Leibniz-like Rule in Lie Groups

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§1 Background

Let M be a manifold, and $p \in M$. Then we have the notion of the tangent space T_pM at the point p. This is the space of all point-derivations at p, i.e. all linear maps $X_p : C_p^\infty(M) \to \mathbb{R}$ such that

$$X_p(fg) = (X_p f)\, g(p) + f(p)\, (X_p g)\,. \tag{1}$$

Here $C_p^\infty(M)$ is the germs of C^∞ functions $p \in M$, i.e. the set of equivalence classes of smooth functions that agree on some neighborhood of p. (1) can be thought of as the analogue of the Leibniz (product) rule for single-variable derivatives:

$$(fg)' = f'g + fg'. \tag{2}$$

Locally speaking, on a coordinate chart $\left(U, x^1, \ldots, x^n\right)$ around p, the tangent vector X_p is a linear combination of the partial derivatives

$$\left.\frac{\partial}{\partial x^1}\right|_p, \left.\frac{\partial}{\partial x^2}\right|_p, \ldots, \left.\frac{\partial}{\partial x^n}\right|_p.$$

Now, given a smooth map $F : N \to M$ and $p \in N$, one can define the differential of F,

$$\begin{aligned} &(dF)_p : T_pN \to T_{F(p)}M \\ &[(dF)_p X_p]\, f = X_p\,(f \circ F)\,, \end{aligned} \tag{3}$$

for $f \in C^\infty_{F(p)}(M)$. Locally, $(dF)_p$ is the Jacobian matrix

$$\left[\frac{\partial F^i}{\partial x^j}(p)\right]_{1 \le i \le m,\ 1 \le j \le n.}$$

The alternate notation for $(dF)_p$ is $f_{*,p}$, which is also used in some literatures. But we will stick to the notation $(dF)_p$, since it manifests the fact that it's the differential of the smooth map F.

$(dF)_p$ takes us from a smooth map of $F : N \to M$ manifolds to a linear map $(dF)_p : T_pN \to T_{F(p)}M$ between vector spaces. So we can expect that it defines a functor from the category of pointed manifolds to the category of vector spaces, given by

$$\begin{aligned} (M,p) &\quad\mapsto\quad T_pM \\ F:(N,p)\to(M,q) &\quad\mapsto\quad (dF)_p : T_pN \to T_{F(p)}M = T_qM. \end{aligned} \tag{4}$$

This, indeed, defines a functor since the differential has the following functorial property:

Theorem 1.1

Let $F : N \to M$ and $G : M \to P$ be smooth maps of manifolds, and $p \in N$. Then,

$$d\left(G \circ F\right)_p = (dG)_{F(p)} \circ (dF)_p. \tag{5}$$

In other words, the following diagram commutes:

$$\begin{tikzcd} T_pN \arrow[r, "(dF)_p"] \arrow[rr, bend right, "d(G\circ F)_p = (dG)_{F(p)}\circ (dF)_p"'] & T_{F(p)}M \arrow[r, "(dG)_{F(p)}"] & T_{G(F(p))}P \end{tikzcd}$$

Furthermore, if $\mathbb{1}_M : M \to M$ is the identity map, then

$$(d\mathbb{1}_M)_p = \mathbb{1}_{T_pM} : T_pM \to T_pM. \tag{6}$$

We won't be proving it here, but the proof can be found in any elementary differential geometry textbook/course notes, for instance here.

§2 Lie group setting

Now suppose the codomain of the smooth map is a Lie group G. Now, using the smooth multiplication in G, we can take the product of two smooth maps into G. In other words, given smooth maps $F_1, F_2 : N \to G$, we can consider the map

$$\begin{aligned} F_1F_2 : N &\to G \\ p &\mapsto F_1(p)F_2(p). \end{aligned} \tag{7}$$

But then, naively, you may be prompted to write the differential $d\left(F_1F_2\right)_p$ like this:

$$d\left(F_1F_2\right)_p \overset{?}{=} \left(dF_1\right)_p F_2(p) + F_1(p)\left(dF_2\right)_p. \tag{8}$$

But if you look at it carefully, (8) makes no sense. $\left(dF_1\right)_p$ is a linear map $T_pN \to T_{F_1(p)}G$, and $F_2(p)$ is a Lie group element. How are we actually multiplying/composing them? Is there some sort of group action going on here? The same can be asked for $F_1(p)$ and $\left(dF_2\right)_p$ as well. But although (8) looks complete nonsense, there's a way to make sense of it. We can define the left/right actions in such a way that (8) actually makes sense. Let

$$\begin{aligned} m : G \times G &\to G \\ (g_1, g_2) &\mapsto g_1 g_2 \end{aligned} \tag{9}$$

be the multiplication map on G, sinc G is a Lie group. Let L_g and R_g be the left and the multiplication by g, respectively. In other words,

$$\begin{aligned} L_g = m\,(g, -) : G &\to G \\ h &\mapsto gh; \end{aligned} \tag{10}$$

$$\begin{aligned} R_g = m\,(-, g) : G &\to G \\ h &\mapsto hg. \end{aligned} \tag{11}$$

The goal of this note is to prove the following:

Theorem 2.1

If G is a Lie group and $F_1, F_2 : N \to G$ are smooth maps, then

$$d\,(F_1F_2)_p = \left(dR_{F_2(p)}\right)_{F_1(p)} \circ (dF_1)_p + \left(dL_{F_1(p)}\right)_{F_2(p)} \circ (dF_2)_p\,. \tag{12}$$

Note that the compositions make sense, because

$$T_pN \xrightarrow{(dF_1)_p} T_{F_1(p)}G \xrightarrow{\left(dR_{F_2(p)}\right)_{F_1(p)}} T_{F_1(p)F_2(p)}G \tag{13}$$

$$T_pN \xrightarrow{(dF_2)_p} T_{F_2(p)}G \xrightarrow{\left(dL_{F_1(p)}\right)_{F_2(p)}} T_{F_1(p)F_2(p)}G \tag{14}$$

Furthermore, the addition is also understood since we are adding two linear maps $T_pN \to T_{F_1(p)F_2(p)}G$.

§3 Proof

Throughout this section, we shall write $g = F_1(p)$ and $h = F_2(p)$.

Lemma 3.1

$T_{(g,h)}\,(G \times G) \cong T_gG \times T_hG$.

Proof. Consider the maps

$$G \xleftarrow{\pi_1} G \times G \xrightarrow{\pi_2} G \tag{15}$$

where $\pi_1(g_1, g_2) = g_1$ and $\pi_2(g_1, g_2) = g_2$. These are smooth maps. So we consider

$$T_gG \xleftarrow{(d\pi_1)_{(g,h)}} T_{(g,h)}(G \times G) \xrightarrow{(d\pi_2)_{(g,h)}} T_hG \tag{16}$$

Now we define

$$\begin{aligned} \Psi : T_{(g,h)}(G \times G) &\to T_gG \times T_hG \\ Z_{(g,h)} &\mapsto \left((d\pi_1)_{(g,h)} Z_{(g,h)}, (d\pi_2)_{(g,h)} Z_{(g,h)}\right). \end{aligned} \tag{17}$$

In the other direction, we consider

$$G \xhookrightarrow{i_1} G \times G \xhookleftarrow{i_2} G \tag{18}$$

where $i_1(g_1) = (g_1, h)$ and $i_2(g_1) = (g, g_1)$. These are clearly smooth maps, since they are inclusions. Now,

$$T_gG \xrightarrow{(di_1)_g} T_{(g,h)}(G \times G) \xleftarrow{(di_2)_h} T_hG \tag{19}$$

Then we define

$$\begin{aligned} \Phi : T_gG \times T_hG &\to T_{(g,h)}(G \times G) \\ (X_g, Y_h) &\mapsto (di_1)_g X_g + (di_2)_h Y_h. \end{aligned} \tag{20}$$

Then one can check that both $\Psi \circ \Phi$ and $\Phi \circ \Psi$ are identity maps. The first one follows from the fact that $\pi_1 \circ i_1 = \pi_2 \circ i_2 = \mathbb{1}_G$ so their differentials are also identity maps. The second one follows using a computation in local coordinates. ■

Proposition 3.2

Consider the smooth map $F = (F_1, F_2) : N \to G \times G$ defined by $p \mapsto (F_1(p), F_2(p))$. Then

$$(dF)_p = \left((dF_1)_p, (dF_2)_p\right), \tag{21}$$

under the identification $T_{(g,h)}(G \times G) \cong T_gG \times T_hG$. More concretely,

$$\Psi \circ (dF)_p = \left((dF_1)_p, (dF_2)_p\right). \tag{22}$$

Proof. Consider this commutative diagram in the category of smooth manifolds and smooth maps:

$$\begin{tikzcd} & N \arrow[dl, "F_1"'] \arrow[d, "F"] \arrow[dr, "F_2"] & \\ G & G \times G \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & G \end{tikzcd} \tag{23}$$

Then we have the following commutative diagram in the category of real vector spaces, by the functorial property of differential (Theorem 1.1):

$$\begin{tikzcd} & & T_pN \arrow[dll, "(dF_1)_p"'] \arrow[d, "(dF)_p"'] \arrow[drr, "(dF_2)_p"] & & \\ T_gG & & T_{(g,h)}(G\times G) \arrow[ll, "(d\pi_1)_{(g,h)}"] \arrow[rr, "(d\pi_2)_{(g,h)}"'] & & T_hG \end{tikzcd} \tag{24}$$

Now, given any $X_p \in T_pN$,

$$\begin{aligned} [\Psi \circ (dF)_p]\, X_p &= \big((d\pi_1)_{(g,h)}(dF)_pX_p, (d\pi_2)_{(g,h)}(dF)_pX_p\big) \\ &= ((dF_1)_pX_p, (dF_2)_pX_p)\,, \end{aligned} \tag{25}$$

by the commutativity of (24). Therefore, $\Psi \circ (dF)_p = \Big((dF_1)_p\,, (dF_2)_p\Big)$. ■

Proposition 3.3

For any $X_g \in T_gG$ and $Y_h \in T_hG$,

$$(dm)_{(g,h)} \circ \Phi\,(X_g, Y_h) = (dR_h)_gX_g + (dL_g)_hY_h. \tag{26}$$

Proof. Consider this commutative diagram in the category of smooth manifolds and smooth maps:

$$\begin{tikzcd} G \arrow[r, "i_1"] \arrow[dr, "R_h=m\circ i_1"'] & G\times G \arrow[d, "m"] & G \arrow[l, "i_2"'] \arrow[dl, "L_g=m\circ i_2"] \\ & G & \end{tikzcd} \tag{27}$$

Then we have the following commutative diagram in the category of real vector spaces, by the functorial property of differential (Theorem 1.1):

$$\begin{tikzcd} T_gG \arrow[r, "(di_1)_g"] \arrow[dr, "(dR_h)_g"'] & T_{(g,h)}(G\times G) \arrow[d, "(dm)_{(g,h)}"] & T_hG \arrow[l, "(di_2)_h"'] \arrow[dl, "(dL_g)_h"] \\ & T_{gh}G & \end{tikzcd} \tag{28}$$

Now,

$$\begin{aligned}(dm)_{(g,h)} \circ \Phi\left(X_g, Y_h\right) &= (dm)_{(g,h)}\left[(di_1)_g X_g + (di_2)_h Y_h\right] \\ &= (dm)_{(g,h)}(di_1)_g X_g + (dm)_{(g,h)}(di_2)_h Y_h \\ &= (dR_h)_g X_g + (dL_g)_h Y_h, \end{aligned} \tag{29}$$

by the commutativity of (28). Therefore, $(dm)_{(g,h)} \circ \Phi\left(X_g, Y_h\right) = (dR_h)_g X_g + (dL_g)_h Y_h$. ■

Now we are ready to finally conclude the proof of Theorem 2.1.

Proof of Theorem 2.1. Note that

$$F_1 F_2 = m \circ (F_1, F_2) = m \circ F. \tag{30}$$

Therefore, by Theorem 1.1,

$$d\left(F_1 F_2\right)_p = (dm)_{gh} \circ (dF)_p\,. \tag{31}$$

$$\begin{tikzcd} & T_{(g,h)}(G \times G) \arrow[dr, "(dm)_{(g,h)}"] \arrow[dd, shift right, "\Psi"'] & \\ T_pN \arrow[ur, "(dF)_p"] & & T_{gh}G \\ & T_gG \times T_hG \arrow[uu, shift right, "\Phi"'] & \end{tikzcd} \tag{32}$$

Now, in light of (32), since $\Phi \circ \Psi$ is identity, we can conclude that

$$\begin{aligned} d\left(F_1 F_2\right)_p = (dm)_{gh} \circ (dF)_p &= (dm)_{gh} \circ (\Phi \circ \Psi) \circ (dF)_p \\ &= (dm)_{gh} \circ \Phi \circ \Big((dF_1)_p\,, (dF_2)_p\Big) \qquad (33) \\ &= (dR_h)_g\,(dF_1)_p + (dL_g)_h\,(dF_2)_p\,. \qquad (34) \end{aligned}$$

Here (33) follows from Proposition 3.2 and (34) follows from Proposition 3.3. Hence, (12) is verified! ■