

## Group Structure on $\tilde{G}$

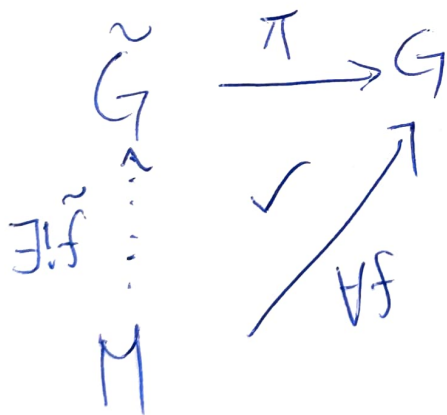
Recall: universal property of cover

$\tilde{G} \xrightarrow{\pi} G$  is universal, in the sense that

$\forall$  simply connected manifold  $M$  and smooth map  $f: M \rightarrow G$ , with

$m_0 \in M$  and  $g_0 \in \pi^{-1}(f(m_0))$ ,

$\exists!$   $\tilde{f}: M \rightarrow \tilde{G}$  smooth s.t.  $\tilde{f}(m_0) = g_0$  &



Now, consider  $s: \tilde{G} \times \tilde{G} \rightarrow G$   
 $(\bar{g}, \bar{h}) \mapsto \pi(\bar{g}) \pi(\bar{h})^{-1}$

$\tilde{G} \times \tilde{G}$  is simply connected, so

$$\begin{array}{ccc} \tilde{G} & \xrightarrow{\pi} & G \\ \uparrow \tilde{s} & \nearrow s & \\ \tilde{G} \times \tilde{G} & & \end{array} \quad \text{Fix } \bar{e} \in \pi^{-1}(e) \text{ and } \tilde{s}(\bar{e}, \bar{e}) = \bar{e}.$$

Define  $\bar{h}^{-1} := \tilde{s}(\bar{e}, \bar{h})$   
 $\bar{g} \cdot \bar{h} := \tilde{s}(\bar{g}, \bar{h}^{-1})$ .

Theorem: This defines a group structure.

Claim 1:  $\pi(\bar{g} \cdot \bar{h}) = \pi(\bar{g}) \cdot \pi(\bar{h})$

Proof:  $\pi(\bar{g} \cdot \bar{h}) = \pi(\tilde{s}(\bar{g}, \bar{h}^{-1}))$   
 $= s(\bar{g}, \bar{h}^{-1}) = \pi(\bar{g}) \pi(\bar{h}^{-1})^{-1}$

$\pi(\bar{h}^{-1})^{-1} = \pi(\tilde{s}(\bar{e}, \bar{h}))^{-1} = s(\bar{e}, \bar{h})^{-1} = (\pi(\bar{e}) \pi(\bar{h})^{-1})^{-1} = \pi(\bar{h})$

Claim 2:  $\bar{e}$  is two-sided identity.

consider  $\cancel{l_{\bar{x}} = \tilde{s}(\bar{e}, \bar{x}^{-1}) : \tilde{G} \rightarrow \tilde{G}}$   
 $\cancel{\text{id}}$

$$l(\bar{x}) = \bar{e} \cdot \bar{x} = \tilde{s}(\bar{e}, \bar{x}^{-1}) : \tilde{G} \rightarrow \tilde{G}$$

$$\text{id}(\bar{x}) = \bar{x} : \tilde{G} \rightarrow \tilde{G}$$

$$\begin{array}{ccc} \tilde{G} & \xrightarrow{\pi} & G \\ \uparrow l & \nearrow \pi & \\ \tilde{G} & & \end{array} \quad \begin{array}{l} \pi(l(\bar{x})) \\ = \pi(\tilde{s}(\bar{e}, \bar{x}^{-1})) \\ = s(\bar{e}, \bar{x}^{-1}) \\ = \pi(\bar{e}) \pi(\bar{x}^{-1})^{-1} \\ = \pi(\bar{x}) \end{array}$$

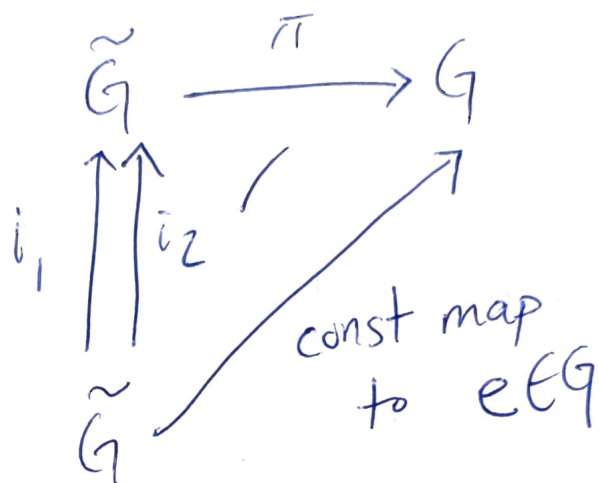
$\Rightarrow$  By uniqueness,  $l = \text{id}$ , i.e.  $\bar{e} \cdot \bar{x} = \bar{x}$   
 $\forall \bar{x} \in \tilde{G}$ .

Similarly,

$$\begin{array}{ccc} \tilde{G} & \xrightarrow{\pi} & G \\ \uparrow \gamma & \nearrow \pi & \\ \tilde{G} & & \end{array} \quad \begin{array}{l} \gamma(\bar{x}) = \bar{x} \cdot \bar{e} \\ \pi(\gamma(\bar{x})) = \pi(\tilde{s}(\bar{x}, \bar{e}^{-1})) \\ = s(\bar{x}, \bar{e}) \\ = \pi(\bar{x}) \end{array}$$

$\Rightarrow \text{id} = \gamma$ . □

Claim 3: Inverses are two-sided.



$$\begin{array}{l|l} i_1(\bar{x}) = \bar{x} \cdot \bar{x}^{-1} & \pi(i_1(\bar{x})) = \pi(\tilde{s}(\bar{x}, (\bar{x}^{-1})^{-1})) \\ i_2(\bar{x}) = \bar{e} & = \pi(\tilde{s}(\bar{x}, \bar{x})) \\ & = s(\bar{x}, \bar{x}) = e \end{array}$$

$\Rightarrow i_1 = i_2$ , since they agree at  $\bar{e}$ .

similarly,

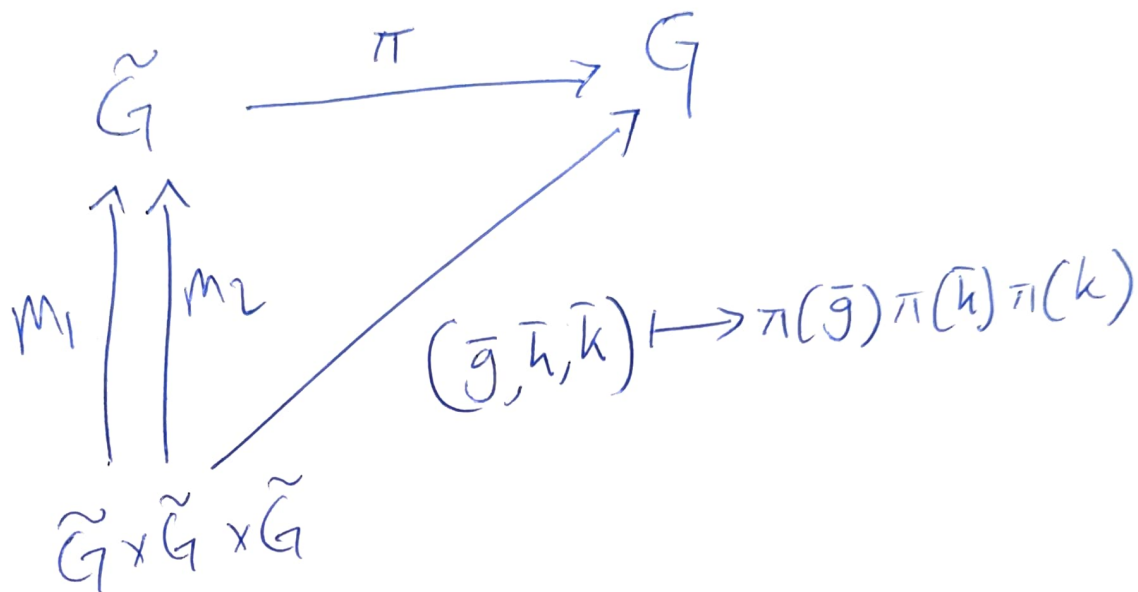
$$\begin{array}{l|l} j_1(\bar{x}) = \bar{x}^{-1} \cdot \bar{x} & \pi(j_1(\bar{x})) = \pi(\tilde{s}(\bar{x}^{-1}, \bar{x}^{-1})) = e \\ j_2(\bar{x}) = \bar{e} & \end{array}$$

$$\pi(j_1(\bar{x})) = \pi(\tilde{s}(\bar{x}^{-1}, \bar{x}^{-1})) = e$$

$$\Rightarrow j_1 = j_2$$

□

Claim 4: Associativity:



$$\begin{aligned}
 m_1(\bar{g}, \bar{h}, \bar{k}) &= (\bar{g} \cdot \bar{h}) \cdot \bar{k} \\
 m_2(\bar{g}, \bar{h}, \bar{k}) &= \bar{g} \cdot (\bar{h} \cdot \bar{k})
 \end{aligned}$$

$$\begin{aligned}
 \pi(m_1(\bar{g}, \bar{h}, \bar{k})) &= \pi(\bar{g} \cdot \bar{h}) \pi(\bar{k}) \\
 &= \pi(\bar{g}) \pi(\bar{h}) \pi(\bar{k}) \\
 &\quad (\text{claim 1})
 \end{aligned}$$

$$\begin{aligned}
 \pi(m_2(\bar{g}, \bar{h}, \bar{k})) &= \pi(\bar{g}) \pi(\bar{h} \cdot \bar{k}) \\
 &= \pi(\bar{g}) \pi(\bar{h}) \pi(\bar{k})
 \end{aligned}$$

$$\Rightarrow m_1 = m_2$$

$\therefore \tilde{G}$  is a group.

