Lie Algebra Associated with a Lie Group

Atonu Roy Chowdhury

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Given a Lie group G, its tangent space at the identity, T_eG, is isomorphic to the vector space $\mathfrak{L}(G)$ of all left-invariant vector fields on G. $\mathfrak{L}(G)$ has the structure of a Lie algebra under the operation $[X, Y] = XY - YX$. In fact, it's a Lie subalgebra of the space of all vector fields $\mathfrak{X}(G)$. We then establish a correspondence between the Lie algebra and the Lie group, namely the exponential map.

§1 Lie algebra

Definition 1.1 (Lie algebra)**.** A **Lie algebra** over a field $\mathbb{K}$ is a $\mathbb{K}$-vector space V equipped with a bilinear operation $[-,-] : V \times V \to V$ (called the **Lie bracket**) such that

(a) $[-,-]$ is antisymmetric, i.e. $[v_1, v_2] = -[v_2, v_1]$ for all $v_1, v_2 \in V$; and

(b) for any $u, v, w \in V$,

$$[[u,v],w] + [[w,u],v] + [[v,w],u] = 0. \tag{1}$$

(1) is also known as the Jacobi identity.

Example 1.1. If V is an associative $\mathbb{K}$-algebra with respect to the product $\cdot$, then we can define the Lie bracket as

$$[v_1, v_2] = v_1 \cdot v_2 - v_2 \cdot v_1. \tag{2}$$

Then V is a Lie algebra with respect to $[-,-]$.

Example 1.2. Let M be a manifold, and $\mathfrak{X}(M)$ is the vector space of all C^∞ vector fields on M. If $X \in \mathfrak{X}(M)$, then X assigns to each point $p \in M$ a tangent vector $X_p \in T_pM$, and this assignment is smooth. Alternatively, $X : C^\infty(M) \to C^\infty(M)$ is a linear map such that

$$X(fg) = fX(g) + X(f)g. \tag{3}$$

(3) is the Leibniz rule. If $X, Y \in \mathfrak{X}(M)$, then $X \circ Y$ is not, in general, a vector field. However,

$X \circ Y - Y \circ X$ is a vector field, since it obeys the Leibniz rule.

$$\begin{aligned}
(X \circ Y - Y \circ X)(fg) &= XY(fg) - YX(fg) \\
&= X(Y(f)g + fY(g)) - Y(X(f)g + fX(g)) \\
&= XY(f)g + Y(f)X(g) + X(f)Y(g) + fXY(g) \\
&\qquad - YX(f)g - X(f)Y(g) - Y(f)X(g) - fYX(g) \\
&= (X \circ Y - Y \circ X)(f)g + f(X \circ Y - Y \circ X)(g).
\end{aligned}$$

Now we define the Lie bracket on $\mathfrak{X}(M)$ as

$$[X, Y] = X \circ Y - Y \circ X, \tag{4}$$

and then $\mathfrak{X}(M)$ becomes a Lie algebra with respect to $[-,-]$.

Note that $\mathfrak{X}(M)$ is, in general, an infinite dimensional Lie algebra. For example, if we take $M = \mathbb{R}$, then

$$\frac{\mathrm{d}}{\mathrm{d}x},\ x\frac{\mathrm{d}}{\mathrm{d}x},\ x^2\frac{\mathrm{d}}{\mathrm{d}x},\ x^3\frac{\mathrm{d}}{\mathrm{d}x}, \ldots$$

are all linearly independent vector fields on $\mathbb{R}$. However, if our manifold is a Lie group, we can actually get a finite dimensional Lie subalgebra of $\mathfrak{X}(M)$.

§2 Invariant vector fields

Let $f : M \to N$ be a smooth map of manifolds. If X is a vector field on M, we can't always push it forward to a vector field f_*X on N. Sure, we can try to define

$$(f_*X)_{f(p)} = (df)_p X_p \in T_{f(p)}N. \tag{5}$$

But the issue with this definition is that not all the points in N can be written as $f(p)$ if f is not surjective. Furthermore, even if f is surjective, if we take $q \in N$, there can be multiple choices for p, i.e. we may have $q = f(p_1) = f(p_2)$ if f is not injective. But we can pushforward a vector field when f is a diffeomorphism.

> **Definition 2.1** (Pushforward of vector field). Let $f : M \to N$ be a diffeomorphism. If $X \in \mathfrak{X}(M)$, the pushforward vector field $f_*X \in \mathfrak{X}(N)$ is defined as
>
> $$(f_*X)_{f(p)} = (df)_p X_p \in T_{f(p)}N. \tag{6}$$

Alternatively, f_*X acting on a smooth function h on N is defined as

$$(f_*X)(h)(q) = X_{f^{-1}(q)}(h \circ f), \text{ or alernatively, } (f_*X)(h)(f(p)) = X_p(h \circ f). \tag{7}$$

Definition 2.2 (Invariant vector field). Let $f : M \to M$ be a diffeomorphism. A vector field $X \in \mathfrak{X}(M)$ is called **f-invariant** if $f_* X = X$. Let $\mathfrak{X}_f(M)$ denote the space of all f-invariant vector fields on M.

Note that

$$
\begin{aligned}
f_* X = X \iff & \left(f_* X\right)_{f(p)} = X_{f(p)} \text{ for all } p \in M \\
\iff & \left(f_* X\right)_{f(p)} h = X_{f(p)} h \text{ for all } p \in M \text{ and } h \in C^\infty(M) \\
\iff & X_p \left(h \circ f\right) = \left(Xh\right)\left(f(p)\right) \text{ for all } p \in M \text{ and } h \in C^\infty(M) \\
\iff & \left(X\left(h \circ f\right)\right)(p) = (Xh) \circ f(p) \text{ for all } p \in M \text{ and } h \in C^\infty(M) \\
\iff & X\left(h \circ f\right) = (Xh) \circ f \text{ for all } h \in C^\infty(M).
\end{aligned} \tag{8}
$$

In other words, in the expresssion $X(h \circ f)$, we can take the composition with f out of the parenthesis. We shall often use (8) as the definition of an invariant vector field.

Proposition 2.1

If $X, Y \in \mathfrak{X}(M)$ are f-invariant, so is $[X, Y]$.

Proof. Take $h \in C^\infty(M)$.

$$
\begin{aligned}
[X, Y]\left(h \circ f\right) &= XY\left(h \circ f\right) - YX\left(h \circ f\right) \\
&= X\left((Yh) \circ f\right) - Y\left((Xh) \circ f\right) \\
&= X\left((Yh)\right) \circ f - Y\left((Xh)\right) \circ f \\
&= \left((XY - YX)\, h\right) \circ f \\
&= \left([X, Y]\, h\right) \circ f.
\end{aligned} \tag{9}
$$

Hence, $[X, Y]$ is also f-invariant. ∎

Therefore, $\mathfrak{X}_f(M)$ is a Lie subalgebra of $\mathfrak{X}(M)$. Now we come back to the context of Lie group, i.e. the manifold $M = G$ is a Lie group. Let $L_g : G \to G$ denote the "left multiplication by g" map, i.e.

$$L_g\left(h\right) = gh. \tag{10}$$

We know L_g is a diffeomorphism.

Definition 2.3 (Left-invariant vector field). A vector field $X \in \mathfrak{X}(G)$ is called **left-invariant** if it is L_g-invariant for every $g \in G$. Then $\mathfrak{L}(G)$ denotes the space of all left-invariant vector fields.

In other words, $X \in \mathfrak{X}(G)$ is left-invariant if $(dL_g)_{g'} X_{g'} = X_{gg'}$ for every $g, g' \in G$. Alternatively, if $X(h \circ L_g) = (Xh) \circ L_g$ for every $h \in C^\infty(G)$ and $g \in G$.

By Proposition 2.1, $\mathfrak{L}(G)$ is a Lie subalgebra of $\mathfrak{X}(G)$. We shall now prove that a left-invariant vector field is completely determined by its value at the identity.

> **Theorem 2.2**
>
> The map $\Phi : \mathfrak{L}(G) \to T_eG$ given by $X \mapsto X_e$ is an isomorphism of vector spaces.

Proof. We define $\Psi : T_eG \to \mathfrak{L}(G)$ as follows: given $X_e \in T_e(G)$,

$$\Psi(X_e)_g = (dL_g)_e X_e \in T_gG. \tag{11}$$

Let's first verify that $X := \Psi(X_e)$ is left-invariant.

$$\begin{aligned}(dL_g)_{g'} X_{g'} &= (dL_g)_{g'} \big(dL_{g'}\big)_e X_e \\ &= \big(d(L_g \circ L_{g'})\big)_e X_e \\ &= \big(dL_{gg'}\big)_e X_e \\ &= X_{gg'}.\end{aligned} \tag{12}$$

Therefore, Ψ is indeed a linear map from T_eG to $\mathfrak{L}(G)$.

$$\begin{array}{ccccc} T_eG & \xrightarrow{\quad\Psi\quad} & \mathfrak{L}(G) & \xrightarrow{\qquad\Phi\qquad} & T_e(G) \\ X_e & \longmapsto & \Psi(X_e) & \longmapsto & \Psi(X_e)_e = (dL_e)_e X_e = X_e, \end{array} \tag{13}$$

since L_e is the identity map. Therefore, $\Phi \circ \Psi = \mathrm{id}_{T_eG}$. On the other hand, if $X \in \mathfrak{L}(G)$, then $(dL_g)_{g'} X_{g'} = X_{gg'}$ for every $g, g' \in G$. Taking $g' = e$, we have $(dL_g)_e X_e = X_g$.

$$\begin{array}{ccccc} \mathfrak{L}(G) & \xrightarrow{\quad\Phi\quad} & T_e(G) & \xrightarrow{\quad\Psi\quad} & \mathfrak{L}(G) \\ X & \longmapsto & X_e & \longmapsto & \Psi(X_e) \end{array} \tag{14}$$

Now,

$$(\Psi(X_e))_g = (dL_g)_e X_e = X_g. \tag{15}$$

Therefore, $\Psi \circ \Phi = \mathrm{id}_{\mathfrak{L}(G)}$. Hence, Φ is a vector space isomorphism. ■

Since T_eG is a finite-dimensional vector space, so is $L(G)$. Since $\mathfrak{L}(G)$ and T_eG are isomorphic as vector spaces, the Lie algebra structure on $\mathfrak{L}(G)$ can be translated to T_eG, by defining the Lie bracket $[-,-]_{T_eG}$ on T_eG

$$[X_e, Y_e]_{T_eG} = \Phi\Big([\Psi(X_e), \Psi(Y_e)] \Big). \tag{16}$$

§3 Lie algebra of a Lie group

Definition 3.1 (Lie algebra associated with a Lie group)**.** Let G be a Lie group. Then $\mathfrak{g} := \mathfrak{L}(G)$ is called the **Lie algebra associated with the Lie group** G.

Note that in general, Lie algebras don't come with a Lie group. However, every Lie group G gives rise to a finite-dimensional Lie algebra $\mathfrak{g}$. It turns out that every finite dimensional Lie algebra is isomorphic to the Lie algebra of a Lie group! This is known as Lie's third theorem.

Now, suppose $f : G \to H$ is a Lie group homomorphism. Then at the tangent space level, we have the linear map $(df)_e : T_eG \to T_{e'}H$. Under the identification $T_eG \cong \mathfrak{g} = \mathfrak{L}(G)$, we get an induced map $\mathfrak{g} \to \mathfrak{h}$.

$$\begin{tikzcd} T_eG \arrow[r, "(df)_e"] & T_{e'}H \arrow[d, "\Psi_H"] \\ \mathfrak{g} \arrow[u, "\Phi_G"] \arrow[r, "\Psi_H \circ (df)_e \circ \Phi_G"'] & \mathfrak{h} \end{tikzcd} \tag{17}$$

Theorem 3.1

$\widehat{f} := \Psi_H \circ (df)_e \circ \Phi_G : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism.

Proof. We have to verify that $\widehat{f}$ preserves Lie bracket. In other words, we need to show that $\widehat{f}\left([X,Y]\right) = \left[\widehat{f}(X), \widehat{f}(Y)\right]$.

$$\begin{aligned} & \widehat{f}\left([X,Y]\right) = \left[\widehat{f}(X), \widehat{f}(Y)\right] \\ \iff\ & (df)_e \circ \Phi_G\left([X,Y]\right) = \Phi_H\left[\widehat{f}(X), \widehat{f}(Y)\right] \\ \iff\ & (df)_e\,[X,Y]_e = \left[\widehat{f}(X), \widehat{f}(Y)\right]_{e'} \\ \iff\ & (df)_e\,[X,Y]_e\,(h) = \left[\widehat{f}(X), \widehat{f}(Y)\right]_{e'}(h) \text{ for any } h \in C^\infty(H) \\ \iff\ & X_e\left(Y(h \circ f)\right) - Y_e\left(X(h\circ f)\right) = \left[\widehat{f}(X), \widehat{f}(Y)\right]_{e'}(h) \text{ for any } h \in C^\infty(H). \end{aligned} \tag{18}$$

Now,

$$\begin{aligned} \left[\widehat{f}(X), \widehat{f}(Y)\right]_{e'}(h) &= \widehat{f}(X)_e\left(\widehat{f}(Y)h\right) - \widehat{f}(Y)_e\left(\widehat{f}(X)h\right) \\ &= (df)_e\,X_e\left(\widehat{f}(Y)h\right) - (df)_e\,Y_e\left(\widehat{f}(X)h\right) \\ &= X_e\left(\widehat{f}(Y)h \circ f\right) - Y_e\left(\widehat{f}(X)h\circ f\right). \end{aligned} \tag{19}$$

Therefore, it suffices to verify that $Y\left(h \circ f\right) = \widehat{f}\left(Y\right) h \circ f$ as functions on G. If we manage to prove it, $X(h \circ f) = \widehat{f}(X)h \circ f$ follows similarly. Now, let's evaluate both sides at $g \in G$.

$$
\begin{aligned}
\text{LHS} &= Y\left(h \circ f\right)(g) = Y_g\left(h \circ f\right) \\
&= \left(dL_g\right)_e Y_e\left(h \circ f\right) = Y_e\left(h \circ f \circ L_g\right) && (20)\\
\text{RHS} &= \left[\widehat{f}\left(Y\right) h \circ f\right](g) = \left[\widehat{f}\left(Y\right) h\right]\left(f(g)\right) \\
&= \left(\widehat{f}\left(Y\right)\right)_{f(g)} h = \left(dL_{f(g)}\right)_{e'} \left(df\right)_e Y_e(h) \\
&= \left(d(L_{f(g)} \circ f)\right)_e Y_e(h) \\
&= Y_e\left(h \circ L_{f(g)} \circ f\right). && (21)
\end{aligned}
$$

Now, $f \circ L_g = L_{f(g)} \circ f$, since f is a group homomorphism. Therefore, the LHS and RHS are equal, and hence we are done! ■

One can easily verify that if $f_1 : G \to H, f_2 : H \to K$ are Lie group homomorphisms, then

$$\widehat{f_2 \circ f_1} = \widehat{f}_2 \circ \widehat{f}_1, \tag{22}$$

$$
\begin{tikzcd}
G \arrow[r, "f_1"] \arrow[rr, bend right, "f_2 \circ f_1"'] & H \arrow[r, "f_2"] & K
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathfrak{g} \arrow[r, "\widehat{f}_1"] \arrow[rr, bend right, "\widehat{f_2 \circ f_1} = \widehat{f}_2 \circ \widehat{f}_1"'] & \mathfrak{h} \arrow[r, "\widehat{f}_2"] & \mathfrak{k}
\end{tikzcd}
\tag{23}
$$

Furthermore, $\widehat{\mathbb{1}}_G = \mathbb{1}_{\mathfrak{g}}$. Therefore, this defines a functor from the category of Lie groups to the category of Lie algebras:

$$
\begin{aligned}
\mathsf{Lie} : \mathbf{LieGrp} &\to \mathbf{LieAlg} \\
G &\mapsto \mathfrak{g} \\
(f : G \to H) &\mapsto (\widehat{f} : \mathfrak{g} \to \mathfrak{h})
\end{aligned}
\tag{24}
$$

Furthermore, Theorem 2.2 defines a natural isomorphism

$$
\begin{tikzcd}
\mathbf{LieGrp} \arrow[r, bend left, "\mathsf{LIVF}"] \arrow[r, bend right, "\mathsf{TSI}"'] \arrow[r, phantom, "\Downarrow\Phi"] & \mathbf{Vect}_{\mathbb{R}}
\end{tikzcd}
$$

where, the functor LIVF takes a Lie group G to the space of all left-invariant vector fields, and TSI takes a Lie group G to the tangent space at identity (hence the names LIVF and TSI).

$$
\begin{aligned}
\mathsf{LIVF} : \mathbf{LieGrp} &\to \mathbf{Vect}_{\mathbb{R}} \\
G &\mapsto \mathfrak{L}(G) \\
(f : G \to H) &\mapsto (\widehat{f} : \mathfrak{L}(G) \to \mathfrak{L}(H))
\end{aligned}
\tag{25}
$$

$$\begin{aligned} \mathsf{TSI} : \mathbf{LieGrp} &\to \mathbf{Vect}_{\mathbb{R}} \\ G &\mapsto T_e G \\ (f : G \to H) &\mapsto ((df)_e : T_e G \to T_{e'} H) \end{aligned} \tag{26}$$

§4 Exponential map

Tangent spaces give us a local linearization of the manifold. This often helps us translate tangent space properties to local properties of manifolds, e.g. inverse function theorem, constant rank theorem, immersion/submersion theorems etc. This "tangent space to manifold" correspondence is formally defined using something called exponential map. In Riemannian geometry, we define the exponential map as follows:

> $\exp_p : T_p M \to M$ is defined in the following way: given $X_p \in T_p M$, let γ be the **unique** geodesic such that $\gamma(0) = p$ and $\gamma'(0) = X_p$. Then we define $\exp_p(X_p) = \gamma(1)$.

(Well, technically $\exp_p$ is defined on an open subset of $T_p M$, which might not be the whole M.) Here, we need the notion of an affine connection for existence and uniqueness of geodesics. Furthermore, one needs to verify that the domain of the geodesic γ contains 1.

Note that we need to impose some further restriction on γ. Because the curve satisfying $\gamma(0) = p$ and $\gamma'(0) = X_p$ only is not unique. The uniqueness is guaranteed by requiring the curve to be a geodesic.

Now suppose $M = G$ is a Lie group. Suppose we want to define $\exp : T_e G \to G$. Given $X_e \in T_e G$, our idea is to take a "unique" curve γ satisfying $\gamma(0) = p$ and $\gamma'(0) = X_p$ along with some other condition. Then we're gonna define $\exp(X_e) = \gamma(1)$.

By Theorem 2.2, every X_e uniquely extends to a left-invariant vector field X. Then we can take γ to be the integral curve of the vector field X, satisfying $\gamma(0) = p$ and $\gamma'(0) = X_p$. By the existence and uniqueness theorem of ODEs, such a γ exists in a neighborhood of 0, and it's unique.

Now, we just know that γ is defined on an interval $(-\varepsilon, \varepsilon)$ containing 0. We don't know how long the interval is, or if it even contains 1. It turns out that the maximal domain of γ is the whole $\mathbb{R}$.

Proposition 4.1

Let X be a left-invariant vector field on G, and γ the maximal integral curve of the vector field X starting at e. Then the domain of γ is $\mathbb{R}$.

Proof. Let α be any integral curve of the vector field X (not necessarily starting at e). Then given $g \in G$, we take the curve

$$\alpha_1(t) = g \cdot \alpha(t) = (L_g \circ \alpha)(t). \tag{27}$$

Then we have

$$
\begin{aligned}
\alpha_1'(t) &= (d\alpha_1)_t \left.\frac{\mathrm{d}}{\mathrm{d}x}\right|_t \\
&= (dL_g)_{\alpha(t)} (d\alpha)_t \left.\frac{\mathrm{d}}{\mathrm{d}x}\right|_t \\
&= (dL_g)_{\alpha(t)} X_{\alpha(t)} = X_{g\alpha(t)} \\
&= X_{\alpha_1(t)}.
\end{aligned} \tag{28}
$$

Therefore, α_1 is also an integral curve. Now let I be the domain of the maximal integral curve γ starting at e. We need to show that $I = \mathbb{R}$.

Fix $t_1 \in I$ with $t_1 \neq 0$, and take $g_1 = \gamma(t_1)$. Now, the curve

$$\alpha_1(t) = g_1 \cdot \gamma(t) \tag{29}$$

is an integral curve for X (as above), starting at g_1. It has domain I. On the other hand, the maximal integral curve of X starting at g_1 is given by

$$\alpha_2(t) = \gamma(t_1 + t). \tag{30}$$

This is maximal because otherwise we'd be able to extend γ to a larger interval. Now, the domain of α_2 is $I - t_1 := (a - t_1, b - t_1)$ if $I = (a, b)$. Since α_2 is maximal, it contains the domain of α_1, i.e.

$$(a, b) \subseteq (a - t_1, b - t_1), \quad \text{or equivalently} \ (a + t_1, b + t_1) \subseteq (a, b). \tag{31}$$

In particular, given any $s \in I$, $s + t_1 \in (a + t_1, b + t_1) \subseteq (a, b)$, i.e. $s + t_1 \in I$. This forces I to be equal to $\mathbb{R}$. ■

Corollary 4.2

$\alpha(t) = g \cdot \gamma(t)$ is the integral curve for X starting at g.

Proof. We have already shown it in the proof for Proposition 4.1. ■

Proposition 4.3

$\gamma : \mathbb{R} \to G$ is a group homomorphism.

Proof. Fix $s \in \mathbb{R}$. Consider the following curves:

$$\alpha_1(t) = \gamma(s)\gamma(t), \tag{32}$$

$$\alpha_2(t) = \gamma(s + t). \tag{33}$$

Both of them are integral curves of the vector field X, and both starting at $\gamma(s)$. Furthermore, both are defined on the whole $\mathbb{R}$. Therefore, by the uniqueness of the maximal integral curve, we have $\alpha_1 = \alpha_2$, i.e.

$$\gamma(s)\gamma(t) = \gamma(s+t). \tag{34}$$

Therefore, $\gamma : \mathbb{R} \to G$ is a group homomorphism. ∎

Now we can define the exponential map.

Definition 4.1 (Exponential map)**.** Let G be a Lie group. The **exponential map**, $\exp : T_eG \to G$ is defined by

$$\exp(X_e) = \gamma(1), \tag{35}$$

where $\gamma : \mathbb{R} \to G$ is the maximal integral curve of the left-invariant vector field X, starting at e. $\gamma : \mathbb{R} \to G$ is also called the **one-parameter subgroup** corresponding to $X_e \in T_eG$.

Lemma 4.4

Under the above notation, $\gamma(s) = \exp(sX_e)$ for any $s \in \mathbb{R}$.

Proof. Fix $s \in \mathbb{R}$. We then define the curve $\alpha : \mathbb{R} \to G$ as follows:

$$\alpha(t) = \gamma(st). \tag{36}$$

Then $\alpha(0) = 0$, and

$$\begin{aligned} \alpha'(t) &= (d\alpha)_t \left.\frac{\mathrm{d}}{\mathrm{d}x}\right|_t \\ &= (d\gamma)_{st}\, s \left.\frac{\mathrm{d}}{\mathrm{d}x}\right|_t \\ &= sX_{\gamma(st)} = sX_{\alpha(t)}. \end{aligned} \tag{37}$$

Therefore, α is the integral curve of the left-invariant vector field sX, starting at e. Therefore,

$$\exp(sX_e) = \alpha(1) = \gamma(s). \tag{38}$$

∎

Lemma 4.5

For $s, t \in \mathbb{R}$, $\exp((s+t)X_e) = \exp(sX_e)\exp(tX_e)$.

Proof. Let $\gamma : \mathbb{R} \to G$ be the one-parameter subgroup corresponding to $X_e \in T_eG$. Then

$$\exp((s+t)X_e) = \gamma(s+t) = \gamma(s)\gamma(t) = \exp(sX_e)\exp(tX_e). \tag{39}$$

■

Remark 4.1. In many literature, exp is defined in a slightly different, albeit equivalent, way. Given $X_e \in T_e G$, they define γ to be the unique Lie group homomorphism $\gamma : \mathbb{R} \to G$ such that $\gamma'(0) = X_e$. We prove the existence and uniqueness of the Lie group homomorphism $\gamma : \mathbb{R} \to G$ in the following proposition.

Proposition 4.6

There is a unique Lie group homomorphism $\gamma : \mathbb{R} \to G$ such that $\gamma'(0) = X_e$.

Proof. Such a γ satisfies

$$\gamma(t+s) = \gamma(t)\gamma(s) = L_{\gamma(t)} \circ \gamma(s). \tag{40}$$

Differentiating with respect to s, we get

$$(d\gamma)_{t+s} = \left(dL_{\gamma(t)}\right)_{\gamma(s)} (d\gamma)_s. \tag{41}$$

This gives us

$$\gamma'(t+s) = \left(dL_{\gamma(t)}\right)_{\gamma(s)} \gamma'(s). \tag{42}$$

At $s = 0$,

$$\gamma'(t) = \left(dL_{\gamma(t)}\right)_e \gamma'(0) = \left(dL_{\gamma(t)}\right)_e X_e. \tag{43}$$

Therefore, γ is the integral curve to the left-invariant vector field X defined as $X_g = (dL_g)_e X_e$. Although we saw a proof of extendability to $\mathbb{R}$ earlier, let's present a different proof now.

We know that γ exists and is unique on $(-\varepsilon, \varepsilon)$. Furthermore, if $|s| + |t| < \varepsilon$, both $\alpha_1(t) := \gamma(s+t)$ and $\alpha_2(t) := \gamma(s)\gamma(t)$ are integral curves to the vector field X with initial point $\gamma(s)$. Therefore,

$$\gamma(s+t) = \gamma(s)\gamma(t), \qquad \text{for } |s| + |t| < \varepsilon. \tag{44}$$

Differentiating with respect to s, we get

$$(d\gamma)_{s+t} = \left(dR_{\gamma(t)}\right)_{\gamma(s)} (d\gamma)_s \tag{45}$$

This gives us

$$\gamma'(t+s) = \left(dR_{\gamma(t)}\right)_{\gamma(s)} \gamma'(s). \tag{46}$$

At $s = 0$,

$$\gamma'(t) = \left(dR_{\gamma(t)}\right)_e \gamma'(0) = \left(dR_{\gamma(t)}\right)_e X_e, \tag{47}$$

proving that $\gamma'(t) = \left(dR_{\gamma(t)}\right)_e X_e = \left(dL_{\gamma(t)}\right)_e X_e$. Now, one can extend γ to $(-2\varepsilon, 2\varepsilon)$ by

$$\gamma(2t) = \gamma(t)^2. \tag{48}$$

Then one can verify that this satisfies $\gamma'(2t) = \left(dL_{\gamma(2t)}\right)_e X_e$. This can be easily done by an application of Leibniz-like formula (see here, [10] for details)

$$\begin{aligned}
2\gamma'(2t) &= \left(dL_{\gamma(t)}\right)_{\gamma(t)} \gamma'(t) + \left(dR_{\gamma(t)}\right)_{\gamma(t)} \gamma'(t) \\
&= \left(dL_{\gamma(t)}\right)_{\gamma(t)} \left(dL_{\gamma(t)}\right)_e X_e + \left(dR_{\gamma(t)}\right)_{\gamma(t)} \left(dR_{\gamma(t)}\right)_e X_e \\
&= \left(dL_{\gamma(t)^2}\right)_e X_e + \left(dR_{\gamma(t)^2}\right)_e X_e \\
&= 2\left(dL_{\gamma(t)^2}\right)_e X_e \\
&= 2\left(dL_{\gamma(2t)}\right)_e X_e. \qquad (49)
\end{aligned}$$

So one can extend γ on $(-2\varepsilon, 2\varepsilon)$. By induction, it can be extended to $(-2^n\varepsilon, 2^n\varepsilon)$, and hence on the whole $\mathbb{R}$. ∎

§5 Flow

Definition 5.1 (The flow of a vector field). Let M be a smooth m-manifold and $X \in \mathfrak{X}(M)$ be a smooth vector field on M. For $p_0 \in M$, the **maximal existence interval** of p_0 is the maximal open interval $I(p_0)$ in which $\gamma : I(p_0) \to M$ is the integral curve of the vector field X, starting at p_0. The **flow** of X is the map $\phi : \mathcal{D} \to M$ defined by

$$\mathcal{D} := \{(t, p_0) \mid p_0 \in M,\ t \in I(p_0)\}, \qquad \text{and } \phi(t, p_0) := \gamma(t),$$

where $\gamma : I(p_0) \to M$ is the integral curve of the vector field X, starting at p_0. If $\mathcal{D} = \mathbb{R} \times M$, the vector field X is called **complete**.

Theorem 5.1

$\mathcal{D}$ is an open subset of $\mathbb{R} \times M$, and ϕ is smooth.

For a proof, interested readers can consult §2.4.2 from the preprint linked in [7]. We have seen in Proposition 4.1 and Corollary 4.2 that if X is a left-invariant vector field on a Lie group G, then it's integral curve starting at g has domain $\mathbb{R}$. Therefore, X is complete.

Proposition 5.2

Let X be a left-invariant vector field on a Lie group G. Then X is complete, and its flow is given by

$$\begin{aligned}
\phi : \mathbb{R} \times G &\to G \\
(t, g) &\mapsto g \exp(tX_e).
\end{aligned} \qquad (50)$$

Proof. By Corollary 4.2, the integral curve for X, starting at g is given by

$$\alpha(t) = g\gamma(t) = g\exp(tX_e), \tag{51}$$

where γ is the integral curve for X, starting at e. Hence, $\phi(t,g) = \alpha(t) = g\exp(tX_e)$. ■

Now, T_eG is a finite-dimensional vector space. We can give it an inner product or a norm and that induces a topology on it. We know that on a finite dimensional vector space, **all** norms are equivalent, i.e. they all generate the same topology. Under this topology, it's homeomorphic to $\mathbb{R}^n$. Therefore, T_eG canonically has a manifold structure, that doesn't depend on the choice of the norm.

> **Theorem 5.3**
>
> $\exp : T_eG \to G$ is smooth.

Proof. Consider the map

$$\begin{aligned} \Phi : \mathbb{R} \times G \times T_eG &\to G \times T_eG \\ (t, g, X_e) &\mapsto (g\exp(tX_e), X_e). \end{aligned} \tag{52}$$

Let X be the left-invariant vector field $X_g = (dL_g)_e X_e$. Consider the vector field $(X, 0)$ on $G \times T_eG$. Then Φ is the flow of $(X, 0)$ on $G \times T_eG$, since $\Phi(-, g, X_e)$ is the integral curve for $(X, 0)$, starting at the point (g, X_e). Therefore, Φ is smooth. It follows that

$$\exp = \pi_1 \circ \Phi|_{\{1\}\times\{e\}\times T_eG}, \tag{53}$$

$$\begin{tikzcd} T_eG \arrow[r, hook] \arrow[rrrr, bend right, "\exp=\pi_1\circ\Phi|_{\{1\}\times\{e\}\times T_eG}"'] & \{1\}\times\{e\}\times T_eG \arrow[r, hook] & \mathbb{R}\times G\times T_eG \arrow[r, "\Phi"] & G\times T_eG \arrow[r, "\pi_1"] & G \end{tikzcd} \tag{54}$$

Therefore, as a composition of smooth maps, $\exp : T_eG \to G$ is smooth. ■

§6 Further properties of exp

Since we have proved that $\exp : T_eG \to G$ is smooth, the next natural question that arises is whether exp is injective/surjective or diffeomorphism. We shall attempt to answer this question by first computing the differential $(d\exp)_{\mathbf{0}}$ at $\mathbf{0} \in T_eG$.

On $(\mathbb{R}_{>0}, \times)$, under the identification $T_1\mathbb{R}_{>0} \cong \mathbb{R}$, the exponential map is $\exp(x) = e^x$. The differential of this map $\exp : \mathbb{R} \to \mathbb{R}_{>0}$ is actually

$$(d\exp)_0 : T_0\mathbb{R} \cong \mathbb{R} \to T_1\mathbb{R}_{>0} \cong \mathbb{R},$$

which can easily be seen to be the identity map $\mathrm{id}_{\mathbb{R}}$ under the aforementioned identifications.

So it's natural to expect that for a general Lie group G, the differential of the exponential map

$$(d\exp)_{\mathbf{0}} : T_{\mathbf{0}}\left(T_e G\right) \to T_e G, \tag{55}$$

is gonna be the identity map under some canonical identification. The identification is actually $T_{\mathbf{0}}\left(T_e G\right) \cong T_e G$.

Lemma 6.1

If V is a finite dimensional $\mathbb{R}$-vector space, $T_{\mathbf{w}}V \cong V$ for any $\mathbf{w} \in V$.

Proof. We define $\Phi : V \to T_{\mathbf{w}}V$ as follows: given $\mathbf{v} \in V$, $\Phi(\mathbf{v}) =: D_{\mathbf{v}} \in T_{\mathbf{w}}V$ is the point-derivation that acts like

$$D_{\mathbf{v}} f = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f\left(t\mathbf{v} + \mathbf{w}\right), \tag{56}$$

for $f \in C^\infty_{\mathbf{w}}(V)$. Clearly $D_{\mathbf{v}}$ is linear, and it's a derivation. In fact, it works exactly like point-derivations, since $D_{\mathbf{v}} f = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}(f \circ c)$, where $c : (-\varepsilon, \varepsilon) \to V, t \mapsto t\mathbf{v}$ is the curve such that $c(0) = \mathbf{w}$ and $c'(0) = D_{\mathbf{v}}$. (Check out Proposition 6.3.3 from [9])

Furthermore, Φ is linear. Also notice that the domain and codomain of Φ has the same dimension. Therefore, we only need to check for injectivity. Suppose $\mathbf{v} \in \operatorname{Ker}\Phi$, i.e. $D_{\mathbf{v}} = 0$. In other words, $D_{\mathbf{v}} f = 0$ for every $f \in C^\infty_0(V)$. Now take $f \in V^*$.

$$0 = D_{\mathbf{v}} f = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f\left(t\mathbf{v}\right) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left(tf\left(\mathbf{v}\right) + f\left(\mathbf{w}\right)\right) = f\left(\mathbf{v}\right). \tag{57}$$

Therefore, $f(\mathbf{v}) = 0$ for every $f \in V^*$. Hence, $\mathbf{v} = 0$, and thus Φ is injective, and we are done! ■

Furthermore, this isomorphism is natural!

$$\begin{tikzcd}
\mathbf{Vect}_{\mathbb{R}}\textbf{-fin}_* \arrow[r, bend left=50, "\mathsf{ID}"{name=U}] \arrow[r, bend right=50, "\mathsf{Tangent}"'{name=D}] & \mathbf{Vect}_{\mathbb{R}} \arrow[Rightarrow, from=U, to=D, "\Phi"]
\end{tikzcd}$$

Here, $\mathbf{Vect}_{\mathbb{R}}\textbf{-fin}_*$ is the category of finite-dimensional *pointed* $\mathbb{R}$-vector spaces. The objects are $(V, \mathbf{w})$ with V being a finite-dimensional $\mathbb{R}$-vector space, and $\mathbf{w} \in V$. The arrows from $(V_1, \mathbf{v}_1)$ to $(V_2, \mathbf{v}_2)$ is a linear map $f : V_1 \to V_2$ such that $f(\mathbf{v}_1) = \mathbf{v}_2$. The functor ID just gives the underlying finite-dimensional vector space, it "forgets" the fixed basepoint.

$$\begin{aligned}
\mathsf{ID} : \mathbf{Vect}_{\mathbb{R}}\textbf{-fin}_* &\to \mathbf{Vect}_{\mathbb{R}} \\
(V, \mathbf{w}) &\mapsto V \\
(f : (V_1, \mathbf{v}_1) \to (V_2, \mathbf{v}_2)) &\mapsto (f : V_1 \to V_2).
\end{aligned} \tag{58}$$

On the other hand, the functor $\mathsf{Tangent}$ just gives the tangent space at the fixed basepoint.

$$\begin{aligned} \mathsf{Tangent} : \mathbf{Vect}_{\mathbb{R}}\text{-}\mathbf{fin}_{*} &\to \mathbf{Vect}_{\mathbb{R}} \\ (V, \mathbf{w}) &\mapsto T_{\mathbf{w}} V \\ (f : (V_1, \mathbf{v}_1) \to (V_2, \mathbf{v}_2)) &\mapsto ((df)_{\mathbf{v}_1} : T_{\mathbf{v}_1} V_1 \to T_{\mathbf{v}_2} V_2). \end{aligned} \tag{59}$$

Proposition 6.2

The isomorphism in Lemma 6.1 is natural.

Proof. Essentially, we have to show that the following diagram commutes for any linear map $f : V \to W$ with $f(\mathbf{v}) = \mathbf{w}$:

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\Phi_{(V,\mathbf{v})}"'] & W \arrow[d, "\Phi_{(W,\mathbf{w})}"] \\ T_{\mathbf{v}} V \arrow[r, "(df)_{\mathbf{v}}"'] & T_{\mathbf{w}} W \end{tikzcd} \tag{60}$$

Take any smooth function $h \in C^{\infty}(W)$. Then given any $\mathbf{u} \in V$,

$$\begin{aligned} \left(\Phi_{(W,\mathbf{w})} \circ f\right)(\mathbf{u})(h) &= \Phi_{(W,\mathbf{w})}[f(\mathbf{u})](h) \\ &= D_{f(\mathbf{u})} h = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} h(t\, f(\mathbf{u}) + \mathbf{w}). \end{aligned} \tag{61}$$

$$\begin{aligned} \left((df)_{\mathbf{v}} \circ \Phi_{(V,\mathbf{v})}\right)(\mathbf{u})(h) &= ((df)_{\mathbf{v}} \circ D_{\mathbf{u}})(h) \\ &= D_{\mathbf{u}}(h \circ f) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} [(h \circ f)(t\mathbf{u} + \mathbf{v})] \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} h(t\, f(\mathbf{u}) + f(\mathbf{v})). \end{aligned} \tag{62}$$

(61) and (62) agree because $f(\mathbf{v}) = \mathbf{w}$. Therefore, $\Phi_{(W,\mathbf{w})} \circ f = (df)_{\mathbf{v}} \circ \Phi_{(V,\mathbf{v})}$, i.e. (60) commutes and we are done! ■

Theorem 6.3

If G is a Lie group, then $(d\exp)_0 \circ \Phi = \mathrm{id}_{T_e G}$.

$$\begin{tikzcd} T_e G \arrow[r, "\Phi"] \arrow[rr, bend right, "(d\exp)_0 \circ \Phi = \mathrm{id}_{T_e G}"'] & T_{\mathbf{0}}(T_e G) \arrow[r, "(d\exp)_0"] & T_e G \end{tikzcd} \tag{63}$$

Proof. Given $X_e \in T_e G$, we have to show that $(d\exp)_0 D_{X_e} = X_e$. Let's evaluate the LHS on an arbitrary $f \in C_e^\infty(G)$.

$$(d\exp)_0 \, D_{X_e} f = D_{X_e} \left(f \circ \exp \right) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f\left(\exp\left(tX_e\right)\right). \tag{64}$$

Now, the curve $\gamma : \mathbb{R} \to G$ given by $\gamma\left(t\right) = \exp(tX_e)$ has the property that $\gamma\left(0\right) = e, \gamma'\left(0\right) = X_e$. Therefore,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f\left(\exp\left(tX_e\right)\right) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left(f \circ \gamma\right) = X_e f. \tag{65}$$

(The last equality follows from Proposition 6.3.3 from [9]) Therefore, $(d\exp)_0 \, D_{X_e} = X_e$. In other words, $(d\exp)_0 \circ \Phi = \mathrm{id}_{T_eG}$. ■

Corollary 6.4

exp is a local diffeomorphism near $\mathbf{0} \in T_eG$.

Proof. Since $(d\exp)_0$ is invertible, by the inverse function theorem, exp is a diffeomorphism in a neighborhood of $\mathbf{0} \in T_eG$. ■

Therefore, there is a neighborhood $U \ni e$ and a neighborhood $\mathfrak{u} \ni \mathbf{0}$ such that exp is a diffeomorphism of $\mathfrak{u}$ and U. Let's call the inverse log.

$$\begin{tikzcd} U \subseteq G \arrow[r, bend left, "\log"] & \mathfrak{u} \subseteq \mathfrak{g} \arrow[l, bend left, "\exp"] \end{tikzcd} \tag{66}$$

(Here, we are using the same symbol $\mathfrak{g}$ for the Lie algebra $\mathfrak{L}(G)$ and T_eG. This abuse of notation will be used often, and the meaning will be clear from the context.) Then the next question arises: is exp globally a diffeomorphism? If so, when?

Example 6.1. For the matrix Lie groups, say $\mathrm{GL}\left(n, \mathbb{R}\right)$ or $\mathrm{SL}\left(n, \mathbb{R}\right)$, the exp map is the matrix exponential

$$\exp\left(A\right) = e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n. \tag{67}$$

The Lie algebra of $\mathrm{GL}(n, \mathbb{R})$ is $\mathfrak{gl}\left(n, \mathbb{R}\right) = \mathbb{R}^{n^2}$. Note that $\mathrm{GL}\left(n, \mathbb{R}\right)$ is not connected, since we can write it as the union of two disjoint open sets:

$$\mathrm{GL}\left(n, \mathbb{R}\right) = \det{}^{-1}\left(\left(0, \infty\right)\right) \cup \det{}^{-1}\left(\left(-\infty, 0\right)\right). \tag{68}$$

Now, given $A \in \mathfrak{gl}\left(n, \mathbb{R}\right) = \mathbb{R}^{n^2}$, $\exp\left(A\right)$ is in the $\det^{-1}\left(\left(0, \infty\right)\right)$ component, since

$$\det e^A = e^{\mathrm{Tr}\, A} > 0. \tag{69}$$

Therefore, $\exp : \mathfrak{gl}\left(n, \mathbb{R}\right) \to \mathrm{GL}\left(n, \mathbb{R}\right)$ is not surjective.

In fact, for any Lie group, $\exp : \mathfrak{g} \to G$ is not surjective, if G is not connected. Indeed, if G° is the connected component of the identity element, then G° is a clopen subset of G, so $T_eG = T_eG^\circ$. Therefore,

$$\exp : T_eG^\circ \to G^\circ,$$

the exp map takes $T_eG = T_eG^\circ$ to G°, not the whole G. So exp can't be surjective if G is not connected. However, this is not a sufficient condition. Even if G is connected, exp can still fail to be surjective.

Example 6.2. Consider $G = \mathrm{SL}\,(n, \mathbb{R})$, it's a connected Lie group. Its Lie algebra is $\mathfrak{sl}\,(n, \mathbb{R})$, all real traceless $n \times n$ matrices. For $A \in \mathfrak{sl}(n, \mathbb{R})$, if $\lambda_1, \ldots, \lambda_n$ are (complex) eigenvalues of A, the eigenvalues of e^A are $e^{\lambda_1}, \ldots, e^{\lambda_n}$. If $n = 2$, either

(i) both λ_1, λ_2 are real, in which case $e^{\lambda_1}, e^{\lambda_2} > 0$, and hence $\operatorname{Tr} e^A > 0$; or

(ii) λ_1, λ_2 are complex conjugates, in which case, we must have λ_1, λ_2 to be purly imaginary, because $\det e^A = 1$. Then,

$$\operatorname{Tr} e^A = e^{\lambda_1} + e^{\overline{\lambda}_1} = e^{ix} + e^{-ix} = 2\cos x \in [-2, 2]. \tag{70}$$

In any case, $\operatorname{Tr} e^A \geq -2$. Now consider the matrix

$$X = \begin{bmatrix} -2 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \in \mathrm{SL}(2, \mathbb{R}). \tag{71}$$

Its trace is $-\frac{5}{2}$, which is less than -2. Therefore, $X \notin \operatorname{im}\exp$, i.e. exp is not surjective.

Example 6.3. Consider $G = S^1$, which is a compact connected Lie group. Its Lie algebra is $T_1S^1 \cong \mathbb{R}$. The exp map is given by

$$\begin{aligned} \exp : T_1S^1 \cong \mathbb{R} &\to S^1 \\ x &\mapsto e^{ix}. \end{aligned} \tag{72}$$

Then exp is a surjective map, although it's not injective.

It's true that for a compact connected Lie group, exp is surjective. The proof amounts to showing that a compact connected Lie group can be given a Riemannian metric, and the two exponentials (the Riemannian one and the Lie group one) agree. Then the fact that exp is surjective is a consequence of a theorem from Riemannian geometry, called the Hopf-Rinow theorem. The proof is outside the scope of this note, so we'll skip it. The interested readers can go through Terence Tao's blog [13] and CUHK's blog [14] for a detailed account.

The answer to the question of when exp is injective is much more nontrivial. Two Stack-Exchange questions [11], [12] give us some classification that G needs to be solvable, simply connected, and $\mathfrak{g}$ does not admit $\mathfrak{e}$ as subalgebra of a quotient. Interested readers are encouraged to read the papers linked in the StackExchange answers.

> **Proposition 6.5** (exp is natural)
>
> Given any Lie group homomorphism $\varphi : G \to H$, the following diagram commutes:
>
> $$\begin{tikzcd} T_eG \arrow[r, "(d\varphi)_e"] \arrow[d, "\exp_G"'] & T_{e'}H \arrow[d, "\exp_H"] \\ G \arrow[r, "\varphi"'] & H \end{tikzcd} \tag{73}$$
>
> i.e. $\varphi \circ \exp_G = \exp_H \circ (d\varphi)_e$.

From a categorical standpoint, this result basically means that exp is a natural transformation between two functors:

$$\begin{tikzcd} \textbf{LieGrp} \arrow[r, bend left, "\mathsf{Lie}"] \arrow[r, bend right, "\mathsf{Forgetful}"'] \arrow[r, phantom, "\Downarrow \exp"] & \textbf{Manifold} \end{tikzcd}$$

The Forgetful functor essentially forgets the group structure, so it takes a Lie group G to its underlying manifold, and a smooth homomorphism map to its underlying smooth map. The functor Lie takes a Lie group G to its Lie algebra $\mathfrak{g} = T_eG$'s underlying manifold, and a Lie group homomorphism φ to its differential map $(d\varphi)_e$.

Proof of Proposition 6.5. Given $X_e \in T_eG$, consider the curve

$$\begin{aligned} \alpha : \mathbb{R} &\to H \\ t &\mapsto \varphi\left(\exp_G\left(tX_e\right)\right), \end{aligned} \tag{74}$$

i.e. $\alpha = \varphi \circ \gamma$, where $\gamma : \mathbb{R} \to G$ is defined by $\gamma(t) = \exp_G(tX_e)$. Then $\alpha(0) = e' \in H$, and

$$\alpha'(0) = (d\varphi)_e \gamma'(0) = (d\varphi)_e X_e. \tag{75}$$

Therefore, by the definition of $\exp_H$,

$$\alpha(1) = \exp_H\left((d\varphi)_e X_e\right), \tag{76}$$

and we are done! ■

If G is connected, any Lie group homomorphism $\varphi : G \to H$ is completely determined by the Lie algabra homomorphism $\widehat{\varphi} = (d\varphi)_e : \mathfrak{g} \to \mathfrak{h}$. Because exp is a diffeomorphism on a neighborhood of the identity $U \ni e$. So $(d\varphi)_e$ completely determines φ on a neighborhood U. But since G is connected, it's generated by any neighborhood of the identity. Hence, $(d\varphi)_e$ completely determines φ on the whole G.

References

[1] J. M. Lee, *Introduction to Smooth Manifolds*, 2nd ed., Graduate Texts in Mathematics, vol. 218, Springer, New York, 2013.

[2] L. W. Tu, *Introduction to Manifolds*, 2nd ed., Springer, New York, 2011.

[3] A. Kirillov Jr., *An Introduction to Lie Groups and Lie Algebras*, Cambridge Studies in Advanced Mathematics, vol. 113, Cambridge University Press, Cambridge, 2008.

[4] P. Etingof, *Lie Groups and Lie Algebras*, Lecture notes, MIT, 2020-2021; arXiv:2201.09397 [math.RT], 2022. Available at: `https://arxiv.org/pdf/2201.09397.pdf`

[5] Z. Q. Wang, *Introduction to Lie Groups – Lecture Notes*, University of Science and Technology of China, 2013. Available at: `https://staff.ustc.edu.cn/~wangzuoq/Courses/13F-Lie/Lie.html`

[6] B. Russo, *Mathematics 199B: Lie Groups and Their Lie Algebras (Winter 2018)*, Department of Mathematics, University of California, Irvine. Available at: `https://www.math.uci.edu/~brusso/199w18.html`

[7] D. A. Salamon and J. W. Robbin, *Introduction to Differential Geometry*, Springer-Verlag, Berlin, 2022. Preprint available at: `https://people.math.ethz.ch/~salamon/PREPRINTS/diffgeo.pdf`

[8] E. P. van den Ban, *Lie Groups — Lecture Notes, Spring 2010*, Department of Mathematics, Utrecht University; available at: `https://webspace.science.uu.nl/~ban00101/lecnotes/lie2010.pdf`.

[9] S. H. H. Chowdhury and A. R. Chowdhury, *MAT 313 – Differential Geometry (Lecture Notes)*, BRAC University, 2022. Available at `https://atonurc.github.io/assets/MAT313_DG.pdf`

[10] A. R. Chowdhury, *Leibniz-like Rule in Lie Groups*. Available at `https://atonurc.github.io/liegrp/leibniz_lie.pdf`.

[11] TheLegend27, "When is the exponential map injective/surjective?", Mathematics StackExchange Question 3859971, Oct 11 2020. `https://math.stackexchange.com/questions/3859971/`

[12] Jonathan Gleason, “Under what conditions is the exponential map on a Lie algebra injective?”, Mathematics StackExchange Q. 475385, Aug 24 2013. https://math.stackexchange.com/questions/475385/

[13] T. Tao, “Two small facts about Lie groups”, Terence Tao’s mathematics blog, 25 June 2011. Available at: https://terrytao.wordpress.com/2011/06/25/two-small-facts-about-lie-groups/

[14] K. K. K. Wong, “Exponential maps of Lie groups,” *Mathematics @ CUHK blog*, June 28, 2011. Available at: https://cuhkmath.wordpress.com/2011/06/28/exponential-maps-of-lie-groups/.