Topology (MAT411)

Lecture Notes

Preface

This series of lecture notes has been prepared for aiding students who took the BRAC University course **Topology (MAT411)** in Fall 2023 semester. These notes were typeset under the supervision of mathematician **Dr. Syed Hasibul Hassan Chowdhury**. The main goal of this typeset is to have an organized digital version of the notes, which is easier to share and handle. I would like to thank Md Mubasshir Chowdhury for his contributions to the chapter on quotient topology. If you see any mistakes or typos, please send me an email at atonuroychowdhury@gmail.com

Atonu Roy Chowdhury

References:

- *Topology*, by **James R. Munkres**

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1 Topological Spaces

§1.1 Basic Definitions

Definition 1.1. Let X be a set. A **topology** on X is a collection $\mathcal{T}$ of subsets of X having the following properties:

1. $\varnothing$ and X are in $\mathcal{T}$.
2. For any subcollection $\{U_\alpha\}_{\alpha \in J}$ of $\mathcal{T}$, the union $\bigcup_{\alpha \in J} U_\alpha$ is in $\mathcal{T}$.
3. For any finite subcoluction $\{U_1, \ldots, U_n\}$ of $\mathcal{T}$, the intersection $\bigcap_{i=1}^{n} U_i$ is in $\mathcal{T}$.

A **topological space** $(X, \mathcal{T})$ is a set X with a given topology $\mathcal{T}$. A subset $U \subset X$ with $U \in \mathcal{T}$ is said to be an open set.

Example 1.1 (Two extreme examples)**.** Let X be a set. Following are 2 examples of topologies on X :

1. (Discrete topology) The discrete Topology on X, denoted by $\mathcal{T}_{\text{disc}}$ is the topology where all subsets $U \subset X$ are defined to be open. Hence, $\mathcal{T}_{\text{disc}} = \mathscr{P}(X)$, the power set of X. One can easily check that $\mathcal{T}_{\text{disc}}$ is indeed a topology.
2. (Indiscrete topology) The indiscrete topology on X, denoted by $\mathcal{T}_{\text{indis}}$ is the topology where only the subsets X and $\varnothing$ are defined to be open sets. In other words, $\mathcal{T}_{\text{indis}} = \{\varnothing, X\}$.

Definition 1.2 (Finite topological space)**.** If X is a finite set and $\mathcal{T}$ is a topology on X, we call $(X, \mathcal{T})$ a **finite topological space**.

Example 1.2. Let X be a 3-element set, $X = \{1, 2, 3\}$. Verify that the following are examples of finite topological spaces:

1. $\mathcal{T} = \{\varnothing, \{1, 2, 3\}\}$.
2. $\mathcal{T} = \{\varnothing, \{1, 2, 3\}, \{1, 2\}, \{2, 3\}, \{2\}\}$.
3. $\mathcal{T} = \{\varnothing, \{1, 2, 3\}, \{1\}, \{1, 2\}\}$.

Non-example: The collection $\mathcal{T} = \{\varnothing, \{1, 2, 3\}, \{1\}, \{2\}\}$ is not a topology on $X = \{1, 2, 3\}$, since it is not closed under union.

Definition 1.3. Let $\mathcal{T}$ and $\mathcal{T}'$ be 2 topologies on the same set X. If $\mathcal{T}' \supseteq \mathcal{T}$, we soy that $\mathcal{T}'$ is **finer** than $\mathcal{T}$, or $\mathcal{T}$ is **coarser** than $\mathcal{T}'$. If the containment above is proper, we say that $\mathcal{T}'$ is strictly finer than $\mathcal{T}$, or $\mathcal{T}$ is strictly coarser than $\mathcal{T}'$.

Example 1.3. In the context of Example 1.2, for the 3-element set $X = \{1, 2, 3\}$, consider the following 4 topologies:

1. $\mathcal{T} = \{\{1, 2, 3\}, \varnothing, \{1, 2\}, \{2\}, \{2, 3\}\}$.
2. $\mathcal{T}_1 = \{\{1, 2, 3\}, \varnothing\}$.
3. $\mathcal{T}_2 = \{\{1, 2, 3\}, \varnothing, \{2\}\}$
4. $\mathcal{T}_3 = \{\{1, 2, 3\}, \varnothing, \{1, 2\}\}$

Observe that $\mathcal{T}$ is strictly finer than all 3 of $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$. Also, one has $\mathcal{T}_1 \subset \mathcal{T}_3$, and $\mathcal{T}_1 \subset \mathcal{T}_2$, i.e. $\mathcal{T}_3$ is strictly finer than $\mathcal{T}_1$, and $\mathcal{T}_2$ is strict finer than $\mathcal{T}_1$.

§1.2 Review of Metric Space

Definition 1.4. A **metric** on a set X is a function $d : X \times X \to \mathbb{R}$ such that:

1. (Non-negativity) $d(x, y) \geq 0$ for any $x, y \in X$, and $d(x, y) = 0$ if and only if $x = y$.
2. (Symmetry) $d(x, y) = d(y, x)$, for any $x, y \in X$.
3. (Triangle inequality) $d(x, z) \leq d(x, y) + d(y, z)$ for any $x, y, z \in X$.

A **metric space** (X, d) is a set X equipped with a metric d.

Example 1.4. The real line $\mathbb{R}$ is a metric space, with distance function $d_{\text{Euc}}(x, y) = |y - x|$. More generally, in $\mathbb{R}^n$, one can define the Euclidean distance

$$d_{\text{Euc}}(\mathbf{x}, \mathbf{y}) = \|\mathbf{y} - \mathbf{x}\| = \sqrt{(y_1 - x_1)^2 + \cdots + (y_n - x_n)^2}, \tag{1.1}$$

for $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$. We call $(\mathbb{R}^n, d_{\text{Euc}})$ the Euclidean n-space.

Definition 1.5. Let (X, d) be a metric space. For each point $x \in X$ and each $\varepsilon < 0$, let

$$B_d(x, \varepsilon) = \{y \in X \mid d(x, y) < \varepsilon\}. \tag{1.2}$$

Then the set $B_d(x, \varepsilon)$ is called ε-ball around x in (X, d).

Definition 1.6 (Metric topology)**.** Let (X, d) be a metric space. The metric topology $\mathcal{T}_d$ on X is the collection of subsets $U \subset X$ such that for each $x \in U$, there exists $\varepsilon > 0$ such that $B_d(x, \varepsilon) \subset U$.

Lemma 1.1

The collection $\mathcal{T}_d$ is a topology on X.

Proof. Observe that $\varnothing$ is vacuously open in metric topology, i.e. $\varnothing \in \mathcal{T}_d$ since there is no element in $\varnothing$ to open the argument with. Also, the whole set $X \in \mathcal{T}_d$, i.e. the whole set X itself is open in the metric topology. This is so because for any $x \in X$, one can choose $B_d(x, 1) = \{y \in X \mid d(x, y) < 1\} \subseteq X$ proving that X is open in the metric topology.

Next, let $\{U_\alpha\}_{\alpha \in J}$ be a subcollection of $\mathcal{T}_d$. Let $W = \bigcup_\alpha U_\alpha$. Consider $x \in W = \bigcup_\alpha U_\alpha$. Hence, there is some $\alpha_0 \in J$ such that $x \in U_{\alpha_0}$. Since $U_{\alpha_0} \in \mathcal{T}_d$, there exists $\varepsilon > 0$ such that

$$B_d(x, \varepsilon) \subset U_{\alpha_0} \subset \bigcup_{\alpha \in J} U_\alpha = W. \tag{1.3}$$

Hence, $\bigcup_{\alpha \in J} U_\alpha \in \mathcal{T}_d$.

Now, let $\{U_1, \ldots, U_n\}$ be a finite subcolution of $\mathcal{T}_d$. Let $V = U_1 \cap \cdots \cap U_n$ and consider $x \in V$. Hence, $x \in U_i$ for each $i \in \{1, \ldots, n\}$. Since, each $U_i \in \mathcal{T}_d$, there exists $\varepsilon_i > 0$, such that $B_d(x, \varepsilon_i) \subset U_i$, for each $i \in \{1, \ldots, n\}$. Choose $\varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_n\} > 0$. Then one has $B_d(x, \varepsilon) \subset B_d(x, \varepsilon_i) \subset U_i$, for any i. Therefore,

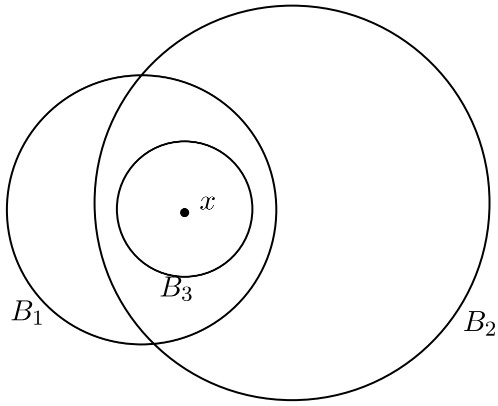
$$B_d(x, \varepsilon) \subset \bigcap_{i=1}^{n} U_i, \tag{1.4}$$

proving that $V = \bigcap_{i=1}^n U_i \in \mathcal{T}_d$. ∎

§1.3 Basis for a Topology

Definition 1.7 (Basis)**.** Let X be a set. A **basis** for a topology on X is a collection $\mathscr{B}$ of subsets of X (called *basis elements*) such that

1. for each $x \in X$, there exists $B \in \mathscr{B}$ such that $x \in B \subset X$;
2. if $x \in B_1 \cap B_2$ with $B_1, B_2 \in \mathscr{B}$, then there exists $B_3 \in \mathscr{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.



Definition 1.8 (Topology generated by a basis)**.** Let $\mathscr{B}$ be a basis for a topology topology on a given set X. The topology $\mathcal{T}$ generated by $\mathscr{B}$ is the collection of subsets $U \subset X$ such that for each $x \in U$, there exists $B \in \mathscr{B}$ with $x \in B \subset U$. In other words, a subset $U \subset X$ is defined to be open in this topology if for each $x \in U$, there exists a basis element $B \subset U$ with $x \in B$.

Lemma 1.2

The collection $\mathcal{T}$ generated by a basis $\mathcal{B}$ as defined above is a topology on X.

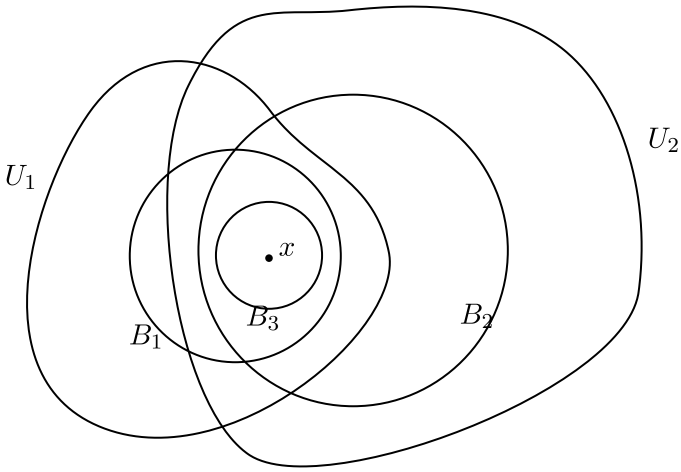
Proof. $\varnothing \in \mathcal{T}$ since there is no element in $\varnothing$ to verify the conditions, and hence $\varnothing$ is vacuously open. By the first condition of basis, for each $x \in X$, there exists $B \in \mathscr{B}$ such that $x \in B \subset X$. Therefore, from the definition of the topology generated by a basis, X is open, i.e. $X \in \mathcal{T}$.

Now, let $\{U_\alpha\}_{\alpha \in J}$ be a subcollection of $\mathcal{T}$. Also, let $\bigcup_{\alpha \in J} U_\alpha = W$. We need to show that $W \in \mathcal{T}$. Consider $x \in W = \bigcup_\alpha U_\alpha$. Hence, there is some $\alpha_0 \in J$ such that $x \in U_{\alpha_0}$. Since $U_{\alpha_0} \in \mathcal{T}$, there exists $B \in \mathscr{B}$ for which $x \in B \subset U_{\alpha_0}$ holds. In other words,

$$x \in B \subset U_{\alpha_0} \subset \bigcup_{\alpha \in J} U_\alpha = W. \tag{1.5}$$

Therefore, $W \in \mathcal{T}$.

Now, let $U_1, U_2 \in \mathcal{T}$. Given $x \in U_1 \cap U_2$, x is in both U_1 and U_2. Since $U_1, U_2 \in \mathcal{T}$, by the definition of topology generated by a basis, there exist basis elements $B_1, B_2 \in \mathscr{B}$ such that $x \in B_1 \subset U_1$ and $x \in B_2 \subset U_2$. Then we have $x \in B_1 \cap B_2$.



By the second condition for a basis, there is some $B_3 \in \mathscr{B}$ such that $x \in B_3 \subset B_1 \cap B_2$. Therefore,

$$x \in B_3 \subset B_1 \cap B_2 \subset U_1 \cap U_2. \tag{1.6}$$

So $U_1 \cap U_2 \in \mathcal{T}$. Now we use induction to prove that $V = \bigcap_{i=1}^{n} U_i \in \mathcal{T}$, where each $U_i \in \mathcal{T}$. The base case $n = 1$ is trivial. Now suppose that this is true for $n-1$, i.e. $\bigcap_{i=1}^{n-1} U_i \in \mathcal{T}$. We also have $U_n \in \mathcal{T}$. We have just proved that the intersection of two elements of $\mathcal{T}$ also belongs to $\mathcal{T}$. Therefore,

$$\left(\bigcap_{i=1}^{n-1} U_i\right) \cap U_n = \bigcap_{i=1}^{n} U_i \in \mathcal{T}. \tag{1.7}$$

Therefore, $\mathcal{T}$ is a topology on X. ■

Lemma 1.3

In any metric space (X, d), the collection of ε-balls

$$\mathscr{B} = \{B_d(x, \varepsilon) \mid x \in X, \varepsilon > 0\}$$

is a basis.

Proof. 1. For each $x \in X$, the 1-ball $B_d(x, 1) \in \mathscr{B}$.

2. Given $B_1 = B_d(x_1, \varepsilon_1)$ and $B_2 = B_d(x_2, \varepsilon_2)$, consider $x \in B_1 \cap B_2$. It is evident that

$$\varepsilon_1 - d(x, x_1) > 0 \text{ and } \varepsilon_2 - d(x, x_2) > 0. \tag{1.8}$$

Let $\varepsilon = \min\{\varepsilon_1 - d(x, x_1), \varepsilon_2 - d(x, x_2)\}$. Then $\varepsilon > 0$. Now we claim that $x \in B_d(x, \varepsilon) =: B_3 \subset B_1 \cap B_2$. Let $y \in B_3 = B_d(x, \varepsilon)$, so that $d(x, y) < \varepsilon$. Then

$$d(x, y) < \varepsilon \leq \varepsilon_1 - d(x, x_1).$$

By the triangle inequality,

$$d(x_1, y) \leq d(x, x_1) + d(x, y) < \varepsilon_1, \tag{1.9}$$

which implies that $y \in B_1 = B_d(x_1, \varepsilon_1)$. So $B_3 \subset B_1$. Similarly, $B_3 \subset B_2$. Therefore, $B_3 = B_d(x, \varepsilon) \subset B_1 \cap B_2$, as required.

■

Proposition 1.4

The metric topology $\mathcal{T}_d$ defined earlier on the metric space coincides with the topology $\mathcal{T}$ on (X, d) generated by the basis of ε-balls as in Lemma 1.3.

Proof. Suppose $U \in \mathcal{T}_d$. Hence, from the definition of metric topology, for each $y \in U$, there exists $\delta > 0$ such that $B_d(y,\delta) \subset U$. Since $B_d(y,\delta) \in \mathscr{B}$, and $y \in B_d(y,\delta) \subset U$, $U \in \mathcal{T}$, the topology on (X,d) generated by the basis $\mathscr{B}$. In other words, $\mathcal{T}_d \subset \mathcal{T}$.

Now conversely, suppose $U \in \mathcal{T}$. Hence, given $y \in U$, there is a basis element $B_d(x,\varepsilon) \in \mathscr{B}$ such that $y \in B_d(x,\varepsilon) \subset U$. Hence, $d(x,y) < \varepsilon$. Define $\delta = \varepsilon - d(x,y) > 0$. Then one immediately finds $B_d(y,\delta) \subset B_d(x,\varepsilon)$. Indeed, if $z \in B_d(y,\delta)$, then $d(y,z) < \delta = \varepsilon - d(x,y)$. By the triangle inequality,

$$d(x,z) \leq d(x,y) + d(y,z) < \varepsilon. \tag{1.10}$$

Therefore, $z \in B_d(x,\varepsilon)$, proving that $y \in B_d(y,\delta) \subset B_d(x,\varepsilon) \subset U$. So we have proved that given $y \in U$, there exists $\delta > 0$ such that $B_d(y,\delta) \subset U$. In other words, $U \in \mathcal{T}_d$, so that $\mathcal{T} \subset \mathcal{T}_d$. Hence, $\mathcal{T} = \mathcal{T}_d$. ∎

Example 1.5. Let $X = \mathbb{R}^2$, and $\mathscr{B}$ be the collection of all circular regions (interior of circles) in the plane. This is the collection of all ε-balls

$$B_{\text{Euc}}(\mathbf{x},\varepsilon) = \left\{\mathbf{y} \in \mathbb{R}^2 \mid d(\mathbf{x},\mathbf{y}) < \varepsilon\right\}$$

with respect to the Euclidean metric $d_{\text{Euc}}(\mathbf{x},\mathbf{y}) = \|\mathbf{x}-\mathbf{y}\| = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2}$, with $\mathbf{x} = (x_1,x_2), \mathbf{y} = (y_1,y_2) \in \mathbb{R}^2$. Indeed, $(\mathbb{R}^2, d_{\text{Euc}})$ is a metric space, and by means of Lemma 1.3 and Proposition 1.4, the collection

$$\mathscr{B} = \left\{B_{\text{Euc}}(\mathbf{x},\varepsilon) \mid \mathbf{x} \in \mathbb{R}^2,\ \varepsilon > 0\right\}$$

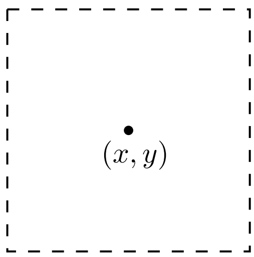
is a basis for the metric topology with respect to the Euclidean metric on $\mathbb{R}^2$.

Example 1.6. Let $X = \mathbb{R}^2$, but in contrast to Example 1.5, here choose $\mathscr{B}'$ to be the collection of all rectangular regions (interior of rectangles) in the plane $\mathbb{R}^2$. This is the collection of all sets of the form

$$(a,b) \times (c,d) \in \mathbb{R} \times \mathbb{R},$$

with $a < b$ and $c < d$. This is the open rectangular area bounded by the vertical lines $x = a$ and $x = b$, and horizontal lines $y = c$ and $y = d$. Let us verify that such a collection, indeed, satisfies the two conditions for a basis:

1. For each $(x,y) \in \mathbb{R}^2$, $(x,y) \in (x-1,x+1)\times(y-1,y+1)$, with $(x-1,x+1)\times(y-1,y+1) \in \mathscr{B}'$.



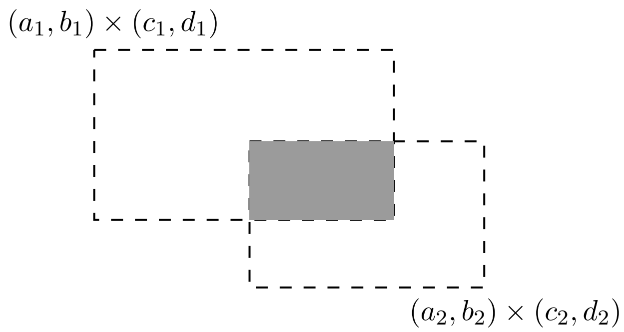
$(x-1,x+1)\times(y-1,y+1)$

2. Consider $B_1 = (a_1,b_1)\times(c_1,d_1)$ and $B_2 = (a_2,b_2)\times(c_2,d_2)$ to be two elements in $\mathscr{B}'$. Take $(x_0,y_0) \in B_1 \cap B_2$. Since $a_1 < x_0 < b_1$ and $a_2 < x_0 < b_2$, one has

$$a := \max\{a_1,a_2\} < x_0 < \min\{b_1,b_2\} =: b,$$

Similarly,

$$c := \max\{c_1,c_2\} < y_0 < \min\{d_1,d_2\} =: d.$$



Then $(x_0, y_0) \in (a, b) \times (c, d) =: B_3 = B_1 \cap B_2$, the shaded open rectangle in the diagram above. The diagram above is the case when $B_1 \cap B_2 \neq \varnothing$. The condition for this to happen is $a < b$ and $c < d$. Otherwise, the intersection is empty, and the second condition for basis is vacuously satisfied.

Proposition 1.5

Let $\mathscr{B}$ be a basis for a topology $\mathcal{T}$ on X, i.e. $\mathcal{T}$ is the topology on X generated by the basis $\mathscr{B}$. Then $\mathcal{T}$ equals the collection of all unions of elements of $\mathscr{B}$.

Proof. Let us first prove that $\mathcal{T}$ is contained in the collection of all unions of elements of $\mathscr{B}$. Let $U \in \mathcal{T}$. For each $x \in U$, there exists $B_x \in \mathscr{B}$ with $x \in B_x \subset U$. Then one easily has $U = \bigcup_{x \in U} B_x$. Indeed, since $x \in B_x \subset U$, taking union over all $x \in U$ gives us

$$\bigcup_{x \in U} \{x\} \subset \bigcup_{x \in U} B_x \subset U.$$

In other word,

$$U \subset \bigcup_{x \in U} B_x \subset U. \tag{1.11}$$

So $U = \bigcup_{x \in U} B_x$. Therefore, any open set on X in the topology $\mathcal{T}$ generated by a basis $\mathscr{B}$ is a union of basis elements from $\mathscr{B}$.

To prove the converse, i.e. any union of basis elements from $\mathscr{B}$ belongs to $\mathcal{T}$, note that every basis element B of $\mathscr{B}$ is open, i.e. it belongs to $\mathcal{T}$. This is because for each $x \in B$, there is a basis element, namely B itself, such that $x \in B \subset B$, proving that $B \in \mathcal{T}$, the topology generated by the basis $\mathscr{B}$. From the definition of topology, it follows that arbitrary union of basis elements from $\mathscr{B}$ will be in $\mathcal{T}$ as well. ■

Proposition 1.6 (Local criterion of open sets)

Let X be a topological space; let A be a subset of X. Then A is open in X if and only if for each $x \in A$ there is an open set U containing x such that $U \subset A$.

Proof. If A is open, then for each $x \in A$, A is an open set containing x and contained in A. Conversely, suppose for every $x \in A$, there exists an open set U_x such that $x \in U_x \subset A$. Then

$$A = \bigcup_{x \in A} \{x\} \subset \bigcup_{x \in A} U_x \subset A. \tag{1.12}$$

So $A = \bigcup_{x \in A} U_x$, i.e. A is a union of open sets, hence open. ■

Example 1.7. If X is any set, the collection of all one-point subsets of X is a basis for the discrete topology $\mathcal{T}_{\text{dis}}$ on X. For example, if $X = \{a, b, c\}$, then

$$\mathcal{T}_{\text{dis}} = \{\{a, b, c\}, \varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}\} = \mathscr{P}(X);$$

and $\mathscr{B} = \{\{a\}, \{b\}, \{c\}\}$. Indeed, $\mathcal{T}_{\text{dis}}$ can be obtained from $\mathscr{B}$ by taking all possible unions. $\varnothing$ is understoor as the union of no basis elements at all.

Lemma 1.7 (Comparing topologies using bases)
Let $\mathscr{B}$ and $\mathscr{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$ on X, respectively. Then the following are equivalent:

1. $\mathcal{T}'$ is finer than $\mathcal{T}$.
2. For each $x \in X$ and any basis element $B \in \mathscr{B}$ containing x, there is a basis element $B' \in \mathscr{B}'$ such that $x \in B' \subset B$.

Proof. **(1⇒2)** Let $x \in X$ and $B \in \mathscr{B}$ with $x \in B$. We have seen in the proof of Proposition 1.5 that, $B \in \mathcal{T}$. By hypothesis, $\mathcal{T} \subset \mathcal{T}'$. Hence, $B \in \mathcal{T}'$. Since $\mathcal{T}'$ is the topology generated by $\mathscr{B}'$, there exists $B' \in \mathscr{B}'$ with $x \in B' \subset B$.

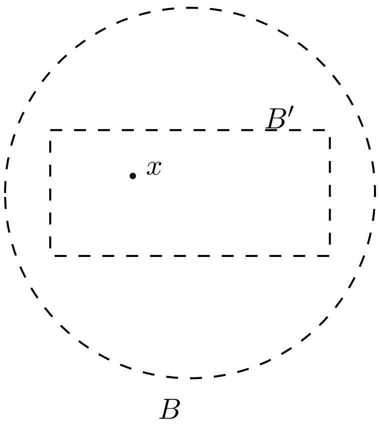
(2⇒1) Let $U \in \mathcal{T}$. Since $\mathcal{T}$ is generated by $\mathscr{B}$, for each $x \in U$, there exists some $B \in \mathscr{B}$ with $x \in B \subset U$. By hypothesis, there exists a $B' \in \mathscr{B}'$ with $x \in B' \subset B$. Therefore, $B' \in U$. We, therefore, have shown that for each $x \in U$, there exists $B' \in \mathscr{B}'$ with $x \in B' \subset U$. Hence, $U \in \mathcal{T}'$, the topology generated by $\mathscr{B}'$. Therefore, $\mathcal{T} \subset \mathcal{T}'$. ■

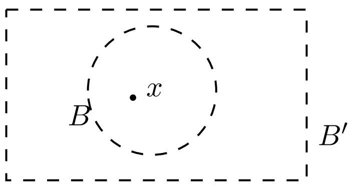
Corollary 1.8
Two bases $\mathscr{B}$ and $\mathscr{B}'$ for topologies on X generate the same topology if and only if

1. for each $x \in B \in \mathscr{B}$, there is a basis element $B' \in \mathscr{B}'$ such that $x \in B' \subset B$; and furthermore,
2. for each $x \in B' \in \mathscr{B}'$, there is a basis element $B \in \mathscr{B}$ such that $x \in B \subset B'$.

Proof. Let $\mathscr{B}$ and $\mathscr{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$ on X, respectively. By Lemma 1.7, $\mathcal{T} \subseteq \mathcal{T}'$ is equivalent to (1). By Lemma 1.7, $\mathcal{T}' \subseteq \mathcal{T}$ is equivalent to (2). ■

Example 1.8. The basis $\mathscr{B}$ of open circular regions in the plane $\mathbb{R}^2$ and the basis $\mathscr{B}'$ of open rectanglular regions generate the same topology on $\mathbb{R}^2$, namely the metric topology.





Example 1.9 (Three important topologies on $\mathbb{R}$)**.** Let $\mathscr{B}$ be the collection of all open intervals in thr real line $\mathbb{R}$:

$$(a,b) = \left(\frac{a+b}{2} - \frac{b-a}{2}, \frac{a+b}{2} + \frac{b-a}{2}\right) = B_{\text{Euc}}\left(\frac{a+b}{2}, \frac{b-a}{2}\right).$$

$\mathscr{B} = \{(a,b) \mid a, b \in \mathbb{R} \text{ with } a < b\}$. This collection $\mathscr{B}$ is a basis on $\mathbb{R}$, and the topology it generates is precisely the Euclidean metric topology on $\mathbb{R}$ by Proposition 1.4. This topology is also called the **standard topology** on $\mathbb{R}$.

Now, let $\mathscr{B}'$ denote the collection of all half-open intervals of the form

$$[a,b) = \{x \in \mathbb{R} \mid a \le x < b\},$$

for $a < b$. In other words, $\mathscr{B}' = \{[a, b) \mid a, b \in \mathbb{R} \text{ with } a < b\}$. The topology on $\mathbb{R}$ generated by the basis $\mathscr{B}'$ is called the **lower-limit topology**. When $\mathbb{R}$ is given the lower-limit topology, the resulting topological space is denoted by $\mathbb{R}_\ell$.

Now, let K denote the set of all numbers of the form $\frac{1}{n}$ for positive integers n. Also, let $\mathscr{B}''$ denote the collection of all open intervals (a, b) along with all sets of the form $(a, b) \setminus K$. The topology on $\mathbb{R}$ generated by $\mathscr{B}''$ is called the **K-topology** on $\mathbb{R}$. $\mathbb{R}$, equipped with the K-topology, is denoted by $\mathbb{R}_K$.

Lemma 1.9

The topologies $\mathbb{R}_\ell$ and $\mathbb{R}_K$ are strictly finer than the standard topology on $\mathbb{R}$.

Proof. Let $\mathcal{T}$, $\mathcal{T}'$ and $\mathcal{T}''$ be the topologies of $\mathbb{R}$, $\mathbb{R}_\ell$ and $\mathbb{R}_K$, respectively. Given a basis element $(a, b) \in \mathscr{B}$ generating $\mathcal{T}$, and $x \in (a, b)$, one finds $[x, b) \in \mathscr{B}'$ generating $\mathcal{T}'$ such that $x \in [x, b) \subset (a, b)$. This proves that $\mathcal{T}'$ is finer than $\mathcal{T}$ using Lemma 1.7.

On the other hand, choose $[x, y) \in \mathscr{B}'$ generating $\mathcal{T}'$. There exists no open interval $(a, b) \in \mathscr{B}$ such that $x \in (a, b) \subset [x, y)$ is satisfied. Hence, $\mathcal{T}'$ is strictly finer than $\mathcal{T}$.

Now, givena basis element $(a, b) \in \mathscr{B}$ generating $\mathcal{T}$, and $x \in (a, b)$, one finds $(a, b) \in \mathscr{B}''$ generating $\mathcal{T}''$ such that $x \in (a, b) \subset (a, b)$. This proves that $\mathcal{T}''$ is finer than $\mathcal{T}$ using Lemma 1.7.

On the other hand, observe that $B = (-2, 2) \setminus K \in \mathscr{B}''$ generating $\mathcal{T}''$, and $0 \in B$. But there exists no open interval $(a, b) \in \mathscr{B}$ such that $0 \in (a, b) \subset B$. Hence, $\mathcal{T}''$ is strictly finer than $\mathcal{T}$. ■

§1.4 Subbasis

Definition 1.9 (Subbasis)**.** A **subbasis** for a topology on X is a collection $\mathscr{S}$ of subsets of X, with union equal to X. One can form the collection $\mathscr{B}$ consisting of all finite intersections of elements of $\mathscr{S}$:

$$B = S_1 \cap \cdots \cap S_n,$$

with $S_1, \ldots, S_n \in \mathscr{S}$, for $n \geq 1$. By the topology $\mathcal{T}$ generated by $\mathscr{S}$, we mean the topology generated by the associated basis $\mathscr{B}$. One clearly has $\mathscr{S} \subset \mathscr{B} \subset \mathcal{T}$.

Lemma 1.10

Let $\mathscr{S}$ be a subbasis on X. The associated collection $\mathscr{B}$ is a basis for a topology.

Proof. There are 2 conditions of a basis to be fulfilled:

1. Since the union of all elements of $\mathscr{S}$ is X by the definition of a subbasis, each $x \in X$ lies in some $S \in \mathscr{S}$. Since S itself is a basis element, the first condition for basis is fulfilled.
2. Suppose $B_1 = S_1 \cap \cdots \cap S_m$ and $B_2 = S'_1 \cap \cdots \cap S'_n$, where $S_i, S'_i \in \mathscr{S}$. Then the intersection
$$B_1 \cap B_2 = (S_1 \cap \cdots \cap S_m) \cap (S'_1 \cap \cdots \cap S'_n) \tag{1.13}$$
is also a finite intersection of elements of $\mathscr{S}$, and hence $B_1 \cap B_2 \in \mathscr{B}$, fulfilling the second condition of basis.

■

Suppose $(X, \mathcal{T})$ is a topological space. We are given $\mathscr{C} \subset \mathcal{T}$, a subcollection of open subsets of X. How do we recognize if $\mathscr{C}$ is a basis for the topology $\mathcal{T}$ on X? The following lemma answers this question.

Lemma 1.11 (Recognition principle)

Let $(X, \mathcal{T})$ be a topological space. Suppose $\mathscr{C} \subset \mathcal{T}$ is a subcollection of open subsets of X, such that for each open $U \in \mathcal{T}$ and each $x \in U$, there exists $C \in \mathscr{C}$ with $x \in C \subset U$. Then $\mathscr{C}$ is a basis for the topology $\mathcal{T}$ on X.

Proof. Let us first check that $\mathscr{C}$ is a basis for *some* topology on X.

1. Observe that $X \in \mathcal{T}$. Hence, by hypothesis, for each $x \in X$, there exists $C \in \mathscr{C}$ with $x \in C \subset X$.
2. Suppose $C_1, C_2 \in \mathscr{C}$ with $x \in C_1 \cap C_2$. Since $\mathscr{C} \subset \mathcal{T}$, C_1 and C_2 are both open and so is their intersection $C_1 \cap C_2$, i.e. $C_1 \cap C_2 \in \mathcal{T}$. By hypothesis, there exists $C_3 \in \mathscr{C}$ such that $x \in C_3 \subset C_1 \cap C_2$. Thus the second condition for basis is also fulfilled.

Let us denote the topology on X generated by $\mathscr{C}$ with $\mathcal{T}'$. We are left to show that $\mathcal{T}' = \mathcal{T}$. Let $U \in \mathcal{T}$ and $x \in U$. By hypothesis, there exists $C \in \mathscr{C}$ with $x \in C \subset U$. Then, by definition of topology generated by a basis, $U \in \mathcal{T}'$. So $\mathcal{T} \subset \mathcal{T}'$.

Now, let $U \in \mathcal{T}'$. By Proposition 1.5, U is a union of elements of $\mathscr{C}$. Since $\mathscr{C} \subset \mathcal{T}$, each element of $\mathscr{C}$ is in $\mathcal{T}$. As $\mathcal{T}$ is a topology, it must be closed under arbitrary union. Hence, $U \in \mathcal{T}$. Therefore, $\mathcal{T}' \subset \mathcal{T}$. ∎

§1.5 The Product Topology on $X \times Y$

Definition 1.10 (Product topology). Let X and Y be topological spaces. The **product topology** on $X \times Y$ is the topology generated by the basis

$$\mathscr{B} = \{U \times V \mid U \text{ open in } X, \text{ and } V \text{ open in Y}\}. \tag{1.14}$$

Lemma 1.12

The collection $\mathscr{B}$ as above is a basis for a topology on $X \times Y$.

Proof. Two conditions for basis need to be checked:

1. Observe that $X \times Y \in \mathscr{B}$.
2. Let $B_1 = U_1 \times V_1$ and $B_2 = U_2 \times V_2$ be two basis elements. Observe that
$$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2). \tag{1.15}$$
From the above set theoretic identity, one verifies that $B_1 \cap B_2 = B_3$, where $B_3 = U_3 \times V_3$ with $U_3 = U_1 \cap U_2$ being open in X and $V_3 = V_1 \cap V_2$ being open in Y. This proves that the second condition for basis is verified.

∎

Lemma 1.12 gives us a basis for the product topology on $X \times Y$ in terms of open sets of X and Y. If we have information about the bases that generate the topologies on X and Y, then the following theorem gives us a basis generating the product topology on $X \times Y$.

Theorem 1.13

If $\mathscr{B}$ is a basis for the topology of X, $\mathscr{C}$ is a basis for the topology of Y, then the collection

$$\mathscr{D} = \{B \times C \mid B \in \mathscr{B} \text{ and } C \in \mathscr{C}\}$$

is a basis for the product topology of X.

Proof. We apply Recognition principle. We know from the definition of the product topology on $X \times Y$ that it is generated by the basis

$$\mathscr{B}_{\text{prod}} \{U \times V \mid U \subset X \text{ and } V \subset Y \text{ are open } \}. \tag{1.16}$$

Now, given an open set $W \subset X \times Y$ and $(x, y) \in W$, there exists a basis element $U \times V \in \mathscr{B}_{\text{prod}}$ with $(x, y) \in U \times V \subset W$. Now, since $U \subset X$ and $V \subset Y$ are open, there are basis elements $B \in \mathscr{B}$ and $C \in \mathscr{C}$ with $x \in B \subset U$ and $y \in C \subset V$. Therefore,

$$(x, y) \in B \times C \subset U \times V \subset W. \tag{1.17}$$

We, thus, have found that for an open set $W \subset X \times Y$ and any $(x, y) \in W$, there exists $B \times C \in \mathscr{D}$ such that $(x, y) \in B \times C \subset W$. So $\mathscr{D}$ is a basis for the product topology on $X \times Y$, by Recognition principle. ∎

Definition 1.11. Let $\pi_1 : X \times Y \to X$ denote the projection onto the first component define by $\pi_1 (x, y) = x$; and let $\pi_2 : X \times Y \to Y$ be the projection onto the second component defined by $\pi_2 (x, y) = y$.

Observe that the preimage of $U \subset X$ under $\pi_1 : X \times Y \to X$ is $\pi_1^{-1} (U) U \times Y$; and the preimage of $V \subset Y$ under $\pi_2 : X \times Y \to Y$ is $\pi_2^{-1} (V) = X \times V$. Note the identity,

$$(U \times Y) \cap (X \times V) = U \times V. \tag{1.18}$$

Since each basis element B for the product topology on $X \times Y$ is of the form $U \times V$ with $U \subset X$ and $V \subset Y$ being open, the basis element $B = U \times V$ can be written as the intersection of $\pi_1^{-1} (U)$ and $\pi_2^{-1} (V)$. It follows that a basis element for the product topology can be written as intersection of subsets from the following collection $\mathscr{S}$:

$$\mathscr{S} = \{U \times Y \mid U \subset X \text{ open}\} \cup \{X \times V \mid V \subset Y \text{ open}\} \quad = \left\{\pi_1^{-1} (U) \mid U \subset X \text{ open}\right\} \cup \left\{\pi_2^{-1} (V) \mid V \subset Y \text{ open}\right\}$$

Theorem 1.14

The collection $\mathscr{S}$ as above is a subbasis for the product topology on $X \times Y$.

Proof. It is immediate that the collection $\mathscr{S}$ is a subbasis for a topology on $X \times Y$ as the union of all elements of $\mathscr{S}$ is equal to $X \times Y$. All that needs to be proved now is that the topology generated by this subbasis is equal to the product topology on $X \times Y$. Let $\mathcal{T}$ denote the product topology on $X \times Y$ and $\mathcal{T}'$ denote topology generated by $\mathscr{S}$. $\mathcal{T}'$ contains arbitrary unions of finite intersections of elements of $\mathscr{S}$. Since each element of $\mathscr{S}$ belongs to $\mathcal{T}$ as well, so do arbitrary unions of finite intersections of elements of $\mathscr{S}$ as $\mathcal{T}$ is a topology. Hence $\mathcal{T}' \subset \mathcal{T}$.

Conversely, since $U \times V$ is a generic basis element from $\mathscr{B}$ generating product topology on $X \times Y$ with U open in X and V open in Y, an arbitrary element W from $\mathcal{T}$ can be written as union of sets of the form $U \times V$. But

$$U \times V = \pi_1^{-1} (U) \cap \pi_2^{-1} (V),$$

with $\pi_1^{-1}(U), \pi_2^{-1}(V) \in \mathscr{S}$ and hence $\pi_1^{-1}(U) \cap \pi_2^{-1}(V) \in \mathcal{T}'$, the topology generated by the subbasis $\mathscr{S}$ so that an arbitrary union of sets of the form $\pi_1^{-1} (U) \cap \pi_2^{-1} (V)$ will also belong to $\mathcal{T}'$ leading to the fact that the arbitrary element $W \in \mathcal{T}$ also belongs to $\mathcal{T}'$. Hence, $\mathcal{T} \subset \mathcal{T}'$. Therefore, $\mathcal{T} = \mathcal{T}'$. ∎

§1.6 Subspace Topology

Definition 1.12 (Subspace topology)**.** Let $(X, \mathcal{T})$ be a topological space. Ass, let $A \subset X$ be a subset. The collection

$$\mathcal{T}_A = \{A \cap U \mid U \in \mathcal{T}\} \tag{1.19}$$

of subsets of A is called the **subspace topology** on A. with this topology, $(A, \mathcal{T}_A)$ is called a subspace of $(X, \mathcal{T})$.

Lemma 1.15

The collection $\mathcal{T}_A$ as defined above is a topology on A.

Proof. Let us first note that $\varnothing$ can be written as $\varnothing = A \cap \varnothing$ with $\varnothing \in \mathcal{T}$ as $\mathcal{T}$ is a topology. Hence by definition (1.19), $\varnothing \in \mathcal{T}_A$. Also, notice that $A = A \cap X$ with $X \in \mathcal{T}$ again by the fact that $\mathcal{T}$ is a topology on X and $A \subseteq X$. Hence, by definition (1.19), $A \in \mathcal{T}_A$.

It remains to show that $\mathcal{T}_A$ is closed under arbitrary union and finite intersection. Let $\{A \cap U_\alpha\}_{\alpha \in J}$ be a subcollection of elements from $\mathcal{T}_A$ as defined in (1.19) associated with the subcollection $\{U_\alpha\}_{\alpha \in J}$ of open subsets from $\mathcal{T}$ indexed by J. Now, observe that

$$\bigcup_{\alpha \in J} (A \cap U_\alpha) = A \cap \left(\bigcup_{\alpha \in J} U_\alpha \right) \tag{1.20}$$

by distributive law in set theory. Hence, using the fact that $\bigcup_{\alpha \in J} U_\alpha \in \mathcal{T}$ holds, one deduces the fact that $\bigcup_{\alpha \in J} (A \cap U_\alpha) \in \mathcal{T}_A$. In other words, $\mathcal{T}_A$ is closed under arbitrary union of elements from it.

Now, let us choose a finite subcollection $\{A \cap U_1, \ldots, A \cap U_n\}$ of elements from $\mathcal{T}_A$ with $\{U_1, \ldots, U_n\} \subset \mathcal{T}$, being a finite subcollection of open subsets of X drawn from $\mathcal{T}$. Again, one observes that

$$\bigcap_{i=1}^{n} (A \cap U_i) = A \cap \left(\bigcap_{i=1}^{n} U_i \right), \tag{1.21}$$

with $\bigcap_{i=1}^{n} U_i \in \mathcal{T}$ as $\mathcal{T}$ a topology. Now, using definition (1.19), one concludes $\bigcap_{i=1}^{n} (A \cap U_i) \in \mathcal{T}_A$. In other words, $\mathcal{T}_A$ is closed under finite intersection of elements from it. ■

Remark 1.1. When $(A, \mathcal{T}_A)$ is a subspace of $(X, \mathcal{T})$, and $V \subset A \subset X$. one cant simply say that "V is open", as there may arise a potential ambiguity. One says "V is open in A" to indicate that $V \in \mathcal{T}_A$ while phrases "V is open in X" to imply that $V \in \mathcal{T}$. The former indicate that $V = A \cap U$ for some $U \in \mathcal{T}$.

Lemma 1.16

If $\mathscr{B}$ is a basis for topology $\mathcal{T}$ on X, and $A \subset X$, then the collection

$$\mathscr{B}_A = \{A \cap B \mid B \in \mathscr{B}\}$$

is a basis for the subspace topology $\mathcal{T}_A$ on A.

Proof. We apply Recognition principle for the collection $\mathscr{B}_A$ in the context of the topological space $(A, \mathcal{T}_A)$. Since each basis element $B \in \mathscr{B}$ is open in X, i.e. $B \in \mathcal{T}$, each subset $A \cap B \in \mathscr{B}_A$ is open in A, i.e. $A \cap B \in \mathcal{T}_A$. Additionally, given $x \in A \cap U \in \mathcal{T}_A$, with $U \in \mathcal{T}$, one has $x \in U \in \mathcal{T}$ so that ther exists $B \in \mathscr{B}$ with $x \in B \subset U$ as $\mathscr{B}$ is a basis for topology $\mathcal{T}$ on X. Also, since $x \in A \cap U$, $x \in A$ so that $x \in B \cap A \subset U \cap A$. Therefore, we have shown that given $A \cap U \in \mathcal{T}_A$ and $x \in A \cap U$, there exists $A \cap B \in \mathscr{B}_A$ such that $x \in A \cap B \subset A \cap U$. Therefore, the collection $\mathscr{B}_A$ meets the criteria as required by the Recognition principle to become a basis for the subspace topology $\mathcal{T}_A$ on A. ■

Example 1.10. Give $\mathbb{R}$ the standard Euclidean metric topology generated by the open intervals (a, b), and let $A = [0, 1)$. According to Lemma 1.16 above, the subspace topology $\mathcal{T}_A$ on A has a basis consisting of intersections $[0, 1) \cap (a, b)$. Observe that the basis consists of all intervals of the form $[0, b)$ and (a, b) with $0 < a < b \leq 1$.

Theorem 1.17

If A is a subspace of X and B is a subspace of Y, then the product topology on $A \times B$ is the same as the subspace topology on $A \times B$ as a subset of $X \times Y$.

Proof. The subspace topology $\mathcal{T}_A$ on A is given by the collection

$$\mathcal{T}_A = \{A \cap U \mid U \in \mathcal{T}\},$$

with $A \subset X$, a subset and $\mathcal{T}$ is the given topology on X. Similarly, the subspace toplogy $\mathcal{T}_B$ on a subset $B \subset Y$ is given by the collection

$$\mathcal{T}_B = \{B \cap V \mid V \in \mathcal{T}'\},$$

Where $\mathcal{T}'$ is the given topology on Y. Hence, by the definition of product topology, the product topology on $A \times B$ is generated by the basis

$$\mathscr{B}_{A\times B} = \{(A \cap U) \times (B \cap V) \mid U \in \mathcal{T},\ V \in \mathcal{T}'\}. \tag{1.22}$$

On the other hand, from the result provided in Lemma 1.16. the collection

$$\mathscr{B}'_{A\times B} = \{(A \times B) \cap (U \times V) \mid U \in \mathcal{T},\ V \in \mathcal{T}'\} \tag{1.23}$$

is a basis for the subspace topology $\mathcal{T}_{A\times B}$ on $A \times B$ as a subset of $X \times Y$. Note that here we use the fact that open subsets $U \times V$ of $X \times Y$, with U open in X and V open in Y, constitute a basis for the product topology on $X \times Y$. In view of the following set theoretic equality,

$$(A \cap U) \times (B \cap V) = (A \times B) \cap (U \times V) \tag{1.24}$$

one concludes that the 2 bases given, by (1.22) and (1.23) are actually the same, i.e. $\mathscr{B}_{A\times B} = \mathscr{B}'_{A\times B}$. Hence, the 2 topologies they generate are the same. ■

Definition 1.13. By an **open subspace** of X, we mean an open subset $A \subset X$ (i.e. A is open in the topology of X) equipped with the subspace topology inherited from X.

Lemma 1.18

Let A be an open subspace of X. Then a subset $V \subset A$ is open if and only if it is open in X.

Proof. Suppose $V \subset A$ is open with respect to subspace topology on A, hence, $V = A \cap U$, for some U open in X. Since A is also open in X, $V = A \cap U$ must also be open in X. Now, let $V \subset A$ and V be open in X. Write $V = A \cap V$ with V open in X. From the definition of subspace topology, V is open in A with respect to subspace topology, as required. ■

2 Closed Sets and Limit Points

§2.1 Closed Sets

Definition 2.1. A subset K of a topological space x is said to be closed if and only if $X \setminus K$ is open.

Example 2.1. The interval $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$ is closed in $\mathbb{R}$ with respect to the standard topology, as $\mathbb{R} \setminus [a, b] = (-\infty, a) \cup (b, \infty)$ is open in $\mathbb{R}$.

Example 2.2. In the discrete topology $\mathcal{T}_{\text{dis}}$ on a set X, every subset is closed. In the indiscrete topology $\mathcal{T}_{\text{indis}}$ on X, only the subsets $\varnothing$ and X are closed.

Theorem 2.1

Let X be a topological space.

(i) $\varnothing$ and X are closed subsets of X.

(ii) The intersection of any collection of closed subsets of X is closed.

(iii) The union of any finite collection of closed subsets of X is closed.

Proof. (i) Let us denote the topology on X by $\mathcal{T}$. From the definition of topology, one knows that $\varnothing, X \in \mathcal{T}$. Hence, $X \setminus X = \varnothing$ is closed. Also, $X \setminus \varnothing = X$ is closed.

(ii) Let $\{K_\alpha\}_{\alpha \in J}$ be any collection of closed subsets of X. Hence, $\{X \setminus K_\alpha\}_{\alpha \in J}$ is a collection of open subsets of X indexed by the same set J. From the definition of topology,

$$\bigcup_{\alpha \in J} (X \setminus K_\alpha) \in \mathcal{T}.$$

By De Morgan's law, one obtains,

$$\bigcup_{\alpha \in J} (X \setminus K_\alpha) = X \setminus \left(\bigcap_{\alpha \in J} K_\alpha \right) \in \mathcal{T}, \quad \text{i.e.,} \quad X \setminus \left(\bigcap_{\alpha \in J} K_\alpha \right) \text{ is open in } X,$$

which in turn implies that

$$\bigcap_{\alpha \in J} K_\alpha \text{ is closed in } X.$$

(iii) Let $\{K_i\}_{i=1}^n$ be a finite collection of closed subsets of X. Then $\{X \setminus K_1, \ldots, X \setminus K_n\}$ is a finite collection of open subsets of X. By the definition of topology, one has,

$$\bigcap_{i=1}^{n} (X \setminus K_i) \in \mathcal{T}.$$

By De Morgan's law, one has,

$$\bigcap_{i=1}^{n} (X \setminus K_i) = X \setminus \left(\bigcup_{i=1}^{n} K_i \right) \in \mathcal{T}, \quad \text{i.e.,} \quad X \setminus \left(\bigcup_{i=1}^{n} K_i \right) \text{ is open in } X,$$

which in turn implies that

$$\bigcup_{i=1}^{n} K_i \text{ is closed in } X.$$

■

When $(A, \mathcal{T}_A)$ is a subspace of $(X, \mathcal{T})$, and $K \subset A \subset X$, we can interpret the phrase "K is closed" in 2 possible ways:

(i) $K \in \mathcal{T}$, or

(ii) $K \cap A \in \mathcal{T}_A$.

In the latter case, we say that K is a closed subset of A or K is closed in the subspace topology on A.

Theorem 2.2

Let A be a subspace of X. A subset $K \subset A$ is closed in A if and only if there exists a closed subset $L \subset X$ with $K = A \cap L$.

Proof. Let $K \subset A$ be closed in A. Then $V = A \setminus K$ is open in A, i.e., $V = A \cap U$ for some $U \in \mathcal{T}$. Hence, by definition of subspace topology, there exists $U \subset X$ open such that $V = A \cap U$.

Let $L = X \setminus U$. Then L is closed in X, and

$$A \cap L = A \cap (X \setminus U) = (A \cap X) \setminus (A \cap U) = A \setminus V = A \setminus (A \setminus K) = K.$$

Conversely, suppose $L \subset X$ is closed and $K = A \cap L$. Then $U = X \setminus L$ is open in X so that $V = A \cap U$ is open in A in subspace topology. Now,

$$A \setminus K = A \setminus (A \cap L) = (A \setminus A) \cup (A \setminus L) = A \cap (X \setminus L) = A \cap U = V,$$

leading to the fact that $A \setminus K$ is open in A in subspace topology. Hence, $K \subset A$ is closed in A as required. ∎

Lemma 2.3

Let $A \subset X$ be closed in X. Then a subset $K \subset A$ is closed in A if and only if it is closed in X.

Proof. Suppose $K \subset A$ is closed in A, in the subspace topology. Then by Theorem 2.2, there exists a closed subset L in X such that $K = A \cap L$. As both A and L are closed in X, so is their intersection $K = A \cap L$.

Conversely, suppose $K \subset A$ is closed in X. One writes $K = A \cap K$. Since K is closed in X, by Theorem 2.2 one concludes that $K \subset A$ is closed in A in subspace topology, as required. ∎

§2.2 Closure and Interior

Definition 2.2. Let X be a topological space and $A \subset X$ a subset. The **closure** $\overline{A}$ (or $\operatorname{Cl} A$) of A is defined to be the intersection of all the closed sets containing A. The **interior** $\operatorname{Int}(A)$ of A is the union of all the open subsets of X that are contained in A.

Remark 2.1. If $x \in \operatorname{Int}(A)$, then x belongs to the union of all open sets contained in A. Hence, by the definition of union, there exists an open set U contained in A such that $x \in U \subset A$.

Lemma 2.4 (Properties of closure and interior)

The followings hold:

(i) $\overline{A}$ is a closed subset of X, and $\operatorname{Int}(A)$ is an open subset of X.

(ii) $A \subset \overline{A}$, and $\operatorname{Int}(A) \subset A$.

Proof. (i) Since the intersection of closed sets is closed, $\overline{A}$ is closed. Similarly, since the union of open sets is open, $\operatorname{Int}(A)$ is open.

(ii) Since A is contained in any closed set containing A, $A \subset \overline{A}$. In a similar manner, since $\mathrm{Int}(A)$ is a union of open sets contained in A, $\mathrm{Int}(A) \subset A$.

■

Example 2.3. Let $X = \mathbb{R}$ with the standard topology and $A = [a, b]$ where $a, b \in \mathbb{R}$ and $a < b$. Here, $\overline{A} = [a, b]$ and $\mathrm{Int}(A) = (a, b)$.

Definition 2.3 (Dense sets)**.** A subset $A \subset X$ is **dense** if $\overline{A} = X$.

Lemma 2.5

Let X be a topological space and $A \subset X$. Then

$$X \setminus \mathrm{Cl}\, A = \mathrm{Int}\,(X \setminus A)\,.$$

Furthermore,

$$X \setminus \mathrm{Int}\, A = \mathrm{Cl}\,(X \setminus A)\,.$$

Proof. Since $\mathrm{Cl}\, A$ is the intersection of all closed sets containing A,

$$\begin{aligned} X \setminus \mathrm{Cl}\, A &= X \setminus \left(\bigcap_{\substack{K \text{ closed,} \\ A \subset K}} K \right) \\ &= \bigcup_{\substack{K \text{ closed,} \\ A \subset K}} (X \setminus K) \\ &= \bigcup_{\substack{U \text{ open,} \\ X \setminus A \supset U}} U \\ &= \mathrm{Int}\,(X \setminus A)\,; \end{aligned}$$

since K being closed is equivalent to $X \setminus K =: U$ being open, and $A \subset K$ is equivalent to $X \setminus A \supset X \setminus K$. The second statement immediately follows from the previous one, since

$$X \setminus \mathrm{Cl}\,(X \setminus A) = \mathrm{Int}\,(X \setminus (X \setminus A)) = \mathrm{Int}\, A. \tag{2.1}$$

Hence, $X \setminus \mathrm{Int}\, A = \mathrm{Cl}\,(X \setminus A)$. ■

Example 2.4. Let $X = \mathbb{R}$ with the standard topology and let $A = \mathbb{Q}$, the set of rational numbers. Given $x \in \mathbb{R}$ and $\varepsilon > 0$, $(x - \varepsilon, x + \varepsilon)$ is not contained in $\mathbb{Q}$, so $\mathrm{Int}\, \mathbb{Q} = \varnothing$. Similarly, given $x \in \mathbb{R} \setminus \mathbb{Q}$ and $\varepsilon > 0$, $(x - \varepsilon, x + \varepsilon)$ is not contained in $\mathbb{R} \setminus \mathbb{Q}$, so $\mathrm{Int}(\mathbb{R} \setminus \mathbb{Q}) = \varnothing$. Then

$$\mathbb{R} \setminus \mathrm{Cl}\, \mathbb{Q} = \mathrm{Int}(\mathbb{R} \setminus \mathbb{Q}) = \varnothing, \tag{2.2}$$

proving that $\mathrm{Cl}\, \mathbb{Q} = \mathbb{R}$.

§2.2.i Closure in subspaces

We use $\mathrm{Cl}_X\, A$ to denote the closure of A in the topology on X.

Theorem 2.6

Let X be a topological space. And $Y \subset X$ is a subspace with $A \subset Y$ a subset. Then the closure of A in Y denoted by $\mathrm{Cl}_Y(A)$, is $\mathrm{Cl}_Y(A) = Y \cap \mathrm{Cl}_X(A)$.

Proof. Let us denote the closure of A in Y by B, i.e., $B = \mathrm{Cl}_Y(A)$. First, we see that $B \subset Y \cap \mathrm{Cl}_X(A)$ holds. Indeed, $\mathrm{Cl}_X(A)$ is closed in X by Lemma 2.4. Hence, by Theorem 2.2, one has $Y \cap \mathrm{Cl}_X(A)$ to be closed in Y. Since $A \subset Y$ is given and since $A \subset \mathrm{Cl}_X(A)$ by Lemma 2.4, one has $A \subset Y \cap \overline{A} \subset Y$. Since $\mathrm{Cl}_Y(A)$ of A in Y is the smallest closed set containing A, by $A \subset Y \cap \mathrm{Cl}_X(A)$, one has $\mathrm{Cl}_Y(A) \subset Y \cap \mathrm{Cl}_X(A)$.

To prove $Y \cap \mathrm{Cl}_X(A) \subset B = \mathrm{Cl}_Y(A)$, note that B is closed in Y. Hence, by Theorem 2.2, $B = Y \cap C$, for some C closed in X. Then $A \subset \mathrm{Cl}_Y(A) \subset C$ implies that C is closed in X and contains A, i.e., $A \subset C \subset X$. Hence, by Lemma 2.4, $\mathrm{Cl}_X(A) \subset C$ and $Y \cap \mathrm{Cl}_X(A) \subset Y \cap C = B = \mathrm{Cl}_Y(A)$. ■

Example 2.5. Let $X = \mathbb{R}$, $Y = \mathbb{Q}$ and $A = \mathbb{Q} \cap [0, \pi)$ in the context of Theorem 2.6 with respect to standard topology. From Example 2.3 and 2.4, we know that $\mathrm{Cl}_X(A) = [0, \pi]$, the closure of A in X, while the closure of A in Y by Theorem 2.6 is

$$\mathrm{Cl}_Y(A) = \mathbb{Q} \cap [0, \pi] = \mathbb{Q} \cap [0, \pi) = A.$$

So $\mathrm{Cl}_Y(A) = A$ is closed in $Y = \mathbb{Q}$, but A is not closed in X.

Definition 2.4 (Neighborhood)**.** Let X be a topological space, $U \subset X$ a subset and $x \in X$. We say that U is a neighbourhood of x if $x \in U$ and U is open in X.

Theorem 2.7

Let A be a subset of a topological space X. A point $x \in X$ lies in the closure $\overline{A}$ if and only if every open set U in X that contains x intersects A. Equivalently, $x \in \overline{A}$ if and only if A intersects U for each neighbourhood U of x.

Proof. Let us take the complement $X \setminus A$ and interior $\mathrm{Int}A$ of $A \subset X$. We have $x \in \mathrm{Int}(X \setminus A)$ if and only if there exists an open set U of X with $x \in U \subset X \setminus A$. Since $X \setminus \mathrm{Cl}\, A = \mathrm{Int}\,(X \setminus A)$, the negation of $x \in \mathrm{Int}(X \setminus A)$ is $x \in \overline{A}$. (We shall use $\overline{A}$ and $\mathrm{Cl}\, A$ interchangeably.)

While the negation of $U \subset X \setminus A$ is $U \cap A \neq \varnothing$. Hence the negation of "there exists an open set U of X with $x \in U \subset X \setminus A$" is therefore "for each open set U of X with $x \in U$, one has $U \cap A \neq \varnothing$". Therefore, one has

$$x \in \overline{A} \iff x \notin \mathrm{Int}\,(X \setminus A) \iff \text{for every open } U \ni x,\ U \cap A \neq \varnothing.$$

■

Theorem 2.8 (Closure in terms of basis)

Let $\mathcal{B}$ be a basis for a topology on X. And let $A \subset X$. A point $x \in X$ lies in $\overline{A}$ if and only if A intersects each basis element $B \in \mathcal{B}$ with $x \in B$.

Proof. Suppose, $x \in \overline{A}$. Then by Theorem 2.7, for every neighbourhood U of x, one has $U \cap A \neq \varnothing$. Since every basis element $B \in \mathcal{B}$ with $x \in B$ is a neighbourhood of x, $B \cap A \neq \varnothing$.

Conversely, suppose each basis element containing x intersects A. Now, choose an open set U of X such that $x \in U$. Since $\mathcal{B}$ is a basis of X, there exists $B \in \mathcal{B}$ with $x \in B \subset U$. By hypothesis, $A \cap B \neq \varnothing$. Since $B \subset U$, one must have $U \cap A \neq \varnothing$. One, therefore, shows that for every neighbourhood U of x, $U \cap A \neq \varnothing$ holds. Then, by Theorem 2.7, $x \in \bar{A}$ as required. ■

Example 2.6. Let $B = \left\{ \frac{1}{n} \mid n \in \mathbb{Z}^+ \right\} \subset \mathbb{R}$. Then $0 \in \overline{B}$. We verify the criteria presented in Theorem 2.8. Consider any basis element (a, b) for the standard topology on $\mathbb{R}$ with $0 \in (a, b)$. One can find $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon) \subset (a, b)$. Now, choose $n \in \mathbb{Z}^+$ such that $\frac{1}{n} < \varepsilon$. For the preferred choice of $n \in \mathbb{Z}^+$, one therefore ensures that $\frac{1}{n} \in (-\varepsilon, \varepsilon) \subset (a, b)$, so that one has $B \cap (a, b) \neq \varnothing$. Hence, by Theorem 2.8, $0 \in \overline{B}$.

In fact, $\overline{B} = \{0\} \cup B$. This is indeed a closed subset of $\mathbb{R}$, since

$$\mathbb{R} \setminus \overline{B} = (-\infty, 0) \cup \left(\bigcup_{n \in \mathbb{Z}^+} \left(\frac{1}{n+1}, \frac{1}{n} \right) \right) \cup (1, \infty),$$

being a union of open intervals, is open.

§2.3 Limit Points

Definition 2.5 (Limit Points). Let A be a subset of a topological space X. A point $x \in X$ is a **limit point** of A if each neighbourhood U of x contains a point of A other than x itself. Phrased differently, x is a limit point of A if x belongs to the closure of $A \setminus \{x\}$. In other words, for every neighbourhood U of x,

$$U \cap (A \setminus \{x\}) \neq \varnothing.$$

Example 2.7. In the case of Example 2.6, $0 \in \mathbb{R}$ is a limit point of B. In this case, there are no other limit points of B.

Theorem 2.9

Let A be a subset of a topological space X, with closure $\overline{A}$ and set of limit points A'. Then $\overline{A} = A \cup A'$.

Proof. Let us prove that $A \cup A' \subset \overline{A}$. We know from Lemma 2.4 that $A \subset \overline{A}$. If $x \in A'$, i.e., if x is a limit point, then for every neighbourhood U of x, one has $U \cap (A \setminus \{x\}) \neq \varnothing$. By Theorem 2.7, $x \in \overline{A}$. Now, $A \subset \overline{A}$ and $A' \subset \overline{A}$ together imply $A \cup A' \subset \overline{A}$.

Now, prove that $\overline{A} \subset A \cup A'$. Let $x \in \overline{A}$. If $x \in A$, then of course $x \in A \cup A'$, and we are done. Otherwise, $x \notin A$, so that $A \setminus \{x\} = A$. Since $x \in \overline{A}$, by Theorem 2.7, every neighbourhood U of x intersects A, i.e., $U \cap A \neq \varnothing$. Since $A \setminus \{x\} = A$, this implies

$$U \cap (A \setminus \{x\}) \neq \varnothing.$$

So $x \in A'$. Thus, $\overline{A} \subset A \cup A'$. ∎

Corollary 2.10

A subset of a topological space is closed if and only if it contains all its limit points.

Proof. $A \subset X$ is closed if and only if $A = \overline{A} = A \cup A'$. This holds if and only if $A' \subset A$, i.e., all the limit points of A are contained in A itself. ∎

Definition 2.6 (Convergence). Let $(x_1, x_2, \ldots) = (x_n)_{n=1}^{\infty}$ be a sequence of points in a topological space X, where $x_n \in X$ for all $n \in \mathbb{N}$. We say that $(x_n)_{n=1}^{\infty}$ converges to a point $y \in X$ if for each neighborhood U of y, there exists an $N \in \mathbb{N}$ such that $x_n \in U$ for all $n > N$. In this situation, we call y a limit of the sequence $(x_n)_{n=1}^{\infty}$ and write this fact as $x_n \to y$ as $y \to \infty$.

Example 2.8. Consider the topology on the three-point set $\{a, b, c\}$ as indicated below:

$$\mathcal{T} = \{\varnothing, \{a\}, \{b, c\}, \{a, b, c\}\}.$$

If we choose the constant sequence $x_n = b$ for each $n \in \mathbb{N}$, then $x_n \to a$, $x_n \to b$, $x_n \to c$. In other words, limit of a sequence is not unique.

For example, there are two neighborhoods of a, namely $\{a, b\}$ and $\{a, b, c\}$. Both of them contain $b = x_n$. So a is a limit of the constant sequence $x_n = b$. Similarly, b and c are also limits of the sequence.

§2.4 Hausdorff Spaces

To obtain unique limits for convergent sequences, we must assume that the topology is sufficiently fine to separate the individual points. Such additional hypotheses are called separation axioms. The most common separation axiom is known as the **Hausdorff property**.

> **Definition 2.7** (Hausdorff space). A topological space X is called a **Hausdorff space** if given any two distinct points $x, y \in X$, there exist open sets U, V with $x \in U$, $y \in V$, and $U \cap V = \varnothing$.

Example 2.9. The 3-point set $X = \{a, b, c\}$ with the topology $\mathcal{T}$ provided in Example 2.8 is not Hausdorff. Take $b, c \in X$. None of the neighborhoods of b is disjoint from any other neighborhoods of c. So X with this topology is not Hausdorff.

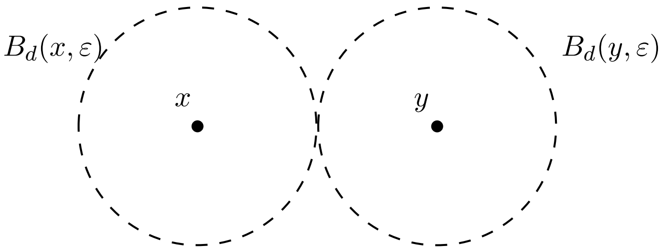
Example 2.10. Take $X = \{a, b\}$ and consider the discrete topology on X:

$$\mathcal{T}_{\text{dis}} = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}.$$

X with the topology $\mathcal{T}_{\text{dis}}$ is a Hausdorff space. The points a and b can be separated, with their respective neighborhoods $\{a\}$ and $\{b\}$.

> **Lemma 2.11**
>
> Every metric space (X, d) is Hausdorff.

Proof. Let $x, y \in X$ with $x \neq y$ so that $d(x, y) > 0$. We choose $\varepsilon = \frac{1}{2} d(x, y)$, and $U = B_d(x, \varepsilon)$ and $V = B_d(y, \varepsilon)$.



U and V are neighborhoods of x and y, respectively. It's easy to see that they are disjoint. Suppose $z \in U \cap V$. Then $z \in U$ implies $d(x, z) < \varepsilon$. $z \in V$ gives us $d(y, z) < \varepsilon$. By the triangle inequality,

$$2\varepsilon = d(x, y) \leq d(x, z) + d(z, y) < \varepsilon + \varepsilon = 2\varepsilon. \tag{2.3}$$

This is clearly a contradiction! Therefore, $U \cap V = \varnothing$, and hence X is a Hausdorff space. ∎

> **Theorem 2.12** (Uniqueness of limit in Hausdorff space)
>
> If X is a Hausdorff space, then a sequence $(x_n)_{n=1}^{\infty}$ of points in X converges to at most one point in X.

Proof. Suppose that $(x_n)_{n=1}^{\infty}$ converges to x and x' in X, where $x \neq x'$. Since X is Hausdorff, there exist neighborhoods U of x and V of x' such that $U \cap V = \varnothing$. Since (x_n) converges to x, there exists an N such that for all $n \geq N$, $x_n \in U$. Similarly, since (x_n) converges to x', there exists an N' such that for all $n \geq N'$, $x_n \in V$. Let $M = \max\{N, N'\}$. Then for all $n \geq M$, it follows that $x_n \in U \cap V$, which is a contradiction since $U \cap V = \varnothing$. Hence, we must have $x = x'$. ∎

> **Definition 2.8.** If X is a Hausdorff space, and a sequence $(x_n)_{n=1}^{\infty}$ of points in x converges to

$y \in X$, we say that y is the limit of $(x_n)_{n=1}^{\infty}$, and write

$$y = \lim_{n\to\infty} x_n.$$

Proposition 2.13

Every singleton set (a one-point set) in a Hausdorff space X is closed.

Proof. Let $x \in X$. We want to prove that $\{x\}$ is closed, i.e. $X \setminus \{x\}$ is open.

Let $y \in X \setminus \{x\}$. Since X is Hausdorff, we can find disjoint open sets U_y and V_y such that $x \in U_y$ and $y \in V_y$. No such V_y contains x. Therefore

$$X \setminus \{x\} = \bigcup_{y \in X\setminus\{x\}} V_y.$$

So $X \setminus \{x\}$ is union of open sets, hence open. So $\{x\}$ is closed. ∎

Corollary 2.14

Every finite set is closed in a Hausdorff space.

Proof. Every finite set can be expressed as union of finitely many singletons. Union of finitely many closed sets is closed. Hence, finite sets are closed. ∎

Definition 2.9 (T_1 axiom)**.** The condition that finite point sets be closed is called the T_1 **axiom**. Topological spaces obeying T_1 axiom are called T_1 **spaces**.

Corollary 2.14 tells us that every Hausdorff space is a T_1 space. However, the converse of Corollary 2.14 is not true. In other words, a T_1 space is not a Hausdorff space, in general. Here is a counterexample: let $\mathbb{N}$, the set of natural numbers, be equipped with the **finite complement topology**. Let us denote the finite complement topology on $\mathbb{N}$ by $\mathcal{T}_f$, i.e.,

$$\mathcal{T}_f = \{U \mid U \subset \mathbb{N} \text{ with } \mathbb{N} \setminus U \text{ is finite or the whole of } \mathbb{N}\}.$$

Choose 2 neighbourhoods U and V of m and n in $\mathbb{N}$, respectively, in finite complement topology such that $m \neq n$. Then, observe that $U \cap V \neq \varnothing$.

Indeed, since $U \in \mathcal{T}_f$, $\mathbb{N} \setminus U$ is a finite set as U is by definition nonempty. Also, since $V \in \mathcal{T}_f$, $\mathbb{N} \setminus V$ is a finite set as V is also by definition nonempty. Now, $(\mathbb{N} \setminus U) \cup (\mathbb{N} \setminus V)$ is also a finite set as the union of two finite sets.

$$(\mathbb{N} \setminus U) \cup (\mathbb{N} \setminus V) = \mathbb{N} \setminus (U \cap V).$$

$\mathbb{N} \setminus (U \cap V)$ is a finite set, so $U \cap V$ is an infinite set. In particular, $U \cap V$ is nonempty. This proves that $\mathbb{N}$ equipped with finite complement topology is not Hausdorff.

Theorem 2.15

Let A be a subset of a T_1 space X. A point $x \in X$ is a limit point of A if and only if each neighborhood U of x intersects A at infinitely many points.

Proof. Suppose that $U \cap A$ consists of infinitely many points. Hence it certainly contains other points than x itself, so that $U \cap (A \setminus \{x\}) \neq \varnothing$, proving that x is a limit point of A.

Conversely, suppose $U \cap A$ is finite. Then

$$U \cap (A \setminus \{x\}) = \{x_1, \dots, x_n\}$$

is closed as X is a T_1 space and hence every finite set is closed in X.

Therefore, $V = U \setminus \{x_1, \dots, x_n\} = U \cap (X \setminus \{x_1, \dots, x_n\})$ is open in X. Since $x \in U$ (as U is a neighborhood of x in X) and $x \notin \{x_1, \dots, x_n\}$, one must have

$$x \in U \cap (X \setminus \{x_1, \dots, x_n\}) = V.$$

Now, V is, by definition, a subset of U that doesn't contain any of the elements in $\{x_1, \dots, x_n\}$.

$$V \cap (A \setminus \{x\}) \subset U \cap (A \setminus \{x\}) = \{x_1, \dots, x_n\}.$$

Since V doesn't contain any of the elements in $\{x_1, \dots, x_n\}$, one must have

$$V \cap (A \setminus \{x\}) = \varnothing,$$

with V open in X, implying that x is not a limit point of A. Hence, we have proved that $U \cap A$ is finite implies that x is not a limit point of A. A contrapositive of the above statement is what we require to hold. ∎

Proposition 2.16

The product of two Hausdorff spaces X and Y is Hausdorff.

Proof. Given two distinct points $(x_1, y_1), (x_2, y_2)$ in $X \times Y$, without loss of generality we may assume that $x_1 \neq x_2$. Since X is Hausdorff, there exist disjoint open sets U_1, U_2 in X such that $x_1 \in U_1$ and $x_2 \in U_2$. Then $U_1 \times Y$ and $U_2 \times Y$ are disjoint neighborhoods of (x_1, y_1) and (x_2, y_2), so $X \times Y$ is Hausdorff. ∎

3 Continuity

§3.1 Definitions

Definition 3.1 (Continuous Maps). Let $f : X \to Y$ be a map of topological spaces. f is said to be continuous if for each open subset V of Y, the set $f^{-1}(V)$ is an open subset of X.

Proposition 3.1

$f : X \to Y$ is continuous if and only if for every closed subset C of Y, the set $f^{-1}(C)$ will be closed in X.

Proof. ($\Rightarrow$) Suppose f is continuous. C is closed, so $Y \setminus C$ is open in Y. Therefore, by the continuity of f, $f^{-1}(Y \setminus C) = X \setminus f^{-1}(C)$ is open in X, so $f^{-1}(C)$ is closed.

($\Leftarrow$) Suppose $f^{-1}(C)$ is closed in X for any closed $C \subset Y$. Take any open set U in Y. Choose $C = Y \setminus U$. Then by the assumption $f^{-1}(Y \setminus U) = X \setminus f^{-1}(U)$ is closed in X. This gives us $f^{-1}(U)$ is open. So f is continuous. ∎

Lemma 3.2

Let X, Y, and Z be topological spaces. If $f : X \to Y$ and $g : Y \to Z$ are continuous functions, then the composite $g \circ f : X \to Z$ is continuous.

Proof. Let $W \subset Z$ be open. Then $g^{-1}(W) \subset Y$ is open as g is a continuous function. Also, $f^{-1}(g^{-1}(W)) \subset X$ is open as f is a continuous function. But from elementary set theory, one has

$$(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W)).$$

Hence, one has $(g \circ f)^{-1}(W) \subset X$ is open, proving that $g \circ f : X \to Z$ is a continuous function. ∎

Lemma 3.3

Let X and Y be topological spaces, and $\mathcal{B}$ is a basis for the topology on Y. Then a function $f : X \to Y$ is continuous if and only if for each basis element $B \in \mathcal{B}$, the preimage $f^{-1}(B)$ is open in X.

Proof. Each basis element $B \in \mathcal{B}$ is open in Y. Hence, continuity of $f : X \to Y$ implies that $f^{-1}(B)$ is open in X. Conversely, suppose that $f^{-1}(B)$ is open in X for each basis element $B \in \mathcal{B}$. Now, pick an open set $V \subset Y$. Since $\mathcal{B}$ is a basis for the topology in Y, V can be expressed as a union of basis elements, i.e.,

$$V = \bigcup_{\alpha \in J} B_\alpha$$

for $\{B_\alpha\}_{\alpha \in J}$ being a collection of basis elements of Y indexed by J. One therefore has

$$f^{-1}(V) = f^{-1}\left(\bigcup_{\alpha \in J} B_\alpha\right) = \bigcup_{\alpha \in J} f^{-1}(B_\alpha).$$

Since each $B_\alpha \in \mathcal{B}$, $f^{-1}(B_\alpha)$ is open in X by hypothesis. Therefore, $f^{-1}(V)$ is also open in X, proving that $f : X \to Y$ is continuous. ∎

Lemma 3.4

Let X and Y be topological spaces, and suppose that $\mathcal{S}$ is a subbasis for the topology on Y. Then a function $f : X \to Y$ is continuous if and only if for each sub-basis element $S \in \mathcal{S}$, the preimage $f^{-1}(S)$ is open in X.

Proof. We form the basis $\mathcal{B}$ from the subbasis $\mathcal{S}$ by taking all possible finite intersections of elements from $\mathcal{S}$ that generate the topology of Y. Now, each subbasis element $S \in \mathcal{S}$ is a basis element belonging to $\mathcal{B}$ by construction, and hence by Lemma 3.3, $f^{-1}(S)$ is open in X if $f : X \to Y$ is a continuous function.

Conversely, by construction, each basis element $B \in \mathcal{B}$ is a finite intersection $B = S_1 \cap \cdots \cap S_n$, for $S_1, \ldots, S_n \in \mathcal{S}$. Then,

$$f^{-1}(B) = f^{-1}(S_1 \cap \cdots \cap S_n) = f^{-1}(S_1) \cap \cdots \cap f^{-1}(S_n).$$

Now, by hypothesis, each element of the finite collection $\{f^{-1}(S_1), \ldots, f^{-1}(S_n)\}$ is open in X as $S_1, \ldots, S_n$ belong to $\mathcal{S}$. Hence,

$$f^{-1}(S_1) \cap \cdots \cap f^{-1}(S_n)$$

is open in X. We have thus shown that for any basis element $B \in \mathcal{B}$ generating the topology of Y, $f^{-1}(B)$ is open in X. Hence, by Lemma 3.3, $f : X \to Y$ is a continuous function. ■

Example 3.1 (Equivalence of topological and calculus definition of continuity)**.** The calculus definition of continuity states that

> Given a function $f : D \to \mathbb{R}$ (with $D \subset \mathbb{R}$ called the domain of f) is said to be continuous at $x_0 \in D$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for each $x \in D$, $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \varepsilon$. If $f : D \to \mathbb{R}$ is continuous at all points of its domain D, then f is called **everywhere continuous** or simply **continuous**.

Then we have the following result

> $f : \mathbb{R} \to \mathbb{R}$ is everywhere continuous if and only if $f^{-1}(U)$ is open whenever U is open. (Here, both the domain and codomain $\mathbb{R}$ of f is taken to be in standard topology.)

($\Rightarrow$) Suppose $f : \mathbb{R} \to \mathbb{R}$ is everywhere continuous as defined in single variable calculus. Also, suppose U is open in $\mathbb{R}$. We have to show that $f^{-1}(U)$ is open in $\mathbb{R}$. Let us pick an arbitrary element $x_0 \in f^{-1}(U) = \{x \in \mathbb{R} \mid f(x) \in U\}$. Now, this tells us that $f(x_0) \in U$. Since U is open set in $\mathbb{R}$ with respect to standard topology which is generated by the basis of all open intervals in $\mathbb{R}$. Hence, there exists $\varepsilon > 0$ such that $B_\varepsilon(f(x_0)) \subset U$ which is the same as $(f(x_0) - \varepsilon, f(x_0) + \varepsilon) \subset U$. Now, find $\delta > 0$ using the continuity of $f : \mathbb{R} \to \mathbb{R}$ at $x_0 \in \mathbb{R}$. Since the calculus definition of continuity holds by hypothesis, there indeed there exists $\delta > 0$ such that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \varepsilon$.

Now, the claim is that $B_\varepsilon(x_0, \delta) \subset f^{-1}(U)$. Suppose that $x \in B_\varepsilon(x_0, \delta) = (x_0 - \delta, x_0 + \delta)$ then $|x - x_0| < \delta$. But from calculus definition of continuity, this leads to

$$|f(x) - f(x_0)| < \varepsilon \implies f(x) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon) = B_\varepsilon(f(x_0)) \subset U,$$

so that one obtains $f(x) \in U$, i.e. $x \in f^{-1}(U)$. This proves the claim that $B_\varepsilon(x_0, \delta) \subset f^{-1}(U)$ with $\delta > 0$ guaranteed to exist by the calculus definition of continuity. This proves that $f^{-1}(U)$ is an open set in $\mathbb{R}$.

($\Leftarrow$) Suppose $f^{-1}(U)$ is open when U is open in $\mathbb{R}$. Suppose we are also given $x_0 \in \mathbb{R}$ and $\varepsilon > 0$. Now, let $U = B_{\text{Euc}}(f(x_0), \varepsilon) = (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$, which is open in $\mathbb{R}$. By hypothesis, $f^{-1}(U)$ is open in $\mathbb{R}$.

Now, $f(x_0) \in U$, so $x_0 \in f^{-1}(U)$. But $f^{-1}(U)$ is open in $\mathbb{R}$. Hence, there exists $\delta > 0$ such that $B_{\text{Euc}}(x_0, \delta) \subset f^{-1}(U)$. Now, we are going to show that this $\delta > 0$ corresponds to the $\varepsilon > 0$ in the calculus definition of continuity at x_0.

For this, suppose $|x - x_0| < \delta$. This implies

$$x \in (x_0 - \delta, x_0 + \delta) = B_{\text{Euc}}(x_0, \delta) \subset f^{-1}(U),$$

i.e., $x \in f^{-1}(U)$. Thus,

$$f(x) \in U = B_{\text{Euc}}(f(x_0), \varepsilon) = (f(x_0) - \varepsilon, f(x_0) + \varepsilon),$$

which implies $|f(x) - f(x_0)| < \varepsilon$.

Example 3.2. Let $\mathbb{R}$ and $\mathbb{R}_f$ be the real numbers with the standard topology and the finite complement topology, respectively. The identity function $\text{id} : \mathbb{R}_f \to \mathbb{R}$, given by $\text{id}(x) = x$, is not continuous since $\text{id}^{-1}((a, b)) = (a, b)$ is not open in the finite complement topology, although (a, b) is open in the standard metric topology. Indeed, (a, b) doesn't belong to the finite complement topology on $\mathbb{R}$ as $\mathbb{R} \setminus (a, b)$ is not finite.

However, the identity function $\text{id}^{-1} : \mathbb{R} \to \mathbb{R}_f$ is continuous, since $\mathbb{R} \setminus F$, with F being a finite subset of $\mathbb{R}$ (a generic open set from the set of real numbers in the finite complement topology), is open in $\mathbb{R}$ in the standard topology.

> **Definition 3.2** (Continuity at a point). We say that $f : X \to Y$ is continuous at $x \in X$ if for each neighbourhood V of $f(x)$, there exists a neighbourhood U of x with $f(U) \subset V$.

> **Theorem 3.5**
>
> Let X and Y be topological spaces, and $f : X \to Y$ a function. Then f is continuous if and only if for each $x \in X$ and each neighbourhood V of $f(x)$, there exists a neighbourhood U of x such that $f(U) \subset V$. In other words, $f : X \to Y$ is contimous if and only if it is continuous at each $x \in X$.

Proof. Suppose $f : X \to Y$ is continuous. Given $x \in X$, $f(x) \in Y$. Suppose also that V is a neighbourhood of $f(x)$ in Y. Then by the continuity of $f : X \to Y$, $f^{-1}(V)$ is a neighbourhood of x in X with $f(f^{-1}(V)) \subset V$. If we write the neighbourhood $f^{-1}(V)$ of x in X by U, we have $f(U) \subset V$.

Conversely, suppose $V \subset Y$ is open. We need to show that $f^{-1}(V)$ is open in X. Choose $x \in f^{-1}(V)$. Since $f(f^{-1}(V)) \subset V$, one has $f(x) \in V$. Hence, V is a neighbourhood of $f(x)$ in Y. Now, by hypothesis, there exists a neighbourhood U_x of x in X such that $f(U_x) \subset V$. But $f(U_x) \subset V$ implies $U_x \subset f^{-1}(V)$. Hence, one has $x \in U_x \subset f^{-1}(V)$.

For every $x \in f^{-1}(V)$, there is an open set U_x such that $x \in U_x \subset f^{-1}(V)$. Therefore, by Proposition 1.6, $f^{-1}(V)$ is open, as required. ∎

> **Theorem 3.6**
>
> Let X and Y be topological spaces. And $f : X \to Y$ a function. The following are equivalent:
>
> 1. f is continuous.
> 2. For every subset $A \subset X$, one has $f(\overline{A}) \subset \overline{f(A)}$.
> 3. For every closed set $K \subset Y$, the preimage $f^{-1}(K)$ is closed in X.

Proof. We have already established the equivalece of 1 and 3 in Proposition 3.1. We shall now prove that 1 implies 2, and 2 implies 3.

(1 ⇒ 2): Suppose $f : X \to Y$ is continuous and $A \subset X$ be a subset. Let $f(x) \in f(\overline{A})$ with $x \in \overline{A}$. We want to show that $f(x) \in \overline{f(A)}$. Let V be a neighbourhood of $f(x)$ in Y. By continuity of f, $f^{-1}(V)$ is open in X. Also, since V is a neighbourhood of $f(x)$, one must have $x \in f^{-1}(V)$. Hence, $f^{-1}(V)$ is a neighbourhood of x in X. Now, since $x \in \overline{A}$, $A \cap f^{-1}(V) \neq \varnothing$ by Theorem 2.7. Choose $y \in A \cap f^{-1}(V)$. Then

$$f(y) \in f(A \cap f^{-1}(V)) \subset f(A) \cap f(f^{-1}(V)) \subset f(A) \cap V,$$

implies that $f(A) \cap V$ is nonempty. Therefore, we have shown that an arbitrary neighbourhood V of $f(x)$ in Y intersects $f(A)$. Then by Theorem 2.7, one concludes that $f(x) \in \overline{f(A)}$, proving that $f(\overline{A}) \subset \overline{f(A)}$.

(2 ⇒ 3): Let $K \subset Y$ be closed. Also, let us denote $f^{-1}(K)$ by A so that $A \subset X$. We will show that $A = \overline{A}$. This will then prove that $A = f^{-1}(K)$ is closed in X. We know from elementary set theory that $f(f^{-1}(K)) \subset K$, i.e., $f(A) \subset K$. Since K is closed, by Lemma 2.4, one obtains $\overline{f(A)} \subset K$. But by hypothesis, $f(\overline{A}) \subset \overline{f(A)} \subset K$. Hence, $f(\overline{A}) \subset K$. Since preimage preserves inclusion from elementary set theory, $f^{-1}(f(\overline{A})) \subset f^{-1}(K)$. But we also know from set theory that $\overline{A} \subset f^{-1}(f(\overline{A}))$ so that one has $\overline{A} \subset f^{-1}(K) = A$. And $A \subset \overline{A}$ by Lemma 2.4, so that one concludes $A = \overline{A}$, as required. ■

§3.2 Homeomorphism

Definition 3.3 (Homeomorphism)**.** A bijective function $f : X \to Y$ between two topological spaces X and Y with the property that both f and $f^{-1} : Y \to X$ are continuous, is called a **homeomorphism**. If there exists a homeomorphism $f : X \to Y$, we say that X and Y are homeomorphic, written as $X \cong Y$.

Definition 3.4 (Imbedding)**.** Suppose that $f : X \to Y$ is an injective continuous map, where X and Y are given topological spaces. Let also that $Z = f(X)$, the image set of f, considered as a subspace of Y. Then the function $f : X \to Z$ obtained by taking the codomain of f to be the range of f. This is immediate that f is bijective. If f happens to be a homeomorphism between X and Z, we say that the map $f : X \to Y$ is a **topological imbedding**, or simply an **imbedding**, of X in Y.

Example 3.3. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 3x + 1$.

$$y = 3x + 1 \implies x = \frac{1}{3}(y - 1).$$

The inverse function $f^{-1} : \mathbb{R} \to \mathbb{R}$ is given by

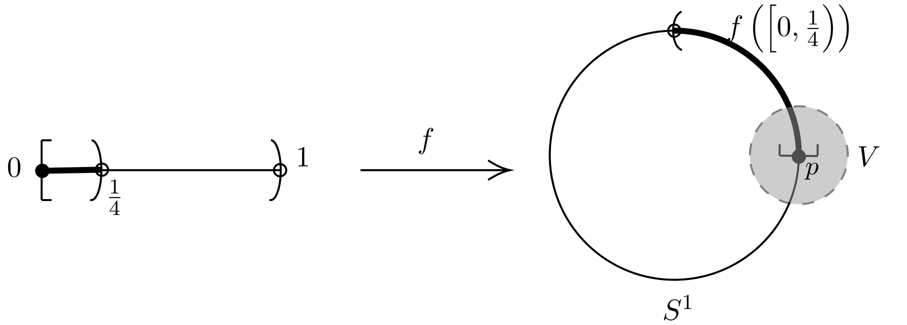
$$f^{-1}(y) = \frac{1}{3}(y - 1).$$

Both f and f^{-1} continuous, and hence f is a homeomorphism.

Example 3.4. Let S^1 denote the unit circle in $\mathbb{R}^2$; that is $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$, considered as a **subspace**[1] of the space $\mathbb{R}^2$. Let $f : [0, 1) \to S^1$ be the

$$f(t) = (\cos 2\pi t, \sin 2\pi t)$$

It is left as an exercise for the reader to show that f is a continuous bijective function. But the function f^{-1} is not continuous.



[1]Subset of $\mathbb{R}^2$ equipped with subspace topology.

$U = \left[0, \frac{1}{4}\right)$ is an open set in $[0, 1)$ according to the subspace topology. We want to show that $f(U)$ is not open in S^1. That would prove the discontinuity of f^{-1}.

Let p be the point $f(0)$. And $p \in f(U)$. We need to find an open set of S^1 in subspace topology containing $p = f(0)$ and contained in $f(U)$ to show that $f(U)$ is open in S^1, i.e we have to find an open set in V of $\mathbb{R}^2$ such that $f(0) = p \in V \cap S^1 \subset f(U)$. But it is impossible as is evident from the figure above. No matter what V we choose, some part of $V \cap S^1$ would lie outside $f(U)$.

Theorem 3.7 (Rules for constructing continuous functions)
Let X, Y, and Z be topological spaces.

(a) (Constant function) If $f.X \to Y$ maps all of X into the single point y_0 of Y, then f is continuous.

(b) (Inclusion) If A is a subspace of X, the inclusion function $\iota : A \to X$ is continuous.

(c) (Restricting the domain) If $f : X \to Y$ is continuous, and if A is a subspace of X, then the restricted function $f\big|_A : A \to Y$ is continuous.

(d) (Restricting or expanding the range) Let $f \cdot X \to Y$ be continuous. If Z is a subspace of Y containing the image set $f(X)$, then the function $g : X \to Z$ obtained by restricting the range of f is continuous. If Z is a space having Y as a subspace, then the function $h : X \to Z$ obtained by expanding the range of f is contınuous.

(e) (Local formulation of continuity) The map $f : X \to Y$ is continuous if X can be written as the union of open sets U_α such that $f\big|_{U_\alpha}$ is continuous for each α.

Proof. (a) Let $f(x) = y_0$ for every $x \in X$. Let $V \subset Y$ be open. Then we have

$$f^{-1}(V) = \begin{cases} X & \text{if } y_0 \in V, \\ \varnothing & \text{if } y_0 \notin V. \end{cases} \tag{3.1}$$

In either case, $f^{-1}(V)$ is open. So f is continuous.

(b) $\iota : A \to X$ is defined as $\iota(a) = a$ for $a \in A \subset X$. Then for U open in X,

$$\iota^{-1}(U) = \{x \in A \mid \iota(x) \in U\} = \{x \in A \mid x \in U\} = A \cap U, \tag{3.2}$$

which is open in the subspace topology on A. So ι is continuous.

(c) Let $\iota : A \to X$ be the inclusion map. The function $f\big|_A$ is the composition of f and ι, i.e.

$$f \circ \iota = f\big|_A : A \to Y, \tag{3.3}$$

so $f\big|_A$ is continuous as a composition of two continuous maps.

(d) Let $f : X \to Y$ be continuous. Also, suppose that $f(X) \subset Z \subset Y$.

We now show that the function $g : X \to Z$ obtained by restricting the codomain is continuous. Let $B \subset Z$ be open. Then by the definition of subspace topology,

$$B = Z \cap U \tag{3.4}$$

for some open $U \subset Y$. Now, from $f(X) \subset Z$, one has

$$f^{-1}(f(X)) \subset f^{-1}(Z) \tag{3.5}$$

But $X \subset f^{-1}(f(X))$ from elementary set theory leading to

$$X \subset f^{-1}(Z). \tag{3.6}$$

Similarly, from $Z \subset Y$, one has

$$f^{-1}(Z) \subset f^{-1}(Y) = X. \tag{3.7}$$

(3.6) and (3.7) together imply that

$$X = f^{-1}(Z). \tag{3.8}$$

Now, notice that by construction of the function g, one has $f(x) = g(x)$ for every $x \in X$. And,

$$g^{-1}(B) = \{x \in X \mid g(x) \in B\} = \{x \in X \mid f(x) \in B\} = f^{-1}(B). \tag{3.9}$$

So that from (3.4), one obtains $f^{-1}(B) = f^{-1}(Z \cap U) = f^{-1}(Z) \cap f^{-1}(U)$.

$$f^{-1}(B) = f^{-1}(Z) \cap f^{-1}(U) = X \cap f^{-1}(U) = f^{-1}(U), \tag{3.10}$$

since $f^{-1}(U) \subset X$. One, therefore, obtains that

$$g^{-1}(B) = f^{-1}(U). \tag{3.11}$$

Since $f : X \to Y$ is continuous and $U \subset Y$ is open, $f^{-1}(U)$ must be open in X. And, hence, $g^{-1}(B)$ is open in X. This way, we see that for every open $B \subset Z$, one finds $g^{-1}(B)$ open in X, which proves that $g : X \to Z$ is continuous.

(e) By hypothesis, one can write $X = \bigcup_{\alpha \in J} U_\alpha$ such that $f|_{U_\alpha} : U_\alpha \to Y$ is continuous for each $\alpha \in J$. Let $V \subset Y$ be open. Then

$$\left(f|_{U_\alpha}\right)^{-1}(V) = \left\{x \in U_\alpha \mid f|_{U_\alpha}(x) \in V\right\} = \{x \in U_\alpha \mid f(x) \in V\} = U_\alpha \cap f^{-1}(V). \tag{3.12}$$

Since $f|_{U_\alpha}$ is continuous, $f^{-1}(V) \cap U_\alpha$ must be open in U_α. Now, $f^{-1}(V) \cap U_\alpha \subset U_\alpha$ open and $U_\alpha \subset X$ open together imply that $f^{-1}(V) \cap U_\alpha \subset X$ open, by Lemma 1.18. Now,

$$\bigcup_{\alpha \in J} (f^{-1}(V) \cap U_\alpha) = f^{-1}(V) \cap \left(\bigcup_{\alpha \in J} U_\alpha\right) = f^{-1}(V) \cap X = f^{-1}(V). \tag{3.13}$$

Hence, $f^{-1}(V)$, being a union of open sets of X, is also open in X, proving that $f : X \to Y$ is continuous.

■

Lemma 3.8 (Pasting Lemma)

Let $X = A \cup B$, where A and B are closed in X. Let $f : A \to Y$ and $g : B \to Y$ be continuous. If $f(x) = g(x)$ for every $x \in A \cap B$, then f and g combine to give a continuous function $h : X \to Y$ defined by

$$h(x) = \begin{cases} f(x) & \text{if } x \in A, \\ g(x) & \text{if } x \in B. \end{cases}$$

Proof. Let C be a closed subset of Y. Now,

$$\begin{aligned} h^{-1}(C) &= \{x \in X \mid h(x) \in C\} \\ &= \{x \in A \cup B \mid h(x) \in C\} \\ &= \{x \in A \mid h(x) \in C\} \cup \{x \in B \mid h(x) \in C\} \\ &= \{x \in A \mid f(x) \in C\} \cup \{x \in B \mid g(x) \in C\} \\ &= f^{-1}(C) \cup g^{-1}(C). \end{aligned}$$

So we have

$$h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C). \tag{3.14}$$

Since f is continuous, $f^{-1}(C)$ is closed in A, hence closed in X. Similarly, $g^{-1}(C)$ is closed in X. So $h^{-1}(C)$ is the union of two closed sets in X, hence it is closed in X. Therefore, h is continuous. ■

Example 3.5. Let us define the function $h : \mathbb{R} \to \mathbb{R}$ by setting

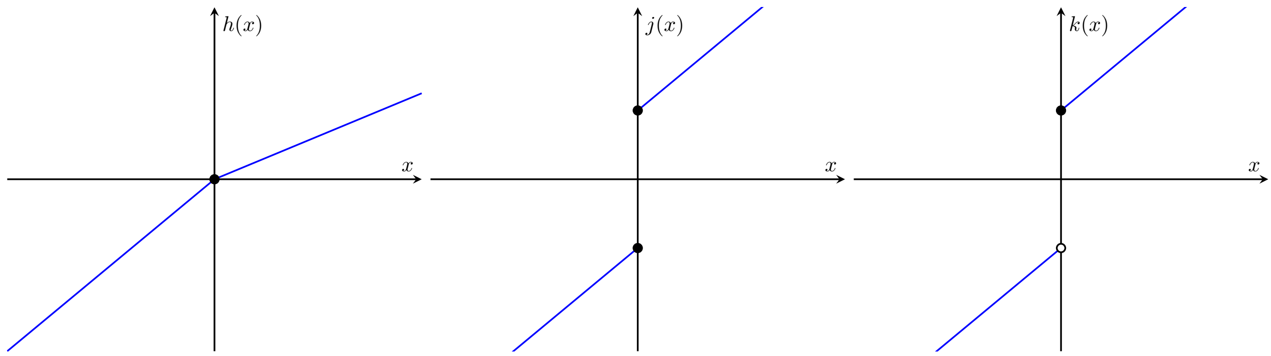
$$h(x) = \begin{cases} x & \text{for } x \leq 0, \\ x/2 & \text{for } x > 0. \end{cases} \tag{3.15}$$

Here, each of the pieces, namely x and $x/2$, are both continuous functions on $x \leq 0$, and $x > 0$, respectively. These two pieces agree with each other on the overlapping part of their domains, which is the one-point set $\{0\}$. The domains of the two pieces $(-\infty, 0]$ and $[0, \infty)$ are closed subsets of $\mathbb{R}$. Hence, the function $h : \mathbb{R} \to \mathbb{R}$ defined piecewise by (3.15) is continuous by Pasting Lemma.

Now, the equation

$$j(x) = \begin{cases} x - 2 & \text{for } x \leq 0 \\ x + 2 & \text{for } x \geq 0 \end{cases} \tag{3.16}$$

do not define a function as the two pieces here do not agree on the overlapping part of their respective domains, namely on the one-point set $\{0\}$. Hence, the pasting lemma doesn't apply here.



The equations given by

$$k(x) = \begin{cases} x - 2 & \text{for } x < 0 \\ x + 2 & \text{for } x \geq 0 \end{cases} \tag{3.17}$$

indeed give a function from $\mathbb{R}$ to itself. For the two pieces involved here there is no nontrivial intersection of their respective domains. But the domains of the two pieces are not both closed subsets of $\mathbb{R}$. Hence the pasting lemma doesn't apply. Indeed, the function $k : \mathbb{R} \to \mathbb{R}$ is not continuous, although the two pieces involved are both continuous on their respective domains. In this case, although $(1, 3)$ is open in $\mathbb{R}$, $k^{-1}((1, 3)) = [0, 1)$ is not open, proving that $k : \mathbb{R} \to \mathbb{R}$ is not continuous.

Example 3.6. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces defined as follows:

$$X = \{R, G, B\}, \quad \mathcal{T}_X = \{\varnothing, X, \{R\}, \{B\}, \{R, G\}, \{R, B\}\}.$$

$$Y = \{1, 2, 3\}, \quad \mathcal{T}_Y = \{\varnothing, \{1\}, \{1, 2\}, Y\}.$$

Let $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ and $g : (Y, \mathcal{T}_Y) \to (X, \mathcal{T}_X)$ be defined by

$$f(R) = 1, \quad f(G) = 2, \quad f(B) = 3$$

$$g(1) = R, \quad g(2) = G, \quad g(3) = B$$

Observe that $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is continuous:

$$f^{-1}(\{0\}) = \varnothing, \quad f^{-1}(\{1\}) = \{R\}, \quad f^{-1}(\{1, 2\}) = \{R, G\}, \quad f^{-1}(Y) = X$$

Now, find that $g : (Y, \mathcal{T}_Y) \to (X, \mathcal{T}_X)$ is also not continuous as $g^{-1}(\{B\}) = \{3\} \notin \mathcal{T}_Y$.

4 Product Topology Revisited

§4.1 Maps Into Products

Let X and Y be topological spaces; we give $X \times Y$ the product topology. Recall that $\pi_1 : X \times Y \to X$ defined by $\pi_1(x, y) = x$ is called the projection onto the first component. Also, $\pi_2 : X \times Y \to Y$ defined by $\pi_2(x, y) = y$ is called the projection onto the second component.

Lemma 4.1

$\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are continuous.

Proof. For a given open set $U \subset X$, $\pi_1^{-1}(U) = U \times Y$ is in the basis $\mathcal{B}$ that generates the product topology on $X \times Y$. In particular, $\pi_1^{-1}(U)$ is open in $X \times Y$, proving that $\pi_1 : X \times Y \to X$ is continuous.

Now, let $V \subset Y$ be open. Then $\pi_2^{-1}(V) = X \times V$ also belongs to the basis $\mathcal{B}$ generating the product topology on $X \times Y$. Hence, $\pi_2^{-1}(V)$ is also open in $X \times Y$, proving that $\pi_2 : X \times Y \to Y$ is continuous. ■

Theorem 4.2 (Maps into products)

Let W be a topological space. A function $f : W \to X \times Y$ is continuous if and only if both of its components $f_1 = \pi_1 \circ f : W \to X$ and $f_2 = \pi_2 \circ f : W \to Y$ are continuous.

Proof. ($\Rightarrow$) It is immediate by the use of the fact that composition of continuous functions is continuous and Lemma 4.1.

($\Leftarrow$) Let $U \subset X$ be open. One then has

$$f_1^{-1}(U) = (\pi_1 \circ f)^{-1}(U) = f^{-1}(\pi_1^{-1}(U)) = f^{-1}(U \times Y). \tag{4.1}$$

On the other hand, let $V \subset Y$ be open. One then has

$$f_2^{-1}(V) = (\pi_2 \circ f)^{-1}(V) = f^{-1}(\pi_2^{-1}(V)) = f^{-1}(X \times V). \tag{4.2}$$

Since, by hypothesis, both f_1 and f_2 are continuous, given $U \subset X$ open, both $f_1^{-1}(U)$ and $f_2^{-1}(V)$ are open in W. And hence their intersection $f_1^{-1}(U) \cap f_2^{-1}(V)$ is also open in W. But

$$\begin{aligned} f_1^{-1}(U) \cap f_2^{-1}(V) &= f^{-1}(U \times Y) \cap f^{-1}(X \times V) \\ &= f^{-1}((U \times Y) \cap (X \times V)) \\ &= f^{-1}(U \times V). \end{aligned}$$

Note that the collection $\mathcal{B} = \{U \times V \mid U \text{ open in } X \text{ and } V \text{ open in } Y\}$ generates the product topology on $X \times Y$. In other words, any open set in $X \times Y$ can be written as a union of open sets in $X \times Y$ of the form $U \times V$. Inverse image of a basic open set $U \times V \in \mathcal{B}$, under f, is open. So, by Lemma 3.3 $f : W \to X \times Y$ is continuous. ■

Example 4.1. Let $W = (a, b) \subset \mathbb{R}$ and $X = Y = \mathbb{R}$. A function $f : (a, b) \to \mathbb{R} \times \mathbb{R}$ is written as $f(t) = (f_1(t), f_2(t))$. Then f is continuous if and only if both the component functions f_1 and f_2 are continuous by Theorem 4.2.

In the circle example (Example 3.4), $f : [0, 1) \to \mathbb{R}^2$ was given by $f(t) = (\cos 2\pi t, \sin 2\pi t)$ where $f([0, 1)) = S^1 \subset \mathbb{R}^2$. Here, both $f_1(t)$ and $f_2(t)$ are continuous functions on $[0, 1)$. Hence by Theorem 4.2, $f : [0, 1] \to \mathbb{R}^2$ is continuous.

Lemma 4.3

The product topology on $X \times Y$ is the coarsest topology for which both the projection maps $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are continuous.

Proof. Both the projection maps $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are continuous for the product topology on $X \times Y$ were proven in Lemma 4.1.

Now, suppose that the projection maps π_1 and π_2 are both continuous with respect to a topology $\mathcal{T}$ on $X \times Y$. Now, let $U \subset X$ be open and $V \subset Y$ be open. From the continuity of the projection maps, $\pi_1 : X \times Y \to X$ with respect to the topology $\mathcal{T}$ implies that $\pi_1^{-1}(U) \in \mathcal{T}$. Also, from the continuity of $\pi_2 : X \times Y \to Y$ with respect to the topology $\mathcal{T}$, one has $X \times V \in \mathcal{T}$. But we have

$$(U \times Y) \cap (X \times V) = U \times V, \tag{4.3}$$

implying that $U \times V \in \mathcal{T}$ for all U open in X and V open in Y. Any open set in the product topology can be written as a union of basic elements of the form $U \times V$. All the basic open sets of the product topology are in the topology $\mathcal{T}$ on $X \times Y$. So $\mathcal{T}$ is finer than the product topology. So we have that any topology on $X \times Y$, with respect to which π_1 and π_2 are continuous, is finer than the product topology. So product topology is the coarsest such that π_1 and π_2 are continuous. ■

§4.2 Product Topology

Let J be an index set. Also, let X be a set. By a J-tuple of elements of X, we mean a function $\mathbf{x} : J \to X$. If $\alpha \in J$, we denote its image under $\mathbf{x}$ simply by x_α instead of $\mathbf{x}(\alpha)$. It's called the α-th coordinate of $\mathbf{x}$. We also often denote the function $\mathbf{x}$ by the symbol $(x_\alpha)_{\alpha\in J}$. The set of all J-tuples of elements of X is denoted by X^J.

Definition 4.1. Let $\{X_\alpha\}_{\alpha\in J}$ be a family of sets indexed by the set J. Take the union $X = \bigcup_{\alpha\in J} X_\alpha$. The Cartesian product of this indexed family, denoted by

$$\prod_{\alpha\in J} X_\alpha,$$

is defined as the set of all J-tuples $(x_\alpha)_{\alpha\in J}$ of elements of X such that $x_\alpha \in X_\alpha$ for all $\alpha \in J$. In other words, $\prod_{\alpha\in J} X_\alpha$ is the set of all functions $\mathbf{x} : J \to \bigcup_{\alpha\in J} X_\alpha = X$ such that $\mathbf{x}(\alpha) \in X_\alpha$ for all $\alpha \in J$.

If all sets X_α are equal to some set Y, then the Cartesian product $\prod_{\alpha\in J} X_\alpha$ is simply the set Y^J of all J-tuples of elements of Y.

Now we add structure to the constituent sets X_α.

Definition 4.2. Suppose $\{X_\alpha\}_{\alpha\in J}$ is a family of topological spaces indexed by J. We introduce a topology on the Cartesian product $\prod_{\alpha\in J} X_\alpha$ in the following way:

Observe that the collection of all sets of the form $\prod_{\alpha\in J} U_\alpha$, with $U_\alpha \subset X_\alpha$ open for all $\alpha \in J$, is a basis for $\prod_{\alpha\in J} X_\alpha$. The topology generated by this basis is called the **box topology**.

Definition 4.3 (Projection map)**.** For each $\beta \in J$, there is a projection function

$$\pi_\beta : \prod_{\alpha\in J} X_\alpha \to X_\beta, \quad (x_\alpha)_{\alpha\in J} \mapsto x_\beta.$$

i.e., $\pi_\beta((x_\alpha)_{\alpha\in J}) = x_\beta$.

Now, suppose $\{X_\alpha\}_{\alpha\in J}$ is a family of topological spaces indexed by J. We wish to equip $\prod_{\alpha\in J} X_\alpha$ with the coarsest topology $\mathcal{T}_{\text{prod}}$ such that each projection π_β is continuous. In other words, for each open set $U_\beta \subset X_\beta$, the inverse image $\pi_\beta^{-1}(U_\beta) \subset \prod_{\alpha\in J} X_\alpha$ must be open. The collection

$$\mathcal{S} = \{\pi_\beta^{-1}(U_\beta) \mid \beta \in J, U_\beta \subset X_\beta \text{ open}\} \tag{4.4}$$

is a subbasis for $\prod_{\alpha\in J} X_\alpha$ and generates the product topology $\mathcal{T}_{\text{prod}}$ on $\prod_{\alpha\in J} X_\alpha$. Consider the basis $\mathcal{B}$ that $\mathcal{S}$ generates. The collection $\mathcal{B}$ consists of all finite intersections of elements of $\mathcal{S}$.

Now, for a given index $\beta \in J$, let us denote by $\mathcal{S}_\beta$ the collection

$$\mathcal{S}_\beta = \{\pi_\beta^{-1}(U_\beta) \mid U_\beta \subset X_\beta \text{ open}\}. \tag{4.5}$$

And then the collection of subbasis reads

$$\mathcal{S} = \bigcup_{\beta\in J} \mathcal{S}_\beta. \tag{4.6}$$

Let us get back to the formation of $\mathcal{B}$ from $\mathcal{S}$ by taking all finite intersections of elements from $\mathcal{S}$. If we intersect elements finitely many times from the same collection $\mathcal{S}_\beta$, we end up with an element again belonging to $\mathcal{S}_\beta$. For example, choose $U_\alpha \subset X_\alpha$ open and $V_\alpha \subset X_\alpha$ open so that $\pi_\alpha^{-1}(U_\alpha), \pi_\alpha^{-1}(V_\alpha)$ both belong to $\mathcal{S}_\alpha$. Then, using elementary set theory,

$$\pi_\alpha^{-1}(U_\alpha) \cap \pi_\alpha^{-1}(V_\alpha) = \pi_\alpha^{-1}(U_\alpha \cap V_\alpha) \in \mathcal{B}_\alpha, \tag{4.7}$$

as $U_\alpha \cap V_\alpha \subset X_\alpha$ is also open.

In other words, by taking finite intersections of elements belonging to a given $\mathcal{S}_\beta$ for some $\beta \in J$, we don't get anything new in the sense that we already have all the elements of $\mathcal{S}_\beta$. Nontrivial things happen when we intersect elements (of course finitely many times) belonging to distinct $\mathcal{S}_\beta$'s. The typical element of $\mathcal{B}$ can thus be described as follows: let $\beta_1, \dots, \beta_n$ be a finite set of distinct indices, then

$$B = \pi_{\beta_1}^{-1}(U_{\beta_1}) \cap \pi_{\beta_2}^{-1}(U_{\beta_2}) \cap \cdots \cap \pi_{\beta_n}^{-1}(U_{\beta_n}) \tag{4.8}$$

is a basic open set in the product topology $\mathcal{T}_{\text{prod}}$.

Now, take a point $\mathbf{x} = (x_\alpha)_{\alpha\in J} \in B$ is given by (4.8). Its β_i-th coordinate x_{β_i} belongs to U_{β_i}. For example, $x_{\beta_1} \in U_{\beta_1}$, $x_{\beta_2} \in U_{\beta_2}$ and so on. As a result, one writes B as a product:

$$B = \prod_{\alpha\in J} U_\alpha \tag{4.9}$$

with $U_{\beta_i} \subset X_{\beta_i}$ are open for $i \in \{1, \dots, n\}$, and $U_\alpha = X_\alpha$ for all $\alpha \in J \setminus \{\beta_1, \dots, \beta_n\}$.

Based on the discussion above, one has the following theorem:

Theorem 4.4 (Comparison of the box and product topologies)

The box topology on $\prod_{\alpha\in J} X_\alpha$ has as basis all sets of the form

$$\prod_{\alpha\in J} U_\alpha$$

with $U_\alpha \subset X_\alpha$ open for all $\alpha \in J$. The product topology on $\prod_{\alpha\in J} X_\alpha$ has as basis all sets of the form

$$\prod_{\alpha\in J} U_\alpha$$

with U_α open in X_α for all $\alpha \in J$ and $U_\alpha = X_\alpha$ for all but finitely many values of α in J.

Remark 4.1. For finite products $\prod_{i=1}^n X_i$, the box topology and the product topology are the same. From Theorem 4.4, by application of Lemma 1.7, one finds that the box topology on a product space $\prod_{\alpha\in J} X_\alpha$ is, in general, finer than the product topology on it.

Indeed, if one denotes by $\mathcal{T}_{\text{prod}}$ the product topology on $\prod_{\alpha\in J} X_\alpha$ and by $\mathcal{T}_{\text{box}}$ the box topology on $\prod_{\alpha\in J} X_\alpha$, then $\mathcal{T}_{\text{prod}} \subset \mathcal{T}_{\text{box}}$. Let us denote the basis of $\prod_{\alpha\in J} X_\alpha$ generating the box topology on it by

$\mathcal{B}_{\text{box}}$ and the basis of $\prod_{\alpha \in J} X_\alpha$ generating the product topology on it by $\mathcal{B}_{\text{prod}}$. The collections $\mathcal{B}_{\text{box}}$ and $\mathcal{B}_{\text{prod}}$ are given by Theorem 4.4. It is evident that for each $\mathbf{x} \in \prod_{\alpha \in J} X_\alpha$ and each $\prod_{\alpha \in J} U_\alpha \in \mathcal{B}_{\text{prod}}$, with $\mathbf{x} \in \prod_{\alpha \in J} U_\alpha$, there exists $\prod_{\alpha \in J} U'_\alpha \in \mathcal{B}_{\text{box}}$ with

$$\mathbf{x} \in \prod_{\alpha \in J} U'_\alpha \subset \prod_{\alpha \in J} U_\alpha,$$

since $\mathcal{B}_{\text{prod}} \subset \mathcal{B}_{\text{box}}$. Therefore, by Lemma 1.7, $\mathcal{T}_{\text{prod}} \subset \mathcal{T}_{\text{box}}$.

Theorem 4.5

Suppose the topology on each space X_α is given by a basis $\mathcal{B}_\alpha$. The collection of all sets of the form

$$\prod_{\alpha \in J} B_\alpha,$$

where $B_\alpha \in \mathcal{B}_\alpha$ for each α, will serve as a basis for the box topology on $\prod_{\alpha \in J} X_\alpha$. The collection of all sets of the same form, where $B_\alpha \in \mathcal{B}_\alpha$ for finitely many indices α and $B_\alpha = X_\alpha$ for all the remaining indices, will serve as a basis for the product topology $\prod_{\alpha \in J} X_\alpha$.

Proof. Given any $\mathbf{x} \in \prod_{\alpha \in J} X_\alpha$ and $\prod_{\alpha \in J} U_\alpha \in \mathcal{T}_{\text{box}}$, there exists basic open sets B_α such that

$$x_\alpha \in B_\alpha \subset U_\alpha, \tag{4.10}$$

for each $\alpha \in J$. So we have

$$\mathbf{x} = (x_\alpha)_{\alpha \in J} \in \prod_{\alpha \in J} B_\alpha \subset \prod_{\alpha \in J} U_\alpha. \tag{4.11}$$

Then by Recognition principle, the collection of all sets of the form

$$\prod_{\alpha \in J} B_\alpha,$$

where $B_\alpha \in \mathcal{B}_\alpha$ for each α, is a basis for the box topology on $\prod_{\alpha \in J} X_\alpha$.

The proof for product topology is analogous. ■

Theorem 4.6

Let A_α be a subspace of X_α, for each $\alpha \in J$. Then $\prod_{\alpha \in J} A_\alpha$ is a subspace of $\prod_{\alpha \in J} X_\alpha$ if both products are given the box topology, or if both products are given the product topology.

Proof. Suppose $\mathcal{B}_\alpha$ is a basis for the topology on X_α. We consider the case of box topology first. By Lemma 1.16 and Theorem 4.5, a basis for the subspace topology on the set $\prod_{\alpha \in J} A_\alpha$ is

$$\begin{aligned} \mathcal{B}_{\text{subspace}} &= \left\{ \left(\prod_{\alpha \in J} B_\alpha \right) \cap \left(\prod_{\alpha \in J} A_\alpha \right) \mid B_\alpha \in \mathcal{B}_\alpha \right\} \\ &= \left\{ \prod_{\alpha \in J} (B_\alpha \cap A_\alpha) \mid B_\alpha \in \mathcal{B}_\alpha \right\}. \end{aligned} \tag{4.12}$$

The collection $\{B_\alpha \cap A_\alpha \mid B_\alpha \in \mathcal{B}_\alpha\}$ is a basis for the subspace A_α, by Lemma 1.16. By Theorem 4.5, a basis for the box topology on $\prod_{\alpha \in J} A_\alpha$ is

$$\mathcal{B}_{\text{box}, \prod_\alpha A_\alpha} = \left\{ \prod_{\alpha \in J} (B_\alpha \cap A_\alpha) \mid B_\alpha \in \mathcal{B}_\alpha \right\}, \tag{4.13}$$

which coincides with the subspace topology basis. So $\prod_{\alpha \in J} A_\alpha$ is a subspace of $\prod_{\alpha \in J} X_\alpha$ if both products are given the box topology.

Now we consider the case where both $\prod_{\alpha\in J} A_\alpha$ and $\prod_{\alpha\in J} X_\alpha$ are given product topologies. The collection $\{B_\alpha \cap A_\alpha \,|B_\alpha \in \mathcal{B}_\alpha\}$ is a basis for the subspace A_α, by Lemma 1.16. By Theorem 4.5, a basis for the product topology on $\prod_{\alpha\in J} A_\alpha$ is

$$\mathcal{B}_{\text{prod},\prod_\alpha A_\alpha} = \left\{ \prod_{\alpha\in J} (B_\alpha \cap A_\alpha) \;\middle|\; B_\alpha \in \mathcal{B}_\alpha \text{ for finitely many } \alpha,\, B_\alpha = X_\alpha \text{ for other } \alpha\text{'s} \right\}. \tag{4.14}$$

A basis for the subspace topology on the set $\prod_{\alpha\in J} A_\alpha$ is

$$\begin{aligned}\mathcal{B}_{\text{subspace}} &= \left\{ \left(\prod_{\alpha\in J} B_\alpha\right) \cap \left(\prod_{\alpha\in J} A_\alpha\right) \;\middle|\; B_\alpha \in \mathcal{B}_\alpha \text{ for finitely many } \alpha,\, B_\alpha = X_\alpha \text{ for other } \alpha\text{'s} \right\} \\ &= \left\{ \prod_{\alpha\in J} (B_\alpha \cap A_\alpha) \;\middle|\; B_\alpha \in \mathcal{B}_\alpha \text{ for finitely many } \alpha,\, B_\alpha = X_\alpha \text{ for other } \alpha\text{'s} \right\}. \end{aligned} \tag{4.15}$$

Since the bases coincide, we can conclude that $\prod_{\alpha\in J} A_\alpha$ is a subspace of $\prod_{\alpha\in J} X_\alpha$ if both products are given the product topology. ■

Theorem 4.7

If each space X_α is a Hausdorff space, then $\prod_{\alpha\in J} X_\alpha$ is a Hausdorff space in both the box and product topologies.

Proof. We shall prove it for product topology only. Since box topology is finer, i.e., it contains more open sets than product topology, if the result is true in product topology, it's true in box topology as well. Take two distince points

$$\mathbf{x} = (x_\alpha)_{\alpha\in J}\,,\ \mathbf{y} = (y_\alpha)_{\alpha\in J} \in \prod_{\alpha\in J} X_\alpha.$$

Since $\mathbf{x} \neq \mathbf{y}$, there exists some index β such that $x_\beta \neq y_\beta$. Since X_β is Hausdorff, there exist disjoint open sets $U_\beta, V_\beta \subset X_\beta$ such that they contain x_β, y_β, respectively. Now we take open sets

$$U = \prod_{\alpha\in J} U_\alpha, V = \prod_{\alpha\in J} V_\alpha, \tag{4.16}$$

where $U_\alpha = V_\alpha = X_\alpha$ for $\alpha \neq \beta$, and for $\alpha = \beta$, we choose $U_\beta, V_\beta \subset X_\beta$ as above. Then U contains $\mathbf{x}$ and V contains $\mathbf{y}$. Furthermore,

$$U \cap V = \left(\prod_{\alpha\in J} U_\alpha\right) \cap \left(\prod_{\alpha\in J} V_\alpha\right) = \prod_{\alpha\in J} (U_\alpha \cap V_\alpha)\,. \tag{4.17}$$

This product is empty, as $U_\beta \cap V_\beta = \varnothing$. So $\prod_{\alpha\in J} X_\alpha$ is a Hausdorff space. ■

Theorem 4.8

Let $\{X_\alpha\}$ be an indexed family of spaces; let $A_\alpha \subset X_\alpha$ for each α. If $\prod X_\alpha$ is given either the product or the box topology, then

$$\prod \overline{A}_\alpha = \overline{\prod A_\alpha}.$$

Proof. Let $\mathbf{x} = (x_\alpha)_{\alpha\in J}$ be a point of $\prod_{\alpha\in J} \overline{A}_\alpha$; we show that $\mathbf{x} \in \overline{\prod A_\alpha}$. Let $U = \prod_{\alpha\in J} U_\alpha$ be a basis element for either the box or product topology that contains $\mathbf{x}$. Since $(x_\alpha)_{\alpha\in J} \in \prod_{\alpha\in J} \overline{A}_\alpha$, $x_\alpha \in \overline{A}_\alpha$ for every $\alpha \in J$. So we can choose a point $y_\alpha \in U_\alpha \cap A_\alpha$ for each α. Then

$$y = (y_\alpha)_{\alpha\in J} \in \prod_{\alpha\in J} (U_\alpha \cap A_\alpha) = \left(\prod_{\alpha\in J} U_\alpha\right) \cap \left(\prod_{\alpha\in J} A_\alpha\right) = U \cap \left(\prod_{\alpha\in J} A_\alpha\right). \tag{4.18}$$

So we have shown that given $\mathbf{x} \in \prod_{\alpha\in J} \overline{A}_\alpha$, every basis element U of $\prod_{\alpha\in J} X_\alpha$ (be it product or box topology) intersects $\prod_{\alpha\in J} A_\alpha$. So it follows that $\mathbf{x}$ belongs to the closure of $\prod A_\alpha$.

Conversely, suppose $\mathbf{x} = (x_\alpha)_{\alpha\in J}$ lies in the closure of $\prod_{\alpha\in J} A_\alpha$, in either topology. We show that $\mathbf{x} \in \prod_{\alpha\in J} \overline{A}_\alpha$, i.e. for any given index β, we have $x_\beta \in \overline{A}_\beta$. Let V_β be an arbitrary open set of X_β containing x_β. Note that $\pi_\beta^{-1}(V_\beta)$ is open in $\prod_{\alpha\in J} X_\alpha$ in either topology. Now, $\pi_\beta^{-1}(V_\beta)$ is a neighborhood of $\mathbf{x} = (x_\alpha)_{\alpha\in J}$, which lies in the closure of $\prod_{\alpha\in J} A_\alpha$. Therefore,

$$\left(\pi_\beta^{-1}(V_\beta)\right) \cap \left(\prod_{\alpha\in J} A_\alpha\right) \neq \varnothing. \tag{4.19}$$

So we take $\mathbf{y} = (y_\alpha)_{\alpha\in J}$ from the intersection. Then y_β belongs to $V_\beta \cap A_\beta$. So given any neighborhood V_β of x_β, it intersects A_β. So it follows that $x_\beta \in \overline{A}_\beta$. ■

Theorem 4.9

Let $f : A \to \prod_{\alpha\in J} X_\alpha$ be given by the equation

$$f(a) = (f_\alpha(a))_{\alpha\in J},$$

where $f_\alpha : A \to X_\alpha$ for each α. Let $\prod_{\alpha\in J} X_\alpha$ have the product topology. Then the function f is continuous if and only if each function f_α is continuous.

Proof. Let π_β be the projection of the product onto its β-th factor. The function π_β is continuous, for if U_β is open in X_β, the set $\pi_\beta^{-1}(U_\beta)$ is a subbasis element for the product topology on X_α. Now suppose that $f : A \to \prod_{\alpha\in J} X_\alpha$ is continuous. The function f_β equals the composite $\pi_\beta \circ f$; being the composite of two continuous functions, it is continuous.

Conversely, suppose that each coordinate function f_α is continuous. To prove that f is continuous, it suffices to prove that the inverse image under f of each subbasis element is open in A, by Lemma 3.4. A typical subbasis element for the product topology on $\prod X_\alpha$ is a set of the form $\pi_\beta^{-1}(U_\beta)$, where β is some index and U_β is open in X_β. Now

$$f^{-1}\left(\pi_\beta^{-1}(U_\beta)\right) = f_\beta^{-1}(U_\beta),$$

because $f_\beta = \pi_\beta \circ f$. Since f_β is continuous, this set is open in A, as desired. ■

§4.3 Metric Topology Revisited

Let (X, d) be a metric space. Recall the associated metric topology $\mathcal{T}_d$ on X.

Definition 4.4. A topological space $(X, \mathcal{T})$ is **metrizable** if there exists a metric d on X such that $\mathcal{T}$ is the topology associated with d, i.e., $\mathcal{T} = \mathcal{T}_d$.

Definition 4.5. Let (X, d) be a metric space. A subset A of X is said to be *bounded* if there is some positive number M such that

$$d(a_1, a_2) < M, \quad \forall a_1, a_2 \in A.$$

If A is bounded and nonempty, then the **diameter** of A is defined to be the number

$$\operatorname{diam} A = \sup\{d(a_1, a_2) \mid a_1, a_2 \in A\}. \tag{8}$$

Remark 4.2. Boundedness of a set is not a topological property, for it depends on the metric d of the underlying metric space (X, d). Given any metric space (X, d), one can construct a metric space $(X, \bar{d})$ on the same set X with respect to a metric $\bar{d}$ relative to which every subset of X is bounded. This metric is defined in the following theorem.

Theorem 4.10

Let (X, d) be a metric space. Define $\bar{d} : X \times X \to \mathbb{R}$ by

$$\bar{d}(x, y) = \min\{d(x, y), 1\}. \tag{9}$$

Then $\bar{d}$ is a metric that induces the same topology as d, i.e., $\mathcal{T}_{\bar{d}} = \mathcal{T}_d$. The metric $\bar{d}$ is called the **standard bounded metric** corresponding to the metric d.

Proof. Let us first verify that the **standard bounded metric** is indeed a metric.

1. If $d(x, y) \geq 1$, then $\bar{d}(x, y) = 1 > 0$. If $d(x, y) < 1$, then $\bar{d}(x, y) = d(x, y) \geq 0$ (since d is a metric). Therefore, $\bar{d}(x, y) \geq 0$, for each $x, y \in X$
2. $\bar{d}(y, x) = \min\{d(y, x), 1\} = \min\{d(x, y), 1\} = \bar{d}(x, y)$.
3. $\bar{d}(x, y) = 0 \iff \min\{d(x, y), 1\} = 0 \iff d(x, y) = 0 \iff x = y$.
4. One needs to verify that for all $x, y, z \in X$,
$$\bar{d}(x, z) \leq \bar{d}(x, y) + \bar{d}(y, z).$$
For $x, y, z \in X$, there are 2 possibilities:
   (a) $d(x, y) \geq 1$ or $d(y, z) \geq 1$. Hence, $\bar{d}(x, y) = 1$ or $\bar{d}(y, z) = 1$, so that one has
   $$\bar{d}(x, y) + \bar{d}(y, z) \geq 1. \tag{4.20}$$
   On the other hand, from the definition of $\bar{d}$,
   $$\bar{d}\,(x, z) \leq 1. \tag{4.21}$$
   So combining (4.20) and (4.21), we get
   $$\bar{d}(x, z) \leq \bar{d}(x, y) + \bar{d}(y, z). \tag{4.22}$$
   (b) $d(x, y) < 1$ or $d(y, z) < 1$. In this case $\bar{d}\,(x, y) = d\,(x, y)$ and $\bar{d}\,(y, z) = d\,(y, z)$. Since d is a metric,
   $$d\,(x, z) \leq d\,(x, y) + d\,(y, z) = \bar{d}\,(x, y) + \bar{d}\,(y, z)\,. \tag{4.23}$$
   Also, from the definition of $\bar{d}$,
   $$\bar{d}\,(x, z) \leq d\,(x, z)\,. \tag{4.24}$$
   So combining (4.23) and (4.24), we get
   $$\bar{d}(x, z) \leq \bar{d}(x, y) + \bar{d}(y, z). \tag{4.25}$$

Next thing to see that given a metric space (X, d), the collection of ε-balls with $\varepsilon < 1$,

$$\mathcal{B} = \{B_d(x, \varepsilon) \mid x \in X, \varepsilon < 1\},$$

is a basis for the metric topology $\mathcal{T}_d$ on X induced by the metric d. But $B_d(x, \varepsilon) = B_{\bar{d}}(x, \varepsilon)$ for $\varepsilon < 1$. Indeed, for $\varepsilon < 1$,

$$\begin{aligned} B_{\bar{d}}(x, \varepsilon) &= \{y \in X \mid \bar{d}(x, y) < \varepsilon\} \\ &= \{y \in X \mid \min\{d(x, y), 1\} < \varepsilon\} \\ &= \{y \in X \mid d(x, y) < \varepsilon\}; \ [\text{since } \varepsilon < 1] \\ &= B_d(x, \varepsilon). \end{aligned}$$

Hence, the basis

$$\mathcal{B}' = \{B_{\bar{d}}(x, \varepsilon) \mid x \in X, \varepsilon < 1\},$$

generating the topology $\mathcal{T}_{\bar{d}}$ on X is the same as the basis

$$\mathcal{B} = \{B_d(x, \varepsilon) \mid x \in X, \varepsilon < 1\},$$

generating the topology $\mathcal{T}_d$ on X. Therefore, the two topologies $\mathcal{T}_{\bar{d}}$ and $\mathcal{T}_d$ induced by the metrics $\bar{d}$ and d, respectively, coincide, i.e., $\mathcal{T}_{\bar{d}} = \mathcal{T}_d$. ■

§4.3.i Euclidean n-space

Let $X = \mathbb{R}^n$ be the set of all real n-tuples $\mathbf{x} = (x_1, \ldots, x_n)$. The Euclidean metric is defined by the Euclidean norm on $\mathbb{R}^n$ defined by

$$d_{\text{Euc}}(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|_{\text{Euc}} = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}, \tag{4.26}$$

where $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ are elements in $\mathbb{R}^n$. The **sup norm** (also known as the **max norm**) on $\mathbb{R}^n$ is defined as

$$\|\mathbf{x}\|_{\max} = \max\{|x_1|, \ldots, |x_n|\}. \tag{4.27}$$

The **square metric** on $\mathbb{R}^n$ is defined by

$$\rho(\mathbf{x}, \mathbf{y}) = \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\} = \|\mathbf{x} - \mathbf{y}\|_{\max}. \tag{4.28}$$

Theorem 4.11

The Euclidean metric d_{Euc}, the square metric ρ, and the product topology on $\mathbb{R}^n$ (as the product of n copies of $\mathbb{R}$ each with standard topology) all define the same topology on $\mathbb{R}^n$.

§4.4 Infnite Cartesian Products

For any index set J, consider the set $\mathbb{R}^J$ of real J-tuples $\mathbf{x} = (x_\alpha)_{\alpha \in J}$. The elements of $\mathbb{R}^J$ are thought of as functions $\mathbf{x} : J \to \mathbb{R}$. For example, for the finite index set $J = \{1, \ldots, n\}$, we can identify $\mathbb{R}^{\{1,\ldots,n\}}$ with $\mathbb{R}^n$. When $J = \{1, 2, \ldots\} = \mathbb{N}$, the set of natural numbers, we write $\mathbb{R}^\omega$ to denote the set of real sequences $\mathbf{x} = (x_n)_{n=1}^{\infty}$. Note that the formulas

$$\|\mathbf{x}\|_{\text{Euc}} = \sqrt{x_1^2 + x_2^2 + \ldots}$$

and

$$\|\mathbf{x}\|_{\max} = \max\{|x_1|, |x_2|, \ldots\}$$

do not make sense for $\mathbf{x} \in \mathbb{R}^\omega$. Replacing the usual metric on $\mathbb{R}$ with the standard bounded metric on it will allow one to generalize the square metric to infinite dimensions in the case of infinite Cartesian products.

Definition 4.6 (Uniform metric on $\mathbb{R}^J$)**.** Let $\mathbf{x} = (x_\alpha)_{\alpha \in J}$ and $\mathbf{y} = (y_\alpha)_{\alpha \in J}$ be two points of $\mathbb{R}^J$. Let us define a metric $\overline{\rho}$ on $\mathbb{R}^J$ by

$$\overline{\rho}(\mathbf{x}, \mathbf{y}) = \sup\left\{\bar{d}(x_\alpha, y_\alpha) \mid \alpha \in J\right\}, \tag{4.29}$$

where $\bar{d}$ is the standard bounded metric on $\mathbb{R}$. This is called the **uniform metric** on $\mathbb{R}^J$, and the topology it induces is called the **uniform topology**.

It is left as an exercise for the reader to verify that $\overline{\rho}$ defined by (4.29) is indeed a metric.

Theorem 4.12

The uniform topology on $\mathbb{R}^J$ is finer than the product topology and coarser than the box topology; these three topologies are all different if J is infinite.

Proof. Take $\mathbf{x} = (x_\alpha)_{\alpha \in J} \in \mathbb{R}^J$. Also, take any basis element $\prod_{\alpha \in J} U_\alpha$ (as given in Theorem 4.4) in product topology on $\mathbb{R}^J$ containing $\mathbf{x}$. Since uniform topology on $\mathbb{R}^J$ is generated by the ε-balls

$B_{\overline{\rho}}(\mathbf{x}, \varepsilon)$, all we need to show is that $\mathbf{x} \in B_{\overline{\rho}}(\mathbf{x}, \varepsilon) \subset \prod_{\alpha \in J} U_\alpha$ for some $\varepsilon > 0$, to prove that the uniform topology is finer than the product topology on $\mathbb{R}^J$ by the use of Lemma 1.7.

We are given the basis element $\prod_{\alpha \in J} U_\alpha$ of product topology on $\mathbb{R}^J$ containing $\mathbf{x} = (x_\alpha)_{\alpha \in J}$. Let $\alpha_1, \ldots, \alpha_n$ be the indices for which $U_{\alpha_i} \neq \mathbb{R}$. And we know that for each $i \in \{1, \ldots, n\}$, U_{α_i} is open in $\mathbb{R}$ in standard topology with $x_{\alpha_i} \in U_{\alpha_i}$. Hence, there exists $\varepsilon_i > 0$ such that $B_{\text{Euc}}(x_{\alpha_i}, \varepsilon_i) \subset U_{\alpha_i}$ for all $i \in \{1, \ldots, n\}$. Here,

$$B_{\text{Euc}}(x_{\alpha_i}, \varepsilon_i) = \{z \in \mathbb{R} \mid |x_{\alpha_i} - z| < \varepsilon_i\}. \tag{4.30}$$

Without loss of generality, choose $\varepsilon_i < 1$ for all $i \in \{1, \ldots, n\}$. Then observe that

$$B_{\bar{d}}(x_{\alpha_i}, \varepsilon_i) \subset B_{\text{Euc}}(x_{\alpha_i}, \varepsilon_i). \tag{4.31}$$

Indeed, let $y \in B_{\bar{d}}(x_{\alpha_i}, \varepsilon_i)$. Then

$$\bar{d}(y, x_{\alpha_i}) < \varepsilon_i \implies \min\{|x_{\alpha_i} - y|, 1\} < \varepsilon_i \implies |x_{\alpha_i} - y| < \varepsilon_i \implies y \in B_{\text{Euc}}(x_{\alpha_i}, \varepsilon_i), \tag{4.32}$$

proving the containment (4.31). But $B_{\text{Euc}}(x_{\alpha_i}, \varepsilon_i) \subset U_{\alpha_i}$. Hence, by (4.31), one has

$$B_{\bar{d}}(x_{\alpha_i}, \varepsilon_i) \subset U_{\alpha_i} \quad \forall i \in \{1, \ldots, n\}, \quad \text{where } \varepsilon_i < 1, \forall i. \tag{4.33}$$

Take $\varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_n\}$. Then we have that

$$B_{\overline{\rho}}(\mathbf{x}, \varepsilon) \subset \prod_{\alpha \in J} U_\alpha. \tag{4.34}$$

Indeed, let $\mathbf{z} \in B_{\overline{\rho}}(\mathbf{z}, \varepsilon)$, $\mathbf{z} = (z_\alpha)_{\alpha \in J}$. Then, one has

$$\overline{\rho}(\mathbf{x}, \mathbf{z}) < \varepsilon \implies \sup\{\bar{d}(x_\alpha, z_\alpha) \mid \alpha \in J\} < \varepsilon \implies \bar{d}(x_\alpha, z_\alpha) < \varepsilon, \quad \forall \alpha \in J \tag{4.35}$$

In particular,

$$\begin{aligned}
& \bar{d}(x_{\alpha_i}, z_{\alpha_i}) < \varepsilon < 1, \quad \forall i \in \{1, \ldots, n\} \\
\implies & \min\{|x_{\alpha_i} - z_{\alpha_i}|, 1\} < \varepsilon, \quad \forall i \\
\implies & |x_{\alpha_i} - z_{\alpha_i}| < \varepsilon_i, \quad \forall i \in \{1, \ldots, n\} \\
\implies & z_{\alpha_i} \in B_{\text{Euc}}(x_{\alpha_i}, \varepsilon_i) \subset U_{\alpha_i} \\
\implies & z_{\alpha_i} \in U_{\alpha_i}, \quad \forall i \in \{1, \ldots, n\} \\
\implies & \mathbf{z} = (z_\alpha)_{\alpha \in J} \in \prod_{\alpha \in J} U_\alpha \\
\implies & B_{\overline{\rho}}(\mathbf{z}, \varepsilon) \subset \prod_{\alpha \in J} U_\alpha.
\end{aligned}$$

Hence, uniform topology on $\mathbb{R}^J$ is finer than the product topology on it.

On the other hand, let us take a basis element $B_{\overline{\rho}}(\mathbf{x}, \varepsilon)$ (WLOG, $\varepsilon < 1$) generating the uniform topology on $\mathbb{R}^J$ containing $\mathbf{x} = (x_\alpha)_{\alpha \in J} \in \mathbb{R}^J$. Observe that the following basis element for the box topology on $\mathbb{R}^J$:

$$U = \prod_{\alpha \in J} \left(x_\alpha - \frac{\varepsilon}{2}, x_\alpha + \frac{\varepsilon}{2} \right), \tag{4.36}$$

containing $\mathbf{x} = (x_\alpha)_{\alpha \in J}$, is contained in $B_{\overline{\rho}}(\mathbf{x}, \varepsilon)$. In other words,

$$\mathbf{x} \in U = \prod_{\alpha \in J} \left(x_\alpha - \frac{\varepsilon}{2}, x_\alpha + \frac{\varepsilon}{2} \right) \subset B_{\overline{\rho}}(\mathbf{x}, \varepsilon). \tag{4.37}$$

Indeed, let $\mathbf{z} = (z_\alpha)_{\alpha\in J} \in U$. Then

$$\begin{aligned}
&|x_\alpha - z_\alpha| < \frac{\varepsilon}{2} < \varepsilon, \quad \forall\alpha \in J \\
\implies &d(x_\alpha, z_\alpha) < \varepsilon, \quad \forall\alpha \in J \\
\implies &\bar{d}(x_\alpha, z_\alpha) < \varepsilon, \quad \forall\alpha \in J \\
\implies &\sup\{\bar{d}(x_\alpha, z_\alpha) \mid \alpha \in J\} \leq \frac{\varepsilon}{2} < \varepsilon \\
\implies &\overline{\rho}(\mathbf{x}, \mathbf{z}) \leq \frac{\varepsilon}{2} < \varepsilon \\
\implies &\mathbf{z} \in B_{\overline{\rho}}(\mathbf{x}, \varepsilon).
\end{aligned}$$

Hence, given $\mathbf{x} \in \mathbb{R}^J$, with $\mathbf{x} \in B_{\overline{\rho}}(\mathbf{x}, \varepsilon)$, a generic basis element from $\mathcal{B} = \{B_{\overline{\rho}}(\mathbf{x}, \varepsilon) \mid \varepsilon \in \mathbb{R}, \varepsilon < 1\}$, we have found a basis element

$$U = \prod_{\alpha\in J} \left(x_\alpha - \frac{\varepsilon}{2}, x_\alpha + \frac{\varepsilon}{2}\right)$$

for the box topology on $\mathbb{R}^J$ satisfying $\mathbf{x} \in U \subset B_{\overline{\rho}}(\mathbf{x}, \varepsilon)$. Then, by Lemma 1.7., the box topology on $\mathbb{R}^J$ is finer than the uniform topology on it. ■

So we have shown that, for $\mathbb{R}^J$,

$$\text{Product topology} \subset \text{Uniform topology} \subset \text{Box topology}. \tag{4.38}$$

The following example illustrates that the three topologies are different for $\mathbb{R}^\omega$.

Example 4.2. Consider the functions $f, g, h : \mathbb{R} \to \mathbb{R}^\omega$ defined as follows:

$$\begin{aligned}
f(t) &= (t, 2t, 3t, \ldots) \\
g(t) &= (t, t, t, \ldots) \\
h(t) &= \left(t, \frac{1}{2}t, \frac{1}{3}t, \ldots\right).
\end{aligned}$$

We give $\mathbb{R}$ the standard topology, as usual. In the product topology for $\mathbb{R}^\omega$, each of these functions are continuous, since the components are continuous (Theorem 4.9). In the box topology, neither of these functions are continuous. Consider the open set

$$U = \prod_{n=1}^{\infty} \left(-\frac{1}{n^2}, \frac{1}{n^2}\right) \tag{4.39}$$

in the box topology. Then

$$\begin{aligned}
f^{-1}(U) &= \left\{t \in \mathbb{R} \mid nt \in \left(-\frac{1}{n^2}, \frac{1}{n^2}\right) \forall n \in \mathbb{Z}^+\right\} \\
&= \left\{t \in \mathbb{R} \mid t \in \left(-\frac{1}{n^3}, \frac{1}{n^3}\right) \forall n \in \mathbb{Z}^+\right\} = \{0\}. \\
g^{-1}(U) &= \left\{t \in \mathbb{R} \mid t \in \left(-\frac{1}{n^2}, \frac{1}{n^2}\right) \forall n \in \mathbb{Z}^+\right\} = \{0\}. \\
h^{-1}(U) &= \left\{t \in \mathbb{R} \mid \frac{t}{n} \in \left(-\frac{1}{n^2}, \frac{1}{n^2}\right) \forall n \in \mathbb{Z}^+\right\} \\
&= \left\{t \in \mathbb{R} \mid t \in \left(-\frac{1}{n}, \frac{1}{n}\right) \forall n \in \mathbb{Z}^+\right\} = \{0\}.
\end{aligned}$$

Although U is open, its preimage under neither of f, g, h is open in $\mathbb{R}$. So neither of these functions are continuous in the box topology. Now consider the uniform topology on $\mathbb{R}^\omega$. We first show that f is not continuous in this topology. Consider the open ball $B_{\overline{\rho}}(\mathbf{0},1)$.

$$\begin{aligned}\mathbf{x}=(x_n)_{n=1}^{\infty}\in B_{\overline{\rho}}(\mathbf{0},1) &\iff \overline{\rho}(\mathbf{0},\mathbf{x})<1\\ &\iff \sup_{n\in\mathbb{Z}^+}\bar{d}(x_n,0)<1\\ &\iff \sup_{n\in\mathbb{Z}^+}|x_n|<1\\ &\implies |x_n|<1 \text{ for all } n\\ &\implies \mathbf{x}=(x_n)_{n=1}^{\infty}\in(-1,1)^{\omega}.\end{aligned}$$

Therefore,

$$\begin{aligned}f^{-1}\left(B_{\overline{\rho}}(\mathbf{0},1)\right)&\subset f^{-1}\left((-1,1)^{\omega}\right)\\ &=\left\{t\in\mathbb{R}\mid nt\in(-1,1)\ \forall n\in\mathbb{Z}^+\right\}\\ &=\left\{t\in\mathbb{R}\mid t\in\left(-\frac{1}{n},\frac{1}{n}\right)\ \forall n\in\mathbb{Z}^+\right\}=\{0\}.\end{aligned}$$

So $f^{-1}\left(B_{\overline{\rho}}(\mathbf{0},1)\right)$ is not open in $\mathbb{R}$. As a result, f is not continuous in the uniform topology on $\mathbb{R}^{\omega}$. Now we shall prove that both g and h are continuous in the uniform topology on $\mathbb{R}^{\omega}$. Consider the basis

$$\mathcal{B}=\left\{B_{\overline{\rho}}(\mathbf{x},\varepsilon)\ \mid \mathbf{x}\in\mathbb{R}^{\omega},\ \varepsilon<1\right\}.$$

Let's take $B_{\overline{\rho}}(\mathbf{x},\varepsilon)\in\mathcal{B}$. We need to show that both $g^{-1}\left(B_{\overline{\rho}}(\mathbf{x},\varepsilon)\right)$ and $h^{-1}\left(B_{\overline{\rho}}(\mathbf{x},\varepsilon)\right)$ are open in $\mathbb{R}$. Consider $t_0\in g^{-1}\left(B_{\overline{\rho}}(\mathbf{x},\varepsilon)\right)$. So $g(t_0)\in B_{\overline{\rho}}(\mathbf{x},\varepsilon)$, i.e. $\overline{\rho}(\mathbf{x},g(t_0))<\varepsilon$. Suppose $\overline{\rho}(\mathbf{x},g(t_0))=\varepsilon_0<\varepsilon$. We claim that

$$\left(t_0-\frac{\varepsilon-\varepsilon_0}{2},t_0+\frac{\varepsilon-\varepsilon_0}{2}\right)\subset g^{-1}\left(B_{\overline{\rho}}(\mathbf{x},\varepsilon)\right). \tag{4.40}$$

Given $t\in\left(t_0-\frac{\varepsilon-\varepsilon_0}{2},t_0+\frac{\varepsilon-\varepsilon_0}{2}\right)$,

$$\overline{\rho}\left(g(t),g(t_0)\right)=\sup_{n\in\mathbb{Z}^+}|g_n(t)-g_n(t_0)|=\sup_{n\in\mathbb{Z}^+}|t-t_0|=|t-t_0|<\frac{\varepsilon-\varepsilon_0}{2}. \tag{4.41}$$

As a result,

$$\overline{\rho}\left(\mathbf{x},g(t)\right)\le\overline{\rho}\left(\mathbf{x},g(t_0)\right)+\overline{\rho}\left(g(t_0),g(t)\right)<\varepsilon_0+\frac{\varepsilon-\varepsilon_0}{2}=\frac{\varepsilon+\varepsilon_0}{2}<\varepsilon. \tag{4.42}$$

So $t\in g^{-1}\left(B_{\overline{\rho}}(\mathbf{x},\varepsilon)\right)$, and hence (4.40) holds. Given any $t_0\in g^{-1}\left(B_{\overline{\rho}}(\mathbf{x},\varepsilon)\right)$, we can find such an open interval around t_0 that is contained in $g^{-1}\left(B_{\overline{\rho}}(\mathbf{x},\varepsilon)\right)$. Therefore, $g^{-1}\left(B_{\overline{\rho}}(\mathbf{x},\varepsilon)\right)$ is open, and hence g is continuous.

A similar construction works for h as well. Consider $t_0\in h^{-1}\left(B_{\overline{\rho}}(\mathbf{x},\varepsilon)\right)$. So $h(t_0)\in B_{\overline{\rho}}(\mathbf{x},\varepsilon)$, i.e. $\overline{\rho}(\mathbf{x},h(t_0))<\varepsilon$. Suppose $\overline{\rho}(\mathbf{x},h(t_0))=\varepsilon_0<\varepsilon$. We claim that

$$\left(t_0-\frac{\varepsilon-\varepsilon_0}{2},t_0+\frac{\varepsilon-\varepsilon_0}{2}\right)\subset h^{-1}\left(B_{\overline{\rho}}(\mathbf{x},\varepsilon)\right). \tag{4.43}$$

Given $t\in\left(t_0-\frac{\varepsilon-\varepsilon_0}{2},t_0+\frac{\varepsilon-\varepsilon_0}{2}\right)$,

$$\overline{\rho}\left(h(t),h(t_0)\right)=\sup_{n\in\mathbb{Z}^+}|h_n(t)-h_n(t_0)|=\sup_{n\in\mathbb{Z}^+}\frac{|t-t_0|}{n}=|t-t_0|<\frac{\varepsilon-\varepsilon_0}{2}. \tag{4.44}$$

As a result,

$$\overline{\rho}\left(\mathbf{x},h(t)\right)\le\overline{\rho}\left(\mathbf{x},h(t_0)\right)+\overline{\rho}\left(h(t_0),h(t)\right)<\varepsilon_0+\frac{\varepsilon-\varepsilon_0}{2}=\frac{\varepsilon+\varepsilon_0}{2}<\varepsilon. \tag{4.45}$$

So $t\in h^{-1}\left(B_{\overline{\rho}}(\mathbf{x},\varepsilon)\right)$, and hence (4.43) holds. Given any $t_0\in h^{-1}\left(B_{\overline{\rho}}(\mathbf{x},\varepsilon)\right)$, we can find such an open interval around t_0 that is contained in $h^{-1}\left(B_{\overline{\rho}}(\mathbf{x},\varepsilon)\right)$. Therefore, $h^{-1}\left(B_{\overline{\rho}}(\mathbf{x},\varepsilon)\right)$ is open, and hence h is continuous.

Now, we compare the three topologies on $\mathbb{R}^\omega$ in light of the continuity of these functions. Suppose $F_1 : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_1)$ and $F_2 : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_2)$ are two maps between topological, whose underlying set-functions are the same (i.e. $F_1(x) = F_2(x)$ for every $x \in X$). Suppose $\mathcal{T}_1$ and $\mathcal{T}_2$ are two topologies on Y, such that $\mathcal{T}_2$ is finer than $\mathcal{T}_1$. If F_2 is continuous, then so is F_1, since call the open sets of $\mathcal{T}_2$ are also in $\mathcal{T}_1$. Similarly, if F_1 is not continuous, then F_2 also can't be continuous. On the other hand, if we discover that F_1 is continuous but F_2 is not, this must mean that $\mathcal{T}_2$ contains an open set whose preimage under F_2 is not open in the topology $\mathcal{T}_X$ of X. But by the continuity of F_1, the preimages of all the open sets of $\mathcal{T}_1$ are open in the topology $\mathcal{T}_X$ of X. Since $\mathcal{T}_2$ is finer than $\mathcal{T}_1$, this means that $\mathcal{T}_2$ contains an open set that is not in $\mathcal{T}_1$. In other words, $\mathcal{T}_2$ is **strictly finer** than $\mathcal{T}_1$.

We have seen earlier in Theorem 4.12 that

$$\text{Product topology} \subset \text{Uniform topology} \subset \text{Box topology}. \tag{4.46}$$

We have proved earlier that $f : \mathbb{R} \to \mathbb{R}^\omega$ is continuous in the product topology on $\mathbb{R}^\omega$, but not continuous in the uniform topology on $\mathbb{R}^\omega$. Therefore, the uniform topology on $\mathbb{R}^\omega$ is strictly finer than the product topology.

$g : \mathbb{R} \to \mathbb{R}^\omega$ and $h : \mathbb{R} \to \mathbb{R}^\omega$ are continuous in the uniform topology on $\mathbb{R}^\omega$, but not continuous in the box topology on $\mathbb{R}^\omega$. Therefore, the box topology on $\mathbb{R}^\omega$ is strictly finer than the uniform topology. Therefore, for $\mathbb{R}^\omega$,

$$\text{Product topology} \subsetneq \text{Uniform topology} \subsetneq \text{Box topology}. \tag{4.47}$$

Theorem 4.13

Let $\bar{d} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by $\bar{d}(a, b) = \min\{|a - b|, 1\}$ be the standard bounded metric on $\mathbb{R}$. If $\mathbf{x} = (x_i)_{i\in\mathbb{N}}$ and $\mathbf{y} = (y_i)_{i\in\mathbb{N}}$ are two points of $\mathbb{R}^\omega$, define

$$D(\mathbf{x}, \mathbf{y}) = \sup\left\{\frac{\bar{d}(x_i, y_i)}{i} \mid i \in \mathbb{N}\right\}. \tag{4.48}$$

Then D is a metric that induces the product topology on $\mathbb{R}^\omega$.

Proof. That D satisfies the first two properties of a metric are trivial. Let us verify that it obeys the triangle inequality. Note that

$$\frac{\bar{d}(x_i, y_i)}{i} \leq \sup\left\{\frac{\bar{d}(x_i, y_i)}{i} \mid i \in \mathbb{N}\right\} = D(\mathbf{x}, \mathbf{y}) \tag{4.49}$$

Then, for $\mathbf{x} = (x_i)_{i\in\mathbb{N}}, \mathbf{y} = (y_i)_{i\in\mathbb{N}},$ and $\mathbf{z} = (z_i)_{i\in\mathbb{N}} \in \mathbb{R}^\omega$, one has

$$\frac{\bar{d}(x_i, z_i)}{i} \leq \frac{\bar{d}(x_i, y_i)}{i} + \frac{\bar{d}(y_i, z_i)}{i} \leq D(\mathbf{x}, \mathbf{y}) + D(\mathbf{y}, \mathbf{z}). \tag{4.50}$$

From (4.50), it follows that

$$\begin{aligned} &\sup\left\{\frac{\bar{d}(x_i, z_i)}{i} \mid i \in \mathbb{N}\right\} \leq D(\mathbf{x}, \mathbf{y}) + D(\mathbf{y}, \mathbf{z}) \\ \implies\ & D(\mathbf{x}, \mathbf{z}) \leq D(\mathbf{x}, \mathbf{y}) + D(\mathbf{y}, \mathbf{z}). \end{aligned} \tag{4.51}$$

So D is indeed a metric.

Let us now prove that D defined by (4.48) gives the product topology on $\mathbb{R}^\omega$. Let us denote the metric topology on $\mathbb{R}^\omega$ induced by the metric D with $\mathcal{T}_D$ and the product topology on $\mathbb{R}^\omega$ by $\mathcal{T}_{\text{prod}}$. We need to show that $\mathcal{T}_D = \mathcal{T}_{\text{prod}}$. Let us first prove that $\mathcal{T}_D \subset \mathcal{T}_{\text{prod}}$.

For this purpose, choose $\mathbf{x} = (x_i)_{i\in\mathbb{N}} \in \mathbb{R}^\omega$ and a basis element $B_D(\mathbf{x}, \varepsilon)$ from the basis set of ε-balls generating the metric topology $\mathcal{T}_D$ on $\mathbb{R}^\omega$, where $\mathbf{x} \in B_D(\mathbf{x}, \varepsilon)$. Now, choose $N \in \mathbb{N}$ large enough such that $\frac{1}{N} < \varepsilon$. With N so chosen, write down the basis element V for the product topology $\mathcal{T}_{\text{prod}}$ as

$$V = (x_1 - \varepsilon, x_1 + \varepsilon) \times \cdots \times (x_N - \varepsilon, x_N + \varepsilon) \times \mathbb{R} \times \mathbb{R} \times \cdots. \tag{4.52}$$

It is immediate that $\mathbf{x} = (x_i)_{i\in\mathbb{N}} \in V$. We now assert that $V \subset B_D(\mathbf{x}, \varepsilon)$.

Observe the following for $\mathbf{y} = (y_i)_{i\in\mathbb{N}} \in \mathbb{R}^\omega$, if $i > N$, then

$$\frac{1}{i} \leq \frac{1}{N}. \tag{4.53}$$

Also,

$$\bar{d}(x_i, y_i) \leq 1. \tag{4.54}$$

Now, (4.53) and (4.54) together imply

$$\frac{\bar{d}(x_i, y_i)}{i} \leq \frac{1}{N}. \tag{4.55}$$

Now,

$$D(\mathbf{x}, \mathbf{y}) = \sup\left\{\frac{\bar{d}(x_i, y_i)}{i} \mid i \in \mathbb{N}\right\} \leq \max\left\{\frac{\bar{d}(x_1, y_1)}{1}, \frac{\bar{d}(x_2, y_2)}{2}, \ldots, \frac{\bar{d}(x_N, y_N)}{N}, \frac{1}{N}\right\}. \tag{4.56}$$

Now, choose $\mathbf{y} = (y_i)_{i\in\mathbb{N}} \in V$. From (4.52), one sees that

$$\begin{aligned} \bar{d}(x_1, y_1) &\leq d(x_1, y_1) < \varepsilon, \\ \frac{\bar{d}(x_2, y_2)}{2} &\leq \frac{d(x_2, y_2)}{2} < \frac{\varepsilon}{2} < \varepsilon \\ &\vdots \\ \frac{\bar{d}(x_N, y_N)}{2} &\leq \frac{d(x_N, y_N)}{N} < \frac{\varepsilon}{N} < \varepsilon. \end{aligned} \tag{4.57}$$

Besides, we have chosen $\frac{1}{N} < \varepsilon$. So we have

$$\max\left\{\frac{\bar{d}(x_1, y_1)}{1}, \frac{\bar{d}(x_2, y_2)}{2}, \ldots, \frac{\bar{d}(x_N, y_N)}{N}, \frac{1}{N}\right\} < \varepsilon. \tag{4.58}$$

Combining (4.56) and (4.58), we get

$$D(\mathbf{x}, \mathbf{y}) < \varepsilon \implies \mathbf{y} \in B_D\left(\mathbf{x}, \varepsilon\right). \tag{4.59}$$

So given any $\mathbf{x} \in \mathbb{R}^\omega$, and a basis element $B_D\left(\mathbf{x}, \varepsilon\right)$ of $\mathcal{T}_D$ containing $\mathbf{x}$, there exists a basis element V of $\mathcal{T}_{\text{prod}}$,

$$\mathbf{y} \in V \subset B_D\left(\mathbf{x}, \varepsilon\right). \tag{4.60}$$

Therefore, by Lemma 1.7,

$$\mathcal{T}_D \subset \mathcal{T}_{\text{prod}}. \tag{4.61}$$

Now, let us prove that $\mathcal{T}_{\text{prod}} \subset \mathcal{T}_D$. For this purpose, consider a basis element

$$U = \prod_{i\in\mathbb{N}} U_i, \tag{4.62}$$

generating $\mathcal{T}_{\text{prod}}$ on $\mathbb{R}^\omega$. Here, U_i is open in $\mathbb{R}$ for $i = \alpha_1, \alpha_2, \ldots, \alpha_n$ and $U_i = \mathbb{R}$, for $i \in \mathbb{N}\setminus\{\alpha_1, \ldots, \alpha_n\}$. Given $\mathbf{x} = (x_i)_{i\in\mathbb{N}} \in \mathbb{R}^\omega$, choose a basis element U as given in (4.62) satisfying $\mathbf{x} \in U$, so that $x_i \in U_i$, for each $i \in \mathbb{N}$. In particular, $x_i \in U_i$ for all $i \in \{\alpha_1, \ldots, \alpha_n\}$, with U_i being open in $\mathbb{R}$, for each $i \in \{\alpha_1, \ldots, \alpha_n\}$. Now choose an interval

$$(x_i - \varepsilon_i, x_i + \varepsilon_i) \subset U_i, \tag{4.63}$$

for each $i \in \{\alpha_1, \ldots, \alpha_n\}$. WLOG, choose $\varepsilon_i < 1$. Then define

$$\varepsilon = \min\left\{\frac{\varepsilon_i}{i} \mid i = \alpha_1, \ldots, \alpha_n\right\}. \tag{4.64}$$

We claim that

$$\mathbf{x} \in B_D(\mathbf{x}, \varepsilon) \subset U. \tag{4.65}$$

Let $\mathbf{y} = (y_i)_{i \in \mathbb{N}} \in B_D(\mathbf{x}, \varepsilon)$. Then $D(\mathbf{x}, \mathbf{y}) < \varepsilon$. Therefore, from the definition of D, one obtains,

$$\frac{\bar{d}(x_i, y_i)}{i} \leq D(\mathbf{x}, \mathbf{y}) < \varepsilon, \quad \forall i \in \{\alpha_1, \ldots, \alpha_n\}. \tag{4.66}$$

Now, from the definition of ε in (4.64), one obtains

$$\varepsilon \leq \frac{\varepsilon_i}{i}, \quad \forall i \in \{\alpha_1, \ldots, \alpha_n\}. \tag{4.67}$$

Combine (4.66) with (4.67) to obtain

$$\frac{\bar{d}(x_i, y_i)}{i} \leq D(\mathbf{x}, \mathbf{y}) < \varepsilon \leq \frac{\varepsilon_i}{i}. \tag{4.68}$$

So we have, for every $i \in \{\alpha_1, \ldots, \alpha_n\}$,

$$\bar{d}(x_i, y_i) < \varepsilon_i. \tag{4.69}$$

Since we assumed WLOG that $\varepsilon_i < 1$, we have

$$\bar{d}(x_i, y_i) < \varepsilon_i < 1. \tag{4.70}$$

In other words,

$$d(x_i, y_i) < \varepsilon_i. \tag{4.71}$$

So we have, for $i \in \{\alpha_1, \ldots, \alpha_n\}$,

$$y_i \in (x_i - \varepsilon_i, x_i + \varepsilon_i) \subset U_i. \tag{4.72}$$

For other i's, $U_i = \mathbb{R}$, so obviously $y_i \in U_i$. Therefore, given $\mathbf{y} = (y_i)_{i \in \mathbb{N}} \in B_D(\mathbf{x}, \varepsilon)$, $y_i \in U_i$. So we have

$$\mathbf{x} \in B_D(\mathbf{x}, \varepsilon) \subset U. \tag{4.73}$$

So given any $\mathbf{x} \in \mathbb{R}^\omega$, and a basis element U of $\mathcal{T}_{\text{prod}}$ containing $\mathbf{x}$, there exists a basis element $B_D(\mathbf{x}, \varepsilon)$ of $\mathcal{T}_D$,

$$\mathbf{x} \in B_D(\mathbf{x}, \varepsilon) \subset U. \tag{4.74}$$

Therefore, by Lemma 1.7,

$$\mathcal{T}_{\text{prod}} \subset \mathcal{T}_D. \tag{4.75}$$

So $\mathcal{T}_D = \mathcal{T}_{\text{prod}}$. ■

Remark 4.3. Theorem 4.11 and Theorem 4.13, imply that both $\mathbb{R}^n$ and $\mathbb{R}^\omega$ equipped with product topology are metrizable. Certain countability axiom and separation axiom as pointed out by Urysohn will render metrizability to to topological space X. It turns out that $\mathbb{R}^J$ endowed with the product topology is only metrizable if J is countable.

5 Quotient Topology

§5.1 Quotient Maps

> **Definition 5.1** (Quotient map). Let X and Y be topological spaces. Also let $p : X \to Y$ be a surjective map. The map p is said to be a quotient map provided $u \subset Y$ open if and only if $p^{-1}(u) \subset X$ is open.

An immediate consequence of this definition is if $p : X \to Y$ is a surjective continuous map that is either open map or a closed map, then p is a quotient map. Suppose $p : X \to Y$ is a surjective continuous map that is open. Let $U \subset Y$ be open. Then $p^{-1}(U) \subset X$ is open by continuity of p. Now, suppose $p^{-1}(U)$ is open for some $U \subset Y$. Then $p(p^{-1}(U))$ is open in Y by openness of p. But from surjectivity of p, one has $p(p^{-1}(U)) = U$. And hence U is open. In other words p is a quotient map. By elementary set theoric argument, one shows that the 'closed map' part of the above consequence holds.

> **Definition 5.2** (Saturated subset). We say that a subset $V \subset X$ is **saturated** with respect to the function $f : X \to Y$ if $f(x) \in f(V) \implies x \in V$.

Or in other words $x \in f^{-1}(f(V)) = \{x \in X \mid f(x) \in f(V)\}$ implies $x \in V$, which is equivalent to claiming that $f^{-1}(f(V)) \subset V$. Since $V \subset f^{-1}(f(V))$ always holds, $V \subset X$ is saturated with respect to the function $f : X \to Y$ if

$$V = f^{-1}(f(V)). \tag{5.1}$$

Equivalently, one can state that a subset $V \subset X$ is saturated with respect to the function $f : X \to Y$ if any point lying outside V cannot have it's image under f to be contained in $f(V)$, i.e, $x \notin V \implies f(x) \notin f(V)$.

> **Lemma 5.1**
>
> Let X and Y be topological spaces. Let $p : X \to Y$ be a surjective map. Then p is a quotient map if and only if p maps saturated open sets of X (or equivalently, saturated closed sets of X) to open sets of Y (or equivalently, closed sets of Y) and p is continuous.

Proof. ($\Rightarrow$) Suppose that $V \subset X$ is open and saturated with respect to the quotient map $p : X \to Y$. Hence,

$$V = p^{-1}(p(V)). \tag{5.2}$$

Hence, $p^{-1}(p(V)) \subset X$ is open. Since, p is a quotient map by hypothesis, one has $p(V) \subset Y$ is open. Also, $p : X \to Y$ being quotient map can easily seen to be continuous.

($\Leftarrow$) Suppose $p : X \to Y$ is a surjective continuous map that takes saturated open sets of X to open sets of Y. We need to show that p is a quotient map.

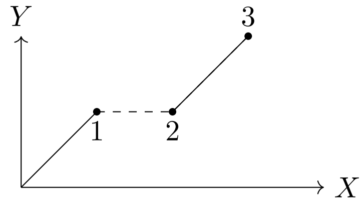
Let $U \subset Y$ be open. Then $p^{-1}(U) \subset X$ is open by continuity of p. On the other hand, let us assume that $p^{-1}(U)$ is open for some $U \in Y$. First, observe that $p^{-1}(U)$ is saturated with respect to p. Since $p(p^{-1}(U)) = U$ as p is surjective,

$$p^{-1}(U) = p^{-1}(p(p^{-1}(U))). \tag{5.3}$$

Now (5.3) tells us that $p^{-1}(U)$ is saturated with respect to p. We therefore see that $p^{-1}(U) \subset X$ is saturated open. Therefore, using the hypothesis, one concludes that $p(p^{-1}(U)) = U$ is open in Y. This proves that p is a quotient map. ■

Example 5.1 (Saturated subsets and Quotient map)**.** Let $X = [0,1] \cup [2,3]$ and $Y = [0,2]$ be subspaces of $\mathbb{R}$. Consider the map $p : X \to Y$ defined by,

$$p(x) = \begin{cases} x & \text{if } x \in [0,1], \\ x-1 & \text{if } x \in [2,3]. \end{cases}$$

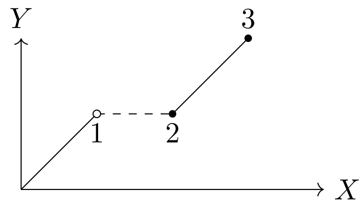


p can be seen to be surjective, continuous and closed. Therefore, it is a quotient map. However, it is not an open map. Indeed, observe that $[0,1]$ is open in X since,

$$[0,1] = \left(-\frac{1}{2}, \frac{3}{2}\right) \cap X.$$

where, $(-\frac{1}{2}, \frac{3}{2})$ is open in $\mathbb{R}$. But $p([0,1]) = [0,1]$, is not open in Y.

Now let $A = [0,1) \cup [2,3]$ be the subspace of X. Denote by $q : A \to Y$ the restriction of p to A, i.e, $q = p\big|_A$.



The map q, being a restriction of the continuous map p is also continuous. Also, q is easily seen to be surjective. But q is not a quotient map. Indeed, $[2,3] \subset A$ is open and

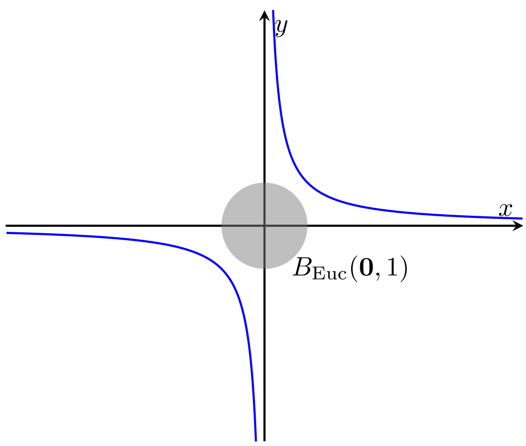
$$q^{-1}(q([2,3])) = q^{-1}([1,2]) = [2,3],$$

implies that $[2,3] \subset A$ is saturated with respect to q. But $q([2,3]) = [1,2] \subset Y$ is not open. Also, note that $[2,3] \subset X$ is not saturated with respect to p. Indeed,

$$p^{-1}(p([2,3])) = p^{-1}([1,2]) = [2,3] \cup \{1\} \neq [2,3]. \tag{5.4}$$

Example 5.2. Let $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ be the projection map onto the first coordinate. Then $\pi_1(x,y) = x$ is easily seen to be a surjective and it is continuous by Lemma 4.1. Also, see that π_1 is an open on a generic basis element of the form $U \times V$, with U, V open in $\mathbb{R}$ (in standard topology), for the product topology $\mathbb{R}^2$. Indeed, $\pi_1(U \times V) = U$ is open in $\mathbb{R}$. Hence, π_1 is a open map.

Let $C = \{(x,y) \in \mathbb{R}^2 \mid xy = 1\}$. C can be seen to be a closed subset of $\mathbb{R}^2$ with the help of the continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x,y) = xy$. Here $\{1\} \subset \mathbb{R}$ is closed. Hence, $C = f^{-1}(\{1\})$ is closed in $\mathbb{R}^2$.



But $f(C) = \mathbb{R} \setminus \{0\}$ is not closed in $\mathbb{R}$. Hence, π_1 is surjective continuous map that is open and hence a quotient map. But π_1 is not a closed map.

Now choose $A \subset \mathbb{R}^2$ with $A = C \cup \{(0,0)\}$. $A = \mathbb{R}^2 \setminus \{(0,0)\}$. Then define $q : A \to \mathbb{R}$ to be the restriction of π_1 to A, i.e, $q = \pi_1|_A$.

q is surjective and continuous. But q is not a quotient map. See that, $\{(0,0)\} \subset A$ is open in the subspace topology it inherits from $\mathbb{R}^2$. Indeed, $\{(0,0)\} = B_{euc}((0,0),1) \cap A$.
Also,

$$q^{-1}(q(\{(0,0)\})) = q^{-1}(\{0\}) = \{(0,0)\} \tag{5.5}$$

implies that $\{(0,0)\} \subset A$ is saturated with respect to q. But $q(\{(0,0)\}) = \{0\} \subset \mathbb{R}$ is not open. Then by Lemma 5.1, q is not a quotient map.

In Example 5.4 we shall see an example of a quotient map which is neither open nor closed.

§5.2 Quotient Topology

Definition 5.3 (Quotient topology). Let $(X, \mathcal{T}_X)$ be a topological space and A is a set. If $p : X \to A$ is a surjective map, then there exists exactly one topology on A with respect to which p is a quotient map. This topology on A is called the **quotient topology** on A induced by p. We denote this topology on A by $\mathcal{T}_Q$. One therefore writes

$$U \in \mathcal{T}_Q \iff p^{-1}(U) \in \mathcal{T}_X. \tag{5.6}$$

Let us first verify it's existence. In other words, we verify that $\mathcal{T}_Q$ defined above is indeed a topology on A. Indeed, $A, \varnothing \in \mathcal{T}_Q$ since $p^{-1}(A) = X \in \mathcal{T}_X$ and $p^{-1}(\varnothing) = \varnothing \in \mathcal{T}_X$.

Now, suppose that $\{U_\alpha\}_{\alpha \in J}$ is a family of elements in $\mathcal{T}_Q$. Hence, one has $p^{-1}(U_\alpha) \in \mathcal{T}_X$ for all $\alpha \in J$. Since, $\mathcal{T}_X$ is a topology on X, one has

$$\bigcup_{\alpha \in J} p^{-1}(U_\alpha) \in \mathcal{T}_X. \tag{5.7}$$

But from elementary set theory, one has

$$\bigcup_{\alpha \in J} p^{-1}(U_\alpha) = p^{-1}\left(\bigcup_{\alpha \in J} U_\alpha\right). \tag{5.8}$$

One therefore obtains, $p^{-1}\left(\bigcup_{\alpha \in J} U_\alpha\right) \in \mathcal{T}_X$ leading to $\bigcup_{\alpha \in J} U_\alpha \in \mathcal{T}_Q$.

Now, let $\{U_i\}_{i=1}^n$ be a finite collection of elements in $\mathcal{T}_Q$. Then, $p^{-1}(U_i) \in \mathcal{T}_X$ for all $i = 1, 2, \ldots, n$. Since $\mathcal{T}_X$ is a topology on X, one has

$$\bigcap_{i=1}^{n} p^{-1}(U_i) \in \mathcal{T}_X. \tag{5.9}$$

But from elementary set theory, one has

$$\bigcap_{i=1}^{n} p^{-1}(U_i) = p^{-1}\left(\bigcap_{i=1}^{n} U_i\right). \tag{5.10}$$

One therefore obtains, $p^{-1}\left(\bigcap_{i=1}^n U_i\right) \in \mathcal{T}_X$ leading to $\bigcap_{i=1}^n U_i \in \mathcal{T}_Q$. (5.8) and (5.10) together imply that $\mathcal{T}_Q$ is a topology on A.

Let us now see that $\mathcal{T}_Q$ is the only topology on A such that $p : (X, \mathcal{T}_X) \to (A, \mathcal{T}_Q)$ is a quotient map. Suppose that $\mathcal{T}_A$ is another topology on A such that $p : (X, \mathcal{T}_X) \to (A, \mathcal{T}_A)$ is a quotient map. We will show that $\mathcal{T}_A = \mathcal{T}_Q$.

Since $p : (X, \mathcal{T}_X) \to (A, \mathcal{T}_A)$ is a quotient map, we have that $U \in \mathcal{T}_A \iff p^{-1}(U) \in \mathcal{T}_X$. Again, since $p : (X, \mathcal{T}_X) \to (A, \mathcal{T}_{\mathrm{Q}})$ is a quotient map, we have that $U \in \mathcal{T}_{\mathrm{Q}} \iff p^{-1}(U) \in \mathcal{T}_X$. Therefore, for a subset $U \subset A$,

$$U \in \mathcal{T}_A \iff p^{-1}(U) \in \mathcal{T}_X \iff U \in \mathcal{T}_{\mathrm{Q}}. \tag{5.11}$$

Therefore, $\mathcal{T}_A = \mathcal{T}_{\mathrm{Q}}$, proving the uniqueness of $\mathcal{T}_{\mathrm{Q}}$.

Example 5.3. Let $p : \mathbb{R} \to A$ where $A = \{a, b, c\}$ be defined by,

$$p(x) = \begin{cases} a & \text{if } x > 0, \\ b & \text{if } x < 0, \\ c & \text{if } x = 0. \end{cases}$$

The quotient topology $\mathcal{T}_{\mathrm{Q}}$ on A is given by

$$\mathcal{T}_{\mathrm{Q}} = \{\varnothing, \{a\}, \{b\}, \{a, b\}, A\}.$$

Indeed, $p^{-1}(\varnothing) = \varnothing$, $p^{-1}(\{a\}) = (0, \infty)$, $p^{-1}(\{b\}) = (-\infty, 0)$, $p^{-1}(\{a, b\}) = (-\infty, 0) \cup (0, \infty)$ $p^{-1}(A) = \mathbb{R}$ are all open in $\mathbb{R}$. In fact, there are no other subset $u \in A$ such that $p^{-1}(u)$ is open in the standard topology on $\mathbb{R}$.

Definition 5.4 (Quotient space)**.** Let X be a topological space, and X^* be a partition of X into disjoint subsets whose union is X. Let $\pi : X \to X^*$ be the surjective map that sends each $x \in X$ to the element (a subset of X) containing it. The set X^* is called the quotient set. In the quotient topology induced by the surjective map $\pi : X \to X^*$, the set X^* becomes a topological space called a **quotient space** of X.

We can see that elements of X^* are subsets of X. We can impose an equivalence relation on X with respect to which each subset of X, being an element of X^*, becomes an equivalence class. In other words, all elements of X belonging to the same subset of X that is an element of X^*, are declared equivalent under this proposed equivalence relation.

Definition 5.5 (Final Topology)**.** Let $\{X_\alpha\}_{\alpha \in J}$ be a J-indexed family of topological spaces and A be a set. Now, let $\{f_\alpha\}_{\alpha \in J}$ be a J-indexed family of functions $f_\alpha : X_\alpha \to A$. The **final topology** on A is the finest topology $\mathcal{T}_{\mathrm{F}}$ on it such that $f_\alpha : X_\alpha \to A$ becomes a continuous function, for all $\alpha \in J$. We say that the final topology on A is induced by the family of functions $\{f_\alpha\}_{\alpha \in J}$.

Lemma 5.2

Let $(X, \mathcal{T}_X)$ be a topological space. Consider the surjective map $\pi : X \to X^*$ as constructed in the definition of quotient space. Now, there is a final topology on X^* induced by a single surjective map π. By definition of final topology, it is the finest topology on X^* that makes the surjective function $\pi : X \to X^*$ continuous. Let us denote this final topology on X^* by $\mathcal{T}_{\mathrm{F}}$. On the other hand, there is the quotient topology on X^* that we denote by $\mathcal{T}_{\mathrm{Q}}$. Then, $\mathcal{T}_{\mathrm{F}} = \mathcal{T}_{\mathrm{Q}}$.

Proof. $\mathcal{T}_{\mathrm{F}}$ is the finest topology such that $\pi : X \to X^*$ is continuous. We know that $\pi : (X, \mathcal{T}_X) \to (X^*, \mathcal{T}_{\mathrm{Q}})$ continuous. Therefore, $\mathcal{T}_{\mathrm{Q}} \subset \mathcal{T}_{\mathrm{F}}$.

Since $\pi : (X, \mathcal{T}_X) \to (X^*, \mathcal{T}_{\mathrm{F}})$ is continuous, given any $U \in \mathcal{T}_{\mathrm{F}}$, $\pi^{-1}(U) \in \mathcal{T}_X$. By the definition of $\mathcal{T}_{\mathrm{Q}}$,

$$V \in \mathcal{T}_{\mathrm{Q}} \iff \pi^{-1}(V) \in \mathcal{T}_X. \tag{5.12}$$

Since $\pi^{-1}(U) \in \mathcal{T}_X$, we have $U \in \mathcal{T}_{\mathrm{Q}}$. As a result, $\mathcal{T}_{\mathrm{F}} \subset \mathcal{T}_{\mathrm{Q}}$. ■

Theorem 5.3

Let $p : X \to Y$ be a quotient map. Also, let A be a subspace of X that is saturated with respect to p. Now, let $q : A \to p(A)$ be the map obtained by restricting p to A. Then the following hold:

1. If A is either open or closed in X, then q is a quotient map.

2. If p is either an open or a closed map, then q is a quotient map.

Proof. ***Step 1.*** We first verify the following 2 equations:

$$q^{-1}(V) = p^{-1}(V), \text{ if } V \subset p(A). \tag{5.13}$$

$$p(U \cap A) = p(U) \cap p(A), \text{ if } U \subset X. \tag{5.14}$$

To check the first equation, note that $V \subset p(A)$ and A is saturated with respect to p. Since, the preimage operation respects inclusion, one has

$$p^{-1}(v) \subset p^{-1}(p(A)) = A. \tag{5.15}$$

Now,

$$p^{-1}(V) = \{x \in X \mid p(x) \in V\} \subset A. \tag{5.16}$$

$$q^{-1}(V) = \{x \in A \mid q(x) \in V\}. \tag{5.17}$$

(5.15) and (5.16) tells us that

$$p^{-1}(V) = \{x \in A \mid p(x) \in V\}. \tag{5.18}$$

Since $p(x) = q(x)$ for all $x \in A$, from (5.17) and (5.18), one has $p^{-1}(V) = q^{-1}(V)$.

For the second equation (5.14), one notes that the following holds using elementary set theory

$$p(U \cap A) \subset p(U) \cap p(A). \tag{5.19}$$

for the function $p : X \to Y$ and U, A being subsets of X. Now, let us show that $p(U) \cap p(A) \subset p(U \cap A)$. Suppose $y \in p(U) \cap p(A)$. Then there exists $u \in U$ and $a \in A$ such that,

$$y = p(a) = p(u). \tag{5.20}$$

From (5.20), one finds that $p(u) \in p(A)$. Hence, $u \in A$. But $u \in U$. Therefore,

$$u \in U \cap A. \tag{5.21}$$

One, therefore obtains using (5.20) and (5.21), that

$$y = p(u) \text{ with } u \in U \cap A. \tag{5.22}$$

Hence, $y \in p(u \cap A)$. Therefore, one concludes that,

$$p(u) \cap p(A) \subset p(u \cap A). \tag{5.23}$$

Now, (5.23) together with (5.19) implies that $p(u) \cap p(A) = p(u \cap A)$.

Step 2: Now suppose that $A \subset X$ is open or $p : X \to Y$ is an open map. Also, suppose that $q^{-1}(V) \subset A$ is open for some $V \subset p(A)$. We need to show that $V \subset p(A)$ is open.

Suppose first the case where $A \subset X$ is open. Also, $q^{-1}(V) \subset A$ is open by hypothesis. Hence, $q^{-1}(V) \subset X$ is open by Lemma 1.18. Now, since $V \subset p(A)$, one has by (5.13), $q^{-1}(V) = p^{-1}(v)$. Therefore, one has $p^{-1}(V) \subset X$ is open. Since $p : X \to Y$ is a quotient map, one has, $V \subset Y$ is open.

Since $V \subset p(A) \subset Y$ one has, V is open in $p(A)$ in the subspace topology it inherits from Y. Indeed, $V = V \cap p(A)$.

We proved that $q^{-1}(V) \subset A$ open implies $V \subset p(A)$ is open. The other direction follows from the fact that $q : A \to p(A)$ being a restriction of the continuous map $p : X \to Y$ is also continuous. Therefore, q is a quotient map.

Now suppose $p : X \to Y$ is open. Since $V \subset p(A)$ holds, by (5.13), one has $q^{-1}(V) = p^{-1}(V)$. By hypothesis, $q^{-1}(V) \subset A$ is open. Hence, $p^{-1}(V) \subset A$ is open. By the definition of subspace topology, there exists $U \subset X$ open such that

$$p^{-1}(V) = U \cap A. \tag{5.24}$$

Since, $p : X \to Y$ is a quotient map, it is surjective. Hence,

$$p(p^{-1}(V)) = V. \tag{5.25}$$

Combining (5.24) and (5.25), one has

$$V = p(U \cap A) = p(U) \cap p(A). \tag{5.26}$$

Since $U \subset X$ is open, $p(U) \subset Y$ is open. From the definition of subspace topology, it then follows from (5.26) that $V \subset p(A)$ is open.

Step 3: The proof for the case when $A \subset X$ is closed or $p : X \to Y$ is closed can be done exactly in the same way as step 2. ■

Example 5.4. Let A be a subset of $\mathbb{R}^2$ defined by

$$A = \left\{(x, y) \in \mathbb{R}^2 \mid x \geq 0 \text{ or } y = 0 \text{ (or both)}\right\}$$

Let $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ be the projection on the first coordinate. Let $q : A \to \mathbb{R}$ be the restriction of π_1, i.e. $q = \pi_1|_A$. Then q is a quotient map. But it is neither open, nor closed.

π_1 is an open map, so by Theorem 5.3, $q = \pi_1|_A$ is a quotient map. It is not an open map, since the image of the open set $(\mathbb{R} \times (0, \infty)) \cap A$ is not open. Indeed,

$$(\mathbb{R} \times (0, \infty)) \cap A = \left\{(x, y) \in \mathbb{R}^2 \mid x \geq 0\right\}. \tag{5.27}$$

So its image under p is $[0, \infty)$, which is not open in $\mathbb{R}$. q is not a closed map either, for the image of the closed set $\{(x, y) \mid xy = 1,\, x > 0\}$ is $(0, \infty)$ is not closed in $\mathbb{R}$.

Lemma 5.4

Composition of quotient maps is also a quotient map.

Proof. Let $p : X \to Y$ and $q : Y \to Z$ be quotient maps. We need to show that $q \circ p : X \to Z$ is a quotient map.

Let $U \subset Z$ be open. Since $q : Y \to Z$ is a quotient map, $U \subset Z$ is open if and only if $q^{-1}(U) \subset Y$ is open. Since $p : X \to Y$ is a quotient map, $q^{-1}(u) \subset Y$ is open if and only if $p^{-1}(q^{-1}(U)) \subset X$ is open. But

$$p^{-1}(q^{-1}(U)) = (q \circ p)^{-1}(U). \tag{5.28}$$

Hence, the quotientness of $q : Y \to Z$ and $p : X \to Y$ amount to the fact that $U \subset Z$ is open if and only if $(q \circ p)^{-1}(U) \subset X$ is open. This then implies that $q \circ p : X \to Z$ is a quotient map. ■

Theorem 5.5

Let $p : X \to Y$ be a quotient map. Let Z be a topological space, and $g : X \to Z$ be a map that is constant on each set $p^{-1}(\{y\})$, for all $y \in Y$. Then g induces a map $f : Y \to Z$ such that $f \circ p = g$. The induced map f is continuous if and only if g is continuous and f is a quotient map if and only if g is a quotient map.

Proof. By hypothesis, for every $y \in Y$, $g(p^{-1}(\{y\}))$ is a singleton. Define $f : Y \to Z$ by

$$f(y) = g(p^{-1}(\{y\})). \tag{5.29}$$

Take any $x \in X$, then (5.29) tells us that $f(p(x)) = g(x)$, i.e, $f \circ p = g$. In other words, the following diagram commutes:

$$\begin{tikzcd} X \arrow[d, "p"'] \arrow[dr, "g"] & \\ Y \arrow[r, "f"'] & Z \end{tikzcd}$$

Now, $p : X \to Y$, being a quotient map, is continuous. If f is continuous, $g = f \circ p$ is continuous. Now, suppose g is continuous. Therefore, if $V \subset Z$ is open, then $g^{-1}(V) \subset X$ is open. Since, $f \circ p = g$, one has

$$g^{-1}(V) = (f \circ p)^{-1}(V) = p^{-1}(f^{-1}(V)) \subset X \text{ is open.} \tag{5.30}$$

Since p is a quotient map, from (5.30), one concludes that $f^{-1}(v) \subset Y$ is open. Hence, f is continuous.

Now, suppose that f is a quotient map. Since p is a quotient map, by Lemma 5.4, $f \circ p = g$ is a quotient map. Now, let g be a quotient map. Then g is surjective. Then surjectivity of g implies that, for every $z \in Z$, there exists $x \in X$ such that, $g(x) = z$. Then, from (5.29), one has

$$f(p(x)) = g(x) = z \tag{5.31}$$

(5.31) tells us that for every $z \in Z$, there exists $y \in Y$, namely $y = p(x)$ such that $f(y) = z$. Hence, f is surjective. Now, since g is continuous (as g is a quotient map by hypothesis), f is continuous by the previous statement of this theorem. Therefore, $V \subset Z$ open implies $f^{-1}(V) \subset Y$ is open. For the other direction, assume $f^{-1}(V) \subset Y$ is open for some $v \subset Z$. As p is a quotient map, $p^{-1}(f^{-1}(V)) \subset X$ is open. But

$$p^{-1}(f^{-1}(v)) = (f \circ p)^{-1}(v) = g^{-1}(v). \tag{5.32}$$

One therefore obtains that $g^{-1}(V) \subset X$ is open. But g is a quotient map. Hence $V \subset Z$ is open. This proves that, f is a quotient map, as required. ■

Corollary 5.6

Let $g : X \to Z$ be a sujective continuous map. Let X^* be the following collection of subsets of X:

$$X^* = \{g^{-1}(\{z\}) \mid z \in Z\}.$$

Give X^* the quotient topology.

(a) The map g induces a bijective continuous map $f : X^* \to Z$, which is a homeomorphism if and only if g is a quotient map.

$$\begin{tikzcd} X \arrow[d, "p"'] \arrow[dr, "g"] & \\ X^* \arrow[r, "f"'] & Z \end{tikzcd}$$

(b) If Z is hausdorff, so is X^*.

Proof. (a) Observe that, X^* is the partition of X into disjoint subsets where elements of each subsets is mapped to a single element of Z by g. In other words, X^* is set of equivalence class of X under the equivalence relation $x \sim y$ if and only if $g(x) = g(y)$. Therefore, g induces a map $f : X^* \to Z$ by bijective map $f(g^{-1}(\{z\})) = z$.

Let p be the quotient map from X to X^* that sends each $x \in X$ to it's equivalence class Now, g is a map that is constant on each set $p^{-1}(\{y\})$. Hence, by Theorem 5.5, f is the induced map from X^* to Z by g such that $f \circ p = g$, which is continuous if and only if g is continuous. But g is continuous by hypothesis. Hence, f is continuous.

Suppose f is a homeomorphism. Then f is also a quotient map. Since p is a quotient map and g is composition of quotient maps, g is a quotient map by Lemma 5.4. Conversly suppose that g is a quotient map. Then f is a quotient map by theorem (5.5). But since f is bijective, f is therefore a homeomorphism.

(b) Suppose that Z is hausdorff. Let $x, y \in X^*$ be distinct. We need to show that there exists disjoint open neighbourhoods of x and y in X^*. Since f is bijective, $f(x) \neq f(y)$. Hence, there exists open sets $U, V \subset Z$ such that $f(x) \in U$, $f(y) \in V$ with $U \cap V = \varnothing$. Hence, f being a continuous, $f^{-1}(U)$, $f^{-1}(V)$ are distinct and open in X^*. Therefore, $f^{-1}(U)$, $f^{-1}(V)$ are disjoint open neighbourhoods of x and y in X^*. Thus, X^* is hausdorff.

■

Example 5.5. Let X be the closed unit ball in $\mathbb{R}^2$:

$$X = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$$

And let X^* be the partition of X into 2 classes:

1. All the one point sets $\{(x, y)\}$ with $x^2 + y^2 < 1$.
2. The set $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$.

Give X^* the quotient topology. Then X^* is homeomorphic to the subspace of $\mathbb{R}^3$ called the unit 2-sphere defined by:

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}.$$

Let $g : X \to S^2$ be the map defined as follows: given $(x, y) \in X = \overline{B(\mathbf{0}, 1)}$, we express $(x, y) = (r\cos\theta, r\sin\theta)$ for $0 \leq r \leq 1$ and $0 \leq \theta < 2\pi$. Then we define

$$g\,(x, y) = g\,(r\cos\theta, r\sin\theta) = (\sin(\pi r)\cos\theta, \sin(\pi r)\sin\theta, -\cos(\pi r)). \tag{5.33}$$

Then g is surjective. g is also continuous, since we can write g as follows:

$$g\,(x, y) = \begin{cases} \left(\frac{\sin\left(\pi\sqrt{x^2+y^2}\right)x}{\sqrt{x^2+y^2}}, \frac{\sin\left(\pi\sqrt{x^2+y^2}\right)y}{\sqrt{x^2+y^2}}, -\cos\left(\pi\sqrt{x^2+y^2}\right)\right) & \text{if } (x, y) \neq (0, 0), \\ (0, 0, -1) & \text{if } (x, y) = (0, 0); \end{cases} \tag{5.34}$$

and each components are continuous on $X \setminus \{(0, 0)\}$. For $(x, y) = (0, 0)$, observe that

$$\left|\frac{\sin\left(\pi\sqrt{x^2+y^2}\right)x}{\sqrt{x^2+y^2}}\right| \leq \left|\sin\left(\pi\sqrt{x^2+y^2}\right)\right|, \tag{5.35}$$

which approaches 0 as $(x, y) \to (0, 0)$. Similarly,

$$\left|\frac{\sin\left(\pi\sqrt{x^2+y^2}\right)y}{\sqrt{x^2+y^2}}\right| \leq \left|\sin\left(\pi\sqrt{x^2+y^2}\right)\right|. \tag{5.36}$$

Therefore,

$$\lim_{(x,y)\to(0,0)} g(x,y) = (0,0,-1), \tag{5.37}$$

so g is continuous at $(0,0)$ as well. So g is continuous on the whole X.

Now we see that X^* is precisely the collection $\{g^{-1}(\{z\}) \mid z \in S^2\}$. From $g(r_1\cos\theta_1, r_1\sin\theta_1) = g(r_2\cos\theta_2, r_2\sin\theta_2)$, comparing the third component, we get

$$\cos\pi r_1 = \cos\pi r_2 \implies r_1 = r_2, \tag{5.38}$$

since cos is injective on $[0,\pi]$. Comparing the first and second components, we get that

$$\text{either } \sin\pi r_1 = \sin\pi r_2 = 0 \text{ or } \cos\theta_1 = \cos\theta_2,\ \sin\theta_1 = \sin\theta_2. \tag{5.39}$$

The first option means that $r_1 = r_2 = 0$ or $r_1 = r_2 = 1$. But there is only one point with $r = 0$. The second option means that $\theta_1 = \theta_2$. Therefore, g is injective on $B(\mathbf{0},1)$, and all the points on S^1 gets mapped to $(0,0,1)$. Therefore, the collection

$$\left\{g^{-1}(\{z\}) \mid z \in S^2\right\} = \left(\bigcup_{x^2+y^2<1} \{(x,y)\}\right) \cup \left\{(x,y) \in X \mid x^2+y^2 = 1\right\} = X^*. \tag{5.40}$$

Now, one can show that g is a quotient map (in fact, it follows readily from Theorem 7.5 once we develop the notion of compact spaces). Then by Corollary 5.6, g induces a homeomorphism $f : X^* \to S^2$.

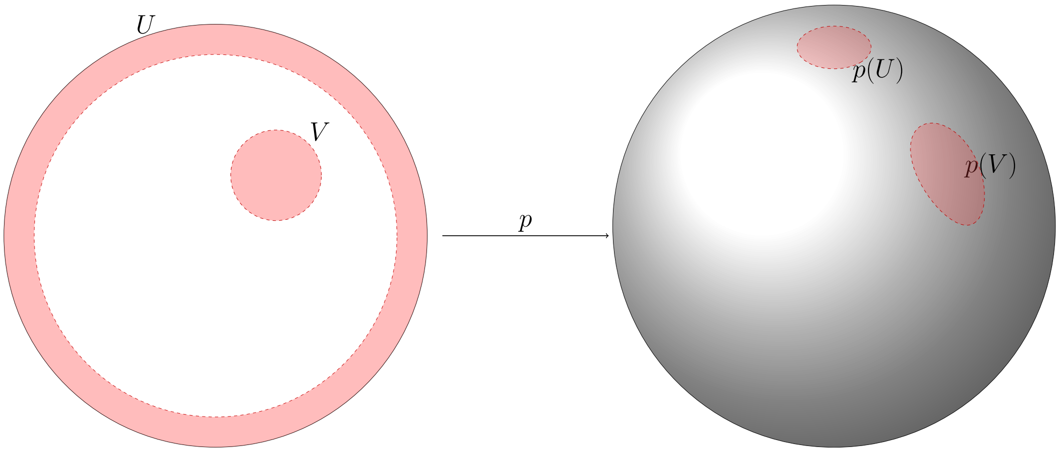
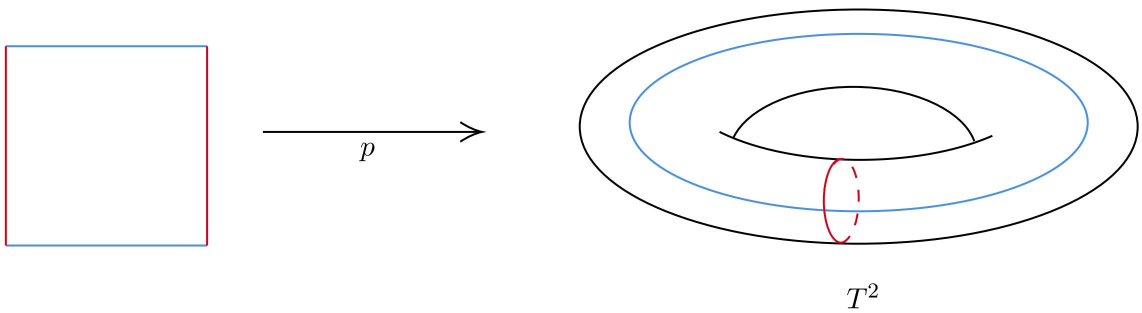


Figure 5.1: Open sets in X that are p-saturated

Example 5.6. Let X be the rectangle $[0,1]\times[0,1]$. Define a partition of X into the following classes:

1. all singletons $\{(x,y)\}$ with $x,y \in (0,1)$;
2. all sets of the form $\{(x,0),(x,1)\}$ with $x \in (0,1)$;
3. all sets of the form $\{(0,y),(1,y)\}$ with $y \in (0,1)$;
4. $\{(0,0),(1,0),(0,1),(1,1)\}$.



Now, use Corollary 5.6. $X = [0,1] \times [0,1]$. Take the continuous map $g : X \to \mathbb{R}^3$ defined by,

$$g(s,t) = ([b + a\cos 2\pi t]\cos 2\pi s, [b + a\cos 2\pi t]\sin 2\pi s, a\sin 2\pi t) \tag{5.41}$$

Deonte by Z, the image of X under g, i.e, $Z = g(X)$. Then $g : X \to Z$ is a continuous surjective map defined by (5.41). Now, check that $X^* = \{g^{-1}(\{z\}) \mid z \in Z\}$, is precisely what is given by the list above. Now, Corollary 5.6 states that there exists a bijective continuous map $f : X^* \to Z$ such that $f \circ p = g$ with $p : X \to X^*$ being the quotient map. One can show that g is a quotient map (in fact, it follows readily from Theorem 7.5 once we develop the notion of compact spaces). Hence, Corollary 5.6 guranatees that f is a homeomorphism. This homeomorphic image of X^* in $\mathbb{R}^3$ is called the 2-torus denoted by T^2.

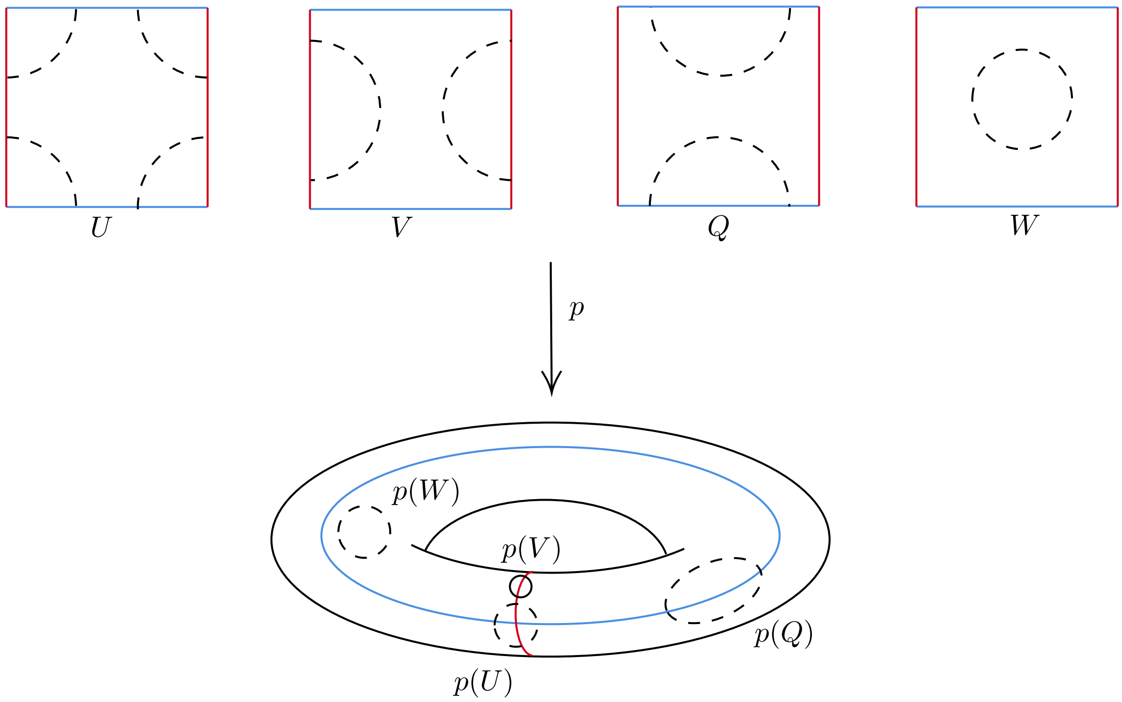


Figure 5.2: Open sets in X that are p-saturated

6 Connectedness

§6.1 Connected spaces

Definition 6.1. Let X be a topological space. A **separation** of X is a pair of disjoint non-empty open subsets U and V of X such that $U \cup V = X$. A topological space X is said to be **connected** if there does not exist a separation of X.

Lemma 6.1

A topological space X is connected if and only if the only subsets of X that are both open and closed (clopen) in X are the empty set and X itself.

Proof. For if A is a nonempty proper subset of X that is both open and closed in X, then the sets $U = A$ and $V = X - A$ are nonempty, open, disjoint, and their union is X. So they constitute a separation of X.

Conversely, if U and V form a separation of X, then U is a nonempty proper subset of X, and it is both open and closed in X, since $V = X \setminus U$ is open. ∎

Lemma 6.2

If Y is a subspace of X, a **separation** of Y is a pair of disjoint nonempty sets A and B whose union is Y, neither of which contains a limit point of the other. The space Y is connected if there exists no separation of Y.

Proof. We have to show that the usual definition of a pair U, V of Y (as a topological space) is equivalent to the one stated above. Then it will follow that the space Y is connected if there exists no separation U, V of Y in the above sense.

Suppose U, V forms a separation of Y. Then U is both open and closed in Y. If we denote the closure of U in X by $\overline{U}$, then the closure of U in Y, denoted by $\mathrm{Cl}_Y(U)$, is as follows:

$$\mathrm{Cl}_Y(U) = \overline{U} \cap Y. \tag{6.1}$$

Since $U \subset Y$ is closed, $\mathrm{Cl}_Y(U) = U$, so that one has

$$U = \overline{U} \cap Y. \tag{6.2}$$

Since the pair U, V forms a separation of Y, $U \cap V = \varnothing$. So

$$\varnothing = U \cap V = \overline{U} \cap Y \cap V = \overline{U} \cap V. \tag{6.3}$$

Suppose U' is the set of limit points of U in X. Then $\overline{U} = U \cup U'$. As a result,

$$\begin{aligned}
\overline{U} \cap V = \varnothing &\implies (U \cup U') \cap V = \varnothing \\
&\implies (U \cap V) \cup (U' \cap V) = \varnothing \\
&\implies \varnothing \cup (U' \cap V) = \varnothing \\
&\implies U' \cap V = \varnothing.
\end{aligned}$$

In other words, V doesn't contain any limit point of U in X. Similarly, U also doesn't contain ant of the limit points of V in X.

Conversely, suppose that U and V are disjoint nonempty sets such that $U \cup V = Y$ and neither of U and V contains a limit point of the other. By hypothesis, one has:

$$U' \cap V = U \cap V' = \varnothing. \tag{6.4}$$

Also by hypothesis, $U \cap V = \varnothing$. Since $\overline{U} = U \cup U'$, and we have $U' \cap V = U \cap V = \varnothing$, we get

$$\overline{U} \cap V = (U \cup U') \cap V = (U \cap V) \cup (U' \cap V) = \varnothing. \tag{6.5}$$

Similarly,

$$\overline{V} \cap U = \varnothing. \tag{6.6}$$

Since $\overline{U} \cap U = U$ and $\overline{U} \cap V = \varnothing$,

$$\overline{U} \cap Y = \overline{U} \cap (U \cup V) = \left(\overline{U} \cap U\right) \cup \left(\overline{U} \cap V\right) = U. \tag{6.7}$$

Similarly,

$$\overline{V} \cap Y = V. \tag{6.8}$$

So both U and V are closed sets in Y. Also, they are complement of each other in Y. So they are both open in Y. Hence, U, V are disjoint nonempty open subsets of Y such that $U \cup V = Y$, i.e. the pair U, V forms a separation of Y. ■

Example 6.1. Let $X = \{a, b\}$ be a 2-point set. Now take the topological space $(X, \mathcal{T}_{\text{indis}})$ by equipping the 2-point set with indiscrete topology so that the only open sets in X are just $\varnothing$ and X itself. In particular, none of the sets $\{a\}$ or $\{b\}$ is open in X with indiscrete topology so that there exists no separation of X. Hence, X equipped with indiscrete topology is connected.

Example 6.2. Let X be the subspace $[-1, 0) \cup (0, 1]$ of $\mathbb{R}$ equipped with standard topology. The sets $[-1, 0)$ and $(0, 1]$ are disjoint and nonempty, and their union is Y. Also, note that $[-1, 0)$ has a limit point in common with the set $(0, 1]$, namely 0. But 0 belongs to none of these sets. In particular, we see that none of the sets $[-1, 0)$ and $(0, 1]$ contains a limit point of the other in $\mathbb{R}$. Hence, by Lemma 6.1, the sets $[-1, 0)$ and $(0, 1]$ form a separation of Y, and hence $Y \subset \mathbb{R}$ is not connected.

Example 6.3. Let X be the subspace $[-1, 1]$ of $\mathbb{R}$ equipped with standard topology. The sets $[-1, 0]$ and $(0, 1]$ are disjoint and nonempty. Also, $[-1, 0] \cup (0, 1] = [-1, 1]$. Yet the pair $[-1, 0]$ and $(0, 1]$ doesn't form a separation of X. This is because $[-1, 0]$ contains 0 which is a limit point of $(0, 1]$. In fact, it will be shown later that there exists no separation of the space $[-1, 1]$.

Example 6.4. Let $\mathbb{Q}$ be the subspace of $\mathbb{R}$ equipped with subspace topology it inherits from $\mathbb{R}$ (with respect to standard topology). Then $\mathbb{Q}$ is not connected. In fact, a set consisting of 2 rational points p, q with $p < q$ is not connected. Here, $Y = \{p, q\}$ is a subspace of $\mathbb{R}$. Then the pair $\{p\}, \{q\}$ forms a separation of Y. Indeed, $\{p\}$ and $\{q\}$ are both open in Y in the subspace topology it inherits from $\mathbb{R}$: choose $a \in \mathbb{R} \setminus \mathbb{Q}$ that lies between p and q, i.e., $p < a < q$. Then write

$$\{p\} = Y \cap (-\infty, a), \;\; \{q\} = Y \cap (a, \infty).$$

Indeed, $\{p\} \cap \{q\} = \varnothing$ and $Y = \{p\} \cup \{q\}$. This proves that Y is not connected. Only singletons containing one rational point are connected.

Lemma 6.3

If the sets C and D form a separation of X, and if Y is a connected subspace of X, then Y lies entirely within either C or D.

Proof. Since C and D are both open in X, from the definition of subspace topology one has $C \cap Y$ and $D \cap Y$ to be open in Y. Since $C \cap Y \subset C$ and $D \cap Y \subset D$ and $C \cap D = \varnothing$, one must have

$$(C \cap Y) \cap (D \cap Y) = \varnothing. \tag{6.9}$$

Also,

$$(C \cap Y) \cup (D \cap Y) = (C \cup D) \cap Y = X \cap Y = Y. \tag{6.10}$$

Hence, the pair $C \cap Y, D \cap Y$ will form a separation of Y if they are both nonempty. But by hypothesis, Y is connected. Hence, one of $C \cap Y$ and $D \cap Y$ is empty. In other words, Y must entirely be contained in C or in D. ■

Theorem 6.4

The union of a collection of connected subspaces of X having a point in common is connected.

Proof. Let $\{A_\alpha\}_{\alpha \in J}$ be a collection of connected subspaces of a topological space X. Also, let $p \in \bigcap_{\alpha \in J} A_\alpha$. We want to show that $\bigcup_{\alpha \in J} A_\alpha$ is connected. We proceed by contradiction. Let $Y = \bigcup_{\alpha \in J} A_\alpha$. Let C, D be a separation of Y, i.e.,

$$C \cup D = Y = \bigcup_{\alpha \in J} A_\alpha, \quad \text{with} \quad C \cap D = \varnothing. \tag{6.11}$$

Now, since $p \in \bigcap_{\alpha \in J} A_\alpha$, it follows that p will be in one of the sets. Suppose $p \in C$. Since A_α is connected for each $\alpha \in J$, it must lie entirely in either C or D. Since $p \in A_\alpha$ and $p \in C$, A_α cannot be contained in D. Therefore, one has $A_\alpha \subset C$ for all $\alpha \in J$. As a result,

$$C \cup D = Y = \bigcup_{\alpha \in J} A_\alpha \subset C. \tag{6.12}$$

$C \cup D \subset C$ implies that $D \subset C$, contradicting the fact that the pair C, D being a separation for Y (and hence they have to be disjoint). ■

Theorem 6.5

Let A be a connected subspace of X. If $A \subset B \subset \overline{A}$, then B is also connected. In other words, if B is formed by adjoining to the connected subspace A some or all of its limit points, then B is connected.

Proof. Let A be connected and let $A \subset B \subset \overline{A}$. Suppose the pair C, D is a separation for B, so that $B = C \cup D$. By Lemma 6.3, A being a connected subspace of B entirely lies in C or in D; suppose that $A \subset C$.

Then we have $\overline{A} \subset \overline{C}$. We know from Lemma 6.1 that $\overline{C} \cap D = \varnothing$. Then we have $B \subset \overline{A} \subset \overline{C}$. Since $B \subset \overline{C}$,

$$B \cap D \subset \overline{C} \cap D = \varnothing. \tag{6.13}$$

So B and D are disjoint. But from $B = C \cup D$, one has $D \subset B$, so that

$$B \cap D = D. \tag{6.14}$$

(6.13) and (6.14) together imply that $D = \varnothing$. This is a contradiction, as C, D is a separation for B, and hence both C and D are nonempty, disjoint, open sets. ■

Theorem 6.6

The image of a connected space under a continuous map is connected.

Proof. Let $f : X \to Y$ be a continuous map; let X be connected. We wish to prove the image space $Z = f(X)$ is connected. Observe that it is sufficient to prove that the image of a connected space under a continuous surjective map is connected. Because if one proves so, then one can easily construct a continuous surjective map g from the given continuous map $f : X \to Y$ by restricting the codomain of f to its range, i.e. a continuous surjective map $g : X \to Z = f(X)$, to prove that $Z = f(X) = g(X)$ is connected. We will, therefore, consider the case of a continuous surjective map $g : X \to Z$.

Suppose that the pair A, B is a separation of Z, i.e. $Z = A \cup B$ and $A \cap B = \varnothing$. Now, we have

$$X = g^{-1}(Z) = g^{-1}(A \cup B) = g^{-1}(A) \cup g^{-1}(B). \tag{6.15}$$

A and B are open in Z, so $g^{-1}(A)$ and $g^{-1}(B)$ are open in X by the continuity of g. Furthermore,

$$g^{-1}(A) \cap g^{-1}(B) = g^{-1}(A \cap B) = g^{-1}(\varnothing) = \varnothing. \tag{6.16}$$

(6.15) and (6.16) together imply that the pair $g^{-1}(A), g^{-1}(B)$ is a separation of X, contradicting the connectedness of X. Therefore, Z must be connected. ■

Corollary 6.7

A topological space X is connected if and only if every continuous map $f : X \to \{0,1\}$ is constant, where $\{0,1\}$ is equipped with the discrete topology.

Proof. If f is connected, then so is $f(X)$. But the only connected subspaces of $\{0,1\}$ are singletons. So f must be constant.

Suppose there is a non-constant continuous function $f : X \to \{0,1\}$. Then $f^{-1}(\{0\})$ and $f^{-1}(\{1\})$ are nonempty, disjoint, open subsets of X, and their union is X. So X is not connected. ■

Theorem 6.8

A finite cartesian product of connected spaces is connected.

Proof. As usual, we shall prove it for the product of two connected spaces. The theorem then follows by induction. Suppose X and Y are connected. We shall prove that $X \times Y$ is connected in the product topology.

Each set of the form $\{x\} \times Y$ and $X \times \{y\}$ is connected, since they are homeomorphic to Y and X, respectively. Let $f : X \times Y \to \{0,1\}$ be continuous. Choose $(x_1, y_1), (x_2, y_2) \in X \times Y$.

$$f_1 = f|_{\{x_1\} \times Y} \quad \text{and} \quad f_2 = f|_{X \times \{y_2\}} \tag{6.17}$$

are continuous as they are restrictions of a continuous map. Since the domains of f_1 and f_2 are connected, they are constant maps. So we have

$$\begin{aligned} f(x_1, y_1) &= f_1(x_1, y_1) \\ &= f_1(x_1, y_2) \\ &= f(x_1, y_2) \\ &= f_2(x_1, y_2) \\ &= f_2(x_2, y_2) \\ &= f(x_2, y_2). \end{aligned} \tag{6.18}$$

Therefore, f is constant. Since every continuous map $f : X \times Y \to \{0,1\}$ is constant, $X \times Y$ is connected by Corollary 6.7. ■

Example 6.5. Consider the countably infinite cartesian product $\mathbb{R}^\omega$ in the box topology. Let us denote by A the subset of $\mathbb{R}^\omega$ consisting of all bounded sequences of real numbers. Also, denote by B the subset of $\mathbb{R}^\omega$ consisting of all unbounded sequences of real numbers. It's easy to verify that $A \cup B = \mathbb{R}^\omega$ and $A \cap B = \varnothing$.

We shall now prove that both A and B are open in the box topology. Note that, a sequence $\mathbf{x} = (x_i)_{i\in\mathbb{N}}$ is **bounded** if there exists some $M \in \mathbb{N}$ such that

$$|x_n| < M, \tag{6.19}$$

for every $n \in \mathbb{N}$. On the other hand, a sequence $\mathbf{x} = (x_i)_{i\in\mathbb{N}}$ is **unbounded** if given any $M \in \mathbb{N}$, there exists some $n_0 \in \mathbb{N}$ (possibly depeding on M), such that

$$|x_{n_0}| > M. \tag{6.20}$$

We now show that A is open in $\mathbb{R}^\omega$ in the box topology. Take $\mathbf{a} = (a_i)_{i\in\mathbb{N}} \in A$. We need to find a basic open set U such that

$$\mathbf{a} \in U \subset A.$$

We choose

$$U = \prod_{i\in\mathbb{N}} (a_i - 1, a_i + 1). \tag{6.21}$$

Clearly, $\mathbf{a} \in U$. Now take $\mathbf{x} = (x_i)_{i\in\mathbb{N}} \in A$. Since $\mathbf{a} = (a_i)_{i\in\mathbb{N}}$ is bounded, there exists some $M \in \mathbb{N}$ such that $|a_n| < M$ for every $n \in \mathbb{N}$. Since $x_n \in (a_n - 1, a_n + 1)$, $|x_n - a_n| < 1$. Therefore,

$$|x_n| \leq |x_n - a_n| + |a_n| < 1 + M. \tag{6.22}$$

Therefore, $(x_i)_{i\in\mathbb{N}}$ is a bounded sequence, i.e. $\mathbf{x} \in A$. So $U \subset A$. Therefore, A is open.

Now we show that B is open in $\mathbb{R}^\omega$ in the box topology. Take $\mathbf{b} = (b_i)_{i\in\mathbb{N}} \in B$. We need to find a basic open set V such that

$$\mathbf{b} \in V \subset B.$$

We choose

$$V = \prod_{i\in\mathbb{N}} (b_i - 1, b_i + 1). \tag{6.23}$$

Clearly, $\mathbf{b} \in V$. Now, take $\mathbf{y} = (y_i)_{i\in\mathbb{N}} \in V$. Since $(b_i)_{i\in\mathbb{N}}$ is unbounded, given any $M \in \mathbb{N}$, there exists $i_0 \in \mathbb{N}$ such that

$$|b_{i_0}| > M + 1.$$

Then, $y_{i_0} \in (b_{i_0} - 1, b_{i_0} + 1)$, so $|y_{i_0} - b_{i_0}| < 1$. As a result,

$$|y_{i_0}| \geq |b_{i_0}| - |b_{i_0} - y_{i_0}| > M + 1 - 1 = M. \tag{6.24}$$

Hence, given any $M \in \mathbb{N}$, there exists $i_0 \in \mathbb{N}$ such that

$$|y_{i_0}| > M.$$

Therefore, $(y_i)_{i\in\mathbb{N}}$ is an unbounded sequence, i.e. $\mathbf{y} \in B$. So $V \subset B$. Therefore, B is open.

Since both A and B are open, and they are disjoint, the pair A, B is a separation for $\mathbb{R}^\omega$. Therefore, $\mathbb{R}^\omega$ is not connected in the box topology. This illustrates that an infinite product of connected spaces in the box topology need not be connected (the fact that $\mathbb{R}$ is connected is to be proved in the next section).

Example 6.6. In this example consider $\mathbb{R}^\omega$ in the product topology. Assuming that $\mathbb{R}$ is connected, we show that $\mathbb{R}^\omega$ is connected. Let $\widetilde{\mathbb{R}}^n \subset \mathbb{R}^\omega$ be a subspace of $\mathbb{R}^\omega$ (in product topology) consisting of all sequences $\mathbf{x} = (x_1, x_2, \ldots)$ such that $x_i = 0$ for $i > n$. In other words,

$$\widetilde{\mathbb{R}}^n = \{(x_1, x_2, \ldots, x_n, 0, 0, 0, \ldots) \mid x_i \in \mathbb{R}\}. \tag{6.25}$$

The topological space $\widetilde{\mathbb{R}}^n$ is clearly seen to be homeomorphic to $\mathbb{R}^n$ being connected by Theorem 6.8. Let us denote by $\mathbb{R}^\infty$, the union of all the $\widetilde{\mathbb{R}}^n$'s, i.e.,

$$\mathbb{R}^\infty = \bigcup_{n=1}^{\infty} \widetilde{\mathbb{R}}^n. \tag{6.26}$$

One immediately sees that $(0, 0, \dots) \in \widetilde{\mathbb{R}}^n$ for each n. Therefore, by Theorem 6.4, $\mathbb{R}^\infty$ is connected. We will now show that the closure of $\mathbb{R}^\infty$ (in product topology) is all of $\mathbb{R}^\omega$. Then it will follow from Theorem 6.5 that $\mathbb{R}^\omega$ is connected.

Let $\mathbf{a} = (a_i)_{i\in\mathbb{N}} = (a_1, a_2, \dots)$ be a point of $\mathbb{R}^\omega$. Also, let

$$U = \prod_{i\in\mathbb{N}} U_i \tag{6.27}$$

be a basis element for the product topology in $\mathbb{R}^\omega$ that contains $\mathbf{a}$. We now show that $\mathbf{a} \in \overline{\mathbb{R}^\infty}$, the closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ in product topology. By Theorem 2.8, all we need to show is that $U \cap \mathbb{R}^\infty \neq \varnothing$. From the definition of product topology on an infinite fold cartesian product of topological spaces, we know that there exists $N \in \mathbb{N}$ such that $U_i = \mathbb{R}$ for every $i > N$. Hence, the point

$$\mathbf{z} = (a_1, \dots, a_N, 0, 0, \dots) \in U = \prod_{i\in\mathbb{N}} U_i, \tag{6.28}$$

as $a_i \in U_i$ for $i \in \{1, \dots, N\}$, and $0 \in U_i = \mathbb{R}$ for $i > N$. Thus $\mathbf{z} \in U \cap \mathbb{R}^\infty$, proving that $U \cap \mathbb{R}^\infty$ is nonempty. Therefore, the closure of $\mathbb{R}^\infty$ is $\mathbb{R}^\omega$, and hence $\mathbb{R}^\omega$ is connected.

Theorem 6.9

If $\{X_\alpha\}_{\alpha\in J}$ is a collection of connected spaces, then their product

$$\prod_{\alpha\in J} X_\alpha$$

is connected in the product topology.

Proof. If J is finite, then we are done by Theorem 6.8. Now suppose J is infinite. Fix $\mathbf{a} = (a_\alpha)_{\alpha\in J} \in \prod_{\alpha\in J} X_\alpha$. Given any finite set $F \subset J$, we define the following subset of $\prod_{\alpha\in J} X_\alpha$:

$$X_F = \left\{ \mathbf{x} = (x_\alpha)_{\alpha\in J} \in \prod_{\alpha\in J} X_\alpha \mid x_\alpha = a_\alpha \text{ for } \alpha \notin F \right\}. \tag{6.29}$$

So we can write X_F as follows:

$$X_F = \prod_{\alpha\in J} U_\alpha, \text{ where } U_\alpha = \begin{cases} X_\alpha & \text{if } \alpha \in F, \\ \{a_\alpha\} & \text{if } \alpha \notin F. \end{cases} \tag{6.30}$$

Then X_F is homeomorphic to the space

$$\prod_{\alpha\in F} X_\alpha. \tag{6.31}$$

Since this is a finite product of connected spaces, by Theorem 6.8, X_F is connected. Now, for any finite set $F \subset J$, $\mathbf{a} \in X_F$. Therefore, the union

$$X := \bigcup_{F\subset J,\ F \text{ finite}} X_F \tag{6.32}$$

is connected by Theorem 6.4. Now we claim that $\overline{X} = \prod_{\alpha\in J} X_\alpha$. Then it will follow from Theorem 6.5 that $\prod_{\alpha\in J} X_\alpha$ is connected.

Choose any $\mathbf{b} = (b_\alpha)_{\alpha\in J} \in \prod_{\alpha\in J} X_\alpha$, and any basic open set

$$U = \prod_{\alpha\in J} U_\alpha \tag{6.33}$$

containing $\mathbf{b}$. We now show that $\mathbf{b} \in \overline{X}$, the closure of X in $\prod_{\alpha\in J} X_\alpha$ in product topology. By Theorem 2.8, all we need to show is that $U \cap X \neq \varnothing$.

By the definition of product topology, $U_\alpha = X_\alpha$ for all but finitely many α's. Let F be the finite subset of J, consisting of all $\alpha \in J$ for which $U_\alpha \neq X_\alpha$. Then we take the point $\mathbf{c} = (c_\alpha)_{\alpha\in J}$ defined by

$$c_\alpha = \begin{cases} b_\alpha & \text{if } \alpha \in F, \\ a_\alpha & \text{if } \alpha \notin F. \end{cases} \tag{6.34}$$

By the definition of X_F given in (6.30), $\mathbf{c} \in X_F$. Furthermore, for $\alpha \in F$, $c_\alpha = b_\alpha \in U_\alpha$, and for $\alpha \notin F$, $c_\alpha = a_\alpha \in X_\alpha = U_\alpha$. Therefore, $\mathbf{c} \in U$ as well. Hence,

$$\mathbf{c} \in U \cap X, \tag{6.35}$$

i.e. $U \cap X$ is nonempty. Therefore, the closure of X is $\prod_{\alpha\in J} X_\alpha$, and hence $\prod_{\alpha\in J} X_\alpha$ is connected. ■

§6.2 Connected subspaces of $\mathbb{R}$

We now use the existence of least upper bound for nonempty bounded subsets of $\mathbb{R}$ to prove that $\mathbb{R}$ is connected.

Definition 6.2 (Convex sets). A subset $C \subset \mathbb{R}$ is convex if for every pair of points $a, b \in C$, the closed interval $[a, b]$ is also contained in C.

Example 6.7. $\mathbb{Q} \subset \mathbb{R}$ is not convex. Also, $(-\infty, 0) \cup (0, \infty)$ is not convex in $\mathbb{R}$ as one can choose $-1 < 1$ in $\mathbb{R}$ with $[-1, 1] \nsubseteq (-\infty, 0) \cup (0, \infty)$.

The convex subsets of $\mathbb{R}$ are $\varnothing$, (a, b), $[a, b)$, $(a, b]$, $[a, b]$, $(-\infty, b)$, $(-\infty, b]$, (a, ∞), $[a, \infty)$, and $\mathbb{R}$ itself.

Theorem 6.10

Each convex subset $C \subset \mathbb{R}$ is connected.

Proof. Suppose the contrary and U, V is a separation of C. Choose $a \in U$ and $b \in V$. We may assume $a < b$. Let $A = [a, b] \cap U$ and $B = [a, b] \cap V$. Then one finds that A, B is a separation of $[a, b]$ with $a \in A$ and $b \in B$. Indeed,

$$A \cap B = [a, b] \cap (U \cap V) = \varnothing, \tag{6.36}$$

since $U \cap V = \varnothing$; and

$$A \cup B = [a, b] \cup (U \cap V) = [a, b] \cap C = [a, b], \tag{6.37}$$

since C is convex, so $[a, b] \subset C$. Now, $A = [a, b] \cap U$ is open in $[a, b]$ in the subspace topology, since U is open in C. Also, since A is bounded, it has a least upper bound. Let $c = \sup A$ be the least upper bound of the elements of A. Now, since $A \subset [a, b]$, b is an upper bound. As c is the least upper bound,

$$c \leq b. \tag{6.38}$$

Also, from $A \subset [a, b]$, every element of A is greater than or equal to a. Since c is an upper bound of the elements of A,

$$a \leq c. \tag{6.39}$$

Therefore,

$$a \leq c \leq b. \tag{6.40}$$

If $c = a$,then $A = \{a\}$, contradicting the fact that A is open in $[a, b]$. If $c = b$, then we use the following property of real line:

Supremum of a subset of $\mathbb{R}$ can be made arbitrarily close to a point in the subset.

It means that given a positive real number $\varepsilon > 0$, there exists $x \in A$ s.t. $c - x < \varepsilon$. Now, A is a closed subset of $[a, b]$ and $[a, b]$ is closed in $\mathbb{R}$ so that A is closed in $\mathbb{R}$. Therefore, $\sup A \in A$ or in other words, $b \in A$ which contradicts the fact that $A \cap B = \varnothing$, since $b \in B$.

One, therefore, is left with the only possibility $a < c < b$. Note that c being the least upper bound for A, is also an upper bound for the elements of A. Hence, $A \cap (c, b] = \varnothing$. As a result,

$$(c, b] \subset B. \tag{6.41}$$

Hence, there are points $y \in (c, b] \subset B$ arbitrarily close to c, i.e., given $\varepsilon > 0$, there exists $y \in (c, b] \subset B$ s.t. $y - c < \varepsilon$. In other words, c is in the closure of B. But B is closed in $[a, b]$ closed in $\mathbb{R}$ so that B is closed in $\mathbb{R}$ so that $c \in B$ leading to $c \in B$. As above, $c = \sup A \in A$. So we have $A \cap B \neq \varnothing$, which leads to a contradiction! Therefore, C is connected. ■

Remark 6.1. In Example 6.7, we saw that $\mathbb{R}$ is a convex subset of itself. Hence, by Theorem 6.10, $\mathbb{R}$ is connected.

Remark 6.2. The converse of Theorem 6.10 also holds. In other words, the only connected subsets of $\mathbb{R}$ are convex sets. Suppose $X \subset \mathbb{R}$ is not convex. Choose $a, b \in X$ such that $[a, b] \not\subset X$. So there exists $c \in (a, b)$ and $c \notin X$. Then choose $U = X \cap (-\infty, c)$ and $V = X \cap (c, \infty)$. Clearly, U and V are open in X, and they are disjoint. Also,

$$U \cup V = (X \cap (-\infty, c)) \cup (X \cap (c, \infty)) = X, \tag{6.42}$$

since $X \subset \mathbb{R} \setminus \{c\}$. U and V are also nonempty, since $a \in U$ and $b \in V$. So U, V is a separation for X, proving that non-convex subsets of $\mathbb{R}$ are not connected.

Theorem 6.11 (Intermediate value theorem)

Let $f : X \to \mathbb{R}$ be a continuous map with X being a connected topological space. If $a, b \in X$, and r lies between $f(a)$ and $f(b)$, then there exists a point $c \in X$ with $f(c) = r$.

Proof. Since X is connected, so is $f(X)$, by Theorem 6.6. By Theorem 6.10 and Remark 6.2, connected is equivalent to convex for $\mathbb{R}$. Therefore, $f(X)$ is convex. Given $a, b \in X$, WLOG, assume $f(a) < f(b)$. Since $f(X)$ is convex, we have that

$$[f(a), f(b)] \subset f(X). \tag{6.43}$$

Therefore, given any r that lies between $f(a)$ and $f(b)$, $r \in f(X)$, i.e. $r = f(c)$ for some $c \in X$. ■

§6.3 Path connected spaces

Definition 6.3 (Path)**.** Given points $x, y \in X$, a **path** in X from x to y is a continuous map $f : [a, b] \to X$ with $f(a) = x$ and $f(b) = y$, where $[a, b] \subset \mathbb{R}$ in the subspace topology.

Definition 6.4 (Path connected space)**.** A space X is said to be **path connected** if every pair of points of X can be joined by a path in X.

Lemma 6.12

A path connected space is connected.

Proof. Let X be path connected, and suppose that the pair U, V is a separation of X. Choose points $x \in U$ and $y \in V$, and a path $f : [a, b] \to X$ from x to y, i.e., $f(a) = x$ and $f(b) = y$. Then one immediately finds that $f^{-1}(U)$ and $f^{-1}(V)$ form a separation for $[a, b]$. Indeed, from $[a, b] = f^{-1}(X)$ follows,

$$[a, b] = f^{-1}(X) = f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V). \tag{6.44}$$

Also,

$$f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = f^{-1}(\varnothing) = \varnothing. \tag{6.45}$$

From the continuity of $f : [a, b] \to X$, it follows that both $f^{-1}(U)$ and $f^{-1}(V)$ are open in $[a, b]$ as U and V are both open in X, forming a separation for X. Hence, we indeed verify that $f^{-1}(U), f^{-1}(V)$ forms a separation for $[a, b]$, contradicting the fact that $[a, b]$ is a connected subspace of $\mathbb{R}$. ■

Lemma 6.13

The continuous image of a path connected space is path connected.

Proof. Let $g : X \to Y$ be a continuous map where X is path connected. We need to show that $g(X) \subset Y$ is path connected. Any two points in $g(X)$ can be written $g(x)$ and $g(y)$ with $x, y \in X$.

Since X is path connected, there exists a path $f : [a, b] \to X$ in X joining x to y, i.e.,

$$f(a) = x \quad \text{and} \quad f(b) = y. \tag{6.46}$$

Then it is easy to see that $g \circ f : [a, b] \to Y$ is a continuous map as a composition of 2 continuous maps and

$$g(f(a)) = g(x) \quad \text{and} \quad g(f(b)) = g(y). \tag{6.47}$$

In other words, $g \circ f : [a, b] \to Y$ is a path in $g(X)$ joining $g(x)$ to $g(y)$. We, therefore, have shown that any two points $g(x)$ and $g(y)$ in $g(X)$ can be joined by a path $g \circ f : [a, b] \to Y$, proving that $g(X)$ is path connected. ■

Definition 6.5. A subset C of a real vector space V is **convex** if for each pair of points $\mathbf{x}, \mathbf{y} \in C$, the straight line path $f : [0, 1] \to V$ defined by

$$f(t) = (1 - t)\mathbf{x} + t\mathbf{y}$$

takes all its values in C, i.e. $f(t) \in C$ for every $t \in [0, 1]$.

Example 6.8. Any convex subset of $\mathbb{R}^n$ is path connected. For example, the n-ball

$$B^n = \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\| \leq 1\} \tag{6.48}$$

is convex for $n \geq 1$, hence path connected.

Example 6.9. The punctured Euclidean space $\mathbb{R}^n \setminus \{0\}$ is path connected for $n \geq 2$. For $n = 1$, the space $\mathbb{R} \setminus \{0\}$ is not connected, and hence not path connected (by the contrapositive of Lemma 6.12). For $n = 0$, the space $\mathbb{R}^0 \setminus \{0\}$ is empty, and hence is path connected, vacuously.

Example 6.10. The $(n - 1)$-dimensional unit sphere

$$S^{n-1} = \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\| = 1\} \tag{6.49}$$

is the continuous image of $g : \mathbb{R}^n \setminus \{\mathbf{0}\} \to S^{n-1}$ given by

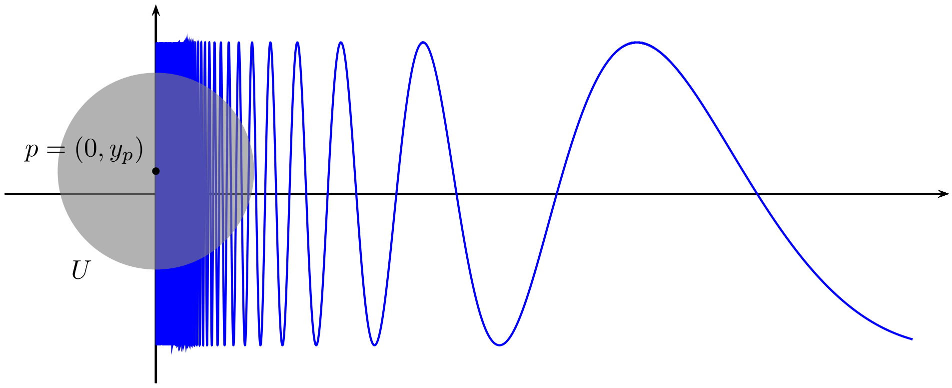
$$g(\mathbf{x}) = \frac{\mathbf{x}}{\|\mathbf{x}\|} \tag{6.50}$$

and hence is path connected by Lemma 6.13, for $n \geq 2$. For $n = 1$, the 0-sphere $S^0 = \{1, -1\}$ is not connected, and hence not path connected. For $n = 0$, the (-1)-sphere S^{-1} is empty, and hence is path connected vacuously.

Example 6.11. The converse of Lemma 6.12) does not hold, in general. Let

$$S = \left\{ \left(x, \sin\left(\frac{1}{x}\right)\right) \mid 0 < x \leq 1 \right\} \subset \mathbb{R}^2. \tag{6.51}$$

It is the image of the connected space $(0,1]$ under the continuous map $g(t) = \left(t, \sin\left(\frac{1}{t}\right)\right)$, and hence is connected by Theorem 6.6.



It oscillates extremely rapidly as t approaches 0. Any point of the vertical interval $V = \{0\} \times [-1,1]$ is a limit point of S. Let us denote the closure of S in $\mathbb{R}^2$ by $\overline{S}$, so that

$$\overline{S} = S \cup V. \tag{6.52}$$

The space $\overline{S}$ is called the **topologist's sine curve**. Since S is connected, it's closure $\overline{S}$ is connected by Theorem 6.5. Now we show that $\overline{S}$ is not path connected. Assume the contrary.

Choose two points $p, q \in \overline{S}$, with $p \in V$ and $q \in S$. Since $\overline{S}$ is path connected, there is a path $f : [a,b] \to \overline{S}$ such that $f(a) = p \in V$ and $f(b) = q \in S$. Now, consider the set

$$A = \{t \in [a,b] \mid f(t) \in V\} = f^{-1}(V). \tag{6.53}$$

This is closed in A, since V is closed in $\overline{S}$ (V is closed in $\mathbb{R}^2$, so $V = \overline{S} \cap V$ is closed in $\overline{S}$). Hence, $\sup A \in A$. Suppose

$$c = \sup A. \tag{6.54}$$

Now consider the subset $[c,b] \subset [a,b]$, and the restriction $f|_{[c,b]}$. Since $c \in A$, $f(c) \in V$. The restriction of a continuous function is continuous, so $f|_{[c,b]}$ is a path from $f(c) \in V$ to $f(b) = q \in S$.

Note that, $f(t) \in S$ for every $t \in (c,b]$. Let us now reparametrize the path $f|_{[c,b]}$ with help of the following homeomorphism

$$\begin{aligned} \alpha : [0,1] &\to [c,b], \\ t &\mapsto (1-t)c + tb. \end{aligned} \tag{6.55}$$

α maps 0 to c and 1 to b. Then we form the composition

$$\widetilde{f} = f|_{[c,b]} \circ \alpha : [0,1] \to \overline{S}. \tag{6.56}$$

Then $\widetilde{f}(0) \in V$, and $\widetilde{f}(t) \in S$ for $t \in (0,1]$. Now write

$$\widetilde{f}(t) = (x(t), y(t)). \tag{6.57}$$

$\widetilde{f}(0) \in V = \{0\} \times [-1,1]$, so $x(0) = 0$. Also since $\widetilde{f}(t) \in S$ for $t \in (0,1]$, we have $x(t) > 0$.

We now show that there is a sequence of points $t_n \in (0,1]$ with $0 < t_n < \frac{1}{n}$, and $y(t_n) = (-1)^n$. $0 < t_n < \frac{1}{n}$ implies that

$$\lim_{n\to\infty} t_n = 0. \tag{6.58}$$

But $y(t_n) = (-1)^n$ does not converge, contradicting the continuity of y. Let us now construct the sequence t_n.

For each $n \in \mathbb{N}$, we choose a point v with

$$v > \frac{1}{x\left(\frac{1}{n}\right)} \quad \text{and} \quad \sin v = (-1)^n . \tag{6.59}$$

Let $u = \frac{1}{v}$, so clearly $u > 0 = x(0)$. Also,

$$u = \frac{1}{v} < x\left(\frac{1}{n}\right). \tag{6.60}$$

u lies between $x(0)$ and $x\left(\frac{1}{n}\right)$. Therefore, by the Intermediate value theorem and the continuity of x, there exists some $t_n \in \left(0, \frac{1}{n}\right)$ such that

$$x(t_n) = u. \tag{6.61}$$

For $t_n \in \left(0, \frac{1}{n}\right)$, we have

$$y(t_n) = \sin\left(\frac{1}{x(t_n)}\right) = \sin\left(\frac{1}{u}\right) = \sin v = (-1)^n . \tag{6.62}$$

Therefore, we have constructed the sequence $(t_n)_{n\in\mathbb{N}}$ which converges to 0, but $(y(t_n))_{n\in\mathbb{N}}$ does not converge, contradicting the continuity of y. So $\overline{S}$ can't be path connected.

§6.4 Components and path components

Definition 6.6 (Components)**.** An equivalence relation $\sim$ can be defined for points in a topological space X in the following way: for a pair of points $x, y \in X$, $x \sim y$ holds if there is a connected subspace C of X with $x, y \in C$. The equivalence classes under $\sim$ are called the **components** of the topological space X.

One can easily check that $\sim$ defined above is indeed an equivalence relation. Symmetry and reflexivity of the relation are obvious. Transitivity follows by noting that if A is a connected subspace containing x and y, and if B is a connected subspace containing y and z, then $A \cup B$ is a subspace containing x and z that is connected because A and B have the point y in common.

Theorem 6.14

The components of X are connected disjoint subspaces of X whose union is X, such that each nonempty connected subspace of X intersects only one of them.

Proof. Since the components are the equivalence classes under the equivalence relation $\sim$ defined above, they must be disjoint nonempty subsets of X whose union is X. If $A \subset X$ is connected and intersects 2 components C_1 and C_2 of X at x and y, respectively, then both x and y belong to A with A being connected. Then by the definition of the equivalence relation stated above, $x \sim y$. Since $x \in C_1$ and $y \in C_2$, one must have $C_1 = C_2$.

Now, we show that each component C is connected. Suppose $x_0 \in C$. Since C is an equivalence class under the defined equivalence relation, for any $x \in C$, one must have $x \sim x_0$. Now, by the definition of the equivalence relation, there exists a connected subset $A_x \subset X$ with $x, x_0 \in A_x$. Since $x \in A_x \cap C$, A_x intersects the component C of X. Then by the result proved in the first paragraph, $A_x \subset C$. One, therefore, obtains

$$C = \bigcup_{x\in C} A_x. \tag{6.63}$$

All the A_x's in have the point x_0 in common and each $A_x \subset C$ is connected. Hence, by Theorem 6.4, C is connected. ■

Definition 6.7. We define another equivalence relation $\sim_{\mathrm{p}}$ on the space X by defining $x \sim_{\mathrm{p}} y$ if there is a path in X from x to y. The equivalence classes are called **path components** of X.

Lemma 6.15

$\sim_{\mathrm{p}}$ defined above is an equivalence relation.

Proof. First observe that if there exists a path $f : [a, b] \to X$ from x to y whose domain is the interval $[a, b]$, then there is a path $g : [c, d] \to X$ from x to y whose domain is $[c, d]$. Indeed, the closed interval $[c, d]$ is homeomorphic to the closed interval $[a, b]$ with $F : [c, d] \to [a, b]$ defined by

$$F(t) = \left(\frac{a-b}{c-d}\right)(t-d) + b \tag{6.64}$$

being the underlying homeomorphism. Then $g : [c, d] \to X$ is given by $g = f \circ F$. Then

$$\begin{aligned} g(c) &= f \circ F(c) = f(a) = x, \text{ and} \\ g(d) &= f \circ F(d) = f(b) = y, \end{aligned} \tag{6.65}$$

that is, g is the required path from x to y.

There is always a path from the point x to itself on X given by the constant continuous function $f : [a, b] \to X$ defined by

$$f(t) = x, \quad \forall\, t \in [a, b]. \tag{6.66}$$

Hence, one has $x \sim_{\mathrm{p}} x$.

Now suppose $x \sim_{\mathrm{p}} y$ in X. Then there is a path $f : [0, 1] \to X$ from x to y in X. Then the reverse path $g : [0, 1] \to X$ defined by $g(t) = f(1-t)$ is a path from y to x, proving that $y \sim_{\mathrm{p}} x$.

Finally, transitivity is proved as follows: Suppose $x \sim_{\mathrm{p}} y$ and $y \sim_{\mathrm{p}} z$. Then there is a path $f : [0, 1] \to X$ from x to y in X and a path $g : [1, 2] \to X$ from y to z in X, so that $f(1) = g(1) = y$. Since f and g agree on $\{y\} = [0, 1] \cap [1, 2]$, one can paste the continuous functions f and g with f to construct a continuous function $h : [0, 2] \to X$, i.e.,

$$h\,(t) = \begin{cases} f\,(t) & \text{if } t \in [0, 1], \\ g\,(t) & \text{if } t \in [1, 2]. \end{cases} \tag{6.67}$$

h is continuous by Pasting Lemma. Hence the continuous function $h : [0, 2] \to X$ yields $h(0) = f(0) = x$ and $h(2) = g(2) = y$. In other words, h is the required path from x to z yielding the fact that $x \sim_{\mathrm{p}} z$. ■

Theorem 6.16

The path components of X are path connected disjoint subspaces of X whose union is X, such that each nonempty path connected subspace of X intersects only one of them.

Proof. Since the path components are the equivalence classes under the equivalence relation $\sim_{\mathrm{p}}$ defined above, they must be disjoint nonempty subsets of X whose union is X. If $A \subset X$ is path connected and intersects 2 path components P_1 and P_2 of X at x and y, respectively, then both x and y belong to A with A being path connected. Then by the definition of the equivalence relation stated above, $x \sim_{\mathrm{p}} y$. Since $x \in P_1$ and $y \in P_2$, one must have $P_1 = P_2$.

Now, we show that each path component P is path connected. Fix $x_0 \in P$. Then for any $x \in P$, there is a path in X from x to x_0. Note that this entire path lies in P. So there is a path in P from x to x_0. Now choose any $x, y \in P$. There is a path in P from x to x_0; and there is a path in P from y to x_0. Joining them (after reversing the second path), we get a path in P from x to y. Therefore, P is path connected. ■

Observe that each component C of X satisfies $\overline{C} = C$, and hence is closed in X. Indeed if $C \neq \overline{C}$, then there is a point $x \in \overline{C}$ that doesn't belong to C. In other words, $x \not\sim y$ for all $y \in C$, i.e., there is no connected subset K of X that contains both x and y. But x and y both belong to $\overline{C}$ and C, being the closure of the connected subset C of X, is also connected. A contradiction! Hence $\overline{C} = C$. And hence each component C of X is closed in X.

If X has finitely many components and C is one of them, then C is also open in X. This is so because the complement of C can be seen to be a finite union of closed sets and hence a closed set. One can say even less about the path components of X, for they do not need to be either open or closed in X, as can be verified from Example 6.13 below.

Example 6.12. Consider the topological space $\mathbb{Q}$ of rational numbers as the subspace of $\mathbb{R}$, equipped with standard topology. Each component of $\mathbb{Q}$ consists of a single point. None of the components of $\mathbb{Q}$ are open in $\mathbb{Q}$. Note that $\mathbb{Q}$ has got infinitely many components.

Example 6.13. The topologist's sine curve $\overline{S}$ that we studied in Example 6.11 is a topological space that has a single component (since it is connected), but it has 2 path components. One path component is the curve S and the other is the vertical interval $V = \{0\} \times [-1, 1]$. It was seen before that V is closed in $\overline{S}$ so that S is open in $\overline{S}$. Also, S is not closed in $\overline{S}$ so that V is not open in S.

§6.5 Local connectedness

Definition 6.8. A space X is locally connected at a point $x \in X$ if for each neighborhood U of x (an open set $U \subset X$ containing x), there is a connected neighborhood V of x contained in U, i.e.

$$x \in V \subset U \subset X.$$

One says that a topological space X is **locally connected** if it is locally connected at each of its points.

Definition 6.9. A space X is locally path connected at a point $x \in X$ if for each neighborhood U of x, there is a path connected neighborhood V of x contained in U, i.e.

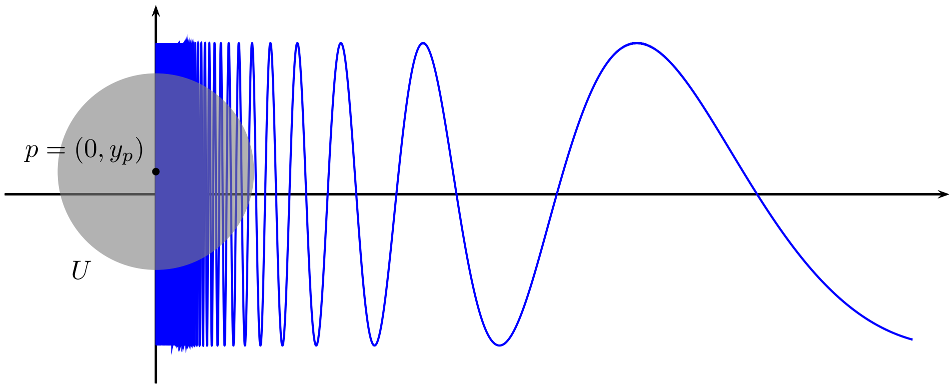
$$x \in V \subset U \subset X.$$

One says that a topological space X is **locally path connected** if it is locally path connected at each of its points.

Example 6.14. $\mathbb{R}$, with respect to standard topology, is locally connected and locally path connected as each neighborhood U of $x \in \mathbb{R}$ contains a connected and path connected basis neighborhood $(x - \varepsilon, x + \varepsilon)$ for some $\varepsilon > 0$.

Example 6.15. Any open subset $\Omega \subset \mathbb{R}^n$ (in which the Euclidean metric, the square metric, and the product topology as product of n copies of $\mathbb{R}$ with respect to standard topology–all yield the same topology) is locally connected and locally path connected, since each neighborhood U of any point $x \in \Omega$ contains a connected and path connected basis neighborhood $B_{\text{Euc}}(x, \varepsilon)$, for some $\varepsilon > 0$.

Example 6.16. Topologist's sine curve $\overline{S}$ is connected. But it is neither locally connected nor locally path connected. Choose a point $p \in V = \{0\} \times [-1, 1] \in S$. Now take an open ball B of radius $\varepsilon > 0$ centered at $p \in V$, which is denoted by $U = B_{\text{Euc}}(p, \varepsilon)$.



Now, consider the neighborhood of $p \in V$ in S in subspace topology $U \cap \overline{S}$. This neighborhood of $\overline{S}$ doesn't contain a connected neighborhood containing p. Also, $U \cap \overline{S}$ doesn't contain any path connected neighborhood containing p. Hence, $\overline{S}$ is neither locally connected nor is it locally path connected.

Example 6.17. The subspace $[-1, 0) \cup (0, 1]$ of $\mathbb{R}$ is not connected, but it is locally connected.

Theorem 6.17

A topological space X is locally connected if and only if for every open set $U \subset X$, each component of U is open in X.

Proof. Suppose first that X is locally connected. Let $U \subset X$ be open. Also let C be a component of U and $x \in C$. Then $x \in U$. Since X is locally connected, there exists a connected neighborhood V of x such that $V \subset U$, i.e., one has $x \in V \subset U$. If there is a point $y \in V$ that is not in C, then that point will not be in the same equivalence class to which x belongs, violating the connectedness of V. Hence, V has to be entirely contained in C. This proves that C is open in X.

Conversely, suppose that components of open sets of X are open in X. Given a point $x \in X$ and a neighborhood U of x, let C be the component of U containing x. Then C is open by hypothesis. Also, C, being a component, is connected. Therefore, given a point $x \in X$ and a neighborhood U of x, there is a connected neighborhood C (namely the component of U containing x) of x such that $C \subset U$. This proves that X is locally connected. ■

Theorem 6.18

A topological space X is locally path connected if and only if for every open set $U \subset X$, each path component of U is open in X.

Proof. Suppose first that X is locally path connected. Let $U \subset X$ be open. Also let P be a path component of U and $x \in P$. Then $x \in U$. Since X is locally path connected, there exists a path connected neighborhood V of x such that $V \subset U$, i.e., one has $x \in V \subset U$. If there is a point $y \in V$ that is not in P, then that point will not be in the same equivalence class to which x belongs, violating the path connectedness of V. Hence, V has to be entirely contained in P. This proves that P is open in X.

Conversely, suppose that path components of open sets of X are open in X. Given a point $x \in X$ and a neighborhood U of x, let P be the path component of U containing x. Then P is open by hypothesis. Also, P, being a component, is path connected. Therefore, given a point $x \in X$ and a neighborhood U of x, there is a connected neighborhood P (namely the component of P containing x) of x such that $P \subset U$. This proves that X is locally path connected. ■

Theorem 6.19

If X is a topological space, each path component of X lies in a component of X. If X is locally path connected, then the components and the path components of X are the same.

Proof. An equivalence class can't be empty, so that each path component P is nonempty. Let $x \in P$. Now take the component C of X containing x. Now the path component P is also connected. If there is a point in P that is not in C, then that point and x can't be in the same equivalence class which violates the connectedness of P. Therefore, $P \subset C$.

We now want to show that if X is locally path connected, then $P = C$. Now suppose that $P \subsetneq C$, i.e., P is properly contained in C. Let Q be the union of all the path components, each of which is different from P and intersects C. Each of these path components lies entirely in C using the argument used in the beginning of the proof. One, therefore, has

$$C = P \cup Q. \tag{6.68}$$

Since X is locally path connected and X is open in itself, the path component P of X is open in X. Also, since each path component of the union Q is open in X by the same reasoning, the union Q is also open in X. Also, it is immediate that they are nonempty as equivalence classes are nonempty. Hence (6.68) reflects the fact that the pair P, Q constitutes a separation of C, contradicting the fact that C is connected. Hence, P can't be properly contained in C. Therefore, $P = C$. ■

7 Compactness

§7.1 Open cover and subcover

Definition 7.1 (Open cover). A collection $\mathcal{C} = \{U_\alpha\}_{\alpha \in J}$ of subsets of X is said to cover X if the union of its elements is equal to X, i.e.,

$$X = \bigcup_{\alpha \in J} U_\alpha. \tag{7.1}$$

If each element in the collection $\mathcal{C}$ is an open subset of X, then we say that $\mathcal{C}$ is an **open cover**.

A subcollection $\mathcal{D} \subset \mathcal{C}$ that also covers X is called a **subcover** of $\mathcal{C}$. If $\mathcal{F}$ is a subcover with finitely many elements in it, then we call it a **finite subcover** of $\mathcal{C}$. In other words, it means that there is a finite subset $\{\alpha_1, \ldots, \alpha_n\} \subset J$ with $\mathcal{F} = \{U_{\alpha_1}, \ldots, U_{\alpha_n}\}$ and $X = U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$.

Definition 7.2 (Compact space). A space X is said to be **compact** if, for each open cover $\mathcal{C} = \{U_\alpha\}_{\alpha \in J}$ of X, there exists a finite subcover $\mathcal{F} = \{U_{\alpha_1}, \ldots, U_{\alpha_n}\}$ of $\mathcal{C}$.

Example 7.1. A finite topological space X is compact as any open cover of X necessarily has finitely many open sets in it.

Example 7.2. The real line $\mathbb{R}$ with respect to the standard topology is not compact since the open cover $\mathcal{C} = \{(n-1, n+1) \mid n \in \mathbb{Z}\}$ doesn't admit a finite subcover.

Example 7.3. The real line $\mathbb{R}$ in the trivial (indiscrete) topology is compact. There are 2 possible open covers: $\{\mathbb{R}\}$ and $\{\mathbb{R}, \varnothing\}$, both of which are finite.

Remark 7.1. When we say that the collection $\mathcal{C}$ of open sets is finite, we only mean that there are finitely many open sets; we do not refer to the finiteness of the open sets involved. For example, in Example 7.3 above, there are two open sets in the open cover $\{\mathbb{R}, \varnothing\}$, of which the open set $\mathbb{R}$ is infinite.

Example 7.4. The subspace $X = \{0\} \cup \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\} \subset \mathbb{R}$ with the standard topology is compact. Given an open covering $\mathcal{C}$ of X, there is an open set $U \in \mathcal{C}$ with $0 \in U$. Now U is open in X in subspace topology that it inherits from $\mathbb{R}$. In other words, U is the intersection of an open set of $\mathbb{R}$ containing 0 and X. Such an intersection will contain all but finitely many elements from the set $\left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$. In other words, there exists $N \in \mathbb{N}$ such that $\frac{1}{n} \in U$, $\forall n > N$. Now, refer back to the open cover $\mathcal{C}$ of X. There are open sets $U_1, U_2, \ldots, U_N \in \mathcal{C}$ with $\frac{1}{n} \in U_n$ for $n = 1, 2, \ldots, N$. One, therefore, immediately finds that the collection $\{U_1, \ldots, U_N, U\}$ covers X. In other words, $\{U, U_1, \ldots, U_N\}$ is a finite subcover of the given open cover $\mathcal{C}$ of X. Since the open cover $\mathcal{C}$ was arbitrary, one finds that X is compact.

Definition 7.3 (Open covering of a subspace). If A is a subspace of a topological space X, a collection $\mathcal{B}$ of subsets of X covers A if the union of the elements of $\mathcal{B}$ contains A. If the elements of the cover of the subspace A are all open sets of X, then the cover is called an **open cover** of the subspace A of X.

Lemma 7.1

Let A be a subspace of X. Then A is compact if and only if each open cover of A (in the light of the above definition) has a finite subcover.

Proof. ($\Rightarrow$) Assume A is compact and let $\{U_\alpha\}$ be an open cover of A in X. This means that $A \subset \bigcup_\alpha U_\alpha$. Hence,

$$A = \left(\bigcup_\alpha U_\alpha\right) \cap A = \bigcup_\alpha (U_\alpha \cap A)$$

Now, $\{U_\alpha \cap A\}_\alpha$ is an open cover of A in A. Since A is compact, every open cover of A in A has a finite subcover. Let the finite sub-cover be $\{U_{\alpha_i} \cap A\}_{i=1}^n$. Thus,

$$A \subset \bigcup_{i=1}^n U_{\alpha_i}$$

which means that $\{U_{\alpha_i}\}_{i=1}^n$ is a finite sub-cover of the open cover $\{U_\alpha\}_\alpha$ of A in X.

($\Leftarrow$) Suppose every open cover of A in X has a finite subcover, and let $\{V_\alpha\}_\alpha$ be an open cover of A in A. Then each V_α is an open set of A in subspace topology. According to the definition of subspace topology, there is an open set U_α in S such that $V_\alpha = U_\alpha \cap A$. Now,

$$A = \bigcup_\alpha V_\alpha = \bigcup_\alpha (U_\alpha \cap A) = \left(\bigcup_\alpha U_\alpha\right) \cap A \subset \bigcup_\alpha U_\alpha$$

Therefore, $\{U_\alpha\}_\alpha$ is an open cover of A in S. By hypothesis, there are finitely many sets $\{U_{\alpha_i}\}_{i=1}^n$ such that $A \subset \bigcup_{i=1}^n U_{\alpha_i}$. Hence,

$$A = \left(\bigcup_{i=1}^n U_{\alpha_i}\right) \cap A = \bigcup_{i=1}^n (U_{\alpha_i} \cap A) = \bigcup_{i=1}^n V_{\alpha_i}$$

So $\{V_{\alpha_i}\}_{i=1}^n$ is a finite subcover of $\{V_\alpha\}$ that covers A in A. Therefore, A is compact. ■

§7.2 Compact and Hausdorff spaces

Theorem 7.2

Every closed subset of a compact space is compact.

Proof. Let X be a compact space and $A \subset X$ be closed. Let $\{U_\alpha\}_\alpha$ be an open cover of A in X. The collection $\{U_\alpha, X \setminus A\}$ is an open cover of X itself. By compactness of X, there is a finite sub-cover $\{U_{\alpha_i}, S \setminus F\}_{i=1}^n$ of X, that is,

$$A \subset X = \left(\bigcup_{i=1}^n U_{\alpha_i}\right) \cup (S \setminus F).$$

Therefore,

$$A \subset \bigcup_{i=1}^n U_{\alpha_i}. \tag{7.2}$$

Therefore, $\{U_{\alpha_i}\}_{i=1}^n$ is a finite subcover of the open cover $\{U_\alpha\}$ of A in X. Hence, A is also compact. ■

Proposition 7.3

Every compact subset of K of a Hausdorff space X is closed.

Proof. We shall prove that $X \setminus K$ is open. Let's take $x \in X \setminus K$. We claim that there is a neighborhood U_x of x that is disjoint from K.

Since X is hausdorff, for each $y \in K$, we can choose disjoint open sets U_y and V_y such that $U_y \ni x$ and $V_y \ni y$. The collection $\{V_y \mid y \in K\}$ is an open cover of K in S. Since K is compact, there exists a finite subcover $\{V_{y_i}\}_{i=1}^n$, i.e.

$$K \subset \bigcup_{i=1}^{n} V_{y_i}. \tag{7.3}$$

Since $U_{y_i} \cap V_{y_i} = \varnothing$ for every i, we have

$$\left(\bigcap_{i=1}^{n} U_{y_i}\right) \cap \left(\bigcup_{i=1}^{n} V_{y_i}\right) = \varnothing.$$

This gives us that

$$U_x \cap K = \varnothing \text{ where } U_x = \bigcap_{i=1}^{n} U_{y_i}. \tag{7.4}$$

U_x is the finite intersection of open sets, hence open. Also, every U_{y_i} contains x, hence their intersection U_x also contains x. So U_x is the desired open set that is disjoint from K, in other words $x \in U_x \subset X \setminus K$.

Given any $x \in X \setminus K$, there exists an open set U_x satisfying $x \in U_x \subset X \setminus K$. Therefore, by Proposition 1.6, $X \setminus K$ is open. So K is closed. ■

Remark 7.2. Recall from Corollary 2.14 that finite subsets of Hausdorff spaces are closed. In this sense, Proposition 7.3 above, compact subspaces generalize finite sets.

Example 7.5. The intervals $(a, b]$, $[a, b)$, and (a, b) are not closed in $\mathbb{R}$ with respect to standard topology. Hence, by Proposition 7.3, they are not compact either in $\mathbb{R}$ in the same topology. We shall prove in the next section that each closed interval $[a, b]$ is compact in $\mathbb{R}$ with respect to the standard topology.

Theorem 7.4

The continuous image of a compact space is compact.

Proof. Let $f : X \to Y$ be a continuous and X compact. Suppose $\{U_\alpha\}$ is an open cover of $f(X)$ by open subsets of Y. Since, f is continuous, the inverse images of $f^{-1}(U_\alpha)$ are all open in X. Moreover,

$$X \subset f^{-1}(f(X)) \subset f^{-1}\left(\bigcup_{\alpha} U_\alpha\right) = \bigcup_{\alpha} f^{-1}(U_\alpha). \tag{7.5}$$

So $\{f^{-1}(U)_\alpha\}$ is an open cover of X. Since X is compact, there is a finite sub-collection $\{f^{-1}(U_{\alpha_i})\}_{i=1}^n$ such that

$$X = \bigcup_{i=1}^{n} f^{-1}(U_{\alpha_i}) = f^{-1}\left(\bigcup_{i=1}^{n} U_{\alpha_i}\right). \tag{7.6}$$

As a result,

$$f(X) = f\left(f^{-1}\left(\bigcup_{i=1}^{n} U_{\alpha_i}\right)\right) \subset \bigcup_{i=1}^{n} U_{\alpha_i}. \tag{7.7}$$

So, given any open cover $\{U_\alpha\}$ of $f(X)$, there is a finite subcover. So $f(X)$ is compact. ■

Theorem 7.5

Let $f : X \to Y$ be a continuous function from a compact space X to a Hausdorff space Y. Then the following hold:

1. If f is a closed map, i.e., it maps closed subsets of X to closed subsets of Y.
2. If f is surjective, then f is a quotient map.
3. If f is bijective, then f is a homeomorphism.

4. If f is injective, then f is an imbedding.

Proof. 1. Let $A \subset X$ be a closed subset. Since X is compact, A being closed in X is also compact. Then by Theorem 7.4, one has $f(A)$ being a compact subspace of Y. Since Y is Hausdorff, by Proposition 7.3, one concludes that $f(A)$ is closed in Y, proving that f is a closed map.

2. $f : X \to Y$ is a surjective continuous function that is closed. Hence f is a quotient map.

3. If $f : X \to Y$ is bijective, then consider the inverse function $f^{-1} : Y \to X$. Then for $A \subset X$ closed, $(f^{-1})^{-1}(A) = f(A)$ is closed in Y by 1, proving that $f^{-1} : Y \to X$ is continuous. Hence, $f : X \to Y$ is a homeomorphism.

4. If $f : X \to Y$ is an injective continuous function from the compact space X to the Hausdorff space Y, then one forms a bijective continuous function $g : X \to f(X)$ by restricting the codomain of f to the range of f. Now, $f(X)$ being a subspace of the Hausdorff space Y is also Hausdorff. Then by 3, $g : X \to f(X)$ is a homeomorphism. Therefore, $f : X \to Y$ is an imbedding.

■

§7.3 Product of compact spaces

Theorem 7.6

The product of finitely many compact spaces is compact.

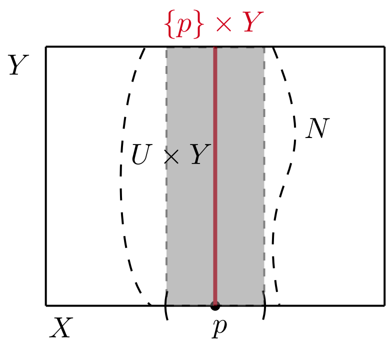
The proof of this theorem requires the following lemma, so we shall prove it first.

Lemma 7.7 (Tube lemma)

Consider the product space $X \times Y$, where Y is compact. Let $p \in X$. Suppose N is an open set of $X \times Y$ containing the "slice" $\{p\} \times Y$ of $X \times Y$. Then there is a neighborhood U of p in X such that $U \times Y \subset N$. The set $U \times Y$ is often called a **tube** about the slice $\{p\} \times Y$.

Proof. For each $q \in Y$, we have $\{p, q\} \in \{p\} \times Y \subset N$. Since $N \subset X \times Y$ is open in the product topology, there is a basis element $U_q \times V_q \subset N$ for the product topology on $X \times Y$, with $p \in U_p$ open in X and $q \in V_q$ open in Y. Now, the collection $\{V_q\}_{q \in Y}$ is an open cover of Y. By compactness of Y, there exists a finite subcover $\{V_{q_1}, \dots, V_{q_n}\}$ of the open cover $\{V_q\}_{q \in Y}$ of Y.

For instance, for $q_i \in Y$, we have the point (p, q_i) in the slice $\{p\} \times Y \subset X \times Y$ with $N \subset X \times Y$ open in the product topology. Thus, the basic element for the product topology on $X \times Y$ that is contained in the open set $N \subset X \times Y$ containing (p, q_i) is $U_{q_i} \times V_{q_i}$.

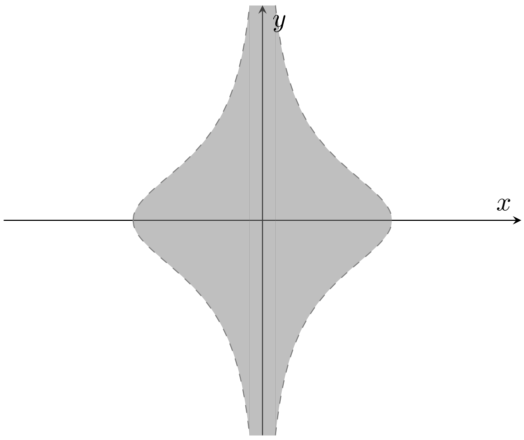


Now, let $U = U_{q_1} \cap \cdots \cap U_{q_n}$. Since $p \in U_{q_i}$ and U being a finite intersection of open sets, is also open. We claim that $U \times Y \subset N$.

Suppose $(x, y) \in U \times Y$. Then $y \in Y$. Since $Y = \bigcup_{i=1}^{n} V_{q_i}$, there exists $i \in \{1, \dots, n\}$ such that $y \in V_{q_i}$. Also, $x \in U \subset U_{q_i}$, so that $(x, y) \in U_{q_i} \times V_{q_i} \subset N$, proving that $U \times Y \subset N$ holds. ■

Remark 7.3. Note that Tube lemma doesn't hold if Y is not compact. Suppose $X = Y = \mathbb{R}$. Consider the following open subset of $\mathbb{R}^2$:

$$N = \left\{ (x, y) \in \mathbb{R}^2 \mid |x| < \frac{1}{1 + y^2} \right\}. \tag{7.8}$$



N contains the y-axis, the slice $\{0\} \times \mathbb{R}$. But N doesn't contain any tube about the y-axis.

Now we get back to the proof of Theorem 7.6.

Proof of Theorem 7.6. Let X and Y be compact spaces. Let $\mathcal{A} = \{A_\alpha\}_{\alpha \in J}$ be an open covering of $X \times Y$. Given $p \in X$, the slice $\{p\} \times Y$ is compact (since it is homeomorphic to Y). Since $\mathcal{A} = \{A_\alpha\}_{\alpha \in J}$ is an open cover of the compact subspace $\{p\} \times Y$ of $X \times Y$, there are finitely many elements $A_1, \ldots, A_m$ from the collection $\mathcal{A} = \{A_\alpha\}_{\alpha \in J}$ such that

$$\{p\} \times Y \subset A_1 \cup \ldots \cup A_m =: N. \tag{7.9}$$

By Tube lemma, the open set N contains a tube $U \times Y$ where $U \subset X$ is open. In other words, one has

$$U \times Y \subset A_1 \cup \ldots \cup A_m. \tag{7.10}$$

Hence, $U \times Y$ is covered by finitely many elements $A_1, \ldots, A_m$.

We now vary $p \in X$. For each $p \in X$, we can choose a neighborhood U_p of p such that the tube $U_p \times Y$ is covered by finitely many elements of $\mathcal{A} = \{A_\alpha\}_{\alpha \in J}$. But the collection $\{U_p\}_{p \in X}$ of neighborhoods of p forms an open covering of X; thus, by compactness of X, there exists a finite subcovering, $\{U_1, \ldots, U_k\}$, covering X. Hence,

$$X = \bigcup_{i=1}^{k} U_i, \tag{7.11}$$

which implies

$$X \times Y = \bigcup_{i=1}^{k} (U_i \times Y). \tag{7.12}$$

Now, each of the finite union on the right side of (7.12), namely $U_i \times Y$, can be covered by finitely many elements of $\mathcal{A} = \{A_\alpha\}_{\alpha \in J}$. Hence, the finite union $\bigcup_{i=1}^{k} (U_i \times Y) = X \times Y$ can also be covered by finitely many elements from the collection $\mathcal{A} = \{A_\alpha\}_{\alpha \in J}$. This provides the required finite subcover. This proves that $X \times Y$ is compact. ■

The infinite version of Theorem 7.6 holds, and it is called Tychonoff's thoerem. We won't prove it in this course.

§7.4 Finite intersection property

There is another criterion of compact sets in terms of closed sets, which we shall explore now. Let $\mathcal{U}$ be a collection of open subsets of a topological space X. Let $\mathcal{C} = \{X \setminus U \mid U \in \mathcal{U}\}$ be the collection of closed complements. To say that $\mathcal{U}$ is a cover of X is equivalent to saying that $\mathcal{C}$ has empty intersection:

$$\bigcap_{C \in \mathcal{C}} C = \bigcap_{U \in \mathcal{U}} (X \setminus U) = X \setminus \left(\bigcup_{U \in \mathcal{U}} U \right). \tag{7.13}$$

So we have

$$\bigcap_{C \in \mathcal{C}} C = \varnothing \iff \bigcup_{U \in \mathcal{U}} U = X. \tag{7.14}$$

Definition 7.4. A collection $\mathcal{C}$ of subsets of X is said to have the **finite intersection property** if for every finite subcollection

$$\{C_1, \dots, C_n\}$$

of $\mathcal{C}$, the intersection $C_1 \cap \cdots \cap C_n$ is nonempty.

Theorem 7.8

Let X be a topological space. Then X is compact if and only if for every collection $\mathcal{C}$ of closed sets in X having the finite intersection property, the intersection $\bigcap_{C \in \mathcal{C}} C$ of all the elements of $\mathcal{C}$ is nonempty.

Proof. ($\Rightarrow$) Suppose X is compact, and a collection $\mathcal{C}$ of closed sets has finite intersection property, we need to show that the whole collection has nonempty intersection. Assume for the sake of contradiction that $\bigcap\limits_{C \in \mathcal{C}} C = \varnothing$.

Each $C \in \mathcal{C}$ is closed in X, so $C = X \setminus U$ for some open U.

$$\varnothing = \bigcap_{C \in \mathcal{C}} C = \bigcap_{X \setminus U \in \mathcal{C}} (X \setminus U) = X \setminus \left(\bigcup_{X \setminus U \in \mathcal{C}} U \right). \tag{7.15}$$

So we have

$$\bigcup_{X \setminus U \in \mathcal{C}} U = X. \tag{7.16}$$

So we got an open cover of X. Since X is compact, there is a finite subcover.

$$X = \bigcup_{i=1}^{n} U_i. \tag{7.17}$$

In other words,

$$\begin{aligned} X \setminus \left(\bigcup_{i=1}^{n} U_i \right) &= \varnothing \\ \implies \bigcap_{i=1}^{n} (X \setminus U_i) &= \varnothing. \end{aligned}$$

$X \setminus U_i$ for $i = 1, \dots, n$ is a finite subcollection of $\mathcal{C}$. But we've just shown that this finite subcollection has empty intersection, contradicting finite intersection property. Therefore, $\bigcap\limits_{C \in \mathcal{C}} C \neq \varnothing$.

($\Leftarrow$) Now assume that for every collection of closed set in X with finite intersection property, the whole collection has nonempty intersection. Assume for the sake of contradiction that X is not compact.

Take an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in J}$ of X that has no finite subcover. Consider the collection of closed sets $\mathcal{C} = \{X \setminus U \ : \ U \in \mathcal{U}\}$. Since X is not compact, no finite subcollection of $\mathcal{U}$ covers X.

Take finitely many closed sets $X \setminus U_1, \ldots, X \setminus U_n$ from the collection of closed sets $\mathcal{C}$. Since no finite subcollection of $\mathcal{U}$ covers X, $\bigcup_{i=1}^n U_i \neq X$. As a result,

$$\bigcap_{i=1}^{n} (X \setminus U_i) = X \setminus \left(\bigcup_{i=1}^{n} U_i \right) \neq \varnothing. \tag{7.18}$$

So $\mathcal{C}$ is a collection of closed sets with finite intersection property. Therefore, the whole collection has nonempty intersection by hypothesis. As $\mathcal{U}$ covers X, $\bigcup\limits_{U \in \mathcal{U}} U = X$.

$$\varnothing \neq \bigcap_{C \in \mathcal{C}} C = \bigcap_{U \in \mathcal{U}} (X \setminus U) = X \setminus \left(\bigcup_{U \in \mathcal{U}} U \right) = X \setminus X = \varnothing$$

Contradiction! So X must be compact. ■

Theorem 7.9

Each closed interval $[a, b]$ is compact.

Proof. Let $\mathcal{O} = \{O_x \mid x \in A\}$ be any Open Cover of $[a, b]$. This means $[a, b] \subset \bigcup\limits_{x \in A} O_x$. Now we shall consider the set

$$C = \{x \in [a, b] \mid [a, x] \text{ has a finite subcover from } \mathcal{O}\}.$$

Note that C is bounded by b. Now we want to show that C is non-empty.

Claim 1: $C \neq \varnothing$

Proof. $\mathcal{O}$ covers $[a, b]$, so there exists an open set in $\mathcal{O}$ that contains a. Let $a \in O_a$. Since O_a is open, there exists $r > 0$ such that $(a - r, a + r) \subset O_a$. Thus,

$$\left[a, a + \frac{r}{2}\right] \subset (a - r, a + r) \subset O_a.$$

So $[a, a + \frac{r}{2}]$ is covered by O_a, in other words $[a, a + \frac{r}{2}]$ has a finite subcover. Thus $[a, a + \frac{r}{2}] \subset C$, our claim is proved. □

C is non-empty and has an upper bound. Therefore $\sup C$ exists. Let $\sup C = s$.

Claim 2: $s = b$

Proof. Assume for the sake of contradiction that $s < b$. So $s \in (a, b)$ and thus there exists an open set in $\mathcal{O}$ that contains s. Let $s \in O_s$. Since O_s is open, there exists $r > 0$ such that $(s - r, s + r) \subset O_s$.
Since s is the supremum, $s - \frac{r}{2}$ cannot be a upper bound of C. Therefore there exists $s' \in C$ such that $s - \frac{r}{2} \leq s' \leq s$. Now,

$$\left[s', s + \frac{r}{2}\right] \subset \left[s - \frac{r}{2}, s + \frac{r}{2}\right] \subset (s - r, s + r) \subset O_s.$$

$s' \in C$ means $[a, s']$ has a finite subcover $\mathcal{O}_{s'}$. Now if we take $\mathcal{O}_{s'} \cup O_s$, it will cover $[a, s'] \cup [s', s + \frac{r}{2}] = [a, s + \frac{r}{2}]$. Thus we can find a finite subcover for $[a, s + \frac{r}{2}]$, therefore $s + \frac{r}{2} \in C$. But s is the supremum, so we arrive at a contradiction. □

Now we want to show that $\sup C = b$ exists in the set C.

Claim 3: $b \in C$.

Proof. $b \in [a, b]$, so there exists an open set in $\mathcal{O}$ that contains b. Let $b \in O_b$. Since O_b is open, there exists $r > 0$ such that $(b - r, b + r) \subset O_b$.
Since b is the supremum, $b - \frac{r}{2}$ cannot be a upper bound of C. Therefore there exists $b' \in C$ such that $b - \frac{r}{2} \leq b' \leq b$. Now,

$$[b', b] \subset \left[b - \frac{r}{2}, b\right] \subset (b - r, b + r) \subset O_b.$$

$b' \in C$ means $[a, b']$ has a finite subcover $\mathcal{O}_{b'}$. Now if we take $\mathcal{O}_{b'} \cup O_b$, it will cover $[a, b'] \cup [b', b] = [a, b]$. Thus we can find a finite subcover for $[a, b]$, therefore $b \in C$. □

Therefore $[a, b]$ has a finite subcover. And this is true for every cover $\mathcal{O}$ of $[a, b]$ since our choice of $\mathcal{O}$ was arbitrary. Hence, for every cover we can find a subcover of $[a, b]$. So $[a, b]$ is compact. ■

Example 7.6. Note that the unit circle $S^1 \subset \mathbb{R}^2$ being a subspace of $\mathbb{R}$ with respect to the subspace topology is Hausdorff, since $\mathbb{R}^2$ with the standard topology is Hausdorff. The surjective continuous map $f : [0, 1] \to S^1$ given by

$$f(t) = (\cos 2\pi t, \sin 2\pi t) \tag{7.19}$$

from the compact space $[0, 1]$ to the Hausdorff space S^1 is a quotient map by statement 2 of Theorem 7.5. Furthermore, $S^1 = f([0, 1])$ is the continuous image of a compact space. So S^1 is compact.

Theorem 7.10 (Heine–Borel theorem)
A subpace A of $\mathbb{R}^n$ is compact if and only if it in closed and bounded in the Euclidean metric d_{Euc} or the square metric ρ.

Proof. Since

$$\rho(\mathbf{x}, \mathbf{y}) \leq d_{\text{Euc}}(\mathbf{x}, \mathbf{y}) \leq \sqrt{n}\rho(\mathbf{x}, \mathbf{y}) \tag{7.20}$$

holds for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, it suffices to consider only the square metric ρ. The inequality above implies that $A \subset \mathbb{R}^n$ is bounded under d_{Euc} if and only if it is bounded under ρ.

($\Rightarrow$) Suppose $A \subset \mathbb{R}^n$ is compact. Since $\mathbb{R}^n$ is Hausdorff, by Proposition 7.3, A is closed. Now consider the collection of open sets

$$\left\{B_\rho(\mathbf{0}, m) \mid m \in \mathbb{Z}^+\right\}, \tag{7.21}$$

whose union is all of $\mathbb{R}^n$. So

$$A \subset \bigcup_{m \in \mathbb{Z}^+} B_\rho(\mathbf{0}, m). \tag{7.22}$$

Since A is compact, there is a finite subcover of this open cover of A. In other words,

$$A \subset \bigcup_{i=1}^{k} B_\rho(\mathbf{0}, m_i). \tag{7.23}$$

Choose $M = \max\{m_1, m_2, \ldots, m_k\}$. Then $A \subset B_\rho(\mathbf{0}, M)$. As a result, for any $\mathbf{x}, \mathbf{y} \in A$,

$$\rho(\mathbf{x}, \mathbf{y}) \leq \rho(\mathbf{x}, \mathbf{0}) + \rho(\mathbf{0}, \mathbf{y}) < 2M. \tag{7.24}$$

This shows that A is bounded.

($\Leftarrow$) Now suppose that $A \subset \mathbb{R}^n$ is closed and bounded under ρ.So $\rho(\mathbf{x}, \mathbf{y}) \leq N$ for any $\mathbf{x}, \mathbf{y} \in A$. Choose some $\mathbf{x}_0 \in A$, and let $\rho(\mathbf{0}, \mathbf{x}_0) = b$. Then by triangle inequality,

$$\rho(\mathbf{0}, \mathbf{x}) \leq \rho(\mathbf{0}, \mathbf{x}_0) + \rho(\mathbf{x}_0, \mathbf{x}) \leq b + N, \tag{7.25}$$

for every $\mathbf{x} \in A$. As a result,

$$A \subset [-b - N, b + N]^n . \tag{7.26}$$

$[-b - N, b + N]^n$ is compact since it's the product of finitely many compact spaces (Theorem 7.6). A is a closed subset of a compact space. Therefore, by Theorem 7.2, A is compact. ■

Theorem 7.11 (Extreme value theorem)
Let $f : X \to \mathbb{R}$ be continuous with X compact. Then there exist points $c, d \in X$ with

$$f(c) \leq f(x) \leq f(d), \tag{7.27}$$

for all $x \in X$.

Proof. X is compact and f is continuous, so $f(X) \subset \mathbb{R}$ is compact. Hence it is closed and bounded. Since it is bounded, it has both infimum and supremum. Suppose

$$m = \inf f(X), \quad \text{and } M = \sup f(X). \tag{7.28}$$

Since m is the infimum of $f(X)$, given any interval $(m - \varepsilon, m + \varepsilon)$, there exists an element $y_0 \in f(X)$ such that

$$m \leq y_0 < m + \varepsilon. \tag{7.29}$$

So $f(X) \cap (m - \varepsilon, m + \varepsilon)$ is nonempty. So $m \in \overline{f(X)}$ by Theorem 2.7. Similarly, since M is the supremum of $f(X)$, given any interval $(M - \varepsilon, M + \varepsilon)$, there exists an element $y_1 \in f(X)$ such that

$$M \geq y_1 > M - \varepsilon. \tag{7.30}$$

So $f(X) \cap (M - \varepsilon, M + \varepsilon)$ is nonempty. So $M \in \overline{f(X)}$ by Theorem 2.7. Since $f(X)$ is closed, it is equal to its closure. So $m, M \in f(X)$, i.e. there are elements c, d such that $m = f(c)$ and $M = f(d)$, so that (7.27) holds. ■

§7.5 The Lebesgue number

Definition 7.5. Let (X, d) be a metric space. Also, let $A \subset X$ be a nonempty subset. For each $x \in X$, define the **distance** from x to A by

$$d(x, A) = \inf\{d(x, a) \mid a \in A\}. \tag{7.31}$$

The **diameter** of A is given by

$$\operatorname{diam}(A) = \sup\{d(a, b) \mid a, b \in A\}. \tag{7.32}$$

Lemma 7.12
The function $x \mapsto d(x, A)$ is continuous.

Proof. Let $x, y \in X$. By the application of the triangle inequality,

$$d(x, A) \leq d(x, a) \leq d(x, y) + d(y, a), \quad \forall a \in A. \tag{7.33}$$

Hence, one obtains,

$$d(x, A) - d(x, y) \leq d(y, a), \quad \forall a \in A. \tag{7.34}$$

By taking the infimum on the right side, one obtains,

$$d(x, A) - d(x, y) \leq \inf\{d(y, a) \mid a \in A\} = d(y, A). \tag{7.35}$$

In other words,

$$d(x, A) - d(y, A) \leq d(x, y). \tag{7.36}$$

Interchanging x with y in (7.36), one obtains

$$d(y, A) - d(x, A) \leq d(y, x) = d(x, y). \tag{7.37}$$

Combining (7.36) and (7.37), we get

$$|d(x, A) - d(y, A)| \leq d(x, y). \tag{7.38}$$

Let $\alpha : (X, d) \to (\mathbb{R}, |\cdot|)$ be the function defined by $\alpha(x) = d(x, A)$. Given $\varepsilon > 0$, we choose $\delta = \varepsilon$. So we have for $d(x, y) < \delta$,

$$|\alpha(x) - \alpha(y)| = |d(x, A) - d(y, A)| \leq d(x, y) < \delta = \varepsilon, \tag{7.39}$$

i.e. $|\alpha(x) - \alpha(y)| < \varepsilon$. Therefore, α is continuous. ■

Lemma 7.13 (Lebesgue number lemma)
Let $\mathcal{C}$ be an open cover of a compact metric space (X, d). There exists a $\delta > 0$ such that for each subset $B \subset X$ of diameter $< \delta$, there exists an element $U \in \mathcal{C}$ with $B \subset U$. The number δ is called the **Lebesgue number** of $\mathcal{C}$.

Proof 1. If $X \in \mathcal{C}$, then any positive real number works as a Lebesgue number. Because for any $B \subset X$ with diameter less than whatever positive real number we choose, there exists $U \in \mathcal{C}_0$, namely, $U = X$ such that $B \subset U = X$.

Now, if $X \notin \mathcal{C}$, then by compactness of X, there is a finite subcollection $\{U_1, \ldots, U_n\}$ of $\mathcal{C}$ that covers X, i.e.,

$$X = U_1 \cup \cdots \cup U_n. \tag{7.40}$$

Let $C_i = X \setminus U_i$ be the i-th closed complement. And each C_i is nonempty as $X \notin \mathcal{C}$ so that $X \notin \{U_1, \ldots, U_n\}$. Define $f : X \to \mathbb{R}$ as

$$f(x) = \frac{1}{n} \sum_{i=1}^{n} d(x, C_i). \tag{7.41}$$

Claim 1: $f(x) > 0$ for all $x \in X$.

Proof. Given $x \in X$, there exists some $i \in \{1, \ldots, n\}$ such that $x \in U_i$. Since $U_i \subset X$ is open, there exists $\varepsilon > 0$ such that $B_d(x, \varepsilon) \subset U_i$. In other words, all the elements of X that are in less than ε-distance from x are inside U_i. As a result, $d(x, C_i) = d(x, X \setminus U_i) \geq \varepsilon$. So we have

$$f(x) = \frac{1}{n} \sum_{i=1}^{n} d(x, C_i) \geq \frac{\varepsilon}{n} > 0. \tag{7.42}$$

□

Since $f : X \to \mathbb{R}$ and X is compact, f attains a minimum by Extreme value theorem. Since $f(x) > 0$ for all x, f achieves a positive minimum, say $\delta > 0$. We now claim that this δ is the desired Lebesgue number.

Claim 2: δ is a Lebesgue number for the finite subcover $\{U_1, \ldots, U_n\}$ of $\mathcal{C}$, and hence for $\mathcal{C}$.

Proof. Let $B \subset X$ with $\operatorname{diam}(B) < \delta$. There is nothing to prove for B empty. Hence, let $B \neq \varnothing$. Choose $p \in B$. Then indeed

$$B \subset B_d(p, \delta). \tag{7.43}$$

Suppose $q \in B$. We are given $p \in B$. And since $\operatorname{diam}(B) < \delta$ by (16), one must have

$$d(p,q) < \delta \Rightarrow q \in B_d(p,\delta). \tag{7.44}$$

Hence, (7.43) follows. Now, consider the real numbers $d(p, C_i)$ for $1 \leq i \leq n$. Suppose

$$\max\{d(p,C_1),\dots,d(p,C_n)\} = d(p,C_m). \tag{7.45}$$

Hence, one obtains

$$\delta \leq f(p) \leq d(p,C_m). \tag{7.46}$$

The second inequality holds, since $f(p)$ is the average of all $d(p,C_i)$'s, whereas $d(p,C_m)$ is the maximum of them. As a result,

$$d(p,C_m) = \inf_{c\in C_m} d(p,c) \geq \delta. \tag{7.47}$$

Suppose there exists some point $c_0 \in C_m$ with $d(p,c_0) < \delta$. Then by the definition of infimum,

$$\inf_{c\in C_m} d(p,c) \leq d(p,c_0) < \delta, \tag{7.48}$$

which contradicts (7.47). Therefore, $C_m \cap B_d(p,\delta) = \varnothing$. So we have

$$B_d(p,\delta) \subset X \setminus C_m = U_m. \tag{7.49}$$

Now, (7.43) and (7.49) together imply

$$B \subset B_d(p,\delta) \subset U_m. \tag{7.50}$$

□

So δ is the required Lebesgue number ■

Proof 2. Take $x \in X$. As $\mathcal{C} = \{U_\alpha\}_{\alpha\in J}$ covers X, we can find $U_\alpha \in \mathcal{C}$ such that $x \in U_\alpha$. Since U_α is open and $x \in U_\alpha$, there exists $r_x > 0$ such that

$$B(x,r_x) \subset U_\alpha. \tag{7.51}$$

We do this for every $x \in X$. So we get an open cover of X

$$X = \bigcup_{x\in X} B\left(x,\frac{r_x}{2}\right). \tag{7.52}$$

Since X is compact, there exists a finite subcover of this open cover. So

$$X = \bigcup_{i=1}^{n} B\left(x_i,\frac{r_{x_i}}{2}\right). \tag{7.53}$$

We define $\delta > 0$ in the following way:

$$\delta = \min\left\{\frac{r_{x_i}}{2} \mid i = 1,2,\dots,n\right\}. \tag{7.54}$$

We claim that this δ is our desired Lebesgue number of the open cover $\mathcal{C}$. Let $A \subset X$ with $\operatorname{diam}(A) < \delta$. Fix $a \in A$. Then there exists $j \in \{1,2,\dots,n\}$ such that

$$a \in B\left(x_j,\frac{r_{x_j}}{2}\right). \tag{7.55}$$

As a result,

$$d(x_j,a) < \frac{r_{x_j}}{2}. \tag{7.56}$$

By the construction of r_{x_j}, there exists $U_\beta \in \mathcal{C}$ such that $B\left(x_j, r_{x_j}\right) \subset U_\beta$. We claim that $A \subset U_\beta$. Take any $b \in A$.

$$d(a,b) \le \operatorname{diam}(A) < \delta \le \frac{r_{x_j}}{2}. \tag{7.57}$$

Hence,

$$d(a,b) < \frac{r_{x_j}}{2}. \tag{7.58}$$

Now,

$$d(x_j,b) \le d(x_j,a) + d(a,b) < \frac{r_{x_j}}{2} + \frac{r_{x_j}}{2} = r_{x_j}. \tag{7.59}$$

So we have $b \in B\left(x_j, r_{x_j}\right)$. For every $b \in A$, we have $b \in B\left(x_j, r_{x_j}\right)$. Therefore,

$$A \subset B\left(x_j, r_{x_j}\right) \subset U_\beta. \tag{7.60}$$

■

Definition 7.6 (Uniform continuity)**.** A function $f : (X, d_X) \to (Y, d_Y)$ between metric spaces (X, d_X) and (Y, d_Y) is said to be **uniformly continuous** if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every pair of points $x_1, x_2 \in X$,

$$d_X(x_1,x_2) < \delta \Rightarrow d_Y(f(x_1), f(x_2)) < \varepsilon. \tag{7.61}$$

Theorem 7.14 (Uniform continuity theorem)

Let $f : (X, d_X) \to (Y, d_Y)$ be a continuous map from the compact metric space (X, d_X) to the metric space (Y, d_Y). Then f is uniformly continuous.

Proof. Given $\varepsilon > 0$, cover Y by the open balls

$$\left\{B_{d_Y}\left(y, \frac{\varepsilon}{2}\right) \mid y \in Y\right\} \tag{7.62}$$

and let $\mathcal{C} = \{f^{-1}\left(B_{d_Y}\left(y, \frac{\varepsilon}{2}\right)\right) \mid y \in Y\}$ be the open covering of X by the preimages of the open balls given by (7.62). Indeed,

$$X = f^{-1}(Y) = f^{-1}\left(\bigcup_{y\in Y} B_{d_Y}\left(y, \frac{\varepsilon}{2}\right)\right) = \bigcup_{y\in Y} f^{-1}\left(B_{d_Y}\left(y, \frac{\varepsilon}{2}\right)\right). \tag{7.63}$$

Since (X, d_X) is a compact metric space, there is a Lebesgue number δ associated with the open cover $\mathcal{C} = \{f^{-1}\left(B_{d_Y}\left(y, \frac{\varepsilon}{2}\right)\right) \mid y \in Y\}$.

Now, let $x_1, x_2 \in X$ such that $d_X(x_1,x_2) < \delta$. Then the 2-point set $\{x_1, x_2\}$ has diameter less than δ. By Lebesgue number lemma, there exists $y \in Y$ s.t.

$$\{x_1, x_2\} \subset f^{-1}\left(B_{d_Y}\left(y, \frac{\varepsilon}{2}\right)\right). \tag{7.64}$$

Then

$$f(\{x_1,x_2\}) = \{f(x_1), f(x_2)\} \subset f\left(f^{-1}\left(B_{d_Y}\left(y, \frac{\varepsilon}{2}\right)\right)\right) \subset B_{d_Y}(y, \frac{\varepsilon}{2}). \tag{7.65}$$

As a result, $d_Y(f(x_1), y), d_Y(f(x_2), y) < \frac{\varepsilon}{2}$, so that $d_Y(f(x_1), f(x_2)) < \varepsilon$ by triangle inequality. Therefore, we've shown that given $\varepsilon > 0$, there exists $\delta > 0$ such that for every pair of points $x_1, x_2 \in X$ with $d_X(x_1,x_2) < \delta$

$$d_Y(f(x_1), f(x_2)) < \varepsilon.$$

Hence, $f : (X, d_X) \to (Y, d_Y)$ is uniformly continuous. ■

§7.6 Limit point and sequential compactness

Definition 7.7. A space X is said to be **limit point compact** if every infinite subset of X has a limit point.

Theorem 7.15

Compactness implies limit point compactness.

Proof. Let X be a compact space. Given $A \subset X$, we want to prove that if A is infinite, then A has a limit point. Let us prove the contrapositive of the above statement, i.e., if $A \subset X$ has no limit point, then A is finite.

So, suppose that $A \subset X$ has no limit point. Then A contains all its limit points (which is an empty set in this case). We then conclude that $A = \overline{A}$, implying that A is closed in X. Since none of the elements of A is a limit point, for each $a \in A$, one can choose a neighbourhood U_a of a such that U_a intersects A at the point a alone. Now, the space X can be covered by $X \setminus A$ (which is open in X since A is closed in X) and the open sets U_a:

$$X = (X \setminus A) \cup \left(\bigcup_{a \in A} U_a \right) \quad \ldots \quad (25)$$

By the compactness of X, there is a finite subcollection of the cover $\{X \setminus A, \{U_a\}_{a\in A}\}$ of X. It means that in the finite subcollection, there are only finitely many of the open sets $\{U_a\}_{a\in A}$ since $(X \setminus A) \cap (\bigcup_{a\in A} U_a) = \varnothing$. Therefore, finitely many open sets from $\{U_a\}_{a\in A}$ each containing only one element of A is all of A. Hence, there are only finitely many elements in A. ■

Example 7.7. The converse of Theorem 7.15 is not true. Let $Y = \{y_1, y_2\}$ and let the topology on Y be the trivial topology consisting of Y and $\varnothing$. Consider $X = \mathbb{Z} \times Y$ with the product topology with $\mathbb{Z}$ being endowed with the discrete topology. Now, given $A \subset X$, with $A \neq \varnothing$, a generic element of A is given by (m, y_i), with $i = 1, 2$ and $m \in \mathbb{Z}$. Any open set in the product topology in X containing (m, y_i) also contains (m, y_{3-i}), for $i \in \{1, 2\}$. Hence, every subset of X, finite or infinite, has a limit point.

This implies that X is limit point compact. However, X is not compact. Since the open cover $\{U_m\}_{m\in\mathbb{Z}}$ with

$$U_m = \{m\} \times Y$$

does not have a subcollection (and hence no finite subcollection) covering X.

Definition 7.8. Let $(x_n)_{n=1}^{\infty}$ be a sequence of points in X. If

$$n_1 < n_2 < \cdots < n_k < \cdots$$

is a strictly increasing sequence of natural numbers, the sequence

$$x_{n_1}, x_{n_2}, \ldots, x_{n_k}, \ldots$$

also denoted by $(x_{n_i})_{i=1}^{\infty}$, is called a **subsequence** of $(x_n)_{n=1}^{\infty}$. It is a **convergent subsequence** if $x_{n_k} \to p$ as $k \to \infty$, for some $p \in X$.

Definition 7.9. A space X is **sequentially compact** if every sequence $(x_n)_{n=1}^{\infty}$ in X has a convergent subsequence $(x_{n_i})_{i=1}^{\infty}$.

Lemma 7.16

The Lebesgue number lemma holds for sequentially compact metric spaces. In other words, if X is a sequentially compact metric space, given an open cover $\mathcal{C} = \{U_\alpha\}_{\alpha \in J}$, there exists $\delta > 0$ such that for each subset $B \subset X$ with $\operatorname{diam} B < \delta$, there exists some $U \in \mathcal{C}$ such that $B \subset U$.

Proof. Assume for the sake of contradiction that $\mathcal{C} = \{U_\alpha\}_{\alpha \in J}$ is an open cover for which no Lebesgue number $\delta > 0$ exists. Then, given $n \in \mathbb{Z}^+$, there exists some a set C_n of diameter less than $\frac{1}{n}$ that is not contained in any $U \in \mathcal{C}$. Then we choose $x_n \in C_n$ for each n. Since X is sequentially compact, this sequence has a convergent subsequence, say $(x_{n_i})_{i=1}^\infty$. Suppose this subsequence converges to $a \in X$.

Since $\mathcal{C} = \{U_\alpha\}_{\alpha \in J}$ covers X, there is some $\beta \in J$ such that $a \in U_\beta$. Since U_β is open, we can find $\varepsilon > 0$ for which

$$B_d(a, \varepsilon) \subset U_\beta. \tag{7.66}$$

We choose i large enough so that $\frac{1}{n_i} < \frac{\varepsilon}{2}$, and $d(x_{n_i}, a) < \frac{\varepsilon}{2}$. C_{n_i} has diameter less than $\frac{1}{n_i}$, by construction. So for any $x \in C_{n_i}$,

$$d(x, x_{n_i}) \leq \sup_{x,y \in C_{n_i}} d(x, y) = \operatorname{diam} C_{n_i} < \frac{1}{n_i} < \frac{\varepsilon}{2}. \tag{7.67}$$

As a result,

$$d(x, a) \leq d(x, x_{n_i}) + d(x_{n_i}, a) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \tag{7.68}$$

So we have $x \in B_d(a, \varepsilon)$. Hence,

$$C_{n_i} \subset B_d(a, \varepsilon) \subset U_\beta. \tag{7.69}$$

But C_{n_i} was constructed so that it is not contained in any $U \in \mathcal{C}$. Contradiction! Therefore, a Lebesgue number must exist. ■

Proposition 7.17

Given a sequentially compact metric space X, for every $\varepsilon > 0$, there exists a finite cover of X using ε-balls.

Proof. Assume for the sake of contradiction that there exists some $\varepsilon > 0$ such that no finite covering of X exists with ε-balls. We construct a sequence $(x_n)_{n=1}^\infty$ as follows. Choose x_1 to be any point of X. Suppose we have constructed up to x_{n-1}. Since no finite covering of X exists with ε-balls,

$$B_d(x_1, \varepsilon) \cup B_d(x_2, \varepsilon) \cup \cdots \cup B_d(x_{n-1}, \varepsilon)$$

is not all of X. So we choose

$$x_n \in X \setminus \left(\bigcup_{i=1}^{n-1} B_d(x_i, \varepsilon) \right). \tag{7.70}$$

By construction, any two elements of this sequence satisfies

$$d(x_m, x_n) \geq \varepsilon. \tag{7.71}$$

Since X is sequentially compact, there is a convergent subsequence $x_{n_i} \to a$. Then there are infinitely many elements of the subsequence in the open ball $B_d\left(a. \frac{\varepsilon}{2}\right)$. Take two such elements x_{n_j} and x_{n_k}. Then we have

$$d\left(x_{n_j}, x_{n_k}\right) \leq d\left(x_{n_j}, a\right) + d(a, x_{n_k}) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \tag{7.72}$$

But then (7.72) contradicts (7.71). Therefore, a finite covering of X with ε-balls must exist. ■

Theorem 7.18

Let X be a metrizable space. Then the following are equivalent:

1. X is compact.
2. X is limit point compact.
3. X is sequentially compact.

Proof. **(1⇒2)** is already done in Theorem 7.15.

(2⇒3) Assume that X is limit point compact. Since X is metrizable, its topology is generated by a metric d. Also metric spaces are Hausdorff. In particular, X is a T_1 space.

Given a sequence $(x_n)_{n\in\mathbb{N}}$ of points of X, consider the set $A = \{x_n \mid n = 1, 2, 3, \ldots\}$. If the set A is finite, then there is a point x such that $x = x_n$ for infinitely many values of n. In this case, the sequence (x_n) has a subsequence that is constant, and therefore converges trivially.

On the other hand, if A is infinite, then A has a limit point x (since A is an infinite subset of a limit point compact space X). We define a subsequence of (x_n) converging to x as follows: First choose n_1 so that

$$x_{n_1} \in B_d(x, 1).$$

Then suppose that the positive integer n_{i-1} is given. Since x is a limit point of $A \subset X$, and X is T_1, by Theorem 2.15, the ball $B_d\left(x, \frac{1}{i}\right)$ intersects A in infinitely many points. So we can choose an index $n_i > n_{i-1}$ such that

$$x_{n_i} \in B_d\left(x, \frac{1}{i}\right).$$

Then the subsequence $x_{n_1}, x_{n_2}, \ldots$ converges to x. Indeed, if $B_d(x, \varepsilon)$ is an open ball about x, there exists some N such that $N\varepsilon > 1$. Then for $j \geq N$,

$$x_{n_j} \in B_d\left(x, \frac{1}{j}\right) \implies d\left(x, x_{n_j}\right) < \frac{1}{j} \leq \frac{1}{N} < \varepsilon. \tag{7.73}$$

So $x_{n_i} \to x$ as $i \to \infty$.

(3⇒1) Suppose $\mathcal{C} = \{U_\alpha\}_{\alpha\in J}$ be an open cover of the sequentially compact metric space X. Let $\delta > 0$ be the Lebesgue number associated with this cover. This exists by Lemma 7.16. Now, suppose $\varepsilon = \frac{\delta}{3}$. By Proposition 7.17, there is a finite covering of X with ε-balls, i.e.,

$$X = B_d(x_1, \varepsilon) \cup B_d(x_2, \varepsilon) \cup \cdots \cup B_d(x_n, \varepsilon). \tag{7.74}$$

Each of these balls are of radius ε, so their diameter is at most $2\varepsilon = \frac{2\delta}{3} < \delta$. By the Lebesgue number lemma for sequentially compact spaces, $B_d(x_i, \varepsilon)$ is contained in some $U_{\alpha_i} \in \mathcal{C}$. So we have

$$X = \bigcup_{i=1}^{n} B_d(x_i, \varepsilon) = \bigcup_{i=1}^{n} U_{\alpha_i}. \tag{7.75}$$

Therefore, a finite subcover of the open cover $\mathcal{C} = \{U_\alpha\}_{\alpha\in J}$ exists. Hence, X is compact. ■

§7.7 Local compactness

Definition 7.10. A space X is said to belocally compact at $x \in X$ if there is a compact subspace C_x of X containing a neighborhood U_x of x:

$$x \in U_x \subset C_x \subset X. \tag{7.76}$$

A space X is said to be **locally compact** if it is locally compact at each of its points.

Example 7.8. If X is compact, then take $U_x = C_x = X$ for a given point $x \in X$ in the equation (7.76), yielding the fact that X is also locally compact.

Example 7.9. The real line $\mathbb{R}$ with respect to the standard topology is locally compact. Each point $x \in \mathbb{R}$ is contained in the compact subspace $C_x = [x-1, x+1]$ containing the neighborhood $U_x = (x-1, x+1)$ of x.

Example 7.10. The set of rational numbers $\mathbb{Q}$ in the subspace topology inherited from $\mathbb{R}$ with respect to the standard topology is **not locally compact**.

A basis neighborhood of x in $\mathbb{Q}$ in subspace topology inherited from $\mathbb{R}$ with respect to standard topology (a basic open set that is a neighborhood of x) is of the form: $\mathbb{Q} \cap (x-\varepsilon, x+\varepsilon)$ for some $\varepsilon > 0$. Now, we will show that any subset $C_x \subset \mathbb{Q}$ containing such a basis neighborhood of x cannot be compact. For example, choose an irrational number $a \in (x-\varepsilon, x+\varepsilon)$ and write a function $f : C_x \to \mathbb{R}$ defined by

$$f(t) = (t-a)^2. \tag{7.77}$$

The function f, being a restriction of the $\mathbb{R} \to \mathbb{R}$ continuous function $t \mapsto (t-a)^2$, is also continuous. But the image of C_x under f can be made arbitrarily close to $0 \in \mathbb{R}$ with $0 \notin f(C_x)$. Hence, the function f doesn't attain a minimum. Hence, by the Extreme value theorem, C_x can't be compact.