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REDUCTION GRAPHS IN THE LAMBDA CALCULUS

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Abstract. In this paper properties of the reduction graphs of lambda terms are studied and some classes of reduction graphs are characterized. Condensed reduction graphs obtained by dividing out 'cyclic equivalence', and spectra, the partially ordered set of all reductions, are also considered. The partial ordering in the spectrum can be seen as a measure for the 'significance' of a reduction; reductions to the normal form and (more generally) cofinal reductions are the most significant reductions. The spectrum is proved to be the completion of the condensed reduction graph.

Introduction

The reduction graph $G(T)$ associated to a λ-term T is the connected pseudodigraph (in the terminology of Harary [6]) obtained by labelling the source node of $G(T)$ by T itself and every other node by a reduct of T such that every reduct of T appears as a label and such that every arc in $G(T)$ corresponds to the contraction of a redex.

The primary motivation for studying reduction graphs derives from the fact that not every connected pseudodigraph can be labelled so as to become a reduction graph. Moreover, except for the graph-theoretical consequences of the Church–Rosser property and of the local finiteness (the second property due to the fact that a λ-term contains only finitely many redexes), few general facts are mentioned in the literature about reduction graphs (see [1, 7, 8]). Therefore, it is worthwhile to establish that some simple digraphs are impossible reduction graphs, so this paper starts by giving, in Section 2, some simple examples of reduction graphs and also some simple impossible cases.

A reduction graph may determine some properties of its source label: e.g., the reduction graph consists of one node only and no arc iff it is the graph of a normal form. But, at the same time, two distinct normal forms are not distinguished by their graphs since the reduction graph is the same for every normal form. Therefore, only reducible terms have nontrivial reduction graphs so that one can try to relate structure and reduction graph of a λ-term to obtain characterizations of some subsets of the set of all reduction graphs. That is what is achieved in Section 3 with respect to *linear* reduction graphs, as defined there, which turn out to be easily characterized as the graphs of *hereditarily multisimple* terms. These terms are also syntactically characterized in the same Section 3 (but the rather tedious proof of the characterization Theorem 3.3 is given in Appendix A).

From Section 4 on we pass to consider also nonlinear graphs, which in general are quite complex. The notion of a *strong component* in a reduction graph, as a maximal set of terms M, N such that M and N can be reduced to each other, called *plane* in [7] and [8], allows us to express a reduction graph in terms of a simpler one, namely the directed acyclic graph whose nodes are planes: the 'condensed reduction graph'.

Now the following question arises: Can we state some general properties of λ-terms in a nonsingleton plane?

Terminal planes (i.e., such that no arc starts from them) are easily seen to be sets of mutually reducible recurrent terms, as defined in [15] and [16].

For nonsingleton nonterminal planes, a special case of a conjecture in Klop [8] is proved.

Bottleneck planes, defined in Section 4, reveal that a condensed reduction graph can also have other nodes distinct from the initial and the terminal ones which are comparable, by the natural order relation induced on the condensed reduction graph, with every other node of the same graph.

Section 5 is devoted to the introduction and study of the *spectrum* of a lambda term T. The elements of the spectrum of T are certain equivalence classes of reductions, finite or infinite, starting with T; these elements are partially ordered by some measure of 'significance' for reductions. For instance, a finite reduction inside an infinite reduction graph is certainly maximally significant in case it reaches a terminal plane, while an infinite reduction which is entirely confined within some nonterminal plane does not reach important parts of the reduction graph it belongs to. Hence the central role of cofinal reductions in a graph.

1. Preliminary notations and terminology

Let us assume some usual notations. R stands for a redex, T, U, V, M, N for terms; A, B stand for collections of redexes, Λ for the set of λ-terms, $\equiv$ stands for identity (up to α-conversion), $=$ for conversion, $\rightarrow$ for one step reduction, $\twoheadrightarrow$ for the reflexive and transitive closure of $\rightarrow$, and $\rightarrow^{+}$ for the transitive closure of $\rightarrow$.

A/R denotes the set of all residuals of redexes in A by contraction of R.

If $\mathcal{S}$ is a reduction from T to M we write

$$\mathcal{S}: T \twoheadrightarrow M \text{ or } T \twoheadrightarrow^{A} M \text{ or, with or without } R_i,\ 1 \leq i \leq n,$$

$\mathcal{S}: T \equiv T^{(0)} \rightarrow^{R_1} T^{(1)} \rightarrow^{R_2} \cdots \rightarrow^{R_n} T^{(n)} \equiv M$ where, as usual, $\rightarrow^{R_i}$ stands for contraction of R_i. Every $T^{(i)}$ is a *stage* of $\mathcal{S}$.

As in [1], $A/\mathscr{D} = (\cdots((A/R_1)/R_2)\cdots)/R_n$. In particular we have $R/\mathscr{D} = (\cdots((R/R_1)/R_2)\cdots)/R_n$.

If $\mathscr{D}_1 : T \twoheadrightarrow^{A_1} M_1$ and $\mathscr{D}_2 : T \twoheadrightarrow^{A_2} M_2$, then $\mathscr{D}_1/\mathscr{D}_2$ is a reduction, that is, $\mathscr{D}_1/\mathscr{D}_2 : M_2 \twoheadrightarrow^{B} N$ where B is the set of residuals of A_1, in M_2, with respect to $\mathscr{D}_2$.

$C[,\ldots,]$ is a multiple context written as $C[T_1,\ldots,T_n]$ when its ith hole is replaced by T_i, $i = 1,\ldots,n$. For a precise definition of multiple context, see [1].

For notions concerning digraphs we mainly refer to [6].

A *pseudodigraph* (directed pseudograph) G consists of a set of nodes and a collection of arcs (oriented edges). A *loop* is an arc joining a node to itself. A *multi-arc* of degree $k \geqslant 1$, which we call k-arc, is composed of k arcs (of the same orientation) between two fixed nodes.

A *source* in G is a node from which all the others can be reached. We denote by $|G|$ the number of all nodes of G and by $\|G\|$ the number of all nodes and of all arcs of G. A *multidigraph* has no loop; moreover, it is a digraph if no multi-arc is in it. A subpseudodigraph of G is a pseudodigraph having all points and arcs in G.

A *path* is an alternating sequence of nodes and arcs starting and ending with nodes; and such that the arcs have the same orientation along the sequence. It is *closed* when the first and last node are the same. A closed path will also be called a *cycle*. Note that a loop is a trivial cycle, consisting of one node and one arc only.

G is *connected* whenever every two nodes are joined by a path. A *component* of G is a maximal connected subgraph of G and a *strong component* is such that every two nodes are mutually reachable. The *condensation* G^0 of G has the strong components C_i of G as its nodes with an arc from node i to node j whenever there is an arc in G from a node in C_i to a node in C_j.

If i is a node in G, node j is called a *successor* of i if j is the endpoint of one of the arcs which leave i. (Hence a node can be its own successor.)

G is *locally finite* if every node in G has only finitely many successors.

2. $\lambda\beta$-pseudodigraphs

We associate to every term T in Λ the pseudodigraph $G(T)$ as follows.

Definition 2.1. Let $T \in \Lambda$. Then $G(T)$, the *$\lambda\beta$-pseudodigraph* of T, or *reduction graph* of T for short, is defined by the following requirements:

(i) M is the label of a node of $G(T)$ iff $T \twoheadrightarrow M$.

(ii) If M_1, M_2 are labels of distinct nodes of $G(T)$, then $M_1 \not\equiv M_2$.

(iii) $n \geqslant 1$ arcs join node M_1 to node M_2, with possibly $M_1 \equiv M_2$, iff M_2 is obtainable from M_1 by contracting $n \geqslant 1$ redex occurrences in M_1.

Evidently $G(T)$ is finite in case $\{M \mid T \twoheadrightarrow M\}$ is finite and infinite, but still locally finite, otherwise. $\|G(T)\| = 1$ iff T is in normal form.

Examples 2.2

(1) $\bullet\, I$ with $I \equiv \lambda x.x$

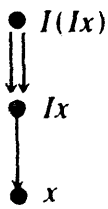
(2) 

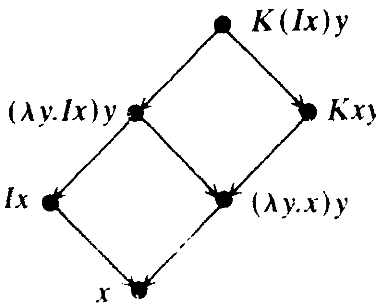
(3) 

where $\Omega \equiv \omega\omega$ and $\omega \equiv \lambda x.xx$;

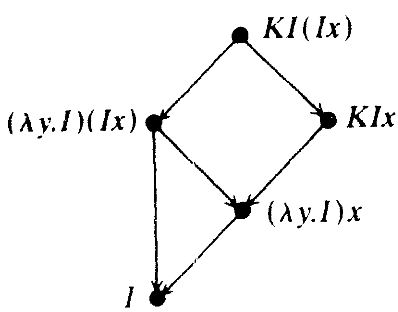

is a loop

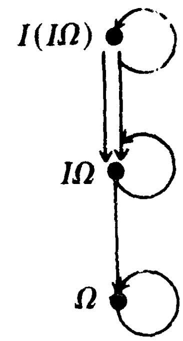
(4) $\omega_3(\lambda y.\omega_3) \bullet \rightleftarrows \bullet\, (\lambda y.\omega_3)(\lambda y.\omega_3)(\lambda y.\omega_3)$
where $\omega_3 \equiv \lambda x.xxx$

(5) 

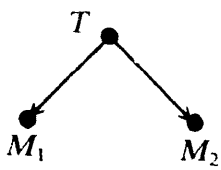
(6) 

where $K \equiv \lambda zu.z$

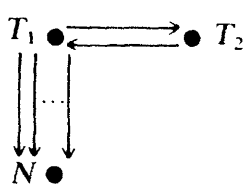
(7) 

(8) 

Because of the Church–Rosser property, a digraph of the form

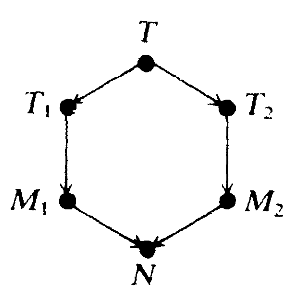


cannot be a reduction graph. Moreover, although the Church–Rosser property allows it, every one of the following kind:



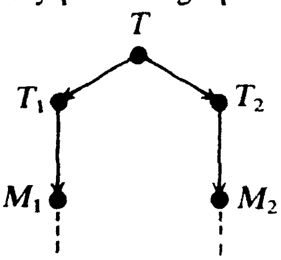
with a k-arc, $k \geq 1$, from T_1 to N, cannot be labelled by λ-terms because if R_1 and R_2 are two redex occurrences in T_1 such that $T_1 \rightarrow^{R_1} N$ and $T_1 \rightarrow^{R_2} T_2$, then the unique redex occurrence in T_2, say R_0, is such that $R_0 \equiv R_1/R_2$. By the 'parallel moves' lemma (see [4]), $T_2 \rightarrow^{R_1/R_2} N$, while $T_2 \rightarrow^{R_0} T_1$.

For the same reasons a digraph with the following shape:



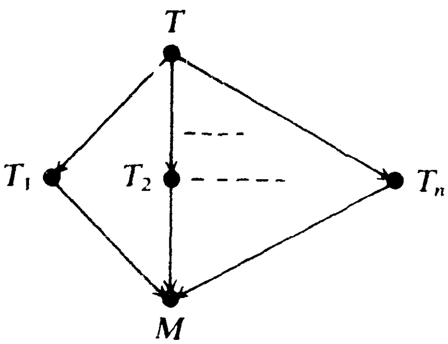
cannot be a $\lambda\beta$-graph. In fact if $T \rightarrow^{R_1} T_1$, $T \rightarrow^{R_2} T_2$, then $T_1 \rightarrow^{R_2/R_1} M_1$, $T_2 \rightarrow^{R_1/R_2} M_2$, with $M_1 \equiv M_2$.

Hence every pseudodigraph starting with

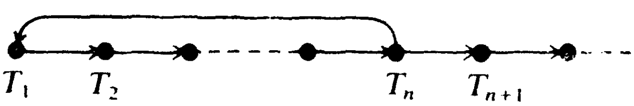


is not a $\lambda\beta$-graph.

Analogously for



with $n \geq 3$ (which is a generalization of [1, Exercise 3.5.6(ii), p. 74]), and likewise for the following graph, already in [7]:



Here loops in every T_i, $i \geq 1$, can also be added but the situation remains the same.

Finally we mention that there exists a universal reduction graph $G(M)$ (see [1, p. 167]) in the sense that, for every N, $G(N)$ is a subgraph of $G(M)$: in fact that is the case if M is the universal generator as defined in [1, p. 166].

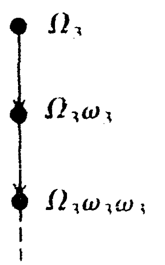
3. Linear reduction graphs

Definition 3.1. A reduction graph is *linear* iff every node has at most one successor.

Examples 2.2(1), (2), (3), (4) and (5) above are linear, while (6), (7) and (8) are not. An example of an infinite $G(T)$ is $G(\Omega_n)$ with $\Omega_n = \omega_n\omega_n$, $n \geq 2$, and $\omega_n = \lambda x.(x\ldots x)_{n\ \text{times}}$, as in the following examples.

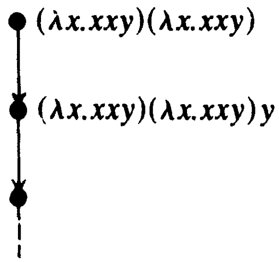
Examples 2.2 (*continued*)

(9)



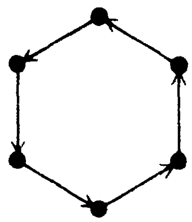
Infinite and linear is also the following example:

(10)



Notice that, according to Definition 3.1, every *pure cycle* as defined in [7] is a linear reduction graph which is a strong component. An example of a pure cycle is

$$MMI \quad \text{where } M = \lambda xy.yyyyxxy$$



Now we need the notions of simple term (introduced in [16, 15]) and multisimple term, defined as follows.

Definition 3.2. (i) T is *simple* iff T contains only one redex. (So in a reduction graph, T has precisely one outgoing arc.)

(ii) T is *multisimple* iff whenever $T \to T_1$ and $T \to T_2$, $T_1 \equiv T_2$ holds. (So in a reduction graph, T has precisely one successor, which may be T itself.)

Thus every simple term is a particular multisimple one. Multisimple but not simple terms are for instance $(\lambda x.(\lambda y.y)x)z$, $(\lambda x.(\lambda x.y)x)x$, $I(I(IK))$ and for every N_1 and N_2 in normal form, the term $\Omega N_1 \Omega N_2$.

It is decidable whether an arbitrary T is multisimple. Moreover, the following syntactic characterization holds for multisimple redexes.

Theorem 3.3. *Let R be a redex. Then*:

(1) *R is simple iff $R \equiv (\lambda x.N_1)N_2$ for some N_1, N_2 in normal form,*

(2) *R is multisimple but not simple iff it has one of the following forms, where $n \geqslant 2$, and N_1, N_2 are in normal form*:

(i) $(\lambda x^n.N_1)N_2^n \equiv \underbrace{(\lambda x.(\lambda x \cdots (\lambda x.}_{n} N_1)\underbrace{N_2 \cdots)N_2)N_2}_{n}$

where x is not in $\mathrm{FV}(N_1) \cup \mathrm{FV}(N_2)$ or $x \equiv N_1 \equiv N_2$,

(ii) $(\lambda x^n.N_1)x^{n-1}N_2 \equiv \underbrace{(\lambda x.(\lambda x \cdots (\lambda x.}_{n} N_1)\underbrace{x \cdots)x)}_{n-1}N_2$

where x is not in $\mathrm{FV}(N_1)$ or $x \equiv N_1$,

(iii) $I''N_1$.

Proof. For the proof, see Appendix A.

From Theorem 3.3 we get the following corollary.

Corollary 3.4. *If R is multisimple not simple and $R \to R'$, then R' is multisimple.*

It is easy to see that multisimple redexes have the following properties:
(a) In a multisimple R, every redex is multisimple; the innermost one is simple.
(b) If R is multisimple not simple and $R \to R'$, then
(b1) all redex occurrences in R' are residuals of redex occurrences in R,
(b2) if R has $n \geq 2$ redex occurrences, then R' has $n-1$ redex occurrences.
(c) If R is multisimple not simple with n redex occurrences, then every reduction from R reaches a normal form by n steps.
(d) No two redex occurrences in R are mutually disjoint.

Remark. We should remark that a multisimple not simple term T, as an irreducible context of multisimple redexes, can have mutually disjoint redex occurrences; but then we must have $T \to T$. This can only be the case if T is an irreducible context of some occurrences of Ω, as follows easily from the result in [12] stating that the only redex R such that $R \to R$, is Ω.

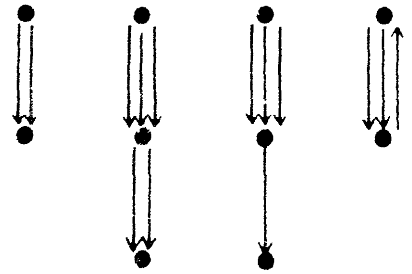
Definition 3.5. T is *hereditarily multisimple* iff every M such that $T \twoheadrightarrow M$ is multisimple.

The notion of hereditarily multisimple term allows the following characterization whose proof is obvious.

Proposition 3.6. *$G(T)$ is linear iff T is hereditarily multisimple.*

Remark 3.7. If a k-arc, $k \geq 2$, joins M to M' in $G(T)$, then $G(T)$ has at least $k+1$ nodes as can easily be proved by induction on k (see Appendix B).

By this remark an infinite number of finite or infinite pseudodigraphs are revealed as impossible reduction graphs, for instance.



So no multisimple not simple redex can be contracted (i.e., in one step reduced) to a normal form.

4. Condensed reduction graphs

In this section we will introduce the notion of 'condensed reduction graph' which is obtained from the reduction graph by dividing out the following equivalence relation.

Definition 4.1. $T \sim V$ (T and V are *cyclic equivalent*) iff $T \twoheadrightarrow V$ and $V \twoheadrightarrow T$.

Since $\sim$ is reflexive, symmetric and transitive, it is an equivalence relation. Let us denote by $T/\sim$ the equivalence class T belongs to, i.e.

$$T/\sim = \{V \mid T \sim V\}.$$

$T/\sim$ is a strong component of $G(T)$, in the terminology of [6].

Every $G(T)$ can be partitioned into $\sim$-classes. Following [8], we call every $T/\sim$ a *plane*, and every element of a plane a *point*. Of course, in particular a plane can be a singleton.

For $T/\sim$, $V/\sim$ in $G(T)/\sim$, the set of planes, we define

$T/\sim \circ\!\!\rightarrow V/\sim$ iff there exist M in $T/\sim$ and N in $V/\sim$ such that $M \to N$ and $M \not\sim N$.

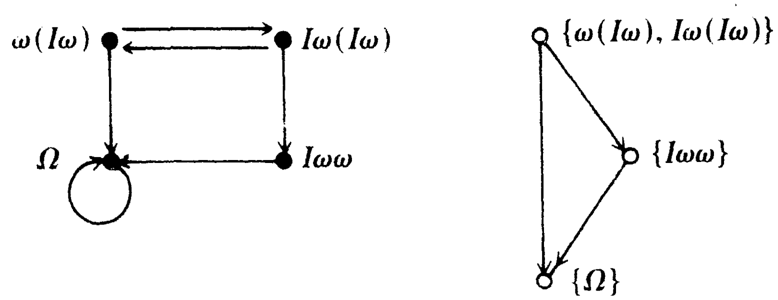
Furthermore, $\circ\!\!\twoheadrightarrow$ will be the transitive reflexive closure of $\circ\!\!\rightarrow$. Note that $\circ\!\!\twoheadrightarrow$ is a partial ordering of $G(T)/\sim$ and that the structure $(G(T)/\sim, \circ\!\!\rightarrow)$, $G^0(T)$, is a directed acyclic labelled graph. In figures, every node in $G^0(T)$ is denoted by '$\circ$' and is labelled on $M/\sim$ for some M such that $T \twoheadrightarrow M$; sometimes we will label a node in $G^0(T)$ only by M, a representant of the plane $M/\sim$.

Clearly, $G^0(T)$ is a countable graph. An interesting but probably hard question is whether $G^0(T)$ is locally finite.

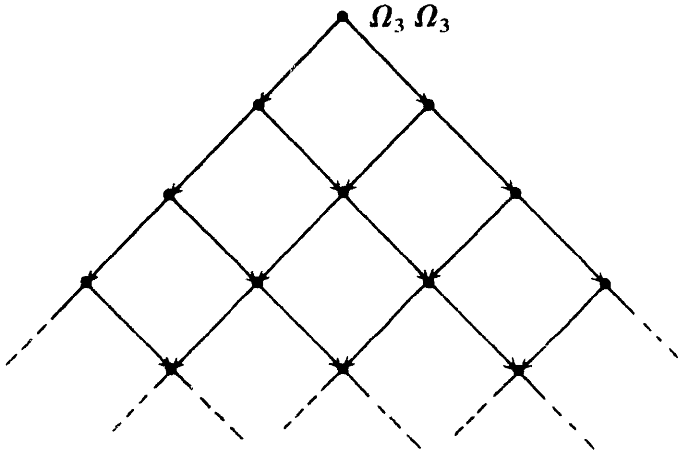
Conjecture 4.2. *$G^0(T)$ is locally finite.*

We will also refer to $G^0(T)$ as the *condensed reduction graph* of T.

An example of a reduction graph and its condensed reduction graph is given below:



It may be that $G^0(T)$ is finite and $G(T)$ is infinite: that is the case when $G(T)$ is infinite only because it contains some infinite plane. Of course in some cases $G^0(T) \equiv G(T)$: for instance, for $G(\Omega_3\Omega_3)$, $G^0(\Omega_3\Omega_3)$,



We shall denote planes by $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ and nodes in $G^0(T)$ shall also be called planes. Let $\mathscr{A} = \{\omega(I\omega), I\omega(I\omega)\}$, $\mathscr{B} = \{I\omega\omega\}$, $\mathscr{C} = \{\Omega\}$; then, in the example above, $\mathscr{C}$ is a terminal plane and $\mathscr{A}$, $\mathscr{B}$ nonterminal ones, according to the following definition.

Definition 4.3. A plane is *terminal* in $G^0(T)$ iff no arc starts from it.

Terminal planes are related to the notion of recurrence, as follows.

Definition 4.4. T is *recurrent* iff for every M such that $T \twoheadrightarrow M$ we have $M \twoheadrightarrow T$.

For properties of recurrent terms, see [11, 15, 16].

Theorem 4.5. *A plane is terminal iff it is the set of mutually reducible recurrent terms in $G(T)$.*

Proof. It is a consequence of the relative definitions.

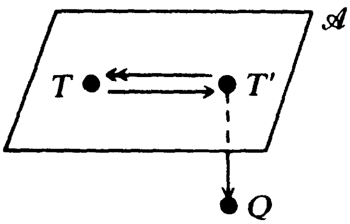
Next we will consider nonterminal planes $\mathscr{A}$. We will call M an *exit* point of $\mathscr{A}$ if for some N we have $M \to N$ and $N \notin \mathscr{A}$.

Theorem 4.6. *Let $\mathscr{A}$ be a nonterminal plane. Then $\mathscr{A}$ does not contain a multisimple term reducing in one step to an exit point of $\mathscr{A}$.*

Proof. Suppose the contrary, that is, $\mathscr{A}$ does contain a term T such that $T \to T'$ where T' is an exit point of $\mathscr{A}$.

Case (a) *T is simple.*

Let $T' \to^R Q$ for some Q not in $\mathscr{A}$, as in the following diagram (where other possible one step reducts of T' are not considered):



and let $\mathscr{D}: T' \twoheadrightarrow T$. By the 'parallel moves' lemma, $T \twoheadrightarrow^{R/\mathscr{D}} V$ for some V not in $\mathscr{A}$, against the hypothesis that T is simple (so $R/\mathscr{D}$ is a singleton) and T' is in $\mathscr{A}$.

Case (b) *T is multisimple but not simple.*

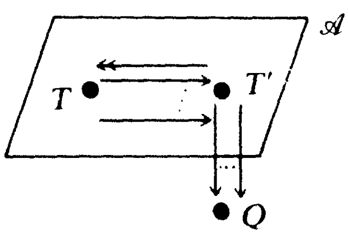
T has $n \geq 2$ redex occurrences.

Case (b1) *$T' \equiv T$ for every T' such that $T \to T'$.*

Then $\mathscr{A}$ is a singleton and terminal, contradiction.

Case (b2) *$T \to T'$ with $T' \not\equiv T$.*

Since T is multisimple, every one step reduct of T is identical to T' and by Corollary 3.4 also T' is multisimple. Thus in case $T' \to Q$, every one step reduct of T' is identical to Q, as shown in the following diagram:



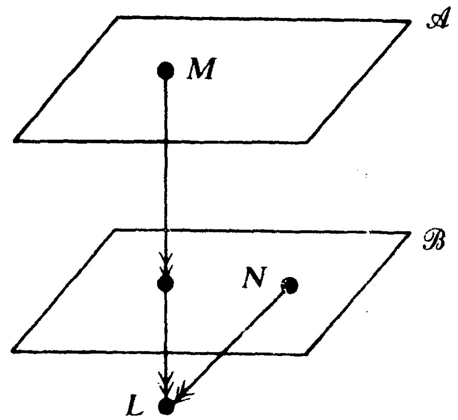
But that is impossible because if T' is multisimple, then every $\mathscr{D}: T' \twoheadrightarrow T$ has Q as its second stage (i.e., $T' \to Q \twoheadrightarrow T$) so that Q turns out to be in $\mathscr{A}$, against the hypothesis.

In [7] it is conjectured that if a plane $\mathscr{A}$ contains an exit point, then every point in $\mathscr{A}$ is an exit point. We note that Theorem 4.6 proves a particular case of this conjecture.

While the reductions in a reduction graph in general 'spread out' in lateral directions, in the notion of *bottleneck*, defined below, we are considering how 'narrow' reduction graphs can be.

Definition 4.7. Let $\mathscr{A}$, $\mathscr{B}$ be in $G(T)$. The relation "$\mathscr{B}$ is a bottleneck for $\mathscr{A}$", written $\mathrm{Bn}(\mathscr{B}, \mathscr{A})$, is defined as follows:

(i) $\mathrm{Bn}(\mathscr{B}, \mathscr{A})$ iff whenever $M \in \mathscr{A}$, $N \in \mathscr{B}$ and $N \twoheadrightarrow L$, then each reduction $\mathscr{D}: M \twoheadrightarrow L$ has some stage in $\mathscr{B}$ (see the diagram below):



(ii) Furthermore we define: $\mathscr{B}$ is a *bottleneck* of $G(T)$ iff $\mathrm{Bn}(\mathscr{B}, T/\sim)$.

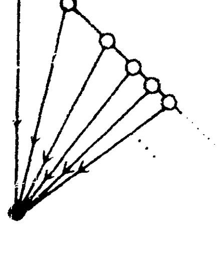
Notation. Instead of $\mathrm{Bn}(\mathscr{B}, \{M\})$ we will write $\mathrm{Bn}(\mathscr{B}, M)$; likewise for $\mathrm{Bn}(\{M\}, \mathscr{A})$.
To bottleneck planes in $G(T)$, bottleneck nodes correspond in $G^0(T)$.

Note that every $G^0(T)$ has at least one bottleneck, namely the node corresponding to the plane $T/\sim$ and in case $G(T)$ has a terminal plane (i.e. in case T has a normal form or, more general, a recurrent reduct), say $T'/\sim$, then to $T'/\sim$ also a bottleneck corresponds in $G^0(T)$. We will call the planes $T/\sim$ and, if it exists, $T'/\sim$ *improper* bottlenecks.

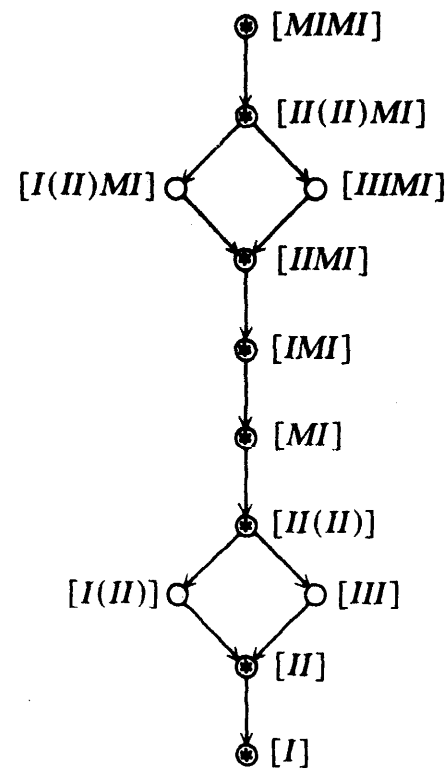
We will now give some examples of condensed reduction graphs containing bottlenecks. The first example contains only improper bottlenecks. The bottlenecks are marked in the diagrams with '⊛'.

Examples 4.8

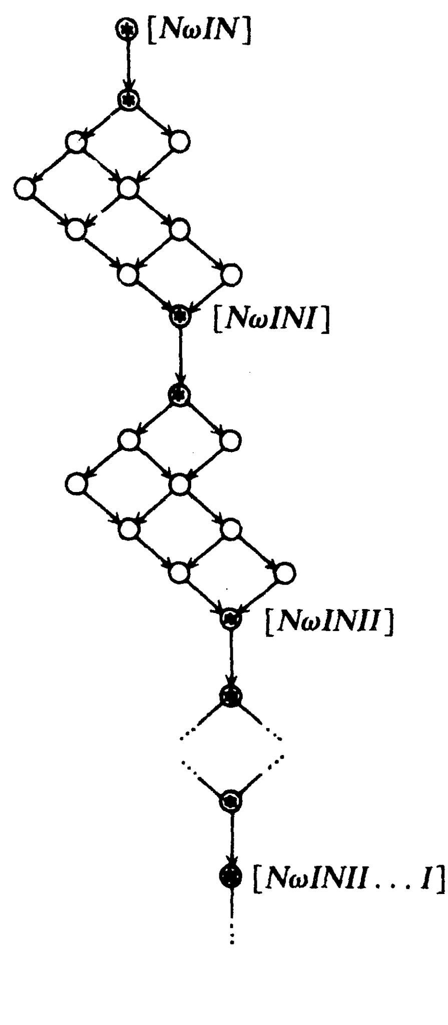
(1) $G^0((\lambda x.I)\Omega_3)$:



(2) $G^0(MIMI)$ where $M \equiv \lambda x.xI(xI)$:



(3) $G^0(N\omega IN)$ where $N \equiv \lambda xyz.xyz\omega IzI$:



In the following theorem we state some basic properties of bottlenecks.

Theorem 4.9. (i) $\mathrm{Bn}(\mathscr{B}, \mathscr{A})$ *is transitive.*

(ii) *If* $\mathrm{Bn}(\mathscr{B}, \mathscr{A})$ *and* $\mathscr{A} \circ\!\!\twoheadrightarrow M/\sim$, *then* $M/\sim \circ\!\!\twoheadrightarrow \mathscr{B}$ *or* $\mathscr{B} \circ\!\!\twoheadrightarrow M/\sim$.

(iii) *For every simple* U *there exists an infinite number of reduction graphs with* $\mathscr{A}$ *such that* $\mathrm{Bn}(U, \mathscr{A})$.

(iv) *Let* $T \twoheadrightarrow U$ *for some simple* U. *Then* $\mathrm{Bn}(U, T)$ *is undecidable.*

Proof. (i) If $\mathrm{Bn}(\mathscr{B}_1, \mathscr{A})$ and $\mathrm{Bn}(\mathscr{B}_2, \mathscr{B}_1)$, then $\mathrm{Bn}(\mathscr{B}_2, \mathscr{A})$; otherwise, for T in $\mathscr{A}$, U_2 in $\mathscr{B}_2$ and some M such that $U_2 \rightarrow^+ M$ there exists a $\mathscr{D}: T \rightarrow^+ M$ where U_2 does not occur as a stage. Then there also exists some $\mathscr{D}': U_1 \rightarrow^+ M$, with U_1 in $\mathscr{B}_1$, where U_2 does not occur as a stage, contrary to the assumption $\mathrm{Bn}(\mathscr{B}_2, \mathscr{B}_1)$.

(ii) By the Church–Rosser property there exists an N such that, for T in $\mathscr{A}$ and U in $\mathscr{B}$,

$$\begin{tikzcd} & T \arrow[dl] \arrow[dr] & \\ U \arrow[dr] & & M \arrow[dl] \\ & N & \end{tikzcd}$$

Since $\mathrm{Bn}(\mathscr{B}, T)$, every $\mathscr{D}: T \twoheadrightarrow N$ can be written as $\mathscr{D}: T \twoheadrightarrow U \twoheadrightarrow N$ for some U in $\mathscr{B}$. By hypothesis $U \not\twoheadrightarrow M$, hence every such $\mathscr{D}$ can be written as $T \twoheadrightarrow M \twoheadrightarrow U \twoheadrightarrow N$.

(iii) Let $R \equiv (\lambda x.N_1)N_2$ with N_1, N_2 in normal form, be the redex of U. Replace R in U by $I^m(\lambda x.N_1)N_2$, $m \geq 1$, and let U_m be the term so obtained. Then $\mathrm{Bn}(U, U_m)$ for every m.

(iv) Suppose the contrary is true for some simple U such that $T \twoheadrightarrow U$. Then $U \equiv C[(\lambda x.N_1)N_2]$, with N_1, N_2 in normal form. Let $\boldsymbol{n}$ represent a numeral and $\boldsymbol{f}$ strongly represent a partial recursive function with a nonrecursive domain:

$$\bar{U} \equiv C[\boldsymbol{fn}I(\lambda x.N_1)N_2] \quad \text{is such that} \quad \bar{U} \twoheadrightarrow C[I^m(\lambda x.N_1)N_2] \twoheadrightarrow U$$

for some $m > 0$ only in case $\boldsymbol{fn} \twoheadrightarrow \boldsymbol{m}$.

Since $\mathrm{Bn}(I^m(\lambda x.N_1)N_2, \bar{U})$ and, by (iii), $\mathrm{Bn}(U, I^m(\lambda x.N_1)N_2)$, we have $\mathrm{Bn}(U, \bar{U})$ iff f is defined on n.

Moreover, some information about T can be obtained by considering the bottlenecks of $G^0(T)$.

Proposition 4.10. *If* T *has normal form, then* $G^0(T)$ *has a finite number of bottlenecks.*

Proof. $G^0(T)$ cannot have an infinite number of bottlenecks because otherwise every maximal reduction from T is infinite. (A reduction is maximal iff it is infinite or reaches the normal form.)

Proposition 4.10 contrasts with the fact that the finiteness property for T of having a normal form, does not in general imply that the (condensed) reduction graph of T is finite.

The next proposition illustrates that the absence of proper bottlenecks in the condensed reduction graph $G^0(T)$ is caused by the possibility of 'independent' reductions from the initial term T.

Proposition 4.11. *Let* $T \equiv \langle M_1, M_2 \rangle$ $(\equiv \lambda x.xM_1M_2)$. *Then*:

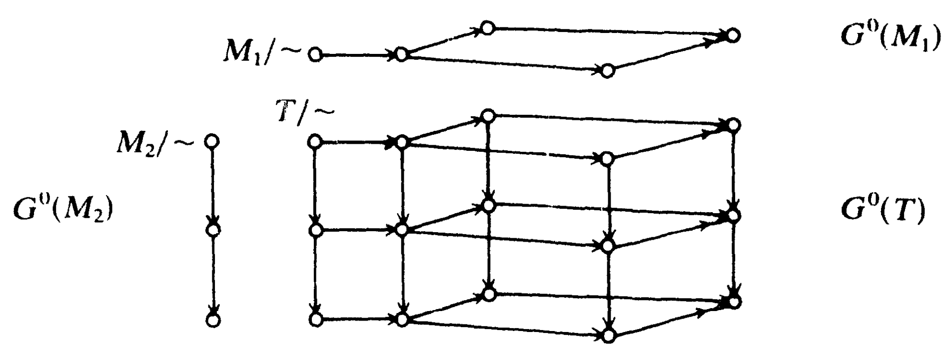
(i) *$G^0(T)$ has a proper bottleneck iff $G^0(M_1)$ has a proper bottleneck and M_2 is recurrent, or $G^0(M_2)$ has a proper bottleneck and M_1 is recurrent.*

(ii) *If $G^0(M_1)$ and $G^0(M_2)$ have no proper bottleneck, then $G^0(T)$ has no proper bottleneck.*

Proof. (ii) immediately follows from (i).

(i) The proof of ($\Leftarrow$) is evident: if, say, $G^0(M_1)$ has a proper bottleneck and M_2 is recurrent, then $G^0(\langle M_1, M_2\rangle)$ is isomorphic to $G^0(M_1)$ and hence has a proper bottleneck too.

($\Rightarrow$) Note that $G^0(\langle M_1, M_2\rangle)$ is isomorphic to $G^0(M_1)\times G^0(M_2)$, the cartesian product of $G^0(M_1)$ and $G^0(M_2)$ (see the diagram below):



From this the result easily follows.

5. Spectra

Let Red(T) be the set of all reductions, finite or infinite, starting from T. We will introduce the notion of the *spectrum* of T, which is a partial ordering of Red(T) where the ordering can be interpreted as measuring the 'significance' of reductions from T. E.g., reductions to the normal form of T, if it exists, are 'best'; in general, even if T has no normal form, cofinal reductions (see definition at the end of this section, Remark 5.10) are 'best'. The partial ordering in the spectrum measures how 'deep' reductions go into the reduction graph. This is consistent with the terminology of 'planes': a reduction which stays in a plane, does not increase the information obtained.

The present definition of 'spectrum' is inspired by [1, Exercise 13.6.5], but differs from the one given there since in [1] only maximal reductions (i.e., infinite or leading to the normal form) are considered. A related definition is found in Lévy [13], viz. of $(R^{\infty}(T), \leqslant)$; however, in [13] the partial ordering $\leqslant$ is defined on the basis of Lévy's notion of equivalence, which is not considered here.

Definition 5.1. For $\mathcal{D}_1$, $\mathcal{D}_2$ in Red(T), define

(i) $\mathcal{D}_1 \leqslant \mathcal{D}_2$ iff for every n there exists an m such that

$$T_1^{(n)} \twoheadrightarrow T_2^{(m)}, \quad \text{where } \mathcal{D}_i : T \equiv T_i^{(0)} \to T_i^{(1)} \to T_i^{(2)} \to \cdots \ (i=1,2).$$

(ii) $\mathcal{D}_1 \simeq \mathcal{D}_2$ iff $\mathcal{D}_1 \leqslant \mathcal{D}_2$ and $\mathcal{D}_2 \leqslant \mathcal{D}_1$.

Let $\tilde{\mathcal{D}} = \{\mathcal{D}' \mid \mathcal{D}' \simeq \mathcal{D}\}$ and $\widetilde{\mathrm{Red}}(T) = \{\tilde{\mathcal{D}} \mid \mathcal{D} \in \mathrm{Red}(T)\}$.

(iii) $\tilde{\mathcal{D}}_1 \leqslant \tilde{\mathcal{D}}_2$ iff $\mathcal{D}_1 \leqslant \mathcal{D}_2$.

Definition 5.2. The *spectrum* of T, Spec(T), is the p.o. $(\widetilde{\mathrm{Red}}(T), \leqslant)$.

Note that cyclic equivalent terms have isomorphic spectra.

Before relating the spectrum of a term to its condensed reduction graph, let us give some examples.

Examples 5.3

(1) Spec(Ω_3):

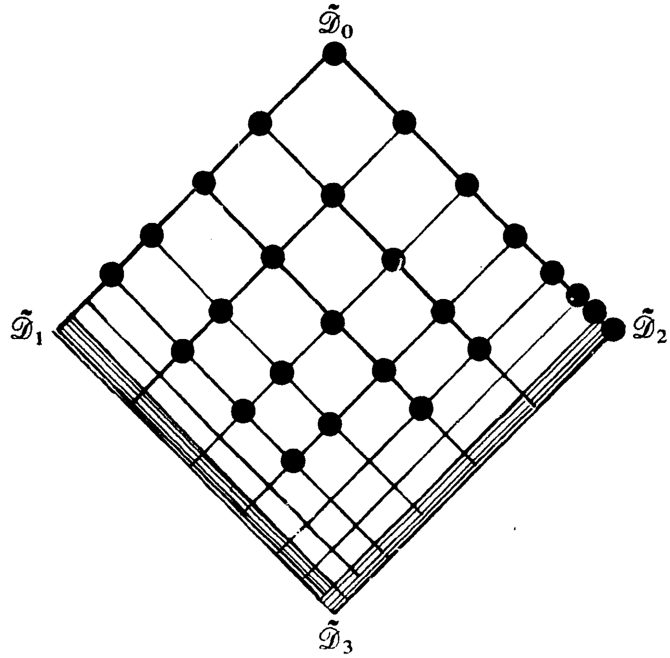
$\tilde{\mathcal{D}}_0$ where $\mathcal{D}_0 = \emptyset$, the empty reduction

$\tilde{\mathcal{D}}_1$ where $\mathcal{D}_1 = \Omega_3 \to \Omega_3\omega_3$

$\tilde{\mathcal{D}}_\omega$ where $\mathcal{D}_\omega = \Omega_3 \to \Omega_3\omega_3 \to \cdots \to \Omega_3\omega_3^n \to \cdots$

(Here MN^n is short for $MN\ldots N$ (n times N).) The spectrum is a p.o. of order type $\omega + 1$.

(2) Spec($\Omega_3\Omega_3$):



Here $\mathscr{D}_0=\emptyset$, $\mathscr{D}_1 = \Omega_3\Omega_3 \to \cdots \to \Omega_3\omega_3^n\Omega_3 \to \cdots$, $\mathscr{D}_2 = \Omega_3\Omega_3 \to \cdots \to \Omega_3(\Omega_3\omega_3^n) \to \cdots$, $\mathscr{D}_3 = \Omega_3\Omega_3 \to \cdots \to \Omega_3\omega_3^n(\Omega_3\omega_3^n) \to \cdots$.

(3) Spec($(\lambda x.I)\Omega_3$):

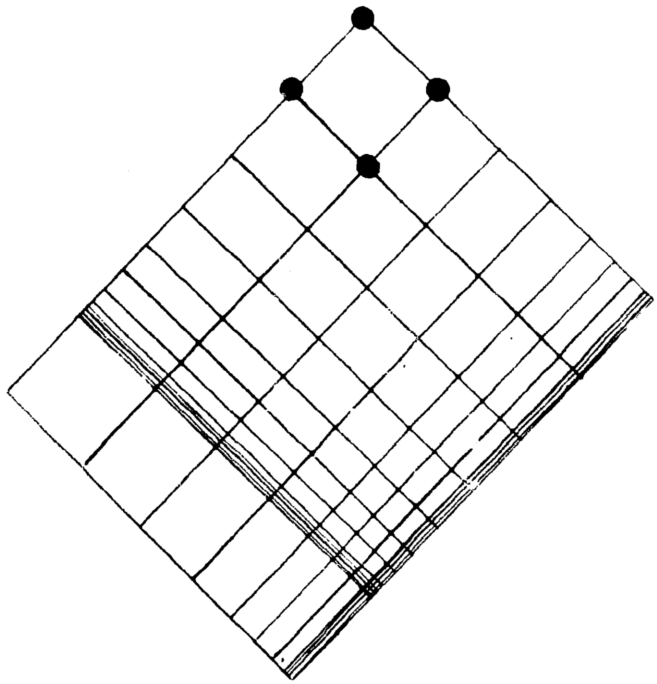
(A p.o. of order type $\omega+2$)

(4) Spec($(\lambda x.I)\Omega_3\omega_3\omega_3$):

(A p.o. of order type $\omega\cdot 2+1$)

Question. Call an ordinal α *linearly representable* by a λ-term T if α is isomorphic to $\mathrm{Spec}(T)$. Which ordinals are linearly representable?

(5) $\mathrm{Spec}((\lambda x.\Omega_3)\Omega_3)$:



In order to relate the spectrum $\mathrm{Spec}(T)$ to the condensed reduction graph $G^0(T)$, we define the *ω-completion* of $G^0(T)$, written as $\overline{G^0(T)}$. To this end, we use the construction in Bloom [2].

Definition 5.4. (i) $\mathrm{Ch}(T)$ is the set of ω-chains in the p.o. $(G^0(T), \circ\!\!\twoheadrightarrow)$.

(ii) Let C, C' be in $\mathrm{Ch}(T)$, where

$$C: M_0/\sim\ \circ\!\!\twoheadrightarrow M_1/\sim\ \circ\!\!\twoheadrightarrow M_2/\sim\ \circ\!\!\twoheadrightarrow \cdots$$

and

$$C': M'_0/\sim\ \circ\!\!\twoheadrightarrow M'_1/\sim\ \circ\!\!\twoheadrightarrow M'_2/\sim\ \circ\!\!\twoheadrightarrow \cdots;$$

here $M_0 \equiv M'_0 \equiv T$.

We define

$$\begin{aligned} C \sqsubseteq C' \quad &\text{iff} \quad \text{for every } n \text{ there exists an } m \text{ such} \\ &\qquad \text{that } M_n/\sim\ \circ\!\!\twoheadrightarrow M'_m/\sim, \\ C \equiv C' \quad &\text{iff} \quad C \sqsubseteq C' \text{ and } C' \sqsubseteq C. \end{aligned}$$

Furthermore, $C/\equiv\ = \{C' \mid C' \equiv C\}$.

(iii) $\overline{G^0(T)} = \{C/\equiv \mid C \in \mathrm{Ch}(T)\}$. Par abus de langage, we will use the notation $\sqsubseteq$ also for the p.o. determined on $\overline{G^0(T)}$ by the p.o. $\sqsubseteq$ on $\mathrm{Ch}(T)$.

Note that $G^0(T)$ is isomorphically embedded in $\overline{G^0(T)}$ by the map $\iota: G^0(T) \to \overline{G^0(T)}$ such that

$$\iota(M/\sim) = C_M/\equiv, \quad \text{where } C_M : T/\sim \;\circ\!\!\twoheadrightarrow M/\sim \;\circ\!\!\twoheadrightarrow M/\sim \;\circ\!\!\twoheadrightarrow \cdots.$$

Intuitively, $\overline{G^0(T)}$ is $G^0(T)$ augmented by limits for ω-chains in $G^0(T)$ in such a way that limits of ω-chains that are equivalent in the sense of Definition 5.4, are identified. We will call the set of 'new' elements in $\overline{G^0(T)}$ (i.e., $\overline{G^0(T)} \setminus G^0(T)$), the *boundary* of $G^0(T)$.

Note also that $\overline{G^0(T)}$ is a cpo (complete partial order), i.e., a partial order where all ω-chains have a 'lub'.

We now have the following theorem.

Theorem 5.5. Spec(T) *is isomorphic to* $\overline{G^0(T)}$ *(as partially ordered structures).*

Proof. We define $\varphi : \text{Spec}(T) \to \overline{G^0(T)}$ as follows. Let $\mathcal{D} \in \text{Red}(T)$.

Case 1. $\mathcal{D}$ is infinite: $\mathcal{D}$ is $M_0 \to M_1 \to M_2 \to \cdots$, where $M_0 \equiv T$. Then

$$\varphi(\mathcal{D}) = C/\equiv, \quad \text{where } C \text{ is } M_0/\sim \;\circ\!\!\twoheadrightarrow M_1/\sim \;\circ\!\!\twoheadrightarrow M_2/\sim \;\circ\!\!\twoheadrightarrow \cdots.$$

Case 2. $\mathcal{D}$ is $M_0 \to M_1 \to \cdots \to M_n$ for some $n \geq 0$. Here $M_0 \equiv T$. Then

$$\varphi(\mathcal{D}) = C/\equiv, \quad \text{where } C \text{ is } M_0/\sim \;\circ\!\!\twoheadrightarrow M_1/\sim \;\circ\!\!\twoheadrightarrow \cdots \;\circ\!\!\twoheadrightarrow M_n/\sim \;\circ\!\!\twoheadrightarrow M_n/\sim \;\circ\!\!\twoheadrightarrow M_n/\sim \;\circ\!\!\twoheadrightarrow \cdots.$$

The proof that φ is indeed an order-preserving isomorphism is routine and left to the reader.

In order to formulate the next proposition, we define a reduction $\mathcal{D}: M^{(0)} \to M^{(1)} \to \cdots$ to be *eventually flat* iff $\mathcal{D}$ is finite or if $\mathcal{D}$ is lying almost entirely in a plane, i.e., there exists an index k such that for all m, n greater than k it holds that $M^{(m)} \sim M^{(n)}$.

Proposition 5.6. (i) $\|\text{Spec}(T)\| = 1$ *iff* T *is recurrent.*

(ii) $G^0(T)$ *is isomorphic to* Spec(T) *iff* $G^0(T)$ *is finite iff all reductions are eventually flat.*

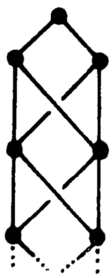
(iii) *If* $G^0(T)$ *has infinitely many bottlenecks, then the boundary of* $G^0(T)$ *is a singleton.*

Proof. (i) and (ii) follow by the relevant definitions.

(iii) Suppose $G^0(T)$ has infinitely many bottlenecks. Then all infinite reductions which are not eventually flat, must pass all these bottlenecks; hence they meet each other infinitely often and therefore they are equivalent.

Conjecture 5.7. *The reverse of Proposition* 5.6 (iii) *also holds: If the boundary of* $G^0(T)$ *is a singleton, then* $G^0(T)$ *has infinitely many bottlenecks.*

Conjecture 5.7 says roughly that a reduction graph which has infinitely many planes either is very 'narrow' (containing infinitely many bottlenecks) or else it must spread out (by the presence of incomparable points in the boundary). E.g., according to the conjecture a 'braid' like



cannot occur as a reduction graph.

Conjecture 5.8 (Klop [9]). *Note (in Examples* 5.3(3), (4), (5) *above) that the elements in the boundary are not always maximal. It is conjectured that in the* λI*-calculus, where there is no possibility of erasing subterms, the elements in the boundary of the spectrum are always maximal.*

In general Spec(T) can be quite complex. In fact one can show the following theorem.

Theorem 5.9 (Klop [10]). *There exists a* λ*-term* T *such that*

(i) $(\mathcal{P}\omega, \subseteq)$, *the powerset of the natural numbers (ordered by inclusion), can be isomorphically embedded in* Spec(T).

(ii) Spec(T) *contains a densely ordered segment.*

Proof. (i) Let T be a term such that $T \twoheadrightarrow \langle II, T\rangle$ where $\langle A, B\rangle$ abbreviates $\lambda z.zAB$. Abbreviate II by 1 and I by 0. Furthermore, let us simply write AB for $\langle A, B\rangle$, ABC for $\langle A, B, C\rangle$, etc. So $T \twoheadrightarrow 1T \twoheadrightarrow 11T \twoheadrightarrow \cdots$, and because $1 \to 0$, we have, for every finite sequence σ of 0's and 1's, $T \twoheadrightarrow \sigma M$.

We want to embed $(\mathcal{P}\omega, \subseteq)$ into Spec(T). Let $X \in \mathcal{P}\omega$, and let k_X be the characteristic function of X:

$$k_X(n) = \begin{cases} 0 & \text{if } n \notin X, \\ 1 & \text{otherwise.} \end{cases}$$

Write k_X as an infinite sequence of 0, 1, and let σ_m be the finite prefix of that sequence up to m. E.g., if $k_X = (0, 0, 1, 1, 0, \ldots)$, then $\sigma_1 = 0$, $\sigma_2 = 00$, $\sigma_3 = 001, \ldots$.

Now we let to $X \in \mathcal{P}\omega$ correspond $\mathcal{D}_X : T \twoheadrightarrow \sigma_1 T \twoheadrightarrow \sigma_2 T \twoheadrightarrow \cdots$ (modulo the spectrum equivalence). Evidently, if $X, X' \in \mathcal{P}\omega$ and $X \supseteq X'$, then the sequence $k_{X'}$ has more 0's. Since $1 \to 0$, we have therefore $\mathcal{D}_X \geq \mathcal{D}_{X'}$ in the spectrum ordering.

Hence $(\mathcal{P}\omega, \subseteq)$ is embedded in Spec(T). Taking $T \equiv (\lambda xz.z1(xx))(\lambda xz.z1(xx))$ we even have that $(\mathcal{P}\omega, \subseteq)$ is isomorphic to the boundary of Spec(T).

(ii) *Claim*: $(\mathcal{P}\omega, \subseteq)$ *contains a densely ordered segment, i.e., a linearly ordered subset* $\mathcal{X} \subseteq \mathcal{P}\omega$ *such that if* $X, Y \in \mathcal{X}$ *and* $X \subsetneq Y$, *then there exists a* $Z \in \mathcal{X}$ *such that*, $X \subsetneq Z \subsetneq Y$.

Together with (i) this claim yields the result in (ii).

Proof of the claim. Let φ be a bijection from $\mathbb{Q}$, the set of rational numbers, to $\mathbb{N}$ $(=\omega)$, the set of natural numbers. To each $r \in \mathbb{Q}$ we associate the set $\{q \in \mathbb{Q} \mid q \leq r\}$; call this set $\mathbb{Q}_r$. Now the collection of all $\varphi(\mathbb{Q}_r)$ is a densely ordered segment in $(\mathcal{P}\omega, \subseteq)$, since $\mathbb{Q}$ is densely ordered.

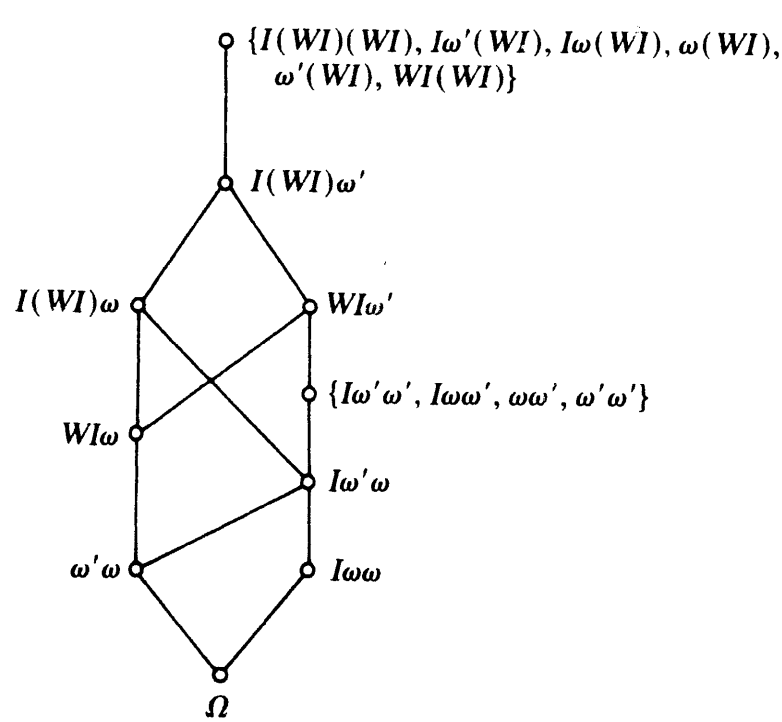
Remark 5.10. By Theorem 5.5 and by the construction of $G^0(T)$, we have that Spec(T) is a cpo, with the empty reduction as 'bottom' element. Moreover, Spec(T) has always a unique maximal element. This is the equivalence class of *cofinal* reductions (a reduction $\mathcal{D}: M_0 \to M_1 \to \cdots$ where $M_0 \equiv T$, is called cofinal if for every U in $G(T)$ there is an M_n such that $U \twoheadrightarrow M_n$. So, in particular, if T has a normal form, then $\mathcal{D}: T \to \cdots$ is cofinal iff it reaches the normal form.)

We conclude this section with the observation that Spec(T) does not need in general to be a lattice (which it is in Examples 5.3(1)–(5) above), as the following example shows.

Examples 5.3 (*continued*)

(6) Spec$(\omega(WI))$ and $G^0(\omega(WI))$, where $W = \lambda xy.xyy$

(In the diagram below $\omega' \equiv \lambda y.Iyy$)



Acknowledgment

I am grateful to Alberto Pettorossi for useful discussions on graphs and to Jan Willem Klop for his helpful suggestions: in particular the section on spectra could never have been written without them.

I would also like to acknowledge some useful comments of an anonymous referee.

Appendix A. Proof of Theorem 3.3(1)

If part. This follows by checking that in all cases R' such that $R \to R'$ does not depend on the contracted redex.

Only if part. This follows by induction on the number of redex occurrences in R.

First Step. $R \equiv (\lambda x.P)Q$ with one redex occurrence in P or in Q (not in both).

Case (a). P is not in normal form.

Then Q is in normal form, say N. By hypothesis, $(\lambda x.P')N \equiv P[N/x]$ with $P \to P'$.

Since P has as leftmost symbol either a free variable, say y, or a bound variable, say λy, there are the following cases only.

Case (a1). P has leftmost symbol y.

Case (a1.1). $y \not\equiv x$.

This case is impossible because identity cannot hold between two terms (both contracta of R) one of which starts by λ and the other one by y.

Case (a1.2). $y \equiv x$.

Then

$$P \equiv xM_1M_2 \cdots M_k \quad \text{for some } M_1, \ldots, M_k, \qquad k \geq 1,$$

and

$$(\lambda x.P')N \equiv (xM_1 \cdots M_k)[N/x] \equiv N(M_1[N/x] \cdots (M_k[N/x]).$$

Since the number of brackets to the left of $(\lambda x.P')$ and to the left of N has to be the same, $k = 1$ and $(\lambda x.P')N \equiv N(M_1(N/x])$ for some reducible $M_1[N/x]$. Then this case is impossible because the normal form N should be identical to a reducible term.

Case (a2). P has leftmost symbol λy.

Case (a2.1). $P \equiv \lambda y.M_1 \cdots M_k$, for some $M_1, \ldots, M_k$, $k \geq 1$.

Then $(\lambda x.P')N \equiv \lambda y.M_1[N/x] \cdots M_k[N/x]$, up to α-conversion, so that such a case is impossible.

Case (a2.2). $P \equiv (\lambda y.N_1)N_2 \cdots N_k$, $k \geq 2$, for some $N_1, \ldots, N_k$ in normal form.

Then $R \equiv (\lambda x.(\lambda y.N_1)N_2 \cdots N_k)N$ and from

(i) $\quad R \to (\lambda y.N_1[N/x])N_2[N/x]N_3[N/x] \cdots N_k[N/x]$,

(ii) $\quad R \to (\lambda x.N_1[N_2/y]N_3 \cdots N_k)N$,

where (i) $\equiv$ (ii) up to α-conversion, it follows that $k = 2$ and

- if x is not in $\mathrm{FV}(N_1) \cup \mathrm{FV}(N_2)$ or $x \equiv N_1 \equiv N_2$ and $N_2 \equiv N$, then $R \equiv (\lambda x.(\lambda x.N_1)N_2)N_2 \equiv (\lambda x^2.N_1)N_2^2$, i.e., R has form (i) of Theorem 3.3,

- if $x \equiv N_2$ and x is not in $\mathrm{FV}(N_1)$ or $x \equiv N_1$, then $R \equiv (\lambda x.(\lambda x.N_1)x)N \equiv \lambda x^2.N_1 xN$, i.e., R has form (ii) of Theorem 3.3.

We can conclude that, in Case (a), R has form (i) or (ii) of Theorem 3.3.

Case (b). Q is reducible.

Then P is in normal form. By hypothesis, $(\lambda x.P)Q' \equiv P[Q/x]$ with $Q \to Q'$ and x is in $\mathrm{FV}(P)$; otherwise $(\lambda x.P)Q' \equiv P$.

Case (b1). $P \equiv yN_1 \ldots N_k$, $k \geqslant 1$, for some $N_1, \ldots, N_k$, if any, in normal form.

Case (b1.1). $y \not\equiv x$.

Follows in the same way as Case (a1.1).

Case (b1.2). $y \equiv x$.

Then $(\lambda x.xN_1 \ldots N_k)Q' \equiv Q(N_1[Q/x]) \ldots (N_k[Q/x])$ so that $k=1$ and $\lambda x.xN_1 \ldots N_k \equiv Q$, while Q is reducible.

There remains the case $P \equiv x$ and then $(\lambda x.x)Q' \equiv Q$ so that

$$R \equiv (\lambda x.P)Q \equiv (\lambda x.x)Q \equiv (\lambda x.x)((\lambda x.x)Q'),$$

where Q' is in normal form because R contains only two redex occurrences.

Hence $R \equiv (\lambda x.x)((\lambda x.x)N)$ for some N in normal form.

We can conclude that, in Case (b), R has form (iii) of Theorem 3.3.

Inductive Step. Let R^i stand for a multisimple (not simple) redex with $i \geqslant 2$ redex occurrences.

$R^{i+1} \equiv (\lambda x.P)Q$, for some reducible P or reducible Q (not both), otherwise $(R^{i+1})' \equiv (\lambda x.P')Q \equiv (\lambda x.P)Q'$, i.e., $P \equiv P'$, $Q \equiv Q'$ and $(R^{i+1})' = R^{i+1}$, while $(R^i)' \not\equiv R^i$ for every i.

Thus there are only Case (a) and Case (b).

Case (a). $R^{i+1} \equiv (\lambda x.C[R^i])N$ for some N in normal form.

Case (a1). $C[R^i]$ has leftmost symbol y.

Case (a1.1). $y \not\equiv x$.

Follows in the same way as Case (a1.1) of the First Step.

Case (a1.2). $y \equiv x$.

Follows in the same way as Case (a1.2) of the First Step.

Case (a2). $C[R^i]$ has leftmost symbol λy.

Case (a2.1). $C[R^i] \equiv \lambda y.C_1[R^i]$.

Then $(R^{i+1})' \equiv (\lambda x.(C[R^i])')N \equiv \lambda y.(C_1[R^i])[N/x]$ up to α-conversion, so that such a subcase is impossible.

Case (a2.2)

$$C[R^i] \equiv \underbrace{(\lambda y.(\ \ldots\ ))N_2}_{R^i}N_3 \ldots N_k, \quad k \geqslant 2,$$

for some $N_2, \ldots, N_k$ in normal form.

Then $R^{i+1} \equiv (\lambda x.(\lambda y.(\ \ldots\ ))N_2N_3 \ldots N_k)N$. From

(1) $\quad R^{i+1} \to (\lambda y.(\ \ldots\ )[N/x])N_2[N/x]N_3[N/x] \ldots N_k[N/x]$,

(2) $\quad R^{i+1} \to (\lambda x.(\ \ldots\ )[N_2/y])N_3 \ldots N_k)N$,

where $(1) \equiv (2)$ up to α-conversion, it follows that $k = 2$, i.e., for $y \equiv x$,

(1) $(R^{i+1})' \equiv (\lambda x.(\dots)[N/x])N_2[N/x]$,

(2) $(R^{i+1})' = (\lambda x.(\dots)[N_2/x])N$,

so that

- if $\lambda x.(\dots) \equiv \lambda x^{i-1} \cdot N_1 N_2^{i-1}$, i.e., it has form (i) of Theorem 3.3, then, for $N \equiv N_2$ and x not in $\mathrm{FV}(N_1) \cup \mathrm{FV}(N_2)$,

$$R^{i+1} \equiv (\lambda x.(\lambda x.(\lambda x^{i-1}.N_1 N_2^{i-1})N_2))N_2 \equiv \lambda x^{i+1}.N_1 N_2^{i+1},$$

i.e., R^{i+1} has form (i) of Theorem 3.3,

- if $\lambda x.(\dots) \equiv \lambda x^{i-1}.N_1 x^{i-1}$, i.e., it has form (ii) of Theorem 3.3, then for $x \equiv N_2$ and x not in $\mathrm{FV}(N_1)$ or $x \equiv N_1$,

$$R^{i+1} \equiv (\lambda x.(\lambda x.(\lambda x^{i-1}.N_1 x^{i-1})x))N \equiv \lambda x^{i+1}.N_1 x^i N,$$

i.e., R^{i+1} has form (ii) of Theorem 3.3,

- $\lambda x.(\dots) \equiv (\lambda x.x)R^{i-2}$ is impossible.

We can conclude that, in Case (a), R^{i+1} has form (i) or (ii) of Theorem 3.3.

Case (b). $R^{i+1} \equiv (\lambda x.N)C[R^i]$ for some N in normal form.

Then $(R^{i+1})' \equiv (\lambda x.N)(C[R^i])' = N[C[R^i]/x]$ is satisfied by $N \equiv x$, so that

$$(R^{i+1})' \equiv (\lambda x.x)(C[R^i])' \equiv C[R^i] \quad \text{and} \quad R^{i+1} \equiv (\lambda x.x)C[R^i].$$

If $C[R^i] \equiv [R^i] \equiv (\lambda x.x)^i N$, then R^{i+1} has form (iii) of Theorem 3.3. Moreover, $C[R^i]$ cannot be as in (i) or (ii) of Theorem 3.3 because in those cases all contracta of R^{i+1} are not the same.

Appendix B. Proof of Remark 3.7

The proof follows by induction on k, $k \geq 2$.

First Step. $k = 2$.

By hypothesis, $M \equiv C[R]$ with a multisimple R having two redex occurrences. So $M \to M'$, $M' \equiv C[R']$, $R' \not\equiv R$, $R \to R'$, R' a simple redex and $R' \not\equiv \Omega$ by Theorem 3.3. Then there exists an M'' such that $M' \to M''$ and $M'' \not\equiv M'$. Therefore, $G(M)$ has at least three nodes.

Inductive Step. By the inductive hypothesis, if M is joined to M' by a $(k-1)$-arc, then $G(M)$ has at least k nodes. Let M be joined to M' by a k-arc. By properties (b) and (d) of multisimple redexes, M' is joined to some M'' by a $(k-1)$-arc. Since the inductive hypothesis holds for $G(M')$, $G(M)$ has at least $k+1$ nodes.

References

[1] H.P. Barendregt, *The Lambda Calculus. Its Syntax and Semantics* (North-Holland, Amsterdam, 1981).

[2] S. Bloom, Varieties of ordered algebras, *CSS* **13** (1976) 200–212.
[3] A. Church, *The Calculi of Lambda-Conversion* (Princeton University Press, 1941).
[4] H.B. Curry and R. Feys, *Combinatory Logic, Vol. I* (North-Holland, Amsterdam, 1958).
[5] H.B. Curry, J.R. Hindley and J.P. Seldin, *Combinatory Logic, Vol. II* (North-Holland, Amsterdam, 1972).
[6] F. Harary, *Graph Theory* (Addison-Wesley, Reading, MA, 1969).
[7] J.W. Klop, Reduction cycles in combinatory logic, in: J.P. Seldin and J.R. Hindley, eds., *To H.B. Curry: Essays on Combinatory Logic, Lambda Calculus and Formalism* (Academic Press, London, 1980).
[8] J.W. Klop, *Combinatory Reduction Systems*, Mathematical Centre Tracts **127** (Mathematical Centre, Amsterdam, 1980).
[9] J.W. Klop, Personal communication, 1982.
[10] J.W. Klop, Personal communication, 1982.
[11] G. Jacopini, and M. Venturini Zilli, Equating for recurrent terms of λ-calculus and combinatory logic, *Quaderno IAC* **III** (85) (1978).
[12] B. Lercher, Lambda-calculus terms that reduce to themselves, *Notre Dame J. Formal Logic* **XVII** (1976) 291–292.
[13] J.J. Lévy, Réductions correctes et optimales dans le lambda-calculi, Ph.D. Thesis, Paris, 1978.
[14] C. St. J.A. Nash-Williams, Infinite graphs: A survey, *J. Combin. Theory* **3** (1967) 286–301.
[15] M. Venturini Zilli, Head recurrent terms in combinatory logic: A generalization of the notion of head normal form, in: G. Ausiello and C. Böhm, eds., *Automata, Languages and Programming* (Springer, Berlin, 1978) pp. 477–493.
[16] M. Venturini Zilli, Recurrence and headrecurrence in combinatory logic and λ-calculus, IAC Intern. Rept., 1981.
[17] M. Venturini Zilli, Cofinality in reduction graphs, in: G. Aussiello and M. Protasi, eds., *Trees in Algebra and Programming* (Springer, Berlin, 1983) pp. 405–416.