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Orders, reduction graphs and spectra[1]

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Abstract

In this paper, we consider Abstract Reduction Systems as the setting where we need to investigate order properties of reduction graphs. To this aim, the main tool is the notion of spectrum, which captures some essential features of reduction sequences. We show that each spectrum is a complete partial order with respect to a suitable information ordering. Then we consider linearly ordered spectra. For those of them which are not well-ordered, we obtain a graph-theoretic characterization in terms of forbidden subgraphs. For well-ordered spectra, we show that they can represent all countable successor ordinals. Then, considering constructive ordinals, we address the well-known problem of knowing which ordinals are lambda representable (that is for which ordinal α there exists a lambda term T such that α is isomorphic to the spectrum of T). We give a partial answer by showing that all successor ordinals α, with $\alpha < \varepsilon_0$, are lambda-representable.

Keywords: Abstract reduction systems; Reduction graphs; Spectra; Ordinals; Constructive ordinals; Lambda-calculi

1. Introduction

Orders naturally arise in computations and reflect different frameworks, both at a syntactical and a semantical level. In fact, they have been used in several contexts in Computer Science, e.g. in term rewriting systems (see [6, 9, 12, 14, 17, 18]). In nondeterministic computations, typical of rewriting, suitable orders can be represented by means of graphs (see, for instance, [1]). In this paper, we first consider Abstract Reduction Systems as introduced by Klop in [12], which include, in particular, all kinds

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of rewriting. In this general setting, we study order properties of reduction graphs. To this aim, *spectra* are used (see [12, 18]), since they turn out to be a necessary tool for the following purposes:

- to eliminate cycles (so getting the *condensed* reduction graphs);
- to take into account also infinite reduction sequences;
- to identify reduction sequences which are, roughly speaking, mutually cofinal.

With the aid of these notions, we are able to distinguish, say, among *linear* and *well-ordered* spectra.

Section 2 contains fundamental notions on reduction graphs, condensed reduction graphs and spectra. Moreover, it contains the basic fact that each spectrum is a complete partial order which is the completion of the partial order associated to the condensed reduction graph. We make use of the proof in [18] which actually works in the ARS setting, although given for the $\lambda\beta$-calculus.

In Section 3 other order properties of spectra of Abstract Reduction Systems are investigated. We prove that:

(1) linearly ordered, not well-founded spectra can be characterized graph-theoretically in terms of forbidden subgraphs;
(2) well-ordered spectra represent exactly all countable successor ordinals.

In that section we somewhere make use of examples from a particular Term Rewriting System, namely the untyped λ-calculus. In Section 4 we focus on such a Term Rewriting System, and address the open problem on lambda-representability of ordinals, [18, 7, 8], that is for which ordinals α there exists a lambda term T such that α is isomorphic to the spectrum of T. We give a partial answer by showing that all successor ordinals α, with $\alpha < \varepsilon_0$, are lambda-representable. Section 4 is more technical in nature and devoted to readers interested on lambda-representability of ordinals.

2. Notations and preliminary notions

We assume the reader is acquainted with the terminology of rewriting and in particular of $\lambda\beta$-calculus (see [2]). The usual notions on ordered structures are also assumed (see e.g. [5]).

Abstract Reduction Systems are defined as follows (see [12] for further investigations).

Definition 1 (*Abstract reduction system*). An *Abstract reduction system* (ARS) is a structure

$$\mathscr{A} = \langle A, \to \rangle$$

where A is a countable set and $\to \subseteq A \times A$ is a relation named *reduction*, whose reflexive and transitive closure is denoted by $\to^*$.

We stress that our definition of ARS, unlike the one in [12], is constrained to the countability of A. Moreover, as in [12], an ARS can be also defined as having finite

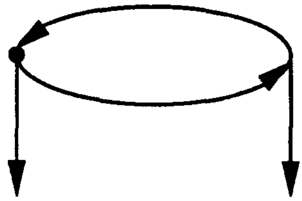

Fig. 1. The Hindley graph.

many distinct reduction relations. Elements of A are noted by $a, b, \ldots$. We say that *a reduces* to b in case $a \to^* b$ and we call b a *reduct* of a. Convertibility is noted by $=$ and identity by $\equiv$. Following [2, 12, 18], for every $a \in A$, we consider the rooted digraph associated to a whose nodes are labelled by elements $b \in A$ that are reducts of a.

Definition 2. Let a be an arbitrary element of an ARS. The reduction graph $G(a)$ of a is the ARS $\langle A_a, \to_a \rangle$, where $A_a = \{b\colon a \to^* b\}$ and $\to_a$ is the restriction of $\to$ to A_a.

A reduction graph can be a very complicated structure. In fact, from the order theoretic point of view it is simply a preorder with respect to $\leqslant$ such that $a \leqslant b$ if and only if $a \to^* b$.

Several examples of reduction graphs can be found in [12, 13]. Here we draw in Fig. 1 the well known reduction graph of Hindley (which shows that the Weak Church–Rosser property does not imply the Church–Rosser property).

In the above graph, as well as in all the others in this paper, we distinguish the root node by a • and we often omit labels. Moreover we will freely say graphs for reduction graphs.

Definition 3 (*Condensed reduction graph*). Let $b, c \in G(a)$. Define first that b and c are *cyclic equivalent*, notation $b \sim c$, if $c \to^* b$ and $b \to^* c$.
The *plane* of b, $b/\sim$, is the equivalence class b belongs to, i.e. the set of all c in $G(a)$ which are cyclic equivalent to b.
The *condensed reduction graph* of $a \in A$, notation $G^0(a)$, is the rooted directed graph whose nodes are planes, labelled by the set of elements of the plane or by whatever element of the plane assumed as its representative element, and with arcs $\to$ defined as follows:
$b/\sim \to d/\sim$ if there exists $c \in b/\sim$ and $c' \in d/\sim$ such that $c \to c'$ and $b \sim d$ does not hold.

It is easily seen that the reduction relation between equivalence classes is well defined.

As an example, Fig. 2 displays the condensed reduction graph of the Hindley graph.

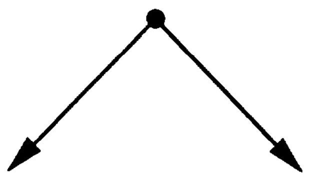

Fig. 2. The condensed Hindley graph.

It is clear that the notion of condensed reduction graph makes the information content of a $G(a)$ more essential from the order-theoretic point of view, since it is a partial order with respect to $\preccurlyeq$.

In the sequel, we assume that all reduction graphs we consider are condensed, if not otherwise stated. We use the simpler notation $G(a)$ instead of $G^0(a)$ whenever possible.

Moreover, we need the following well-known notion from general graph theory (see [19]).

Definition 4. Graphs G and G' are said to be *homeomorphic* if and only if there exists a graph G'' such that both G and G' can be obtained from G'' by inserting new nodes of degree two into arcs of G''.

Now we pass to consider spectra of ARSs. Whilst in a reduction graph finite reductions only are considered, in a spectrum infinite countable reductions are also taken into account. The notion of spectrum assumed here for an ARS is essentially the one in [18] for the $\lambda\beta$-calculus, which was inspired by the one in [2] where maximal (under the assumed ordering) reductions only are considered. A related, but different, definition is given in [14].

Definition 5 (*Reduction*). A *reduction* D is a finite or infinite sequence

$$a_0 \to a_1 \to a_2 \to \cdots$$

Now we define a preorder on reductions.

Definition 6. Let D, D' be reductions. We set:
- $D \preccurlyeq D'$ if every node in D reduces to some node in D'.
- $D \simeq D'$ if $D \preccurlyeq D'$ and $D' \preccurlyeq D$.
- Let $\tilde{D} = \{D' : D' \simeq D\}$.
  We put $\tilde{D}_1 \preccurlyeq \tilde{D_2}$ iff $D_1 \preccurlyeq D_2$.

The relation $\preccurlyeq$ between equivalence classes is easily seen to be well defined.

Definition 7. A reduction D is in $G(a)$ for some a if its first node is the root a.

Definition 8 (*Cofinality*). A reduction D in $G(a)$ is *cofinal* in $G(a)$ if $D' \preccurlyeq D$ for every D' in $G(a)$.

In particular, in the presence of the Church–Rosser property, if a has a normal form then every reduction to the normal form is cofinal.

Definition 9 (*Spectrum*). Let $a \in A$. Let $Red(a)$ be the set of all reductions starting from a. The *spectrum* of a, notation $Spec(a)$, is $\langle \tilde{Red}(a), \preccurlyeq \rangle$, where $\tilde{Red}(a) = \{\tilde{D} : D \in Red(a)\}$.

Spectra are essential to formulate order relationships between reductions. The next theorem states that a spectrum is always a cpo. In the proof of this theorem we use the following known result (see for instance [5, p. 54]). Recall ([5, p. 42]) that a subset P of a cpo Q is said to be *join-dense* in case for every $a \in P$ there exists a subset S of P such that $a = sup\, S$.

Lemma 10. *A partial order with a countable join-dense subset is a cpo if and only if every countable chain has a sup.*

Remark 11. The previous lemma applies to spectra since equivalence classes of finite reductions do give rise to a countable join-dense subset.

Theorem 12. *Let a be an arbitrary element of an ARS. Then Spec(a) is a cpo.*

Proof. See [18]. It follows from the definition that $\langle \tilde{Red}(a), \preccurlyeq \rangle$ is a partial ordering. In order to relate $Spec(a)$ to $G^0(a)$, as in [18], we define the ω-*completion* of $G^0(a)$, notation $\overline{G^0(a)}$, by using the construction and the notation in [3]. To this aim we need the following notions:

- the set $Ch(a)$ of ω-*chains* in the partial order $\langle G^0(a), \rightarrow \rangle$, with typical elements C, C';
- the ordering $\sqsubseteq$ defined as
  - $C \sqsubseteq C'$ if every node in C 'reduces' to some node in C';
  - $C \equiv C'$ if $C \sqsubseteq C'$ and $C' \sqsubseteq C$. Let $C/\!\equiv\, = \{C' : C' \equiv C\}$.
  - $\overline{G^0(a)} = \{C/\equiv\, : C \in Ch(a)\}$.

Let us still use $\sqsubseteq$ for the partial ordering induced on $\overline{G^0(a)}$ by the partial ordering on $Ch(a)$.

We prove the theorem by showing that

(1) $\overline{G^0(a)}$ is a cpo, i.e. a partial order where all $\sqsubseteq$-chains have a *sup*;

(2) $Spec(a)$ is isomorphic, as a partially ordered structure, to $\overline{G^0(a)}$.

To prove the first point, let us consider an $\sqsubseteq$-chain $C_0 \sqsubseteq C_1 \sqsubseteq C_2 \cdots$. Choose an arbitrary $n_0 > 0$. Consider the initial segment C_{i,n_0} of length n_0 of chain C_i, with $0 \leqslant i \leqslant n_0$. By the definition of the ordering between chains, we have in C_{n_0+1} an initial segment $C_{n_0+1,m}$ of length m, for some $m > n_0$, which dominates all the C_{i,n_0}. Let a_{n_0} be the last element in $C_{n_0+1,m}$. So whatever the reduction ends in, a_{n_0} dominates all the C_{i,n_0}. Assume a_{n_0} to be the first element of the chain to be constructed as the sup, with respect to $\sqsubseteq$. Take m as n_1. Repeat the previous argument with n_1 in place of n_0. We

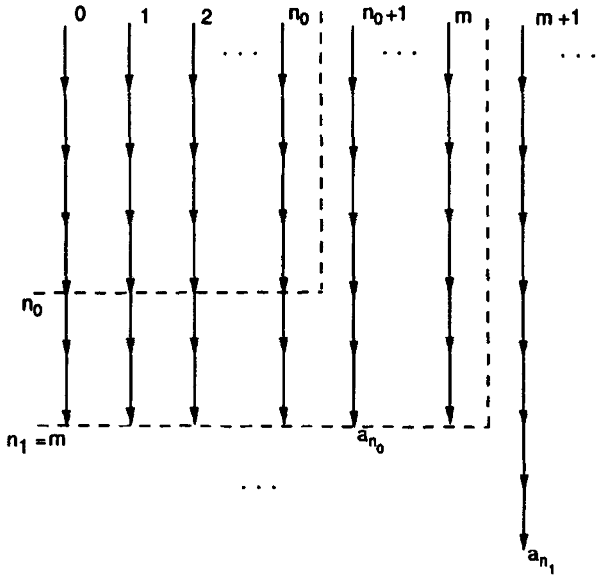

Fig. 3. The first two 'dominant' elements.

thus get a sequence of elements:

$$a_{n_0}, a_{n_1}, \ldots$$

Observe that $a_{n_j} \to^* a_{n_{j+1}}$, so that

$$C^* : a \to^* a_{n_0} \to^* a_{n_1} \to^* \cdots$$

is the required sup. Fig. 3 gives an insight of the diagonalization we make use of.

To prove the second point, observe that every reduction D in $G(a)$ is naturally associated to an ω-chain C_D in $G^0(a)$ by collapsing points of the same plane and possibly by repeating the last plane ω-times. Then define $\varphi : Spec(a) \to \overline{G^0(a)}$ by $\varphi(D) = C_D/\equiv$. The proof that φ is indeed an order-preserving isomorphism is standard and so it is left to the reader. □

Notice that a direct proof of Theorem 12 can be obtained from the proof we have given since one can arrange the argument of the sup construction to work directly on (equivalence classes of) reductions. Nevertheless we prefer the given proof because from it we immediately obtain the following noteworthy fact:

Corollary 13. *For every a, $G(a)$ and $G^0(a)$ have isomorphic spectra.*

Notice that in *Spec*(a) the empty reduction is the bottom element. For what concerns the existence of the maximum, we have the following result (analogous to 5.14 in [12]).

Proposition 14. *Spec*(a) *has the maximum element if and only if the condensed reduction graph* $G^0(a)$ *is Church–Rosser.*

Proof. The proof follows from the fact that the maximum is composed of cofinal reductions, if any, and from the equivalence, for countable ARSs, of the Church–Rosser property and the cofinality property. □

3. Order properties of spectra

In this section we investigate other order properties of spectra. We first consider linear non-well-founded spectra and then well-ordered spectra.

3.1. Linearly ordered spectra

We recall (see [15]) that a *linear order* is an order under which all elements are comparable; a linear order is *well-ordered* in case there is no infinite descending chain. We observe first that linearly ordered spectra can be characterized as follows.

Definition 15. $G^0(a)$ *contains the condensed Hindley Graph* in case it has a subgraph G homeomorphic to the one in Fig. 2 and the two nodes in G, corresponding to the leaves in the Figure, are such that neither of them reduces to the other.

We recall that the notion of homeomorphic graphs has been given in Definition 4.

Proposition 16. *Spec*(a) *is linearly ordered if and only if* $G^0(a)$ *does not contain the condensed Hindley Graph.*

Proof. Assume that $G^0(a)$ contains the condensed Hindley graph. Consider the derivations D and D', from the root of $G^0(a)$, which correspond, via the homeomorphism, to the two arcs from the root to the leaves in the condensed Hindley graph. Within *Spec*(a), the corresponding equivalence classes are not comparable under the order $\preccurlyeq$.

Conversely, if *Spec*(a) is not linearly ordered, then there exist two finite not comparable reductions D and D'. There must exist one last node a_H common to D and D'. The final segments of D and D' starting from a_H gives rise to a subgraph of $G^0(a)$ which is easily seen to be homeomorphic to the condensed Hindley graph. □

3.2. Non-well-founded spectra

For linear and non-well-founded spectra, we have two basic examples whose reduction graphs are the following.

(1) *Finite branching* – Let G_1 be the graph in Fig. 4, where ∘ stands for a limit node.

(2) *Not finite branching* – Let G_2 be the graph in Fig. 5.

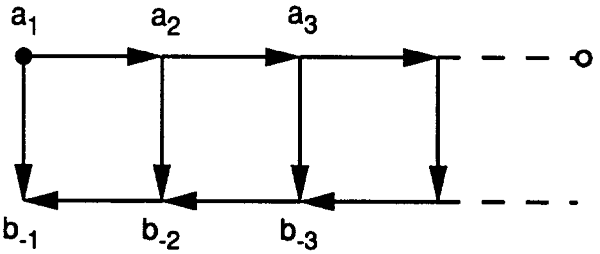

Fig. 4. Finite branching.

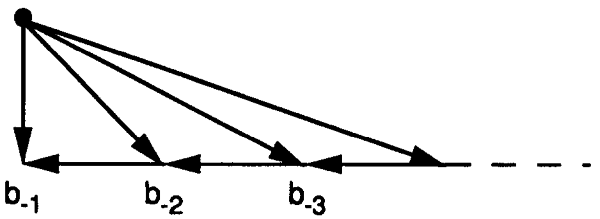

Fig. 5. Not finite branching.

In the following lemma we use *Spec*(G) with an obvious abuse of notation.

Lemma 17. *Spec*(G_1) *and Spec*(G_2) *are linearly ordered by* $\preccurlyeq$ *and their order types are* $\omega \oplus 1 \oplus \omega^*$ *and* $1 \oplus \omega^*$, *respectively, where* $\oplus$ *stands for the linear sum and* ω^* *stands for* ω *in the reverse order.*

Proof.

(1) Let a_1 be the root of G_1 and let $a_1, a_2, \ldots, a_n \ldots$ be the elements of the descending chain of $\rightarrow$; moreover, let $b_{-1}, b_{-2}, \ldots, b_{-n} \ldots$ be the elements of the descending chain of $\leftarrow$.
In order to make the argument more perspicuous, we describe the elements of *Spec*(G_1) by words on the alphabet $\{\mathbf{r}, \mathbf{d}\}$, where '**r**' stands for 'right' and '**d**' stands for 'down'. Then
- the language $\mathbf{r}^*$ represents all finite reductions starting from the root a_1 to some a_n;
- $\mathbf{r}^\omega$ represents the infinite reduction $a_1 \rightarrow a_2 \rightarrow \cdots$;
- the language $\mathbf{r}^*\mathbf{d}$ represents all finite reductions starting from the root a_1 to some b_{-n} via a_n.

These reductions (which include the empty one) are all the elements of *Spec*(G_1). Then the stated order type follows since for every n, m, with $m > n \geqslant 0$:
- $\mathbf{r}^n \prec \mathbf{r}^m \prec \mathbf{r}^\omega$,
- $\mathbf{r}^n\mathbf{d} \succ \mathbf{r}^m\mathbf{d}$.

Moreover, for arbitrary h and k, $\mathbf{r}^h\mathbf{d} \succ \mathbf{r}^k$, where k is possibly also ω.

(2) Let a be the root of G_2 and let $b_{-1}, b_{-2}, \ldots, b_{-n}$ be the elements of the descending chain of $\leftarrow$. Let $\tilde{D}_{-n}$ be the class of $D_{-n} : a \rightarrow b_{-n}$. The equivalence classes of

Fig. 6. $Spec(HH(\lambda z.\mathbf{I}))$.

these reductions (together with the empty one) are all the elements of $Spec(G_2)$. Then the stated order type follows since for every n, m, with $m > n > 0$, $D_{-n} \succ D_{-m}$ holds. $\square$

Theorem 18. *Let $\mathscr{A}$ be an ARS an let $a \in A$ be given. Let $Spec(a)$ be linear. Then $Spec(a)$ is well-founded if and only if $G^0(a)$ contains no subgraph homeomorphic to G_1 or G_2.*

Proof. We prove the non-trivial direction of the implication. So let $Spec(a)$ be linear and not well-founded. We claim that there exists an infinite descending chain $C : b_{-1} \leftarrow b_{-2} \leftarrow \cdots$ of elements of $G(a)$. Let $D_{-1} \succ D_{-2} \succ \cdots$ be a descending chain of reductions. Choose $b_{-1} \in D_{-1}$ such that b_{-1} reduces to no element of D_{-2}, choose $b_{-2} \in D_{-2}$ such that b_{-2} reduces to no element of D_{-3} and so on. Now, let $D_{b_{-1}}$, $D_{b_{-2}}$ be reductions from a to b_{-1}, b_{-2} respectively. Since $Spec(a)$ is linear, it follows that $D_{b_{-2}} \prec D_{b_{-1}}$, the other direction being impossible because of the choice of b_{-1}. It follows that b_{-2} reduces to b_{-1}. More generally, we obtain that $b_{-(n+1)}$ reduces to b_{-n}, and the claim is proved.
Now, two cases are possible.

(1) There are infinitely many disjoint reductions from some node a' to elements of C. Then a subgraph homeomorphic to G_2 is immediately obtained.
(2) The previous case does not occur. Hence, starting from the root a, a reduct a_2 of a must exist such that infinitely many elements of C can be reached from it. Since this cannot be done by disjoint reductions, a reduct a_3 of a_2 must again exist such that infinitely many elements of C can be reached from it. So we obtain a chain $C' : a \to a_2 \to \cdots$ with each node of C' such that infinitely many elements of C can be reached from it. Then a subgraph homeomorphic to G_1 is easily obtained as follows: choose first a reduction $D_1 : a \to^* b_{-1}$ and then, iteratively, choose a reduction $D_i : a_i \to^* b_{-j_i}$, where b_{-j_i} is such that $j_i > j_{i-1}$. Finally, transform the arcs of G_1 into the corresponding reductions by adding nodes of degree two. $\square$

We exemplify the theorem with a lambda term whose spectrum is linearly ordered and not well-founded. The reader will recognize G_1 inside the reduction graph in Fig. 6 of $Spec(HH(\lambda z.\mathbf{I}))$ with $H \equiv \lambda xy.y(y\mathbf{I}xxy)$.

3.3. Well-ordered spectra

Since well-ordered spectra represent ordinals, the following general question naturally arises: which ordinals are representable by the spectra of ARS?

A related question can be asked when constructive ARS are considered, e.g. an ARS generated by a Term Rewriting System. This second aspect has the specific interest in getting information about (classes of) constructive ordinals. Let us mention the open problem concerning which ordinals are representable by spectra of $\lambda\beta\mathbf{K}$-calculus [18, 8, 9].

Now we address the first question. In the next Section we mention some partial results concerning the second one.

Lemma 19. *Let Spec*(a) *be a well-ordered spectrum of an arbitrary ARS* $\mathscr{A} = \langle A, \rightarrow \rangle$. *Let Spec*($a$)$=\alpha$ *and let* $\beta < \alpha$. *Then there exists an ARS* $\mathscr{B} = \langle B, \rightarrow \rangle$, *with* $B \subset A$ *and* $\rightarrow_B$ *the restriction of* $\rightarrow_A$, *such that Spec*(a)$=\beta + 1$ *in* $\mathscr{B}$.

Proof. Let $\tilde{D}_\beta$ be the element of *Spec*(a) corresponding to β and let D_β be a representative. Take $\mathscr{B}$ as the set of all elements b in $G(a)$ such that $b \rightarrow^* c$ for some $c \in D_\beta$. It is enough to notice that all equivalence classes of *Spec*(a) in $\mathscr{A}$ to $\tilde{D}_\beta$ included, are in *Spec*(a) (with *Spec*(a) considered in $\mathscr{B}$). □

Theorem 20 (Ordinals).

(1) *Every well-ordered spectrum of a countable ARS represents a countable successor ordinal.*

(2) *Every countable successor ordinal can be represented as the spectrum of a finite branching ARS.*

Proof.

(1) Let *Spec*(a) be a well-ordered spectrum of an arbitrary ARS. We show that its order type is a countable successor ordinal α. Since *Spec*(a) is well ordered, $G(a)$ turns out to be Church–Rosser because of comparability of reductions. By Proposition 14 it follows that α must be a successor ordinal, so we have just to prove that α is countable. By contradiction, assume that α is the minimum uncountable ordinal of some spectrum *Spec*(a) of some ARS $\mathscr{A}$. Without loss of generality, we may assume that $G(a)$ is without cycles, for otherwise we can replace $G(a)$ with the condensed graph $G^0(a)$ getting the same spectrum, by Corollary 2. *Spec*(a) must have a maximum element $\tilde{D}_{\max}$, by Proposition 14.
   - Let $\tilde{D}_{\max}$ have a finite representative $D_{\max}$ and let $a_{\max}$ be the last element of $D_{\max}$. Observe that any other representative of $\tilde{D}_{\max}$ must have $a_{\max}$ as its last element (otherwise we obtain cycles). Consider the ARS $\mathscr{A}'$ obtained from $\mathscr{A}$ by eliminating $a_{\max}$ and by restricting the reduction relation accordingly. It is easy to see that a is still an element of $\mathscr{A}'$ and that *Spec*(a) is still well-ordered. So let β be the order type of *Spec*(a) in $\mathscr{A}'$. Then we must have $\alpha = \beta + 1$. Hence β has to be uncountable, contradicting the minimality of α.

- Let $\tilde{D}_{\max}$ have infinite representatives only. Let $D_{\max}$ be a representative of the form $a_1 \to a_2 \to \cdots$
  Let $\tilde{D}$ be any element of the spectrum different from $\tilde{D}_{\max}$. We claim that, for some n, $\tilde{D} \prec \tilde{D}_{\max,n}$ where $\tilde{D}_{\max,n}$ is the equivalence class of $a_1 \to a_2 \to \cdots \to a_n$. The claim easily follows observing that if, for every n, $\tilde{D}_{\max,n} \preccurlyeq \tilde{D}$ then $\tilde{D}_{\max} \preccurlyeq \tilde{D}$, which is impossible. It follows that $Spec(a) \setminus \tilde{D}_{\max}$ is the countable union of classes $C_n = \{\tilde{D} : \tilde{D} \prec \tilde{D}_{\max,n}\}$, then some C_n must be uncountable, say when $n = n_0$. Now, consider the subgraph $G(a_{n_0})$ of $G(a)$ restricted to the elements b such that b reduces to a_{n_0}, and let $\mathscr{A}'$ be the corresponding ARS. It follows that $Spec(a)$ in this restricted ARS is still well-ordered with an uncountable order type $\beta < \alpha$. This again contradicts the minimality of α.

(2) We have to show that every countable successor ordinal α is the order type of some $Spec(a)$ with a an element of some finite branching ARS $\mathscr{A}$. Let $\mathscr{F}$ be the family of ordinals which can be so represented. Clearly, the ordinal number 1 is in $\mathscr{F}$ since it corresponds to a one point ARS with the empty reduction relation. To prove our claim, we show that $\mathscr{F}$ is closed under successor and under successor of countable *sup*.

- Let a be an element of some finite branching ARS $\mathscr{A}$, such that $Spec(a) = \alpha$. Extend $\mathscr{A}$ with a new element b, and set $a' \to b$ for every a' such that a reduces to a'. The new ARS is still finite branching. Moreover $Spec(a)$ is still well-ordered and $Spec(a) = \alpha + 1$.
- Let a countable increasing sequence S of countable ordinals $\{\alpha_0, \alpha_1, \ldots\}$ be given. Assume that there exists a sequence $S_1 = \{a_0, a_1, \ldots\}$ where each a_n is an element of some finite branching ARS A_n, and $Spec(a_n) = \alpha_n$. We may of course assume that the A_n are pairwise disjoint. Define a new ARS $\mathscr{A}$, putting $A = \bigcup_n G(a_n)$, and setting $a'_n \to a_{n+1}$ for every $a'_n \in G(a_n)$. It is easy to see that $\mathscr{A}$ is still finite branching and that $Spec(a_0)$ is still well-ordered in $\mathscr{A}$, with $Spec(a_0) \geqslant Sup(\alpha_n) + 1$. Now in case $Spec(a_0) > Sup(\alpha_n) + 1$ we can use Lemma 19. $\square$

Remark 21. If the finite branching property is not required, a more direct proof of item 2 of the just stated Proposition can be given as follows.
First observe that a countable ordinal $\alpha > 0$ can be considered as an ARS by setting $\beta \to \gamma$ if $\beta < \gamma$. Then in the ARS α, $Spec(0)$ is well-ordered and represents $\alpha + 1$ by the following argument.

- Two finite reductions ending with the same ordinal are identified; two infinite reductions with the same sup are identified. Map every finite reduction ending in an ordinal β:
  to β if β is finite;
  to $\beta + 1$ otherwise.
- Map every infinite reduction to its sup.

Such a mapping is well defined with respect to the equivalence relation between reductions.

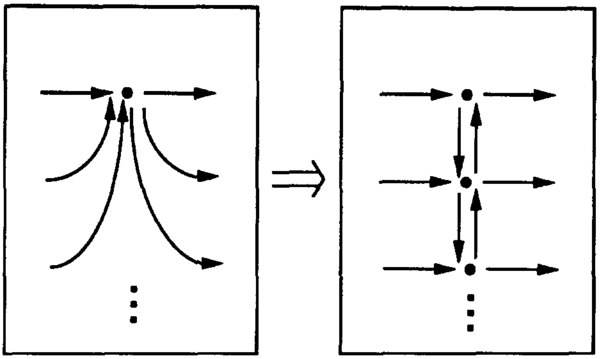

Fig. 7. Transformation 1.

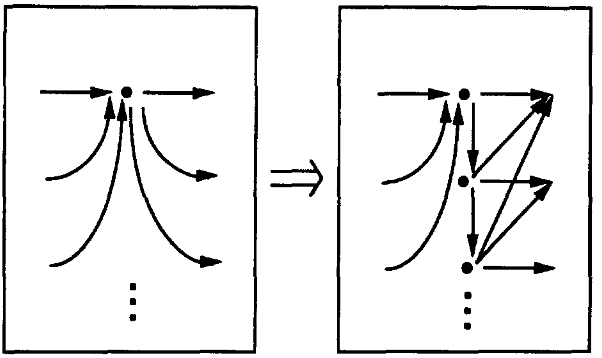

Fig. 8. Transformation 2.

Remark 22. For what concerns the finite branching property in the just stated Remark 21, one referee suggested the following interesting facts:

(1) for every ARS, the finite branching (as well as the finite branching for $\leftarrow$) can be obtained preserving the spectrum, at the cost of introducing cycles, by the transformation displayed in Fig. 7.

(2) In the well-ordered case, the transformation displayed in Fig. 8 keeps the spectrum well-ordered, at the cost of possibly increasing the order type, but then Lemma 19 can be applied;

(3) Moreover in Remark 21 one does not actually need all reductions $\beta \to \gamma$, with $\beta < \gamma$: in fact, since α is infinite countable, there is a bijection $\psi : \omega \to \alpha$ and so, for every $\beta \in \alpha$, one can define

- if $\beta + 1 \in \alpha$ then $\beta \to \beta + 1$.
- for every $m < \psi^{-1}(\beta)$ such that $\beta < \psi(m)$ define $\beta \to \psi(m)$.

The ordinal α within such a reduction relation gets a finite branching ARS with $\mathrm{Spec}(0) = \alpha + 1$. The key point in the construction is that if $\gamma < \beta < \alpha$ then $\gamma \to^* \beta$ via $\gamma \to^* \gamma + n \to^* \beta$ for some n that must exist since $\psi^{-1}(\beta) > \psi^{-1}(\gamma')$ just for a finite number of γ'.

Remark 23. We observe that, for an arbitrary ARS, *Spec*(a) is in general not countable (see for instance Theorem 5.9 in [18]). The fact that *Spec*(a) is always countable is a particular property of well-ordered spectra.

We now address a more constructive issue.

From [12] we recall the notion of increasing ARS, a notion which implies well-foundedness (and moreover is a consequence of well-foundedness plus finite branching of $\leftarrow$).

Definition 24. An ARS is *increasing* iff there is a map $|\ |: \mathscr{A} \to \mathbf{N}$ such that for all $a, b \in A$, $a \to b$ implies $|a| < |b|$.

Proposition 25. *Let Spec*(a) *be a well-ordered spectrum of an increasing ARS. Then the order type of Spec*(a) *is bounded by* $\omega + 1$.

Proof. *Spec*(a) being well-ordered, a $\tilde{D}_{\max}$ must exist by Proposition 14: let $D_{\max} \in \tilde{D}_{\max}$. Suppose, by contradiction, that the order type of *Spec*(a) is greater than $\omega + 1$. Then an infinite reduction $D: b_1 \to b_2 \to \cdots$ must exist such that $\tilde{D} \prec \tilde{D}_{\max}$, otherwise all reductions, except at most $D_{\max}$, are finite and, because of the well-ordering hypothesis, the order type is not greater than $\omega + 1$.

Now let $D_{\max,n}: a_1 \to a_2 \to \cdots \to a_n$ be the initial segment of $D_{\max}$ of length n. That $\tilde{D} \prec \tilde{D}_{\max,n}$ for some n must hold. Then every b_i reduces to a_n, and so $|b_i| < |a_n|$ for every i. This is a contradiction since, by the choice of D, there are infinitely many b_i. $\square$

4. Ordinals representable by lambda terms

Here we address the question of which ordinals are representable by lambda terms. We assume acquaintance with λ-calculus (see [2]). Moreover, we assume acquaintance with basic results on the theory of ordinals; all the needed notions can be found in [15]. Obviously, all finite ordinals greater than zero are representable; so we consider infinite ordinals only. Moreover, observe that, by Proposition 25, the $\lambda\beta\mathbf{I}$-calculus gives rise to ordinals bounded by $\omega + 1$. So we will consider the full $\lambda\beta\mathbf{K}$-calculus. We start with some examples of spectra representing ordinals. The first of them is in [18]. We make use of the known combinators $\mathbf{K} \equiv \lambda xy.x$, $\mathbf{I} \equiv \lambda x.x$, $\mathbf{O} \equiv \lambda xy.y$, $\omega_3 \equiv \lambda x.xxx$, $\Omega_3 \equiv \omega_3\omega_3$.

Example 26. (1) *Spec*$((\lambda x.\mathbf{I})\Omega_3\omega_3\omega_3)$ has order type $\omega \cdot 2 + 1$ whose graph representation is in Fig. 9:

(2) *Spec*$(HH\mathbf{O})$ with $H \equiv \lambda xyz.z(y(y\mathbf{I}\omega_3\omega_3)xxy)$ has order type $\omega^2 + 1$ (see Fig. 10).

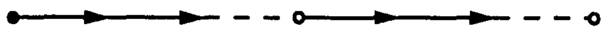

Fig. 9. $\omega \cdot 2 + 1$.

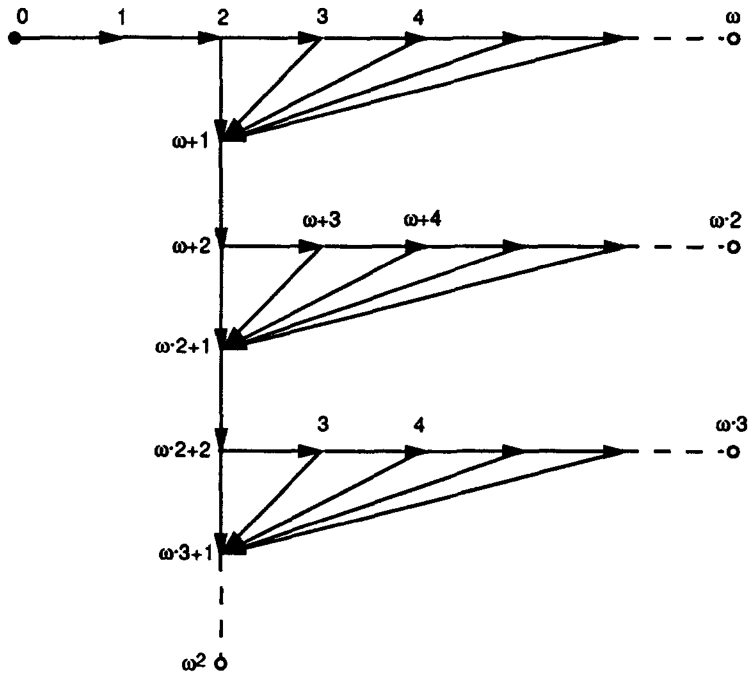

Fig. 10. $\omega^2 + 1$.

Now we give some general representation results. We recall that ordinals we are dealing with are all infinite and consequently ordinal functions F are interesting only on infinite ordinals. So from now on 'ordinal' stands for 'infinite ordinal' and ordinal function F stands for $F : On \setminus \omega \rightarrow On \setminus \omega$, where $On \setminus \omega$ is the class of all infinite ordinals.

To motivate the next definition, we stress that in a term having a well-ordered spectrum, redexes must always be in a 'hierarchical structure'. This means that, in every pair of redexes, one of them, once reduced, reaches a node of the reduction graph 'lower' than the node reached by the other. The need of obtaining terms with such a structure motivates the use of the normal forms below.

Definition 27. Let α be an ordinal and $\mathcal{N}$ be the set of all closed normal forms N with $N \equiv \lambda y.yN_1 \cdots N_k$ for some $N_1 \cdots N_k$ where $k \geqslant 0$ and y may occur in $N_1, \ldots, N_k$.

We say that α is *nf-representable* by N_α (also N_α *represents* α) if $N_\alpha \in \mathcal{N}$ and $Spec(N_\alpha \mathbf{O})$ has order type α.

Definition 28. Let F be an ordinal function.

We say that F is *normal form representable* (*nf-representable* for short) if there exists a closed term $\underline{F}$ such that

- $\underline{F}$ is in the normal form;
- for every nf-representable ordinal α and for every normal form N_α, representing α, $\underline{F}N_\alpha$ one step reduces to a normal form representing $F(\alpha)$.

Lemma 29. *The successor function $F(\alpha) = \alpha + 1$ is nf-representable by $\underline{F} = \lambda zy.y(y\mathbf{I}zy)$.*

Proof. In fact, $\underline{F}N_\alpha$ one step reduces to $\lambda y.y(y\mathbf{I}N_\alpha y)$. We claim that $\lambda y.y(y\mathbf{I}N_\alpha y)$ is a normal form representing $\alpha + 1$. Indeed $(\lambda y.y(y\mathbf{I}N_\alpha y))\mathbf{O} \to \mathbf{O}(\mathbf{O}\mathbf{I}N_\alpha\mathbf{O}) \to \mathbf{O}((\lambda z.z) N_\alpha\mathbf{O}) \to \mathbf{O}(N_\alpha\mathbf{O})$. So its spectrum has order type $4 + \alpha + 1 = \alpha + 1$, α being infinite. In fact, for every term M, if $Spec(M)$ has order type α, for some ordinal α, then $Spec(\mathbf{O}M)$ has order type $\alpha + 1$. To see this, observe that all reductions D starting from $\mathbf{O}M$ are as follows:

- $D : \mathbf{O}M \to \mathbf{O}M_1 \to \cdots \to \mathbf{O}M_n \to \cdots$ with $D' : M \to M_1 \to \cdots \to M_n \to \cdots$ is a finite (possibly of length 0) or infinite reduction sequence from M.
- $D : \mathbf{O}M \to \mathbf{O}M_1 \to \cdots \to \mathbf{O}M_n \to \mathbf{I}$ with $D' : M \to M_1 \to \cdots \to M_n$ is a finite reduction sequence from M.

All reductions in the second item are strictly greater than the ones in the first item; moreover, they are in the same equivalence class, which therefore is the maximum element of the spectrum of $\mathbf{O}M$. On the other hand, the equivalence classes generated by the reductions in the first item, give rise to an ordered set isomorphic to the spectrum of M. □

Lemma 30. *Let β be fixed and nf-representable by N_β. Then the function $F(\alpha) = \beta + \alpha$ is nf-representable by $\underline{F} = \lambda zy.y(y\mathbf{I}N_\beta y)(\lambda uvx.x(vu))yz$.*

Proof. In fact, $\underline{F}N_\alpha$ reduces after one step to $T \equiv \lambda y.y(y\mathbf{I}N_\beta y)(\lambda uvx.x(vu))yN_\alpha$. We claim that T is a normal form representing $\beta + \alpha$. Indeed, $Spec(T\mathbf{O})$ is, roughly speaking, isomorphic to a copy of β followed by a copy of α. □

Lemma 31. *The function $F(\alpha) = \alpha \cdot \omega + 1$ is nf-representable.*

Proof. Let $H \equiv \lambda xyz.z(y(y\mathbf{I}wy)xxy)$ where variable w is free. Take $\underline{F} \equiv \lambda wy.y\mathbf{I}HHy$. Let H_α stand for $H[N_\alpha/w]$. Notice that $\underline{F}N_\alpha\mathbf{O}$ reduces to $H_\alpha H_\alpha\mathbf{O}$. Now $H_\alpha H_\alpha\mathbf{O}$ reduces to $\lambda z.z(\mathbf{O}(\mathbf{O}\mathbf{I}N_\alpha\mathbf{O})H_\alpha H_\alpha\mathbf{O})$. The internal redex gives rise, roughly speaking, to a copy of α. Once such a redex is erased, a more nested occurrence of $H_\alpha H_\alpha\mathbf{O}$ is obtained and so on. Thus, ω disjoint copies of α are obtained and, taking into account the cofinal reductions, we get $\alpha \cdot \omega + 1$. □

Definition 32. Let F be an ordinal function. We define F^ω as follows: $F^\omega(\alpha) = \sup_{n<\omega}(\sum_{i=0}^{n} F^i(\alpha))$.

Proposition 33. *Let F be an nf-representable ordinal function. Then $G(\alpha) = F^{\omega}(\alpha)+1$ is nf-representable.*

Proof. We need to represent the iterations of F. Let $\underline{F}$ be a normal form representing F. By the notation $\underline{F}[w]$ we display the context of the fresh variable w in the term obtained by reducing $\underline{F}w$. By our nf-representability hypothesis in Definition 28, if N_α is a normal form representing α then $\underline{F}[N_\alpha/w]$ represents $F(\alpha)$. Let $\hat{F} \equiv \lambda wz.z(\underline{F}[w])$. Consider the term $Q \equiv \lambda zuwy.y\mathbf{I}uwMzuwy$ with $M \equiv \lambda x_1x_2x_3x_4x_5z.z(x_5(x_5\mathbf{I}x_4x_5)x_2x_2x_3x_1x_5)$.

Let $\underline{G} \equiv \lambda wy.y\mathbf{I}QQ\hat{F}wy$. Then $\underline{G}N_\alpha\mathbf{O} \to^* QQ\hat{F}N_\alpha\mathbf{O}$. By the definition of $\hat{F}$, we obtain $M\underline{F}[N_\alpha/w]Q\hat{F}N_\alpha\mathbf{O}$. By the definition of M, we obtain $\lambda z.z(\mathbf{O}(\mathbf{O}\mathbf{I}N_\alpha\mathbf{O})QQ\hat{F}(\underline{F}[N_\alpha/w])\mathbf{O})$.

The internal redex gives rise, roughly speaking, to a copy of α. Once such a redex is erased, a nested occurrence of $QQ\hat{F}(\underline{F}[N_\alpha/w])\mathbf{O}$ is obtained. By further reducing, $\underline{F}[N_\alpha/w]$ takes the place of N_α, so we obtain $\lambda z.z(\mathbf{O}(\mathbf{O}\mathbf{I}(\underline{F}[N_\alpha/w])\mathbf{O})QQ\hat{F}(\underline{F}[\underline{F}[N_\alpha/w]/w])\mathbf{O})$. By continuing, we get all finite sums $\alpha + F(\alpha) + F^2(\alpha) + \cdots + F^n(\alpha)$. At the limit (and also considering the cofinal reductions), we get $F^{\omega}(\alpha)+1$. □

The proof of the previous proposition makes use of techniques introduced in [4].

Recall (see [15, Definition 15.4]) that an ordinal number α is said to be a *γ-number* if $\beta + \alpha = \alpha$ for every $\beta < \alpha$. Recall also that each ω^β is a γ-number.

Theorem 34. *The ordinal $\omega^\omega + 1$ is nf-representable.*

Proof. Let $\alpha \geqslant \omega$. By Lemma 31, $F(\alpha) = \alpha \cdot \omega + 1$ is nf-representable. By Proposition 33, $G(\alpha) = F^{\omega}(\alpha) + 1$ is nf-representable.

Now let $\alpha_0 = \omega + 1$. Then by iteration of F we have

$$G(\alpha_0) = \sup_n \left(\sum_{i=0}^{n} F^i(\alpha_0) \right) + 1.$$

Observe that $F^{n+1}(\alpha_0) = \omega^{n+2} + 1$. In fact, by induction with respect to n, $(\omega^{n+1}+1)\cdot\omega = \sup_m(\omega^{n+1}+1)\cdot m = \sup_m((\omega^{n+1}+1) + \cdots + (\omega^{n+1}+1) m\text{-times}) = \sup_m(\omega^{n+1}\cdot m + 1) = \omega^{n+2}$. Therefore, $\sum_{i=0}^{n} F^i(\alpha_0) = \omega^{n+1} + 1$, since every ω^n is a γ-number. It follows that $G(\alpha_0) = \omega^\omega + 1$. □

By Proposition 33 we have shown how to perform the transformation from F to $F^\omega + 1$. So, starting from $F(\alpha) = \alpha\cdot\omega + 1$, we obtain $F^\omega(\alpha) + 1 = G(\alpha) = \alpha\cdot\omega^\omega + 1$. Now $G^\omega(\alpha)+1 = H(\alpha) = \alpha\cdot\omega^{\omega^2}+1$. By a further step we get $H^\omega(\alpha)+1 = \alpha\cdot\omega^{\omega^3}+1$. Such an iterative process is dominated by the ordinal ω^{ω^ω}.

To get such an ordinal we must iterate the *transformation* $F \to F^\omega + 1$. Call Φ such a transformation. We observe that Φ transforms ordinal functions into (faster growing) ordinal functions, so the order type of Φ is: $(On \to On) \to (On \to On)$. As just noticed, a finite number of iterations of Φ are not enough to get ω^{ω^ω}. This means that we need

Φ^{ω}. In fact, once Φ^{ω} is available and F is as just stated, i.e. $F(\alpha)=\alpha\cdot\omega+1$, we obtain $(\Phi^{\omega}F)(\alpha)=\alpha\cdot\omega^{\omega^{\omega}}+1$. This relies on the fact, intuitive for now, that $\Phi^{\omega}F=F^{\omega^{\omega}}$.

To make the giant step to ε_0, we need to consider all higher-order functionals of finite type on On as well as all transfinite iterations, below ε_0, of ordinal functions. To this aim, we proceed as follows:

- in Section 4.1 we put all the required mathematical definitions;
- in Section 4.2 we show how to represent all the functionals we have introduced by suitable λ-terms.

4.1. Higher-order functionals on ordinals

We need to define F^{α}, with α a countable transfinite ordinal. We distinguish between the following two aspects.

(1) From an abstract mathematical point of view, we use transfinite recursion [15, Theorem 13.6]. In particular, when the recursion is done w.r.t. a limit ordinal α, it is to be shown that whatever sequence to α will do. It is easy to check that this is the case for our definitions.

(2) From a constructive point of view, we stress that all notions we are going to introduce have an effective character. This can be seen for instance by using the following notation system in [16] (see also [10]), to reach limit ordinals.

Definition 35. Each limit ordinal $\gamma<\varepsilon_0$ can be written in the Cantor normal form (see e.g. [15]) as

$$\omega^{\beta_0}+\omega^{\beta_1}+\cdots+\omega^{\beta_k}$$

with $\gamma>\beta_0\geqslant\beta_1\geqslant\cdots\geqslant\beta_k>0$. A *fundamental sequence* $\gamma[n]$ for γ is

(1) $\gamma[n]=\omega^{\beta_0}+\omega^{\beta_1}+\cdots+\omega^{\beta_{k-1}}+\omega^{\beta_k-1}\cdot n$ if β_k is a successor ordinal;

(2) $\gamma[n]=\omega^{\beta_0}+\omega^{\beta_1}+\cdots+\omega^{\beta_{k-1}}+\omega^{\beta_k[n]}$ if β_k is a limit ordinal.

Another proof of the effectiveness of our constructions is given in the next subsection, where we show their lambda-representability.

Definition 36. Let $F:On\to On$ be such that $F(\alpha)\geqslant\alpha$ for every α. Let $\gamma<\varepsilon_0$. We put

- $F^0(\alpha)=\alpha$,
- $F^{\gamma+1}(\alpha)=F(F^{\gamma}(\alpha))$,
- $F^{\gamma}(\alpha)=\sup_m(F^{\gamma[m]}(\alpha))$ for γ a limit ordinal and $\gamma[m]$ a fundamental sequence for γ.

Now we turn to higher-order functionals on ordinals.

Definition 37 (*Higher-order ordinal domains*). $\tau_0=On$, $\tau_{n+1}=\tau_n\to\tau_n$, for every $n\geqslant 0$.

Remark 38. Notice that every τ_n, $n>0$, is a cpo once ordered componentwise. Moreover, for every $n>0$, we call an element $F\in\tau_n$ *progressive* if for every $f\in\tau_{n-1}$, $\mathrm{F}(f)\geqslant f$.

Observe that every τ_n restricted to its progressive elements is still a cpo. From now on we restrict ourselves to *progressive functions and functionals*, if not otherwise stated.

Let $\mathbf{F}$ be a functional in τ_n, $n \geqslant 2$. We define the exponentiation of $\mathbf{F}$ to an ordinal $\gamma < \varepsilon_0$ in a way analogous to that in Definition 36.

Definition 39. Let $\mathbf{F} \in \tau_n$, with $n \geqslant 2$. Let $\gamma < \varepsilon_0$. We put

- $\mathbf{F}^0(f) = f$,
- $\mathbf{F}^{\gamma+1}(f) = \mathbf{F}(\mathbf{F}^{\gamma}(f))$,
- $\mathbf{F}^{\gamma}(f) = \sup_m(\mathbf{F}^{\gamma[m]}(f))$ for γ a limit ordinal, with $\gamma[m]$ a fundamental sequence for γ, and where the sup is within τ_{n-1} as a cpo.

Now we define inside each τ_n, with $n > 1$, a particular kind of functionals, namely *iterators*, with the notation Φ_n, which iterate their arguments ω times. This makes sense when arguments are functions only, i.e. for $n > 1$.

Definition 40. For each $f \in \tau_{n-1} : \Phi_n(f) = f^{\omega}$, with $f^{\omega} = \sup_m f^m$.

We write functionally $\Phi_n f$ for $\Phi_n(f)$.

Proposition 41. *For every* $\gamma < \varepsilon_0$, $\Phi_n^{\gamma} f = f^{\omega^{\gamma}}$.

Proof. By transfinite induction:

- $\Phi_n^0 f = f = f^{\omega^0}$,
- $\Phi_n^{\gamma+1} f = \Phi_n f^{\omega^{\gamma}} = f^{\omega^{\gamma}\cdot\omega} = f^{\omega^{\gamma+1}}$,
- for a limit $\gamma : \Phi_n^{\gamma} f = \sup_m(\Phi_n^{\gamma[m]} f) = \sup_m(f^{\omega^{\gamma[m]}}) = f^{\omega^{\gamma}}$. □

Also by transfinite induction we can prove the following proposition.

Proposition 42. *For every* $\alpha, \beta < \varepsilon_0$, $\Phi_n^{\alpha} \circ \Phi_n^{\beta} = \Phi_n^{\alpha+\beta}$, *where* $\circ$ *stands for composition.*

Now we construct a sequence of functions in τ_1 starting from a function F_1 such that $F_1(\alpha) > \alpha$ for every α, by defining

$$F_{n+1} = \Phi_{n+1}\Phi_n \cdots \Phi_2 F_1.$$

Proposition 43. $F_{n+1} = F_1^{\omega^{\cdot^{\cdot^{\omega}}}}$, *i.e.* F_1 *exponentiated to* $\omega^{\cdot^{\cdot^{\omega}}}$ *where the exponent is taken* n *times.*

Proof. Using Proposition 41 several times, we obtain

$$F_{n+1} = \Phi_{n+1}\Phi_n \cdots \Phi_2 F_1 = \Phi_n^{\omega} \cdots \Phi_2 F_1 = \Phi_{n-1}^{\omega^{\omega}} \cdots \Phi_2 F_1 = \cdots = F_1^{\omega^{\cdot^{\cdot^{\omega}}}}. \qquad \square$$

If we start from F_1, with $F_1(\alpha)=\alpha\cdot\omega$, as previously done, we obtain $F_{n+1}(\alpha)=\alpha\cdot\omega^{\omega^{\cdot^{\cdot^{\omega}}}}$ where the exponent is taken n times. We call

$$S_\omega=\{F_n\}_{n\in\mathbf{N}} \tag{1}$$

the sequence of the so defined functions.

4.2. Lambda-terms for higher-order functionals on ordinals

Now we define a sequence of terms $\underline{\Phi}_n$ which nf-represent functionals Φ_n in the sense we are going to make precise.

Definition 44 (*Terms representing functionals*). Let:

- $\underline{\Phi}_2=\lambda x_1x_0y.\,y\mathbf{I}Q_1Q_1x_1x_0y$ where $Q_1\equiv\lambda ux_1x_0y.\,y\mathbf{I}x_1x_0M_1ux_0x_1xy$ and $M_1\equiv\lambda xux_1x_0yz.z(y(y\mathbf{I}x_0y)uux_1y)$.
- for $n\geqslant 2, \underline{\Phi}_{n+1}=\lambda x_nx_{n-1}\cdots x_1x_0y.y\mathbf{I}Q_nQ_nx_nx_{n-1}\cdots x_1x_0y$ where $Q_n\equiv\lambda ux_nx_{n-1}\cdots x_1x_0y.\,y\mathbf{I}x_nx_{n-1}M_nux_nx_{n-1}\cdots x_1x_0y$ and $M_n\equiv\lambda xux_nx_{n-1}x_{n-2}\cdots x_1x_0yz.z(y(y\mathbf{I}x_{n-1}x_{n-2}\mathbf{I}\cdots x_1\mathbf{I}x_0\mathbf{I}y)uux_nxx_{n-2}\cdots x_1x_0y)$.

We define moreover:
$\hat{\Phi}_{n+1}=\lambda\bar{x}_nz_n.z_n(\lambda\bar{x}_{n-1}z_{n-1}.z_{n-1}(\cdots(\lambda\bar{x}_1z_1.z_1(\lambda\bar{x}_0z_0.z_0(\underline{\Phi}_{n+1}[\bar{x}_n/x_n,\bar{x}_{n-1}/x_{n-1},\ldots,\bar{x}_1/x_1,\bar{x}_0/x_0])))\cdots))$ and, for a term $\underline{F}$ representing an ordinal function, we define $\hat{F}=\lambda\bar{x}_0z_0.z_0(\underline{F}[\bar{x}_0/x_0])$.

To help reading the previous terms, we notice that for every i, variable x_i, also when renamed as $\bar{x}_i$, ranges over terms representing elements of the domain τ_i. Now we define the term $\underline{F}_n$ which nf-represents the nth element F_n of the sequence S_ω defined in Eq. (1) above. More precisely, we represent $F_n(\alpha)+1$, because of the cofinal reductions.

Definition 45. Let $\underline{F}_1$ be the term representing $\alpha\cdot\omega+1$. We put $\underline{F}_{n+1}=\underline{\Phi}_{n+1}\hat{\Phi}_n\cdots\hat{\Phi}_2\hat{F}_1$.

Proposition 46. *$\underline{F}_n$ nf-represents the function $F_n(\alpha)+1$.*

Proof. We simply sketch the proof. First we explain the role of the transformation $\hat{\Phi}_n$, which is needed to allow $\underline{\Phi}_n$ to be applied to its arguments without giving rise to a redex until $\mathbf{O}$ takes the heading position. The trick consists of 'enveloping' $\underline{\Phi}_n$ into $n+1$ nested 1-tuples. Terms are structured in such a way that when $\hat{\Phi}_{n-1}$ substitutes $\bar{x}_{n-1}$ in $\hat{\Phi}_n$, it remains duly enveloped into n nested 1-tuples. So in the sequel, when we write $\hat{\Phi}_n[\hat{\Phi}_{n-1}/\bar{x}_{n-1}]$ we mean not only that the substitution has been performed but also that the enveloping has been reduced by 1.

The reduction flow is roughly the following: clearly $\underline{\Phi}_2$ represents the iteration of whatever nf-representable function. For $n>2$, $\underline{\Phi}_n$ applied to its arguments $\hat{\Phi}_{n-1},\ldots,\hat{\Phi}_2,\hat{F}_1$, N_α, $\mathbf{O}$, gives rise to a redex of the form

$$\mathbf{O}(\mathbf{O}\mathbf{I}\hat{\Phi}_{n-1}\hat{\Phi}_{n-2}\mathbf{I}\cdots\hat{\Phi}_2\mathbf{I}\hat{F}_1\mathbf{I}N_\alpha\mathbf{I}\mathbf{O})Q_{n-1}Q_{n-1}\hat{\Phi}_{n-1}(\hat{\Phi}_{n-1}[\hat{\Phi}_{n-2}/\bar{x}_{n-2}])\cdots\hat{F}_1N_\alpha\mathbf{O}.$$

Now observe that:
- The internal redex corresponds to the basis step in the iteration of $\underline{\Phi}_{n-1}$.
- If this internal redex is skipped by **O**, the term $Q_{n-1}Q_{n-1}\hat{\Phi}_{n-1}(\hat{\Phi}_{n-1}[\hat{\Phi}_{n-2}/\overline{x}_{n-2}])$ $\cdots \hat{F}_1 N_\alpha \mathbf{O}$ is obtained where $\hat{\Phi}_{n-1}[\hat{\Phi}_{n-2}/\overline{x}_{n-2}]$ has taken the place of $\hat{\Phi}_{n-2}$.

By continuing the iteration, at the successive step the place of $\hat{\Phi}_{n-2}$ is taken by $\hat{\Phi}_{n-1}[\hat{\Phi}_{n-1}[\hat{\Phi}_{n-2}/\overline{x}_{n-2}]/\overline{x}_{n-2}]$. This means that $\underline{\Phi}_n$ iterates composition of $\underline{\Phi}_{n-1}$. □

Theorem 47. *All successor ordinals less than ε_0 are nf-representable.*

Proof. Observe first that representability of composition of functionals is not a problem. Then, by transfinite induction, by using the Cantor normal form, Propositions 41 and 42, it is possible to prove that for every $\gamma < \varepsilon_0$ and for every n there is a term representing Φ_n^γ. From this result and the other representability results the theorem follows easily. □

4.2.1. Towards $\varepsilon_0 + 1$

We have shown that the ordinals $\omega+1$, $\omega^\omega+1$, $\omega^{\omega^\omega}+1, \ldots$ are all nf-representable. It is clear from the methods used above that the (successor of the) limit can be obtained once the corresponding sequence of normal forms can be generated in a uniform way, that is by a specified term, in such a way that the spectrum of the resulting terms turns out to be well-ordered. The lack of uniformity in the normal forms which have been actually set up, makes this task dramatically laborious, if it can be performed at all. So we leave the problem of linearly representing the ordinal $\varepsilon_0 + 1$ open, though we believe that it is also nf-representable.

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