chapter 1

Floating Point representation

- IEEE format (64 bits)
- rounding error
- machine epsilon
- loss of significance

$$fl(x \pm y) = (x \pm y)\left(1 + \frac{x\delta_1 \pm y\delta_2}{x \pm y}\right)$$

Polynomials

function

variable, coefficient $a_i x^i$

i → non-negative integers 0, 1, 2, 3, ..., n

$$f(x) = \underbrace{a_0 x^0 + a_1 x^1 + a_2 x^2 + \dots + a_n x^n}_{\text{finite sum}}$$

example:

polynomial

$1 + 2x + 3x^2$

$3 + 9x + 17x^2 + 81x^3$

NOT polynomial

$1 + 2x + 3x^2 + 4x^3 + 5x^4 + \dots$ (infinite sum)

$7x^{-1}$, $\sqrt{x} = x^{1/2}$

$7x + 9x^{2/3}$ X

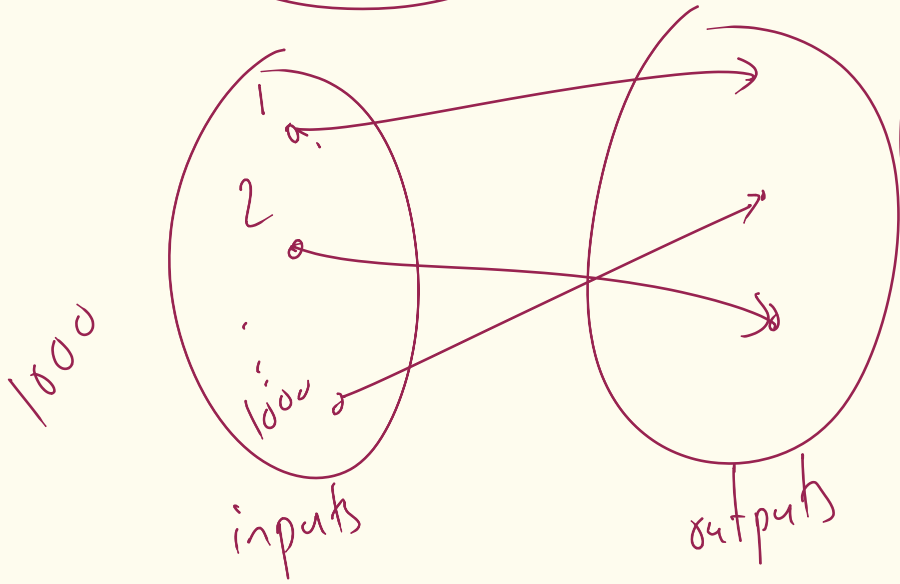
$$f(x) = a_0 + a_1 x + \cdots + a_n x^n = \sum_{i=0}^{n} a_i x^i$$

coefficients

storing a polynomial = $\boxed{\text{storing its coefficients.}}$

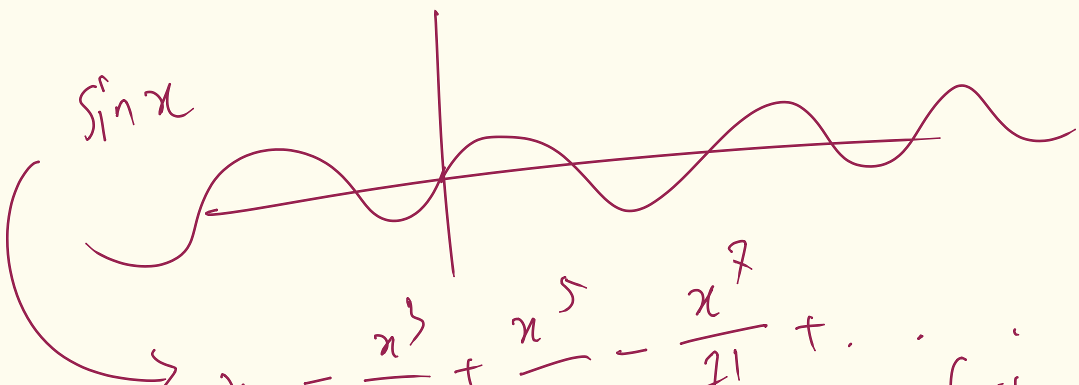
list/array

function



if input == 1 :

if input == 2 :



$$x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{7!} + \cdots$$

infinite sum

$\cos x$, $\tan x$, e^x

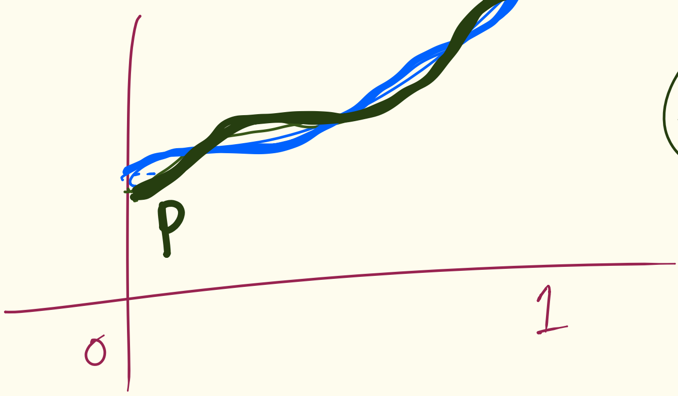
Weierstrass Approximation thm:

suppose f is a continuous function on $[0,1]$. If I choose ε = very small number, then there is a polynomial $p(x)$ s.t.

$$\max |f(x) - p(x)| < \varepsilon.$$

choose $= 10^{-20}$

$f = \cancel{ex} e^x \sin x^2 \cdot \tan x^3$



$f \neq p$

very close

$\sin(0.1)$

approximate sin using a polynomial p.

$p(0.1) \to \sum a_i x^i$

Taylor Series

↳ infinite sum.

$$f(x) = f(0) + x \cdot f'(0) + \frac{x^2}{2} \cdot f''(0) + \frac{x^3}{3!} \cdot f'''(0) + \cdots$$

Mclaurin series

$f(x) = \sin x.$

$f'(x) = \cos x$

$f''(x) = -\sin x$

$f'''(x) = -\cos x$

$f''''(x) = \sin x$

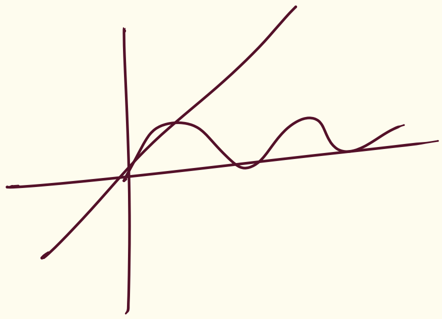
$$f(x) = 0 + x + \cancel{\frac{x^2}{2} \cdot 0} + \frac{x^3}{3!} \times (-1) + \frac{x^4}{4!} \times 0 + \cdots$$

$$= \boxed{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots}$$

$$\boxed{f(0),\ f'(0),\ f''(0),\ f'''(0),\ \ldots}$$

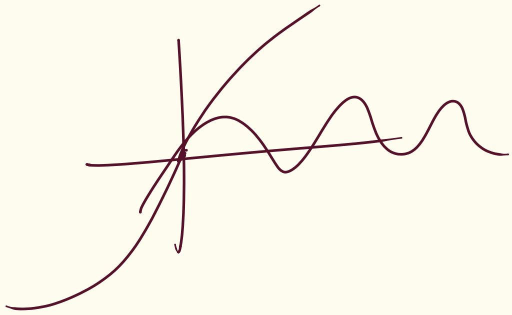
$$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

infinite series



1 term: x

2 terms: $x - \frac{x^3}{6}$



$\sin(0.1) = 0.0998334\underline{1}\ldots$ 7 digits

$x \longrightarrow 0.1$

$x - \frac{x^3}{6} \longrightarrow 0.099833$

$x - \frac{x^3}{6} + \frac{x^5}{120} \longrightarrow 0.0998334\underline{1}$

Truncation error:

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

Taylor series

$$P_n(x) = \underbrace{a_0 + a_1 x + \dots + a_n x^n}$$

$$f \neq P_n$$

Taylor's theorem:

$$\boxed{f(x) - P_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}}$$ for some c.

cos cos

-sin

-cos

sin

$$f(x) = \sin x$$

$$P_6(x) = x - \frac{x^3}{6} + \frac{x^5}{120}$$

$$\boxed{f(x) - P_6(x) = \frac{f^{(7)}(c)}{7!} x^7}$$

choose $x = 0.1$

$$f(x) - P_6(x)$$

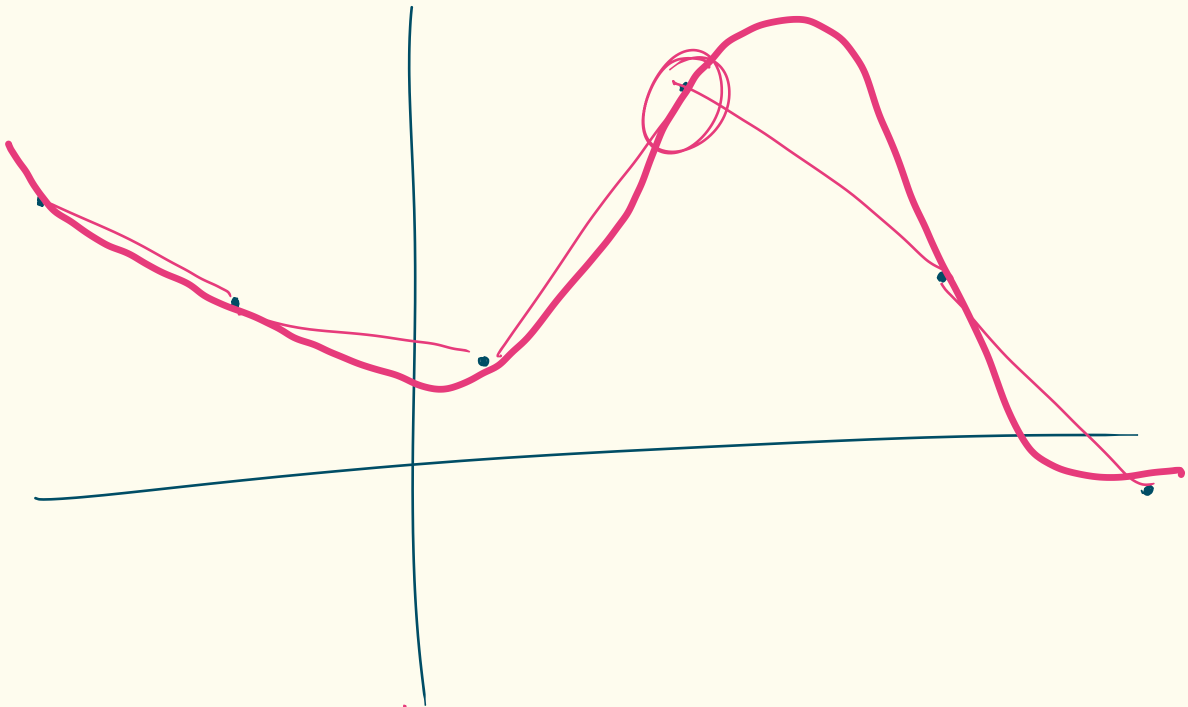
$$\left| \sin(0.1) - P_6(0.1) \right| = \left| \frac{-\cos c}{7!} \cdot (0.1)^7 \right|$$

$$\leq \frac{(0.1)^7}{7!} = 1.9 \times 10^{-11}$$

error is at most 1.9×10^{-11}

Interpolation:

connecting dots (is a smooth way)

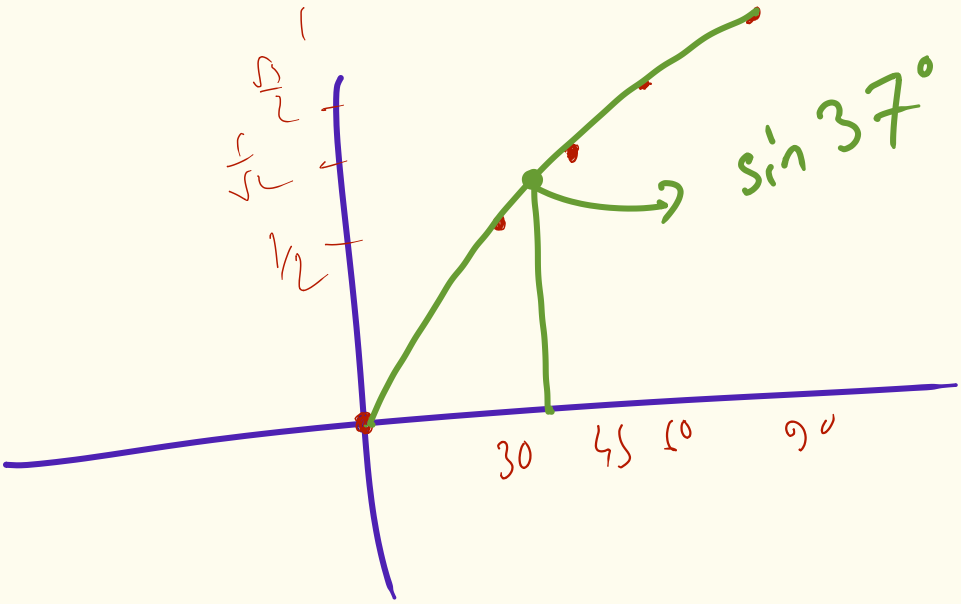


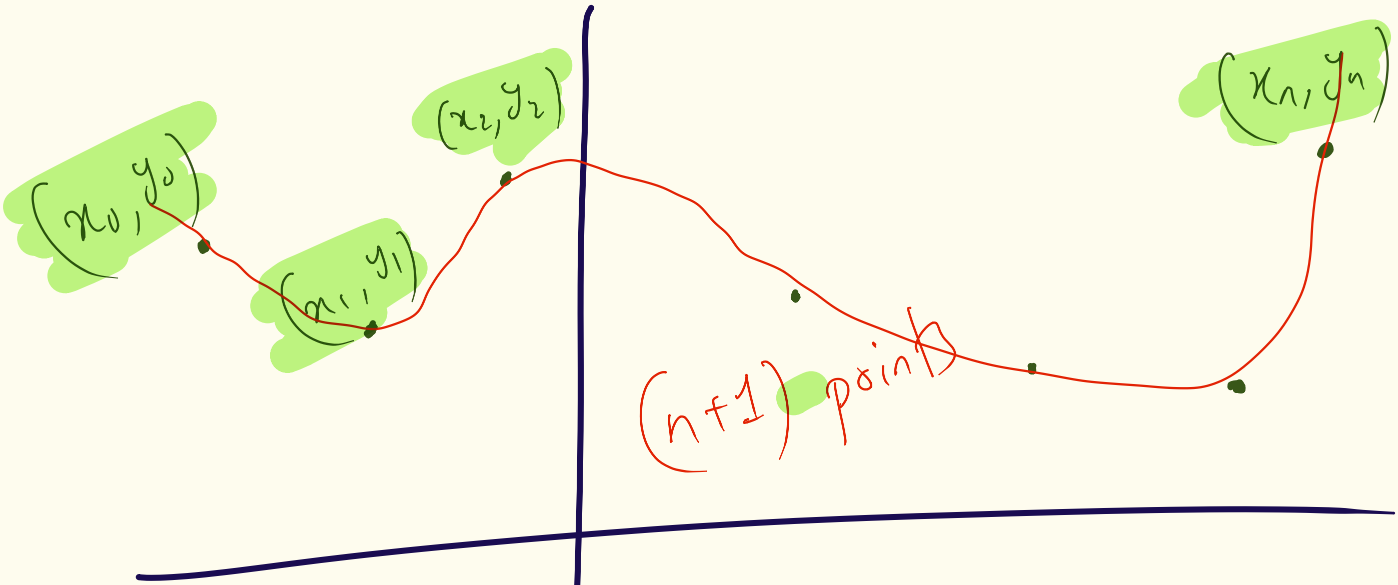
Polynomial interpolation:

$\sin \rightarrow 0^\circ, 30^\circ, 45^\circ, 60^\circ, 90^\circ$

$\sin(37^\circ) = ?$

$\approx \cancel{\dfrac{\sin(30^\circ) + \sin(45^\circ)}{2}}$ linear function





Poly $p(x)$

$p(x_0) = y_0$ | . . . . | $p(x_n) = y_n$

$p(x_1) = y_1$

Poly degree $= n$:

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots + a_n x^n = p(x)$$

$$\begin{aligned}
p(x_0) = y_0 \quad &\Rightarrow \quad a_0 + a_1 x_0 + a_2 x_0^2 + \dots + a_n x_0^n = y_0 \\
p(x_1) = y_1 \quad &\Rightarrow \quad a_0 + a_1 x_1 + a_2 x_1^2 + \dots + a_n x_1^n = y_1 \\
&\quad \vdots \\
p(x_n) = y_n) \quad &\Rightarrow \quad a_0 + a_1 x_n + a_2 x_n^2 + \dots + a_n x_n^n = y_n
\end{aligned}$$

$$\underbrace{\begin{pmatrix} 1 & x_0 & x_0^2 & \dots & x_0^n \\ 1 & x_1 & x_1^2 & \dots & x_1^n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^n \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix}}_{a} = \underbrace{\begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix}}_{y}$$

$$\Rightarrow \quad A a = y.$$

$$\Rightarrow \quad a = ? \quad A^{-1} \cdot y$$

$(0,0)\ ,\ \left(\frac{\pi}{6}, \frac{1}{2}\right), \left(\frac{\pi}{2}, 1\right)$

$x_0 = 0 \qquad y_0 = 0$

$x_1 = \frac{\pi}{6} \qquad y_1 = \frac{1}{2}$

$x_2 = \frac{\pi}{2} \qquad y_2 = 1$

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & \frac{\pi}{6} & \left(\frac{\pi}{6}\right)^2 \\ 1 & \frac{\pi}{2} & \left(\frac{\pi}{2}\right)^2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{2} \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \quad \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \frac{1}{2} \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 0.76 \\ 2.54 \\ 3.29 \end{pmatrix}}_{\text{random}}$$

Polynomial $\boxed{0.76 + 2.54x + 3.29x^2}$

$x = 0.3$