in \pi_1\left(S^1, b_0\right)$. Let e_1 and e_2 respectively denote the identity elements of $\pi_1\left(B^2, b_0\right)$ and $\pi_1\,(X, x_0)$, where $x_0 = h(b_0)$. We've just shown that $j_*\,(g) = e_1$. Group homomorphisms map identity elements to identity elements. So we have

$$h_*\,(g) = k_*\,(j_*\,(g)) = k_*\,(e_1) = e_2$$

So h_* is the trivial homomorphism.

(3)⇒(1). Let $p : \mathbb{R} \to S^1$ be the standard covering map given by $p(x) = (\cos 2\pi x, \sin 2\pi x)$. Also, let p_0 be the restriction of p to I, *i.e.* $p_0 = p\big|_I$.

$h_* : \pi_1(S^1, b_0) \to \pi_1(X, x_0)$ with $h(b_0) = x_0$. Since S^1 is path connected, $\pi_1(S^1, b_0)$ is isomorphic to $\pi_1(S^1, b_1)$ for any $b_0, b_1 \in S^1$. So we can assume without loss of generality that $b_0 = (1, 0)$.

Claim — $[p_0]$ generates $\pi_1(S^1, b_0)$.

Proof. We proved in Theorem 4.2.2 that $\pi_1(S^1, b_0)$ is isomorphic to $\mathbb{Z}$. In context of the proof of the same theorem, $\phi : \pi_1(S^1, b_0) \to \mathbb{Z}$ is a group isomorphism. So $\pi_1(S^1, b_0)$ is also an infinite cyclic group generated by $\phi^{-1}(1)$. Now we need to show that $\phi^{-1}(1) = [p_0]$, *i.e.* $\phi([p_0]) = 1$.

p_0 is a path in S^1 beginning at $(1, 0)$, and $p : \mathbb{R} \to S^1$ is a covering map. So p_0 has a unique lifting $\widetilde{p_0} : I \to \mathbb{R}$ such that beginning at 0, and $\widetilde{p_0}(1) = \phi([p_0])$.

$$(\cos 2\pi s, \sin 2\pi s) = p_0(s) = p(\widetilde{p_0}(s)) = (\cos 2\pi\widetilde{p_0}(s), \sin 2\pi\widetilde{p_0}(s))$$

This gives us

$$2\pi\widetilde{p_0}(s) = 2\pi s + 2\pi n\ , \text{ for some } n \in \mathbb{Z} \implies \widetilde{p_0}(s) = s + n$$

Since $\widetilde{p_0}$ begins at 0, $\widetilde{p_0}(s) = s$. So $\phi([p_0]) = \widetilde{p_0}(1) = 1$, and $\pi_1(S^1, b_0)$ is generated by $[p_0]$. □

The inverse element of $[p_0]$ is $[\overline{p_0}]$, where $\overline{p_0}(s) = p_0(1 - s)$ is the reverse loop of p_0. We denote $[\overline{p_0}] = [p_0^{-1}]$. So all the elements of $\pi_1(S^1, b_0)$ can be expressed as

$$\ldots, [p_0^{-3}], [p_0^{-2}], [p_0^{-1}], [p_0^{0}], [p_0], [p_0^{2}], [p_0^{3}], \ldots$$

where $[p_0^0]$ is the identity element, the constant loop based at b_0. And

$$p_0^n = \underbrace{p_0 * p_0 * \cdots * p_0}_{n \text{ times}} \quad \text{and} \quad p_0^{-n} = \underbrace{p_0^{-1} * p_0^{-1} * \cdots * p_0^{-1}}_{n \text{ times}}\ , \quad \text{for } n \in \mathbb{N}$$

Let $f = h \circ p_0$. h_* is the trivial homomorphism, so $[h \circ p_0] = h_*([p_0]) = [e_{x_0}]$, where e_{x_0} is the constant loop based at x_0. So $f \simeq_p e_{x_0}$ and let $F : I \times I \to X$ be a path homotopy between them.

Consider the map $p_0 \times \mathrm{id} : I \times I \to S^1 \times I$ defined by $(p_0 \times \mathrm{id})(s, t) = (p_0(s), t)$.

Claim — $p_0 \times \mathrm{id}$ is a quotient map.

Proof. $(s, t) \mapsto s \mapsto p_0(s)$ is continuous, $(s, t) \mapsto t$ is continuous. Both coordinates of $p_0 \times \mathrm{id}$ are continuous, so $p_0 \times \mathrm{id}$ is continuous.

p_0 is a surjective map. So for any $x \in S^1$, we can find $s_0 \in I$ such that $p_0(s_0) = x$. Now for any $(x, t) \in S^1 \times I$, $(p_0 \times \mathrm{id})(s_0, t) = (x, t)$, so $p_0 \times \mathrm{id}$ is surjective.

$I \times I$ is a bounded closed subset of $\mathbb{R}^2$, so it's compact by Heine-Borel Theorem. $S^1 \times I$ is a subspace of $\mathbb{R}^3$, thus hausdorff. $p_0 \times \mathrm{id}$ is a continuous map from a compact space to a hausdorff space. Therefore, by Proposition 0.7.5, $p_0 \times \mathrm{id}$ is a closed map.

$p_0 \times \mathrm{id}$ is continuous, surjective, closed. By Lemma 0.10.1, it is a quotient map. □

$p_0(0) = b_0 = p_0(1)$, othewise p_0 is injective on $(0, 1)$. So $p_0 \times \mathrm{id}$ is injective on $(0, 1) \times I$. Each point of the form (b_0, t) has two preimages under $p_0 \times \mathrm{id}$, namely $(0, t)$ and $(1, t)$. Since F is a path homotopy between f and e_{x_0},

$$F(0, t) = x_0 = F(1, t)$$

So F is constant on the preimage of (b_0, t). For all other points (x, t), the preimage of (x, t) is a singleton set. So F is trivially constant on the preimage.

Therefore, by Theorem 0.10.3, F induces a map $H : S^1 \times I \to X$ such that $H \circ (p_0 \times \mathrm{id}) = F$. Since F is continuous, by the same theorem, H is continuous.

$$\begin{tikzcd} I \times I \arrow[d, "p_0 \times \mathrm{id}"'] \arrow[dr, "F"] & \\ S^1 \times I \arrow[r, "H"'] & X \end{tikzcd}$$

Since F is a path homotopy between f and e_{x_0},

$$F(s,0) = f(s) = h(p_0(s)) \quad \text{and} \quad F(s,1) = e_{x_0}(s) = x_0 \ \forall s \in I$$

If $x \in S^1$, there exists some $s_0 \in I$ such that $p_0(s_0) = x$.

$$H(x,0) = H(p_0(s_0), 0) = H((p_0 \times \mathrm{id})(s_0, 0)) = F(s_0, 0) = h(p_0(s_0)) = h(x)$$

This is true for every $x \in S^1$. On the other hand, let $\widetilde{e_{x_0}} : S^1 \to X$ be the continuous map that maps all of S^1 to $x_0 \in X$.

$$H(x,1) = H(p_0(s_0), 1) = H((p_0 \times \mathrm{id})(s_0, 1)) = F(s_0, 1) = x_0 = \widetilde{e_{x_0}}(x)$$

Therefore, H is a homotopy between h and $\widetilde{e_{x_0}}$. This proves that h is nulhomotopic.

We've proved that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1). So the three statements are equivalent. ■

Corollary 5.1.4

The inclusion map $j : S^1 \to \mathbb{R}^2 \setminus \{\mathbf{0}\}$ is not nulhomotopic. Also, the identity map $i : S^1 \to S^1$ is not nulhomotopic.

Proof. There is a retraction of $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ onto S^1 given by $r(x) = \frac{x}{\|x\|}$ (check that this is continuous). By Lemma 5.1.1, the induced homomorphism $j_* : \pi_1(S^1, b_0) \to \pi_1(\mathbb{R}^2 \setminus \{\mathbf{0}\}, b_0)$ is injective.

Assume for the sake of contradiction that j is nulhomotopic. Then by Lemma 5.1.3, j_* is a trivial homomorphism. But we've just shown that j_* is injective. As $\pi_1(S^1, b_0)$ is nontrivial, the image of j_* can't be just the identity element of $\pi_1(\mathbb{R}^2 \setminus \{\mathbf{0}\}, b_0)$. So we arrive at a contradiction! Therefore, j is not nulhomotopic.

Similarly, the induced homomorphism i_* of the identity map $i : (S^1, b_0) \to (S^1, b_0)$ is the identity group homomorphism. i_* can't be trivial since $\pi_1(S^1, b_0)$ is nontrivial. Hence, using Lemma 5.1.3 again, i is not nulhomotopic. ■

Theorem 5.1.5

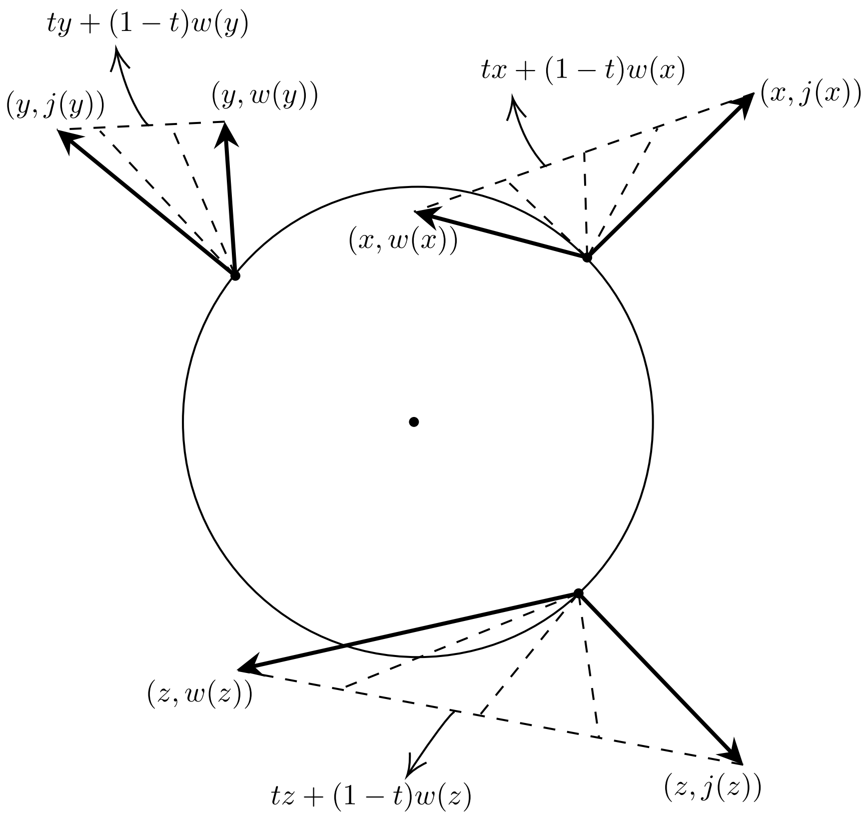
Given a nonvanishing vector field on B^2, there exists a point of S^1 where the vector field points directly inward, and a point of S^1 where it points directly outward.

Proof. A **vector field** on B^2 is an ordered pair $(x, v(x))$, where $x \in B^2$ and $v : B^2 \to \mathbb{R}^2$ is a continuous map. To say that a vector field is **nonvanishing** means that $v(x) \neq \mathbf{0}$ for every $x \in B^2$. In such a case, v actually maps B^2 to $\mathbb{R}^2 \setminus \{\mathbf{0}\}$.

Assume for the sake of contradiction that $v(x)$ doesn't directly point inward at any point $x \in S^1$. Let w be the restriction of v on S^1, *i.e.* $w = v\big|_{S^1}$. In other words, $w : S^1 \to \mathbb{R}^2 \setminus \{\mathbf{0}\}$ has a continuous extension given by $v : B^2 \to \mathbb{R}^2 \setminus \{\mathbf{0}\}$. Then by Lemma 5.1.3, w is nulhomomtopic.

Now observe that w is homotopic to the inclusion map $j : S^1 \to \mathbb{R}^2 \setminus \{\mathbf{0}\}$; the homotopy $F : S^1 \times I \to \mathbb{R}^2 \setminus \{\mathbf{0}\}$ is given by

$$F(x,t) = t\, j(x) + (1-t)\, w(x) = tx + (1-t)\, w(x)\ , \quad \text{for } x \in S^1, t \in I$$



But we are yet to show that $F(x,t) \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$. From the fact that $w(x) \neq \mathbf{0}$ and $j(x) \neq \mathbf{0}$, it immediately follows that $F(x,1) \neq \mathbf{0}$ and $F(x,0) \neq \mathbf{0}$. If for some $t \in (0,1)$, $F(x,t) = \mathbf{0}$ then

$$tx + (1-t)\,w(x) = \mathbf{0} \implies w(x) = \frac{-t}{1-t}x$$

This means $w(x)$ is a negative scalar multiple of x. Geometrically this interprets as $w(x)$ pointing directly inwards, which is a contradiction! So $F(x,t) \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$ for every $x \in S^1, t \in I$.

So we have proved that $F : S^1 \times I \to \mathbb{R}^2 \setminus \{\mathbf{0}\}$ is a homotopy between w and j. In other words, $w \simeq j$. We've also proved that w is nulhomotopic. Since $\simeq$ is an equivalence relation, $j : S^1 \to \mathbb{R}^2 \setminus \{\mathbf{0}\}$ is nulhomotopic. But this contradicts with Corollary 5.1.4. Hence, there must exist a point $x \in S^1$ such that $v(x)$ points directly inward.

If we had started the proof with the vector field $(x, -v(x))$ and carried out the same arguments, we would have reached the conclusion that $-v(x)$ points directly inward for some $x \in S^1$. In other words, $v(x)$ points directly outward for some $x \in S^1$. ∎

§5.2 Fixed Point Theorems

Lemma 5.2.1

Let $f : X \to X$ be continuous. If $X = [0,1]$ then there exists $x \in X$ such that $f(x) = x$. This x is called a fixed point of f.

Proof. Let $g : [0,1] \to [-1,1]$ be defined by $g(x) = f(x) - x$. Then g is continuous. $f(x) \in [0,1]$, so $f(0) \geq 0$. In other words,

$$g(0) = f(0) - 0 \geq 0$$

Similarly, $f(x) \in [0,1]$, gives us $f(1) \leq 1$. In other words,

$$g(1) = f(1) - 1 \leq 0$$

Combining these two inequalities, we get $g(1) \leq 0 \leq g(0)$. By *Intermediate Value Theorem*, there exists $y \in [0,1]$ such that $g(y) = 0$. For that y, $0 = g(y) = f(y) - y$. Hence $f(y) = y$. ∎

Theorem 5.2.2 (Brouwer fixed point theorem)
If $f : B^2 \to B^2$ is continuous, then there exists $x \in B^2$ such that $f(x) = x$.

Proof. We shall proceed by contradiction. Suppose $f(x) \neq x$ for every $x \in B^2$. We define $v(x) = f(x) - x$. Clearly v is continuous. So we obtain a nonvanishing vector field $(x, v(x))$ on B^2.

Using Theorem 5.1.5, we get that there is a point $x \in S^1$ at which $v(x)$ points directly outward. In other words, $v(x) = ax$ for some positive real number a.

$$f(x) - x = v(x) = ax \implies f(x) = (1+a)x \implies \|f(x)\| = |1+a|\,\|x\| = 1 + a > 1$$

But f is a map from B^2 to itself. So $\|f(x)\| \leq 1$ for every $x \in B^2$. Contradiction! Therefore, there must exist some $x \in B^2$ such that $f(x) = x$. ■

Now we shall prove a linear algebra result using topological techniques.

Corollary 5.2.3
Let A be a 3×3 matrix with positive real entries. Then A has a positive real eigenvalue.

Proof. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation whose matrix representation (with reference to the standard basis for $\mathbb{R}^3$) is A. Let F denote the first octant of $\mathbb{R}^3$. In other words,

$$F = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \;:\; x_1 \geq 0\,,\; x_2 \geq 0\,,\; x_3 \geq 0\right\}$$

Let B be the intersection of F with S^2, *i.e.* $B = F \cap S^2$. It's easy to see that B is homeomorphic to the unit ball B^2 (left as an exercise for the reader). Now we claim that Brouwer fixed point theorem holds for continuous maps from B to itself.

Claim — Let $f : B \to B$ be continuous, then there exists $y \in B$ such that $f(y) = y$.

Proof. Let $g : B \to B^2$ be a homeomorphism. Then $g \circ f \circ g^{-1} : B^2 \to B^2$ is a continuous map from B^2 to itself. According to Brouwer fixed point theorem, it has a fixed point $x \in B^2$.

$$g\left(f\left(g^{-1}(x)\right)\right) = x \implies f\left(g^{-1}(x)\right) = g^{-1}(x) \implies f(y) = y$$

where $y = g^{-1}(x) \in B$. □

Take $x = (x_1, x_2, x_3) \in B$. Then all the components of x are nonnegative. Since $\|x\| = 1$, at least one of the components must be positive. A is a 3×3 matrix with positive real entries. So $T(x) = Ax$ has all components positive. As a result, $\frac{T(x)}{\|T(x)\|} \in B$.

Consider that map $f : B \to B$ defined by $f(x) = \frac{T(x)}{\|T(x)\|}$. T is a linear map from a finite dimensional vector space to itself, so it's continuous. Furthermore, the map $x \mapsto \frac{x}{\|x\|}$ is also continuous. Therefore, their composition f must also be continuous.

f is a continuous map from B to itself. By the claim stated above, there exists $x_0 \in B$ such that $f(x_0) = x_0$.

$$\frac{T(x_0)}{\|T(x_0)\|} = f(x_0) = x_0 \implies Ax_0 = T(x_0) = \|T(x_0)\|\, x_0$$

Therefore, x_0 is an eigenvector of the linear transformation T, with eigenvalue $\|T(x_0)\|$, which is necessarily positive. Thus we've proved that A has a positive real eigenvalue. ■

§5.3 Fundamental Theorem of Algebra

In this section we shall prove the **Fundamental Theorem of Algebra (FTA)** using topological techniques.

Theorem 5.3.1 (Fundamental theorem of algebra)
A polynomial equation $x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$ with complex coefficients has at least one complex root.

Proof. ***Step 1.*** Consider thet map $f : S^1 \to S^1$ given by $f(z) = z^n$, where z is a complex number with unit modulus. In this step we shall prove that the induced homomorphism f_* of fundamental groups is injective.

Let $p_0 : I \to S^1$ be the standard loop in S^1. Here $p\big|_I = p_0$, with the familiar covering map $p : \mathbb{R} \to S^1$ given by $p(x) = (\cos 2\pi x, \sin 2\pi x)$. So we have

$$p_0(s) = e^{2\pi i s} \equiv (\cos 2\pi s, \sin 2\pi s)$$

Let $g = f \circ p_0$. Then $f_*([p_0]) = [g]$.

$$g(s) = f(p_0(s)) = f\left(e^{2\pi i s}\right) = \left(e^{2\pi i s}\right)^n = e^{2\pi i n s} \equiv (\cos 2\pi n s, \sin 2\pi n s)$$

We've proved before that the unique lifting of p_0 starting at 0 is given by $\widetilde{p_0}(s) = s$. g is also a loop in S^1 based at $b_0 = (1, 0)$. Using Lemma 4.1.1, g has a unique lifting $\widetilde{g} : I \to \mathbb{R}$ beginning at 0.

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ {[0,1]} \arrow[ur, "\widetilde{g}"] \arrow[r, "g"'] & S^1 \end{tikzcd}$$

$$\begin{aligned} g(s) = p(\widetilde{g}(s)) &\implies (\cos 2\pi n s, \sin 2\pi n s) = (\cos 2\pi \widetilde{g}(s), \sin 2\pi \widetilde{g}(s)) \\ &\implies 2\pi \widetilde{g}(s) = 2\pi n s + 2\pi k \ , \text{ for some } k \in \mathbb{Z} \\ &\implies \widetilde{g}(s) = ns + k \end{aligned}$$

Since $\widetilde{g}$ begins at 0, $\widetilde{g}(s) = ns$. If we consider the isomorphism $\phi : \pi_1\left(S^1, b_0\right) \to \mathbb{Z}$ given in the proof of Theorem 4.2.2, then

$$n = \widetilde{g}(1) = \phi([g]) = \phi([f \circ p_0]) = \phi(f_*([p_0]))$$

while $\phi([p_0]) = 1$.

We've seen before that all elements of $\pi_1\left(S^1, b_0\right)$ can be expressed as $[p_0^m]$, where $m \in \mathbb{Z}$. To show that f_* is injective, suppose $f_*([p_0^x]) = f_*([p_0^y])$ for $x, y \in \mathbb{Z}$. We need to show that $x = y$.

$$\begin{aligned} f_*([p_0^x]) = f_*([p_0^y]) &\implies f_*([p_0])^x = f_*([p_0])^y \implies \phi(f_*([p_0])^x) = \phi(f_*([p_0])^y) \\ &\implies x\ \phi(f_*([p_0])) = y\ \phi(f_*([p_0])) \\ &\implies xn = yn \implies x = y \end{aligned}$$

So f_* is injective.

Step 2. In this step, we shall prove that if $g : S^1 \to \mathbb{R}^2 \setminus \{\mathbf{0}\}$ is the map given by $g(z) = z^n$, then g is not nulhomotopic.

In Step 1, we had the map $f : S^1 \to S^1$ defined by $f(z) = z^n$. If $j : S^1 \to \mathbb{R}^2 \setminus \{\mathbf{0}\}$ is the inclusion map, then we have

$$j \circ f = g \implies j_* \circ f_* = g_*$$

We've shown in the proof of Corollary 5.1.4 that S^1 is a retract of $\mathbb{R}^2 \setminus \{\mathbf{0}\}$. As a result, j_* is injective by Lemma 5.1.1. By step 1, f_* is injective. As a composition of two injective maps, g_* is injective.

We, therefore, have proved that $g_* : \pi_1\left(S^1, b_0\right) \to \pi_1\left(\mathbb{R}^2 \setminus \{\mathbf{0}\}, b_0\right)$ is injective. As $\pi_1\left(S^1, b_0\right)$ is nontrivial, the image of g_* can't be just the identity element of $\pi_1\left(\mathbb{R}^2 \setminus \{\mathbf{0}\}, b_0\right)$. So g_* is not trivial. Hence, by Lemma 5.1.3, g is not nulhomotopic.

Step 3. In this step, we shall prove a special case of FTA. We assume that $|a_{n-1}| + |a_{n-2}| + \cdots + |a_1| + |a_0| < 1$. Then we shall prove that the polynomial equation $x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$ has a root lying in B^2.

Assume the contrary. Then we can define $k : B^2 \to \mathbb{R}^2 \setminus \{\mathbf{0}\}$ by

$$k(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 = 0$$

Let h be the restriction of k to S^1, *i.e.* $k\big|_{S^1} = h$. In other words, $h : S^1 \to \mathbb{R}^2 \setminus \{\mathbf{0}\}$ extends to a continuous map $k : B^2 \to \mathbb{R}^2 \setminus \{\mathbf{0}\}$. By Lemma 5.1.3, h is nulhomotopic.

We shall now define a homotopy F between h and g (as defined in step 2). Consider $F : S^1 \times I \to \mathbb{R}^2 \setminus \{\mathbf{0}\}$ defined by

$$F(z,t) = z^n + t\left(a_{n-1}z^{n-1} + \cdots + a_1 z + a_0\right)$$

However, we are yet to show that $F(z,t) \neq \mathbf{0}$ for any $z \in S^1, t \in I$. For $z \in S^1$, $|z| = 1$. Using triangle inequality,

$$\begin{aligned}
\left|t\left(a_{n-1}z^{n-1} + \cdots + a_1 z + a_0\right)\right| &\le t\left(\left|a_{n-1}z^{n-1}\right| + |a_1 z| + |a_0|\right) \\
&= t\left(|a_{n-1}| + \cdots + |a_1| + |a_0|\right) \\
1 - \left|t\left(a_{n-1}z^{n-1} + \cdots + a_1 z + a_0\right)\right| &\ge 1 - t\left(|a_{n-1}| + \cdots + |a_1| + |a_0|\right) \\
&\ge 1 - \left(|a_{n-1}| + \cdots + |a_1| + |a_0|\right) > 0
\end{aligned}$$

$$\begin{aligned}
z^n &= F(z,t) - t\left(a_{n-1}z^{n-1} + \cdots + a_1 z + a_0\right) \\
|z^n| &= \left|F(z,t) - t\left(a_{n-1}z^{n-1} + \cdots + a_1 z + a_0\right)\right| \\
&\le |F(z,t)| + \left|t\left(a_{n-1}z^{n-1} + \cdots + a_1 z + a_0\right)\right| \\
|F(z,t)| &\ge |z^n| - \left|t\left(a_{n-1}z^{n-1} + \cdots + a_1 z + a_0\right)\right| \\
&= 1 - \left|t\left(a_{n-1}z^{n-1} + \cdots + a_1 z + a_0\right)\right| > 0
\end{aligned}$$

So $F(z,t) \neq \mathbf{0}$, F is a map from $S^1 \times I$ to $\mathbb{R}^2 \setminus \{\mathbf{0}\}$. So F is indeed a homotopy between h and g, *i.e.* $h \simeq g$. We've seen that h is nulhomotopic. By the equivalence of $\simeq$, g is nulhomotopic. But we proved in step 2 that g is not nulhomotopic. Contradiction! So the polynomial equation must have a root in B^2.

Step 4. In this step we shall prove the general result.

Consider the polynomial equation $x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$. We choose $c \in \mathbb{R}$ with $c > 0$, and substitute $x = cy$.

$$\begin{aligned}
&(cy)^n + a_{n-1}(cy)^{n-1} + \cdots + a_1(cy) + a_0 = 0 \\
\implies\ & c^n y^n + c^{n-1}a_{n-1}y^{n-1} + \cdots + ca_1 y + a_0 = 0 \\
\implies\ & y^n + \frac{a_{n-1}}{c}y^{n-1} + \cdots + \frac{a_1}{c^{n-1}}y + \frac{a_0}{c^n} = 0
\end{aligned}$$

If this equation can be shown to admit a root y_0, then the original equation is easily seen to have a root $x_0 = cy_0$. Now we choose large enough c such that

$$\left|\frac{a_{n-1}}{c}\right| + \left|\frac{a_{n-2}}{c^2}\right| + \cdots + \left|\frac{a_1}{c^{n-1}}\right| + \left|\frac{a_0}{c^n}\right| < 1$$

Then using step 3, $y^n + \frac{a_{n-1}}{c}y^{n-1} + \cdots + \frac{a_1}{c^{n-1}}y + \frac{a_0}{c^n} = 0$ has a root $y_0 \in B^2$. Therefore, $x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$ has a root $x_0 = cy_0$. ■

6 Lecture 6

§6.1 Simplices

Definition 6.1.1 (Geometric independence). Given a set $\{\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ of points of $\mathbb{R}^N$, *i.e.* each $\mathbf{a}_i$ is an N-tuple, this set is said to be **geometrically independent** if for any scalars $t_i \in \mathbb{R}$, the equations

$$\sum_{i=0}^{n} t_i = 0 \quad \text{and} \quad \sum_{i=0}^{n} t_i \mathbf{a}_i = 0$$

imply that $t_0 = t_1 = \cdots = t_n = 0$.

It is immediate that a one-point set $\{\mathbf{a}_0\}$ is always geometrically independent.

Lemma 6.1.1

$\{\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ is geometrically independent if and only if the vectors

$$\mathbf{a}_1 - \mathbf{a}_0, \mathbf{a}_2 - \mathbf{a}_0, \ldots, \mathbf{a}_n - \mathbf{a}_0 \ \text{ are linearly independent.}$$

Proof. ($\Rightarrow$) Suppose $\{\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ is geometrically independent. Consider $t_i \in \mathbb{R}$ for $i = 1, 2, \ldots, n$ such that

$$\sum_{i=1}^{n} t_i (\mathbf{a}_i - \mathbf{a}_0) = 0$$

Take $t_0 \in \mathbb{R}$ such that $\sum\limits_{i=0}^{n} t_i = 0$. Now we have

$$\begin{aligned}\sum_{i=0}^{n} t_i \mathbf{a}_i &= \sum_{i=1}^{n} t_i \mathbf{a}_i + t_0 \mathbf{a}_0 = \sum_{i=1}^{n} t_i (\mathbf{a}_i - \mathbf{a}_0) + \sum_{i=1}^{n} t_i \mathbf{a}_0 + t_0 \mathbf{a}_0 \\ &= \sum_{i=1}^{n} t_i (\mathbf{a}_i - \mathbf{a}_0) + \mathbf{a}_0 \sum_{i=0}^{n} t_i = 0\end{aligned}$$

Since $\{\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ is geometrically independent, $\sum\limits_{i=0}^{n} t_i = 0$ and $\sum_{i=0}^{n} t_i \mathbf{a}_i$ together gives us $t_i = 0$ for every $i = 0, 1, 2, \ldots, n$. Therefore, the vectors $\mathbf{a}_1 - \mathbf{a}_0, \mathbf{a}_2 - \mathbf{a}_0, \ldots, \mathbf{a}_n - \mathbf{a}_0$ are linearly independent.

($\Leftarrow$) Conversely, suppose the vectors $\mathbf{a}_1 - \mathbf{a}_0, \mathbf{a}_2 - \mathbf{a}_0, \ldots, \mathbf{a}_n - \mathbf{a}_0$ are linearly independent. Consider $t_i \in \mathbb{R}$ for $i = 0, 1, 2, \ldots, n$ such that

$$\sum_{i=0}^{n} t_i = 0 \quad \text{and} \quad \sum_{i=0}^{n} t_i \mathbf{a}_i = 0$$

Using these, we get

$$0 = \sum_{i=0}^{n} t_i \mathbf{a}_i = \sum_{i=1}^{n} t_i (\mathbf{a}_i - \mathbf{a}_0) + \mathbf{a}_0 \sum_{i=0}^{n} t_i = \sum_{i=1}^{n} t_i (\mathbf{a}_i - \mathbf{a}_0) \implies \sum_{i=1}^{n} t_i (\mathbf{a}_i - \mathbf{a}_0) = 0$$

Since $\mathbf{a}_1 - \mathbf{a}_0, \mathbf{a}_2 - \mathbf{a}_0, \ldots, \mathbf{a}_n - \mathbf{a}_0$ are linearly independent, all the t_i's must be 0, for $i = 1, 2, \ldots, n$. As $\sum\limits_{i=0}^{n} t_i = 0$, t_0 is also 0. Therefore, $t_0 = t_1 = \cdots = t_n = 0$, and thus $\{\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ is geometrically independent. ■

Definition 6.1.2 (Simplex). Let $\{\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_n\}$ be a geometrically independent set in $\mathbb{R}^N$. We define the n**-simplex** σ spanned by $\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_n$ to be the set of all points $\mathbf{x}$ of $\mathbb{R}^N$ such that

$$\mathbf{x} = \sum_{i=0}^{n} t_i \mathbf{a}_i \quad \text{where} \quad \sum_{i=0}^{n} t_i = 1 \text{ and } t_i \geq 0 \ \ \forall\, i$$

The numbers t_i are uniquely determined by $\mathbf{x}$; they arre called the **barycentric coordinates** of the point $\mathbf{x}$ of σ with respect to $\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_n$.

Example 6.1.1 (Simplices in low dimensions)

A 0-simplex is a point. The 1-simplex spanned by $\mathbf{a}_0$ and $\mathbf{a}_1$ consists of all points $\mathbf{x}$ of the form

$$\mathbf{x} = t\mathbf{a}_0 + (1-t)\,\mathbf{a}_1 \quad \text{where} \quad t \in [0,1]$$

This is just the line segment joining $\mathbf{a}_0$ and $\mathbf{a}_1$.

The 2-simplex spanned by $\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2$ equals the triangle (geometric independece guarantees non-collinearity of the points) having these three points as vertices. This can be seen in the following way:

$$\mathbf{x} = \sum_{i=0}^{2} t_i \mathbf{a}_i = t_0\mathbf{a}_0 + (1-t_0)\left(\frac{t_1}{1-t_0}\mathbf{a}_1 + \frac{t_2}{1-t_0}\mathbf{a}_2\right) = t_0\mathbf{a}_0 + (1-t_0)\left(\frac{t_1}{\lambda}\mathbf{a}_1 + \frac{t_2}{\lambda}\mathbf{a}_2\right)$$

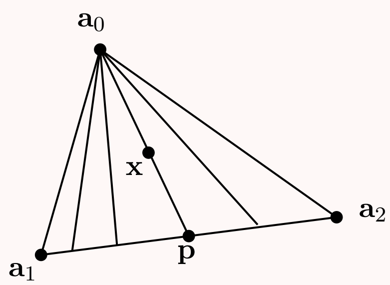
where $\lambda = 1 - t_0$. Note that, $t_0 + t_1 + t_2 = 1$ gives us $\frac{t_1}{\lambda} + \frac{t_2}{\lambda} = \frac{t_1+t_2}{1-t_0} = 1$. So $\mathbf{p}$ is a point on the line joining $\mathbf{a}_1$ and $\mathbf{a}_1$ given by

$$\mathbf{p} = \frac{t_1}{\lambda}\mathbf{a}_1 + \frac{t_2}{\lambda}\mathbf{a}_2 \quad \text{because} \quad \frac{t_1}{\lambda} + \frac{t_2}{\lambda} = 1 \text{ and } \frac{t_i}{\lambda} \geq 0$$

Any point lying on the line segment joining $\mathbf{a}_0$ and $\mathbf{p}$ is given by

$$t_0\mathbf{a}_0 + (1-t_0)\,\mathbf{p} = t_0\mathbf{a}_0 + (1-t_0)\left(\frac{t_1}{\lambda}\mathbf{a}_1 + \frac{t_2}{\lambda}\mathbf{a}_2\right)$$

Therefore, $\mathbf{x}$ represents a point in the triangular region formed by joining the points $\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2$. See the figure below:



Definition 6.1.3. A subset $A \subseteq \mathbb{R}^N$ is said to be **convex** if for each pair $\mathbf{x}, \mathbf{y}$ of points of A, the line segment joining them lies in A.

Definition 6.1.4. The points $\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_n$ that span a are called the **vertices** of σ; the number n is called the **dimension** of σ. Any simplex spanned by a subset of $\{\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_n\}$ is called a **face** of σ. In particular, the face spanned by $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ is called the **face opposite to** $\mathbf{a}_0$. The faces of σ different from σ itself are called the **proper faces** of σ; their union is called the

boundary of σ and denoted $\operatorname{Bd}\sigma$. The **interior** of σ is defined by the equation $\operatorname{Int}\sigma = \sigma \setminus \operatorname{Bd}\sigma$. The set $\operatorname{Int}\sigma$ is sometimes called an open simplex.

Let us list some basic properties of simplices. Throughout, let P be the n-plane determined by the points of the geometrically independent set $\{\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_n\}$; and let σ be the n-simplex spanned by these points. If $\mathbf{x} \in \sigma$, let $\{t_i(\mathbf{x})\}$ be the barycentric coordinates of $\mathbf{x}$; they are determined uniquely by the conditions

$$\mathbf{x} = \sum_{i=0}^{n} t_i \mathbf{a}_i \quad \text{and} \quad \sum_{i=0}^{n} t_i = 1$$

Then the following properties hold:

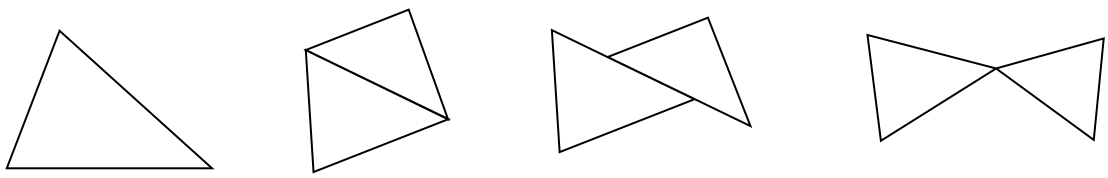
1. The barycentric coordinates $t_i(\mathbf{x})$ with respect to $\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_n$ are continuous functions of $\mathbf{x}$.
2. σ equals the union of all line segments joining $\mathbf{a}_0$ to points of the simplex s spanned by $\mathbf{a}_1, \ldots, \mathbf{a}_n$. Two such line segments intersect only in the point $\mathbf{a}_0$.
3. σ is a compact, convex set in $\mathbb{R}^N$, which equals the intersection of all convex sets in $\mathbb{R}^N$ containing $\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_n$.
4. Given a simplex σ, there is one and only one geometrically independent set of points spanning σ.
5. $\operatorname{Int}\sigma$ is convex and is open in the plane P; its closure is σ. Furthermore, $\operatorname{Int}\sigma$ equals the union of all open line segments joining $\mathbf{a}_0$ to points of $\operatorname{Int} s$, where s is the face of σ opposite to $\mathbf{a}_0$.
6. There is a homeomorphism of σ with the unit ball B^n that carries $\operatorname{Bd}\sigma$ onto the unit sphere S^{n-1}.

§6.2 Simplicial Complexes

Definition 6.2.1. A **simplicial complex** K in $\mathbb{R}^N$ is a collection of simplices in $\mathbb{R}^N$ such that:

i. Every face of a simplex of K is in K.

ii. The intersection of any two simplices of K is a face of each of them.



Simplicial Complex — Simplicial Complex — NOT Simplicial Complex — Simplicial Complex

Lemma 6.2.1

A collection K of simplices is a simplicial complex if and only if the following hold:

i. Every face of a simplex of K is in K.

ii. Every pair of distinct simplices of K have disjoint interiors.

Definition 6.2.2. If L is a subcollection of K that contains all faces of its elements, then L is a simplicial complex in its own right; it is called a **subcomplex** of K. One subcomplex of K is the collection of all simplices of K of dimension at most p; it is called the p-**skeleton** of K and is denoted $K^{(p)}$. The points of the collection $K^{(0)}$ are called the **vertices** of K.

Definition 6.2.3. Let $|K|$ be the subset of $\mathbb{R}^N$ that is the union of the simplices of K. Giving each simplex its natural topology as a subspace of $\mathbb{R}^N$, we then *topologize* $|K|$ by declaring a subset A of $|K|$ to be closed in $|K|$ if and only if $A \cap \sigma$ is closed in σ, for each σ in K. The space $|K|$ is called the **underlying space** of K, or the **polytope** of K.

In general, the topology of $|K|$ is finer (more open sets) than the topology $|K|$ inherits as a subspace of $\mathbb{R}^N$. If A is closed in the subspace topology, then according to the definition of subspace topology, $A = B \cap |K|$ for some closed set B in $\mathbb{R}^N$. Since B is closed in $\mathbb{R}^N$, $B \cap \sigma$ is closed in σ in the subspace topology σ inherits from $\mathbb{R}^N$.

Therefore, for each σ in K, $B \cap \sigma$ is closed in σ. σ is a subset of $|K|$, so $|K| \cap \sigma = \sigma$.

$$\therefore B \cap \sigma = B \cap (|K| \cap \sigma) = (B \cap |K|) \cap \sigma = A \cap \sigma$$

As a result, $A \cap \sigma$ is closed in σ for every σ in K. Hence, by the definition of topology on $|K|$, A is closed in $|K|$.

For each closed set A in the subspace topology on $|K|$, A is also closed in $|K|$ with respect to the topology defined earlier. So, the subspace topology is contained in the topology of $|K|$.

The two topologies on $|K|$ are, in general, different as we will see using some examples. In fact, if K is finite, then they are the same. We will prove the containment in the other direction for the case of K being finite.

Suppose K is finite and A is closed in $|K|$. Then $A \cap \sigma_i$ is closed in σ_i for each $i \in \{1, 2, \ldots, n\}$ with $|K| = \bigcup\limits_{i=1}^{n} \sigma_i$.

Now, $A \cap \sigma_i$ is closed in σ_i for every i, and σ_i is closed in $\mathbb{R}^N$. Hence, by Lemma 0.2.2 (the lemma is true if you replace "open" by "closed"), each $A \cap \sigma_i$ is closed in $\mathbb{R}^N$.

$$A \subseteq K = \bigcup_{i=1}^{n} \sigma_i \implies A = \bigcup_{i=1}^{n} (A \cap \sigma_i)$$

Finite union of closed sets is closed. Since each $A \cap \sigma_i$ is closed in $\mathbb{R}^N$, A is also closed in $\mathbb{R}^N$. As $A \subseteq |K|$, $A = A \cap |K|$. Therefore, A is closed in $|K|$ in the subspace topology inherited by $|K|$ from $\mathbb{R}^N$.

Example 6.2.1

Let K be the collection of all 1-simplices in $\mathbb{R}$ of the form $[m, m+1]$, where $m \in \mathbb{Z} \setminus \{0\}$; along with all the 1-simplices of the form $\left[\frac{1}{n+1}, \frac{1}{n}\right]$ where $n \in \mathbb{N}$.

We claim that the set $X = \left\{\frac{1}{n} \ : \ n \in \mathbb{N}\right\}$ is closed in $|K|$. $X \cap [m, m+1]$ is $\{1\}$ when $m = 1$, and $\varnothing$ when $m \neq 1$. Empty set is obviously closed. $\mathbb{R}$ is hausdorff, so is its subspace $[1, 2]$. Singleton sets are closed in hausdorff space (Proposition 0.4.1). So $\{1\}$ is closed in $[1, 2]$.

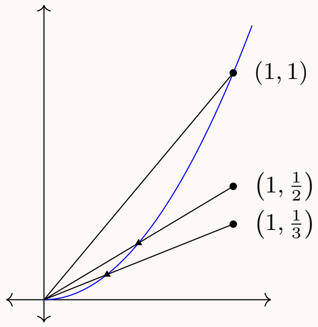
If we take a simplex of the form $\left[\frac{1}{n+1}, \frac{1}{n}\right]$, then $X \cap \left[\frac{1}{n+1}, \frac{1}{n}\right] = \left\{\frac{1}{n+1}, \frac{1}{n}\right\}$. $\left[\frac{1}{n+1}, \frac{1}{n}\right]$ is hausdorff as a subspace of $\mathbb{R}$, so both $\left\{\frac{1}{n+1}\right\}$ and $\left\{\frac{1}{n}\right\}$ are closed in $\left[\frac{1}{n+1}, \frac{1}{n}\right]$. Therefore, their union $\left\{\frac{1}{n+1}, \frac{1}{n}\right\}$ is closed in $\left[\frac{1}{n+1}, \frac{1}{n}\right]$.

For each simplex σ, $X \cap \sigma$ is closed in σ. Therefore, X is closed in $|K|$.

But X has an accumulation point 0. But $0 \notin X$, so X is not closed in $\mathbb{R}$. Therefore, X is not closed in the subspace topology of $|K|$. Hence, the topology of $|K|$ is strictly finer than the subspace topology.

Example 6.2.2

Let K be the collection of simplices $\sigma_1, \sigma_2, \ldots$, where σ_i is the 1-simplex in $\mathbb{R}^2$ having vertices $(0,0)$ and $\left(1, \frac{1}{i}\right)$. Then K is a simplicial complex.



We take $X = |K| \cap \left\{\left(x, x^2\right) \ : \ x > 0\right\}$. The 1-simplex σ_i is given by $y = \frac{1}{i}x$ for $x \in [0,1]$. Its intersection with the open parabolic arc $\left\{\left(x, x^2\right) \ : \ x > 0\right\}$ is the singleton $\left(\frac{1}{i}, \frac{1}{i^2}\right)$. So $X = \left\{\left(\frac{1}{i}, \frac{1}{i^2}\right) \ : \ i \in \mathbb{N}\right\}$.

$X \cap \sigma_i$ is singleton for every i. σ_i is a subspace of Hausdorff space $\mathbb{R}^2$, so singleton sets are closed. Therefore, $X \cap \sigma_i$ is closed in σ_i for every i. Hence, X is closed in the topology on $|K|$.

X has an accumulation point $(0,0)$. But $(0,0) \notin X$, so X is not closed in $\mathbb{R}^2$. Therefore, X is not closed in the subspace topology of $|K|$. Hence, the topology of $|K|$ is strictly finer than the subspace topology.

§6.3 Abstract Simplicial Complex

Definition 6.3.1 (Abstract Simplicial Complex). An **abstract simplicial complex** is a collection $\mathcal{S}$ of finite nonempty sets, such that if A is an element of $\mathcal{S}$, so is every nonempty subset of A.

Definition 6.3.2. Suppose $\mathcal{S}$ is an abstract simplicial complex. The element A of $\mathcal{S}$ is called a **simplex** of $\mathcal{S}$; its dimension is one less than the number of its elements. Each nonempty subset of A is called a **face** of A.

The **dimension of** $\mathcal{S}$ is the largest dimension of one of its simplices, or is infinite if there is no such largest dimension. The **vertex set** V of $\mathcal{S}$ is the union of the one-point elements of $\mathcal{S}$. An element $v \in V$ is called a **vertex** and it will be considered the same as the 0-simplex $\{v\} \in \mathcal{S}$. A subcollection of $\mathcal{S}$ that is itself a complex is called a **subcomplex** of $\mathcal{S}$.

Definition 6.3.3 (Isomorphism). Two abstract complexes $\mathcal{S}$ and $\mathcal{T}$ are said to be **isomorphic** if there is a bijective correspondence f mapping the vertex set of $\mathcal{S}$ to the vertex set of $\mathcal{T}$ such that

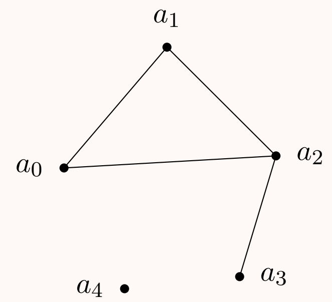
$$\{a_0, a_1, \ldots, a_n\} \in \mathcal{S} \iff \{f(a_0), f(a_1), \ldots, f(a_n)\} \in \mathcal{T}$$

Example 6.3.1

Consider the following collection of finite nonempty sets

$$\mathcal{S} = \{\, \{a_0, a_1, a_2\}, \{a_0, a_1\}, \{a_1, a_2\}, \{a_0, a_2\}, \{a_2, a_3\}, \{a_0\}, \{a_1\}, \{a_2\}, \{a_3\}, \{a_4\} \,\}$$

Then $\mathcal{S}$ is an abstract simplicial complex. It can be visualized as



$V = \{a_0, a_1, a_2, a_3, a_4\}$ and $V \notin \mathcal{S}$. In general, the vertex set is not a simplex. The nonempty set $\{a_0, a_1, a_2\}$ is a simplex of dimension $3 - 1 = 2$. The dimension of $\mathcal{S}$ is also 2. Here $\{a_0, a_1, a_2\}$ is a subcomplex of $\mathcal{S}$.

Definition 6.3.4 (Vertex Scheme)**.** If K is a geometric simplicial complex, let V be the vertex set of K. Let $\mathcal{K}$ be the collection of all subsets $\{a_0, a_1, \ldots, a_n\}$ of K such that the vertices $a_0, a_1, \ldots, a_n$ span a simplex of K. The collection $\mathcal{K}$ is called the **vertex scheme** of K.

The collection $\mathcal{K}$ is an example of an abstract simplicial complex.

Remark. Whenever we refer to a simplicial complex, we mean a *geometric* simplicial complex, which is a collection of simplices in $\mathbb{R}^N$. We can take the vertex set V of the geometric simplicial complex, and we can construct an abstract simplicial complex $\mathcal{K}$ out of it, which we call the *vertex scheme*.

Theorem 6.3.1

Every abstract simplicial complex $\mathcal{S}$ is isomorphic to the vertex scheme of some geometric simplicial complex.

Lemma 6.3.2

Let K and L be simplicial complexes, and let $f : K^{(0)} \to L^{(0)}$ be a map. Suppose that whenever the vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_n$ of K span a simplex of K, the points $f(\mathbf{v}_0), f(\mathbf{v}_1), \ldots, f(\mathbf{v}_n)$ are vertices of a simplex of L. Then f can be extended to a continuous map $g : |K| \to |L|$ such that

$$\mathbf{x} = \sum_{i=0}^{n} t_i \mathbf{v}_i \implies g(\mathbf{x}) = \sum_{i=0}^{n} t_i f(\mathbf{v}_i)$$

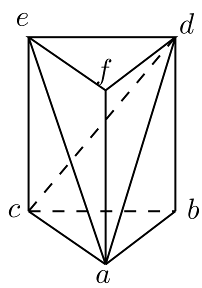
We call g the **simplicial map** induced by the vertex map f.

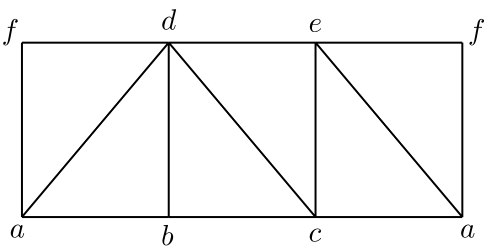
Lemma 6.3.3

Let K and L be simplicial complexed. Suppose $f : K^{(0)} \to L^{(0)}$ is a bijective correspondence such that the vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_n$ of K span a simplex of K if and only if $f(\mathbf{v}_0), f(\mathbf{v}_1), \ldots, f(\mathbf{v}_n)$ span a simplex of L. Then the induced simplicial map $g : |K| \to |L|$ is a homeomorphism.

Definition 6.3.5 (Geometric Realization)**.** It is guaranteed by Theorem 6.3.1 that every abstract simplicial complex $\mathcal{S}$ is isomorphic to the vertex scheme of some simplicial complex K. We call K a **geometric realization** of $\mathcal{S}$.

Let's see an example that illustrates these concepts. Suppose we wish to identify a simplicial complex K whose underlying space $|K|$ is homeomorphic to the cylinder $S^1 \times I$. Let us draw the simplicial complex K consisting of six 2-simplices, twelve 1-simplices and six 0-simplices.





K L

The abstract simplicial complex $\mathcal{S}$ has the vertex set $V = \{a, b, c, d, e, f\}$ while

$$\begin{aligned}\mathcal{S} = \{ & \{a, f, d\}, \{a, b, d\}, \{b, c, d\}, \{c, d, e\}, \{a, c, e\}, \{a, e, f\}, \\ & \{a, f\}, \{f, d\}, \{a, d\}, \{a, b\}, \{b, d\}, \{b, c\}, \\ & \{c, d\}, \{d, e\}, \{c, e\}, \{a, c\}, \{a, e\}, \{e, f\}, \\ & \{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{f\} \}\end{aligned}$$

Of course this abstract simplicial complex $\mathcal{S}$ is isomorphic to the vertex scheme of the simplicial complex K pictured above.

Let $f : L^{(0)} \to K^{(0)}$ be the map that assigns each vertex of L to the correspondingly labeled vertex of K. It is immediate that f is surjective. Then f extends to a simplicial map $g : |L| \to |K|$. This map is continuous surjective. Since $|L|, |K|$ are compact hausdorff, the map g is a quotient map (Proposition 0.10.6). This quotient map g is also called "pasting map". It identifies the right edge of $|L|$ linearly with the left edge of $|L|$.

§6.4 Abelian Group Essentials

Here we shall write abelian groups additively. Then 0 denotes the neutral element, and $-g$ denotes the inverse of g. If n is a positive integer, then ng denotes the n-fold sum $g + \cdots + g$, and $(-n)g$ denotes $n(-g)$.

Definition 6.4.1 (Free Abelian Group)**.** An abelian group G is **free** if it has a *basis* – that is, if there is a family $\{g_\alpha\}_{\alpha\in J}$ of elements of G such that each $g \in G$ can be written *uniquely* as a finite sum

$$g = \sum_\alpha n_\alpha g_\alpha\ , \qquad \text{(there are finitely many summands here)}$$

Uniqueness implies that each element g_α has infinite order; *i.e.* g_α generates an infinite cyclic subgroup of G. Suppose the contrary, *i.e.* $\exists\, n \in \mathbb{N}$ such that g_α has order n. This means $ng_\alpha = 0$. But then

$$g_\alpha = 1g_\alpha \ \text{ and } g_\alpha = 0 + g_\alpha = ng_\alpha + g_\alpha = (n+1)g_\alpha$$

Since $n \in \mathbb{N}$, $n + 1 \neq 1$, so the representation of g_α is not unique. Contradiction! Hence, there is no $n \in \mathbb{N}$ for which g_α is annihilated, meaning that g_α has infinite order.

More generally, if each $g \in G$ can be written as the finite sum $\sum_\alpha n_\alpha g_\alpha$, but not necessarily uniquely, then we say that the family $\{g_\alpha\}_{\alpha\in J}$ **generates** G. In particular, if $\{g_\alpha\}_{\alpha\in J}$ is finite, we say that G is **finitely generated**.

If G is free and has a basis consisting of n elements, say $g_1, g_2, \ldots, g_n$, then every other basis for G consists of precisely n elements.

Now $2G = \{2g \ : \ g \in G\}$ is easily seen to be a subgroup of G. An arbitrary element $2g \in 2G$ can be expressed uniquely as

$$2g = 2\sum_{i=1}^n m_i g_i\ , \qquad \text{with } m_i \in \mathbb{Z}$$

Now, the group $G/2G$ consists of cosets, the elements of which are of the form

$$\sum_{i=1}^{n} (2m_i + \varepsilon_i)\, g_i \ , \qquad \text{with } m_i \in \mathbb{Z} \text{ and } \varepsilon_i \in \{0,1\}$$

If all the ε_i's are 0, then the element belongs to the subgroup $2G$. In other words, when all the ε_i's are 0, one gets the representative of the coset $0 + 2G$. Since ε_i can be either 0 or 1, and $i \in \{1, 2, \ldots, n\}$, there are 2^n distinct cosets. Hence $|G/2G| = 2^n$.

Choosing a different basis, with m elements $g_1', g_2', \ldots, g_m'$, we get that $|G/2G| = 2^m$. Therefore, $2^n = 2^m$, so $m = n$.

> **Definition 6.4.2** (Rank)**.** The number of elements in a basis for G is called the **rank** of G.

There is a specific way of constructing free abelian groups. Given a set S, we define the **free abelian group G generated by S** to be the set of all functions $\phi : S \to \mathbb{Z}$ such that

$$\phi(x) = 0 \ \text{ for all but finitely many values of } x \in S$$

The addition of two such functions ϕ and ψ is defined as

$$(\phi + \psi)(x) = \phi(x) + \psi(x)$$

Given $x \in S$, the characteristic function ϕ_x for x is defined as

$$\phi_x(y) = \begin{cases} 0 & \text{if } y \neq x \\ 1 & \text{if } y = x \end{cases}$$

The functions $\{\phi_x \ : \ x \in S\}$ form a basis for G. Because, each $\phi \in G$ can be written as

$$\phi = \sum_{x \in S,\, \phi(x) \neq 0} n_x \phi_x \ , \qquad \text{where } n_x = \phi(x)$$

Since $\phi(x) \neq 0$ for only finitely many x, the sum is a finite sum.

> **Definition 6.4.3** (Torsion Subgroup)**.** If G is an abelian group, an element $g \in G$ has finite order if $ng = 0$ for some $n \in \mathbb{N}$. The set of all elements of finite order in G is a subgroup T of G, called the **torsion subgroup**. If T vanishes, we say G is **torsion-free**.

We've seen earlier that in a free abelian group, no nonzero element has finite order. In other words, a free abelian group is necessarily torsion-free. But the converse is not true in general.

> **Definition 6.4.4** (Direct Sum)**.** Suppose G is an abelian group and suppose $\{G_\alpha\}_{\alpha \in J}$ is a collection of subgroups of G. Suppose each $g \in G$ can be written uniquely as a finite sum $g = \sum g_\alpha$, where $g_\alpha \in G_\alpha$ for every $\alpha \in J$. Then G is called the direct sum of the groups G_α. It is written as
>
> $$G = \bigoplus_{\alpha \in J} G_\alpha$$
>
> If the collection $\{G_\alpha\}$ is finite, say $\{G_\alpha\} = \{G_1, G_2, \ldots, G_n\}$, we write the direct sums as
>
> $$G = G_1 \oplus G_2 \oplus \cdots \oplus G_n$$

More generally, if each $g \in G$ can be written as a finite sum $g = \sum g_\alpha$, where $g_\alpha \in G_\alpha$ for every $\alpha \in J$. But the finite sum expression is not necessarily unique. Then G is called the sum of the groups G_α. It is written as

$$G = \sum_{\alpha \in J} G_\alpha$$

If the collection $\{G_\alpha\}$ is finite, say $\{G_\alpha\} = \{G_1, G_2, \dots, G_n\}$, we write the direct sums as

$$G = G_1 + G_2 + \cdots + G_n$$

If $G = \sum_{\alpha \in J} G_\alpha$, then this sum will be direct if and only if for every fixed index $\alpha_0 \in J$,

$$G_{\alpha_0} \cap \left(\sum_{\alpha \in J, \alpha \neq \alpha_0} G_\alpha \right) = \{0\}$$

If G is free, then it has a basis, say $\{g_\alpha\}_{\alpha \in J}$. Then all these basis elements are of infinite order. The subgroup G_α that g_α generates is infinite cyclic, and G is the direct sum of these subgroups $\{G_\alpha\}_{\alpha \in J}$. Conversely, if G is a direct sum of infinite cyclic groups, then G is a free abelian group.

Theorem 6.4.1 (Fundamental Theorem of Finitely Generated Abelian Groups)
Let G be a finitely generated abelian group. Let T be its torsion subgroup.

(a) There is a free abelian subgroup H of G having finite rank β such that $G = H \oplus T$.

(b) There are finite cyclic groups $T_1, T_2, \dots, T_k$ where T_i has order $t_i > 1$, such that $t_i | t_{i+1}$, *i.e.* t_{i+1} is divisible by t_i for each $i \in \{1, 2, \dots, k-1\}$ and

$$T = T_1 \oplus T_2 \oplus \cdots \oplus T_k$$

(c) The numbers β and $t_1, t_2, \dots, t_k$ are uniquely determined by G.

The number β is called the **Betti number** of G; the numbers $t_1, t_2, \dots, t_k$ are called the **torsion coefficients** of G.

The group $\mathbb{Z}/m\mathbb{Z}$ is often written as $\mathbb{Z}_m$. It's a well-known result from group theory that if m and n are coprime positive integers, then

$$\mathbb{Z}_m \oplus \mathbb{Z}_n \cong \mathbb{Z}_{mn}$$

Using this, one can refine the statement of Fundamental Theorem of Finitely Generated Abelian Groups. H is a free abelian group having rank β, so

$$H \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$$

which is the β-fold direct sum of $\mathbb{Z}$. Also, each T_i is finite cyclic group of order t_i. So $T_i \cong \mathbb{Z}_{t_i}$.

$$T = T_1 \oplus T_2 \oplus \cdots \oplus T_k \cong \mathbb{Z}_{t_1} \oplus \mathbb{Z}_{t_2} \oplus \cdots \mathbb{Z}_{t_k}$$

Each t_i can be prime power factorized, thus we get

$$T \cong \mathbb{Z}_{a_1} \oplus \mathbb{Z}_{a_2} \oplus \cdots \mathbb{Z}_{a_s}$$

where each a_i is power of some prime. Combining the results,

$$G = H \oplus T \cong (\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}) \oplus (\mathbb{Z}_{a_1} \oplus \mathbb{Z}_{a_2} \oplus \cdots \mathbb{Z}_{a_s})$$

These a_i's are called **invariant factors** of G.

7 Lecture 7

§7.1 Notion of Orientation

> **Definition 7.1.1.** Let σ be a simplex. Define two orderings of its vertex set to be equivalent if they differ from one another by an even permutation. If $\dim \sigma > 0$, the orderings of the vertices can be classified in two ways; in other words, there are two equivalence classes. Each of these classes is called an orientation of σ. (If σ is a 0-simplex, then there is only one class and hence only one orientation of σ.) An **oriented simplex** is a simplex σ together with an orientation of σ.

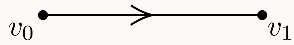
If the points $v_0, v_1, \ldots, v_p$ are geometrically independent, we shall use the symbol $v_0 v_1 \ldots v_p$ (without comma) to denote the simplex they span; and we use the symbol

$$[v_0, v_1, \ldots, v_p]$$

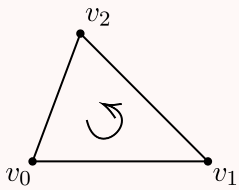
to denote the oriented simplex consisting of the simplex $v_0 v_1 \ldots v_p$ with the given ordering $(v_0, v_1, \ldots, v_p)$
.

Example 7.1.1

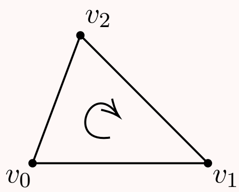
We attach an arrow to a given 1-simplex to prescribe an orientation. The oriented 1-simplex $[v_0, v_1]$ is pictured with the arrow directed from v_0 to v_1.



An orientation of a 2-simplex is depicted by a circular arrow. The oriented 2-simplex $[v_0, v_1, v_2]$ is pictured as



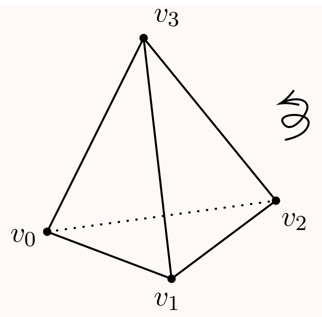
The 2-simplex $v_0 v_1 v_2$ can be given a clockwise orientation to obtain the oriented 2-simplex $[v_0, v_2, v_1]$.



$[v_0, v_2, v_1]$, $[v_1, v_0, v_2]$ and $[v_2, v_1, v_0]$ all represent the same oriented 2-simplex; because they differ from each other by an even permutation.

Example 7.1.2

The oriented 3-simplex $[v_0, v_1, v_2, v_3]$ is drawn by attching a spiral arrow to the simplex $v_0 v_1 v_2 v_3$.

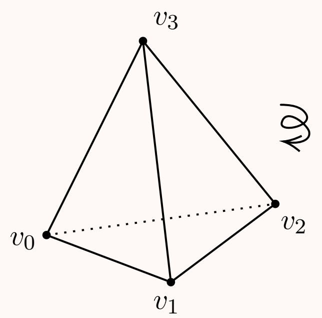


The orientation thus prescribed obeys the "right hand screw" rule. If one curls their right hand fingers in the direction from v_0 to v_1 to v_2, then the thumb will be pointed towards the direction of v_3.

It can also be verified that the oriented 3-simplex $[v_0, v_2, v_3, v_1]$ obeys the right hand screw rule. Indeed, if one curls their right hand fingers in the direction from v_0 to v_2 to v_3, then the thumb will be pointed towards the direction of v_1.

The ordering of the vertex set of the oriented 3-simplex $[v_0, v_1, v_2, v_3]$ differs from that of $[v_0, v_2, v_3, v_1]$ by an even permutation. Indeed, by permuting the vertices of $[v_0, v_2, v_3, v_1]$ twice ($v_2 \leftrightarrow v_3$ followed by $v_3 \leftrightarrow v_1$) one arrives at $[v_0, v_1, v_2, v_3]$.

Consider the 3-simplex $[v_0, v_2, v_1, v_3]$ which differs from $[v_0, v_1, v_2, v_3]$ by a single permutation, *i.e.* an odd permutation. Hence, $[v_0, v_2, v_1, v_3]$ is oppositely oriented. It is represeted with a downwards spiral arrow:



Definition 7.1.2 (p-chain)**.** Let K be a simplicial complex. A **p-chain** on K is a function c from the set of oriented p-simplices of K to the integers, such that:

1. $c(\sigma) = -c(\sigma')$, if σ and σ' are opposite orientations of the same simplex.
2. $c(\sigma) = 0$ for all but finitely many oriented p-simplices σ.

We add two p-chains c and c' pointwise

$$\left(c + c'\right)(\sigma) := c(\sigma) + c'(\sigma) \ , \quad \text{where } \sigma \text{ is an oriented } p\text{-simplex of } K$$

With this addition operation between p-chains, the set of all p-chains on the simpicial complex K, becomes a group. We call this **the group of oriented p-chains of K**, and denote it by $C_p(K)$.

If $p < 0$ or $p > \dim K$, we let $C_p(K)$ denote the trivial group, the group consisting of the identity element only.

Definition 7.1.3 (Elementary chain)**.** If σ is a oriented simplex, then the **elementary chain** c corresponding to σ is the function defined as:

1. $c(\sigma) = 1$.
2. $c(\sigma') = -1$; where σ' is the same simplex as σ, but with opposite orientation.

3. $c(\tau) = 0$ for every other oriented simplex τ.

Abuse of Notation. We often use the symbol σ to denote not only a simplex, or an oriented simplex, but also to denote the elementary p-chain c corresponding to the oriented simplex σ. With this convention, if σ and σ are opposite orientations of the same simplex, then we can write $\sigma' = -\sigma$ instead of writing $c(\sigma) = -c(\sigma')$.

Lemma 7.1.1

$C_p(K)$ is free abelian; a basis for $C_p(K)$ can be obtained by orienting each p-simplex and using the corresponding elementary chains as a basis.

Proof. The proof is straightforward. First, orient the p simplices of K arbitrarily. Then each p-chain on K can be written as a finite linear combination

$$c = \sum n_i \sigma_i$$

of the corresponding elementary chains σ_i. The chain c takes the integer value n_i on the oriented p-simplex σ_i, the value $-n_i$ on the same p-simplex as σ_i but with opposite orientation, and 0 on all oriented p-simplices not appearing in the summation. ■

Note that, the free abelian group $C_0(K)$ has a natural basis, since a 0-simplex has only one orientation. But $C_p(K)$ has no “natural” basis if $p > 0$. Because one must orient the p-simplices arbitrarily in order to obtain a basis.

Corollary 7.1.2

Any function f from the oriented p-simplices of K to an abelian group G extends uniquely to a homomorphism $C_p(K) \to G$ provided that $f(-\sigma) = -f(\sigma)$ for all oriented p-simplices σ.

Corollary 7.1.2 is a consequence of Lemma 7.1.1 and the universal property of free abelian groups, which states that:

Theorem 7.1.3 (Universal Property of Free Abelian Groups)

If S is a set and $\mathbb{Z}^{(S)}$ is the free abelian group on S, and given an abelian group G, $f : S \to G$ is a mapping; then there exists a **unique group homomorphism** $g : \mathbb{Z}^{(S)} \to G$ with $g \circ \iota = f$. In other words, the following diagram commutes:

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "\iota"'] & G \\ \mathbb{Z}^{(S)} \arrow[ur, "\exists!\, g"'] & \end{tikzcd}$$

In Theorem 7.1.3, $\mathbb{Z}^{(S)}$ refers to the group of all functions $\phi : S \to \mathbb{Z}$ such that $\phi(s) = 0$ for all but finitely many $s \in S$, and ι is the map that maps $x \in S$ to $\phi_x \in \mathbb{Z}^{(S)}$.

In the present context, S is the set of all oriented p-simplices of K and $\mathbb{Z}^{(S)}$ is the free abelian group $C_p(K)$. Given $c = \sum n_i \sigma_i \in C_p(K)$, define $g : C_p(K) \to G$ by

$$g\left(\sum n_i \sigma_i\right) = \sum n_i f(\sigma_i)$$

Here $n_i \in \mathbb{Z}$ for each i, and $f(\sigma_i) \in G$. This g is our desired unique homomorphism $C_p(K) \to G$, that is an extension of f.

§7.2 Homology Groups

Definition 7.2.1. We define a homomorphism

$$\partial_p : C_p(K) \to C_{p-1}(K)$$

called the **boundary operator**. If $\sigma = [v_0, v_1, \ldots, v_p]$ is an oriented p-simplex ($p > 0$), we define

$$\partial_p \sigma = \partial_p [v_0, v_1, \ldots, v_p] = \sum_{i=0}^{p} (-1)^i [v_0, \ldots, \widehat{v_i}, \ldots, v_p]$$

$\widehat{v_i}$ means that the vertex v_i is removed from the given array. Since $C_p(K)$ is the trivial group for $p < 0$, the operator ∂_p is the trivial group homomorphism for $p \leq 0$.

We need to check the well-definedness of ∂_p and $\partial_p(-\sigma) = -\partial_p(\sigma)$. It suffices to check that in case of exchange of two adjacent vertices of $[v_0, v_1, \ldots, v_p]$, ∂_p changes sign. In other words, we need to show that

$$\partial_p [v_0, \ldots, v_j, v_{j+1}, \ldots, v_p] = -\partial_p [v_0, \ldots, v_{j+1}, v_j, \ldots, v_p]$$

For $i \neq j, j+1$, the i-th terms in these two expressions differ precisely by a sign; the terms are identical except that v_j and v_{j+1}, have been interchanged. Now we need to consider the terms corresponding to $i = j$ and $i = j+1$ separately.

For the first case $\partial_p [v_0, \ldots, v_j, v_{j+1}, \ldots, v_p]$, the $i = j$ and $i = j+1$ terms are:

$$(-1)^j [v_0, \ldots, v_{j-1}, \widehat{v_j}, v_{j+1}, \ldots, v_p] + (-1)^{j+1} [v_0, \ldots, v_{j-1}, v_j, \widehat{v_{j+1}}, v_{j+2}, \ldots, v_p]$$

On the other hand, the $i = j$ and $i = j+1$ terms for $\partial_p [v_0, \ldots, v_j, v_{j+1}, \ldots, v_p]$ are:

$$(-1)^j [v_0, \ldots, v_{j-1}, \widehat{v_{j+1}}, v_j, \ldots, v_p] + (-1)^{j+1} [v_0, \ldots, v_{j-1}, v_{j+1}, \widehat{v_j}, v_{j+2}, \ldots, v_p]$$

It's straightforward to see that

$$-(-1)^j [v_0, \ldots, v_{j-1}, \widehat{v_j}, v_{j+1}, \ldots, v_p] = (-1)^{j+1} [v_0, \ldots, v_{j-1}, v_{j+1}, \widehat{v_j}, v_{j+2}, \ldots, v_p] \text{ and}$$

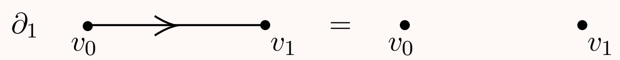
$$-(-1)^j [v_0, \ldots, v_{j-1}, \widehat{v_{j+1}}, v_j, \ldots, v_p] = (-1)^{j+1} [v_0, \ldots, v_{j-1}, v_j, \widehat{v_{j+1}}, v_{j+2}, \ldots, v_p]$$

So $\partial_p [v_0, \ldots, v_j, v_{j+1}, \ldots, v_p]$ and $\partial_p [v_0, \ldots, v_{j+1}, v_j, \ldots, v_p]$ have opposite signs.

Example 7.2.1

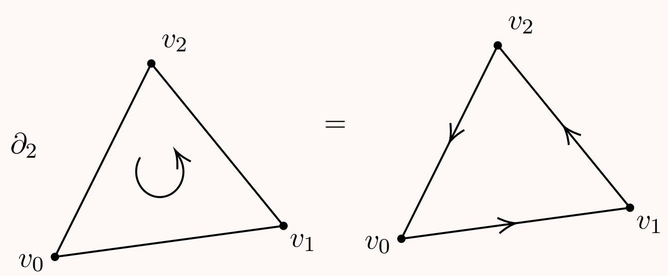
For an oriented 1-simplex $[v_0, v_1]$,

$$\partial_1 [v_0, v_1] = v_1 - v_0$$



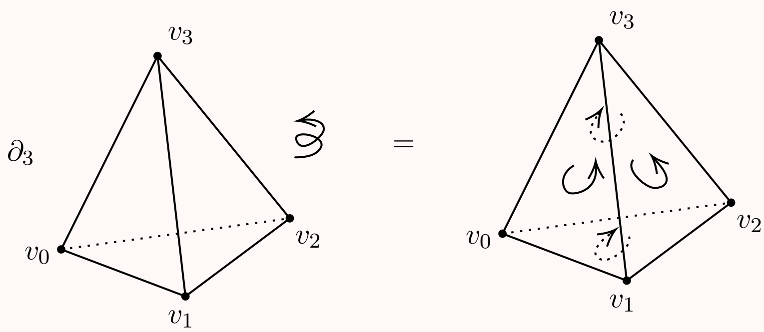
For an oriented 2-simplex $[v_0, v_1, v_2]$,

$$\partial_2 [v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$$



For an oriented 3-simplex $[v_0, v_1, v_2, v_3]$,

$$\partial_3 \, [v_0, v_1, v_2, v_3] = [v_1, v_2, v_3] - [v_0, v_2, v_3] + [v_0, v_1, v_3] - [v_0, v_1, v_2]$$



If we apply the ∂_1 operator on $\partial_2 \, [v_0, v_1, v_2]$, we get

$$\begin{aligned}\partial_1 \, \partial_2 \, [v_0, v_1, v_2] &= \partial_1 \, [v_1, v_2] - \partial_1 \, [v_0, v_2] + \partial_1 \, [v_0, v_1] \\ &= (v_2 - v_1) - (v_2 - v_0) + (v_1 - v_0) = 0\end{aligned}$$

If we apply the ∂_2 operator on $\partial_3 \, [v_0, v_1, v_2 v_3]$, we get

$$\begin{aligned}\partial_2 \, \partial_3 \, [v_0, v_1, v_2, v_3] &= \partial_2 \, [v_1, v_2, v_3] - \partial_2 \, [v_0, v_2, v_3] + \partial_2 \, [v_0, v_1, v_3] - \partial_2 \, [v_0, v_1, v_2] \\ &= [v_2, v_3] - [v_1, v_3] + [v_1, v_2] - [v_2, v_3] + [v_0, v_3] - [v_0, v_2] + [v_1, v_3] \\ &\qquad - [v_0, v_3] + [v_0, v_1] - [v_1, v_2] + [v_0, v_2] - [v_0, v_1] \\ &= 0\end{aligned}$$

These computations illustrate a general fact.

Lemma 7.2.1

$\partial_{p-1} \circ \partial_p = 0.$

Proof. Let $\sigma = [v_0, \ldots, v_p]$.

$$\begin{aligned}\partial_{p-1} \, \partial_p \sigma &= \sum_{i=0}^{p} (-1)^i \, \partial_{p-1} \, [v_0, \ldots, \widehat{v_i}, \ldots, v_p] \\ &= \sum_{i=0}^{p} (-1)^i \left(\sum_{j<i} (-1)^j \, [\ldots, \widehat{v_j}, \ldots, \widehat{v_i}, \ldots] + \sum_{j>i} (-1)^{j-1} \, [\ldots, \widehat{v_i}, \ldots, \widehat{v_j}, \ldots] \right) \\ &= \sum_{i=0}^{p} \sum_{j<i} (-1)^i \, (-1)^j \, [\ldots, \widehat{v_j}, \ldots, \widehat{v_i}, \ldots] + \sum_{i=0}^{p} \sum_{j>i} (-1)^i \, (-1)^{j-1} \, [\ldots, \widehat{v_i}, \ldots, \widehat{v_j}, \ldots] \\ &= \sum_{i=0}^{p} \sum_{j<i} (-1)^{i+j} \, [\ldots, \widehat{v_j}, \ldots, \widehat{v_i}, \ldots] - \sum_{i=0}^{p} \sum_{j>i} (-1)^{i+j} \, [\ldots, \widehat{v_i}, \ldots, \widehat{v_j}, \ldots]\end{aligned}$$

Now let's fix two positive integers i_0 and j_0 with $j_0 < i_0$. Now, the term corresponding to $i = i_0 \leq p$ and $j = j_0 < i_0$ in the first summand is:

$$(-1)^{i_0+j_0} \, [\ldots, \widehat{v_{j_0}}, \ldots, \widehat{v_{i_0}}, \ldots]$$

The term corresponding to $i = j_0 \leq p$ and $j = i_0 > j_0$ in the second summand is:

$$(-1)^{i_0+j_0} \, [\ldots, \widehat{v_{j_0}}, \ldots, \widehat{v_{i_0}}, \ldots]$$

These two terms cancel each other. This way, all the terms get canceled. So $\partial_{p-1} \, \partial_p \, \sigma = 0$. ■

Definition 7.2.2. The kernel of the homomorphism $\partial_p : C_p(K) \to C_{p-1}(K)$ is a subgroup of $C_p(K)$. It is called **the group of p-cycles** and is denoted by $Z_p(K)$.

The image of the homomorphism $\partial_{p+1} : C_{p+1}(K) \to C_p(K)$ is a subgroup of $C_p(K)$. It is called **the group of p-cycles** and is denoted by $B_p(K)$.

By Lemma 7.2.1, $\partial_p \circ \partial_{p+1} = 0$. As a result, the image of the boundary operator ∂_{p+1} is contained in the kernel of the boundary operator ∂_p. In other words, each boundary of a $(p+1)$-chain is automatically a p-cycle. So $B_p(K) \subseteq Z_p(K)$. $Z_p(K)$ is a abelian group, so all its subgroups are normal. Thus $B_p(K)$ is a normal subgroup of $Z_p(K)$, and so the quotient group $Z_p(K)/B_p(K)$ is defined.

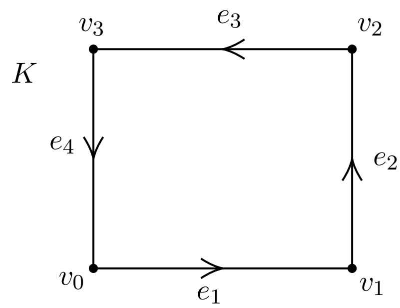
Definition 7.2.3 (Homology Group)**.** The p-th homology group $H_p(K)$ of K is defined as

$$H_p(K) = Z_p(K)/B_p(K)$$

§7.3 Homology Group Computation

Let's now do a few computation of homology groups.

Example 7.3.1. Consider the following 1-dimensional simplicial complex K. The underlying space $|K|$ of this complex K is the boundary of a square with edges e_1, e_2, e_3, e_4.



The group $C_1(K)$ is free abelian of rank 4. It is generated by $\{e_1, e_2, e_3, e_4\}$. To put it more concretely, it is generated by the elementary 1-chains corresponding to the oriented 1-simplices e_1, e_2, e_3, e_4. A generic element of $C_1(K)$ is a $\mathbb{Z}$-linear combination of e_1, e_2, e_3, e_4. So, if c is a general 1-chain,

$$\begin{aligned} c &= \sum_{i=1}^{4} n_i e_i = n_1 e_1 + n_2 e_2 + n_3 e_3 + n_4 e_4 \ ; \qquad n_i \in \mathbb{Z} \\ \partial_1 c &= n_1 (v_1 - v_0) + n_2 (v_2 - v_1) + n_3 (v_3 - v_2) + n_4 (v_0 - v_3) \\ &= (n_4 - n_1) v_0 + (n_1 - n_2) v_1 + (n_2 - n_3) v_2 + (n_3 - n_4) v_3 \end{aligned}$$

Therefore, c is a 1-cycle, *i.e.* $\partial_1 c = 0$ if and only if $n_1 = n_2 = n_3 = n_4$. Hence, a generic 1-cycle can be written as

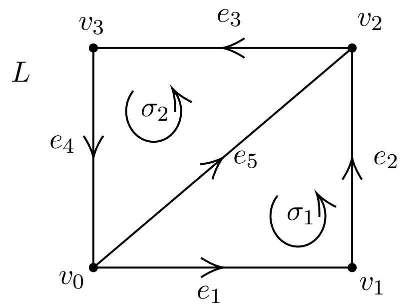
$$c = n_1 (e_1 + e_2 + e_3 + e_4) \quad \text{with } n_1 \in \mathbb{Z}$$

Therefore, $Z_1(K)$ is infinite cyclic and generated by the 1-chain $e_1 + e_2 + e_3 + e_4$. So $Z_1(K) \cong \mathbb{Z}$.

Since there are no 2-simplex in K, $C_2(K)$ is, by definition, trivial. And hence ∂_2 is the trivial homomorphism. As a result, $B_1(K)$ is trivial. Therefore,

$$H_1(K) = Z_1(K)/B_1(K) = Z_1(K) \cong \mathbb{Z}$$

Example 7.3.2. Now consider the following complex L.



The underlying space $|L|$ is a square. So, if c is a general 1 chain,

$$\begin{aligned} c &= \sum_{i=1}^{4} n_i e_i = n_1 e_1 + n_2 e_2 + n_3 e_3 + n_4 e_4 + n_5 e_5 \ ; \qquad n_i \in \mathbb{Z} \\ \partial_1 c &= n_1 (v_1 - v_0) + n_2 (v_2 - v_1) + n_3 (v_3 - v_2) + n_4 (v_0 - v_3) + n_5 (v_2 - v_0) \\ &= (n_4 - n_1 - n_5) v_0 + (n_1 - n_2) v_1 + (n_2 - n_3 + n_5) v_2 + (n_3 - n_4) v_3 \end{aligned}$$

Hence, c is a 1-cycle if the values of $\partial_1 c$ on the vertices are all 0. In other words,

$$n_4 - n_1 - n_5 = 0 \ , \ n_1 - n_2 = 0 \ , \ n_2 - n_3 + n_5 = 0 \ , \ n_3 - n_4 = 0$$

These equations give us $n_1 = n_2$; $n_3 = n_4$; $n_5 = n_3 - n_2$. Here we have 2 degrees of freedom. Because we can assign the values of n_2 and n_3 arbitrarily to obtain the values of n_1, n_4 and n_5. Thus a generic 1-cycle c can be written as

$$c = n_2 e_1 + n_2 e_2 + n_3 e_3 + n_3 e_4 + (n_3 - n_2) e_5 = n_2 (e_1 + e_2 - e_5) + n_3 (e_3 + e_4 + e_5)$$

So $Z_1(L)$ is free abelian with rank 2. A basis for $Z_1(L)$ consists of the 1-chains $e_1 + e_2 - e_5$ and $e_3 + e_4 + e_5$. Hence, $Z_1(L) \cong \mathbb{Z}^2$.

On the other hand,

$$\partial_2 \sigma_1 = e_1 + e_2 - e_5 \quad \text{and} \quad \partial_2 \sigma_2 = e_3 + e_4 + e_5$$

So, a generic 1-boundary can be written as a $\mathbb{Z}$-linear combination of $e_1 + e_2 - e_5$ and $e_3 + e_4 + e_5$. Hence, $B_1(L) \cong \mathbb{Z}^2$.

$$H_1(L) = Z_1(L) / B_1(L) \cong \mathbb{Z}^2 / \mathbb{Z}^2 \implies H_1(L) = 0$$

Since there are no 3-simplex in L, $C_3(L)$ is, by definition, trivial. And hence ∂_3 is the trivial homomorphism. As a result, $B_2(L)$ is trivial. Now, if c_2 is a generic 2-chain,

$$c_2 = m_1 \sigma_1 + m_2 \sigma_2 \implies \partial_2 c_2 = m_1 (e_1 + e_2 - e_5) + m_2 (e_3 + e_4 + e_5)$$

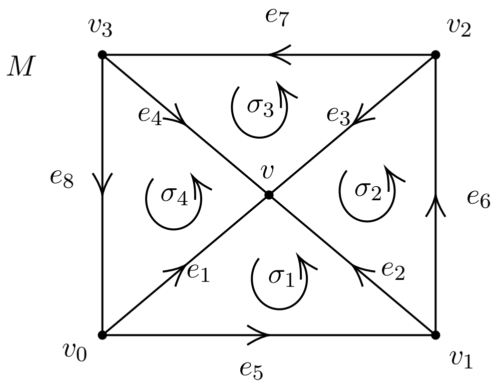
If c_2 is a 2-cycle, we must have $m_1 = m_2 = 0$. Hence $Z_2(L)$ is trivial. Therefore,

$$H_2(L) = Z_2(L) / B_2(L) = 0$$

At this stage, we need some terminologies.

Definition 7.3.1. We shall say that a chain c is **carried by a subsimplex** L of K, if c has value 0 on every simplex that is not in L. And we say two p-chains c and c' are **homologous** if $c - c' = \partial_{p+1} d$ for some $(p+1)$-chain d. In particular, if $c = \partial_{p+1} d$, we say that c is homologous to 0, or simply c **bounds.**

Example 7.3.3. Consider the following complex M whose underlying space is a square.



Instead of computing the group of 1-cycles, we reason as follows: Given a 1-chain c, let a be its value on e_1. Then we claim that the chain

$$c_1 = c + \partial_2 (a\sigma_1)$$

has value 0 on e_1. To prove it, we write c as

$$c = ae_1 + \sum_{i \neq 1} n_i e_i \ , \quad \text{with } n_i \in \mathbb{Z}$$

Since $\partial_2 (\sigma_1) = e_5 + e_2 - e_1$, we get

$$c_1 = ae_1 + \sum_{i \neq 1} n_i e_i + a (e_5 + e_2 - e_1) = \sum_{i \neq 1} n_i' e_i$$

So, the value of c_1 on e_1 is indeed 0.

This can be interpreted as follows: after modifying the 1-chain c by a boundary, one "pushes it off e_1". Our next goal is to "push c_1 off e_2". Let b be the value of c_1 on e_2. So we have

$$c_1 = be_2 + \sum_{i \neq 2} n_i e_i \ , \quad \text{with } n_i \in \mathbb{Z}$$

We take the chain $c_2 = c_1 + \partial_2 (b\sigma_2)$, and we claim that its value on e_2 is 0. Using $\partial_2 (\sigma_2) = e_6 + e_3 - e_2$, we get

$$c_2 = be_2 + \sum_{i \neq 2} n_i e_i + b (e_6 + e_3 - e_2) = \sum_{i \neq 2} n_i' e_i$$

So, the value of c_2 on e_2 is indeed 0. In other words, by modifying c_1 by a boundary, we "pushed c_1 off e_2".

Now our goal is to "push c_2 off e_3". Let d be the value of c_2 on e_3. So we have

$$c_2 = de_3 + \sum_{i \neq 3} n_i e_i \ , \quad \text{with } n_i \in \mathbb{Z}$$

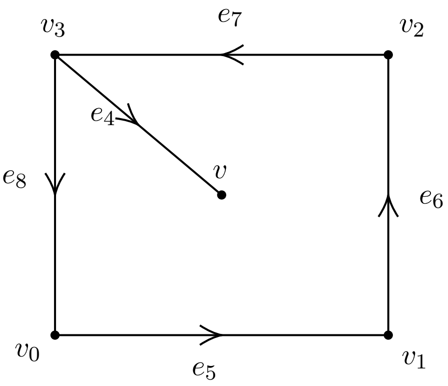
We take the chain $c_3 = c_2 + \partial_2 (d\sigma_3)$, and we claim that its value on e_3 is 0. Using $\partial_2 (\sigma_3) = e_7 + e_4 - e_3$, we get

$$c_3 = de_3 + \sum_{i \neq 3} n_i e_i + d (e_7 + e_4 - e_3) = \sum_{i \neq 3} n_i' e_i$$

So, the value of c_3 on e_3 is indeed 0. In other words, by modifying c_1 by a boundary, we "pushed c_2 off e_3".

$$\begin{aligned} c_3 &= c_2 + \partial_2 (d\sigma_3) = c_1 + \partial_2 (b\sigma_2) + \partial_2 (d\sigma_3) = c + \partial_2 (a\sigma_1) + \partial_2 (b\sigma_2) + \partial_2 (d\sigma_3) \\ &= c + \partial_2 (a\sigma_1 + b\sigma_2 + d\sigma_3) \end{aligned}$$

Therefore, given a 1-chain c, we can find a chain c_3 that is homologous to c, and carried by the following subcomplex:



If c happens to be a cycle, then c_3 is also a cycle. Let $c_3 = n_4e_4 + n_5e_5 + n_6e_6 + n_7e_7 + n_8e_8$, with each $n_i \in \mathbb{Z}$.

$$\begin{aligned}\partial_2 c_3 &= n_4 (v - v_3) + n_5 (v_1 - v_0) + n_6 (v_2 - v_1) + n_7 (v_3 - v_2) + n_8 (v_0 - v_3)\\ &= n_4 v + (n_8 - n_5) v_0 + (n_5 - n_6) v_1 + (n_6 - n_7) v_2 + (n_7 - n_8 - n_4) v_3\end{aligned}$$

The only way $\partial_2 c_3$ can be 0 is $n_4 = 0$ and $n_5 = n_6 = n_7 = n_8$. Thus, for the 1-chain c_3 to be a cycle, it must vanish on e_4. In other words, every 1-cycle of M is homologous to a 1-cycle carried by the boundary of the square.

$$\begin{aligned}\partial_2\sigma_1 + \partial_2\sigma_2 + \partial_2\sigma_3 + \partial_2\sigma_4 &= (e_5 + e_2 - e_1) + (e_6 + e_3 - e_2) + (e_7 + e_4 - e_3) + (e_8 + e_1 - e_4)\\ &= e_5 + e_6 + e_7 + e_8\\ \implies c_3 &= n_5 (e_5 + e_6 + e_7 + e_8) = \partial_2 (n_5\sigma_1 + n_5\sigma_2 + n_5\sigma_3 + n_5\sigma_4)\end{aligned}$$

Therefore, the 1-cycle c_3 is homologous to 0, *i.e.* c_3 bounds. Since the given 1-cycle c is homologous to c_3, c also bounds. Hence, every 1-cycle of M bounds; so $H_1(M) = 0$.

A generic 2-chain on M is given by

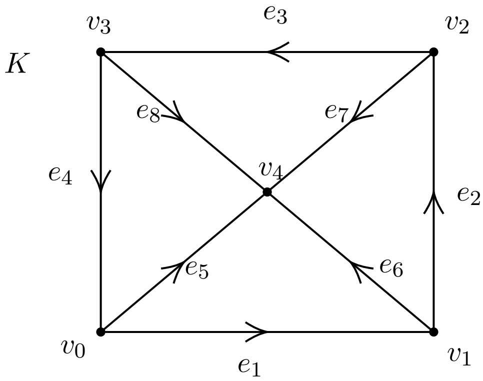
$$c = m_1\sigma_1 + m_2\sigma_2 + m_3\sigma_3 + m_4\sigma_4\ ; \quad \text{with } m_i \in \mathbb{Z}$$

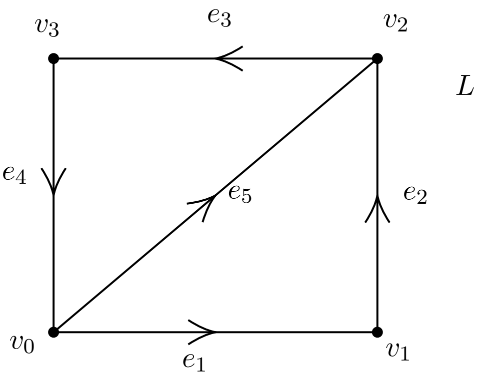
$$\begin{aligned}\partial_2 c &= m_1 (e_5 + e_2 - e_1) + m_2 (e_6 + e_3 - e_2) + m_3 (e_7 + e_4 - e_3) + m_4 (e_8 + e_1 - e_4)\\ &= (m_4 - m_1) e_1 + (m_1 - m_2) e_2 + (m_2 - m_3) e_3 + (m_3 - m_4) e_4\\ &\qquad + m_1e_5 + m_2e_6 + m_3e_7 + m_4e_8\end{aligned}$$

$\partial_2 c = 0$ if and only if $m_1 = m_2 = m_3 = m_4 = 0$. This means that the group $Z_2(M)$ of 2-cycles on M is trivial. Hence, $H_2(M) = 0$.

Note that, the underlying spaces for the complex L and M, dealt in Example 7.3.2 and Example 7.3.3, respectively, are both square. Using the result of the respective examples, we find that the homology groups of L and M are exactly the same. This validates the fact that the homology groups of a complex depend only on the underlying space, which will be proved in Algebraic Topology II.

Example 7.3.4. In this example, we are going to compute the first homology groups of the following complexes and interpret the results geometrically:





First let's focus on $H_1(K)$. There are no 2-simplices in K. So $C_2(K) = 0$, and hence ∂_2 is trivial. Therefore,

$$H_1(K) = \operatorname{Ker}\partial_1 / \operatorname{im}\partial_2 = \operatorname{Ker}\partial_1$$

A general 1-chain of K is of the form $c = \sum_{i=1}^{8} n_i e_i$, where $n_i \in \mathbb{Z}$.

$$\begin{aligned}\partial_1 c &= n_1(v_1 - v_0) + n_2(v_2 - v_1) + n_3(v_3 - v_2) + n_4(v_0 - v_3) + n_5(v_4 - v_0) + n_6(v_4 - v_1) \\ &\quad + n_7(v_4 - v_2) + n_8(v_4 - v_3) \\ &= v_0(-n_1 + n_4 - n_5) + v_1(n_1 - n_2 - n_6) + v_2(n_2 - n_3 - n_7) + v_3(n_3 - n_4 - n_8) \\ &\quad + v_4(n_5 + n_6 + n_7 + n_8)\end{aligned}$$

If c is a 1-cycle, then $\partial_1 c = 0$, so we must have

$$-n_1 + n_4 - n_5 = n_1 - n_2 - n_6 = n_2 - n_3 - n_7 = n_3 - n_4 - n_8 = n_5 + n_6 + n_7 + n_8 = 0$$

So $n_5 = n_4 - n_1$, $n_6 = n_1 - n_2$, $n_7 = n_2 - n_3$ and $n_8 = n_3 - n_4$.

$$n_5 + n_6 + n_7 + n_8 = (n_4 - n_1) + (n_1 - n_2) + (n_2 - n_3) + (n_3 - n_4) = 0$$

Therefore,

$$\begin{aligned}c &= n_1e_1 + n_2e_2 + n_3e_3 + n_4e_4 + (n_4 - n_1)e_5 + (n_1 - n_2)e_6 + (n_2 - n_3)e_7 + (n_3 - n_4)e_8 \\ &= n_1(e_1 - e_5 + e_6) + n_2(e_2 - e_6 + e_7) + n_3(e_3 - e_7 + e_8) + n_4(e_4 + e_5 - e_8)\end{aligned}$$

So, if $c \in \operatorname{Ker}\partial_1$, it can be expressed as an integral linear combination of $e_1 - e_5 + e_6$, $e_2 - e_6 + e_7$, $e_3 - e_7 + e_8$ and $e_4 + e_5 - e_8$. So $\operatorname{Ker}\partial_1$ is free abelian of rank 4. Hence,

$$H_1(K) = \operatorname{Ker}\partial_1 \cong \mathbb{Z}^4$$

Now we are going to compute $H_1(L)$. As before, there are no 2-simplices in L. So $C_2(L) = 0$, and hence ∂_2 is trivial. Therefore,

$$H_1(L) = \operatorname{Ker}\partial_1 / \operatorname{im}\partial_2 = \operatorname{Ker}\partial_1$$

A general 1-chain of L is of the form $c = \sum_{i=1}^{5} n_i e_i$, where $n_i \in \mathbb{Z}$.

$$\begin{aligned}\partial_1 c &= n_1(v_1 - v_0) + n_2(v_2 - v_1) + n_3(v_3 - v_2) + n_4(v_0 - v_3) + n_5(v_2 - v_0) \\ &= v_0(-n_1 + n_4 - n_5) + v_1(n_1 - n_2) + v_2(n_2 - n_3 + n_5) + v_3(n_3 - n_4)\end{aligned}$$

If c is a 1-cycle, then $\partial_1 c = 0$, so we must have

$$-n_1 + n_4 - n_5 = n_1 - n_2 = n_2 - n_3 + n_5 = n_3 - n_4 = 0$$

So, $n_1 = n_2$, $n_3 = n_4$, and $n_5 = -n_1 + n_4 = n_3 - n_1$. Therefore,

$$\begin{aligned}c &= n_1e_1 + n_1e_2 + n_3e_3 + n_3e_4 + (n_3 - n_1)e_5 \\ &= n_1(e_1 + e_2 - e_5) + n_3(e_3 + e_4 + e_5)\end{aligned}$$

So, if $c \in \operatorname{Ker}\partial_1$, it can be expressed as an integral linear combination of $e_1 + e_2 - e_5$ and $e_3 + e_4 + e_5$. So $\operatorname{Ker}\partial_1$ is free abelian of rank 2. Hence,

$$H_1(L) = \operatorname{Ker}\partial_1 \cong \mathbb{Z}^2$$

Now, how can we interpret these results geometrically? Notice that K has 4 "holes". A ***n*-dimensional hole** can be mathematically defined as an n-cycle which is not a boundary of any $(n+1)$-dimensional object, in our case $(n+1)$-chain. By this definition, $e_1 + e_6 - e_5$ is a 1-dimensional hole of K. Similarly, $e_2 + e_7 - e_6$, $e_3 + e_8 - e_7$ and $e_4 + e_5 - e_8$ are also 1-dimensional holes of K. And all of them generate $H_1(K)$. So, we can say that the **Betti number** of $H_1(K)$ correspond to the number of 1-dimensional holes of K. In general, it is true that the Betti number of $H_p(K)$ denotes the number of p-dimensional holes of K. It is not so hard to prove, and interested readers are encouraged to prove it.

Similarly, for L, since $H_1(L) \cong \mathbb{Z}^2$, the Betti number of $H_1(L)$ is 2. And L has exactly two 1-dimensional holes: they are $e_3 + e_4 + e_5$ and $e_1 + e_2 - e_5$. It is of no surprise that they are the generator of $H_1(L)$. So the Betti number of $H_1(L)$ is equal to the number of 1-dimensional holes.

A Category Theory Basics

Lecture videos and lecture notes of Category Theory course.

§A.1 What is a Category?

Definition A.1.1 (Category)**.** A category $\mathcal{C}$ consists of

- A collection $\mathcal{C}_0$ whose elements are called the objects of $\mathcal{C}$. Elements of $\mathcal{C}_0$ are denoted by uppercase letters $X, Y, Z, \ldots$
- A collection $\mathcal{C}_1$ whose elements are called the morphisms or arrows of $\mathcal{C}$. Elements of $\mathcal{C}_1$ are denoted by lowercase letters $f, g, h, \ldots$

such that the following hold:

(i) Each morphism assigns two objects called source (or domain) and target (or codomain). We denote them by $s(f)$ and $t(f)$, respectively for a given arrow f. If $s(f) = X \in \mathcal{C}_0$ and $t(f) = Y \in \mathcal{C}_0$ for a given $f \in \mathcal{C}_1$, we write

$$f : X \to Y \;, \text{ or } \; X \xrightarrow{f} Y$$

(ii) Each object $X \in \mathcal{C}_0$ has a distinguished morphism $\mathrm{id}_X : X \to X$.

(iii) For each pair of morphisms $f, g \in \mathcal{C}_1$ such that $t(f) = s(g)$, there exist specified morphisms $g \circ f$ called composite morphisms such that

$$s\,(g \circ f) = s(f) \quad \text{and} \quad t\,(g \circ f) = t(g)$$

In other words, $X \xrightarrow{f} Y \xrightarrow{g} Z$ implying $g \circ f : X \to Z$.

These structures need to satisfy the following axioms:

(a) (Unitality) For each morphism $f : X \to Y$,

$$f \circ \mathrm{id}_X = \mathrm{id}_Y \circ f = f$$

Warning: $\mathrm{id}_X \circ f$ doesn't make sense, so doesn't $f \circ \mathrm{id}_Y$.

(b) (Associativity) For $X, Y, Z, W \in \mathcal{C}_0$ and $f, g, h \in \mathcal{C}_1$ satisfying

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$$

one must have

$$h \circ (g \circ f) = (h \circ g) \circ f$$

The two morphisms that are set equal using associativity axiom can be understood more clearly using the following diagrams:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] \arrow[rrr, bend right=50, "h \circ (g \circ f)"'] & Y \arrow[r, "g"] & Z \arrow[r, "h"] & W \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[rrr, bend right=50, "(h \circ g) \circ f"'] & Y \arrow[r, "g"] \arrow[rr, bend right, "h \circ g"'] & Z \arrow[r, "h"] & W \end{tikzcd}$$

Example A.1.1

We can take a collection of groups in $\mathcal{C}_0$ and the group homomorphisms between them in $\mathcal{C}_1$. In other words, the objects are groups and the morphisms are group homomorphisms. This forms a category.

Similarly, one can form a category of topological spaces too. In that case, the morphisms will be continuous functions between the spaces.

§A.2 Functor and Functorial Properties

Definition A.2.1 (Functor). Let $\mathcal{C} = (\mathcal{C}_0, \mathcal{C}_1)$ and $\mathcal{D} = (\mathcal{D}_0, \mathcal{D}_1)$ be two categories. A functor $\mathcal{F}$ from $\mathcal{C}$ to $\mathcal{D}$, denoted by $\mathcal{F} : \mathcal{C} \to \mathcal{D}$, is a map that has the following properties:

i. $\mathcal{F}$ maps objects of $\mathcal{C}_0$ to objects of $\mathcal{D}_0$.

ii. $\mathcal{F}$ maps morphisms of $\mathcal{C}_1$ to morphisms of $\mathcal{D}_1$, such that for $f \in \mathcal{C}_1$ and $X, Y \in \mathcal{C}_0$

$$f : X \to Y \implies \mathcal{F}(f) : \mathcal{F}(X) \to \mathcal{F}(Y)$$

iii. For every $X \in \mathcal{C}_0$,

$$\mathcal{F}(\mathrm{id}_X) = \mathrm{id}_{\mathcal{F}(X)}$$

iv. For $f, g \in \mathcal{C}_1$ with $t(f) = s(g)$ (in other words, $g \circ f$ makes sense), condition ii guarantees that $t(\mathcal{F}(f)) = s(\mathcal{F}(g))$ (so $\mathcal{F}(g) \circ \mathcal{F}(f)$ makes sense). Then we must have

$$\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f)$$

The definition of functor can be visualized using the following diagram:

Category $\mathcal{C}$

$$\begin{tikzcd} \cdots \arrow[r] & X \arrow[loop above, "\mathrm{id}_X"] \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r] & \cdots \end{tikzcd}$$

$$\Big\downarrow \mathcal{F}$$

Category $\mathcal{D}$

$$\begin{tikzcd} \cdots \arrow[r] & \mathcal{F}(X) \arrow[loop above, "\mathrm{id}_{\mathcal{F}(X)}"] \arrow[r, "\mathcal{F}(f)"] & \mathcal{F}(Y) \arrow[r, "\mathcal{F}(g)"] & \mathcal{F}(Z) \arrow[r] & \cdots \end{tikzcd}$$

Question. Why is Theorem 2.3.2 called functorial properties of induced homomorphism? Is it somehow related to functors?

The answer is, yes. It is indeed related to functors. We shall use the same notations as Theorem 2.3.2 here.

Recall from Example A.1.1 that, we can make a category of topological spaces. Pointed topological spaces are not an exception; we can make a category of pointed topological spaces with the spaces (X, x_0), (Y, y_0) and (Z, z_0). These three spaces are gonna be the objects of the category.

But what are the morphisms of this category? We've seen in the aforementioned example that the morphisms are continuous functions. Here we already have two continuous maps,

$$h_{x_0} : (X, x_0) \to (Y, y_0) \quad \text{and} \quad k_{y_0} : (Y, y_0) \to (Z, z_0)$$

Also, composition of continuous maps is indeed continuous, so $k_{y_0} \circ h_{x_0} : (X, x_0) \to (Z, z_0)$ is also a morphism of this category.

The identity maps are also continuous, so we can take the identity maps $i_{x_0} : (X, x_0) \to (X, x_0)$, $i_{y_0} : (Y, y_0) \to (Y, y_0)$ and $i_{z_0} : (Z, z_0) \to (Z, z_0)$ as morphisms. Thus this category of pointed topological spaces is formed.

$$\begin{tikzcd} (X, x_0) \arrow[loop above, "i_{x_0}"] \arrow[r, "h_{x_0}"] & (Y, y_0) \arrow[loop above, "i_{y_0}"] \arrow[r, "k_{y_0}"] & (Z, z_0) \arrow[loop above, "i_{z_0}"] \end{tikzcd}$$

We can also make a category of fundamental groups. For that, the fundamental groups $\pi_1(X, x_0)$, $\pi_1(Y, y_0)$ and $\pi_1(Z, z_0)$ are gonna be the objects of the category. And the morphisms are groups homomorphisms between them. In this case, we can take the induced homomorphisms between fundamental groups. Also, we take the identity homomorphisms to make it a category.

$$\begin{tikzcd} \pi_1(X, x_0) \arrow[loop above, "\mathrm{id}_{\pi_1(X,x_0)}"] \arrow[r, "(h_{x_0})_*"] & \pi_1(Y, y_0) \arrow[loop above, "\mathrm{id}_{\pi_1(Y,y_0)}"] \arrow[r, "(k_{y_0})_*"] & \pi_1(Z, z_0) \arrow[loop above, "\mathrm{id}_{\pi_1(Z,z_0)}"] \end{tikzcd}$$

Now, if there is a functor $\mathcal{F}$ between these two categories, it must take each topological spaces to the respective fundamental groups.

$$\mathcal{F}\big((X, x_0)\big) = \pi_1(X, x_0) \ , \ \mathcal{F}\big((Y, y_0)\big) = \pi_1(Y, y_0) \ , \ \mathcal{F}\big((Z, z_0)\big) = \pi_1(Z, z_0)$$

To satisfy the second criterion of functor, as $h_{x_0} : (X, x_0) \to (Y, y_0)$, we must have

$$\begin{aligned} \mathcal{F}(h_{x_0}) : \mathcal{F}\big((X, x_0)\big) \to \mathcal{F}\big((Y, y_0)\big) &\implies \mathcal{F}(h_{x_0}) : \pi_1(X, x_0) \to \pi_1(Y, y_0) \\ &\implies \mathcal{F}(h_{x_0}) = (h_{x_0})_* \end{aligned}$$

Similarly for other continuous functions, $\mathcal{F}$ maps them to their respective induced homomorphisms. For $\mathcal{F}$ to satisfy the 3rd property of functor, we must have

$$\mathcal{F}\left(\mathrm{id}_{(X,x_0)}\right) = \mathrm{id}_{\mathcal{F}\left((X,x_0)\right)} \implies \mathcal{F}(i_{x_0}) = \mathrm{id}_{\pi_1(X,x_0)} \implies (i_{x_0})_* = \mathrm{id}_{\pi_1(X,x_0)}$$

So, for $\mathcal{F}$ to be a functor, $(i_{x_0})_*$ must be the identity homomorphism of $\pi_1(X, x_0)$. This is guaranteed by Theorem 2.3.2.

Now, For $\mathcal{F}$ to satisfy the 4th property of functor, we must have

$$\mathcal{F}(k_{y_0} \circ h_{x_0}) = \mathcal{F}(k_{y_0}) \circ \mathcal{F}(h_{x_0}) \implies (k_{y_0} \circ h_{x_0})_* = (k_{y_0})_* \circ (h_{x_0})_*$$

which is, again, guaranteed by Theorem 2.3.2.

Category of Pointed Topological Spaces

$$
\begin{tikzcd}
(X, x_0) \arrow[loop above, "i_{x_0}"] \arrow[r, "h_{x_0}"] \arrow[rr, bend right, "k_{y_0} \circ h_{x_0}"'] & (Y, y_0) \arrow[loop above, "i_{y_0}"] \arrow[r, "k_{y_0}"] & (Z, z_0) \arrow[loop above, "i_{z_0}"]
\end{tikzcd}
$$

$$
\Big\downarrow \mathcal{F}
$$

Category of Groups

$$
\begin{tikzcd}
\pi_1 (X, x_0) \arrow[loop above, "(i_{x_0})_*"] \arrow[r, "(h_{x_0})_*"] \arrow[rr, bend right, "(k_{y_0})_* \circ (h_{x_0})_*"'] & \pi_1 (Y, y_0) \arrow[loop above, "(i_{y_0})_*"] \arrow[r, "(k_{y_0})_*"] & \pi_1 (Z, z_0) \arrow[loop above, "(i_{z_0})_*"]
\end{tikzcd}
$$

To summarize, Theorem 2.3.2 ensures that $\mathcal{F}$ is a functor between the category of pointed topological spaces and the category groups. That's why this theorem is called **functorial properties** of induced homomorphism.