Group Structure on the Universal Cover $\widetilde{G}$ of a Connected Lie Group G

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Let G be a connected Lie group. Then it has a unique (upto diffeomorphism) (universal) simply connected covering manifold $\widetilde{G}$. The purpose of this note is to give a group structure on $\widetilde{G}$. First, let's recall the "universal property" of this covering $p : \widetilde{G} \to G$:

> For any simply connected manifold M and a smooth map $f : M \to G$, suppose that for some $m_0 \in M$, we fix $g_0 \in p^{-1}(f(m))$. Then there is **unique** smooth map $\widetilde{f} : M \to \widetilde{G}$ such that $\widetilde{f}(m_0) = g_0$, and $p \circ \widetilde{f} = f$. In other words, the following diagram commutes:

$$\begin{tikzcd} \widetilde{G} \arrow[r, "p"] & G \\ M \arrow[u, dashed, "\widetilde{f}"] \arrow[ur, "f"'] & \end{tikzcd}$$

This universal property will be our best friend in order to prove that $\widetilde{G}$ can be given a Lie-group structure. We will use the following equivalent form of the universal property:

> Let M be a simply connected manifold and $f : M \to G$ be a smooth map. Suppose there are two smooth maps $f_1, f_2 : M \to \widetilde{G}$, such that $p \circ f_1 = p \circ f_2$, and $f_1(m_0) = f_2(m_0)$ for some $m_0 \in M$. Then $f_1 = f_2$.

$$\begin{tikzcd} \widetilde{G} \arrow[r, "p"] & G \\ M \arrow[u, shift left, "f_1"] \arrow[u, shift right, "f_2"'] \arrow[ur, "f"'] & \end{tikzcd}$$

This clearly follows from the **uniqueness** of $\widetilde{f}$, as we are fixing $f_1(m_0) = f_2(m_0)$. So both f_1 and f_2 fit in place of $\widetilde{f}$, and hence, they must be equal.

§1 Defining the binary operation

Consider the map

$$\begin{aligned} s : \widetilde{G} \times \widetilde{G} &\to G \\ \left(\overline{g}, \overline{h}\right) &\mapsto p\left(\overline{g}\right) p\left(\overline{h}\right)^{-1}. \end{aligned} \tag{1}$$

$\widetilde{G} \times \widetilde{G}$ is simply connected, because $\pi_1\left(\widetilde{G} \times \widetilde{G}\right) = \pi_1\left(\widetilde{G}\right) \times \pi_1\left(\widetilde{G}\right) = \{1\}$. Let's fix some $\overline{e} \in p^{-1}(e)$ (this will be our identity element). Notice that

$$s\left(\overline{e}, \overline{e}\right) = p\left(\overline{e}\right) p\left(\overline{e}\right)^{-1} = e. \tag{2}$$

Therefore, there exists a unique smooth map $\widetilde{s} : \widetilde{G} \times \widetilde{G} \to \widetilde{G}$ such that $\widetilde{s}\left(\overline{e}, \overline{e}\right) = \overline{e}$, and the following diagram commutes:

$$\begin{tikzcd} \widetilde{G} \arrow[r, "p"] & G \\ \widetilde{G} \times \widetilde{G} \arrow[u, dashed, "\widetilde{s}"] \arrow[ur, "s"'] & \end{tikzcd}$$

Then we define

$$\overline{h}^{-1} := \widetilde{s}\left(\overline{e}, \overline{h}\right), \tag{3}$$

$$\overline{g} \cdot \overline{h} := \widetilde{s}\left(\overline{g}, \overline{h}^{-1}\right). \tag{4}$$

We claim that this defines a group structure on $\widetilde{G}$. Clearly, the multiplication map and inversion maps are smooth as they are made out of smooth maps. Hence, it would prove that $\widetilde{G}$ is a Lie group.

> **Lemma 1.1**
>
> For any $\overline{g}, \overline{h} \in \widetilde{G}$,
>
> (a) $\overline{e}^{-1} = \overline{e}$.
>
> (b) $\overline{e} \cdot \overline{e} = \overline{e}$.
>
> (c) $p\left(\overline{h}^{-1}\right) = p\left(\overline{h}\right)^{-1}$.
>
> (d) $p\left(\overline{g} \cdot \overline{h}\right) = p\left(\overline{g}\right) p\left(\overline{h}\right)$, i.e. $p : \widetilde{G} \to G$ is (going to be) a Lie group homomorphism (once we prove that $\widetilde{G}$ is a Lie group).

Proof. (a)

$$\overline{e}^{-1} = \widetilde{s}\left(\overline{e}, \overline{e}\right) = \overline{e}. \tag{5}$$

(b)

$$\overline{e} \cdot \overline{e} = \widetilde{s}\left(\overline{e}, \overline{e}^{-1}\right) = \widetilde{s}\left(\overline{e}, \overline{e}\right) = \overline{e}. \tag{6}$$

(c) Using $p \circ \widetilde{s} = s$,

$$p\left(\overline{h}^{-1}\right) = p\left(\widetilde{s}\left(\overline{e}, \overline{h}\right)\right) = s\left(\overline{e}, \overline{h}\right) = p(\overline{e})p(\overline{h})^{-1} = p(\overline{h})^{-1}. \tag{7}$$

(d) Using $p \circ \widetilde{s} = s$,

$$p\left(\overline{g} \cdot \overline{h}\right) = p\left(\widetilde{s}\left(\overline{g}, \overline{h}^{-1}\right)\right) = s\left(\overline{g}, \overline{h}^{-1}\right) = p(\overline{g})p\left(\overline{h}^{-1}\right)^{-1} = p\left(\overline{g}\right)p\left(\overline{h}\right). \tag{8}$$

∎

Proposition 1.2

The operation defined in (4) is associative.

Proof. Consider the map

$$\begin{aligned} m : \widetilde{G} \times \widetilde{G} \times \widetilde{G} &\to G \\ \left(\overline{g}, \overline{h}, \overline{k}\right) &\mapsto p\left(\overline{g}\right)p\left(\overline{h}\right)p\left(\overline{k}\right). \end{aligned} \tag{9}$$

Note that this is well-defined since the multiplication in G is associative.

$$\begin{tikzcd} \widetilde{G} \arrow[r, "p"] & G \\ \widetilde{G} \times \widetilde{G} \times \widetilde{G} \arrow[u, shift left, "m_1"] \arrow[u, shift right, "m_2"'] \arrow[ur, "m"'] & \end{tikzcd} \tag{10}$$

where the maps m_1 and m_2 are defined as follows:

$$\begin{aligned} m_1 : \widetilde{G} \times \widetilde{G} \times \widetilde{G} &\to \widetilde{G} \\ \left(\overline{g}, \overline{h}, \overline{k}\right) &\mapsto \left(\overline{g} \cdot \overline{h}\right) \cdot \overline{k}. \end{aligned} \tag{11}$$

$$\begin{aligned} m_2 : \widetilde{G} \times \widetilde{G} \times \widetilde{G} &\to \widetilde{G} \\ \left(\overline{g}, \overline{h}, \overline{k}\right) &\mapsto \overline{g} \cdot \left(\overline{h} \cdot \overline{k}\right). \end{aligned} \tag{12}$$

We need to show that m_1 and m_2 are equal. Clearly, they agree on $(\overline{e}, \overline{e}, \overline{e})$, because by (b) of Lemma 1.1,

$$\begin{aligned} (\overline{e} \cdot \overline{e}) \cdot \overline{e} = \overline{e} \cdot \overline{e} = \overline{e}; \\ \overline{e} \cdot (\overline{e} \cdot \overline{e}) = \overline{e} \cdot \overline{e} = \overline{e}. \end{aligned}$$

Both m_1 and m_2 make (10) commutative, since by (d) of Lemma 1.1,

$$\begin{aligned}
(p \circ m_1)\left(\overline{g}, \overline{h}, \overline{k}\right) &= p\left(\left(\overline{g} \cdot \overline{h}\right) \cdot \overline{k}\right) \\
&= p\left(\overline{g} \cdot \overline{h}\right) p\left(\overline{k}\right) \\
&= p\left(\overline{g}\right) p\left(\overline{h}\right) p\left(\overline{k}\right) = m\left(\overline{g}, \overline{h}, \overline{k}\right). \qquad (13) \\
(p \circ m_2)\left(\overline{g}, \overline{h}, \overline{k}\right) &= p\left(\overline{g} \cdot \left(\overline{h} \cdot \overline{k}\right)\right) \\
&= p\left(\overline{g}\right) p\left(\left(\overline{h} \cdot \overline{k}\right)\right) \\
&= p\left(\overline{g}\right) p\left(\overline{h}\right) p\left(\overline{k}\right) = m\left(\overline{g}, \overline{h}, \overline{k}\right). \qquad (14) \\
& \qquad (15)
\end{aligned}$$

Since $\widetilde{G} \times \widetilde{G} \times \widetilde{G}$ is simply connected, by the uniqueness parrt of the universal property, $m_1 = m_2$, i.e. (4) is associative. ∎

> **Proposition 1.3**
>
> $\overline{e}$ is a two-sided identity.

Proof. Consider the map

$$\begin{aligned}
\ell : \widetilde{G} &\to \widetilde{G} \\
\overline{g} &\mapsto \overline{e} \cdot \overline{g} = \widetilde{s}\left(\overline{e}, \overline{g}^{-1}\right).
\end{aligned} \qquad (16)$$

We then have the following diagram:

$$\begin{tikzcd}
\widetilde{G} \arrow[r, "p"] & G \\
\widetilde{G} \arrow[u, shift left, "\mathrm{id}_{\widetilde{G}}"] \arrow[u, shift right, "\ell"'] \arrow[ur, "p"'] &
\end{tikzcd} \qquad (17)$$

Note that $\mathrm{id}_{\widetilde{G}}$ and ℓ agree on $\overline{e}$. Furthermore,

$$(p \circ \ell)\left(\overline{g}\right) = p\left(\overline{e} \cdot \overline{g}\right) = p\left(\overline{e}\right) p\left(\overline{g}\right) = p\left(\overline{g}\right). \qquad (18)$$

So $p \circ \mathrm{id}_{\widetilde{G}} = p \circ \ell$. Therefore, both ℓ and $\mathrm{id}_{\widetilde{G}}$ make (17) commutative. Hence, by the uniqueness parrt of the universal property, $\ell = \mathrm{id}_{\widetilde{G}}$, i.e. $\overline{e} \cdot \overline{g} = \overline{g}$.

Analogously, we can define

$$\begin{aligned}
r : \widetilde{G} &\to \widetilde{G} \\
\overline{g} &\mapsto \overline{g} \cdot \overline{e} = \widetilde{s}\left(\overline{g}, \overline{e}^{-1}\right).
\end{aligned} \qquad (19)$$

We then have the following diagram:

$$
\begin{tikzcd}
\widetilde{G} \arrow[r, "p"] & G \\
\widetilde{G} \arrow[u, shift left, "r"'] \arrow[u, shift right, "\mathrm{id}_{\widetilde{G}}"] \arrow[ur, "p"'] &
\end{tikzcd}
\tag{20}
$$

Note that $\mathrm{id}_{\widetilde{G}}$ and r agree on $\overline{e}$. Furthermore,

$$
(p \circ r)(\overline{g}) = p(\overline{g} \cdot \overline{e}) = p(\overline{g})\, p(\overline{e}) = p(\overline{g})\,. \tag{21}
$$

So $p \circ \mathrm{id}_{\widetilde{G}} = p \circ r$. Therefore, both r and $\mathrm{id}_{\widetilde{G}}$ make (20) commutative. Hence, by the uniqueness parrt of the universal property, $r = \mathrm{id}_{\widetilde{G}}$, i.e. $\overline{g} \cdot \overline{e} = \overline{g}$. ■

Proposition 1.4

The inverse defined on (3) is two-sided inverse.

Proof. Consider the maps

$$
\begin{aligned}
i_1 : \widetilde{G} &\to \widetilde{G} \\
\overline{g} &\mapsto \overline{g} \cdot \overline{g}^{-1}.
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
\mathrm{const}_{\overline{e}} : \widetilde{G} &\to \widetilde{G} \\
\overline{g} &\mapsto \overline{e}.
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
\mathrm{const}_{e} : \widetilde{G} &\to G \\
\overline{g} &\mapsto e.
\end{aligned}
\tag{24}
$$

We then have the following diagram:

$$
\begin{tikzcd}
\widetilde{G} \arrow[r, "p"] & G \\
\widetilde{G} \arrow[u, shift left, "i_1"'] \arrow[u, shift right, "\mathrm{const}_{\overline{e}}"] \arrow[ur, "\mathrm{const}_e"'] &
\end{tikzcd}
\tag{25}
$$

Note that $\mathrm{const}_{\overline{e}}$ and i_1 agree on $\overline{e}$. Furthermore,

$$
(p \circ i_1)(\overline{g}) = p\left(\overline{g} \cdot \overline{g}^{-1}\right) = p(\overline{g})\, p\left(\overline{g}^{-1}\right) = p(\overline{g})\, p(\overline{g})^{-1} = e. \tag{26}
$$

So $p \circ \mathrm{const}_{\overline{e}} = p \circ i_1$. Therefore, both i_1 and $\mathrm{const}_{\overline{e}}$ make (25) commutative. Hence, by the uniqueness parrt of the universal property, $i_1 = \mathrm{const}_{\overline{e}}$, i.e. $\overline{g} \cdot \overline{g}^{-1} = \overline{e}$.

Furthermore, consider the maps

$$\begin{aligned} i_2 : \widetilde{G} &\to \widetilde{G} \\ \overline{g} &\mapsto \overline{g}^{-1} \cdot \overline{g}. \end{aligned} \tag{27}$$

We then have the following diagram:

$$\begin{tikzcd} \widetilde{G} \arrow[r, "p"] & G \\ \widetilde{G} \arrow[u, shift left, "\mathrm{const}_{\overline{e}}"] \arrow[u, shift right, "i_2"'] \arrow[ur, "\mathrm{const}_e"'] & \end{tikzcd} \tag{28}$$

Note that $\mathrm{const}_{\overline{e}}$ and i_1 agree on $\overline{e}$. Furthermore,

$$(p \circ i_1)(\overline{g}) = p\left(\overline{g}^{-1} \cdot \overline{g}\right) = p\left(\overline{g}^{-1}\right) p(\overline{g}) = p(\overline{g})^{-1} p(\overline{g}) = e. \tag{29}$$

So $p \circ \mathrm{const}_{\overline{e}} = p \circ i_1$. Therefore, both i_2 and $\mathrm{const}_{\overline{e}}$ make (28) commutative. Hence, by the uniqueness parrt of the universal property, $i_2 = \mathrm{const}_{\overline{e}}$, i.e. $\overline{g}^{-1} \cdot \overline{g} = \overline{e}$. ■

By combining Proposition 1.2, Proposition 1.3 and Proposition 1.4, we finally conclude that

Theorem 1.5

$\widetilde{G}$ is a group with respect to the operations defined on (3) and (4).